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Geometric embedding properties of Bestvina–Brady subgroups

HUNG CONG TRAN

We compute the relative divergence of right-angled Artin groups with respect to their Bestvina–Brady subgroups and the subgroup distortion of Bestvina–Brady subgroups. We also show that for each integer $n \geq 3$, there is a free subgroup of rank n of some right-angled Artin group whose inclusion is not a quasi-isometric embedding. The corollary answers the question of Carr about the minimum rank n such that some right-angled Artin group has a free subgroup of rank n whose inclusion is not a quasi-isometric embedding. It is well known that a right-angled Artin group A_Γ is the fundamental group of a graph manifold whenever the defining graph Γ is a tree with at least three vertices. We show that the Bestvina–Brady subgroup H_Γ in this case is a horizontal surface subgroup.

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1 Introduction

For each Γ a finite simplicial graph, the associated *right-angled Artin group* A_Γ has generating set S the vertices of Γ, and relations $st = ts$ whenever s and t are adjacent vertices. If Γ is nonempty, there is a homomorphism from A_Γ onto the integers that takes every generator to 1. The *Bestvina–Brady subgroup* H_Γ is defined to be the kernel of this homomorphism.

Bestvina–Brady subgroups were introduced by Bestvina and Brady [2] to study the finiteness properties of subgroups of right-angled Artin groups. One result in [2] is that the Bestvina–Brady subgroup H_Γ is finitely generated if and only if the graph Γ is connected. This fact is a motivation to study the geometric connection between a right-angled Artin group and its Bestvina–Brady subgroup. More precisely, we examine the relative divergence of right-angled Artin groups with respect to their Bestvina–Brady subgroups and the subgroup distortion of Bestvina–Brady subgroups.

Theorem 1.1 *Let Γ be a connected, finite, simplicial graph with at least two vertices. Let A_Γ be the associated right-angled Artin group and H_Γ the Bestvina–Brady subgroup. Then the relative divergence $\mathrm{Div}(A_\Gamma, H_\Gamma)$ and the subgroup distortion $\mathrm{Dist}_{A_\Gamma}^{H_\Gamma}$ are both linear if Γ is a join graph. Otherwise, they are both quadratic.*

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In the above theorem, we can see that the relative divergence $\mathrm{Div}(A_\Gamma, H_\Gamma)$ and the subgroup distortion $\mathrm{Dist}_{A_\Gamma}^{H_\Gamma}$ are equivalent. In general, we show that the relative divergence is always dominated by the subgroup distortion for any pair of finitely generated groups (G, H), where H is a normal subgroup of G such that the quotient group G/H is an infinite cyclic group see Proposition 4.3.

Carr [3] proved that nonabelian two-generator subgroups of right-angled Artin groups are quasi-isometrically embedded free groups. In his paper, he also showed an example of a distorted free subgroup of a right-angled Artin group. However, the minimum rank n such that some right-angled Artin group has a free subgroup of rank n whose inclusion is not a quasi-isometric embedding was still unknown; see [3]. The following corollary of Theorem 1.1 answers this question.

Corollary 1.2 *For each integer $n \geq 3$, there is a right-angled Artin group containing a free subgroup of rank n whose inclusion is not a quasi-isometric embedding.*

We remark that a special case of Theorem 1.1 can also be derived as a consequence of previous work by Hruska and Nguyen [6] on distortion of surfaces in graph manifolds. They showed that every virtually embedded horizontal surface in a 3–dimensional graph manifold has quadratic distortion. This led us to prove the following theorem, which implies that many Bestvina–Brady subgroups are also horizontal surface subgroups.

Theorem 1.3 *If Γ is a finite tree with at least three vertices, then the associated right-angled Artin group A_Γ is a fundamental group of a graph manifold, and the Bestvina–Brady subgroup H_Γ is a horizontal surface subgroup.*

It is well known that a right-angled Artin group A_Γ is the fundamental group of a graph manifold whenever the defining graph Γ is a tree with at least three vertices. However, the fact that the Bestvina–Brady subgroup H_Γ is a horizontal subgroup does not seem to be recorded in the literature. With the use of Theorem 1.3, we see that Theorem 1.1 can be viewed as a generalization of a special case of the quadratic distortion theorem of Hruska and Nguyen. Moreover, Theorem 1.3 combined with the Hruska–Nguyen theorem gives an alternative proof of Corollary 1.2.

