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DAHA and iterated torus knots

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The theory of DAHA-Jones polynomials is extended from torus knots to their arbitrary iterations (for any reduced root systems and weights), which includes the polynomiality, duality and other properties of the DAHA superpolynomials. Presumably they coincide with the reduced stable Khovanov–Rozansky polynomials in the case of nonnegative coefficients. The new theory matches well the classical theory of algebraic knots and (unibranch) plane curve singularities; the Puiseux expansion naturally emerges. The corresponding DAHA superpolynomials are expected to coincide with the reduced ones in the Oblomkov–Shende–Rasmussen conjecture upon its generalization to arbitrary dominant weights. For instance, the DAHA uncolored superpolynomials at $a = 0$, $q = 1$ are conjectured to provide the Betti numbers of the Jacobian factors (compactified Jacobians) of the corresponding singularities.

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0 Introduction

The theory of *DAHA-Jones polynomials* of torus knots [12; 22; 13] is fully extended in this paper to arbitrary iterated knots for any reduced root systems and dominant weights, which includes the polynomiality, duality and other properties of the *DAHA superpolynomials*. We conjecture the coincidence of the latter with the stable reduced *Khovanov–Rozansky polynomials* [27; 28; 26; 37] for *pseudoalgebraic* knots, including all algebraic knots.

A similar connection is also expected with the physics superpolynomials associated with the *BPS states* (see eg [15; 1; 16]) and those related to the Hilbert schemes of plane curve singularities and *rational DAHA*; see [33; 23; 22] and references therein. We note that using rational DAHA here is restricted only to the torus knots (and adding colors is generally an open problem).

The new theory matches well the classical theory of algebraic knots and unibranch *plane curve singularities*, though we see no *a priori* reasons for this. The *Newton pairs* in the theory of *Puiseux expansion* naturally emerge in our approach; see eg [17]. For instance, the iterations that are trivial topologically result in interesting algebraic

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symmetries of the DAHA superpolynomials, for instance, in uniform formulas for the torus knots in the form $T(\mathsf{r}, m\mathsf{r}+\mathsf{s})$ for *any* $m>0$.

We expect that our superpolynomials add t to the *Oblomkov–Shende conjecture* [34; 30] and add weights to [33, Conjecture 2] for algebraic knots; see [30, Section 1.5]. The *reduced ORS conjecture* connects the Poincaré polynomial $\mathcal{P}_{KhR}$ of the triply graded *reduced* Khovanov–Rozansky homology of an algebraic link with $\mathcal{P}_{alg}$ describing the cohomology of the nested Hilbert scheme of the corresponding plane curve singularity under the *weight filtration*. We conjecture them to coincide with the uncolored DAHA superpolynomial $\mathcal{P}_{DAHA}$.

The coincidence with $\mathcal{P}_{KhR}$ is expected for any *pseudoalgebraic* knots (see above and Conjecture 2.4). The reduction of $\mathcal{P}_{DAHA}$ to the *HOMFLYPT* polynomials is conjectured for any iterated knots and weights (we prove it for sl_2 and checked in all the examples). The link $\mathcal{P}_{DAHA}\sim\mathcal{P}_{KhR}$ is a challenge (our examples are mainly for the Khovanov polynomials).

Finding $\mathcal{P}_{KhR}$ is an involved task; one of the reasons is that the *skein relations* generally cannot be extended from the HOMFLYPT polynomials to the HOMFLYPT homology. The polynomial $\mathcal{P}_{alg}$ is very sophisticated too. For instance, its portion of minimal a–degree requires knowing the so called *perverse filtration* in the cohomology of the *Jacobian factors*. Even the Betti numbers of the latter are known only in few cases. See [36, Theorem 22, Conjecture 23] concerning the torus knots and those for *Puiseux exponents* $(4,2u,v)$.

By contrast, $\mathcal{P}_{DAHA}$ can be calculated in an entirely formal way without any (theoretical) limitations; we expect that this construction will eventually include arbitrary iterated *links*. Thus, the main obstacle with stating parts (ii) and (iii) of Conjecture 2.4 below for arbitrary weights is the absence of $\mathcal{P}_{KhR}$ and $\mathcal{P}_{alg}$ in such a generality. Concerning $\mathcal{P}_{DAHA}\sim\mathcal{P}_{alg}$, the connection of DAHA with the K–theory of the *Hilbert schemes* of $\mathbb{C}^2$ [41] and the *affine Springer fibers* [45; 49] can be mentioned; the Jacobian factors of isolated plane curve singularities are certain affine Springer fibers [29].

In the examples we calculated, the t–coefficients of the DAHA superpolynomials evaluated at $a=0, q=1$ coincide with the Betti numbers (when known) of the corresponding Jacobian factors (compactified Jacobians), which are the ranks of the corresponding cohomology groups. Their sum, the *Euler number*, can be calculated via the HOMFLYPT polynomial; see [34, Section 7] and [30].

We provide conjectural Betti numbers in the case of Puiseux exponents $(6,8,9)$ and $(6,9,10)$ and quite a few examples beyond the technique used in [36]. In the case of $(6,8,9)$, we calculate *formal* DAHA counterparts of such Betti numbers for $2\omega_1$ and

ω_2. Also, we do this for several uncolored *nonalgebraic* iterated knots; for instance, we extend the family $(4,6,v\geq 7)$ from [36] to *pseudoalgebraic* $v=3,5$.

In all considered examples (not only for the pseudoalgebraic iterated knots), our superpolynomials reduce to the corresponding HOMFLYPT polynomials under $t\mapsto q$, $a\mapsto -a$ for arbitrary dominant weights and any iterated knots. They match well the (uncolored) Khovanov polynomials (for sl_2) in the sense of [12, Conjecture 3.7] for all *pseudoalgebraic* knots we considered; recall that the latter are defined as those with nonnegative coefficients of (uncolored) $\mathcal{P}_{DAHA}$. It is possible that such a connection can be extended to any iterated knots; see Section 4.1.

We note that our paper was triggered by paper [40], though it is restricted to A_1 and we do not quite understand its approach. The Newton pairs and other features of our construction do not appear in [40]. Also, the polynomials J_n there have significant q,t–denominators (even in the uncolored case); the polynomiality of DAHA-Jones polynomials is the key in our theory. The J_2–polynomials for $Cab(\pm 5,2)T(3,2)$ from [40, Section 5.2] are very different from our ones in these cases (and we do not understand how they were obtained); see (4-3) below. The first part of [40] is devoted to the *skein modules*, which continues [5]; this direction is of obvious importance.

1 DAHA and Macdonald polynomials

1.1 Affine root systems

Let $R=\{\alpha\}\subset\mathbb{R}^n$ be a root system of type $A_n,\dots,G_2$ with respect to a euclidean form (z,z') on $\mathbb{R}^n\ni z,z'$, W the Weyl group generated by the reflections s_α, R_+ the set of positive roots corresponding to fixed simple roots $\alpha_1,\dots,\alpha_n$, and $R_-=-R_+$. The form is normalized by the condition $(\alpha,\alpha)=2$ for *short* roots. The weight lattice is $P=\bigoplus_{i=1}^n\mathbb{Z}\omega_i$, where $\{\omega_i\}$ are fundamental weights: $(\omega_i,\alpha_j^\vee)=\delta_{ij}$ for the coroots $\alpha^\vee=2\alpha/(\alpha,\alpha)$. The root lattice is $Q=\bigoplus_{i=1}^n\mathbb{Z}\alpha_i$. Replacing $\mathbb{Z}$ by $\mathbb{Z}_\pm=\{m\in\mathbb{Z},\pm m\geq 0\}$, we obtain $P_\pm,Q_\pm$. See eg [7] or [10].

Setting $\nu_\alpha:=(\alpha,\alpha)/2$, the vectors $\widetilde{\alpha}=[\alpha,\nu_\alpha j]\in\mathbb{R}^n\times\mathbb{R}\subset\mathbb{R}^{n+1}$ for $\alpha\in R$, $j\in\mathbb{Z}$ form the *twisted affine root system* $\widetilde{R}\supset R$ ($z\in\mathbb{R}^n$ are identified with $[z,0]$). We add $\alpha_0:=[-\vartheta,1]$ to the simple roots for the *maximal short root* $\vartheta\in R_+$. The corresponding set $\widetilde{R}_+$ of positive roots is $R_+\cup\{[\alpha,\nu_\alpha j],\alpha\in R,j>0\}$.

The set of the indices of the images of α_0 by all automorphisms of the affine Dynkin diagram will be denoted by O ($O=\{0\}$ for E_8,F_4,G_2). Let $O':=\{r\in O,r\neq 0\}$. The elements ω_r for $r\in O'$ are *minuscule weights*, defined by the inequalities $(\omega_r,\alpha^\vee)\leq 1$ for all $\alpha\in R_+$. We set $\omega_0=0$ for the sake of uniformity.

Affine Weyl groups Given $\widetilde{\alpha}=[\alpha,\nu_\alpha j]\in\widetilde{R}$ and $b\in P$, let

$$s_{\widetilde{\alpha}}(\widetilde{z})=\widetilde{z}-(z,\alpha^\vee)\widetilde{\alpha},\quad b'(\widetilde{z})=[z,\zeta-(z,b)] \tag{1-1}$$

for $\widetilde{z}=[z,\zeta]\in\mathbb{R}^{n+1}$. The *affine Weyl group* $\widetilde{W}=\langle s_{\widetilde{\alpha}},\ \widetilde{\alpha}\in\widetilde{R}_+\rangle$ is the semidirect product $W\ltimes Q$ of its subgroups $W=\langle s_\alpha,\alpha\in R_+\rangle$ and Q, where α is identified with

$$s_\alpha s_{[\alpha,\nu_\alpha]}=s_{[-\alpha,\nu_\alpha]}s_\alpha\quad\text{ for }\alpha\in R.$$

The *extended Weyl group* $\widehat{W}$ is $W\ltimes P$, where the corresponding action is

$$(wb)([z,\zeta])=[w(z),\zeta-(z,b)]\quad\text{ for } w\in W, b\in P. \tag{1-2}$$

It is isomorphic to $\widetilde{W}\ltimes\Pi$ for $\Pi:=P/Q$. The latter group consists of $\pi_0=\mathrm{id}$ and the images π_r of minuscule ω_r in P/Q.

The group Π is naturally identified with the subgroup of $\widehat{W}$ of the elements of the length zero; the *length* is defined as

$$l(\widehat{w})=|\lambda(\widehat{w})|\quad\text{ for }\lambda(\widehat{w}):=\widetilde{R}_+\cap\widehat{w}^{-1}(-\widetilde{R}_+).$$

One has $\omega_r=\pi_r u_r$ for $r\in O'$, where u_r is the element $u\in W$ of minimal length such that $u(\omega_r)\in P_-$.

Setting $\widehat{w}=\pi_r\widetilde{w}\in\widehat{W}$ for $\pi_r\in\Pi$ and $\widetilde{w}\in\widetilde{W}$, $l(\widehat{w})$ coincides with the length of any reduced decomposition of $\widetilde{w}$ in terms of the simple reflections s_i for $0\le i\le n$.

1.2 Definition of DAHA

We follow [13; 12; 10]. Let m be the least natural number such that $(P,P)=(1/\mathsf{m})\mathbb{Z}$. Thus $\mathsf{m}=|\Pi|$ with the following exceptions: $\mathsf{m}=2$ for D_{2k}, and $\mathsf{m}=1$ for B_{2k} and C_k.

The double affine Hecke algebra, *DAHA*, depends on the parameters q and t_ν $(\nu\in\{\nu_\alpha\})$ and will be defined over the ring $\mathbb{Z}_{q,t}:=\mathbb{Z}[q^{\pm1/\mathsf{m}},t_\nu^{\pm1/2}]$ formed by polynomials in terms of $q^{\pm1/\mathsf{m}}$ and $\{t_\nu^{1/2}\}$. Note that the coefficients of the Macdonald polynomials will belong to $\mathbb{Q}(q,t_\nu)$.

For $\widetilde{\alpha}=[\alpha,\nu_\alpha j]\in\widetilde{R}$, $0\le i\le n$, we set

$$t_{\widetilde{\alpha}}=t_\alpha=t_{\nu_\alpha}=q_\alpha^{k_\nu},\quad q_{\widetilde{\alpha}}=q^{\nu_\alpha},\quad t_i=t_{\alpha_i},\quad q_i=q_{\alpha_i}.$$

Also, using here (and below) *sht* and *lng* instead of ν, we set

$$\rho_k:=\frac{1}{2}\sum_{\alpha>0}k_\alpha\alpha=k_{\mathrm{sht}}\rho_{\mathrm{sht}}+k_{\mathrm{lng}}\rho_{\mathrm{lng}},\quad \rho_\nu=\frac{1}{2}\sum_{\nu_\alpha=\nu}\alpha=\sum_{\nu_i=\nu,i>0}\omega_i.$$

For pairwise commutative $X_1,\dots,X_n$,

$$X_{\widetilde{b}} := \prod_{i=1}^{n} X_i^{l_i} q^j \quad \text{if } \widetilde{b}=[b,j], \quad \widehat{w}(X_{\widetilde{b}}) = X_{\widehat{w}(\widetilde{b})}, \quad \text{where } b=\sum_{i=1}^{n} l_i\omega_i \in P, \quad j\in\frac{1}{m}\mathbb{Z}, \quad \widehat{w}\in\widehat{W}. \tag{1-3}$$

For instance, $X_0 := X_{\alpha_0} = qX_{\vartheta}^{-1}$.

Recall that $\omega_r = \pi_r u_r$ for $r\in O'$ (see above). We will use that π_r^{-1} is $\pi_{\iota(i)}$, where ι is the standard involution of the nonaffine Dynkin diagram, induced by $\alpha_i\mapsto -w_0(\alpha_i)$. Generally, $\iota(b)=-w_0(b)=b^{\iota}$, where w_0 is the longest element in W. Finally, we set $m_{ij}=2,3,4,6$ when the number of links between α_i and α_j in the affine Dynkin diagram is $0,1,2,3$, respectively.

Definition 1.1 *The double affine Hecke algebra $\mathcal{H}$ is generated over $\mathbb{Z}_{q,t}$ by the elements $\{T_i \mid 0\le i\le n\}$, pairwise commutative $\{X_b \mid b\in P\}$ satisfying (1-3), and the group Π, where the following relations are imposed:*

(o)	$(T_i - t_i^{1/2})(T_i + t_i^{-1/2}) = 0$	*for* $0\le i\le n$.
(i)	$T_iT_jT_i\cdots = T_jT_iT_j\cdots$	*with* m_{ij} *factors on each side.*
(ii)	$\pi_r T_i \pi_r^{-1} = T_j$	*if* $\pi_r(\alpha_i)=\alpha_j$.
(iii)	$T_iX_b = X_bX_{\alpha_i}^{-1}T_i^{-1}$	*if* $(b,\alpha_i^{\vee})=1$, *for* $0\le i\le n$.
(iv)	$T_iX_b = X_bT_i$ if $(b,\alpha_i^{\vee})=0$	*for* $0\le i\le n$.
(v)	$\pi_r X_b \pi_r^{-1} = X_{\pi_r(b)} = X_{u_r^{-1}(b)} q^{(\omega_{\iota(r)},b)}$	*for* $r\in O'$.

Given $\widetilde{w}\in\widetilde{W}$ and $r\in O$, the product

$$T_{\pi_r\widetilde{w}} := \pi_r T_{i_l}\cdots T_{i_1}, \quad \text{where } \widetilde{w}=s_{i_l}\cdots s_{i_1} \text{ for } l=l(\widetilde{w}), \tag{1-4}$$

does not depend on the choice of the reduced decomposition. Moreover,

$$T_{\widehat{v}}T_{\widehat{w}} = T_{\widehat{v}\widehat{w}} \quad \text{whenever } l(\widehat{v}\widehat{w}) = l(\widehat{v}) + l(\widehat{w}) \text{ for } \widehat{v},\widehat{w}\in\widehat{W}. \tag{1-5}$$

In particular, we arrive at the pairwise commutative elements

$$Y_b := \prod_{i=1}^{n} Y_i^{l_i}, \quad \text{if } b=\sum_{i=1}^{n} l_i\omega_i\in P, \quad \text{where } Y_i := T_{\omega_i}, b\in P. \tag{1-6}$$

When acting in the polynomial representation $\mathcal{V}$ (see below), they are called *difference Dunkl operators*. Their orbit sums (symmetrizations) $\sum_{w\in W} Y_{w(b)}$ then become the Macdonald operators and their generalizations upon acting on symmetric Laurent polynomials.

1.3 The automorphisms

The following maps can be (uniquely) extended to automorphisms of $\mathcal{H}$, where $q^{1/(2m)}$ must be added to $\mathbb{Z}_{q,t}$ (see [10, (3.2.10–15)]):

$$\text{(1-7)}\qquad \tau_+\colon X_b \mapsto X_b,\quad T_i \mapsto T_i\ (i>0),\quad Y_r \mapsto X_r Y_r q^{-(\omega_r,\omega_r)/2},$$
$$\tau_+\colon T_0 \mapsto q^{-1} X_\vartheta T_0^{-1},\qquad \pi_r \mapsto q^{-(\omega_r,\omega_r)/2} X_r \pi_r\ (r\in O'),$$
$$\text{(1-8)}\qquad \tau_-\colon Y_b \mapsto Y_b,\quad T_i \mapsto T_i\ (i\geq 0),\quad X_r \mapsto Y_r X_r q^{(\omega_r,\omega_r)/2},$$
$$\tau_-(X_\vartheta) = q T_0 X_\vartheta^{-1} T_{s_\vartheta}^{-1};\qquad \sigma := \tau_+\tau_-^{-1}\tau_+ = \tau_-^{-1}\tau_+\tau_-^{-1},$$
$$\text{(1-9)}\qquad \sigma(X_b) = Y_b^{-1},\quad \sigma(Y_b) = T_{w_0}^{-1} X_{b^\iota}^{-1} T_{w_0},\quad \sigma(T_i) = T_i\ (i>0).$$

These automorphisms fix t_v, q and their fractional powers, as well as the *anti-involution*

$$\text{(1-10)}\qquad \varphi\colon X_b \mapsto Y_b^{-1},\quad Y_b \mapsto X_b^{-1},\quad T_i \mapsto T_i\quad (1\leq i\leq n).$$

We will also need the involution

$$\text{(1-11)}\qquad \eta\colon T_i \mapsto T_i^{-1},\quad X_b \mapsto X_b^{-1},\quad \pi_r \mapsto \pi_r\quad (0\leq i\leq n),$$

which "conjugates" t and q, namely $t_v^{1/(2m)} \mapsto t_v^{-1/(2m)}$ and $q^{1/(2m)} \mapsto q^{-1/(2m)}$. As above, $b\in P$ and $r\in O'$. The involution η extends the *Kazhdan–Lusztig involution* in the affine Hecke theory; see [10, (3.2.19–22)]. Note that

$$\varphi\tau_\pm\varphi = \tau_\mp = \sigma\tau_\pm^{-1}\sigma^{-1},\quad \eta\tau_\pm\eta = \tau_\pm^{-1}.$$

Let us list the matrices corresponding to the automorphisms and anti-automorphisms above upon the natural projection onto $\mathrm{GL}_2(\mathbb{Z})$, corresponding to $t_v^{1/(2m)} = 1 = q^{1/(2m)}$. The matrix $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ will represent the map

$$X_b \mapsto X_b^\alpha Y_b^\gamma,\quad Y_b \mapsto X_b^\beta Y_b^\delta$$

for $b\in P$. One has

$$\tau_+ \mapsto \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\quad \tau_- \mapsto \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix},\quad \sigma \mapsto \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad \varphi \mapsto \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix},\quad \eta \mapsto \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}.$$

Projective $\mathbf{GL_2}$ We define the *projective* $\mathrm{GL}_2^\wedge(\mathbb{Z})$ as the group generated by $\tau_\pm$ and η subject to the relations $\tau_+\tau_-^{-1}\tau_+ = \tau_-^{-1}\tau_+\tau_-^{-1}$, $\eta^2=1$ and $\eta\tau_\pm\eta = \tau_\pm^{-1}$. The span of $\tau_\pm$ is the projective $\mathrm{PSL}_2^\wedge(\mathbb{Z})$ (due to Steinberg), which is isomorphic to the braid group B_3.

1.4 Macdonald polynomials

Following [10], we use the PBW theorem to express any $H \in \mathcal{H}$ in the form $\sum_{a,w,b} c_{a,w,b} X_a T_w Y_b$ for $w \in W$ and $a, b \in P$ (this presentation is unique). Then we substitute as follows:

$$\{\ \}_{ev}\colon X_a \mapsto q^{-(\rho_k,a)}, \quad Y_b \mapsto q^{(\rho_k,b)}, \quad T_i \mapsto t_i^{1/2}. \tag{1-12}$$

Polynomial representation The functional $\mathcal{H} \ni H \mapsto \{H\}_{ev}$, called *coinvariant*, acts via the projection $H \mapsto H\Downarrow := H(1)$ of $\mathcal{H}$ onto the *polynomial representation* $\mathcal{V}$, which is the $\mathcal{H}$–module induced from the one-dimensional character $T_i(1) = t_i^{-1/2} = Y_i(1)$ for $1 \le i \le n$ and $T_0(1) = t_0^{-1/2}$. Recall that $t_0 = t_{\mathrm{sht}}$; see [10; 12; 13] here and below.

In detail, the polynomial representation $\mathcal{V}$ is isomorphic to $\mathbb{Z}_{q,t}[X_b]$ as a vector space, and the action of T_i $(0 \le i \le n)$ there is given by the *Demazure–Lusztig operators*:

$$T_i = t_i^{1/2} s_i + (t_i^{1/2} - t_i^{-1/2})(X_{\alpha_i} - 1)^{-1}(s_i - 1) \quad \text{for } 0 \le i \le n. \tag{1-13}$$

The elements X_b become the multiplication operators and π_r $(r \in O')$ act via the general formula $\widehat{w}(X_b) = X_{\widehat{w}(b)}$ for $\widehat{w} \in \widehat{W}$. Note that τ_- and η naturally act in the polynomial representation. See formula [13, (1.37)]. For the latter,

$$\eta(f) = f^\star, \quad \text{where } X_b^\star = X_{-b},\ (q^\upsilon)^\star = q^{-\upsilon},\ (t^\upsilon)^\star = t^{-\upsilon} \text{ for } \upsilon \in \mathbb{Q}. \tag{1-14}$$

Concerning the action of τ_-, see eg formula [13, (1.37)]. Also,

$$\{\varphi(H)\}_{ev} = \{H\}_{ev}, \quad \{\eta(H)\}_{ev} = \{H\}_{ev}^\star. \tag{1-15}$$

Macdonald polynomials The Macdonald polynomials $P_b(X)$ for $b \in P_+$ (they are due to Kadell for the classical root systems) are uniquely defined as follows. For $c \in P$, let c_+ be a unique element such that $c_+ \in W(c) \cap P_+$. Given $b \in P_+$ and assuming that $b_+ \ne c_+ \in b - Q_+$,

$$P_b - \sum_{a \in W(b)} X_a \in \bigoplus_{c} \mathbb{Q}(q, t_v) X_c \quad \text{and} \quad CT(P_b X_{c'} \mu(X; q, t)) = 0, \tag{1-16}$$

where

$$\mu(X; q, t) := \prod_{\alpha \in R_+} \prod_{j=0}^{\infty} \frac{(1 - X_\alpha q_\alpha^j)(1 - X_\alpha^{-1} q_\alpha^{j+1})}{(1 - X_\alpha t_\alpha q_\alpha^j)(1 - X_\alpha^{-1} t_\alpha q_\alpha^{j+1})}.$$

Here, CT is the constant term; μ is considered a Laurent series of X_b with the coefficients expanded in terms of positive powers of q. The coefficients of P_b belong

to the field $\mathbb{Q}(q,t_v)$. One has

$$P_b(X^{-1}) = P_{b^\iota}(X) = P_b^\star(X), \quad P_b(q^{-\rho_k}) = P_b(q^{\rho_k}) = (P_b(q^{-\rho_k}))^\star, \tag{1-17}$$

where

$$(P_b(q^{-\rho_k}))^\star = q^{-(\rho_k,b)} \prod_{\alpha>0} \prod_{j=0}^{(\alpha^\vee,b)-1} \left(\frac{1-q_\alpha^j t_\alpha X_\alpha(q^{\rho_k})}{1-q_\alpha^j X_\alpha(q^{\rho_k})} \right).$$

Recall that $\iota(b) = b^\iota = -w_0(b)$ for $b \in P$ and see formula [10, (3.3.23)]. We set $P_b^\circ := P_b / P_b(q^{-\rho_k})$ for $b \in P_+$; this is the so called *spherical normalization*.

DAHA provides an important alternative (operator) approach to P–polynomials; namely, they satisfy the (defining) relations

$$L_f(P_b) = f(q^{-\rho_k - b}) P_b, \quad \text{where } b \in P_+,\ L_f := f(X_a \mapsto Y_a), \tag{1-18}$$

for any symmetric (W–invariant) polynomial $f \in \mathbb{C}[X_a, a \in P]^W$.

2 DAHA-Jones theory

2.1 Iterated torus knots

The torus knots $T(\mathsf{r},\mathsf{s})$ are defined for any integers assuming that $\gcd(\mathsf{r},\mathsf{s}) = 1$. One has the symmetries $T(\mathsf{r},\mathsf{s}) = T(\mathsf{s},\mathsf{r}) = T(-\mathsf{r},-\mathsf{s})$, where we use "=" for the ambient isotopy equivalence. Also $T(\mathsf{r},\mathsf{s}) = \bigcirc$ if $|\mathsf{r}| \le 1$ or $|\mathsf{s}| \le 1$ for the *unknot*, denoted by $\bigcirc$ here and below. See eg [38; 17] or the Knot Atlas for the details and the corresponding theory of the invariants.

The *iterated torus knots*, denoted in this paper by $\mathcal{T}(\vec{\mathsf{r}},\vec{\mathsf{s}})$, will be associated with two sequences of integers *of any signs*:

$$\vec{\mathsf{r}} = \{\mathsf{r}_1, \dots \mathsf{r}_\ell\},\ \vec{\mathsf{s}} = \{\mathsf{s}_1, \dots \mathsf{s}_\ell\} \quad \text{such that } \gcd(\mathsf{r}_i, \mathsf{s}_i) = 1. \tag{2-1}$$

We will denote by ℓ the *length* of $\vec{\mathsf{r}}$ and $\vec{\mathsf{s}}$. The pairs $\{\mathsf{r}_i, \mathsf{s}_i\}$ will become *characteristic* or *Newton pairs* for algebraic knots. We will use this name, but generally consider arbitrary iterated knots in this paper.

These are combinatorial data. The (topological) definition of iterated torus knots requires one more sequence:

$$\mathsf{a}_1 = \mathsf{s}_1, \quad \mathsf{a}_i = \mathsf{a}_{i-1}\mathsf{r}_{i-1}\mathsf{r}_i + \mathsf{s}_i \ (i = 2, \dots, m). \tag{2-2}$$

See eg [17] and [36]. Then, in terms of the cabling defined below,

$$\mathcal{T}(\vec{\mathsf{r}},\vec{\mathsf{s}}) := Cab(\vec{\mathsf{a}},\vec{\mathsf{r}})(\bigcirc) = (Cab(\mathsf{a}_\ell,\mathsf{r}_\ell)\cdots Cab(\mathsf{a}_2,\mathsf{r}_2))(T(\mathsf{a}_1,\mathsf{r}_1)). \tag{2-3}$$

Note that the first iteration (application of Cab) is for $\{\mathsf{a}_1, \mathsf{r}_1\}$ (not for the last pair!). Recall that $\bigcirc$ is the *unknot* and knots are considered up to *ambient isotopy* (we use "$=$" for it).

