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Relations between Witten–Reshetikhin–Turaev and nonsemisimple $\mathfrak{sl}(2)$ 3–manifold invariants

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The Witten–Reshetikhin–Turaev (WRT) invariants extend the Jones polynomials of links in S^3 to invariants of links in 3–manifolds. Similarly, the authors constructed two 3–manifold invariants N_r and N_r^0 which extend the Akutsu–Deguchi–Ohtsuki (ADO) invariant of links in S^3 colored by complex numbers to links in arbitrary manifolds. All these invariants are based on the representation theory of the quantum group $U_q\mathfrak{sl}_2$, where the definition of the invariants N_r and N_r^0 uses a nonstandard category of $U_q\mathfrak{sl}_2$–modules which is not semisimple. In this paper we study the second invariant, N_r^0, and consider its relationship with the WRT invariants. In particular, we show that the ADO invariant of a knot in S^3 is a meromorphic function of its color, and we provide a strong relation between its residues and the colored Jones polynomials of the knot. Then we conjecture a similar relation between N_r^0 and a WRT invariant. We prove this conjecture when the 3–manifold M is not a rational homology sphere, and when M is a rational homology sphere obtained by surgery on a knot in S^3 or a connected sum of such manifolds.

57N10; 57R56

Introduction

In [18], Witten proposed a program to construct a topological invariant of 3–manifolds from the viewpoint of quantum mathematical physics. Reshetikhin and Turaev [16] gave a rigorous construction of these invariants, which have become known as quantum invariants of 3–manifolds. These invariants are defined via surgery presentations of a 3–manifold, and the best known example is a weighted sum of colored Jones polynomials. The invariants of Reshetikhin and Turaev generalize to the setting of modular categories. Some of the common obstructions to applying this construction to any ribbon tensor category $\mathcal{D}$ include the following facts: (i) the simple objects may have zero "quantum dimension", (ii) there might be infinitely many isomorphism classes of simple objects in $\mathcal{D}$ and (iii) $\mathcal{D}$ might be nonsemisimple. In [6] we derived

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a general categorical setting where these obstructions can be overcome. In particular, we showed that the category $\mathscr{C}$ of nilpotent representations of a generalized version of quantized $\mathfrak{sl}(2)$ at a primitive r^{th}–ordered root of unity gives rise to two related invariants, N_r and N_r^0. In this paper we investigate the invariant N_r^0.

Let $\mathscr{C}$ be the category mentioned above and defined in Section 1.2. This category has a complex family of weight modules, divided into typical and atypical modules. Here all the atypical modules have integral weights.

Let F be the usual Reshetikhin–Turaev invariant of $\mathscr{C}$–colored framed oriented links in S^3 arising from $\mathscr{C}$. The invariant F has the following properties:

- If L is a framed oriented link whose components are all colored by simple modules of $\mathscr{C}$ with integral weights, then F is determined by the Kauffman bracket and so is a version of the colored Jones polynomial.
- If L is a framed oriented link with a component colored by a typical module, then $F(L)=0$.

In [9], Geer, Patureau-Mirand and Turaev gave an extension of F to framed oriented links colored with modules in $\mathscr{C}$ with nonintegral weights; see also Costantino and Murakami [8]. This extension is an invariant F' defined on $\mathscr{C}$–colored framed oriented links with at least one component colored by a typical module. F' is a generalization of the framed oriented link invariants defined by Akutsu, Deguchi and Ohtsuki in [1]. We have the relation

$$F'(L_1 \sqcup L_2) = F'(L_1)F(L_2),$$

where L_1 is in the domain of F' and L_2 is any $\mathscr{C}$–colored framed oriented link. From this relation it follows that F' recovers F: if L is any $\mathscr{C}$–colored framed oriented link then

$$F(L) = \frac{F'(L \sqcup o)}{F'(o)}, \tag{1}$$

where o is an unknot colored by any typical module. Thus F' is a kind of extension of the colored Jones polynomial to complex colors. Furthermore, as we show in Corollary 18, the invariant $F'(K_\alpha)$ of a knot $K \subset S^3$ colored by a typical module of weight $\alpha \in \mathbb{C}$ is a meromorphic function of α whose residues at the integers are proportional to the colored Jones polynomials of K evaluated at $q = \mathrm{e}^{i\pi/r}$. This relation allows us to reprove the well-known symmetry principle (see Kirby and Melvin [13]) for the colored Jones polynomials of K using a mainly graphical argument detailed in Corollary 16; see Remark 17.