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2 Right-angled Artin groups and Bestvina–Brady subgroups

Definition 2.1 Given a finite simplicial graph Γ, the associated *right-angled Artin group* A_Γ has generating set S the vertices of Γ, and relations $st = ts$ whenever s and t are adjacent vertices.

Let S_1 be a subset of S. The subgroup of A_Γ generated by S_1 is a right-angled Artin group A_{Γ_1}, where Γ_1 is the induced subgraph of Γ with vertex set S_1 (ie Γ_1 is the union of all edges of Γ with both endpoints in S_1). The subgroup A_{Γ_1} is called a *special subgroup* of A_Γ.

Definition 2.2 Let Γ be a finite simplicial graph with the set S of vertices. Let T be a torus of dimension $|S|$ with edges labeled by the elements of S. Let X_Γ denote the subcomplex of T consisting of all faces whose edge labels span a complete subgraph in Γ (or equivalently, mutually commute in A_Γ). X_Γ is called the *Salvetti complex*.

Remark 2.3 The fundamental group of X_Γ is A_Γ. The universal cover $\widetilde{X}_\Gamma$ of X_Γ is a CAT(0) cube complex with a free, cocompact action of A_Γ. Obviously, the 1–skeleton of $\widetilde{X}_\Gamma$ is the Cayley graph of A_Γ with respect to the generating set S.

Definition 2.4 Let Γ be a finite simplicial graph. Let $\Phi\colon A_\Gamma \to \mathbb{Z}$ be an epimorphism which sends all the generators of A_Γ to 1 in $\mathbb{Z}$. The kernel H_Γ of Φ is called the *Bestvina–Brady subgroup*.

Remark 2.5 There is a natural continuous map $f\colon X_\Gamma \to S^1$ which induces the homomorphism $\Phi\colon A_\Gamma \to \mathbb{Z}$. Moreover, it is not hard to see that the lifting map $\widetilde{f}\colon \widetilde{X}_\Gamma \to \mathbb{R}$ is an extension of Φ.

Theorem 2.6 (Bestvina and Brady [2] and Dicks and Leary [4]) *Let Γ be a finite simplicial graph. The Bestvina–Brady subgroup H_Γ is finitely generated if and only if Γ is connected. Moreover, the set T of all elements of the form st^{-1} whenever s and t are adjacent vertices form a finite generating set for H_Γ. Furthermore, if Γ is a tree with n edges, then the Bestvina–Brady subgroup H_Γ is a free group of rank n.*

Definition 2.7 Let Γ_1 and Γ_2 be two graphs; the *join* of Γ_1 and Γ_2 is a graph obtained by connecting every vertex of Γ_1 to every vertex of Γ_2 by an edge.

Let J be a complete subgraph of Γ which decomposes as a nontrivial join. We call A_J a *join subgroup* of A_Γ.

Let Γ be a finite simplicial graph with vertex set S, and let g an element of A_Γ. A *reduced word* for g is a minimal-length word in the free group $F(S)$ representing g.

Given an arbitrary word representing g, one can obtain a reduced word by a process of "shuffling" (ie interchanging commuting elements) and canceling inverse pairs. Any two reduced words for g differ only by shuffling. For an element $g \in A_\Gamma$, a cyclic reduction of g is a minimal-length element of the conjugacy class of g. If w is a reduced word representing g, then we can find a cyclic reduction $\overline{g}$ by shuffling commuting generators in w to get a maximal-length word u such that $w = u\bar{w}u^{-1}$. In particular, g itself is *cyclically reduced* if and only if every shuffle of w is cyclically reduced as a word in the free group $F(S)$.

3 Relative divergence, geodesic divergence and subgroup distortion

Before we define the concepts of relative divergence, geodesic divergence and subgroup distortion, we need to build the tools to measure them, namely the notions of domination and equivalence.

Definition 3.1 Let $\mathcal{M}$ be the collection of all functions from $[0,\infty)$ to $[0,\infty]$. Let f and g be arbitrary elements of $\mathcal{M}$. *The function f is dominated by the function g*, denoted by $f \preceq g$, if there are positive constants A, B, C and D such that $f(x) \leq Ag(Bx) + Cx$ for all $x > D$. Two functions f and g are *equivalent*, denoted by $f \sim g$, if $f \preceq g$ and $g \preceq f$.