Cabling The *cabling* $Cab(\mathsf{a},\mathsf{b})(K)$ of any oriented knot K in (oriented) S^3 is defined as follows; see eg [32; 17] and references therein. We consider a small 2–dimensional torus around K and put there the torus knot $T(\mathsf{a},\mathsf{b})$ in the direction of K, which is $Cab(\mathsf{a},\mathsf{b})(K)$ (up to ambient isotopy). We will sometimes set $Cab(\vec{\mathsf{a}},\vec{\mathsf{r}}) = Cab(\vec{\mathsf{a}},\vec{\mathsf{r}})(\bigcirc)$.

This procedure depends on the order of a,b and orientation of K. We choose the latter in the standard way (compatible with almost all sources, including the Mathematica package "KnotTheory"); the parameter a gives the number of turns around K. This construction also depends on the framing of the cable knots; we take the natural one, associated with the parallel copy of the torus where a given cable knot sits (its parallel copy has zero linking number with this knot).

By construction, $Cab(\mathsf{a},0)(K) = \bigcirc$ and $Cab(\mathsf{a},1)(K) = K$ for any knot K and $\mathsf{a} \neq 0$. Accordingly, we have the following *reduction cases*:

(2-4) When $\mathsf{r}_i = 0$ $(i < \ell)$,

$$\mathcal{T}(\vec{\mathsf{r}},\vec{\mathsf{s}}) = \mathcal{T}(\{\mathsf{r}_{i+1},\dots,\mathsf{r}_\ell\},\{\mathsf{s}_{i+1},\dots,\mathsf{s}_\ell\}) \quad \text{and} \quad \mathcal{T}(\vec{\mathsf{r}},\vec{\mathsf{s}}) = \bigcirc \text{ for } i = \ell.$$

(2-5) When $\mathsf{r}_i = 1$, $\mathsf{s}_i = 0$,

$$\mathcal{T}(\vec{\mathsf{r}},\vec{\mathsf{s}}) = \mathcal{T}(\{\mathsf{r}_1,\dots,\mathsf{r}_{i-1},\mathsf{r}_{i+1},\dots,\mathsf{r}_\ell\},\{\mathsf{s}_1,\dots,\mathsf{s}_{i-1},\mathsf{s}_{i+1},\dots,\mathsf{s}_\ell\}).$$

(2-6) When $\mathsf{r}_i = 1$, $\mathsf{s}_i \neq 0$,

$$\mathcal{T}(\vec{\mathsf{r}},\vec{\mathsf{s}}) = \mathcal{T}(\{\mathsf{r}_1,\dots,\mathsf{r}_{i-1},\mathsf{r}_{i+1},\dots,\mathsf{r}_\ell\},\{\dots,\mathsf{s}_{i-1},\mathsf{s}'_{i+1},\mathsf{s}_{i+2},\dots\}),$$

where $\mathsf{s}'_{i+1} = \mathsf{s}_{i+1} + \mathsf{s}_i\mathsf{r}_{i+1}$ if $i < \ell$ (no $\mathsf{s}'_{\ell+1}$ for $i = \ell$).

Let us comment on the last relation; see (2-2). Since $r_i = 1$, one has

$$\begin{aligned}\mathsf{a}_i = \mathsf{a}_{i-1}\mathsf{r}_{i-1} + \mathsf{s}_i, \quad \mathsf{a}_{i+1} &= \mathsf{a}_i\mathsf{r}_{i+1} + \mathsf{s}_{i+1} \\ &= \mathsf{a}_{i-1}\mathsf{r}_{i-1}\mathsf{r}_{i+1} + (\mathsf{s}_{i+1} + \mathsf{s}_i\mathsf{r}_{i+1}).\end{aligned}$$

The pairs $\{\mathsf{r}_i,\mathsf{a}_i\}$ are sometimes called topological (cable); the isotopy equivalence of algebraic links generally can be seen only at the level of r,a–parameters (not at the level of the Newton or Puiseux pairs).

Also, $T(\mathsf{r},\mathsf{s}) = T(\mathsf{s},\mathsf{r})$, which results in the *transposition property*:

(2-7) $$\mathcal{T}(\vec{\mathsf{r}},\vec{\mathsf{s}}) = \mathcal{T}(\{\mathsf{s}_1,\mathsf{r}_2,\dots,\mathsf{r}_\ell\},\{\mathsf{r}_1,\mathsf{s}_2,\dots,\mathsf{s}_\ell\}).$$

The counterparts of these properties hold for the DAHA-Jones polynomials, though due to entirely algebraic reasons.

Switching from K to its *mirror image*, denoted by $K^\star$, one has

$$\begin{aligned} Cab(\mathsf{a},\mathsf{b})(K^\star) &= (Cab(-\mathsf{a},\mathsf{b})(K))^\star \quad \text{for any } \mathsf{a},\mathsf{b} \text{ with } \gcd(\mathsf{a},\mathsf{b})=1,\\ Cab(-\vec{\mathsf{a}},\vec{\mathsf{r}}) &= (Cab(\vec{\mathsf{a}},\vec{\mathsf{r}}))^\star,\\ \mathcal{T}(\vec{\mathsf{r}},-\vec{\mathsf{s}}) &= (\mathcal{T}(\vec{\mathsf{r}},\vec{\mathsf{s}}))^\star. \end{aligned} \tag{2-8}$$

Changing the *orientation*, denoted by "$-$", at the i^{th} step, we obtain that for any $1\le i\le \ell$,

$$\begin{aligned} -Cab(\vec{\mathsf{a}},\vec{\mathsf{r}}) &= Cab(\{\dots,\mathsf{a}_{i-1},-\mathsf{a}_i,\mathsf{a}_{i+1},\dots\},\{\dots,\mathsf{r}_{i-1},-\mathsf{r}_i,\mathsf{r}_{i+1},\dots\}),\\ \pm\mathcal{T}(\vec{\mathsf{r}},\vec{\mathsf{s}}) &= \mathcal{T}(\{\dots,\mathsf{r}_{i-1},-\mathsf{r}_i,\mathsf{r}_{i+1},\dots\},\{\dots,\mathsf{s}_{i-1},-\mathsf{s}_i,\mathsf{s}_{i+1},\dots\}). \end{aligned} \tag{2-9}$$

We note that the Jones and HOMFLYPT polynomials for $K^\star$ are obtained from those for K by the formal conjugation of the parameters, namely $q\mapsto q^{-1}$ and $a\mapsto a^{-1}$. This will hold for the DAHA-Jones polynomials and DAHA superpolynomials (the conjugation of t will be added to that of q and a). The orientation does not influence our construction.

2.2 Algebraic knots

There exists a deep connection of the iterated torus knots for strictly positive r_i and s_i with the germs of isolated *plane curve singularities*. Its origin is the Newton's successive approximations for y in terms of x satisfying a polynomial equation $f(x,y)=0$ in a neighborhood of $0=(x=0,y=0)$. The main claim is that the knot $\mathcal{T}(\vec{\mathsf{r}},\vec{\mathsf{s}})$ for the *characteristic pairs* $\{\mathsf{r}_i,\mathsf{s}_i\}$ with $\mathsf{r}_i,\mathsf{s}_i>0$ is the link of the *germ of the singularity*

$$y = x^{\mathsf{s}_1/\mathsf{r}_1}(c_1 + x^{\mathsf{s}_2/(\mathsf{r}_1\mathsf{r}_2)}(c_2+\cdots+x^{\mathsf{s}_\ell/(\mathsf{r}_1\mathsf{r}_2\cdots\mathsf{r}_\ell)})) \quad \text{at } 0, \tag{2-10}$$

which is the intersection of the corresponding plane curve $f(x,y)=0$ with a small 3–dimensional sphere in $\mathbb{C}^2$ around 0. We will always assume that this germ is *unibranch*.

The inequality $\mathsf{s}_1<\mathsf{r}_1$ is commonly imposed here (otherwise x and y can be switched). Formula (2-10) is the celebrated *Newton–Puiseux expansion*, though Puiseux performed the multiplication here,

$$y = b_1x^{\mathsf{m}_1/\mathsf{r}_1} + b_2x^{\mathsf{m}_2/(\mathsf{r}_1\mathsf{r}_2)} + b_3x^{\mathsf{m}_3/(\mathsf{r}_1\mathsf{r}_2\mathsf{r}_3)} + \cdots,$$

for $\mathsf{m}_1=\mathsf{s}_1$ and $\mathsf{m}_i=\mathsf{s}_i+\mathsf{r}_i\mathsf{m}_{i-1}$, where $\{\mathsf{r}_i,\mathsf{m}_i\}$ are called the *Puiseux pairs* (which we will not use).

We will denote the germ of the curve in (2-10) by $\mathcal{C}_{\vec{r},\vec{s}}$. If it has more than one branch (near 0), then its complete topological description requires knowing the sequences $\{\vec{r},\vec{s}\}$ for all components together with the pairwise linking numbers. See eg [31] and [17, page 49]. All algebraic knots can be obtained in such a way; the positivity of r_i and s_i is necessary for $\mathcal{T}(\vec{r},\vec{s})$ to be an algebraic link.

Jacobian factors One can associate with a unibranch $\mathcal{C}_{\vec{r},\vec{s}}$ the *Jacobian factor* $J(\mathcal{C}_{\vec{r},\vec{s}})$. Up to a homeomorphism, it can be introduced as the canonical compactification of the *generalized Jacobian* of an integral *rational* planar curve with $\mathcal{C}_{\vec{r},\vec{s}}$ as its *only* singularity [29]; there is a purely local definition as well. Its dimension is the δ*–invariant* of the singularity $\mathcal{C}_{\vec{r},\vec{s}}$, also called its *arithmetic genus*.

The Jacobian factors are in the focus of many studies; see eg [4; 18, Theorem 1; 46; 36; 29]. The latter paper, where the connection with the *affine Springer fibers* was established, provides a certain link to the theory of DAHA modules [45; 49], though we cannot relate this to our using DAHA. An important connection of $J(\mathcal{C}_{\vec{r},\vec{s}})$ and the *rational DAHA* was established in [33; 23] (see references therein), which must be mentioned, but this is not connected with our approach or that from [12; 13] so far.

Calculating the *Euler number* $e(J(\mathcal{C}_{\vec{r},\vec{s}}))$, the topological Euler characteristic of $J(\mathcal{C}_{\vec{r},\vec{s}})$, and the corresponding *Betti numbers* in terms of $\vec{r}$ and $\vec{s}$ is a challenging problem. For torus knots $T(r,s)$, one has $e(J(\mathcal{C}_{r,s})) = (1/r+s)\binom{r+s}{r}$ due to [4]. This formula is related to the perfect modules of rational DAHA; see eg [21].

The Euler numbers of $J(\mathcal{C}_{\vec{r},\vec{s}})$ were calculated in [36, Main theorem] for the following triples of *Puiseux characteristic exponents*:

$$(4,2u,v),\quad (6,8,v)\quad\text{and}\quad (6,10,v)\quad\text{for odd } u,v>0, \tag{2-11}$$

where $4<2u<v$, $8<v$ and $10<v$, respectively. Here, $\delta=\dim J(\mathcal{C}_{\vec{r},\vec{s}})$ is $(r-1)(s-1)/2$ for $T(r,s)$ and $2u+(v-1)/2-1$ for the series $(4,2u,v)$. Generally, δ equals the cardinality $|\mathbb{N}\setminus\Gamma|$, where Γ is the *valuation semigroup* associated with $\mathcal{C}_{\vec{r},\vec{s}}$; see [36] and [50]. The Euler numbers of the Jacobian factors (any isolated plane curve singularities) can be also calculated now via the HOMFLYPT polynomials of the corresponding links (see below) due to [34; 30].

Concerning the Betti numbers for the torus knots and the series $(4,2u,v)$, the odd (co)homology of $J(\mathcal{C}_{\vec{r},\vec{s}})$ vanishes, and the formulas for the Betti numbers $h^{(2k)}=\mathrm{rk}(H^{2k}(J(\mathcal{C}_{\vec{r},\vec{s}})))$ are known for some values of k (those sufficiently close to 0 or δ), where $0\le k\le\delta$. Not much is known/expected beyond these two series. Paper [36] contains several results and conjectures for the Betti numbers of Jacobian factors; for instance, there are quite a few formulas there for $T(r,s)$ with $\min(r,s)\le 4$.

What DAHA provides According to Conjecture 2.4, the DAHA superpolynomial $\mathcal{H}_{\{\vec{r},\vec{s}\}}$ (defined in Theorem 2.1) at $a=0$, $q=1$ in the uncolored case is $\sum_{k=0}^{\delta} h^{(2k)}t^k$ for *any* unibranch $\mathcal{C}_{\vec{r},\vec{s}}$, which matches the formulas from [36]. This includes/implies the *van Straten–Warmt conjecture*, which claims that all odd Betti numbers of $J(\mathcal{C}_{\vec{r},\vec{s}})$ vanish since the DAHA superpolynomials contain only integral powers of t. In the opposite direction, such an interpretation of $\mathcal{H}_{\{\vec{r},\vec{s}\}}(\Box; q=1,t,a=0)$ would apply its positivity. The whole (uncolored) $\mathcal{H}_{\{\vec{r},\vec{s}\}}$ describes the geometry of Hilbert schemes of $\mathcal{C}_{\vec{r},\vec{s}}$ due to [33, Conjecture 2].

We conjecture that the coefficients of the whole $\mathcal{H}_{\{\vec{r},\vec{s}\}}$ are positive for any rectangle Young diagrams and algebraic knots; let us mention [6] in this respect. If such a positivity holds for one box, the corresponding knots will be called *pseudoalgebraic*; then $\mathcal{H}_{\{\vec{r},\vec{s}\}}$ is expected to coincide with the uncolored *stable reduced Khovanov–Rozansky polynomial* of $\mathcal{T}(\vec{r},\vec{s})$, producing (by definition) the Khovanov–Rozansky polynomials for sl_N $a=t^N\sqrt{q/t}$ for sufficiently large N. The substitution $a=-1$ here is expected to result in the *Heegaard–Floer homology*. This is stated in the DAHA parameters q,t,a; see parts (ii) and (iii) of Conjecture 2.4. We restrict ourselves there to the uncolored case, because this is needed in [33] and in the definition of the Khovanov–Rozansky triply graded homology. However, see [47] and other papers on the *categorification* (via quantum groups).

2.3 DAHA-Jones polynomials

The following theorems are mainly an extension of [13, Theorem 1.2] on DAHA-Jones polynomials from torus knots to arbitrary iterated torus knots.

Torus knots $T(r,s)$ are naturally represented by $\gamma_{r,s}\in \mathrm{GL}_2(\mathbb{Z})$ with the first column $(r,s)^{\mathrm{tr}}$ (tr is the transposition) for $r,s\in\mathbb{Z}$ assuming that $\gcd(r,s)=1$. Then let $\widehat{\gamma}_{r,s}\in \mathrm{GL}_2^{\wedge}(\mathbb{Z})$ be any pullback of $\gamma_{r,s}$.

Note that (r,s) can be obviously lifted to γ of determinant 1 and, accordingly, to $\widehat{\gamma}$ from the subgroup $\mathrm{PSL}_2^{\wedge}(\mathbb{Z})$ generated by $\{\tau_{\pm}\}$, ie without using η. This is actually sufficient for the construction of the DAHA-Jones polynomials below.

For a polynomial F in terms of fractional powers of q and t_v, the *tilde-normalization* $\widetilde{F}$ will be the result of the division of F by the lowest q,t_v–monomial, assuming that it is well defined. We put $q^{\bullet}t^{\bullet}$ for a monomial factor (possibly fractional) in terms of q and t_v.

Theorem 2.1 *Let R be a reduced irreducible root system. Recall that $H\mapsto H\Downarrow := H(1)$, where the action of $H\in\mathcal{H}$ in $\mathcal{V}$ is used. Given two sequences $\vec{r},\vec{s}$ of length*

ℓ as in (2-1), we lift $(\mathsf{r}_i,\mathsf{s}_i)^{\mathrm{tr}}$ to γ_i and then to $\widehat{\gamma}_i\in \mathrm{GL}_2^{\wedge}(\mathbb{Z})$ as above. For a weight $b\in P_+$, the DAHA-Jones polynomial is

$$JD^{R}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t)=JD_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t) \tag{2-12}$$
$$:=\big\{\widehat{\gamma}_1\big(\cdots\big(\widehat{\gamma}_{\ell-1}\big(\big(\widehat{\gamma}_\ell(P_b)/P_b(q^{-\rho_k})\big)\Downarrow\big)\Downarrow\big)\cdots\big)\big\}_{ev}.$$

It does not depend on the particular choice of the lifts $\gamma_i\in \mathrm{GL}_2(\mathbb{Z})$ and $\widehat{\gamma}_i\in \mathrm{GL}_2^{\wedge}(\mathbb{Z})$ for $1\le i\le \ell$ and is indeed a polynomial in terms of q and t_v. Assuming that $\mathsf{r}_i,\mathsf{s}_i>0$ for all i, ie that the knot $\mathcal{T}(\vec{\mathsf{r}},\vec{\mathsf{s}})$ is algebraic, the tilde-normalization $\widetilde{JD}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t)$ is well defined and is a polynomial in terms of q and t_v with constant term 1.

Proof The proofs of this and the next theorem almost exactly follow those in [13] (for torus knots). We will provide here only the justification of the polynomiality. If $JD^{R}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t)$ for $b\in P_+$ is not a polynomial in terms of q and t_v, then $P_b^\circ:=P_b/P_b(q^{-\rho_k})$ has a pole at $\epsilon=0$, where $\epsilon=(1-q^j t_{\mathrm{sht}}^r t_{\mathrm{lng}}^s)$ for certain $j>0, r,s\ge 0,\ r+s>0$. We can assume that ϵ is a maximal such binomial, ie that there is no singularity here at $(1-q^{jv}t_{\mathrm{sht}}^{rv}t_{\mathrm{lng}}^{sv})/\epsilon=0$ for any $\mathbb{Z}\ni v>1$.

Let us localize and complete the ring of coefficients of $\mathcal{HH}$ and the polynomial representation $\mathcal{V}$, which is $\mathbb{Z}_{q,t}=\mathbb{Z}[q^{\pm 1/(2\mathsf{m})},t_v^{\pm 1/2}]$, by this ϵ, ie with respect to the ideal $(1-q^j t_{\mathrm{sht}}^r t_{\mathrm{lng}}^s)$; the notations will be $\mathbb{Z}^{(\epsilon)}_{q,t}$, $\mathcal{HH}^{(\epsilon)}$ and $\mathcal{V}^{(\epsilon)}$. Note that we added $q^{\pm 1/(2\mathsf{m})}$ to $\mathbb{Z}_{q,t}$. We will use the *evaluation pairing*, defined as

$$\{E,F\}_{ev}=E(Y^{-1})(F(X))(q^{-\rho_k}),\quad E,F\in\mathcal{V}^{(\epsilon)}.$$

See [11, Theorem 11.8] and [13, Theorem 1.2]. We set $RAD_{\epsilon,\ell}=\{F\in\mathcal{V}^{(\epsilon)}\mid\{F,\mathcal{V}\}_{ev}\subset\epsilon^\ell\mathbb{Z}^{(\epsilon)}_{q,t}\}$ for $\ell\in\mathbb{Z}_+$. Switching from $\mathcal{V}$ to the whole $\mathcal{HH}$, we define

$$RAD_{\epsilon,\ell}:=\{H\in\mathcal{HH}^{(\epsilon)}\mid\{\mathcal{HH}H\mathcal{HH}\}_{ev}\in\epsilon^\ell\mathbb{Z}^{(\epsilon)}_{q,t}\}\quad\text{for }\ell\in\mathbb{Z}_+. \tag{2-13}$$

Equivalently, $RAD_{\epsilon,\ell}=\{H\in\mathcal{HH}^{(\epsilon)}\mid H(\mathcal{V}^{(\epsilon)})\subset Rad_{\epsilon,\ell}\}$, since $Rad_{\epsilon,\ell}=\{F\in\mathcal{V}^{(\epsilon)}\mid\{\mathcal{HH}(F)\}_{ev}\subset\epsilon^\ell\mathbb{Z}^{(\epsilon)}_{q,t}\}$; see [11, Lemma 11.3].

Here, q is not a root of unity. Therefore, any Y–invariant submodule of $\mathcal{V}$ is invariant with respect to the natural action of τ_- in $\mathcal{V}$. For instance, $\tau_-(P_b)=q^{-(b,b)/2-(b,\rho_k)}P_b$, assuming that P_b for $b\in P_+$ is well defined. We conclude that ψ and τ_- preserve $RAD_{\epsilon,p}$ for any $p\in\mathbb{Z}_+$ (and generic q). The same holds for η; see (1-11). Thus, the whole $\mathrm{GL}_2^{\wedge}(\mathbb{Z})$ preserves each $RAD_{\epsilon,p}$.

Let $\widetilde{P}_b:=\epsilon^l P_b^\circ\in\mathcal{V}^{(\epsilon)}$ for minimal such $l\in\mathbb{N}$. Then $\widetilde{P}_b(q^{-\rho_k})\in\epsilon^l\mathbb{Z}^{(\epsilon)}_{q,t}$ due to the normalization of P_b°. Since $\widetilde{P}_b$ is an eigenfunction of $\{L_f\}$ from (1-18), one has that $\widetilde{P}_b\in Rad_{\epsilon,l}$; see [11, Lemmas 11.4–5].

This implies that $\widehat{\gamma}(\widetilde{P}_b)\in RAD_{\epsilon,l}$ for any $\gamma\in \mathrm{PSL}_2(\mathbb{Z})$. The projection $\widetilde{P}'=\widehat{\gamma}(\widetilde{P}_b)\Downarrow = \widehat{\gamma}(\widetilde{P}_b)(1)$ then belongs to $Rad_{\epsilon,l}$. Hence, $\widehat{\gamma}'(\widetilde{P}')\in RAD_{\epsilon,l}$ for any $\gamma'\in \mathrm{PSL}_2(\mathbb{Z})$ and we can continue. One automatically obtains that $\{\widetilde{P}''\}_{ev}$ is divisible by ϵ^l for $\widetilde{P}'':=\widehat{\gamma}'(\widetilde{P}')\Downarrow$ and for any further such polynomials obtained by this procedure for γ'',γ''', and so on.

We conclude that $JD_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t)$ has actually no singularity at $\epsilon=0$, which is a contradiction sufficient to claim polynomiality of $JD_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t)$ for any $b\in P_+$, $\vec{\mathsf{r}}$ and $\vec{\mathsf{s}}$. We mention that the nonsymmetric Macdonald polynomials are used in the corresponding reasoning from [13], which are actually more convenient here than P_b. □

Theorem 2.2 (i) *The polynomials $JD_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t)$ coincide up to $q^\bullet t^\bullet$ for the pairs from 2.1–2.1, ie for $\{\vec{\mathsf{r}},\vec{\mathsf{s}}\}$ with isotopic $\mathcal{T}(\vec{\mathsf{r}},\vec{\mathsf{s}})$. Also, (2-7) corresponds to*

$$JD_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t)=JD_{\{\mathsf{s}_1,\mathsf{r}_2,\dots,\mathsf{r}_\ell\},\{\mathsf{r}_1,\mathsf{s}_2,\dots,\mathsf{s}_\ell\}}(b;q,t), \tag{2-14}$$

and the following DAHA counterparts of (2-8) and (2-9) hold:

$$JD_{\vec{\mathsf{r}},-\vec{\mathsf{s}}}(b;q,t)=(JD_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t))^\star \quad \text{for } \star \text{ from (1-14)}, \tag{2-15}$$

$$JD_{\vec{\mathsf{r}},\vec{\mathsf{s}}}=JD_{\{\dots,\mathsf{r}_{i-1},-\mathsf{r}_i,\mathsf{r}_{i+1},\dots\},\{\dots,\mathsf{s}_{i-1},-\mathsf{s}_i,\mathsf{s}_{i+1},\dots\}} \quad \text{for } 1\le i\le \ell.$$

(ii) *Let us assume that, for $b,c\in P_+$ and certain $w\in W$ with $w\ne \mathrm{id}$,*

$$q^{(b+\rho_k-w(\rho_k)-w(c),\alpha)}=1=q_\alpha^{(b-w(c),\alpha^\vee)}t_{\mathrm{sht}}^{(\rho^w_{\mathrm{sht}},\alpha^\vee)}t_{\mathrm{lng}}^{(\rho^w_{\mathrm{lng}},\alpha^\vee)} \tag{2-16}$$

for any $\alpha\in R_+$, where we set $\rho^w_v:=w(\rho_v)-\rho_v$. Then

$$JD_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t)=JD_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(c;q,t) \quad \text{for such } q,\{t_v\} \tag{2-17}$$

and for any $\vec{\mathsf{r}}$ and $\vec{\mathsf{s}}$. Also, for $b=\sum_{i=1}^n b_i\omega_i$,

$$JD_{\vec{\mathsf{r}},\vec{\mathsf{s}}}\big(b;q=1,t\big)=\prod_{i=1}^n JD_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\omega_i;q=1,t)^{b_i} \quad \text{for any } \vec{\mathsf{r}} \text{ and } \vec{\mathsf{s}}. \tag{2-18}$$

Justifying the symmetries from part (i) is essentially parallel to those in [13, Theorem 1.2]. Let us comment on 2.1. Essentially, one needs to check here that the torus knot $T(m\mathsf{r}+\mathsf{s},\mathsf{r})$ results in the same DAHA-Jones polynomial as the "2–cable" corresponding to the Newton pairs $\{1,m\},\{\mathsf{r},\mathsf{s}\}$. Topologically, $T(m\mathsf{r}+\mathsf{s},\mathsf{r})$ is isotopic to $Cab(m\mathsf{r}+\mathsf{s},\mathsf{r})T(m,1)$, since $T(m,1)$ is the unknot. Algebraically, such a coincidence is an interesting DAHA calculation.

Extending the connection conjectures from [12], we expect that $JD_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t_v\mapsto q_v)$ coincide up to $q^\bullet$ with the reduced quantum group (WRT) invariants for the corresponding $\mathcal{T}(\vec{\mathsf{r}},\vec{\mathsf{s}})$ and any $b\in P_+$. The quantum group is associated with the root

system $\widetilde{R}$; see [12]. The *shift operator* was used there to deduce this coincidence from [43] in the case of A_n and torus knots; quite a few confirmations were provided for other root systems (including special ones).

Using the shift operator for the iterated knots is almost exactly the same as for the torus knots. The QG invariants of iterated knots are less studied than those for the torus knots, though the Rosso–Jones formula can be potentially used for them and any root systems. In the case of A_1, we will prove (i) using the formulas from [38; 32].

2.4 DAHA superpolynomials

Following [12; 22; 13], Theorems 2.1 and 2.2 can be extended to the *DAHA superpolynomials*, which includes the stabilization of $JD^{A_n}(b;q,t)$ with respect to n. This stabilization was announced in [12]; its proof was published in [22]. Both approaches use [41] and can be extended to arbitrary iterated knots; the DAHA duality conjecture proposed in [12] was proven in [22].

We mainly omit the justifications in this paper; with few reservations, they are similar to those in [22; 13] (for torus knots). The focus of this paper is on the connections with the Khovanov–Rozansky theory and the geometry of (germs of) unibranch plane curve singularities.

The sequences $\vec{\mathsf{r}}$ and $\vec{\mathsf{s}}$ will be as above (ℓ is the length), as well as the iterated knots $\mathcal{T}(\vec{\mathsf{r}},\vec{\mathsf{s}})$. We will also use the DAHA-Jones polynomials $\widetilde{JD}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}$ and (later) the Jacobian factors $J(\mathcal{C}_{\vec{\mathsf{r}},\vec{\mathsf{s}}})$ of $\mathcal{C}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}$, which are the germs of unibranch isolated plane curve singularities; see Section 2.2.