In [6], the authors laid out a relationship between N_r and N_r^0 analogous to that outlined above between F' and F; we now briefly recall this relation. The invariants N_r and N_r^0 are WRT-type 3–manifold invariants which consist of certain weighted sums of $F'(L)$, where L is a surgery presentation of M. These invariants are topological invariants of triples (M,T,ω), where M is a closed oriented 3–manifold, T is a $\mathscr{C}$–colored framed oriented link in M, and ω is an element in $H^1(M\setminus T;\mathbb{C}/2\mathbb{Z})$. For N_r the triples must satisfy some requirements of "typicality" as in the case of F'. The invariant N_r^0 is zero unless ω is in the image of the natural map $H^1(M;\mathbb{Z}/2\mathbb{Z})\to H^1(M;\mathbb{C}/2\mathbb{Z})$ induced by the universal coefficient theorem. (Compare this with the above statement that $F(L)$ is zero if at least one component of L is colored by an atypical module.) Finally, N_r recovers N_r^0 (compare with (1)) since

$$\mathsf{N}_r^0(M,T,\omega)=\frac{\mathsf{N}_r\big((M,T,\omega)\,\#\,(M',T',\omega')\big)}{\mathsf{N}_r(M',T',\omega')}, \tag{2}$$

where (M',T',ω') is a triple for which N_r does not vanish; for further details on the notion of connected sum, see [6].

Since F is essentially the colored Jones polynomial, the above analogy leads us to the question: Is N_r^0 related to the WRT-invariant? The purpose of this paper is to answer this question positively for certain types of triples (M,T,ω). To formulate this properly we must define the WRT-invariant of a triple (M,T,ω). Kirby and Melvin [13] and Blanchet [2] considered WRT-type invariants of (M,ω), where $\omega\in H^1(M,\mathbb{Z}/2\mathbb{Z})$. In Theorem 11 we give a slight generalization of their invariants to triples of the form (M,T,ω), where T is a colored framed oriented link in M and $\omega\in H^1(M\setminus T,\mathbb{Z}/2\mathbb{Z})$. We denote this invariant by $\mathrm{WRT}_r(M,T,\omega)$. The question above can be formulated as the following conjecture.

If G is a finite abelian group, let $\mathrm{ord}(G)$ be the order of G, ie the number of elements in the set underlying G. If G is an infinite abelian group, set $\mathrm{ord}(G)=0$.

Conjecture 1 *Let (M,T,ω) be a compatible triple where ω takes values in $\mathbb{Z}/2\mathbb{Z}\subset\mathbb{C}/2\mathbb{Z}$. Then*

$$\mathsf{N}_r^0(M,T,\omega)=\mathrm{ord}(H_1(M;\mathbb{Z}))\,\mathrm{WRT}_r(M,T,\omega).$$

Note that if an abelian group G has a square presentation matrix $A\in\mathcal{M}_n(\mathbb{Z})$, then $\mathrm{ord}(G)=|\det(A)|$. In particular, if a 3–manifold is obtained by surgery on a framed oriented link in S^3 whose linking matrix is A, then A is a presentation matrix for $H_1(M;\mathbb{Z})$ and thus $\mathrm{ord}(H_1(M;\mathbb{Z}))=|\det(A)|$.

In Sections 3 and 4 we prove Conjecture 1 in the following two cases: (i) when M is an empty rational homology sphere obtained by surgery on a knot in S^3 (or more in general a connected sum of manifolds of this type) and (ii) when the first Betti number of M is greater than zero.

It should be noticed that the invariant N_r does not reduce to N_r^0. For example, the invariant WRT_r is trivial for $r=2$, and N_r^0 should only depend on $H_1(M,\mathbb{Z})$. But for $r=2$, the invariant $\mathsf{N}_r(M,\varnothing,\omega)$ is a canonical renormalization of the Reidemeister torsion of M associated to ω, and in particular it classifies lens spaces. This is shown by Blanchet and the authors in [3], where the invariant N_r is extended to manifolds with boundary using the setting of topological quantum field theory (TQFT). The results in this paper have some consequences for the TQFTs of [3] and open the bigger question of their relations with the Witten–Reshetikhin–Turaev TQFTs.