Remark 3.2 A function f in $\mathcal{M}$ is *linear, quadratic or exponential* if f is respectively equivalent to a degree-one polynomial, a degree-two polynomial or a function of the form a^{bx+c}, where $a > 1$ and $b > 0$.

Definition 3.3 Let $\{\delta^n_\rho\}$ and $\{\delta'^n_\rho\}$ be two families of functions of $\mathcal{M}$, indexed over $\rho \in (0,1]$ and positive integers $n \geq 2$. *The family $\{\delta^n_\rho\}$ is dominated by the family $\{\delta'^n_\rho\}$*, denoted by $\{\delta^n_\rho\} \preceq \{\delta'^n_\rho\}$, if there exists a constant $L \in (0,1]$ and a positive integer M such that $\delta^n_{L\rho} \preceq \delta'^{Mn}_\rho$ for all ρ and n. Two families $\{\delta^n_\rho\}$ and $\{\delta'^n_\rho\}$ are *equivalent*, denoted by $\{\delta^n_\rho\} \sim \{\delta'^n_\rho\}$, if $\{\delta^n_\rho\} \preceq \{\delta'^n_\rho\}$ and $\{\delta'^n_\rho\} \preceq \{\delta^n_\rho\}$.

Remark 3.4 A family $\{\delta^n_\rho\}$ is dominated by (or dominates) a function f in $\mathcal{M}$ if $\{\delta^n_\rho\}$ is dominated by (or dominates) the family $\{\delta'^n_\rho\}$ where $\delta'^n_\rho = f$ for all ρ and n. The equivalence between a family $\{\delta^n_\rho\}$ and a function f in $\mathcal{M}$ can be defined similarly. Thus, a family $\{\delta^n_\rho\}$ is linear, quadratic, exponential, etc if $\{\delta^n_\rho\}$ is equivalent to a function f with said property.

Definition 3.5 Let X be a geodesic space and A a subspace of X. Let r be any positive number.

(1) $N_r(A) = \{x \in X \mid d_X(x, A) < r\}$.
(2) $\partial N_r(A) = \{x \in X \mid d_X(x, A) = r\}$.
(3) $C_r(A) = X - N_r(A)$.
(4) Let $d_{r,A}$ be the induced length metric on the complement of the r–neighborhood of A in X. If the subspace A is clear from context, we use the notation d_r instead of $d_{r,A}$.

Definition 3.6 Let (X, A) be a pair of metric spaces. For each $\rho \in (0, 1]$ and positive integer $n \geq 2$, we define a function $\delta^n_\rho\colon [0, \infty) \to [0, \infty]$ as follows:

For each r, let $\delta^n_\rho(r) = \sup d_{\rho r}(x_1, x_2)$ where the supremum is taken over all $x_1, x_2 \in \partial N_r(A)$ such that $d_r(x_1, x_2) < \infty$ and $d(x_1, x_2) \leq nr$. The family of functions $\{\delta^n_\rho\}$ is *the relative divergence* of X with respect to A, denoted by $\mathrm{Div}(X, A)$.

We now define the concept of relative divergence of a finitely generated group with respect to a subgroup.

Definition 3.7 Let G be a finitely generated group with subgroup H. We define *the relative divergence* of G with respect to H, denoted by $\mathrm{Div}(G, H)$, to be the relative divergence of the Cayley graph $\Gamma(G, S)$ with respect to H for some finite generating set S.

Remark 3.8 The concept of relative divergence was introduced by the author [9] with the name upper relative divergence. The relative divergence of geodesic spaces is a pair quasi-isometry invariant concept. This implies that the relative divergence of a finitely generated group does not depend on the choice of finite generating sets.

Definition 3.9 The *divergence* of a bi-infinite geodesic α, denoted by Div_α, is a function $g\colon (0, \infty) \to (0, \infty)$ such that for each positive number r, the value $g(r)$ is the infimum on the lengths of all paths, outside the open ball about $\alpha(0)$ with radius r, connecting $\alpha(-r)$ and $\alpha(r)$.

The following lemma is deduced from the proof of Corollary 4.8 in [1].