Theorem 2.3 *We switch to the root system A_n for sl_{n+1}, setting $t=t_{\mathrm{sht}}=q^k$. Let us consider $P_+\ni b=\sum_{i=1}^n b_i\omega_i$ as a (dominant) weight for any A_m (for sl_{m+1}) with $m\geq n-1$, where we set $\omega_n=0$ upon the restriction to A_{n-1}.*

(i) **Stabilization** *Given $\mathcal{T}(\vec{\mathsf{r}},\vec{\mathsf{s}})$, there exists $\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t,a)$, a polynomial from $\mathbb{Z}[q,t^{\pm1},a]$, satisfying the relations*

$$\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t,a=-t^{m+1})=\pm q^{\bullet}t^{\bullet}JD^{A_m}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t)\quad \text{for any } m\geq n-1, \tag{2-19}$$

and normalized as follows. The polynomial $\mathcal{H}(a=0)$ is assumed minimal, ie not divisible by q,t or any prime number; the coefficient of the minimal power of t in $\mathcal{H}(q=0,a=0)$ is selected positive.

(ii) **Symmetries** *The corresponding normalization of JD–polynomials, denoted by $\widetilde{JD}$, extends the tilde-normalization for algebraic knots from Theorem 2.1. For sufficiently large m (for all $m\geq n-1$ if $\mathsf{r}_i,\mathsf{s}_i>0$),*

$$\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t,a=-t^{m+1})=\widetilde{JD}^{A_m}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t). \tag{2-20}$$

The symmetries from part (i) of Theorem 2.2 hold for $\mathcal{H}_{\vec{r},\vec{s}}$, except for (2-15); the latter holds for $a^\star = a^{-1}$ and up to $a^\bullet q^\bullet t^\bullet$.

(iii) **Color exchange** *Imposing (2-16), we consider w as an element of $\mathbf{S}_{m+1}$ (the Weyl group for A_m) for every $m \geq n$, naturally acting in the corresponding P. Then for any $\vec{r}$ and $\vec{s}$,*

$$\mathcal{H}_{\vec{r},\vec{s}}(b;q,t,a) = \mathcal{H}_{\vec{r},\vec{s}}(c;q,t,a) \quad \text{for such } q,t. \tag{2-21}$$

In particular, let $w = s_i = (i, i+1)$ with $i < n$ and $b = c - (k + (c,\alpha_i))\alpha_i$ for a dominant c. Then the inequalities $c_i/2 \leq -k \leq c_i + \min\{c_{i\pm 1>0}\}$ for $k \in -\mathbb{Z}_+$ imply (2-16).

(iv) **Specialization and $\deg_a$** *Making $q = 1$, one has*

$$\mathcal{H}_{\vec{r},\vec{s}}(b;q=1,t,a) = \prod_{i=1}^{n} \mathcal{H}_{\vec{r},\vec{s}}(\omega_i;q=1,t,a)^{b_i} \quad \text{for } b = \sum_{i=1}^{n} b_i\omega_i. \tag{2-22}$$

Assuming that $|r_1| > |s_1|$ and $r_i \neq 0$ for $j > 1$, $\deg_a \mathcal{H}_{\vec{r},\vec{s}}(b;q,t,a)$ (the $a-$degree) is no greater than $(\max\{1,|s_1|\}|r_2\cdots r_\ell|-1)$ times $\mathrm{ord}(\lambda_b)$, the number of boxes in the Young diagram λ_b associated with $b \in P_+$.

(v) **Superduality** [25; 12; 22] *Up to a power of q and t,*

$$\mathcal{H}_{\vec{r},\vec{s}}(\lambda;q,t,a) = q^\bullet t^\bullet \mathcal{H}_{\vec{r},\vec{s}}(\lambda^{\mathrm{tr}};t^{-1},q^{-1},a), \tag{2-23}$$

where we switch from (arbitrary) dominant weights b to the corresponding Young diagrams $\lambda = \lambda_b$, and λ^{tr} is the transposition of λ.

Concerning the stabilization and duality, we mainly follow here [22], switching from P_λ to the *modified Macdonald polynomials*. Since the projective action of $\mathrm{PSL}(2,\mathbb{Z})$ is compatible with such a passage (see [22; 41]), the transition to the cables of torus knots is relatively direct (with modest deviations for nonalgebraic knots). Note that the DAHA of type gl_{n+1} and that for sl_{n+1} (of type A_n) result in *coinciding DAHA-Jones polynomials* up to $q^\bullet t^\bullet$, which is not too difficult to justify. The compatibility of DAHA and the Macdonald polynomials of types gl_{n+1} and sl_{n+1} is used here.

Color exchange See [13] for a systematic consideration of the color exchange in (2-21). In terms of the Young diagrams it is as follows. We associate with $c = \sum_{i=1}^n c_i\omega_i$ in part (iii) the Young diagram

$$\lambda_c = \{m_1 = c_1 + \cdots + c_n, m_2 = c_1 + \cdots + c_{n-1}, \ldots, m_n = c_n, 0, 0, \ldots\}.$$

Then, we switch to $\lambda'_c = \{m'_i = m_i - k(i-1)\}$, apply $w \in W$ to λ'_c, and finally obtain

$$\lambda_b = \{m'_{w(i)} + k(i-1)\} = \{m_{w(i)} + k(i - w(i))\}. \tag{2-24}$$

Here, w transforms the rows of λ, and we set

$$w\{m_1, m_2, \dots, m_n\} = \{m_{w(1)}, m_{w(2)}, \dots, m_{w(n)}\}.$$

Given $k < 0$ (it can be fractional), λ_b *must be a Young diagram*; this condition determines which w can be used.

Toward $C^\vee C$ The second part of [12] is devoted to the DAHA-Jones polynomials for the root system $C^\vee C_1$, which prepared the extension of the DAHA-Jones theory to all $C^\vee C_n$. Furthermore, we expect here a relatively smooth switch to arbitrary iterated knots. There is an interesting application of DAHA-Jones polynomials for $C^\vee C_1$ to the superpolynomials of type A for knots $T(2\mathsf{m}+1, 2)$ (ie any A_n). We see relations of this kind for certain iterated knots, say $Cab(2\mathsf{m}+1, 2)T(3, 2)$, but we do not know how far this can go; cf [16].

The a–degree Due to (2-22), it suffices to check the equality for the fundamental weights only. We expect that, for algebraic knots, $\deg_a \mathcal{H}_{\vec{r},\vec{s}}(b; q, t, a)$ from part (iv) equals $(\mathsf{s}_1\mathsf{r}_2\cdots\mathsf{r}_\ell - 1)\mathrm{ord}(\lambda_b)$ (there is no such coincidence in general). This equality is likely to hold for the HOMFLYPT polynomials of algebraic knots (the inequality from part (iv) holds for all iterated knots), which would formally result in such an equality for $\mathcal{H}$ assuming part (i) of Conjecture 2.4 below.

2.5 Connection conjecture

Let us briefly discuss the HOMFLYPT and Khovanov–Rozansky polynomials. The stable Khovanov–Rozansky homology is the sl_N homology from [27; 28] in the range of N where the isomorphism in [37, Theorem 1] holds; see also [26]. Thus, they can be recovered from the *triply graded HOMFLYPT homology*. The cabling formula for the quantum group invariants can be found in [32, Main theorem]; it provides the formulas for the HOMFLYPT polynomials of torus iterated knots. See also [3; 2] and [30] for the HOMFLYPT skein relations.

We need to restrict part (ii) of the conjecture below to the uncolored case, since the Khovanov–Rozansky triply graded homology requires this. The corresponding *reduced* Poincaré polynomial, the so called *stable Khovanov–Rozansky polynomial*, is denoted by $KhR_{\mathrm{stab}}(q, t, a)$ in what will follow. The passage to the *stable* Khovanov–Rozansky polynomials for sl_N, denoted by KhR_N, is $a \mapsto t^N\sqrt{q/t}$. Equivalently, $a_{st} \mapsto q_{st}^N$ in the standard parameters from the conjecture below. We always consider them under the tilde-normalization; the notation will be $\widetilde{KhR}_N$. Note the relation to the *Heegaard–Floer homology* for $N = 0$.

We mention that *categorification theory* can generally deal with arbitrary weights, but this is not sufficiently understood in such a generality. See eg [26; 48; 39; 47] and references therein. Also, our construction is *reduced* (the normalization is by 1 at $\circ$, necessary for the polynomiality of DAHA-Jones polynomials), which creates additional difficulties for the categorification.

The polynomials KhR_{stab} are expected to coincide with the (reduced) physics *superpolynomials* based on the *BPS states* [15; 1; 19; 20] (their theory is not mathematically rigorous) and those obtained in terms of *rational DAHA* [23; 22] for torus knots. The latter are known only in the uncolored case so far; the case of symmetric powers of the fundamental representation is in progress, see [20].

We note that using the Macdonald polynomials instead of Schur functions in the construction of the so called *knot operators* was suggested in [1], which triggered [12]. These operators naturally appear in the approach to the Jones and WRT invariants of torus knots via the Verlinde algebras. This method was actually justified mathematically only for the root systems A and D. It requires roots of unity q and the formula for the Verlinde S–operator, which becomes very involved in the Macdonald setting (even for A_1). Also, the final output must be uniform in terms of q, which cannot be really explained in this approach; this seems the main reason why only the simplest superpolynomials were discussed in [1]. These problems were resolved (actually bypassed) in [12] for any reduced root systems and dominant weights.

Conjecture 2.4 (Connection conjecture) *Let $\mathcal{H}^{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t,a)_{st}$ denote $\mathcal{H}^{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t,a)$ expressed in terms of the standard parameters (in the Khovanov–Rozansky theory; see [12] and [33, Section 1]):*

$$\begin{aligned} t=q_{st}^2,\quad q=(q_{st}t_{st})^2,\quad a=a_{st}^2t_{st},\\ q_{st}^2=t,\quad t_{st}=\sqrt{q/t},\quad a_{st}^2=a\sqrt{t/q}. \end{aligned} \tag{2-25}$$

(i) *We conjecture that for any $\vec{\mathsf{r}}$ and $\vec{\mathsf{s}}$ as above (possibly negative),*

$$\mathcal{H}^{\vec{\mathsf{r}},\vec{\mathsf{s}}}(b;q,t\mapsto q,a\mapsto -a)=\widetilde{HOM}^{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\lambda_b;q,a), \tag{2-26}$$

where the latter is the tilde-normalization of the (reduced) HOMFLYPT polynomial; b is an arbitrary dominant weight represented by the Young diagram λ_b. The authors conjecture that the a–degrees of these polynomials are equal to $(\mathsf{s}_1\mathsf{r}_2\cdots\mathsf{r}_\ell-1)\,\mathrm{ord}(\lambda_b)$ provided the positivity of s_i and r_i, and that $\mathsf{s}_1<\mathsf{r}_1$.

(ii) *For uncolored pseudoalgebraic knots (those with positive coefficients of uncolored DAHA superpolynomials), one has*

$$\mathcal{H}^{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\Box;q,t,a)_{st}=\widetilde{KhR}_{\mathrm{stab}}(q_{st},t_{st},a_{st}),\qquad \text{where } \Box=\omega_1. \tag{2-27}$$

We denote by $\widetilde{KhR}_{stab}$ the (reduced) KhR_{stab} divided by the smallest power of a_{st} and by $q_{st}^{\bullet}t_{st}^{\bullet}$ such that $\widetilde{KhR}_{stab}(a_{st}=0)\in\mathbb{Z}_+[q_{st},t_{st}]$ with the constant term 1.

(iii) Following Section 2.2, let $\mathcal{C}_{\vec{r},\vec{s}}$ be the (unibranch) germ at 0 of the plain curve singularity corresponding to $\{r_i,s_i>0\}$ and $h^{(k)}=\mathrm{rk}(H^k(J(\mathcal{C}_{\vec{r},\vec{s}})))$ the Betti numbers of the Jacobian factor $J(\mathcal{C}_{\vec{r},\vec{s}})$. Then [33, Conjecture 2] and claim (ii) above result in the coincidence of the tilde-normalization of reduced $\mathcal{P}_{alg}$ there with $\mathcal{H}_{\vec{r},\vec{s}}(\Box;q,t,a)_{st}$; see formula (4-24) below for some details. We also conjecture that

$$\mathcal{H}_{\vec{r},\vec{s}}(\Box;q=1,t,a=0)=\sum_{i=0}^{2\delta}h^{(i)}t^{i/2}\qquad \text{for } \delta=\dim J(\mathcal{C}_{\vec{r},\vec{s}}), \tag{2-28}$$

which implies that $h^{odd}=0$ (the van Straten–Warmt conjecture).

We hope that we will be able to justify part (i) in the Connection conjecture following [12, Proposition 2.3], where we used [43] (for torus knots), and Proposition 4.2 below, which states that

$$\mathcal{H}(m\omega_1;q,q,-q^2)=\widetilde{HOM}(m\omega_1;q,q^2)=\widetilde{JD}^{A_1}(m\omega_1;q,q). \tag{2-29}$$

The latter is the reduced tilde-normalized Jones polynomial for any iterated knot $\mathcal{T}(\vec{r},\vec{s})$. Using [38; 32], this approach can be generally extended to any root systems.

Concerning part (ii), it is generally difficult to calculate KhR_{n+1} except for $n=1$ (the celebrated Khovanov polynomials), especially for cables; KhR_{stab} are of a more algebraic nature, but not too much simpler. Even if KhR_{stab} is known, the problem of recovering all individual KhR_{n+1} from it is quite a challenge. This is generally provided by the theory of *differentials* ∂_{n+1} from [26; 37]. The corresponding homology associated with ∂_{n+1} gives KhR_{n+1} for any $n\geq 1$.

These differentials are involved, but their certain algebraic simplification, suggested in [37; 15] and developed further in [12], works surprisingly well for sufficiently small knots. The assumption in [12] is that the actual ∂_{n+1} are "as surjective as possible" beginning with $a=0$. See [12, Conjecture 2.7; Section 3.6]; the "smallest" torus knot when KhR_2 cannot be obtained this way is $T(12,7)$. The output of this reduction procedure will be denoted by $DAHA'KhR_{n+1}$.

For example, if $\mathcal{H}$ (in the DAHA-parameters) contains $q^i(t^ja^m+t^{j-n-1}a^{m+1}+t^{j-2n-2}a^{m+2})$, where $q^it^ia^m$ was not involved in the reductions for smaller $\deg_a$, then $q^it^{j-2n-2}a^{m+2}$ will go to $DAHA'KhR_{n+1}$ from this triple (subject to possible further reductions). However if the actual (*topological*) ∂_{n+1} is not *onto* when acting from the space associated with $q^it^{j-n-1}a^{m+1}$ to that for $q^it^ja^m$, then $q^it^ja^m$ is the

right term to pick for KhR_{n+1}. Thus, there can be *topological corrections*, which makes the above recovery algorithm insufficient. Nevertheless, even if these corrections are present, one can expect that $\widetilde{KhR}_2 - DAHA' KhR_2$ is a linear combination of $q_{st}^i t_{st}^j(1+t_{st})$ (the case of $\partial_2 = 0$), $q_{st}^i t_{st}^j(1-t_{st}^2)$ and so on *with nonnegative coefficients*. If this is the case, then such a positivity can be considered a confirmation of part (ii), which always holds in the examples of pseudoalgebraic knots we considered.

We extend this procedure to arbitrary iterated knots changing all "-1" by t^{n+1}/a in $\mathcal{H}(\square; q, t, a)$ followed by the reduction as above. Practically, one deletes in $\mathcal{H}$ all pairs $\pm t^i q^j a^m (t^{n+1} + a)$ and $t^i q^j a^m (t^{2n+2} - a^2)$ beginning with $m = 0$, then switches to the standard parameters and substitutes $-1 \mapsto 1/t_{st}$. We will also use $DAHA^+ KhR$, where -1 is replaced by a/t^{n+1} before applying the reduction procedure. In all (uncolored) examples we considered, one of these two procedures gives correct $\widetilde{KhR}_{n+1}$ for stable n, or else results in the corrections as above for any n. Presumably $DAHA^+ KhR$ must be used here when the a–leading term of $\mathcal{H}(\square; q, t, a)$ is positive, but the evidence is limited so far.

3 Numerical confirmations

We will consider examples (mainly numerical) confirming the Connection conjecture. We selected only really necessary, relatively simple and instructional ones. They are expected to contain a lot of geometric-topological information (including the Betti numbers of Jacobian factors), and are important confirmations of our Connection conjecture. We do not discuss torus knots here; see [12; 13].

3.1 Simplest algebraic iterations

(1) $\vec{r} = \{3, 2\}, \quad \vec{s} = \{2, 1\}, \quad \mathcal{T} = Cab(13, 2)T(3, 2);$

$$\begin{aligned}\mathcal{H}_{\vec{r},\vec{s}}(\square; q, t, a) = {} & 1 + qt + q^2 t + q^3 t + q^2 t^2 + q^3 t^2 + 2q^4 t^2 + q^3 t^3 + q^4 t^3 + 2q^5 t^3 \\ & + q^4 t^4 + q^5 t^4 + 2q^6 t^4 + q^5 t^5 + q^6 t^5 + q^7 t^5 + q^6 t^6 + q^7 t^6 + q^7 t^7 + q^8 t^8 \\ & + a^3 \big(q^6 + q^7 t + q^8 t^2\big) \\ & + a^2 \big(q^3 + q^4 + q^5 + q^4 t + 2q^5 t + 2q^6 t + q^5 t^2 + 2q^6 t^2 + 2q^7 t^2 \\ & + q^6 t^3 + 2q^7 t^3 + q^8 t^3 + q^7 t^4 + q^8 t^4 + q^8 t^5\big) \\ & + a\big(q + q^2 + q^3 + q^2 t + 2q^3 t + 3q^4 t + q^5 t + q^3 t^2 + 2q^4 t^2 + 4q^5 t^2 \\ & + q^6 t^2 + q^4 t^3 + 2q^5 t^3 + 4q^6 t^3 + q^7 t^3 + q^5 t^4 + 2q^6 t^4 \\ & + 3q^7 t^4 + q^6 t^5 + 2q^7 t^5 + q^8 t^5 + q^7 t^6 + q^8 t^6 + q^8 t^7\big).\end{aligned}$$

Betti numbers We will systematically provide the values of the DAHA superpolynomials at $a=0, q=1$; the corresponding Betti numbers (conjecturally coinciding with their t–coefficients) are known only in 2 cases from the eight uncolored algebraic knots we consider. We will also give the formula for such a *DAHA-Betti polynomial* for $2\omega_1$ and $\mathcal{T}_{\{3,2\},\{2,1\}} = Cab(13,2)T(3,2)$, which is the square of $\mathcal{H}(q=1,t,a=0)$ in (3-1), that for ω_2, and calculate the *DAHA-Betti polynomials* for nonalgebraic knots. As for the latter, the coefficients in (3-11) and (3-12) are clear predecessors of the Betti numbers for $Cab(13,2)T(3,2)$, the first algebraic knot in the family $Cab(2\mathsf{m}+1,2)T(3,2)$ for $\mathsf{m}\in\mathbb{Z}$.

For $\vec{\mathsf{r}}=\{3,2\}, \vec{\mathsf{s}}=\{2,1\}$, the DAHA-Betti polynomial equals

$$\mathcal{H}(q=1,t,a=0) = 1+3t+4t^2+4t^3+4t^4+3t^5+2t^6+t^7+t^8, \tag{3-1}$$

which gives exactly the first line of Betti numbers from the table before [36, Conjecture 23]; the corresponding Puiseux exponents are $(4,6,7)$. The next example gives the second line in this table, for the exponents $(4,6,9)$; see formula (3-2) below. The corresponding two $\mathcal{C}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}$ are represented by the local rings $\mathcal{R}=\mathbb{C}[\![z^4, z^6+z^{7,9}]\!]$; $\delta=8,9$. Note that *the arithmetic genus δ always coincides with the t–degree of the Betti polynomial* (in the examples below). Any other $\mathcal{H}(q=1,t,a=0)$ provided below are beyond known (or conjectured) formulas for the Betti numbers of $J(\mathcal{C}_{\vec{\mathsf{r}},\vec{\mathsf{s}}})$ (as far as we know).

HOMFLYPT polynomial It coincides with the corresponding DAHA-generated tilde-normalized HOMFLYPT polynomial:

$$\begin{aligned}
&\mathcal{H}(q,t\mapsto q,a\mapsto -a)\\
&=1+q^2+q^3+2q^4+q^5+3q^6+q^7+3q^8+q^9+3q^{10}+q^{11}+2q^{12}+q^{13}+q^{14}+q^{16}\\
&\qquad -a^3\big(q^6+q^8+q^{10}\big)\\
&\qquad +a^2\big(q^3+q^4+2q^5+2q^6+3q^7+2q^8+3q^9+2q^{10}+2q^{11}+q^{12}+q^{13}\big)\\
&\qquad -a\big(q+q^2+2q^3+2q^4+4q^5+3q^6+5q^7+3q^8+5q^9\\
&\qquad +3q^{10}+4q^{11}+2q^{12}+2q^{13}+q^{14}+q^{15}\big).
\end{aligned}$$

Up to $q^{\bullet}$, this polynomial is equal to the one considered in [34, Section 7] upon its division by a^{16} and the substitution $z=q-q^{-1}$. We need to change a^2 and q^2 there by our a and q, ie switch from the *standard parameters* to ours.

All superpolynomials we obtained (we provide here only some), including nonalgebraic iterated knots, satisfy the HOMFLYPT part of the Connection conjecture. The HOMFLYPT formulas will be omitted below. We used software by S Artamonov for calculating colored HOMFLYPT polynomials, connected with papers [3; 2].

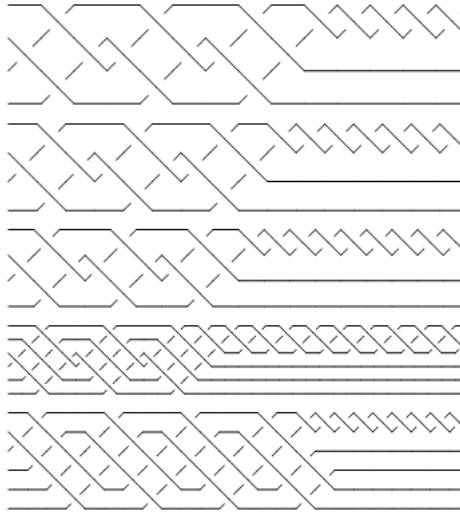

Figure 1: Cables $(11,2)$, $(13,2)$, $(15,2)$ of $T(3,2)$, $Cab(19,3)T(3,2)$ and $Cab(25,2)T(4,3)$.

Figure 1 contains the braid for the knot $Cab(13,2)T(3,2)$ and further iterated knots we will consider below. These presentations are used to calculate the HOMFLYPT polynomials and Khovanov polynomials.

Khovanov–Rozansky polynomials We provide the corresponding DAHA-Khovanov polynomials for all examples below in *the standard parameters*; for instance, q and t in the next two formulas are q_{st} and t_{st} from the conjecture. If the algorithm from [12] results in $DAHA'Kh := DAHA'KhR_2$ matching the Khovanov polynomial (for sl_2), then this is an indication that $DAHA'KhR_{n+1}$ can match $\widetilde{KhR}_{n+1}$ for all n.

For this and the next knot, we will provide $DAHA'KhR_3$, but will omit them in all other examples (they were calculated for any n); $\widetilde{KhR}_{\geq 3}$ are unknown for the knots we consider here.

(1) $$\begin{aligned} DAHA'Kh(Cab(13,2)T(3,2)) &= 1+q^4t^2+q^6t^3+q^6t^4+q^{10}t^5+q^8t^6 \\ &\quad +q^{12}t^7+2q^{12}t^8+2q^{14}t^9+q^{16}t^{10}+2q^{18}t^{11}+2q^{20}t^{12}+q^{22}t^{13}, \\ DAHA'KhR_3(Cab(13,2)T(3,2)) &= 1+q^4t^2+q^8t^3+q^6t^4+q^8t^4+q^{10}t^5 \\ &\quad +q^{12}t^5+q^8t^6+q^{10}t^6+2q^{14}t^7+q^{16}t^7+2q^{12}t^8+q^{14}t^8+q^{18}t^8 \\ &\quad +2q^{16}t^9+2q^{18}t^9+2q^{16}t^{10}+q^{22}t^{10}+3q^{20}t^{11}+q^{22}t^{11}+2q^{20}t^{12} \\ &\quad +q^{24}t^{12}+q^{26}t^{12}+3q^{24}t^{13}+q^{24}t^{14}+q^{28}t^{14}+2q^{28}t^{15}+q^{32}t^{16}. \end{aligned}$$

This knot, the next one, and the last two knots (6 and 7 below) have their DAHA-Khovanov polynomials $DAHA'Kh$ $(n=2)$ coinciding with the actual Khovanov polynomials (under the tilde-normalization).

$\mathcal{H}_{\{3,2\},\{2,3\}}$ switch from $Cab(13,2)$ to the "next" $Cab(15,2)$.

(2) $\vec{\mathsf{r}}=\{3,2\},\quad \vec{\mathsf{s}}=\{2,3\},\quad \mathcal{T}=Cab(15,2)T(3,2);$

$$\begin{aligned}\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\square;q,t,a) = 1 &+ qt + q^2t + q^3t + q^2t^2 + q^3t^2 + 2q^4t^2 + q^3t^3 + q^4t^3\\ &+ 2q^5t^3 + q^4t^4 + q^5t^4 + 2q^6t^4 + q^5t^5 + q^6t^5 + 2q^7t^5\\ &+ q^6t^6 + q^7t^6 + q^8t^6 + q^7t^7 + q^8t^7 + q^8t^8 + q^9t^9\\ &+ a^3\big(q^6 + q^7t + q^8t^2 + q^9t^3\big)\\ &+ a^2\big(q^3 + q^4 + q^5 + q^4t + 2q^5t + 2q^6t + q^5t^2 + 2q^6t^2 + 2q^7t^2 + q^6t^3\\ &+ 2q^7t^3 + 2q^8t^3 + q^7t^4 + 2q^8t^4 + q^9t^4 + q^8t^5 + q^9t^5 + q^9t^6\big)\\ &+ a\big(q + q^2 + q^3 + q^2t + 2q^3t + 3q^4t + q^5t + q^3t^2 + 2q^4t^2 + 4q^5t^2 + q^6t^2 + q^4t^3\\ &+ 2q^5t^3 + 4q^6t^3 + q^7t^3 + q^5t^4 + 2q^6t^4 + 4q^7t^4 + q^8t^4 + q^6t^5 + 2q^7t^5\\ &+ 3q^8t^5 + q^7t^6 + 2q^8t^6 + q^9t^6 + q^8t^7 + q^9t^7 + q^9t^8\big).\end{aligned}$$

Betti numbers

(3-2) $\mathcal{H}(q=1,t,a=0) = 1+3t+4t^2+4t^3+4t^4+4t^5+3t^6+2t^7+t^8+t^9.$

Khovanov–Rozansky polynomials

(2) $$\begin{aligned}DAHA'Kh(Cab(15,2)T(3,2)) &= 1 + q^4t^2 + q^6t^3 + q^6t^4\\ &\quad + q^{10}t^5 + q^8t^6 + q^{12}t^7 + 2q^{12}t^8 + 2q^{14}t^9 + q^{18}t^{11} + q^{20}t^{12},\\ DAHA'KhR_3(Cab(15,2)T(3,2)) &= 1 + q^4t^2 + q^8t^3 + q^6t^4 + q^8t^4\\ &\quad + q^{10}t^5 + q^{12}t^5 + q^8t^6 + q^{10}t^6 + 2q^{14}t^7 + q^{16}t^7 + 2q^{12}t^8 + q^{14}t^8 + q^{18}t^8\\ &\quad + 2q^{16}t^9 + 2q^{18}t^9 + q^{16}t^{10} + q^{22}t^{10} + 2q^{20}t^{11} + q^{22}t^{11} + q^{24}t^{12} + q^{26}t^{12}.\end{aligned}$$

3.2 Degree 5 and beyond

The a–degree was 3 in the previous examples; let us consider two examples of uncolored DAHA superpolynomials of degree 5 (which is $\mathsf{s}_1\mathsf{r}_2-1$ in these cases) and then some further examples of DAHA-Betti polynomials, which are beyond known or conjectured formulas. We note that $DAHA'KhR$ do not coincide with the reduced (tilde-normalized) Khovanov polynomials $\widetilde{Kh}$ in these two examples, but the structure of the difference matches our conjecture.