The referee pointed out a similar conjecture by Kerler that we discuss now. The reduced quantum group associated to $\mathfrak{sl}(2)$ when q is a root of unity of order r is a finite-dimensional Hopf algebra. As a consequence, the dual of this Hopf algebra has a nonzero right integral (see Hennings [10] and Kauffman and Radford [11]) from which one can define a Hennings–Kauffman–Radford invariant of 3–manifolds HKR_r. The following theorem due to Chen, Kuppum and Srinivasan was first suggested and conjectured by Ohtsuki [15] and Kerler (see [12]; note that Kerler's conjecture is more general).

Theorem 2 [5, Theorem 1.1] *If q is a root of unity of odd order $r\geq 3$, then for any closed oriented 3–manifold M we have*

$$\mathrm{HKR}_r(M)=\mathrm{ord}(H_1(M;\mathbb{Z}))\,\mathrm{WRT}_r^{\mathsf{SO}(3)}(M),$$

where $\mathrm{WRT}_r^{\mathsf{SO}(3)}(M)$ denotes the $\mathrm{SO}(3)$ version of the Witten–Reshetikhin–Turaev invariant.

This suggests that N_r^0 could be equal to the Hennings–Kauffman–Radford invariant. However, a direct comparison between HKR_r and N_r^0 is difficult for the following reasons: the Hopf algebra $U_q^H\mathfrak{sl}(2)$ whose representations are used to define N_r^0 is infinite-dimensional, it is not a ribbon Hopf algebra, and it is not known if it has a right integral. So it is not possible to apply directly the Hennings–Kauffman–Radford construction to $U_q^H\mathfrak{sl}(2)$. On the other hand, we only know how to compute N_r^0 using a set of representations $\{V_\alpha\}_{\alpha\in\mathbb{C}\setminus\mathbb{Z}}$ (see Section 1.2) that do not exist for the reduced quantum group involved in the definition of HKR_r.

Recently, Murakami [14] combined the Hennings–Kauffman–Radford construction and the so-called logarithmic knot invariants to define a generalized Kashaev invariant GK_r of nonempty framed oriented links in 3–manifolds (which also has an $\mathrm{SO}(3)$ version).

The relation between HKR_r and GK_r is very similar to the relation between N_r^0 and N_r: if M is an empty manifold then

$$\mathrm{GK}_r\big((M,\varnothing)\#(M',L)\big)=\mathrm{HKR}_r(M)\,\mathrm{GK}_r(M',L).$$

(Compare with (2).) Hence another approach to establish a link between N_r^0 and HKR_r might be to study the relation between N_r and GK_r.

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1 Preliminaries

1.1 Notation

All manifolds in the present paper are oriented, connected and compact unless explicitly stated, and all tangles are framed and oriented. Given a set Y, a graph is said to be *Y–colored* if it is equipped with a map from the set of its edges to Y.

Let r be an integer greater than or equal to 2 and let $q=e^{i\pi/r}$. For $x\in\mathbb{C}$, we write q^x for $e^{xi\pi/r}$ and set $\{x\}=q^x-q^{-x}$. Let $X_r=\mathbb{Z}\setminus r\mathbb{Z}\subset\mathbb{C}$ and define the *modified dimension* $\mathsf{d}\colon\mathbb{C}\setminus X_r\to\mathbb{C}$ by

$$\mathsf{d}(\alpha)=(-1)^{r-1}\prod_{j=1}^{r-1}\frac{\{j\}}{\{\alpha+r-j\}}=(-1)^{r-1}\frac{r\{\alpha\}}{\{r\alpha\}}. \tag{3}$$