Lemma 3.10 *Let Γ be a connected, finite, simplicial graph with at least two vertices. Assume that Γ is not a join. Let g be a cyclically reduced element in A_Γ that does not lie in any join subgroup. Then the divergence of the bi-infinite geodesic $\cdots ggggg \cdots$ is at least quadratic.*

Definition 3.11 Let G be a group with a finite generating set S and H a subgroup of G with a finite generating set T. The *subgroup distortion* of H in G is the function

$$\mathrm{Dist}^H_G\colon (0, \infty) \to (0, \infty), \quad \mathrm{Dist}^H_G(r) = \max\{|h|_T : h \in H,\ |h|_S \leq r\}.$$

Remark 3.12 It is well known that the concept of distortion does not depend on the choice of finite generating sets.

4 Connection between subgroup distortion and relative divergence

Lemma 4.1 *Let H be a finitely generated group with finite generating set T and ϕ in* $\mathrm{Aut}(H)$*. Let $G = \langle H, t / tht^{-1} = \phi(h)\rangle$ and $S = T \cup \{t\}$.*

(1) *Each element in G can be written uniquely in the form ht^n, where h is a group element in H.*

(2) *The set S is a finite generating set of G, and*

$$d_S(ht^m, h't^n) \geq |m-n| \quad \text{and} \quad d_S(ht^m, Ht^n) = |m-n|.$$

Proof The statement (1) is well known, and we only need to prove statement (2). Let ψ be the map from G to $\mathbb{Z}$ by sending element t to 1 and each generator in T to 0. It is not hard to see that ψ is a group homomorphism. We first show that the absolute value of $\psi(g)$ is at most the length of g with respect to S for each group element g in G. In fact, let $w_1 t^{n_1} w_2 t^{n_2} \cdots w_k t^{n_k}$ be the shortest word in S that represents g, where each w_i is a word in T. Therefore,

$$\psi(g) = n_1 + n_2 + \cdots + n_k$$

and

$$|g|_S = (\ell(w_1) + \ell(w_2) + \cdots + \ell(w_k)) + (|n_1| + |n_2| + \cdots + |n_k|).$$

This implies that the absolute value of $\psi(g)$ is at most the length of g with respect to S. The distance between two elements ht^m and $h't^n$ is the length of the group element $g = (ht^m)^{-1}h't^n$. Obviously, $\psi(g) = n - m$. Therefore, the distance between two elements ht^m and $h't^n$ is at least $|m-n|$. This fact directly implies that the distance between ht^m and any element in Ht^n is at least $|m-n|$. Also, ht^n is an element in Ht^n, and the distance between ht^m and ht^n is at most $|m-n|$. Therefore, the distance between ht^m and Ht^n is exactly $|m-n|$. □

Lemma 4.2 *Let H be a finitely generated group with finite generating set T and ϕ in $Aut(H)$. Let $G = \langle H, t / tht^{-1} = \phi(h)\rangle$ and $S = T \cup \{t\}$. Let n be an arbitrary positive integer, and let x and y be two points in $\partial N_n(H)$. Then there is a path outside $N_n(H)$ connecting x and y if and only if the pair (x, y) is of either the form $(h_1 t^n, h_2 t^n)$ or $(h_1 t^{-n}, h_2 t^{-n})$, where h_1 and h_2 are elements in H.*

Proof By Lemma 4.1, the pair (x, y) must be of the form $(h_1 t^{m_1}, h_2 t^{m_2})$, where $|m_1| = |m_2| = n$. We first assume that $m_1 m_2 < 0$. Let γ be an arbitrary path connecting x and y. By Lemma 4.1, we observe that if two vertices ht^m and $h't^{m'}$ of γ are consecutive, then $|m-m'| \leq 1$. Therefore, there exists a vertex of γ that belongs to H. Thus, there is no path outside $N_n(H)$ connecting x and y.

If $m_1 = m_2$, then x and y both lie in the same coset $t_{m_1}H$. Therefore, there is a path α with all vertices in $t_{m_1}H$ connecting x and y. By Lemma 4.1 again, α must lie outside $N_n(H)$. Therefore, the pair (x, y) is of either the form (h_1t^n, h_2t^n) or (h_1t^{-n}, h_2t^{-n}). □

Proposition 4.3 *Let H be a finitely generated group and $G = \langle H, t / tht^{-1} = \phi(h)\rangle$, where ϕ in $Aut(H)$. Then $\mathrm{Div}(G, H) \preceq \mathrm{Dist}_G^H$.*

Proof Let T be a finite generating set of H, and let $S = T \cup \{t\}$. Then S is a finite generating set of G. Suppose that $\mathrm{Div}(G, H) = \{\delta_\rho^n\}$. We will show that $\delta_\rho^n(r) \leq \mathrm{Dist}_G^H(nr)$ for every positive integer r.