(3) $\vec{\mathsf{r}}=\{3,3\},\quad \vec{\mathsf{s}}=\{2,1\},\quad \mathcal{T}=Cab(19,3)T(3,2);$

$$\begin{aligned}\mathcal{H}_{\tilde{r},\tilde{s}}(\square;q,t,a) = {} & 1+qt+q^2t+q^3t+q^4t+q^5t+q^2t^2+q^3t^2+2q^4t^2+2q^5t^2+3q^6t^2\\
&+2q^7t^2+q^8t^2+q^3t^3+q^4t^3+2q^5t^3+3q^6t^3+4q^7t^3+4q^8t^3+4q^9t^3+q^{10}t^3+q^4t^4\\
&+q^5t^4+2q^6t^4+3q^7t^4+5q^8t^4+5q^9t^4+6q^{10}t^4+4q^{11}t^4+q^{12}t^4+q^5t^5+q^6t^5\\
&+2q^7t^5+3q^8t^5+5q^9t^5+6q^{10}t^5+7q^{11}t^5+6q^{12}t^5+3q^{13}t^5+q^6t^6+q^7t^6+2q^8t^6\\
&+3q^9t^6+5q^{10}t^6+6q^{11}t^6+8q^{12}t^6+7q^{13}t^6+3q^{14}t^6+q^{15}t^6+q^7t^7+q^8t^7+2q^9t^7\\
&+3q^{10}t^7+5q^{11}t^7+6q^{12}t^7+8q^{13}t^7+8q^{14}t^7+3q^{15}t^7+q^8t^8+q^9t^8+2q^{10}t^8\\
&+3q^{11}t^8+5q^{12}t^8+6q^{13}t^8+8q^{14}t^8+7q^{15}t^8+3q^{16}t^8+q^9t^9+q^{10}t^9+2q^{11}t^9\\
&+3q^{12}t^9+5q^{13}t^9+6q^{14}t^9+8q^{15}t^9+6q^{16}t^9+q^{17}t^9+q^{10}t^{10}+q^{11}t^{10}+2q^{12}t^{10}\\
&+3q^{13}t^{10}+5q^{14}t^{10}+6q^{15}t^{10}+7q^{16}t^{10}+4q^{17}t^{10}+q^{11}t^{11}+q^{12}t^{11}+2q^{13}t^{11}\\
&+3q^{14}t^{11}+5q^{15}t^{11}+6q^{16}t^{11}+6q^{17}t^{11}+q^{18}t^{11}+q^{12}t^{12}+q^{13}t^{12}+2q^{14}t^{12}\\
&+3q^{15}t^{12}+5q^{16}t^{12}+5q^{17}t^{12}+4q^{18}t^{12}+q^{13}t^{13}+q^{14}t^{13}+2q^{15}t^{13}+3q^{16}t^{13}\\
&+5q^{17}t^{13}+4q^{18}t^{13}+q^{19}t^{13}+q^{14}t^{14}+q^{15}t^{14}+2q^{16}t^{14}+3q^{17}t^{14}+4q^{18}t^{14}\\
&+2q^{19}t^{14}+q^{15}t^{15}+q^{16}t^{15}+2q^{17}t^{15}+3q^{18}t^{15}+3q^{19}t^{15}+q^{16}t^{16}+q^{17}t^{16}\\
&+2q^{18}t^{16}+2q^{19}t^{16}+q^{20}t^{16}+q^{17}t^{17}+q^{18}t^{17}+2q^{19}t^{17}+q^{20}t^{17}+q^{18}t^{18}\\
&+q^{19}t^{18}+q^{20}t^{18}+q^{19}t^{19}+q^{20}t^{19}+q^{20}t^{20}+q^{21}t^{21}\\
&+a^5\big(q^{15}+q^{16}t+q^{17}t+q^{17}t^2+q^{18}t^2+q^{19}t^2+q^{18}t^3+q^{19}t^3+q^{19}t^4+q^{20}t^4+q^{20}t^5+q^{21}t^6\big)\\
&+a^4\big(q^{10}+q^{11}+q^{12}+q^{13}+q^{14}+q^{11}t+2q^{12}t+3q^{13}t+3q^{14}t+3q^{15}t+q^{16}t+q^{12}t^2\\
&+2q^{13}t^2+4q^{14}t^2+5q^{15}t^2+5q^{16}t^2+3q^{17}t^2+q^{18}t^2+q^{13}t^3+2q^{14}t^3+4q^{15}t^3\\
&+6q^{16}t^3+6q^{17}t^3+3q^{18}t^3+q^{19}t^3+q^{14}t^4+2q^{15}t^4+4q^{16}t^4+6q^{17}t^4+6q^{18}t^4\\
&+3q^{19}t^4+q^{15}t^5+2q^{16}t^5+4q^{17}t^5+6q^{18}t^5+5q^{19}t^5+q^{20}t^5+q^{16}t^6+2q^{17}t^6\\
&+4q^{18}t^6+5q^{19}t^6+3q^{20}t^6+q^{17}t^7+2q^{18}t^7+4q^{19}t^7+3q^{20}t^7+q^{21}t^7+q^{18}t^8\\
&+2q^{19}t^8+3q^{20}t^8+q^{21}t^8+q^{19}t^9+2q^{20}t^9+q^{21}t^9+q^{20}t^{10}+q^{21}t^{10}+q^{21}t^{11}\big)\\
&+a^3\big(q^6+q^7+2q^8+2q^9+2q^{10}+q^{11}+q^{12}+q^7t+2q^8t+4q^9t+6q^{10}t+7q^{11}t\\
&+4q^{13}t+2q^{14}t+q^8t^2+2q^9t^2+5q^{10}t^2+8q^{11}t^2+12q^{12}t^2+12q^{13}t^2+10q^{14}t^2\\
&+5q^{15}t^2+2q^{16}t^2+q^9t^3+2q^{10}t^3+5q^{11}t^3+9q^{12}t^3+14q^{13}t^3+17q^{14}t^3\\
&+15q^{15}t^3+9q^{16}t^3+3q^{17}t^3+q^{18}t^3+q^{10}t^4+2q^{11}t^4+5q^{12}t^4+9q^{13}t^4+15q^{14}t^4\\
&+19q^{15}t^4+18q^{16}t^4+10q^{17}t^4+3q^{18}t^4+q^{11}t^5+2q^{12}t^5+5q^{13}t^5+9q^{14}t^5\\
&+15q^{15}t^5+20q^{16}t^5+18q^{17}t^5+9q^{18}t^5+2q^{19}t^5+q^{12}t^6+2q^{13}t^6+5q^{14}t^6\\
&+9q^{15}t^6+15q^{16}t^6+19q^{17}t^6+15q^{18}t^6+5q^{19}t^6+q^{13}t^7+2q^{14}t^7+5q^{15}t^7\\
&+9q^{16}t^7+15q^{17}t^7+17q^{18}t^7+10q^{19}t^7+2q^{20}t^7+q^{14}t^8+2q^{15}t^8+5q^{16}t^8\\
&+9q^{17}t^8+14q^{18}t^8+12q^{19}t^8+4q^{20}t^8+q^{15}t^9+2q^{16}t^9+5q^{17}t^9+9q^{18}t^9\\
&+12q^{19}t^9+6q^{20}t^9+q^{21}t^9+q^{16}t^{10}+2q^{17}t^{10}+5q^{18}t^{10}+8q^{19}t^{10}+7q^{20}t^{10}\\
&+q^{21}t^{10}+q^{17}t^{11}+2q^{18}t^{11}+5q^{19}t^{11}+6q^{20}t^{11}+2q^{21}t^{11}+q^{18}t^{12}+2q^{19}t^{12}\\
&+4q^{20}t^{12}+2q^{21}t^{12}+q^{19}t^{13}+2q^{20}t^{13}+2q^{21}t^{13}+q^{20}t^{14}+q^{21}t^{14}+q^{21}t^{15}\big)\end{aligned}$$

$$
\begin{aligned}
&+a^2\big(q^3+q^4+2q^5+2q^6+2q^7+q^8+q^9+q^4t+2q^5t+4q^6t+6q^7t+8q^8t+7q^9t+6q^{10}t\\
&\quad+3q^{11}t+q^{12}t+q^5t^2+2q^6t^2+5q^7t^2+8q^8t^2+13q^9t^2+15q^{10}t^2+15q^{11}t^2+10q^{12}t^2\\
&\quad+5q^{13}t^2+q^{14}t^2+q^6t^3+2q^7t^3+5q^8t^3+9q^9t^3+15q^{10}t^3+20q^{11}t^3+24q^{12}t^3+19q^{13}t^3\\
&\quad+11q^{14}t^3+4q^{15}t^3+q^{16}t^3+q^7t^4+2q^8t^4+5q^9t^4+9q^{10}t^4+16q^{11}t^4+22q^{12}t^4+29q^{13}t^4\\
&\quad+27q^{14}t^4+16q^{15}t^4+6q^{16}t^4+q^{17}t^4+q^8t^5+2q^9t^5+5q^{10}t^5+9q^{11}t^5+16q^{12}t^5+23q^{13}t^5\\
&\quad+31q^{14}t^5+30q^{15}t^5+19q^{16}t^5+6q^{17}t^5+q^{18}t^5+q^9t^6+2q^{10}t^6+5q^{11}t^6+9q^{12}t^6+16q^{13}t^6\\
&\quad+23q^{14}t^6+32q^{15}t^6+30q^{16}t^6+16q^{17}t^6+4q^{18}t^6+q^{10}t^7+2q^{11}t^7+5q^{12}t^7+9q^{13}t^7\\
&\quad+16q^{14}t^7+23q^{15}t^7+31q^{16}t^7+27q^{17}t^7+11q^{18}t^7+q^{19}t^7+q^{11}t^8+2q^{12}t^8+5q^{13}t^8\\
&\quad+9q^{14}t^8+16q^{15}t^8+23q^{16}t^8+29q^{17}t^8+19q^{18}t^8+5q^{19}t^8+q^{12}t^9+2q^{13}t^9+5q^{14}t^9\\
&\quad+9q^{15}t^9+16q^{16}t^9+22q^{17}t^9+24q^{18}t^9+10q^{19}t^9+q^{20}t^9+q^{13}t^{10}+2q^{14}t^{10}+5q^{15}t^{10}\\
&\quad+9q^{16}t^{10}+16q^{17}t^{10}+20q^{18}t^{10}+15q^{19}t^{10}+3q^{20}t^{10}+q^{14}t^{11}+2q^{15}t^{11}+5q^{16}t^{11}\\
&\quad+9q^{17}t^{11}+15q^{18}t^{11}+15q^{19}t^{11}+6q^{20}t^{11}+q^{15}t^{12}+2q^{16}t^{12}+5q^{17}t^{12}+9q^{18}t^{12}+13q^{19}t^{12}\\
&\quad+7q^{20}t^{12}+q^{21}t^{12}+q^{16}t^{13}+2q^{17}t^{13}+5q^{18}t^{13}+8q^{19}t^{13}+8q^{20}t^{13}+q^{21}t^{13}+q^{17}t^{14}\\
&\quad+2q^{18}t^{14}+5q^{19}t^{14}+6q^{20}t^{14}+2q^{21}t^{14}+q^{18}t^{15}+2q^{19}t^{15}+4q^{20}t^{15}+2q^{21}t^{15}+q^{19}t^{16}\\
&\quad+2q^{20}t^{16}+2q^{21}t^{16}+q^{20}t^{17}+q^{21}t^{17}+q^{21}t^{18}\big)\\
&+a\big(q+q^2+q^3+q^4+q^5+q^2t+2q^3t+3q^4t+4q^5t+5q^6t+4q^7t+2q^8t+q^9t+q^3t^2+2q^4t^2\\
&\quad+4q^5t^2+6q^6t^2+9q^7t^2+10q^8t^2+9q^9t^2+5q^{10}t^2+2q^{11}t^2+q^4t^3+2q^5t^3+4q^6t^3+7q^7t^3\\
&\quad+11q^8t^3+14q^9t^3+16q^{10}t^3+13q^{11}t^3+6q^{12}t^3+2q^{13}t^3+q^5t^4+2q^6t^4+4q^7t^4+7q^8t^4\\
&\quad+12q^9t^4+16q^{10}t^4+20q^{11}t^4+20q^{12}t^4+13q^{13}t^4+4q^{14}t^4+q^{15}t^4+q^6t^5+2q^7t^5+4q^8t^5\\
&\quad+7q^9t^5+12q^{10}t^5+17q^{11}t^5+22q^{12}t^5+24q^{13}t^5+17q^{14}t^5+7q^{15}t^5+q^{16}t^5+q^7t^6+2q^8t^6\\
&\quad+4q^9t^6+7q^{10}t^6+12q^{11}t^6+17q^{12}t^6+23q^{13}t^6+26q^{14}t^6+18q^{15}t^6+7q^{16}t^6+q^{17}t^6+q^8t^7\\
&\quad+2q^9t^7+4q^{10}t^7+7q^{11}t^7+12q^{12}t^7+17q^{13}t^7+23q^{14}t^7+26q^{15}t^7+17q^{16}t^7+4q^{17}t^7\\
&\quad+q^9t^8+2q^{10}t^8+4q^{11}t^8+7q^{12}t^8+12q^{13}t^8+17q^{14}t^8+23q^{15}t^8+24q^{16}t^8+13q^{17}t^8\\
&\quad+2q^{18}t^8+q^{10}t^9+2q^{11}t^9+4q^{12}t^9+7q^{13}t^9+12q^{14}t^9+17q^{15}t^9+22q^{16}t^9+20q^{17}t^9\\
&\quad+6q^{18}t^9+q^{11}t^{10}+2q^{12}t^{10}+4q^{13}t^{10}+7q^{14}t^{10}+12q^{15}t^{10}+17q^{16}t^{10}+20q^{17}t^{10}+13q^{18}t^{10}\\
&\quad+2q^{19}t^{10}+q^{12}t^{11}+2q^{13}t^{11}+4q^{14}t^{11}+7q^{15}t^{11}+12q^{16}t^{11}+16q^{17}t^{11}+16q^{18}t^{11}+5q^{19}t^{11}\\
&\quad+q^{13}t^{12}+2q^{14}t^{12}+4q^{15}t^{12}+7q^{16}t^{12}+12q^{17}t^{12}+14q^{18}t^{12}+9q^{19}t^{12}+q^{20}t^{12}+q^{14}t^{13}\\
&\quad+2q^{15}t^{13}+4q^{16}t^{13}+7q^{17}t^{13}+11q^{18}t^{13}+10q^{19}t^{13}+2q^{20}t^{13}+q^{15}t^{14}+2q^{16}t^{14}+4q^{17}t^{14}\\
&\quad+7q^{18}t^{14}+9q^{19}t^{14}+4q^{20}t^{14}+q^{16}t^{15}+2q^{17}t^{15}+4q^{18}t^{15}+6q^{19}t^{15}+5q^{20}t^{15}+q^{17}t^{16}\\
&\quad+2q^{18}t^{16}+4q^{19}t^{16}+4q^{20}t^{16}+q^{21}t^{16}+q^{18}t^{17}+2q^{19}t^{17}+3q^{20}t^{17}+q^{21}t^{17}+q^{19}t^{18}\\
&\quad+2q^{20}t^{18}+q^{21}t^{18}+q^{20}t^{19}+q^{21}t^{19}+q^{21}t^{20}\big).
\end{aligned}
$$

We note that examples involving torus knots and cables different from $T(2m+1,2)$, $Cab(2m+1,2)(\cdot)$ are important, since the knots/cables of types $(*,2)$ and their invariants are generally very special.

Betti numbers (Conjecturally)

$$\begin{aligned}(3\text{-}3)\quad \mathcal{H}(q=1,t,a=0) = 1+5t+12t^2+20t^3+28t^4+34t^5+37t^6\\ +37t^7+36t^8+33t^9+29t^{10}+25t^{11}+21t^{12}+17t^{13}+13t^{14}\\ +10t^{15}+7t^{16}+5t^{17}+3t^{18}+2t^{19}+t^{20}+t^{21}.\end{aligned}$$

The germ $\mathcal{C}_{\vec{r},\vec{s}}$ is given here by the local ring $\mathcal{R}=\mathbb{C}[\![z^6,z^9+z^{10}]\!]$ with the *Puiseux exponents* $(6,9,10)$, the valuation semigroup Γ generated by $6,9,19$ and $\delta=|\mathbb{N}\setminus\Gamma|=21$, which matches the top t–degree in the DAHA-Betti polynomial from (3-3). Thus, the Euler number of $J(\mathcal{C}_{\vec{r},\vec{s}})$ is 365 (the value in (3-3) at $t=1$), which follows from [34; 30], and we predict that the Betti numbers of $J(\mathcal{C}_{\vec{r},\vec{s}})$ are the coefficients of this polynomial. This example is from the table after [36, Theorem 13] with some cases where the approach there is not applicable (including counting the Euler number).

Khovanov polynomial

$$\begin{aligned}(3)\quad DAHA'Kh(Cab(19,3)T(3,2)) = 1+q^4t^2+q^6t^3+q^6t^4+q^{10}t^5+q^8t^6\\ +q^{12}t^7+q^{10}t^8+q^{12}t^8+2q^{14}t^9+q^{12}t^{10}+q^{14}t^{10}+2q^{16}t^{11}+q^{18}t^{11}\\ +3q^{16}t^{12}+q^{20}t^{12}+2q^{18}t^{13}+2q^{20}t^{13}+2q^{18}t^{14}+q^{22}t^{14}+4q^{22}t^{15}\\ +q^{20}t^{16}+2q^{22}t^{16}+q^{24}t^{16}+q^{26}t^{16}+4q^{24}t^{17}+3q^{24}t^{18}+q^{28}t^{18}\\ +3q^{26}t^{19}+2q^{28}t^{19}+q^{26}t^{20}+q^{28}t^{20}+2q^{30}t^{20}+4q^{30}t^{21}+2q^{30}t^{22}\\ +2q^{32}t^{22}+3q^{32}t^{23}+q^{32}t^{24}+q^{36}t^{24}+2q^{36}t^{25}+q^{34}t^{26}\\ +q^{36}t^{26}+2q^{38}t^{26}+q^{36}t^{27}+q^{38}t^{27}+q^{40}t^{27}+q^{40}t^{29}+q^{42}t^{30}.\end{aligned}$$

This one is now different from the actual Khovanov polynomial:

$$\begin{aligned}\widetilde{Kh}-DAHA'Kh = q^{34}t^{23}(1+2t-t^3)+q^{36}t^{22}(1-t^2)\\ +q^{40}t^{27}(1-t^2)+q^{42}t^{28}(1-t^2).\end{aligned}$$

There is no coincidence, but the t–degree corrections involve only top terms and all are in the expected direction. As it was discussed after the Connection conjecture, if the difference $\widetilde{Kh}-DAHA'Kh$ is a sum of the terms $q^it^j(1+t)$ and $q^it^j(1-t^2)$ with positive coefficients, then this is a confirmation of part (ii). Here and below (where Khovanov polynomials are discussed), q and t are actually q_{st} and t_{st}. This is what can be expected, taking into consideration the nature of the procedure from [12] (and the example of $T(12,7)$ there).

$\mathcal{H}_{\{4,2\},\{3,1\}}$ The next example will be of the same $\deg_a=5$. Recall that $\deg_a\le(\mathsf{s}_1\mathsf{r}_2-1)$ in the uncolored case with 2 iterations; conjecturally the coincidence holds here for algebraic knots. Generally, $(\mathsf{s}_1\mathsf{r}_2-1)$ must be multiplied by $\mathrm{ord}(\lambda_b)$.

(4) $\vec{\mathsf{r}}=\{4,2\},\quad \vec{\mathsf{s}}=\{3,1\},\quad \mathcal{T}=Cab(25,2)T(4,3);$

$$\begin{aligned}
\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\square;q,t,a) &= 1+qt+q^2t+q^3t+q^4t+q^5t+q^2t^2+q^3t^2+2q^4t^2+2q^5t^2+3q^6t^2+q^7t^2\\
&+q^8t^2+q^3t^3+q^4t^3+2q^5t^3+3q^6t^3+4q^7t^3+3q^8t^3+3q^9t^3+q^4t^4+q^5t^4+2q^6t^4+3q^7t^4\\
&+5q^8t^4+4q^9t^4+5q^{10}t^4+q^{11}t^4+q^5t^5+q^6t^5+2q^7t^5+3q^8t^5+5q^9t^5+5q^{10}t^5+6q^{11}t^5\\
&+q^{12}t^5+q^6t^6+q^7t^6+2q^8t^6+3q^9t^6+5q^{10}t^6+5q^{11}t^6+7q^{12}t^6+q^{13}t^6+q^7t^7+q^8t^7\\
&+2q^9t^7+3q^{10}t^7+5q^{11}t^7+5q^{12}t^7+6q^{13}t^7+q^{14}t^7+q^8t^8+q^9t^8+2q^{10}t^8+3q^{11}t^8\\
&+5q^{12}t^8+5q^{13}t^8+5q^{14}t^8+q^9t^9+q^{10}t^9+2q^{11}t^9+3q^{12}t^9+5q^{13}t^9+4q^{14}t^9+3q^{15}t^9\\
&+q^{10}t^{10}+q^{11}t^{10}+2q^{12}t^{10}+3q^{13}t^{10}+5q^{14}t^{10}+3q^{15}t^{10}+q^{16}t^{10}+q^{11}t^{11}+q^{12}t^{11}\\
&+2q^{13}t^{11}+3q^{14}t^{11}+4q^{15}t^{11}+q^{16}t^{11}+q^{12}t^{12}+q^{13}t^{12}+2q^{14}t^{12}+3q^{15}t^{12}+3q^{16}t^{12}\\
&+q^{13}t^{13}+q^{14}t^{13}+2q^{15}t^{13}+2q^{16}t^{13}+q^{17}t^{13}+q^{14}t^{14}+q^{15}t^{14}+2q^{16}t^{14}+q^{17}t^{14}\\
&+q^{15}t^{15}+q^{16}t^{15}+q^{17}t^{15}+q^{16}t^{16}+q^{17}t^{16}+q^{17}t^{17}+q^{18}t^{18}\\
&+a^5\big(q^{15}+q^{16}t+q^{17}t^2+q^{18}t^3\big)\\
&+a^4\big(q^{10}+q^{11}+q^{12}+q^{13}+q^{14}+q^{11}t+2q^{12}t+2q^{13}t+2q^{14}t+2q^{15}t+q^{12}t^2+2q^{13}t^2\\
&+3q^{14}t^2+3q^{15}t^2+2q^{16}t^2+q^{13}t^3+2q^{14}t^3+3q^{15}t^3+3q^{16}t^3+2q^{17}t^3+q^{14}t^4\\
&+2q^{15}t^4+3q^{16}t^4+2q^{17}t^4+q^{18}t^4+q^{15}t^5+2q^{16}t^5+2q^{17}t^5+q^{18}t^5+q^{16}t^6\\
&+2q^{17}t^6+q^{18}t^6+q^{17}t^7+q^{18}t^7+q^{18}t^8\big)\\
&+a^3\big(q^6+q^7+2q^8+2q^9+2q^{10}+q^{11}+q^{12}+q^7t+2q^8t+4q^9t+5q^{10}t+6q^{11}t+4q^{12}t\\
&+3q^{13}t+q^{14}t+q^8t^2+2q^9t^2+5q^{10}t^2+7q^{11}t^2+10q^{12}t^2+7q^{13}t^2+5q^{14}t^2+q^{15}t^2+q^9t^3\\
&+2q^{10}t^3+5q^{11}t^3+8q^{12}t^3+12q^{13}t^3+9q^{14}t^3+6q^{15}t^3+q^{16}t^3+q^{10}t^4+2q^{11}t^4+5q^{12}t^4\\
&+8q^{13}t^4+13q^{14}t^4+9q^{15}t^4+5q^{16}t^4+q^{17}t^4+q^{11}t^5+2q^{12}t^5+5q^{13}t^5+8q^{14}t^5+12q^{15}t^5\\
&+7q^{16}t^5+3q^{17}t^5+q^{12}t^6+2q^{13}t^6+5q^{14}t^6+8q^{15}t^6+10q^{16}t^6+4q^{17}t^6+q^{18}t^6+q^{13}t^7\\
&+2q^{14}t^7+5q^{15}t^7+7q^{16}t^7+6q^{17}t^7+q^{18}t^7+q^{14}t^8+2q^{15}t^8+5q^{16}t^8+5q^{17}t^8+2q^{18}t^8\\
&+q^{15}t^9+2q^{16}t^9+4q^{17}t^9+2q^{18}t^9+q^{16}t^{10}+2q^{17}t^{10}+2q^{18}t^{10}+q^{17}t^{11}+q^{18}t^{11}+q^{18}t^{12}\big)\\
&+a^2\big(q^3+q^4+2q^5+2q^6+2q^7+q^8+q^9+q^4t+2q^5t+4q^6t+6q^7t+7q^8t+6q^9t+5q^{10}t\\
&+2q^{11}t+q^{12}t+q^5t^2+2q^6t^2+5q^7t^2+8q^8t^2+12q^9t^2+12q^{10}t^2+11q^{11}t^2+5q^{12}t^2\\
&+2q^{13}t^2+q^6t^3+2q^7t^3+5q^8t^3+9q^9t^3+14q^{10}t^3+17q^{11}t^3+17q^{12}t^3+8q^{13}t^3+3q^{14}t^3\\
&+q^7t^4+2q^8t^4+5q^9t^4+9q^{10}t^4+15q^{11}t^4+19q^{12}t^4+20q^{13}t^4+9q^{14}t^4+3q^{15}t^4+q^8t^5\\
&+2q^9t^5+5q^{10}t^5+9q^{11}t^5+15q^{12}t^5+20q^{13}t^5+20q^{14}t^5+8q^{15}t^5+2q^{16}t^5+q^9t^6+2q^{10}t^6\\
&+5q^{11}t^6+9q^{12}t^6+15q^{13}t^6+19q^{14}t^6+17q^{15}t^6+5q^{16}t^6+q^{17}t^6+q^{10}t^7+2q^{11}t^7+5q^{12}t^7\\
&+9q^{13}t^7+15q^{14}t^7+17q^{15}t^7+11q^{16}t^7+2q^{17}t^7+q^{11}t^8+2q^{12}t^8+5q^{13}t^8+9q^{14}t^8\\
&+14q^{15}t^8+12q^{16}t^8+5q^{17}t^8+q^{12}t^9+2q^{13}t^9+5q^{14}t^9+9q^{15}t^9+12q^{16}t^9+6q^{17}t^9+q^{18}t^9\\
&+q^{13}t^{10}+2q^{14}t^{10}+5q^{15}t^{10}+8q^{16}t^{10}+7q^{17}t^{10}+q^{18}t^{10}+q^{14}t^{11}+2q^{15}t^{11}+5q^{16}t^{11}\\
&+6q^{17}t^{11}+2q^{18}t^{11}+q^{15}t^{12}+2q^{16}t^{12}+4q^{17}t^{12}+2q^{18}t^{12}+q^{16}t^{13}+2q^{17}t^{13}+2q^{18}t^{13}\\
&+q^{17}t^{14}+q^{18}t^{14}+q^{18}t^{15}\big)
\end{aligned}$$

$$+a\big(q+q^2+q^3+q^4+q^5+q^2t+2q^3t+3q^4t+4q^5t+5q^6t+3q^7t+2q^8t+q^9t+q^3t^2+2q^4t^2$$
$$+4q^5t^2+6q^6t^2+9q^7t^2+8q^8t^2+7q^9t^2+3q^{10}t^2+q^{11}t^2+q^4t^3+2q^5t^3+4q^6t^3+7q^7t^3$$
$$+11q^8t^3+12q^9t^3+13q^{10}t^3+7q^{11}t^3+2q^{12}t^3+q^5t^4+2q^6t^4+4q^7t^4+7q^8t^4+12q^9t^4$$
$$+14q^{10}t^4+17q^{11}t^4+10q^{12}t^4+3q^{13}t^4+q^6t^5+2q^7t^5+4q^8t^5+7q^9t^5+12q^{10}t^5+15q^{11}t^5$$
$$+19q^{12}t^5+11q^{13}t^5+3q^{14}t^5+q^7t^6+2q^8t^6+4q^9t^6+7q^{10}t^6+12q^{11}t^6+15q^{12}t^6+19q^{13}t^6$$
$$+10q^{14}t^6+2q^{15}t^6+q^8t^7+2q^9t^7+4q^{10}t^7+7q^{11}t^7+12q^{12}t^7+15q^{13}t^7+17q^{14}t^7+7q^{15}t^7$$
$$+q^{16}t^7+q^9t^8+2q^{10}t^8+4q^{11}t^8+7q^{12}t^8+12q^{13}t^8+14q^{14}t^8+13q^{15}t^8+3q^{16}t^8+q^{10}t^9$$
$$+2q^{11}t^9+4q^{12}t^9+7q^{13}t^9+12q^{14}t^9+12q^{15}t^9+7q^{16}t^9+q^{17}t^9+q^{11}t^{10}+2q^{12}t^{10}$$
$$+4q^{13}t^{10}+7q^{14}t^{10}+11q^{15}t^{10}+8q^{16}t^{10}+2q^{17}t^{10}+q^{12}t^{11}+2q^{13}t^{11}+4q^{14}t^{11}+7q^{15}t^{11}$$
$$+9q^{16}t^{11}+3q^{17}t^{11}+q^{13}t^{12}+2q^{14}t^{12}+4q^{15}t^{12}+6q^{16}t^{12}+5q^{17}t^{12}+q^{14}t^{13}+2q^{15}t^{13}$$
$$+4q^{16}t^{13}+4q^{17}t^{13}+q^{18}t^{13}+q^{15}t^{14}+2q^{16}t^{14}+3q^{17}t^{14}+q^{18}t^{14}+q^{16}t^{15}$$
$$+2q^{17}t^{15}+q^{18}t^{15}+q^{17}t^{16}+q^{18}t^{16}+q^{18}t^{17}\big).$$

Betti numbers (Conjecturally)

$$\text{(3-4)}\quad \mathcal{H}(q=1,t,a=0)=1+5t+11t^2+17t^3+22t^4+24t^5+25t^6$$
$$+24t^7+22t^8+19t^9+16t^{10}+12t^{11}+10t^{12}$$
$$+7t^{13}+5t^{14}+3t^{15}+2t^{16}+t^{17}+t^{18}.$$

The corresponding (germ of the) curve is given by the ring $\mathcal{R}=\mathbb{C}[\![z^6,z^8+z^9]\!]$ in this case, with the Puiseux exponents $(6,8,9)$ and the valuation semigroup Γ generated by $6,8,25$. Accordingly, $\delta=|\mathbb{N}\setminus\Gamma|=18$ in the considered case, which matches the top t–degree in (3-4).