Finally, let

$$H_r=\{1-r,3-r,\dots,r-3,r-1\}. \tag{4}$$

1.2 A quantization of $\mathfrak{sl}(2)$ and some of its modules

Here we give a slightly generalized version of quantum $\mathfrak{sl}(2)$, for more details see [7]. Let $U_q^H\mathfrak{sl}(2)$ be the $\mathbb{C}$–algebra given by generators E, F, K, K^{-1}, H and relations

$$HK=KH,\qquad [H,E]=2E,\qquad [H,F]=-2F,$$
$$KEK^{-1}=q^2E,\quad KFK^{-1}=q^{-2}F,\quad [E,F]=\frac{K-K^{-1}}{q-q^{-1}}.$$

The algebra $U_q^H\mathfrak{sl}(2)$ is a Hopf algebra where the coproduct and counit are defined by

$$\begin{aligned} \Delta(E) &= 1\otimes E + E\otimes K, & \varepsilon(E) &= 0,\\ \Delta(F) &= K^{-1}\otimes F + F\otimes 1, & \varepsilon(F) &= 0,\\ \Delta(H) &= H\otimes 1 + 1\otimes H, & \varepsilon(H) &= 0,\\ \Delta(K) &= K\otimes K, & \varepsilon(K) &= 1. \end{aligned}$$

Define $\overline{U}_q^H\mathfrak{sl}(2)$ to be the Hopf algebra $U_q^H\mathfrak{sl}(2)$ modulo the relations $E^r = F^r = 0$.

Let V be a finite-dimensional $\overline{U}_q^H\mathfrak{sl}(2)$–module. An eigenvalue $\lambda\in\mathbb{C}$ of the operator $H\colon V\to V$ is called a *weight* of V and the associated eigenspace is called a *weight space*. We call V a *weight module* if V splits as a direct sum of weight spaces and $q^H = K$ as operators on V. Let $\mathscr{C}$ be the category of finite-dimensional weight $\overline{U}_q^H\mathfrak{sl}(2)$–modules. The category $\mathscr{C}$ is a ribbon Ab–category, see Geer, Patureau-Mirand and Turaev [9], Murakami [14] and Ohtsuki [15].

We now recall the definition of the duality morphisms and the braiding of the category $\mathscr{C}$. Let V and W be objects of $\mathscr{C}$. Let $\{v_i\}$ be a basis of V and $\{v_i^*\}$ be a dual basis of $V^* = \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})$. Then

$$\begin{aligned} b_V\colon \mathbb{C}\to V\otimes V^* &\quad \text{given by } 1\mapsto \sum v_i\otimes v_i^*,\\ d_V\colon V^*\otimes V\to\mathbb{C} &\quad \text{given by } f\otimes w\mapsto f(w),\\ b'_V\colon \mathbb{C}\to V^*\otimes V &\quad \text{given by } 1\mapsto \sum K^{r-1}v_i\otimes v_i^*,\\ d'_V\colon V\otimes V^*\to\mathbb{C} &\quad \text{given by } v\otimes f\mapsto f(K^{1-r}v) \end{aligned}$$

are duality morphisms of $\mathscr{C}$. In [15] Ohtsuki introduced an R–matrix operator defined on $V\otimes W$ by

$$R = q^{H\otimes H/2}\sum_{n=0}^{r-1}\frac{\{1\}^{2n}}{\{n\}!}q^{n(n-1)/2}E^n\otimes F^n, \tag{5}$$

where $q^{H\otimes H/2}$ is the operator given by

$$q^{H\otimes H/2}(v\otimes v') = q^{\lambda\lambda'/2}v\otimes v',$$

for weight vectors v and v' of weights λ and λ'. The braiding $c_{V,W}\colon V\otimes W\to W\otimes V$ on $\mathscr{C}$ is defined by $v\otimes w\mapsto\tau(R(v\otimes w))$, where τ is the permutation $x\otimes y\mapsto y\otimes x$.

For each $n\in\{0,\dots,r-1\}$, let S_n be the usual $(n+1)$–dimensional irreducible highest weight $\overline{U}_q^H\mathfrak{sl}(2)$–module with highest weight n. The module S_n has a basis $\{s_i = F^i s_0 \mid i = 0,\dots,n\}$ determined by $H.s_i = (n-2i)s_i$, $E.s_0 = 0 = F^{n+1}.s_0$

and $E.s_i = (\{i\}\{n+1-i\}/\{1\}^2)s_{i-1}$. Its quantum dimension is given by the trace of the action of K^{1-r}, and so $\operatorname{qdim}(S_n) = (-1)^n\{n+1\}/\{1\}$.