Indeed, let x and y be arbitrary points in $\partial N_r(H)$ such that $d_{r,H}(x, y) < \infty$ and $d_S(x, y) \leq nr$. By Lemma 4.2, x and y both lie in the same coset t^mH, where $|m| = r$. Therefore, there is a path γ with all vertices in t^mH connecting x and y, and the length of γ is at most $\mathrm{Dist}_G^H(nr)$. By Lemma 4.1 again, the path γ must lie outside $N_r(H)$. Therefore, $d_{\rho r,H}(x, y) \leq \mathrm{Dist}_G^H(nr)$. Thus, $\delta_\rho^n(r) \leq \mathrm{Dist}_G^H(nr)$. This implies that $\mathrm{Div}(G, H) \preceq \mathrm{Dist}_G^H$. □

5 Relative divergence of right-angled Artin groups with respect to Bestvina–Brady subgroups and subgroup distortion of Bestvina–Brady subgroups

From now on, we let Γ be a finite, connected, simplicial graph with at least two vertices. Let A_Γ be the associated right-angled Artin group and H_Γ its Bestvina–Brady subgroup. Let X_Γ be the associated Salvetti complex and $\widetilde{X}_\Gamma$ its universal covering. We consider the 1–skeleton of $\widetilde{X}_\Gamma$ as a Cayley graph of A_Γ, and the vertex set S of Γ as a finite generating set of A_Γ. By Theorem 2.6, we can choose the set T of all elements of the form st^{-1} whenever s and t are adjacent vertices as a finite generating set for H_Γ. Let Φ and $\widetilde{f}$ be the group homomorphism and continuous map as in Remark 2.5.

Lemma 5.1 *Let M be the diameter of Γ. Let a and b be arbitrary vertices in S. For each integer m, the length of $a^m b^{-m}$ with respect to T is at most $M|m|$.*

Proof Since the diameter of Γ is M, we can choose a positive integer $n \leq M$ and $n+1$ generators $s_0, s_1, \ldots, s_n$ in S such that the following conditions hold:

(1) $s_0 = a$ and $s_n = b$.

(2) s_i and s_{i+1} commute for $i \in \{0, 1, 2, \dots, n-1\}$.

Obviously,

$$\begin{aligned} a^m b^{-m} = s_0^m s_n^{-m} &= (s_0^m s_1^{-m})(s_1^m s_2^{-m})(s_2^m s_3^{-m}) \cdots (s_{n-2}^m s_{n-1}^{-m})(s_{n-1}^m s_n^{-m}) \\ &= (s_0 s_1^{-1})^m (s_1 s_2^{-1})^m (s_2 s_3^{-1})^m \cdots (s_{n-2} s_{n-1}^{-1})^m (s_{n-1} s_n^{-1})^m. \end{aligned}$$

Also, $s_{i-1} s_i^{-1}$ belongs to T. Therefore, the length of $a^m b^{-m}$ with respect to T is at most $n|m|$. This implies the length of $a^m b^{-m}$ with respect to T is at most $M|m|$. □

Proposition 5.2 *The subgroup distortion $\mathrm{Dist}_{A_\Gamma}^{H_\Gamma}$ is dominated by a quadratic function. Moreover, $\mathrm{Dist}_{A_\Gamma}^{H_\Gamma}$ is linear when Γ is a join.*

Proof We first show that $\mathrm{Dist}_{A_\Gamma}^{H_\Gamma}$ is dominated by a quadratic function. Let n be an arbitrary positive integer and h be an arbitrary element in H_Γ such that $|h|_S \le n$. We can write $h = s_1^{m_1} s_2^{m_2} s_3^{m_3} \cdots s_k^{m_k}$ such that:

(1) Each s_i lies in S, $|m_i| \ge 1$ and $|m_1| + |m_2| + |m_3| + \cdots + |m_k| \le n$.

(2) $m_1 + m_2 + m_3 + \cdots + m_k = 0$.