Furthermore, $\mathcal{H}(q=1,t=1,a=0)=227$, which does coincide with the specialization at $v=9$ of the general formula for the Euler number $e(J(\mathcal{C}_{\vec{r},\vec{s}}))=229/2+25v/2$ in the case of the Puiseux exponents $(6,8,v)$. This formula is from [36, Main theorem]; its proof was technically involved and was not extended to the Betti numbers. We conjecture that they are the t–coefficients in (3-4) in this case ($v=9$).

Khovanov polynomial

$$\text{(4)}\quad DAHA'Kh(Cab(25,2)T(4,3))=1+q^4t^2+q^6t^3+q^6t^4+q^{10}t^5+q^8t^6$$
$$+q^{12}t^7+q^{10}t^8+q^{12}t^8+2q^{14}t^9+q^{12}t^{10}+q^{14}t^{10}+2q^{16}t^{11}+q^{18}t^{11}$$
$$+3q^{16}t^{12}+q^{20}t^{12}+2q^{18}t^{13}+2q^{20}t^{13}+q^{18}t^{14}+q^{20}t^{14}+q^{22}t^{14}$$
$$+4q^{22}t^{15}+q^{20}t^{16}+q^{22}t^{16}+q^{24}t^{16}+q^{26}t^{16}+2q^{24}t^{17}+q^{26}t^{17}$$
$$+2q^{24}t^{18}+q^{28}t^{18}+2q^{26}t^{19}+q^{28}t^{19}+2q^{28}t^{20}+q^{30}t^{20}+3q^{30}t^{21}$$
$$+q^{32}t^{22}+q^{34}t^{23}+q^{32}t^{24}+q^{36}t^{24}+q^{34}t^{25}+q^{36}t^{26}+q^{38}t^{27}.$$

This one is different from the corresponding Khovanov polynomial as well as in the previous example. Namely,

$$\widetilde{Kh} - DAHA'Kh = q^{32}t^{22}(1-t^2) + q^{34}t^{23}(1-t^2) + q^{36}t^{24}(1-t^2) + q^{38}t^{25}(1-t^2).$$

Once again, the t–degrees increase from $\widetilde{Kh}$ to $DAHA'Kh$; also, the number of q,t–monomials is the same in both, but this seems accidental. In spite of the nonzero difference, this confirms Conjecture 2.4.

Further Betti numbers First of all, our DAHA-Betti polynomials (as far as we calculated them) fully match the table before [36, Conjecture 23]. For instance, the coefficients of $\mathcal{H}_{\{5,2\},\{2,1\}}(q=1,t,a=0)$ for $Cab(21,2)T(5,2)$ constitute the line there for the Puiseux exponents $(4,10,11)$. Without posting the (known) formulas for the superpolynomials, let us provide below only the DAHA-Betti polynomials, which conjecturally give the corresponding Betti numbers.

$\boldsymbol{\mathcal{H}_{\{4,2\},\{3,1\}}}$ In this example, the knot, the ring of the singularity, the valuation semigroup and the arithmetic genus are as follows:

$$\mathcal{T} = Cab(20,3)T(3,2), \quad \mathcal{R} = C[\![t^6, t^9+t^{11}]\!], \quad \Gamma = \langle 6,9,20\rangle, \quad \delta = 22.$$

The counterpart of the DAHA-Betti polynomial (3-3) reads

$$\begin{aligned}(3\text{-}5)\quad \mathcal{H}(q=1,t,a=0) = {} & 1+5t+12t^2+20t^3+28t^4+35t^5+39t^6+40t^7\\ & +39t^8+37t^9+33t^{10}+29t^{11}+25t^{12}+21t^{13}+17t^{14}\\ & +13t^{15}+10t^{16}+7t^{17}+5t^{18}+3t^{19}+2t^{20}+t^{21}+t^{22}.\end{aligned}$$

The arithmetic genus of $\mathcal{C} = \mathcal{C}_{\{3,3\},\{2,2\}}$ equals $\delta = 22$ and $e(J(\mathcal{C})) = 423$ in this case; the Puiseux exponents are $(6,9,11)$, and $\deg_a$=5.

$\boldsymbol{\mathcal{H}_{\{5,2\},\{3,2\}}}$ The Puiseux exponents are $(6,15,17)$, and $\deg_a = 5$. One has

$$\mathcal{T} = Cab(32,3)T(5,2), \quad \mathcal{R} = \mathbb{C}[\![z^6, z^{15}+z^{17}]\!], \quad \Gamma = \langle 6,15,32\rangle, \quad \delta = 37,$$

and the DAHA-Betti polynomial is

$$\begin{aligned}(3\text{-}6)\quad \mathcal{H}(q=1,t,a=0) = {} & 1+5t+15t^2+32t^3+55t^4+81t^5+108t^6\\ & +134t^7+157t^8+175t^9+186t^{10}+192t^{11}+192t^{12}+189t^{13}\\ & +181t^{14}+172t^{15}+159t^{16}+147t^{17}+132t^{18}+120t^{19}+105t^{20}\\ & +93t^{21}+79t^{22}+68t^{23}+56t^{24}+47t^{25}+37t^{26}+30t^{27}+23t^{28}\\ & +18t^{29}+13t^{30}+10t^{31}+7t^{32}+5t^{33}+3t^{34}+2t^{35}+t^{36}+t^{37}.\end{aligned}$$

The Euler number of $J(\mathcal{C}_{\{5,2\},\{3,2\}})$ is 3031.

$\mathcal{H}_{\{3,4\},\{2,1\}}$ In this case, $\deg_a = 7$ and the Puiseux exponents are $(8, 12, 13)$. One has

$$\mathcal{T} = Cab(25,4)T(3,2), \quad \mathcal{R} = \mathbb{C}[\![z^8, z^{12} + z^{13}]\!], \quad \Gamma = \langle 8, 12, 25\rangle, \quad \delta = 40,$$

and the DAHA-Betti polynomial is

$$\begin{aligned}(3\text{-}7)\quad \mathcal{H}(q=1,t,a=0) = 1 &+ 7t + 24t^2 + 56t^3 + 104t^4 + 165t^5 + 232t^6 \\ &+ 297t^7 + 355t^8 + 402t^9 + 435t^{10} + 454t^{11} + 461t^{12} + 456t^{13} \\ &+ 442t^{14} + 420t^{15} + 394t^{16} + 362t^{17} + 330t^{18} + 295t^{19} \\ &+ 262t^{20} + 229t^{21} + 199t^{22} + 168t^{23} + 143t^{24} + 118t^{25} \\ &+ 97t^{26} + 78t^{27} + 63t^{28} + 48t^{29} + 38t^{30} + 28t^{31} + 21t^{32} \\ &+ 15t^{33} + 11t^{34} + 7t^{35} + 5t^{36} + 3t^{37} + 2t^{38} + t^{39} + t^{40}.\end{aligned}$$

The Euler number of $J(\mathcal{C}_{\{3,4\},\{2,1\}})$ is 7229. The latter follows from [34; 30]; the Betti numbers are conjectured.

Double iteration: $\vec{\mathsf{r}} = \{3,2,2\}, \vec{\mathsf{s}} = \{2,1,1\}$ The corresponding knot is $\mathcal{T} = Cab(53,2)Cab(13,2)T(3,2)$ and $\deg_a = 7$. In this case,

$$\mathcal{R} = \mathbb{C}[\![z^8, z^{12} + z^{14} + z^{15}]\!], \quad \Gamma = \langle 8, 12, 26, 53\rangle, \quad \delta = 42,$$

and the DAHA-Betti polynomial is

$$\begin{aligned}(3\text{-}8)\quad \mathcal{H}(q=1,t,a=0) = 1 &+ 7t + 24t^2 + 56t^3 + 104t^4 + 166t^5 \\ &+ 236t^6 + 306t^7 + 370t^8 + 424t^9 + 465t^{10} + 492t^{11} + 507t^{12} \\ &+ 510t^{13} + 504t^{14} + 488t^{15} + 466t^{16} + 437t^{17} + 406t^{18} + 370t^{19} \\ &+ 335t^{20} + 298t^{21} + 264t^{22} + 230t^{23} + 199t^{24} + 168t^{25} + 143t^{26} \\ &+ 118t^{27} + 97t^{28} + 78t^{29} + 63t^{30} + 48t^{31} + 38t^{32} + 28t^{33} \\ &+ 21t^{34} + 15t^{35} + 11t^{36} + 7t^{37} + 5t^{38} + 3t^{39} + 2t^{40} + t^{41} + t^{42}.\end{aligned}$$

The Euler number of $J(\mathcal{C}_{\{3,2,2\},\{2,1,1\}})$ is 8512. This case is listed in [36] (right before Section 4) as an example beyond the technique there.

We omit the complete DAHA superpolynomials and the DAHA-Khovanov polynomials in this and the previous examples (they are all known). Quite a few interesting relations for the (conjectural) Betti numbers can be seen from these and other examples of the DAHA-Betti polynomials. For instance, we note that b_2 of $J(\mathcal{C})$ coincides with the *multiplicity* of the corresponding singularity minus 1 (Rego) and therefore exactly with $\deg_a = \mathsf{s}_1\mathsf{r}_2\cdots\mathsf{r}_\ell$ of the DAHA-superpolynomial ($\mathsf{s}_1 < \mathsf{r}_1$).

3.3 The case of two boxes

We will now consider the iterated knot from the first example with the weight $2\omega_1$. We calculated the one for ω_2 too; it is superdual to the one we provide and will be omitted. Also, we will omit the corresponding DAHA-Khovanov polynomial (calculated using the algorithm from [12]), since the Khovanov polynomials are uncolored.

(5) $$\vec{r} = \{3,2\}, \quad \vec{s} = \{2,1\}, \quad \mathcal{T} = Cab(13,2)T(3,2);$$

$$\begin{aligned}
\mathcal{H}_{\vec{r},\vec{s}}(\square\square;q,t,a) = {} & 1+t^{16}q^{32}+t^{15}q^{31}+t^{14}q^{31}+t^{13}q^{31}+t^{15}q^{30}+2t^{14}q^{30}+2t^{13}q^{30}\\
&+2t^{12}q^{30}+t^{14}q^{29}+3t^{13}q^{29}+4t^{12}q^{29}+3t^{11}q^{29}+t^{14}q^{28}+2t^{13}q^{28}+5t^{12}q^{28}+6t^{11}q^{28}\\
&+4t^{10}q^{28}+t^{13}q^{27}+3t^{12}q^{27}+7t^{11}q^{27}+8t^{10}q^{27}+4t^{9}q^{27}+t^{13}q^{26}+2t^{12}q^{26}+5t^{11}q^{26}\\
&+10t^{10}q^{26}+9t^{9}q^{26}+2t^{8}q^{26}+t^{12}q^{25}+3t^{11}q^{25}+7t^{10}q^{25}+12t^{9}q^{25}+7t^{8}q^{25}+t^{7}q^{25}\\
&+t^{12}q^{24}+2t^{11}q^{24}+5t^{10}q^{24}+10t^{9}q^{24}+14t^{8}q^{24}+4t^{7}q^{24}+t^{11}q^{23}+3t^{10}q^{23}+7t^{9}q^{23}\\
&+12t^{8}q^{23}+11t^{7}q^{23}+t^{6}q^{23}+t^{11}q^{22}+2t^{10}q^{22}+5t^{9}q^{22}+10t^{8}q^{22}+13t^{7}q^{22}+6t^{6}q^{22}\\
&+t^{10}q^{21}+3t^{9}q^{21}+7t^{8}q^{21}+12t^{7}q^{21}+10t^{6}q^{21}+t^{5}q^{21}+t^{10}q^{20}+2t^{9}q^{20}+5t^{8}q^{20}\\
&+10t^{7}q^{20}+13t^{6}q^{20}+3t^{5}q^{20}+t^{9}q^{19}+3t^{8}q^{19}+7t^{7}q^{19}+11t^{6}q^{19}+9t^{5}q^{19}+t^{9}q^{18}\\
&+2t^{8}q^{18}+5t^{7}q^{18}+10t^{6}q^{18}+11t^{5}q^{18}+2t^{4}q^{18}+t^{8}q^{17}+3t^{7}q^{17}+7t^{6}q^{17}+10t^{5}q^{17}\\
&+4t^{4}q^{17}+t^{8}q^{16}+2t^{7}q^{16}+5t^{6}q^{16}+9t^{5}q^{16}+8t^{4}q^{16}+t^{7}q^{15}+3t^{6}q^{15}+7t^{5}q^{15}+8t^{4}q^{15}\\
&+2t^{3}q^{15}+t^{7}q^{14}+2t^{6}q^{14}+5t^{5}q^{14}+8t^{4}q^{14}+3t^{3}q^{14}+t^{6}q^{13}+3t^{5}q^{13}+6t^{4}q^{13}+5t^{3}q^{13}\\
&+t^{6}q^{12}+2t^{5}q^{12}+5t^{4}q^{12}+6t^{3}q^{12}+t^{2}q^{12}+t^{5}q^{11}+3t^{4}q^{11}+5t^{3}q^{11}+t^{2}q^{11}+t^{5}q^{10}\\
&+2t^{4}q^{10}+4t^{3}q^{10}+3t^{2}q^{10}+t^{4}q^{9}+3t^{3}q^{9}+3t^{2}q^{9}+t^{4}q^{8}+2t^{3}q^{8}+3t^{2}q^{8}+t^{3}q^{7}+2t^{2}q^{7}\\
&+tq^{7}+t^{3}q^{6}+2t^{2}q^{6}+tq^{6}+t^{2}q^{5}+tq^{5}+t^{2}q^{4}+tq^{4}+tq^{3}+tq^{2}\\
&+a^{6}\big(t^{4}q^{35}+t^{3}q^{34}+t^{3}q^{33}+t^{2}q^{33}+t^{2}q^{32}+t^{2}q^{31}+tq^{30}+tq^{29}+q^{27}\big)\\
&+a^{5}\big(t^{7}q^{35}+t^{6}q^{35}+t^{5}q^{35}+t^{7}q^{34}+2t^{6}q^{34}+2t^{5}q^{34}+t^{4}q^{34}+2t^{6}q^{33}+4t^{5}q^{33}+4t^{4}q^{33}\\
&+t^{3}q^{33}+t^{6}q^{32}+4t^{5}q^{32}+5t^{4}q^{32}+3t^{3}q^{32}+2t^{5}q^{31}+5t^{4}q^{31}+5t^{3}q^{31}+t^{2}q^{31}+t^{5}q^{30}\\
&+4t^{4}q^{30}+6t^{3}q^{30}+3t^{2}q^{30}+2t^{4}q^{29}+6t^{3}q^{29}+5t^{2}q^{29}+t^{4}q^{28}+4t^{3}q^{28}+5t^{2}q^{28}+tq^{28}\\
&+2t^{3}q^{27}+5t^{2}q^{27}+3tq^{27}+t^{3}q^{26}+4t^{2}q^{26}+3tq^{26}+2t^{2}q^{25}+3tq^{25}+q^{25}+t^{2}q^{24}+3tq^{24}\\
&+q^{24}+2tq^{23}+q^{23}+tq^{22}+q^{22}+q^{21}+q^{20}\big)\\
&+a^{4}\big(t^{9}q^{35}+t^{8}q^{35}+t^{7}q^{35}+t^{10}q^{34}+2t^{9}q^{34}+4t^{8}q^{34}+3t^{7}q^{34}+2t^{6}q^{34}+2t^{9}q^{33}+5t^{8}q^{33}\\
&+8t^{7}q^{33}+6t^{6}q^{33}+3t^{5}q^{33}+t^{9}q^{32}+5t^{8}q^{32}+11t^{7}q^{32}+13t^{6}q^{32}+8t^{5}q^{32}+2t^{4}q^{32}\\
&+2t^{8}q^{31}+9t^{7}q^{31}+16t^{6}q^{31}+15t^{5}q^{31}+6t^{4}q^{31}+t^{3}q^{31}+t^{8}q^{30}+5t^{7}q^{30}+16t^{6}q^{30}\\
&+21t^{5}q^{30}+14t^{4}q^{30}+3t^{3}q^{30}+2t^{7}q^{29}+9t^{6}q^{29}+22t^{5}q^{29}+20t^{4}q^{29}+8t^{3}q^{29}+t^{7}q^{28}\\
&+5t^{6}q^{28}+16t^{5}q^{28}+25t^{4}q^{28}+14t^{3}q^{28}+2t^{2}q^{28}+2t^{6}q^{27}+9t^{5}q^{27}+21t^{4}q^{27}+20t^{3}q^{27}\\
&+5t^{2}q^{27}+t^{6}q^{26}+5t^{5}q^{26}+16t^{4}q^{26}+22t^{3}q^{26}+10t^{2}q^{26}+2t^{5}q^{25}+9t^{4}q^{25}+20t^{3}q^{25}\\
&+13t^{2}q^{25}+2tq^{25}+t^{5}q^{24}+5t^{4}q^{24}+15t^{3}q^{24}+17t^{2}q^{24}+4tq^{24}+2t^{4}q^{23}+9t^{3}q^{23}
\end{aligned}$$

$$+15t^2q^{23}+6tq^{23}+t^4q^{22}+5t^3q^{22}+13t^2q^{22}+8tq^{22}+q^{22}+2t^3q^{21}+8t^2q^{21}+9tq^{21}$$
$$+q^{21}+t^3q^{20}+5t^2q^{20}+8tq^{20}+2q^{20}+2t^2q^{19}+6tq^{19}+2q^{19}+t^2q^{18}+4tq^{18}+3q^{18}$$
$$+5t^2q^{20}+8tq^{20}+2q^{20}+2t^2q^{19}+2tq^{17}+2q^{17}+tq^{16}+2q^{16}+q^{15}+q^{14})$$
$$+a^3(t^{10}q^{35}+t^{12}q^{34}+2t^{11}q^{34}+3t^{10}q^{34}+3t^9q^{34}+t^8q^{34}+t^{12}q^{33}+3t^{11}q^{33}+6t^{10}q^{33}$$
$$+8t^9q^{33}+6t^8q^{33}+2t^7q^{33}+2t^{11}q^{32}+8t^{10}q^{32}+14t^9q^{32}+16t^8q^{32}+10t^7q^{32}+3t^6q^{32}$$
$$+t^{11}q^{31}+5t^{10}q^{31}+15t^9q^{31}+25t^8q^{31}+22t^7q^{31}+11t^6q^{31}+2t^5q^{31}+2t^{10}q^{30}+10t^9q^{30}$$
$$+26t^8q^{30}+36t^7q^{30}+26t^6q^{30}+9t^5q^{30}+t^4q^{30}+t^{10}q^{29}+5t^9q^{29}+18t^8q^{29}+39t^7q^{29}$$
$$+42t^6q^{29}+21t^5q^{29}+4t^4q^{29}+2t^9q^{28}+10t^8q^{28}+29t^7q^{28}+50t^6q^{28}+39t^5q^{28}+12t^4q^{28}$$
$$+t^3q^{28}+t^9q^{27}+5t^8q^{27}+18t^7q^{27}+43t^6q^{27}+52t^5q^{27}+25t^4q^{27}+4t^3q^{27}+2t^8q^{26}$$
$$+10t^7q^{26}+29t^6q^{26}+53t^5q^{26}+41t^4q^{26}+10t^3q^{26}+t^8q^{25}+5t^7q^{25}+18t^6q^{25}+43t^5q^{25}$$
$$+50t^4q^{25}+20t^3q^{25}+t^2q^{25}+2t^7q^{24}+10t^6q^{24}+29t^5q^{24}+50t^4q^{24}+32t^3q^{24}+5t^2q^{24}$$
$$+t^7q^{23}+5t^6q^{23}+18t^5q^{23}+41t^4q^{23}+39t^3q^{23}+10t^2q^{23}+2t^6q^{22}+10t^5q^{22}+28t^4q^{22}$$
$$+41t^3q^{22}+18t^2q^{22}+tq^{22}+t^6q^{21}+5t^5q^{21}+18t^4q^{21}+36t^3q^{21}+24t^2q^{21}+3tq^{21}$$
$$+2t^5q^{20}+10t^4q^{20}+26t^3q^{20}+27t^2q^{20}+6tq^{20}+t^5q^{19}+5t^4q^{19}+17t^3q^{19}+26t^2q^{19}$$
$$+9tq^{19}+2t^4q^{18}+10t^3q^{18}+21t^2q^{18}+12tq^{18}+q^{18}+t^4q^{17}+5t^3q^{17}+15t^2q^{17}+13tq^{17}$$
$$+q^{17}+2t^3q^{16}+9t^2q^{16}+12tq^{16}+2q^{16}+t^3q^{15}+5t^2q^{15}+10tq^{15}+3q^{15}+2t^2q^{14}+7tq^{14}$$
$$+3q^{14}+t^2q^{13}+4tq^{13}+3q^{13}+2tq^{12}+3q^{12}+tq^{11}+2q^{11}+q^{10}+q^9)$$
$$+a^2(t^{13}q^{34}+t^{12}q^{34}+t^{11}q^{34}+t^{14}q^{33}+2t^{13}q^{33}+4t^{12}q^{33}+4t^{11}q^{33}+3t^{10}q^{33}+2t^{13}q^{32}$$
$$+5t^{12}q^{32}+10t^{11}q^{32}+10t^{10}q^{32}+6t^9q^{32}+t^8q^{32}+t^{13}q^{31}+5t^{12}q^{31}+12t^{11}q^{31}+20t^{10}q^{31}$$
$$+18t^9q^{31}+8t^8q^{31}+t^7q^{31}+2t^{12}q^{30}+9t^{11}q^{30}+21t^{10}q^{30}+32t^9q^{30}+24t^8q^{30}+9t^7q^{30}$$
$$+t^6q^{30}+t^{12}q^{29}+5t^{11}q^{29}+17t^{10}q^{29}+35t^9q^{29}+45t^8q^{29}+25t^7q^{29}+7t^6q^{29}+2t^{11}q^{28}$$
$$+9t^{10}q^{28}+27t^9q^{28}+49t^8q^{28}+49t^7q^{28}+20t^6q^{28}+3t^5q^{28}+t^{11}q^{27}+5t^{10}q^{27}+17t^9q^{27}$$
$$+42t^8q^{27}+62t^7q^{27}+44t^6q^{27}+11t^5q^{27}+t^4q^{27}+2t^{10}q^{26}+9t^9q^{26}+27t^8q^{26}+57t^7q^{26}$$
$$+63t^6q^{26}+28t^5q^{26}+3t^4q^{26}+t^{10}q^{25}+5t^9q^{25}+17t^8q^{25}+42t^7q^{25}+70t^6q^{25}+48t^5q^{25}$$
$$+11t^4q^{25}+2t^9q^{24}+9t^8q^{24}+27t^7q^{24}+56t^6q^{24}+66t^5q^{24}+24t^4q^{24}+2t^3q^{24}+t^9q^{23}$$
$$+5t^8q^{23}+17t^7q^{23}+42t^6q^{23}+67t^5q^{23}+42t^4q^{23}+6t^3q^{23}+2t^8q^{22}+9t^7q^{22}+27t^6q^{22}$$
$$+55t^5q^{22}+54t^4q^{22}+15t^3q^{22}+t^8q^{21}+5t^7q^{21}+17t^6q^{21}+41t^5q^{21}+60t^4q^{21}+26t^3q^{21}$$
$$+2t^2q^{21}+2t^7q^{20}+9t^6q^{20}+27t^5q^{20}+50t^4q^{20}+38t^3q^{20}+5t^2q^{20}+t^7q^{19}+5t^6q^{19}$$
$$+17t^5q^{19}+39t^4q^{19}+43t^3q^{19}+12t^2q^{19}+2t^6q^{18}+9t^5q^{18}+26t^4q^{18}+41t^3q^{18}+18t^2q^{18}$$
$$+tq^{18}+t^6q^{17}+5t^5q^{17}+17t^4q^{17}+34t^3q^{17}+25t^2q^{17}+2tq^{17}+2t^5q^{16}+9t^4q^{16}+24t^3q^{16}$$
$$+25t^2q^{16}+5tq^{16}+t^5q^{15}+5t^4q^{15}+16t^3q^{15}+25t^2q^{15}+8tq^{15}+2t^4q^{14}+9t^3q^{14}+19t^2q^{14}$$
$$+10tq^{14}+t^4q^{13}+5t^3q^{13}+14t^2q^{13}+11tq^{13}+q^{13}+2t^3q^{12}+8t^2q^{12}+11tq^{12}+q^{12}+t^3q^{11}$$
$$+5t^2q^{11}+9tq^{11}+2q^{11}+2t^2q^{10}+6tq^{10}+2q^{10}+t^2q^9+4tq^9+3q^9+2tq^8+2q^8+tq^7$$
$$+2q^7+q^6+q^5)$$