Since q is a root of unity and $F^r = 0$, we can consider a larger class of finite-dimensional highest weight modules: For each $\alpha \in \mathbb{C}$ we let V_α be the r–dimensional highest weight $\overline{U}_q^H\mathfrak{sl}(2)$–module of highest weight $\alpha + r - 1$. The module V_α has a basis $\{v_0, \dots, v_{r-1}\}$ whose action is given by

$$(6)\qquad H.v_i = (\alpha + r - 1 - 2i)v_i, \quad E.v_i = \frac{\{i\}\{i-\alpha\}}{\{1\}^2}v_{i-1}, \quad F.v_i = v_{i+1}.$$

To describe these modules we need the following definitions. A module is *irreducible* or *simple* if it has no proper submodule. A nonzero module is *indecomposable* if it cannot be written as a direct sum of two proper submodules. A module V is *absolutely irreducible* if $\operatorname{End}_{\mathscr{C}}(V) = \mathbb{C}\operatorname{Id}_V$.

All the modules V_α have vanishing quantum dimension. They are divided into:

Atypical modules If $k \in X_r = \mathbb{Z} \setminus r\mathbb{Z} \subset \mathbb{C}$ then V_k is indecomposable but not irreducible, however it is still absolutely irreducible (since any endomorphism must map the highest weight vector v_0 to a multiple of itself). In particular, if $k \in \{0, \dots, r-1\}$ then the assignment sending the highest weight vector s_0 of S_{r-1-k} to the vector v_k of V_k determines an injective homomorphism $\iota\colon S_{r-1-k} \to V_k$. Here the submodule S_{r-1-k} in V_k is not a direct summand. Also, if $j \in \{1-r, \dots, 0\}$ then the assignment sending the highest weight vector v_0 of V_j to the highest weight vector s_0 of S_{r-1+j} induces a surjective homomorphism $\pi\colon V_j \to S_{r-1+j}$.

Typical modules If $\alpha \in \mathbb{C} \setminus X_r$ then V_α is irreducible, and so absolutely irreducible. We call such modules *typical*.

Let A be the set of typical modules. For $g \in \mathbb{C}/2\mathbb{Z}$, define $\mathscr{C}_g$ as the full subcategory of weight modules with weights congruent to g mod 2. Then it is easy to see that $\{\mathscr{C}_g\}_{g\in\mathbb{C}/2\mathbb{Z}}$ is a $\mathbb{C}/2\mathbb{Z}$–grading in $\mathscr{C}$; see [6].

1.3 The link invariants F and F'

The well-known Reshetikhin–Turaev construction defines a $\mathbb{C}$–linear functor F from the category of $\mathscr{C}$–colored ribbon graphs with coupons to $\mathscr{C}$. When L is a $\mathscr{C}$–colored framed oriented link, $F(L)$ can be identified with a complex number. When L is a framed oriented link whose components are all colored by S_n, $F(L)$ is the Kauffman bracket with variable specialization $A = q^{1/2} = e^{i\pi/2r}$, so it is a version of the colored Jones polynomial specialized at the root of unity $q = e^{i\pi/r}$. (For details, see Section 1.4.)

Vanishing quantum dimensions make the functor F trivial on any closed $\mathscr{C}$–colored ribbon graph that has at least one edge colored by a typical module. In [9], the definition of F is extended to a nontrivial map F' defined on closed $\mathscr{C}$–colored ribbon graphs with at least one edge colored by a typical module. We now recall the definition of F'.

Let T_W be any $\mathscr{C}$–colored $(1,1)$–ribbon graph with both ends colored by the same element W of $\mathscr{C}$. If W is absolutely irreducible, then $F(T_W)$ is an endomorphism of W that is determined by a scalar $\langle T_W \rangle$:

$$F(T_W) = \langle T_W \rangle \, \mathrm{Id}_W .$$

Theorem 3 [9] *Let L be a closed $\mathscr{C}$–colored ribbon graph with at least one edge colored by a typical module V_α. Cutting this edge, we obtain a $\mathscr{C}$–colored $(1,1)$–ribbon graph T_{V_α} whose closure is L. Then*

$$F'(L) = \mathsf{d}(\alpha)(V_\alpha)\langle T_{V_\alpha} \rangle \in \mathbb{C}$$

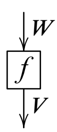
is independent of the choice of the edge to be cut and yields a well-defined invariant of L.