Obviously, we can rewrite h as follows:

$$h = (s_1^{m_1} s_2^{-m_1})(s_2^{(m_1+m_2)} s_3^{-(m_1+m_2)}) \cdots (s_{k-1}^{(m_1+m_2+\cdots+m_{k-1})} s_k^{-(m_1+m_2+\cdots+m_{k-1})}).$$

Let M be the diameter of Γ. By Lemma 5.1, we have

$$\begin{aligned} |h|_T &\le M|m_1| + M|m_1+m_2| + \cdots + M|m_1+m_2+\cdots+m_{k-1}| \\ &\le M|m_1| + M(|m_1|+|m_2|) + \cdots + M(|m_1|+|m_2|+\cdots+|m_{k-1}|) \\ &\le M(k-1)n \le Mn^2. \end{aligned}$$

Therefore, the distortion function $\mathrm{Dist}_{A_\Gamma}^{H_\Gamma}$ is bounded above by Mn^2.

We now assume that Γ is a join of Γ_1 and Γ_2. We need to prove that the distortion $\mathrm{Dist}_{A_\Gamma}^{H_\Gamma}$ is linear. Let n be an arbitrary positive integer and h be an arbitrary element in H_Γ such that $|h|_S \le n$. Since A_Γ is the direct product of A_{Γ_1} and A_{Γ_2}, we can write $h = (a_1^{m_1} a_2^{m_2} \cdots a_k^{m_k})(b_1^{n_1} b_2^{n_2} \cdots b_\ell^{n_\ell})$ such that:

(1) Each a_i is a vertex of Γ_1 and each b_j is a vertex of Γ_2.

(2) $(|m_1| + |m_2| + \cdots + |m_k|) + (|n_1| + |n_2| + \cdots + |n_\ell|) \le n$.

(3) $(m_1 + m_2 + \cdots + m_k) + (n_1 + n_2 + \cdots + n_\ell) = 0$.

Let $m = m_1 + m_2 + \cdots + m_k$. Then $n_1 + n_2 + \cdots + n_\ell = -m$ and $|m| \le n$. Let a be a vertex in Γ_1 and b a vertex in Γ_2. Since a commutes with each b_j, b commutes with each a_i, and a and b commute, we can rewrite h as follows:

$$\begin{aligned} h &= (a_1^{m_1} a_2^{m_2} \cdots a_k^{m_k} b^{-m})(b^m a^{-m})(a^m b_1^{n_1} b_2^{n_2} \cdots b_\ell^{n_\ell}) \\ &= (a_1 b^{-1})^{m_1} (a_2 b^{-1})^{m_2} \cdots (a_k b^{-1})^{m_k} (ba^{-1})^m (ab_1^{-1})^{-n_1} (ab_2^{-1})^{-n_2} \cdots (ab_\ell^{-1})^{-n_\ell}. \end{aligned}$$

Also, ab_j^{-1}, $a_i b^{-1}$ and ba^{-1} all belong to T. Therefore,

$$|h|_T \le (|m_1| + |m_2| + \cdots + |m_k|) + (|n_1| + |n_2| + \cdots + |n_\ell|) + |m| \le 2n.$$

Therefore, the distortion function $\mathrm{Dist}_{A_\Gamma}^{H_\Gamma}$ is bounded above by $2n$. □

Proposition 5.3 *If* Γ *is not a join graph, then the relative divergence* $\mathrm{Div}(A_\Gamma, H_\Gamma)$ *is at least quadratic.*

Proof Let J be a maximal join in Γ, and let v be a vertex not in J. Let g in A_J be the product of all vertices in J. Let $n = \Phi(g)$ and let $h = gv^{-n}$. Then h is an element in H_Γ. Since J is a maximal join in Γ and v is a vertex not in J, we see that h does not lie in any join subgroup. Also, h is a cyclically reduced element. Therefore, the divergence of the bi-infinite geodesic $\alpha = \cdots hhhhh \cdots$ is at least quadratic by Lemma 3.10.

Let t be an arbitrary generator in S and $k = |h|_S$. We can assume that $\alpha(0) = e$, $\alpha(km) = h^m$ and $\alpha(-km) = h^{-m}$. In order to prove that the relative divergence $\mathrm{Div}(A_\Gamma, H_\Gamma)$ is at least quadratic, it is sufficient to prove each function δ_ρ^n dominates the divergence function of α for each $n \ge 2k + 2$.