$$
\begin{aligned}
&+a\big(t^{15}q^{33}+t^{14}q^{33}+t^{13}q^{33}+t^{15}q^{32}+2t^{14}q^{32}+3t^{13}q^{32}+3t^{12}q^{32}+t^{11}q^{32}+2t^{14}q^{31}\\
&+5t^{13}q^{31}+8t^{12}q^{31}+7t^{11}q^{31}+2t^{10}q^{31}+t^{14}q^{30}+4t^{13}q^{30}+9t^{12}q^{30}+14t^{11}q^{30}\\
&+11t^{10}q^{30}+3t^{9}q^{30}+2t^{13}q^{29}+7t^{12}q^{29}+16t^{11}q^{29}+22t^{10}q^{29}+15t^{9}q^{29}+3t^{8}q^{29}\\
&+t^{13}q^{28}+4t^{12}q^{28}+12t^{11}q^{28}+24t^{10}q^{28}+29t^{9}q^{28}+14t^{8}q^{28}+2t^{7}q^{28}+2t^{12}q^{27}\\
&+7t^{11}q^{27}+19t^{10}q^{27}+34t^{9}q^{27}+32t^{8}q^{27}+11t^{7}q^{27}+t^{6}q^{27}+t^{12}q^{26}+4t^{11}q^{26}\\
&+12t^{10}q^{26}+28t^{9}q^{26}+42t^{8}q^{26}+27t^{7}q^{26}+5t^{6}q^{26}+2t^{11}q^{25}+7t^{10}q^{25}+19t^{9}q^{25}\\
&+38t^{8}q^{25}+43t^{7}q^{25}+17t^{6}q^{25}+t^{5}q^{25}+t^{11}q^{24}+4t^{10}q^{24}+12t^{9}q^{24}+28t^{8}q^{24}+46t^{7}q^{24}\\
&+34t^{6}q^{24}+6t^{5}q^{24}+2t^{10}q^{23}+7t^{9}q^{23}+19t^{8}q^{23}+38t^{7}q^{23}+46t^{6}q^{23}+17t^{5}q^{23}+t^{4}q^{23}\\
&+t^{10}q^{22}+4t^{9}q^{22}+12t^{8}q^{22}+28t^{7}q^{22}+45t^{6}q^{22}+31t^{5}q^{22}+4t^{4}q^{22}+2t^{9}q^{21}+7t^{8}q^{21}\\
&+19t^{7}q^{21}+37t^{6}q^{21}+42t^{5}q^{21}+11t^{4}q^{21}+t^{9}q^{20}+4t^{8}q^{20}+12t^{7}q^{20}+28t^{6}q^{20}+42t^{5}q^{20}\\
&+21t^{4}q^{20}+t^{3}q^{20}+2t^{8}q^{19}+7t^{7}q^{19}+19t^{6}q^{19}+35t^{5}q^{19}+32t^{4}q^{19}+5t^{3}q^{19}+t^{8}q^{18}\\
&+4t^{7}q^{18}+12t^{6}q^{18}+27t^{5}q^{18}+35t^{4}q^{18}+11t^{3}q^{18}+2t^{7}q^{17}+7t^{6}q^{17}+19t^{5}q^{17}+31t^{4}q^{17}\\
&+18t^{3}q^{17}+t^{2}q^{17}+t^{7}q^{16}+4t^{6}q^{16}+12t^{5}q^{16}+25t^{4}q^{16}+23t^{3}q^{16}+3t^{2}q^{16}+2t^{6}q^{15}\\
&+7t^{5}q^{15}+18t^{4}q^{15}+24t^{3}q^{15}+7t^{2}q^{15}+t^{6}q^{14}+4t^{5}q^{14}+12t^{4}q^{14}+21t^{3}q^{14}+10t^{2}q^{14}\\
&+2t^{5}q^{13}+7t^{4}q^{13}+16t^{3}q^{13}+13t^{2}q^{13}+tq^{13}+t^{5}q^{12}+4t^{4}q^{12}+11t^{3}q^{12}+14t^{2}q^{12}\\
&+2tq^{12}+2t^{4}q^{11}+7t^{3}q^{11}+12t^{2}q^{11}+3tq^{11}+t^{4}q^{10}+4t^{3}q^{10}+9t^{2}q^{10}+5tq^{10}+2t^{3}q^{9}\\
&+6t^{2}q^{9}+6tq^{9}+t^{3}q^{8}+4t^{2}q^{8}+5tq^{8}+2t^{2}q^{7}+4tq^{7}+q^{7}+t^{2}q^{6}+3tq^{6}+q^{6}+2tq^{5}\\
&+q^{5}+tq^{4}+q^{4}+q^{3}+q^{2}\big).
\end{aligned}
$$

Betti numbers (Presumably)

$$
\begin{aligned}
(3\text{-}9)\quad \mathcal{H}(q=1,t,a=0) &= 1+6t+17t^2+32t^3+48t^4+62t^5+70t^6+70t^7\\
&\quad +64t^8+54t^9+41t^{10}+28t^{11}+18t^{12}\\
&\quad +10t^{13}+5t^{14}+2t^{15}+t^{16}\\
&= (1+3t+4t^2+4t^3+4t^4+3t^5+2t^6+t^7+t^8)^2.
\end{aligned}
$$

We put here (and below) "presumably", since at the moment it is not clear which Betti numbers this polynomial calculates (even conjecturally). The last relation is a particular case of part (iv) of Theorem 2.3 (the specialization at $q=1$). The full specialization formula in this case reads as follows:

$$
\begin{aligned}
\mathcal{H}_{\{3,2\},\{2,1\}}(2\omega_1;q=1,t,a) &= \big(1+3a+3a^2+a^3+3t+7at+5a^2t\\
&\quad +a^3t+4t^2+8at^2+5a^2t^2+a^3t^2+4t^3+8at^3\\
&\quad +4a^2t^3+4t^4+6at^4+2a^2t^4+3t^5+4at^5\\
&\quad +a^2t^5+2t^6+2at^6+t^7+at^7+t^8\big)^2\\
&= \mathcal{H}_{\{3,2\},\{2,1\}}(\omega_1;q=1,t,a)^2.
\end{aligned}
$$

$\mathcal{H}_{\{3,2\},\{2,1\}}(\omega_2)$ The superpolynomial for the same iterated knot and the weight ω_2 is connected with the one for $2\omega_1$ by the superduality, so we will not list it. However, let us provide its value at $a=0, q=1$ (presumably a sequence of Betti numbers):

$$\begin{aligned}(3\text{-}10)\quad \mathcal{H}_{\{3,2\},\{2,1\}}(\omega_2; q=1,t,a=0)\\ =1+3t+7t^2+11t^3+18t^4+23t^5+29t^6+31t^7+36t^8+35t^9+37t^{10}\\ +34t^{11}+34t^{12}+31t^{13}+31t^{14}+25t^{15}+25t^{16}+21t^{17}\\ +19t^{18}+15t^{19}+15t^{20}+10t^{21}+10t^{22}+7t^{23}+6t^{24}\\ +4t^{25}+4t^{26}+2t^{27}+2t^{28}+t^{29}+t^{30}+t^{32}.\end{aligned}$$

A geometric meaning of formulas (3-9) and (3-10) in terms of certain Betti numbers remains to be found. Generally, we expect that a variant of the construction from [30; 14] can be used here.

3.4 Pseudoalgebraic knots

This section is a theoretical challenge. It indicates that some nonalgebraic iterated knots are similar to the algebraic ones, especially the *pseudoalgebraic*, defined above as those with positive uncolored DAHA superpolynomials. This class conjecturally includes all algebraic knots and their mirror images.

The two knots below possess such a positivity and all other DAHA properties of algebraic knots; for instance, the existence of the tilde-normalization from Theorem 2.1 for any A_N holds, as does the strict equality for $\deg_a$ in (2-20). We give below some other examples of pseudoalgebraic knots that do *not* have these two features; see (4-1) and (4-7).

In spite of the presence of negative s_i there, we think that the coefficients of the DAHA-Betti polynomials in (3-11) and (3-12) can be expected to be Betti numbers of Jacobian factors for certain (germs of) curves. Hopefully partial normalizations of $C_{\vec{\mathsf{r}},\vec{\mathsf{s}}}$ will emerge here, however this is unknown at the moment.

$$\begin{aligned}&(\mathbf{6})\qquad \vec{\mathsf{r}}=\{3,2\},\quad \vec{\mathsf{s}}=\{2,-3\},\quad \mathcal{T}=Cab(9,2)T(3,2);\\ &\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\square;q,t,a)=1+a^3q^6+qt+q^2t+q^3t+q^2t^2+q^3t^2+2q^4t^2+q^3t^3+q^4t^3+q^5t^3\\ &\qquad +q^4t^4+q^5t^4+q^5t^5+q^6t^6\\ &\quad +a^2\big(q^3+q^4+q^5+q^4t+2q^5t+q^6t+q^5t^2+q^6t^2+q^6t^3\big)\\ &\quad +a\big(q+q^2+q^3+q^2t+2q^3t+3q^4t+q^5t+q^3t^2+2q^4t^2\\ &\qquad +3q^5t^2+q^4t^3+2q^5t^3+q^6t^3+q^5t^4+q^6t^4+q^6t^5\big).\end{aligned}$$

Betti numbers (Presumably)

$$\mathcal{H}(q=1,t,a=0)=1+3t+4t^2+3t^3+2t^4+t^5+t^6. \tag{3-11}$$

This sequence of coefficients and the one in the next example do not come from algebraic knots. However they fit the table of Betti numbers from [36] for the Puiseux exponents $(4,6,v)$. The first two entries there (our examples (1) and (2)) are for $(4,6,7)$ and $(4,6,9)$, respectively; the knots from (6) and (7) *formally* correspond to nonalgebraic $(4,6,3)$ and $(4,6,5)$, respectively.

Khovanov polynomial

$$DAHA'Kh(Cab(9,2)T(3,2))=1+q^4t^2+q^6t^3+q^6t^4+q^{10}t^5+q^8t^6+q^{12}t^7 \\ +2q^{12}t^8+2q^{14}t^9+q^{18}t^{11}+q^{20}t^{12}. \tag{6}$$

This polynomial, as well as the next $DAHA'Kh$, coincide with the corresponding (reduced) Khovanov polynomials. This confirms the Connection conjecture(ii) and also demonstrates that the differentials that are necessary to extract the Khovanov polynomials from the stable ones satisfy assumptions (2.13) from [12, Section 3.6].

$\boldsymbol{\mathcal{H}_{\{3,2\},\{2,-1\}}}$ This knot is pseudoalgebraic but nonalgebraic as well. In a sense, it is the greatest nonalgebraic, since $Cab(13,2)T(3,2)$ from (1) above is the first algebraic one in this series.

$$\vec{r}=\{3,2\},\quad \vec{s}=\{2,-1\},\quad \mathcal{T}=Cab(11,2)T(3,2); \tag{7}$$

$$\begin{aligned}\mathcal{H}_{\vec{r},\vec{s}}(\square;q,t,a)=1&+qt+q^2t+q^3t+q^2t^2+q^3t^2+2q^4t^2+q^3t^3+q^4t^3+2q^5t^3+q^4t^4\\&+q^5t^4+q^6t^4+q^5t^5+q^6t^5+q^6t^6+q^7t^7+a^3\big(q^6+q^7t\big)\\&+a^2\big(q^3+q^4+q^5+q^4t+2q^5t+2q^6t+q^5t^2+2q^6t^2+q^7t^2+q^6t^3+q^7t^3+q^7t^4\big)\\&+a\big(q+q^2+q^3+q^2t+2q^3t+3q^4t+q^5t+q^3t^2+2q^4t^2+4q^5t^2+q^6t^2+q^4t^3\\&+2q^5t^3+3q^6t^3+q^5t^4+2q^6t^4+q^7t^4+q^6t^5+q^7t^5+q^7t^6\big).\end{aligned}$$

Betti numbers (Presumably)

$$\mathcal{H}(q=1,t,a=0)=1+3t+4t^2+4t^3+3t^4+2t^5+t^6+t^7. \tag{3-12}$$

The general formula is not very difficult to calculate because $q=1$ is the *trivial central charge* (when there are significant simplifications):

$$\mathcal{H}_{\{3,2\},\{2,2n-15\}}(q=1,t,a=0)=1+3t+4(t^2+\cdots+t^{n-4}) \\ +3t^{n-3}+2t^{n-2}+t^{n-1}+t^n \tag{3-13}$$

for $Cab(2n-3,2)T(3,2)$, $n\geq 6$.

Khovanov polynomial

$$DAHA'Kh(Cab(11,2)T(3,2)) = 1+q^4t^2+q^6t^3+q^6t^4+q^{10}t^5+q^8t^6+q^{12}t^7 \\ +2q^{12}t^8+2q^{14}t^9+q^{16}t^{10}+2q^{18}t^{11}+q^{20}t^{12}. \tag{7}$$

This polynomial and the previous one coincide with the corresponding Khovanov polynomials (under the tilde-normalization); recall that we use the *standard parameters* in all formulas involving Khovanov polynomials. Thus, q,t above are actually q_{st},t_{st}.

4 Further aspects

4.1 Nonalgebraic knots

Continuing the previous section, let us discuss nonalgebraic knots that are not *pseudoalgebraic*. Though the first knot we will consider is *pseudoalgebraic*, it has some features that are not expected in algebraic knots. For instance, the corresponding $\deg_a$ is strictly smaller than $(|\mathsf{s}_1\mathsf{r}_2|-1)=3$. The coincidence holds for the similar pseudoalgebraic knots from (6) and (7) above and is conjectured to be true for all algebraic knots.

(a) $\vec{\mathsf{r}}=\{3,-2\},\quad \vec{\mathsf{s}}=\{2,5\},\quad \mathcal{T}=Cab(7,2)T(3,2);$

$$\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\square;q,t,a) = 1+qt+q^2t+q^3t+q^2t^2+q^3t^2+q^4t^2+q^3t^3+q^4t^3+q^4t^4+q^5t^5 \\ +a^2\big(q^3+q^4+q^4t+q^5t+q^5t^2\big) \\ +a\big(q+q^2+q^3+q^2t+2q^3t+2q^4t+q^3t^2+2q^4t^2+q^5t^2 \\ +q^4t^3+q^5t^3+q^5t^4\big).$$

$$\mathcal{H}(q=1,t,a=0) = 1+3t+3t^2+2t^3+t^4+t^5, \tag{4-1}$$

$$DAHA'Kh(Cab(7,2)T(3,2)) = 1+q^4t^2+q^6t^3+q^6t^4+q^{10}t^5+q^8t^6 \\ +q^{12}t^7+q^{12}t^8+q^{14}t^9+q^{18}t^{11}+q^{20}t^{12}.$$

We see that the first formula is quite similar to (3-13). The DAHA-Khovanov polynomial coincides here with the actual Khovanov one (reduced and under the tilde normalization); recall that we use the standard parameters in the formulas for the DAHA-Khovanov polynomials.

(b) $\vec{\mathsf{r}}=\{3,-2\},\quad \vec{\mathsf{s}}=\{2,7\},\quad \mathcal{T}=Cab(5,2)T(3,2);$

$$\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\square;q,t,a) = 1-\tfrac{1}{t}a^3q^5+qt+q^2t+q^2t^2+q^3t^2+q^3t^3+q^4t^4 \\ +a^2\big(q^3-q^5-\tfrac{1}{t}q^4+q^4t\big) \\ +a\big(q+q^2-q^4+q^2t+2q^3t+q^3t^2+q^4t^2+q^4t^3\big).$$

$$\mathcal{H}(q=1,t,a=0)=1+2t+2t^2+t^3+t^4,$$
$$DAHA'Kh(Cab(5,2)T(3,2))=1+q^4t^2+q^6t^3+q^6t^4+q^{10}t^5+q^{10}t^6+q^{12}t^7+q^{12}t^8+q^{14}t^9+q^{18}t^{11}+q^{20}t^{12}. \tag{4-2}$$

The knot $Cab(5,2)T(3,2)$ is the first with nonpositive $\mathcal{H}_{\vec{r},\vec{s}}(\square;q,t,a)$ in the considered family. The actual Khovanov polynomials coincide with $DAHA'Kh$.

The corresponding DAHA-Jones polynomial for A_1 is

$$\widetilde{JD}^{A_1}_{\{3,-2\},\{2,7\}}(\omega_1;q,t)=\mathcal{H}(q,t,a=-t^2)=1+qt+q^2t-qt^2+q^3t^2+q^4t^2-q^2t^3-q^3t^3-q^4t^3-q^5t^4+q^5t^5. \tag{4-3}$$

Its value at $t=q$ is the (reduced) *Jones polynomial* $\widetilde{J}=1+q^2-q^7-q^9+q^{10}$ (for our q and under the tilde normalization); the (reduced) one produced by the KnotTheory package is $q^4+q^6-q^{11}-q^{13}+q^{14}$.

We note that (4-3) has little to do with the polynomial obtained in [40, Section 5.2], (for the same knot). We cannot comment on this and the general approach used in [40]. The latter stimulated our paper, but the construction there remains unclear to us.

(c) $\vec{r}=\{3,-2\},\quad \vec{s}=\{2,9\},\quad \mathcal{T}=Cab(3,2)T(3,2);$

$$\mathcal{H}_{\vec{r},\vec{s}}(\square;q,t,a)=1-q^2+qt+q^2t-q^3t+q^2t^2+q^3t^3+a^3\big(-\tfrac{1}{t^2}q^4-\tfrac{1}{t}q^5\big)+a^2\big(q^3-q^4-q^5-\tfrac{1}{t^2}q^3-\tfrac{1}{t}q^3-\tfrac{1}{t}2q^4\big)+a\big(q+q^2-2q^3-q^4-\tfrac{1}{t}q^2-\tfrac{1}{t}q^3+q^2t+q^3t-q^4t+q^3t^2\big).$$

$$\mathcal{H}(q=1,t,a=0)=t+t^2+t^3,$$
$$DAHA'Kh(Cab(3,2)T(3,2))=1+q^4t^2+q^4t^3+q^6t^3+2q^6t^4+q^{10}t^5+q^{10}t^6+q^{12}t^7+q^{12}t^8+q^{14}t^9+q^{18}t^{11}+q^{20}t^{12}. \tag{4-4}$$

Here, $\widetilde{Kh}-DAHA'Kh=q^{20}t^{12}(1-t^2)$, which matches our expectations.

(d) $\vec{r}=\{3,-2\},\quad \vec{s}=\{2,11\},\quad \mathcal{T}=Cab(1,2)T(3,2);$

$$\mathcal{H}_{\vec{r},\vec{s}}(\square;q,t,a)=-q+t-2q^2t+qt^2-q^3t^2+q^2t^3+a^3\big(-q^5-\tfrac{1}{t^2}q^3-\tfrac{1}{t}q^4\big)+a^2\big(-2q^3-2q^4-\tfrac{1}{t^2}q^2-\tfrac{1}{t}q^2-\tfrac{1}{t}2q^3-q^4t-q^5t\big)+a\big(-3q^2-q^3-\tfrac{1}{t}q-\tfrac{1}{t}q^2+qt-3q^3t-q^4t+q^2t^2-q^4t^2\big).$$

(4-5) $$\mathcal{H}(q=1,t,a=0)=-1-t+t^3,$$

$$\begin{aligned}\widetilde{Kh}(Cab(1,2)T(3,2)) &= DAHA'Kh(Cab(1,2)T(3,2))\\ &= 1+t+q^2t^2+q^4t^2+q^4t^3+q^6t^3+2q^6t^4+q^{10}t^5\\ &\quad +q^{10}t^6+q^{12}t^7+q^{12}t^8+q^{14}t^9+q^{18}t^{11}+q^{20}t^{12}.\end{aligned}$$

(e) $\vec{\mathsf{r}}=\{3,-2\}$, $\vec{\mathsf{s}}=\{2,13\}$, $\mathcal{T}=Cab(-1,2)T(3,2)$;

$$\begin{aligned}\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\square;q,t,a) &= 1+2qt-t^2+qt^2+2q^2t^2-qt^3+q^3t^3\\ &\quad +a^3\big(q^4+\tfrac{1}{t^2}q^2+\tfrac{1}{t}q^3+q^5t\big)\\ &\quad +a^2\big(2q^2+2q^3+\tfrac{1}{t^2}q+\tfrac{1}{t}q+\tfrac{1}{t}2q^2+q^2t+2q^3t+2q^4t+q^4t^2+q^5t^2\big)\\ &\quad +a\big(3q+q^2+\tfrac{1}{t}1+\tfrac{1}{t}q+qt+4q^2t+q^3t-qt^2+q^2t^2+3q^3t^2+q^4t^2+q^4t^3\big).\end{aligned}$$

Note that we have $\mathcal{H}(q=0,t,a=0)=1+a/t$ for $\mathsf{m}<0$ in the family of knots $Cab(2\mathsf{m}+1,2)T(3,2)$; it was 1 or t before.

(4-6) $$\mathcal{H}(q=1,t,a=0)=1+2t+2t^2,$$

$$\begin{aligned}\widetilde{Kh}(Cab(-1,2)T(3,2)) &= DAHA^+Kh(Cab(-1,2)T(3,2))\\ &= 1+q^2t+q^4t+q^4t^2+q^6t^3+q^8t^3+q^8t^4\\ &\quad +q^{10}t^4+2q^{10}t^5+q^{14}t^6+q^{14}t^7+q^{16}t^8\\ &\quad +q^{16}t^9+q^{18}t^{10}+q^{22}t^{12}+q^{24}t^{13}.\end{aligned}$$

Note the use of the procedure $DAHA^+Kh$ here and in the next example (which starts with the substitution $-1\mapsto a/t^2$ in DAHA parameters).

(f) $\vec{\mathsf{r}}=\{3,-2\}$, $\vec{\mathsf{s}}=\{2,15\}$, $\mathcal{T}=Cab(-3,2)T(3,2)$;

$$\begin{aligned}\mathcal{H}_{\vec{\mathsf{r}},\vec{\mathsf{s}}}(\square;q,t,a) &= 1+2qt+qt^2+2q^2t^2-qt^3+q^2t^3+2q^3t^3+q^4t^4\\ &\quad +a^3\big(q^4+\tfrac{1}{t^2}q^2+\tfrac{1}{t}q^3+q^5t+q^6t^2\big)\\ &\quad +a^2\big(2q^2+2q^3+\tfrac{1}{t^2}q+\tfrac{1}{t}q+\tfrac{1}{t}2q^2+q^2t+2q^3t+2q^4t\\ &\qquad +q^3t^2+2q^4t^2+2q^5t^2+q^5t^3+q^6t^3\big)\\ &\quad +a\big(3q+q^2+\tfrac{1}{t}1+\tfrac{1}{t}q+qt+4q^2t+q^3t+2q^2t^2\\ &\qquad +4q^3t^2+q^4t^2+q^3t^3+3q^4t^3+q^5t^3+q^5t^4\big).\end{aligned}$$

(4-7) $$\mathcal{H}(q=1,t,a=0)=1+2t+3t^2+2t^3+t^4,$$

$$\begin{aligned}DAHA^+Kh(Cab(-3,2)T(3,2)) &= 1+q^2t+q^8t+q^4t^2+q^6t^3+q^8t^4\\ &\quad +q^{10}t^5+q^{12}t^6+q^{14}t^6+q^{12}t^7\\ &\quad +2q^{14}t^7+q^{18}t^8+q^{18}t^9+q^{20}t^{10}\\ &\quad +q^{20}t^{11}+q^{22}t^{12}+q^{26}t^{14}+q^{28}t^{15}.\end{aligned}$$

Here, $\widetilde{Kh}-DAHA^{+}Kh=q^{12}(t^5-t^7)$. Note the use of $DAHA^{+}$ here and that there is only one negative term in $\mathcal{H}(\square;q,t,a)$, namely $-qt^3$. Using the procedure $DAHA'Kh$ would result in the extra term $-q^8(t-t^3)$ in this difference, where -1 does not agree with the (topological) ∂_2.

(g) $\vec{r}=\{3,-2\},\quad \vec{s}=\{2,17\},\quad \mathcal{T}=Cab(-5,2)T(3,2);$

$$\begin{aligned}\mathcal{H}_{\vec{r},\vec{s}}(\square;q,t,a)&=1+2qt+qt^2+2q^2t^2+q^2t^3+2q^3t^3+q^3t^4+2q^4t^4+q^5t^5\\&\quad+a^3\big(q^4+\tfrac{1}{t^2}q^2+\tfrac{1}{t}q^3+q^5t+q^6t^2+q^7t^3\big)\\&\quad+a^2\big(2q^2+2q^3+\tfrac{1}{t^2}q+\tfrac{1}{t}q+\tfrac{1}{t}2q^2+q^2t+2q^3t+2q^4t+q^3t^2\\&\qquad+2q^4t^2+2q^5t^2+q^4t^3+2q^5t^3+2q^6t^3+q^6t^4+q^7t^4\big)\\&\quad+a\big(3q+q^2+\tfrac{1}{t}1+\tfrac{1}{t}q+qt+4q^2t+q^3t+2q^2t^2+4q^3t^2+q^4t^2+q^2t^3\\&\qquad+2q^3t^3+4q^4t^3+q^5t^3+q^4t^4+3q^5t^4+q^6t^4+q^6t^5\big).\end{aligned}$$

(4-8)
$$\mathcal{H}(q=1,t,a=0)=1+2t+3t^2+3t^3+3t^4+t^5,$$

$$\begin{aligned}DAHA'Kh(Cab(-5,2)T(3,2))&=1+q^2t+q^4t^2+q^6t^3+q^8t^4+q^{10}t^5\\&\quad+q^{12}t^6+q^{12}t^7+q^{14}t^7+q^{16}t^8+q^{18}t^8\\&\quad+q^{16}t^9+2q^{18}t^9+q^{22}t^{10}+q^{22}t^{11}+q^{24}t^{12}\\&\quad+q^{24}t^{13}+q^{26}t^{14}+q^{30}t^{16}+q^{32}t^{17}.\end{aligned}$$

One has that $\widetilde{Kh}-DAHA'Kh=q^{12}(t^5-t^7)+q^{16}(t^7-t^9)$, matching our expectations, though this is obviously not in "top" t–degrees.

This knot is *pseudoalgebraic*, as well as all further knots in the considered family. However, they are quite different numerically from the patterns in (3-11) and (3-12). For example, the polynomial $\mathcal{H}(q=1,t,a=0)$ becomes $1+2t+3t^2+3t^3+4t^4+3t^5+t^6$ for $Cab(-7,2)T(3,2)$. Generally,

(4-9)
$$\begin{aligned}\mathcal{H}_{\{3,-2\},\{2,2n+7\}}(q=1,t,a=0)&=1+2t+3t^2+3t^3\\&\quad+4(t^4+\cdots+t^{n-2})+3t^{n-1}+t^n\end{aligned}$$

for $Cab(-2n+5,2)T(3,2)$, $n\geq 5$. Let us provide a *pseudoalgebraic* knot of type **(4)** (for $T(4,3)$).