We will use the following proposition later.

Proposition 4 *Let T be a $(1,1)$–tangle formed from a closed $\mathscr{C}$–colored ribbon graph and a single open uncolored component. Let T_W be T with the open component colored by W. We have the following equalities of scalars:*

$$\langle T_{S_{r-1-k}} \rangle = \langle T_{V_k} \rangle \quad \text{for } k \in \{0, \dots, r-1\},$$
$$\langle T_{S_{r-1+j}} \rangle = \langle T_{V_j} \rangle \quad \text{for } j \in \{1-r, \dots, 0\}.$$

Proof In this proof we use the language of coupons; for more details see [17]. In particular, a morphism $f\colon V \to W$ can be represented by a coupon $c(f)$, which is a box with arrows:

$$c(f) =$$


By definition of F, we have $F(c(f)) = f$. By fusing this coupon to the bottom of the $(1,1)$–tangle T_W we obtain a ribbon graph which we denote by $T_W \circ c(f)$. Similarly, we can fuse $c(f)$ to the top of the tangle T_V to obtain a ribbon graph $c(f) \circ T_V$.

From the discussion about atypical modules presented above, we have the injection $\iota\colon S_{r-1-k} \to V_k$ and the surjection $\pi\colon V_j \to S_{r-1+j}$, for $k \in \{0, \dots, r-1\}$ and

$j \in \{1-r, \dots, 0\}$. Thus, as explained in the previous paragraph, we can consider the ribbon graphs $T_{V_k} \circ c(\imath)$ and $c(\imath) \circ T_{S_{r-1-k}}$. Since the category of $\mathscr{C}$–colored ribbon graphs is a ribbon category, we have that $T_{V_k} \circ c(\imath)$ and $c(\imath) \circ T_{S_{r-1-k}}$ are equal as ribbon graphs, so their images are equal under F. Combining this with the fact that $F(T_{V_k})$ and $F(T_{S_{r-1-k}})$ are scalar endomorphisms, we have

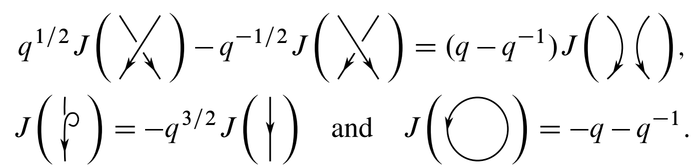
$$\langle T_{V_k} \rangle \imath = \langle T_{V_k} \rangle F(c(\imath)) = F(T_{V_k} \circ c(\imath)) = F(c(\imath) \circ T_{S_{r-1-k}}) = \imath \langle T_{S_{r-1-k}} \rangle.$$

Thus, we have $\langle T_{V_k} \rangle = \langle T_{S_{r-1-k}} \rangle$. Similarly, we have $\langle T_{S_{r-1+j}} \rangle = \langle T_{V_j} \rangle$. □

1.4 Comparison with the Jones polynomials

In this paper, by the colored Jones polynomial, we mean the Kauffman bracket version, which is an invariant of framed oriented links independent of their orientation. Let $L = L_1 \sqcup \cdots \sqcup L_k \subset S^3$ be a framed oriented link and $J(L) \in \mathbb{C}[q^{\pm 1/2}]$ be its *Jones polynomial*, determined by the following skein relations:

(7)
(8)