Indeed, let r be an arbitrary positive integer. Let $x = h^{-r} t^r$ and $y = h^r t^r$. By a similar argument as in Lemmas 4.1 and 4.2, the two points x and y both lie in $\partial N_r(H_\Gamma)$, and $d_{r,H_\Gamma}(x, y) < \infty$. Moreover,

$$d_S(x, y) \le d_S(x, h^{-r}) + d_S(h^{-r}, h^r) + d_S(h^r, y) \le r + 2kr + r \le (2k+2)r \le nr.$$

Let γ be an arbitrary path outside $N_{\rho r}(H)$ connecting x and y. Obviously, the path γ must lie outside the open ball $B(\alpha(0), \rho r)$. It is obvious that we can connect x and h^{-r} by a path γ_1 of length r which lies outside $B(\alpha(0), \rho r)$. Similarly, we can connect y and h^r by a path γ_2 of length r which lies outside $B(\alpha(0), \rho r)$. Let γ_3 be the subsegment of α connecting $\alpha(-\rho r)$ and h^{-r}. Let γ_4 be the subsegment of α connecting $\alpha(\rho r)$ and h^r. It is not hard to see the lengths of γ_3 and γ_4 are both $(k - \rho)r$.

Let $\overline{\gamma} = \gamma_3 \cup \gamma_1 \cup \gamma \cup \gamma_2 \cup \gamma_4$. Then $\overline{\gamma}$ is a path that lies outside $B(\alpha(0), \rho r)$ connecting $\alpha(-\rho r)$ and $\alpha(\rho r)$. Therefore, the length of $\overline{\gamma}$ is at least $\mathrm{Div}_\alpha(\rho r)$. Also,

$$\ell(\overline{\gamma}) = \ell(\gamma_3) + \ell(\gamma_1) + \ell(\gamma) + \ell(\gamma_2) + \ell(\gamma_5) = \ell(\gamma) + 2(k - \rho + 1)r.$$

Thus,

$$\ell(\gamma) \ge \mathrm{Div}_\alpha(\rho r) - 2(k - \rho + 1)r.$$

This implies that

$$d_{\rho r, H_\Gamma}(x,y) \geq \mathrm{Div}_\alpha(\rho r) - 2(k-\rho+1)r.$$

Therefore,

$$\delta^n_\rho(r) \geq \mathrm{Div}_\alpha(\rho r) - 2(k-\rho+1)r.$$

Thus, the relative divergence $\mathrm{Div}(A_\Gamma, H_\Gamma)$ is at least quadratic. □

The following theorem is deduced from Propositions 4.3, 5.2 and 5.3.

Theorem 5.4 *Let Γ be a connected, finite, simplicial graph with at least two vertices. Let A_Γ be the associated right-angled Artin group and H_Γ the Bestvina–Brady subgroup. Then the relative divergence $\mathrm{Div}(A_\Gamma, H_\Gamma)$ and the subgroup distortion $\mathrm{Dist}_{A_\Gamma}^{H_\Gamma}$ are both linear if Γ is a join graph. Otherwise, they are both quadratic.*

Corollary 5.5 *For each integer $n \geq 3$, there is a right-angled Artin group containing a free subgroup of rank n whose inclusion is not a quasi-isometric embedding.*

Proof For each positive integer $n \geq 3$, let Γ be a tree with n edges such that Γ is not a join graph. By the above theorem, the distortion of H_Γ in the right-angled Artin group A_Γ is quadratic. Also, H_Γ is the free group of rank n by Theorem 2.6. □

6 Connection to horizontal surface subgroups

Definition 6.1 A *graph manifold* is a compact, irreducible, connected orientable 3–manifold M that can be decomposed along $\mathcal{T}$ into finitely many Seifert manifolds, where $\mathcal{T}$ is the canonical toric decomposition of Johannson [8] and of Jaco and Shalen [7]. We call the collection $\mathcal{T}$ its JSJ-decomposition in M, and each element in $\mathcal{T}$ its JSJ-torus.

Definition 6.2 If M is a Seifert manifold, a properly immersed surface $g\colon S \looparrowright M$ is *horizontal* if $g(S)$ is transverse to the Seifert fibers everywhere. In the case where M is a graph manifold, a properly immersed surface $g\colon S \looparrowright M$ *horizontal* if $g(S) \cap P_v$ is horizontal for every Seifert component P_v.