(h) $\vec{r}=\{4,-2\},\quad \vec{s}=\{3,7\},\quad \mathcal{T}=Cab(17,2)T(4,3);$

$$\begin{aligned}\mathcal{H}_{\vec{r},\vec{s}}(\square;q,t,a)&=1+qt+q^2t+q^3t+q^4t+q^5t+q^2t^2+q^3t^2+2q^4t^2+2q^5t^2+3q^6t^2+q^7t^2\\&\quad+q^8t^2+q^3t^3+q^4t^3+2q^5t^3+3q^6t^3+4q^7t^3+3q^8t^3+2q^9t^3+q^4t^4+q^5t^4+2q^6t^4\\&\quad+3q^7t^4+5q^8t^4+4q^9t^4+3q^{10}t^4+q^5t^5+q^6t^5+2q^7t^5+3q^8t^5+5q^9t^5+4q^{10}t^5\\&\quad+2q^{11}t^5+q^6t^6+q^7t^6+2q^8t^6+3q^9t^6+5q^{10}t^6+3q^{11}t^6+q^{12}t^6+q^7t^7+q^8t^7+2q^9t^7\end{aligned}$$

$$
\begin{aligned}
&+3q^{10}t^7+4q^{11}t^7+q^{12}t^7+q^8t^8+q^9t^8+2q^{10}t^8+3q^{11}t^8+3q^{12}t^8+q^9t^9+q^{10}t^9\\
&+2q^{11}t^9+2q^{12}t^9+q^{13}t^9+q^{10}t^{10}+q^{11}t^{10}+2q^{12}t^{10}+q^{13}t^{10}+q^{11}t^{11}+q^{12}t^{11}\\
&+q^{13}t^{11}+q^{12}t^{12}+q^{13}t^{12}+q^{13}t^{13}+q^{14}t^{14}\\
&+a^4\big(q^{10}+q^{11}+q^{12}+q^{11}t+2q^{12}t+q^{13}t+q^{12}t^2+2q^{13}t^2+q^{14}t^2+q^{13}t^3+q^{14}t^3+q^{14}t^4\big)\\
&+a^3\big(q^6+q^7+2q^8+2q^9+2q^{10}+q^{11}+q^7t+2q^8t+4q^9t+5q^{10}t+5q^{11}t+2q^{12}t+q^8t^2\\
&+2q^9t^2+5q^{10}t^2+7q^{11}t^2+7q^{12}t^2+2q^{13}t^2+q^9t^3+2q^{10}t^3+5q^{11}t^3+7q^{12}t^3+5q^{13}t^3\\
&+q^{14}t^3+q^{10}t^4+2q^{11}t^4+5q^{12}t^4+5q^{13}t^4+2q^{14}t^4+q^{11}t^5+2q^{12}t^5+4q^{13}t^5+2q^{14}t^5\\
&+q^{12}t^6+2q^{13}t^6+2q^{14}t^6+q^{13}t^7+q^{14}t^7+q^{14}t^8\big)\\
&+a^2\big(q^3+q^4+2q^5+2q^6+2q^7+q^8+q^9+q^4t+2q^5t+4q^6t+6q^7t+7q^8t+6q^9t+4q^{10}t\\
&+q^{11}t+q^5t^2+2q^6t^2+5q^7t^2+8q^8t^2+12q^9t^2+11q^{10}t^2+7q^{11}t^2+q^{12}t^2+q^6t^3+2q^7t^3\\
&+5q^8t^3+9q^9t^3+14q^{10}t^3+14q^{11}t^3+7q^{12}t^3+q^{13}t^3+q^7t^4+2q^8t^4+5q^9t^4+9q^{10}t^4+14q^{11}t^4\\
&+11q^{12}t^4+4q^{13}t^4+q^8t^5+2q^9t^5+5q^{10}t^5+9q^{11}t^5+12q^{12}t^5+6q^{13}t^5+q^{14}t^5+q^9t^6\\
&+2q^{10}t^6+5q^{11}t^6+8q^{12}t^6+7q^{13}t^6+q^{14}t^6+q^{10}t^7+2q^{11}t^7+5q^{12}t^7+6q^{13}t^7+2q^{14}t^7\\
&+q^{11}t^8+2q^{12}t^8+4q^{13}t^8+2q^{14}t^8+q^{12}t^9+2q^{13}t^9+2q^{14}t^9+q^{13}t^{10}+q^{14}t^{10}+q^{14}t^{11}\big)\\
&+a\big(q+q^2+q^3+q^4+q^5+q^2t+2q^3t+3q^4t+4q^5t+5q^6t+3q^7t+2q^8t+q^9t+q^3t^2+2q^4t^2\\
&+4q^5t^2+6q^6t^2+9q^7t^2+8q^8t^2+6q^9t^2+2q^{10}t^2+q^4t^3+2q^5t^3+4q^6t^3+7q^7t^3+11q^8t^3\\
&+12q^9t^3+10q^{10}t^3+3q^{11}t^3+q^5t^4+2q^6t^4+4q^7t^4+7q^8t^4+12q^9t^4+13q^{10}t^4+10q^{11}t^4\\
&+2q^{12}t^4+q^6t^5+2q^7t^5+4q^8t^5+7q^9t^5+12q^{10}t^5+12q^{11}t^5+6q^{12}t^5+q^{13}t^5+q^7t^6+2q^8t^6\\
&+4q^9t^6+7q^{10}t^6+11q^{11}t^6+8q^{12}t^6+2q^{13}t^6+q^8t^7+2q^9t^7+4q^{10}t^7+7q^{11}t^7+9q^{12}t^7\\
&+3q^{13}t^7+q^9t^8+2q^{10}t^8+4q^{11}t^8+6q^{12}t^8+5q^{13}t^8+q^{10}t^9+2q^{11}t^9+4q^{12}t^9+4q^{13}t^9\\
&+q^{14}t^9+q^{11}t^{10}+2q^{12}t^{10}+3q^{13}t^{10}+q^{14}t^{10}+q^{12}t^{11}+2q^{13}t^{11}+q^{14}t^{11}+q^{13}t^{12}\\
&+q^{14}t^{12}+q^{14}t^{13}\big).
\end{aligned}
$$

The corresponding Khovanov polynomial equals

$$
\begin{aligned}
DAHA'Kh(Cab(17,2)T(4,3)) = 1 &+ q^4t^2 + q^6t^3 + q^6t^4 + q^{10}t^5 + q^8t^6 + q^{12}t^7\\
&+ q^{10}t^8 + q^{12}t^8 + 2q^{14}t^9 + q^{12}t^{10} + q^{14}t^{10} + 2q^{16}t^{11} + q^{18}t^{11} + 3q^{16}t^{12}\\
&+ q^{20}t^{12} + 2q^{18}t^{13} + 2q^{20}t^{13} + q^{18}t^{14} + q^{20}t^{14} + q^{22}t^{14} + 4q^{22}t^{15}\\
&+ q^{20}t^{16} + q^{22}t^{16} + q^{24}t^{16} + q^{26}t^{16} + 2q^{24}t^{17} + q^{26}t^{17} + q^{24}t^{18} + q^{28}t^{18}\\
&+ q^{30}t^{20} + q^{26}t^{19} + q^{28}t^{19} + q^{28}t^{20} + 2q^{30}t^{21} + q^{32}t^{22} + q^{36}t^{24} + q^{34}t^{25}.
\end{aligned}
$$

Here, $\widetilde{Kh} - DAHA'Kh = q^{34}(t^{23} - t^{25})$, which agrees with part (ii) of Conjecture 2.4. The corresponding (reduced and tilde-normalized) HOMFLYPT polynomial coincides with $\mathcal{H}(\square; q, q, -a)$, as well as in *all* examples we considered (not only those posted in the paper).

The last example will be a *non-pseudoalgebraic* knot with a relatively involved relation between $DAHA'Kh$ and $\widetilde{Kh}$.

(i) $$\vec{r}=\{4,-2\},\quad \vec{s}=\{3,17\},\quad \mathcal{T}=Cab(7,2)T(4,3);$$

$$\begin{aligned}
\mathcal{H}_{\vec{r},\vec{s}}(\square;q,t,a) &= -q^2-2q^3t-q^4t-q^3t^2-4q^4t^2-q^5t^2-2q^4t^3-6q^5t^3-q^6t^3+t^4-q^4t^4\\
&\quad -4q^5t^4-7q^6t^4-q^7t^4+qt^5+q^2t^5+q^3t^5+q^4t^5-q^5t^5-5q^6t^5-7q^7t^5-q^8t^5+q^2t^6+q^3t^6\\
&\quad +2q^4t^6+q^5t^6-q^6t^6-4q^7t^6-6q^8t^6-q^9t^6+q^3t^7+q^4t^7+2q^5t^7+q^6t^7-q^7t^7-2q^8t^7\\
&\quad -4q^9t^7-q^{10}t^7+q^4t^8+q^5t^8+q^6t^8-q^9t^8-2q^{10}t^8+q^5t^9+q^6t^9-q^{11}t^9+q^6t^{10}+q^7t^{11}\\
&+a^5\left(-q^{11}-\tfrac{1}{t^3}q^8-\tfrac{1}{t^2}q^9-\tfrac{1}{t}q^{10}-q^{12}t-q^{13}t^2-q^{14}t^3\right)\\
&+a^4\left(-q^7-3q^8-3q^9-2q^{10}-\tfrac{1}{t^3}q^6-\tfrac{1}{t^3}q^7-\tfrac{1}{t^2}q^6-\tfrac{1}{t^2}2q^7-\tfrac{1}{t^2}2q^8-\tfrac{1}{t}2q^7-\tfrac{1}{t}3q^8-\tfrac{1}{t}2q^9\right.\\
&\quad -2q^8t-3q^9t-3q^{10}t-2q^{11}t-q^8t^2-2q^9t^2-3q^{10}t^2-3q^{11}t^2-2q^{12}t^2-q^9t^3-2q^{10}t^3\\
&\quad \left.-3q^{11}t^3-3q^{12}t^3-2q^{13}t^3-q^{11}t^4-2q^{12}t^4-2q^{13}t^4-q^{14}t^4-q^{13}t^5-q^{14}t^5\right)\\
&+a^3\left(-q^5-7q^6-8q^7-6q^8-q^9-\tfrac{1}{t^3}q^5-\tfrac{1}{t^2}q^4-\tfrac{1}{t^2}2q^5-\tfrac{1}{t^2}3q^6-\tfrac{1}{t^2}q^7-\tfrac{1}{t}3q^5-\tfrac{1}{t}5q^6-\tfrac{1}{t}5q^7\right.\\
&\quad -\tfrac{1}{t}q^8-3q^6t-11q^7t-9q^8t-6q^9t-q^{10}t-q^6t^2-6q^7t^2-13q^8t^2-9q^9t^2-6q^{10}t^2-q^{11}t^2\\
&\quad -3q^7t^3-8q^8t^3-13q^9t^3-9q^{10}t^3-6q^{11}t^3-q^{12}t^3+q^6t^4-3q^8t^4-6q^9t^4-11q^{10}t^4\\
&\quad -8q^{11}t^4-5q^{12}t^4-q^{13}t^4+q^7t^5-q^9t^5-3q^{10}t^5-7q^{11}t^5-5q^{12}t^5-3q^{13}t^5-q^{11}t^6\\
&\quad \left.-3q^{12}t^6-2q^{13}t^6-q^{14}t^6-q^{13}t^7\right)\\
&+a^2\left(-3q^4-10q^5-7q^6-3q^7-\tfrac{1}{t^2}q^3-\tfrac{1}{t^2}q^4-\tfrac{1}{t^2}q^5-\tfrac{1}{t}q^3-\tfrac{1}{t}4q^4-\tfrac{1}{t}4q^5-\tfrac{1}{t}2q^6-q^4t-8q^5t\right.\\
&\quad -16q^6t-9q^7t-3q^8t-3q^5t^2-14q^6t^2-20q^7t^2-9q^8t^2-3q^9t^2-q^5t^3-8q^6t^3-19q^7t^3\\
&\quad -21q^8t^3-9q^9t^3-3q^{10}t^3+q^3t^4+q^4t^4+2q^5t^4-q^6t^4-11q^7t^4-19q^8t^4-20q^9t^4-9q^{10}t^4\\
&\quad -3q^{11}t^4+q^4t^5+2q^5t^5+3q^6t^5-q^7t^5-8q^8t^5-14q^9t^5-16q^{10}t^5-7q^{11}t^5-2q^{12}t^5+q^5t^6\\
&\quad +2q^6t^6+2q^7t^6-q^8t^6-3q^9t^6-8q^{10}t^6-10q^{11}t^6-4q^{12}t^6-q^{13}t^6+q^6t^7+q^7t^7-q^{10}t^7\\
&\quad \left.-3q^{11}t^7-4q^{12}t^7-q^{13}t^7+q^7t^8-q^{12}t^8-q^{13}t^8\right)\\
&+a\left(-3q^3-4q^4-2q^5-\tfrac{1}{t}q^2-\tfrac{1}{t}q^3-\tfrac{1}{t}q^4-q^3t-7q^4t-8q^5t-3q^6t-3q^4t^2-13q^5t^2-10q^6t^2\right.\\
&\quad -3q^7t^2-q^4t^3-7q^5t^3-18q^6t^3-11q^7t^3-3q^8t^3+qt^4+q^2t^4+q^3t^4+q^4t^4-2q^5t^4-12q^6t^4\\
&\quad -20q^7t^4-11q^8t^4-3q^9t^4+q^2t^5+2q^3t^5+3q^4t^5+3q^5t^5-2q^6t^5-12q^7t^5-18q^8t^5-10q^9t^5\\
&\quad -3q^{10}t^5+q^3t^6+2q^4t^6+4q^5t^6+3q^6t^6-2q^7t^6-7q^8t^6-13q^9t^6-8q^{10}t^6-2q^{11}t^6+q^4t^7\\
&\quad +2q^5t^7+3q^6t^7+q^7t^7-q^8t^7-3q^9t^7-7q^{10}t^7-4q^{11}t^7-q^{12}t^7+q^5t^8+2q^6t^8+q^7t^8\\
&\quad \left.-q^{10}t^8-3q^{11}t^8-q^{12}t^8+q^6t^9+q^7t^9-q^{12}t^9+q^7t^{10}\right).
\end{aligned}$$

In this example, applying the $DAHA'$ reduction procedure goes essentially as follows. Starting with $m=0$, one cancels consecutively the pairs in the form

$$\pm q^x t^y(a^m+a^{m+1}/t^2)\quad\text{and}\quad q^x t^y(a^m-a^{m+2}/t^4)$$

in the superpolynomial $\mathcal{H}_{\{4,-2\},\{3,17\}}$. The output is

$$\begin{aligned}-q^4+a^4\big(-q^{11}-\tfrac{1}{t^2}q^8-\tfrac{1}{t^2}q^9-\tfrac{1}{t}q^9-\tfrac{1}{t}2q^{10}\big)-q^5t+t^2+qt^3+q^2t^3+q^3t^3\\ +q^4t^3+q^5t^3+q^4t^4+q^5t^4+2q^6t^4+q^7t^5\\ +a^3\big(-q^{10}-\tfrac{1}{t^2}q^6-\tfrac{1}{t^2}2q^7-\tfrac{1}{t}2q^8-\tfrac{1}{t}q^9-q^9t^2-q^{10}t^2-q^{11}t^3\big)\\ +a^2\big(-\tfrac{1}{t^2}q^6-\tfrac{1}{t}q^7-q^6t-q^8t^2+q^5t^3+q^6t^3+q^7t^3+q^7t^4+q^8t^4\big)\\ +a\big(-q^5-\tfrac{1}{t}q^4-q^6t+qt^2+q^2t^3+q^3t^3+2q^4t^3+2q^5t^3\\ +q^6t^3+q^5t^4+2q^6t^4+4q^7t^4+q^8t^5\big).\end{aligned}$$

Then we substitute $a\mapsto q_{st}^4t_{st}$, $q\mapsto(q_{st}t_{st})^2$ and $t\mapsto q_{st}^2$ then divide by the smallest power of q_{st}, and, finally, replace $-1\mapsto 1/t_{st}$. Recall that q and t mean actually q_{st} and t_{st}, respectively, in *all* formulas for *KhR*–polynomials. Finally,

$$\begin{aligned}DAHA'Kh(Cab(7,2)T(4,3))=1+q^{32}t^{22}+q^{36}t^{24}+q^{34}t^{25}+q^4(t^2+t^7)\\ +q^6(t^3+t^4+t^8)+q^8(t^6+t^9)+q^{10}(t^5+t^8+t^{10})+q^{14}(2t^9+t^{10}+t^{12})\\ +q^{12}(t^7+t^8+t^{10}+t^{13})+q^{20}(t^{12}+2t^{13}+t^{14})+q^{16}(2t^{11}+2t^{12}+t^{14}+t^{15})\\ +q^{18}(t^{11}+2t^{13}+2t^{16})+q^{22}(t^{14}+4t^{15}+2t^{18})+q^{24}(t^{16}+t^{17}+t^{19}+t^{20})\\ +q^{26}(t^{16}+t^{17}+t^{21})+q^{28}(t^{18}+t^{21}+t^{22})+q^{30}(t^{20}+2t^{23}).\end{aligned}$$

One has

$$\begin{aligned}\widetilde{Kh}(Cab(7,2)T(4,3))-DAHA'Kh=q^{12}t^{11}(1-t^2)\\ \times\big(1+2q^6t^3+2q^{10}t^5+q^{14}t^8+2q^{18}t^{10}+q^{22}t^{12}+(1+t)(q^4t+q^{12}t^6+q^{16}t^8)\big),\end{aligned}$$

where the positivity of the coefficients matches our expectations (based on our assumptions concerning ∂_2) and holds for *all* (uncolored) nonalgebraic knots we calculated so far, where $DAHA'Kh$ is taken if the a–leading term in $\mathcal{H}(\square;q,t,a)$ is negative (in our examples, the q,t–monomials there are all positive or all negative) and $DAHA^+Kh$ otherwise. We hesitate to conjecture this in general.

4.2 The case of A_1

We are going to prove part (i) of the Connection conjecture in the A_1 case, ie the equality

$$\mathcal{H}(b\omega_1;q,q,-q^2)=\widetilde{HOM}(b\omega_1;q,q^2)=\widetilde{JD}^{A_1}(b\omega_1;q,q),\quad b\in\mathbb{Z}_+. \tag{4-10}$$

We need the simplest case of the theory of the difference *shift operators*; see [9, Theorem 2.4] and [12].

For $t_\alpha = q_\alpha^{k_{v\alpha}}$, let $k = \{k_v = 0, 1\}$. Generally, we define

$$\mathcal{X}_k = \prod_{\alpha \in R_+} \prod_{j=0}^{k_\alpha - 1} \big((q_\alpha^j X_\alpha)^{1/2} - (q_\alpha^j X_\alpha)^{-1/2}\big). \tag{4-11}$$

Recall that $q_\alpha = q^{v_\alpha}$, where $v_\alpha := (\alpha, \alpha)/2$.

We put $H = H^{(k)}$ for elements H from the double affine Hecke algebra $\mathcal{H}\!\mathcal{l}^{(k)}$ with the structural parameters q and $t_v = q_v^{k_v}$, as well as their images in the corresponding polynomial representation, denoted by $\mathcal{V}^{(k)}$. Accordingly, $\tau_\pm^{(k)}$ will be automorphisms of $\mathcal{H}\!\mathcal{l}^{(k)}$.

We set $\circ = (\{k_v = 0\})$. Then the operator H° is obtained from H by replacing every Y_b $(b \in P)$ by the difference operators b^{-1}, and T_w $(w \in W)$ by w. One has

$$\tau_+^\circ(Y_b) = q^{-(b,b)/2} X_b Y_b, \tau_-^\circ(X_b) = q^{(b,b)/2} Y_b X_b \quad \text{for } b \in P. \tag{4-12}$$

Lemma 4.1 *Let H be an algebraic expression with complex coefficients in terms of the standard W–symmetrizations of monomials with respect to $\{X_b\}$ and those with respect to $\{Y_b\}$. The restriction of the corresponding operator to the subspace $(\mathcal{V}^{(k)})^W$ of W–invariant elements in $\mathcal{V}^{(k)}$ will be denoted by $H_{sym}^{(k)}$. Then*

$$H_{sym}^{(k)} = \mathcal{X}_k^{-1} H_{sym}^\circ \mathcal{X}_k, \quad (\tau_\pm^{(k)}(H^{(k)}))_{sym} = \mathcal{X}_k^{-1} (\tau_\pm^\circ(H^\circ))_{sym} \mathcal{X}_k.$$

Let us *formally* set for $\lambda, \lambda' \in \mathbb{C}^n$,

$$X_\lambda = q^{x_\lambda}, \quad x_\lambda = (\lambda, x), \quad X_{\lambda+\lambda'} = X_\lambda X_{\lambda'}, \quad x^2 = (x,x) = \sum_{i=1}^n (x, \omega_i)(x, \alpha_i^\vee).$$

We will treat $\tau_\pm^\circ$ and any elements $\gamma \in \mathrm{SL}(2, \mathbb{Z})$ as automorphisms of the linear span of $X_\lambda q^{Mx^2/2}$ for all $\lambda \in \mathbb{C}^n$ and M from a sufficiently general subset of $\mathbb{C}$ (see the inequalities for M, N, M' below). Given λ, only $c\lambda$ for certain $c \in \mathbb{C}$ will be actually needed in (4-13) and (4-14).

This linear span will be denoted by $\widetilde{\mathcal{V}}$. The action of $\widehat{W}$ in $\mathcal{V}$ is naturally extended to $\widetilde{\mathcal{V}}$ through its action on $\{x_\lambda\}$; $q^{Mx^2/2}$ are W–invariant and satisfy the standard difference equations for $P \subset \widehat{W}$.

The main formulas we need are as follows:

$$\sigma^\circ(X_\lambda q^{-Mx^2/2}) = \frac{q^{\lambda^2/(2M)}}{M^{1/2}} X_{\lambda/M} q^{+x^2/(2M)}, \tag{4-13}$$

$$(\tau_+^\circ)^N(X_\lambda q^{-Mx^2/2}) = X_\lambda q^{(N-M)x^2/2},$$

$$(\tau_-^\circ)^N(X_\lambda q^{-Mx^2/2}) = ((\sigma^\circ)^{-1}(\tau_+^\circ)^{-N}\sigma^\circ)(X_\lambda e^{-Mx^2/2})$$

$$= \frac{1}{(1-MN)^{1/2}} q^{\frac{\lambda^2 N}{2(1-MN)}} X_{\lambda/(1-MN)} q^{-x^2\frac{M}{2(1-MN)}}.$$

Here, $N \in \mathbb{Z}$ and the parameters M are sufficiently general complex numbers; for instance, $MN \neq 1$ in the last formula. We also assume that $(MM')^{1/2} = M^{1/2}(M')^{1/2}$ for $M, M' \in \mathbb{C}$.

These formulas can be readily extended to the following *free action* of the whole $\mathrm{SL}(2,\mathbb{Z})$ on $X_\lambda q^{zx^2/2}$,

$$\gamma^\circ(X_\lambda q^{zx^2/2}) = \frac{1}{(cz+d)^{1/2}} q^{-\frac{\lambda^2 c}{2(cz+d)}} X_{\frac{\lambda}{cz+d}} q^{\frac{az+b}{cz+d}x^2/2} \tag{4-14}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, $\lambda \in \mathbb{C}^n$, and for generic $z \in \mathbb{C}$.

DAHA-Jones polynomials We will now switch to the A_1 case. Let $\alpha = \alpha_1$, $s = s_1$ and $\omega = \omega_1$; then $\alpha = \alpha_1 = 2\omega$ and $\rho = \omega$. The extended affine Weyl group is $\widehat{W} = \langle s, \omega \rangle$. The weights $b\omega$ $(b \in \mathbb{Z})$ will be denoted simply by b.

The *double affine Hecke algebra* $\mathcal{H}$ is generated by invertible elements $Y = Y_\omega$, $T = T_1$, and $X = X_\omega$, subject to the quadratic relation $(T - t^{1/2})(T + t^{-1/2}) = 0$ and the cross relations

$$TXT = X^{-1}, \quad T^{-1}YT^{-1} = Y^{-1}, \quad Y^{-1}X^{-1}YXT^2q^{1/2} = 1. \tag{4-15}$$

The field of definition will be $\mathbb{Q}(q^{1/4}, t^{1/2})$. Here, $q^{\pm 1/4}$ is needed for the automorphisms $\tau_\pm$:

$$\tau_+(X) = X, \quad \tau_+(T) = T, \quad \tau_+(Y) = q^{-1/4}XY, \tag{4-16}$$

$$\tau_-(Y) = Y, \quad \tau_-(T) = T, \quad \tau_-(X) = q^{1/4}YX.$$

We will prove the Connection conjecture for the *nonreduced* version of the DAHA-Jones polynomials. Namely, we modify (2-12) as

$$JD^\sharp_{\vec{r},\vec{s}}(b;q,t) := \{\hat{\gamma}_1\big(\cdots\big(\hat{\gamma}_{\ell-1}\big((\hat{\gamma}_\ell(P_b^{(k)}))\Downarrow\big)\Downarrow\big)\cdots\big)\}^{(k)}, \tag{4-17}$$

for $t := q^k$ and $\gamma = \gamma_{r,s}$ lifted to $\hat{\gamma} \in \mathrm{PSL}_2^\wedge(\mathbb{Z})$, where we show explicitly the dependence of the P–polynomial on the parameter k and also use $^{(k)}$ for the corresponding

coinvariant. Generally, the nonreduced DAHA-Jones polynomials are inconvenient because of their nontrivial q,t–denominators. However, they become q–polynomials as $t=q$ and simplify the considerations.

4.3 Jones polynomials

We are going to use the relation for the colored Jones polynomials of the cable $Cab(\mathsf{a},\mathsf{r})K$ with that of the initial knot K. See [32; 44] and references therein. Note that t in [44] is our $q^{1/4}$, and $Cab(r,s)$ there is our $Cab(\mathsf{a},\mathsf{r})$ (in this order!).

Assuming here and below that the j–summation has step 1, one has

$$\mathcal{J}^{\sharp}_{Cab(\mathsf{a},\mathsf{r})K}(b;q)=q^{-\mathsf{arb}(b+2)/4}\sum_{j=-b/2}^{b/2}q^{\mathsf{a}j(j\mathsf{r}+1)}\mathcal{J}^{\sharp}_{K}(2\mathsf{r}j;q). \tag{4-18}$$

This formula gives a recursive definition of the Jones polynomials of iterated knots; the unknot is normalized as

$$\mathcal{J}^{\sharp}_{\bigcirc}(b;q)=(q^{(b+1)/2}-q^{-(b+1)/2})/(q^{1/2}-q^{-1/2}). \tag{4-19}$$

Proposition 4.2 *For any (possibly negative) $\{\mathsf{r},\mathsf{s}\}$ from Theorem 2.1,*

$$\begin{aligned}JD^{\sharp}_{\{\mathsf{r}_1,\dots,\mathsf{r}_\ell\},\{\mathsf{s}_1,\dots,\mathsf{s}_\ell\}}(b;q,t\mapsto q)&\\ =q^{-\mathsf{a}_\ell\mathsf{r}_\ell b(b+2)/4}\mathcal{J}^{\sharp}_{T(\{\mathsf{r}_1,\dots,\mathsf{r}_\ell\},\{\mathsf{s}_1,\dots,\mathsf{s}_\ell\})}(b;q\mapsto q^{-1}),&\end{aligned} \tag{4-20}$$

where $T(\{\mathsf{r}_1,\dots,\mathsf{r}_\ell\},\{\mathsf{s}_1,\dots,\mathsf{s}_\ell\})=Cab(\vec{\mathsf{a}},\vec{\mathsf{r}})=(Cab(\mathsf{a}_\ell,\mathsf{r}_\ell)\cdots Cab(\mathsf{a}_1,\mathsf{r}_1))(\bigcirc)$, and a_i are recursively defined by relations $\mathsf{a}_{i+1}=\mathsf{r}_i\mathsf{r}_{i+1}\mathsf{a}_i+\mathsf{s}_{i+1}$ for $1\le i\le\ell-1$ and the initial condition $\mathsf{a}_1=\mathsf{s}_1$.