More generally, if each L_i is colored by an integer $n_i \geq 0$ then, roughly speaking, one defines the $\vec{n}^{\text{th}}$–colored Jones polynomial $J_{\vec{n}}(L)$ as a linear combination of Jones polynomials of framed oriented links obtained by taking parallels of each component of L at most n_i times. More precisely, one identifies the tubular neighborhood of each component L_i with the product $S^1 \times [-1,1] \times [-1,1]$ (using the framing of L_i and an arbitrary orientation) and defines framed oriented links

$$L_i^k = S^1 \times \left\{ \frac{0}{k}, \frac{1}{k}, \dots, \frac{k-1}{k} \right\} \times \{0\},$$

adopting the notation $L_i^k \cdot L_i^h = L_i^{k+h}$. Then one recursively defines a linear combination of framed oriented links parallel to L_i by

$$T_n(L_i) := L_i^1 \cdot T_{n-1}(L_i) - T_{n-2}(L_i) \quad \text{and} \quad T_0(L_i) = \varnothing,\ T_1(L_i) = L_i. \tag{9}$$

Finally, $J_{\vec{n}}(L)$ is defined as the linear combination of the Jones polynomials of the framed oriented links obtained by replacing L_i with $T_{n_i}(L_i)$. Clearly, the Jones polynomial defined above corresponds to the case when $n_i = 1$ for all i. The following holds.

Proposition 5 *Let $L = L_1 \sqcup \cdots \sqcup L_k \subset S^3$ be a framed oriented link, and let $\vec{n} = (n_1, \dots, n_k)$ be a tuple of integers all in the set $\{0, \dots, r-1\}$. Let $L_{\vec{n}}$ be the* *framed oriented link L such that L_i is colored by n_i, for all i. Similarly, let L_S be the framed oriented link L such that L_i is colored by S_{n_i}, where S_{n_i} is the simple module defined in Section 1.2. Then*

$$J_{\vec{n}}(L_{\vec{n}})|_{q=e^{i\pi/r}} = F(L_S).$$

Proof First, assume that $n_i = 1$, for all i. In this case, we will prove that the relations of (7) and (8) hold. We start by recalling that S_1 is spanned by two vectors s_0, s_1 with $H(s_i) = 1-2i$, $K(s_i) = q^{1-2i}s_i$ and $E(s_1) = 0 = F(s_2)$ while $E(s_2) = s_1$ and $F(s_1) = s_2$. The second relation of (8) is a consequence of the formula for the quantum dimension $\operatorname{qdim}(S_1) = -q-q^{-1}$ given above. The first relation follows from the fact that the inverse of the twist on S_1 is given by the action of

$$\theta = K^{r-1}\sum_{n=0}^{r-1}\frac{\{1\}^{2n}}{\{n\}!}q^{n(n-1)/2}(-KF)^n q^{-H^2/2}E^n;$$

see [6]. To see that (7) holds, recall that the braiding c_{S_1,S_1} is defined by $v\otimes w\mapsto \tau(R(v\otimes w))$, where R is the R–matrix and τ is the permutation $x\otimes y\mapsto y\otimes x$. Since E^2 and F^2 act by zero on S_1, we have that c_{S_1,S_1} and $c^{-1}_{S_1,S_1}$ are determined by

$$\tau\circ\big(q^{H\otimes H/2}(\mathrm{Id}+(q-q^{-1})E\otimes F)\big) \quad\text{and}\quad (\mathrm{Id}-(q-q^{-1})E\otimes F)q^{-H\otimes H/2}\circ\tau,$$

respectively. Thus, (7) follows from the direct computations

$$(q^{1/2}\tau\circ R - q^{-1/2}R^{-1}\circ\tau)(s_0\otimes s_0) = (q-q^{-1})s_0\otimes s_0,$$
$$(q^{1/2}\tau\circ R - q^{-1/2}R^{-1}\circ\tau)(s_0\otimes s_1) = (q-q^{-1})s_0\otimes s_1,$$
$$(q^{1/2}\tau\circ R - q^{-1/2}R^{-1}\circ\tau)(s_1\otimes s_1) = (q-q^{-1})s_1\otimes s_1,$$
$$(q^{1/2}\tau\circ R - q^{-1/2}R^{-1}\circ\tau)(s_1\otimes s_0) = (q-q^{-1})s_1\otimes s_0.$$