Theorem 6.3 *If Γ is a finite tree with at least three vertices, then the associated right-angled Artin group A_Γ is a fundamental group of a graph manifold, and the Bestvina–Brady subgroup H_Γ is a horizontal surface subgroup.*

Proof First, we construct the graph manifold M whose fundamental group is A_Γ. Let v be a vertex of Γ of degree $k \geq 2$. Let $u_1, u_2, \dots, u_k$ be all elements in $\ell k(v)$. Let Σ_v be a punctured disk with k holes whose boundaries are labeled by elements

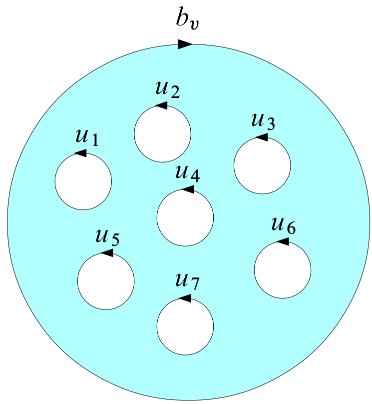

Figure 1: A punctured disk Σ_v when the degree of v in Γ is 7

in $\ell k(v)$. We also label the outside boundary component of Σ_v by b_v; see Figure 1. Obviously, $\pi_1(\Sigma_v)$ is the free group generated by $u_1, u_2, \dots, u_k$.

Let $P_v = \Sigma_v \times S^1_v$, where we label the circle factor in P_v by v. Obviously, each P_v is a Seifert manifold. Moreover, for each u_i in $\ell k(v)$, the Seifert manifold P_v contains the torus $S^1_{u_i} \times S^1_v$ as a component of its boundary.

We construct the graph manifold by gluing pairs of Seifert manifolds (P_{v_1}, P_{v_2}) along their tori $S^1_{v_1} \times S^1_{v_2}$ whenever v_1 and v_2 are adjacent vertices in Γ. We observe that the pair of such regions are glued together by switching fiber and base directions. It is not hard to see that the fundamental group of M is the right-angled Artin group A_Γ.

We now construct the horizontal surface S in M with the Bestvina–Brady subgroup H_Γ as its fundamental group. We first construct the horizontal surface S_v on each Seifert piece $P_v = \Sigma_v \times S^1_v$, where v is a vertex of Γ of degree $k \geq 2$.

We remind the reader that Σ_v is a punctured disk with k holes whose boundaries are labeled by the elements $u_1, u_2, \dots, u_k$ in $\ell k(v)$. We also label the outside boundary component of Σ_v by b_v; see Figure 1. We label the circle factor in P_v by v.

Let S_v be a copy of the punctured disk Σ_v. However, we relabel all inside circles by $c_1, c_2, \dots, c_k$ and the outside circle by c_v. We will construct a map $(g, h)\colon S_v \to \Sigma_v \times S^1_v$ as follows:

(1) The map g is the identity map that maps each c_i to u_i and c_v to b_v.

(2) The map h has degree -1 on boundary component c_i and degree k on c_v.

We now construct the map h with the above properties. We observe that the fundamental group of S_v is generated by $c_1, c_2, \dots, c_k$ and c_v with a unique relator $c_1 c_2 c_3 \cdots c_k c_v = e$. Here we abused notation for the presentation of $\pi_1(S_v)$. By that presentation of $\pi_1(S_v)$, we can see that there is a group homomorphism ϕ from $\pi_1(S_v)$ to $\mathbb{Z}$ that maps each c_i to -1 and c_v to k. By [5, Proposition 1B.9], the group homomorphism ϕ is induced by a map h from S_v to S^1_v. Therefore, we constructed the desired map h.

Finally, we identify the surface S_v with its image via the map (g,h). By construction, $\pi_1(S_v)$ is the subgroup of $\pi_1(P_v)$ generated by elements $u_1v^{-1}, u_2v^{-1}, \dots, u_kv^{-1}$. We observe that if we glue pair of Seifert manifolds (P_{v_1}, P_{v_2}) along their tori $S^1_{v_1} \times S^1_{v_2}$, the pair of horizontal surfaces (S_{v_1}, S_{v_2}) will be matched up along their boundaries in $S^1_{v_1} \times S^1_{v_2}$. Therefore, we constructed a horizontal surface S in M. By the Van Kampen theorem, the fundamental group of S is generated by all elements of the form st^{-1} whenever s and t are adjacent vertices in Γ. In other words, $\pi_1(S)$ is the Bestvina–Brady subgroup by Theorem 2.6. □

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Department of Mathematics, The University of Georgia
Athens, GA, United States

hung.tran@uga.edu

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