Proof As above, we formally set $X_\lambda=q^{\lambda x}$ for $\lambda\in\mathbb{C}$, which extends $X_b=X_{b\omega_1}=X^b$ for b in $\mathbb{Z}$ (identified with the lattice $P=\mathbb{Z}\omega_1$). The *free action* of $\mathrm{SL}(2,\mathbb{Z})$ on $X_\lambda q^{Mx^2}$ from (4-13) will be used for $\gamma=\left(\begin{smallmatrix}r&\star\\ s&\star\end{smallmatrix}\right)$. We need the formula

$$\gamma^{\circ}\big(X_\lambda(\gamma^{\circ})^{-1}(X_\mu)\big)=q^{-rs\lambda^2/4-s\lambda\mu/2}X_{r\lambda+\mu}, \tag{4-21}$$

which results in

$$\begin{aligned}\gamma^{\circ}(P_b^{(1)}(\gamma^{\circ})^{-1}(X-X^{-1}))&=\gamma^{\circ}\Big(\sum_{j=-b/2}^{b/2}X^{2j}(\gamma^{\circ})^{-1}(X-X^{-1})\Big)\\ &=\sum_{j=-b/2}^{b/2}q^{-(rj+1)sj}X^{2rj+1}-\sum_{j=-b/2}^{b/2}q^{-(rj-1)sj}X^{2rj-1}\end{aligned}$$

$$= \sum_{j=-b/2}^{b/2} q^{-(rj+1)sj} X^{2rj+1} - \sum_{j=-b/2}^{b/2} q^{-(rj+1)sj} X^{-2rj-1}$$
$$= \sum_{j=-b/2}^{b/2} q^{-(rj+1)sj} (X - X^{-1}) P^{(1)}_{2rj},$$

and then implies the relation

$$(\widehat{\gamma}^{(1)}_\ell P^{(1)}_b (\widehat{\gamma}^{(1)}_\ell)^{-1}) \Downarrow = (\mathcal{X}_1^{-1} \widehat{\gamma}^{\circ}_\ell \mathcal{X}_1 P^{(1)}_b \mathcal{X}_1^{-1} (\widehat{\gamma}^{\circ}_\ell)^{-1} \mathcal{X}_1) \Downarrow$$
$$= \sum_{j=-b/2}^{b/2} q^{-(\mathsf{r}_\ell j+1)\mathsf{s}_\ell j} P^{(1)}_{2\mathsf{r}_\ell j} \tag{4-22}$$

for $\mathcal{X}_1 = X - X^{-1}$. Thus, we can switch from $k = 1$ to the case of $k = 0$, where the free action of $\mathrm{PSL}(2, \mathbb{Z})$ can be used.

Using the latter relation we can proceed by induction as follows:

$$\begin{aligned}
&JD^{\sharp}_{T(\{\mathsf{r}_1,\dots,\mathsf{r}_\ell\},\{\mathsf{s}_1,\dots,\mathsf{s}_\ell\})}(b; q, t \mapsto q) \\
&\quad = \{\widehat{\gamma}^{(1)}_1 \big(\cdots \big(\widehat{\gamma}^{(1)}_{\ell-1} \big(\widehat{\gamma}^{(1)}_\ell P^{(1)}_b (\widehat{\gamma}^{(1)}_\ell)^{-1}\big) \Downarrow (\widehat{\gamma}^{(1)}_{\ell-1})^{-1}\big) \Downarrow\big) \cdots\big) (\widehat{\gamma}^{(1)}_1)^{-1}\}^{(1)} \\
&\quad = \sum_{j=-b/2}^{b/2} q^{-(\mathsf{r}_\ell j+1)\mathsf{s}_\ell j} \{\widehat{\gamma}^{(1)}_1 \big(\cdots \big(\widehat{\gamma}^{(1)}_{\ell-1} P^{(1)}_{2\mathsf{r}_\ell j} (\widehat{\gamma}^{(1)}_{\ell-1})^{-1}\big) \Downarrow\big) \cdots\big) (\widehat{\gamma}^{(1)}_1)^{-1}\}^{(1)} \\
&\quad = \sum_{j=-b/2}^{b/2} q^{-(\mathsf{r}_\ell j+1)\mathsf{s}_\ell j} JD^{\sharp}_{T(\{\mathsf{r}_1,\dots,\mathsf{r}_{\ell-1}\},\{\mathsf{s}_1,\dots,\mathsf{s}_{\ell-1}\})}(2\mathsf{r}_\ell j; q, t \mapsto q) \\
&\quad = \sum_{j=-b/2}^{b/2} q^{-(\mathsf{r}_\ell j+1)\mathsf{s}_\ell j - \mathsf{a}_{\ell-1}\mathsf{r}_{\ell-1}(2\mathsf{r}_\ell j)(2\mathsf{r}_\ell j+2)/4} \\
&\qquad\qquad \times \mathcal{J}^{\sharp}_{T(\{\mathsf{r}_1,\dots,\mathsf{r}_{\ell-1}\},\{\mathsf{s}_1,\dots,\mathsf{s}_{\ell-1}\})}(2\mathsf{r}_\ell j; q \mapsto q^{-1}) \\
&\quad = \sum_{j=-b/2}^{b/2} q^{-(\mathsf{r}_\ell j+1)(\mathsf{s}_\ell + \mathsf{a}_{\ell-1}\mathsf{r}_\ell \mathsf{r}_{\ell-1})j} \mathcal{J}^{\sharp}_{T(\{\mathsf{r}_1,\dots,\mathsf{r}_{\ell-1}\},\{\mathsf{s}_1,\dots,\mathsf{s}_{\ell-1}\})}(2\mathsf{r}_\ell j; q \mapsto q^{-1}) \\
&\quad = \sum_{j=-b/2}^{b/2} q^{-(\mathsf{r}_\ell j+1)\mathsf{a}_\ell j} \mathcal{J}^{\sharp}_{T(\{\mathsf{r}_1,\dots,\mathsf{r}_{\ell-1}\},\{\mathsf{s}_1,\dots,\mathsf{s}_{\ell-1}\})}(2\mathsf{r}_\ell j; q \mapsto q^{-1}) \\
&\quad = q^{-\mathsf{a}_\ell \mathsf{r}_\ell b(b+2)/4} \mathcal{J}^{\sharp}_{T(\{\mathsf{r}_1,\dots,\mathsf{r}_\ell\},\{\mathsf{s}_1,\dots,\mathsf{s}_\ell\})}(b; q \mapsto q^{-1}).
\end{aligned}$$

Finally,

$$JD^{\sharp}_{\bigcirc}(b;q,t\mapsto q)=\frac{q^{(b+1)/2}-q^{-(b+1)/2}}{q^{1/2}-q^{-1/2}}=\mathcal{J}^{\sharp}_{\bigcirc}(b;q\mapsto q^{-1}), \tag{4-23}$$

which completes the proof. □

4.4 Concluding remarks

Concerning part (iii) of the Connection conjecture, *adding colors* (arbitrary Young diagrams λ) to [33, Conjecture 1.2] is a natural challenge. This would provide a t–extension of [30, Theorem 1.2]. Then, our construction (which is for arbitrary colors) will be employed at full potential; we hope to extend it from iterated torus knots (the setup of this paper) to iterated *links*.

The switch to the curves C_λ from [30, Section 1.3] supported on $C=\mathcal{C}_{\vec{r},\vec{s}}$ and associated with Young diagrams λ is natural here. This approach was initiated in [14] (via the so called *resolved conifold*). Using (the germs of) such curves can be hopefully combined with considering the *weight filtration* in the cohomology of the corresponding nested Hilbert schemes and Jacobian factors.

On the other hand, DAHA uniformly manages arbitrary root systems and weights via the Macdonald polynomials; changing the curves $\mathcal{C}_{\vec{r},\vec{s}}$ for incorporating the colors does not seem really necessary from this perspective. Instead, the spaces $\mathrm{Bun}_G(\mathcal{C}_{\vec{r},\vec{s}})$ of G–bundles over $\mathcal{C}_{\vec{r},\vec{s}}$ for the simple Lie group G associated with the root system R can be expected. This is related to the interpretation of the Jacobian factors as Springer fibers due to [29]; $\mathcal{C}_{\vec{r},\vec{s}}$ would then become *spectral curves*.

See also [8, Corollary 2.3] (and Theorem 4.10), where this connection was established for torsion free sheaves over arbitrary (possibly singular) coverings of P^1 or elliptic curves E, assuming the semistability (stability for E) of their direct images, the sheaves over P^1 or E. The connection with *Baker functions* and the "group" τ–function was established there (Proposition 2.4), which makes this construction a variant of the *Hitchin system* for *factorizable Lie group schemes* over E or P^1. These schemes are associated with arbitrary (nonunitary) classical r–matrices; see [8] and [10, Section 1.7].

This approach potentially leads to certain cell decompositions of Jacobian factors, different from those in [36] and defined via the Kac–Moody groups. The latter act as Hecke transformations in the fibers of the coverings above; the output, but not the procedure, is entirely local (only the germ $\mathcal{C}$ is really necessary).

Khovanov–Rozansky homology On the topological side, the *skein relations*, the key in [30, Theorem 1.2], are missing (by now) upon the switch from the HOMFLYPT polynomials to their homology from [27; 28; 26; 48; 39] (generalizing the Khovanov homology for the Jones polynomials). Adding colors (dominant weights) here is generally a difficult problem, though the *categorification theory* (based on Quantum Groups) provides tools for this.

In contrast to the Khovanov polynomials $(n = 2)$, not many formulas for Khovanov–Rozansky polynomials KhR_{stab} are known. Let us mention some recent progress here (by Oblomkov and Rozansky) based on further development of the theory of matrix factorization. This may provide much better tools for finding the KhR–polynomials (at least for torus knots and in the uncolored case).

Recall that Khovanov–Rozansky polynomials are defined as the Poincaré polynomials of the corresponding triply graded homology. They exist for *any links*; such an universality is one of the reasons why their theory is so complicated. For us, the Khovanov–Rozansky and Khovanov polynomials are the only way so far to interpret geometrically (topologically) the DAHA superpolynomials of *arbitrary* iterated torus knots. They are generally nonalgebraic and their connection with the algebraic geometry is not clear at the moment, though there is some partial progress with finding the geometric meaning of *pseudoalgebraic knots*.

As the examples of Section 4.1 indicate, it is possible that part (ii) of Conjecture 2.4 can be extended to any (uncolored) *non-pseudoalgebraic knots* using a special procedure of replacing the negative terms in the DAHA superpolynomials by certain positive ones. Such knots are defined exactly as those with some negative terms. Algebraically this procedure seems somewhat artificial, but it worked for all non-pseudoalgebraic knots we considered upon the reduction from superpolynomials to the Khovanov polynomials (where we can compare our polynomials vs. the topological ones). However, the evidence is insufficient so far.

Betti numbers The examples above confirm our conjecture that the DAHA superpolynomials split the Euler numbers of the Jacobian factors (compactified Jacobians) of unibranch plane curve singularities into the corresponding Betti numbers. Recall that this is under the substitution $a \mapsto 0$ and $q \mapsto 1$ in DAHA superpolynomials. See (2-22) concerning $q = 1$, the case of trivial central charge.

Such a specialization is the simplest demonstration of the power of adding "t" to the theory, *the refinement*; our conjecture on Betti numbers is presumably related to the expected connection of our superpolynomials to $\mathcal{P}_{\mathrm{alg}}$, which is a reduced modification

of that from [33, Conjecture 2]:

$$\mathcal{P}_{\mathrm{alg}} = \Big(\frac{q_{st}}{a_{st}}\Big)^{\mu} \frac{1-q_{st}^2}{1+a_{st}^2 t_{st}} \sum_{l,m\geq 0} q_{st}^{2l} a_{st}^{2m} t_{st}^{m^2} \mathfrak{w}(C_{\vec{\mathbf{r}},\vec{\mathbf{s}}}^{[l\leq l+m]}). \tag{4-24}$$

Here, μ is the Milnor number ($\mu = 2\delta$ in the unibranch case) and $\mathfrak{w}$ is the *weight filtration* in the compactly supported cohomology of the corresponding scheme; its existence was justified by Deligne. See the Overview and [33, Section 4]. The parameter t_{st} is associated to it; $t_{st}^{m^2}$ there is a necessary normalization.

The spectral sequence E_1 in terms of the weight filtration on the compactly supported cohomology over $\mathbb{Q}$, which converges to the whole cohomology and degenerates at E_2 (for smooth not necessarily compact manifolds), can potentially connect (4-24) with the Betti numbers. Here, one needs to know that the odd cohomology of the Jacobian factors vanish for unibranch plane curve singularities (the van Straten–Warmt conjecture) and that the corresponding *mixed Hodge filtration* is *pure* (a conjecture). We thank V Shende for the explanations; see [33] (though the Betti numbers are not explicitly discussed there).

Generally, not too much is known concerning the Betti numbers of the general Jacobian factors, though they were calculated for many torus knots and those for the series $(4, 2u, v)$ of Puiseux exponents, where $4 < 2u < v$ for both u and v odd. See [36] and above. Actually there is not even a general conjectural formula for them in the case of arbitrary torus knots, which can be potentially produced in our approach (to be considered elsewhere).

We checked quite a few examples (beyond those posted above), including torus knots. For instance, our DAHA-Betti polynomial for $T(12,7)$ is

$$\begin{aligned}\mathcal{H}_{12,7}(\Box; q=1, t, a=0) = 1 &+ 6t + 19t^2 + 41t^3 + 70t^4 + 102t^5 + 133t^6 + 159t^7 \\ &+ 178t^8 + 190t^9 + 194t^{10} + 193t^{11} + 184t^{12} + 174t^{13} + 159t^{14} + 145t^{15} \\ &+ 127t^{16} + 112t^{17} + 95t^{18} + 81t^{19} + 66t^{20} + 55t^{21} + 43t^{22} + 35t^{23} \\ &+ 26t^{24} + 20t^{25} + 14t^{26} + 11t^{27} + 7t^{28} + 5t^{29} + 3t^{30} + 2t^{31} + t^{32} + t^{33}.\end{aligned}$$

Its coefficients match those given before [36, Theorem 21]. We will omit the corresponding DAHA superpolynomial; note that it can be also obtained by E Gorsky's construction (via rational DAHA).

Semigroups and semimodules Apart from the torus knots, we established the coincidence of the DAHA-Betti numbers with the actual ones for many entries in the table before [36, Conjecture 23]. The examples in this table are more involved that those

for torus knots, since not all modules Δ over the valuation semigroup Γ contribute to the cell decompositions of $J(\mathcal{C}_{\vec{r},\vec{s}})$ from [36]. See Definition 8 of *admissible* modules there, as well as Theorem 13 (we use his notation). Let us provide an example, which demonstrates how surprising our conjectural connection with DAHA is and the challenges that we face.

The set of Puiseux exponents $(4, 2u, v) = (4, 14, 17)$ corresponds to the Newton's pairs $\{r_1 = 7, s_1 = 2\}$ and $\{r_2 = 2, s_2 = 3\}$. They encode the planar singularity

$$\mathcal{C}_{\vec{r},\vec{s}} : y = x^{s_1/r_1}(c + x^{s_2/(r_1 r_2)}) = x^{2/7}(c + x^{3/14}) \quad \text{with } \mathcal{R} = \mathbb{C}[\![z^4, z^{14} + z^{17}]\!].$$

Recall that the latter is the local ring of this singularity, and its topological link is $Cab(31, 2)T(7, 2)$.

The semigroup is $\Gamma = \langle 4, 14, 31\rangle$ (41 is the last missing valuation for this $\mathcal{C}$); $\delta = 21$ and $e(J(\mathcal{C}_{\vec{r},\vec{s}})) = 194$. The total number of semimodules Δ (which is relatively simple to calculate) is 222. Thus, not all contribute to the cell decomposition; 28 are not present. Their list is involved combinatorially, as well as the calculation of the dimensions of the cells corresponding to *admissible* Δ (though there are explicit formulas for the dimensions and the admissibility). It is not impossible that semimodules Δ have some connection to the DAHA-procedure. However DAHA also produces the correct number of missing (nonadmissible) ones, which seems more surprising (at the moment).

The corresponding γ–matrices, needed in the construction of the DAHA superpolynomial, are $\gamma_{7,2} = \tau_+^3\tau_-^2$ and $\gamma_{2,3} = \tau_-\tau_+\tau_-$. Note the order of $2, 3$ in the last γ. Its Betti-reduction $\mathcal{H}_{\{7,2\},\{2,3\}}(\square; q = 1, t, a = 0)$ matches the corresponding entry in [36]. It reads

$$1 + 3t + 6t^2 + 10t^3 + 13t^4 + 15t^5 + 16t^6 + 16t^7 + 16t^8 + 16t^9 + 15t^{10} + 14t^{11} + 12t^{12} + 10t^{13} + 8t^{14} + 7t^{15} + 5t^{16} + 4t^{17} + 3t^{18} + 2t^{19} + t^{20} + t^{21}.$$

For the sake of completeness, let us provide the whole DAHA superpolynomial. Its a–degree is $s_1 r_2 - 1 = 3$;

$$\vec{r} = \{7, 2\}, \quad \vec{s} = \{2, 3\}, \quad \mathcal{T} = Cab(31, 2)T(7, 2);$$

$$\begin{aligned}\mathcal{H}_{\{7,2\},\{2,3\}}(\square; q, t, a) = {} & 1+qt+q^2t+q^3t+q^2t^2+q^3t^2+2q^4t^2+q^5t^2+q^6t^2+q^3t^3+q^4t^3 \\ & +2q^5t^3+2q^6t^3+2q^7t^3+q^8t^3+q^9t^3+q^4t^4+q^5t^4+2q^6t^4+2q^7t^4+3q^8t^4+2q^9t^4 \\ & +2q^{10}t^4+q^5t^5+q^6t^5+2q^7t^5+2q^8t^5+3q^9t^5+3q^{10}t^5+3q^{11}t^5+q^6t^6+q^7t^6+2q^8t^6 \\ & +2q^9t^6+3q^{10}t^6+3q^{11}t^6+4q^{12}t^6+q^7t^7+q^8t^7+2q^9t^7+2q^{10}t^7+3q^{11}t^7+3q^{12}t^7 \\ & +4q^{13}t^7+q^8t^8+q^9t^8+2q^{10}t^8+2q^{11}t^8+3q^{12}t^8+3q^{13}t^8+4q^{14}t^8+q^9t^9+q^{10}t^9 \\ & +2q^{11}t^9+2q^{12}t^9+3q^{13}t^9+3q^{14}t^9+4q^{15}t^9+q^{10}t^{10}+q^{11}t^{10}+2q^{12}t^{10}+2q^{13}t^{10}\end{aligned}$$

$$
\begin{aligned}
&+3q^{14}t^{10}+3q^{15}t^{10}+3q^{16}t^{10}+q^{11}t^{11}+q^{12}t^{11}+2q^{13}t^{11}+2q^{14}t^{11}+3q^{15}t^{11}+3q^{16}t^{11}\\
&+2q^{17}t^{11}+q^{12}t^{12}+q^{13}t^{12}+2q^{14}t^{12}+2q^{15}t^{12}+3q^{16}t^{12}+2q^{17}t^{12}+q^{18}t^{12}+q^{13}t^{13}\\
&+q^{14}t^{13}+2q^{15}t^{13}+2q^{16}t^{13}+3q^{17}t^{13}+q^{18}t^{13}+q^{14}t^{14}+q^{15}t^{14}+2q^{16}t^{14}+2q^{17}t^{14}\\
&+2q^{18}t^{14}+q^{15}t^{15}+q^{16}t^{15}+2q^{17}t^{15}+2q^{18}t^{15}+q^{19}t^{15}+q^{16}t^{16}+q^{17}t^{16}+2q^{18}t^{16}+q^{19}t^{16}\\
&+q^{17}t^{17}+q^{18}t^{17}+2q^{19}t^{17}+q^{18}t^{18}+q^{19}t^{18}+q^{20}t^{18}+q^{19}t^{19}+q^{20}t^{19}+q^{20}t^{20}+q^{21}t^{21}\\
&+a^3\big(q^6+q^7t+q^8t+q^9t+q^8t^2+q^9t^2+2q^{10}t^2+q^{11}t^2+q^{12}t^2+q^9t^3+q^{10}t^3+2q^{11}t^3\\
&+2q^{12}t^3+2q^{13}t^3+q^{10}t^4+q^{11}t^4+2q^{12}t^4+2q^{13}t^4+3q^{14}t^4+q^{11}t^5+q^{12}t^5+2q^{13}t^5\\
&+2q^{14}t^5+3q^{15}t^5+q^{12}t^6+q^{13}t^6+2q^{14}t^6+2q^{15}t^6+3q^{16}t^6+q^{13}t^7+q^{14}t^7+2q^{15}t^7\\
&+2q^{16}t^7+3q^{17}t^7+q^{14}t^8+q^{15}t^8+2q^{16}t^8+2q^{17}t^8+2q^{18}t^8+q^{15}t^9+q^{16}t^9+2q^{17}t^9\\
&+2q^{18}t^9+q^{19}t^9+q^{16}t^{10}+q^{17}t^{10}+2q^{18}t^{10}+q^{19}t^{10}+q^{17}t^{11}+q^{18}t^{11}+2q^{19}t^{11}\\
&+q^{18}t^{12}+q^{19}t^{12}+q^{20}t^{12}+q^{19}t^{13}+q^{20}t^{13}+q^{20}t^{14}+q^{21}t^{15}\big)\\
&+a^2\big(q^3+q^4+q^5+q^4t+2q^5t+3q^6t+2q^7t+q^8t+q^5t^2+2q^6t^2+4q^7t^2+4q^8t^2+4q^9t^2\\
&+2q^{10}t^2+q^{11}t^2+q^6t^3+2q^7t^3+4q^8t^3+5q^9t^3+6q^{10}t^3+5q^{11}t^3+3q^{12}t^3+q^7t^4+2q^8t^4\\
&+4q^9t^4+5q^{10}t^4+7q^{11}t^4+7q^{12}t^4+5q^{13}t^4+q^8t^5+2q^9t^5+4q^{10}t^5+5q^{11}t^5+7q^{12}t^5\\
&+8q^{13}t^5+6q^{14}t^5+q^9t^6+2q^{10}t^6+4q^{11}t^6+5q^{12}t^6+7q^{13}t^6+8q^{14}t^6+6q^{15}t^6+q^{10}t^7\\
&+2q^{11}t^7+4q^{12}t^7+5q^{13}t^7+7q^{14}t^7+8q^{15}t^7+6q^{16}t^7+q^{11}t^8+2q^{12}t^8+4q^{13}t^8+5q^{14}t^8\\
&+7q^{15}t^8+8q^{16}t^8+5q^{17}t^8+q^{12}t^9+2q^{13}t^9+4q^{14}t^9+5q^{15}t^9+7q^{16}t^9+7q^{17}t^9+3q^{18}t^9\\
&+q^{13}t^{10}+2q^{14}t^{10}+4q^{15}t^{10}+5q^{16}t^{10}+7q^{17}t^{10}+5q^{18}t^{10}+q^{19}t^{10}+q^{14}t^{11}+2q^{15}t^{11}\\
&+4q^{16}t^{11}+5q^{17}t^{11}+6q^{18}t^{11}+2q^{19}t^{11}+q^{15}t^{12}+2q^{16}t^{12}+4q^{17}t^{12}+5q^{18}t^{12}+4q^{19}t^{12}\\
&+q^{16}t^{13}+2q^{17}t^{13}+4q^{18}t^{13}+4q^{19}t^{13}+q^{20}t^{13}+q^{17}t^{14}+2q^{18}t^{14}+4q^{19}t^{14}+2q^{20}t^{14}\\
&+q^{18}t^{15}+2q^{19}t^{15}+3q^{20}t^{15}+q^{19}t^{16}+2q^{20}t^{16}+q^{21}t^{16}+q^{20}t^{17}+q^{21}t^{17}+q^{21}t^{18}\big)\\
&+a\big(q+q^2+q^3+q^2t+2q^3t+3q^4t+2q^5t+q^6t+q^3t^2+2q^4t^2+4q^5t^2+4q^6t^2+4q^7t^2\\
&+2q^8t^2+q^9t^2+q^4t^3+2q^5t^3+4q^6t^3+5q^7t^3+6q^8t^3+5q^9t^3+4q^{10}t^3+q^{11}t^3+q^5t^4\\
&+2q^6t^4+4q^7t^4+5q^8t^4+7q^9t^4+7q^{10}t^4+7q^{11}t^4+2q^{12}t^4+q^6t^5+2q^7t^5+4q^8t^5+5q^9t^5\\
&+7q^{10}t^5+8q^{11}t^5+9q^{12}t^5+3q^{13}t^5+q^7t^6+2q^8t^6+4q^9t^6+5q^{10}t^6+7q^{11}t^6+8q^{12}t^6\\
&+10q^{13}t^6+3q^{14}t^6+q^8t^7+2q^9t^7+4q^{10}t^7+5q^{11}t^7+7q^{12}t^7+8q^{13}t^7+10q^{14}t^7+3q^{15}t^7\\
&+q^9t^8+2q^{10}t^8+4q^{11}t^8+5q^{12}t^8+7q^{13}t^8+8q^{14}t^8+10q^{15}t^8+3q^{16}t^8+q^{10}t^9+2q^{11}t^9\\
&+4q^{12}t^9+5q^{13}t^9+7q^{14}t^9+8q^{15}t^9+9q^{16}t^9+2q^{17}t^9+q^{11}t^{10}+2q^{12}t^{10}+4q^{13}t^{10}\\
&+5q^{14}t^{10}+7q^{15}t^{10}+8q^{16}t^{10}+7q^{17}t^{10}+q^{18}t^{10}+q^{12}t^{11}+2q^{13}t^{11}+4q^{14}t^{11}+5q^{15}t^{11}\\
&+7q^{16}t^{11}+7q^{17}t^{11}+4q^{18}t^{11}+q^{13}t^{12}+2q^{14}t^{12}+4q^{15}t^{12}+5q^{16}t^{12}+7q^{17}t^{12}+5q^{18}t^{12}\\
&+q^{19}t^{12}+q^{14}t^{13}+2q^{15}t^{13}+4q^{16}t^{13}+5q^{17}t^{13}+6q^{18}t^{13}+2q^{19}t^{13}+q^{15}t^{14}+2q^{16}t^{14}\\
&+4q^{17}t^{14}+5q^{18}t^{14}+4q^{19}t^{14}+q^{16}t^{15}+2q^{17}t^{15}+4q^{18}t^{15}+4q^{19}t^{15}+q^{20}t^{15}+q^{17}t^{16}\\
&+2q^{18}t^{16}+4q^{19}t^{16}+2q^{20}t^{16}+q^{18}t^{17}+2q^{19}t^{17}+3q^{20}t^{17}+q^{19}t^{18}+2q^{20}t^{18}+q^{21}t^{18}\\
&+q^{20}t^{19}+q^{21}t^{19}+q^{21}t^{20}\big).
\end{aligned}
$$

Versal deformations Last, but not least, we refer the reader to [24] for some relatively recent developments concerning the Hilbert schemes of (locally) plane curves, including their role in the theory of the Gopakumar–Vafa BPS invariants (see [35]) and various generalizations.

This paper is helpful to put our construction into perspective and to link it to the *versal deformations of plane curve singularities*, at least in the case of usual (nonnested) Hilbert schemes of curves. We do not touch this important direction here. However, let us mention that the theory of *adjacent singularities* of the pairs $\{R, V\}$ for finite-dimensional irreducible representations V of the simple Lie groups G for (arbitrary) root systems R is expected to be related to the theory of the DAHA superpolynomials.

When $t_{st} = 1$, and for the minimal possible degree of a_{st}, the sum in (4-24) essentially reduces to the following summation:

$$\sum_{n\geq 0} q_{st}^{n+1-\delta} e(\mathcal{C}^{[n]}) = \sum_{0\leq i\leq \delta} n_{\mathcal{C}}(i)\Big(\frac{q_{st}}{(1-q_{st})^2}\Big)^{i+1-\delta}, \quad \mathcal{C} = \mathcal{C}_{\vec{r},\vec{s}}, \tag{4-25}$$

for the Euler numbers of Hilbert schemes $\mathcal{C}^{[n]}$ of n points on $\mathcal{C}$ of arithmetic genus δ (subschemes in $\mathcal{C}$ of length n, to be more exact). See [35; 24]. Importantly, $n_{\mathcal{C}}(i) \in \mathbb{Z}_+$ due to [18] and [42], because these numbers are the multiplicities of $\mathcal{C}$ in the closures of the strata in the space of its versal deformation, stratified with respect to the *geometric* genus of the deformation curves.

In our notations, $t_{st} = 1$ corresponds to $q = t$, and we can therefore use our $\sqrt{q}$ instead of q_{st} here upon $a = 0$. Hopefully, [33, Conjecture 2] and our construction lead to a similar deformation interpretation of the generalization of the right-hand side of (4-25) to the nested Hilbert-schemes (which adds the parameter a_{st}) under the weight filtration (associated with t_{st}) and for arbitrary Young diagrams.

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