Finally, to prove the statement in general, it is sufficient to note that the standard tensor decomposition of $S_1^{\otimes n}$ as a sum of copies of S_i with $i\le n$ is still valid for $n<r$ in $\mathscr{C}$. To prove this it is sufficient to note that if $2\le n<r$ then $S_{n-1}\otimes S_1\simeq S_n\oplus S_{n-2}$ and argue by induction. Hence the formula (9) expressing $T_n(L)$ translates this decomposition algebraically expressing $F(L)$ (with L colored by n) as a linear combination of invariants of cables of L whose components are all colored by S_1. Thus, the theorem follows. □

1.5 The 3–manifold invariants N_r^0 and WRT

The Witten–Reshetikhin–Turaev invariant uses certain weighted sums of colored Jones polynomials of framed oriented links in S^3 to define invariants of framed oriented links in 3–manifolds. Such a weighted sum can be described using Kirby colors. In [6] an analogous procedure uses F' to define an invariant of colored framed oriented links in arbitrary manifolds. In this section we recall some of the results of [6] and discuss a refined WRT invariant.

In this section, we fix an integer $r \geq 2$ with $r \notin 4\mathbb{Z}$. Let M be a compact connected oriented 3–manifold, T be a $\mathscr{C}$–colored ribbon graph in M and $\omega \in H^1(M \setminus T, \mathbb{C}/2\mathbb{Z})$. For any embedding of T in S^3, a *surgery presentation* for (M,T) is an oriented framed link L in $S^3 \setminus T$ such that (M,T) is diffeomorphic to $(S^3(L),T)$, where $S^3(L)$ is the closed 3–manifold obtained by performing surgery on L. If L is a surgery presentation of (M,T), the map g_ω on the set of edges of $L \cup T$ with values in $\mathbb{C}/2\mathbb{Z}$ defined by $g_\omega(e_i) = \omega(m_i)$, where m_i is a meridian of the edge e_i, is called the $\mathbb{C}/2\mathbb{Z}$*–coloring* of $L \cup T$ induced by ω.

Definition 6 Let M, T and ω be as above.

(1) We say that (M,T,ω) is a *compatible triple* if for each edge e of T its coloring is in $\mathscr{C}_{g_\omega(m_e)}$, where m_e is a meridian of e.

(2) A framed oriented link $L \subset S^3$ which is a surgery presentation for (M,T) is a *computable presentation* of the compatible triple (M,T,ω) if one of the two following conditions holds:

(a) $L \neq \varnothing$ and $g_\omega(L_i) \in (\mathbb{C}/2\mathbb{Z}) \setminus (\mathbb{Z}/2\mathbb{Z})$ for all components L_i of L.

(b) $L = \varnothing$ and there exists an edge of T colored by $V_\alpha \in \mathsf{A}$.

Next we explain when a given compatible triple (M,T,ω) has a computable surgery presentation. The inclusion $\mathbb{Z}/2\mathbb{Z} \subset \mathbb{C}/2\mathbb{Z}$ induces an injective map

$$H^1(M \setminus T, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}\big(H_1(M \setminus T, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}\big) \hookrightarrow \mathrm{Hom}\big(H_1(M \setminus T, \mathbb{Z}), \mathbb{C}/2\mathbb{Z}\big) \cong H^1(M \setminus T, \mathbb{C}/2\mathbb{Z}).$$

We say that a cohomology class $\omega \in H^1(M \setminus T, \mathbb{C}/2\mathbb{Z})$ is *integral* if it is in the image of this map.

Proposition 7 *Let M be a 3–manifold which is not diffeomorphic to the sphere S^3. A compatible triple (M,T,ω) has computable surgery presentation if and only if the cohomology class ω is not integral.*

Proof If ω is not integral then from [6, Proposition 1.5] there exists a computable surgery presentation of (M,T,ω). On the other hand, assume that (M,T,ω) has a computable surgery presentation. Since M is not S^3, $L \neq \varnothing$, so from part (2a) of Definition 6 we have that ω is not integral. □

Recall the set $H_r = \{1-r, 3-r, \dots, r-1\}$ defined in (4). For $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ we define the Kirby color Ω_α as the formal linear combination

$$\Omega_\alpha = \sum_{k \in H_r} \mathsf{d}(\alpha + k) V_{\alpha+k}. \tag{10}$$

If $\overline{\alpha}$ is the image of α in $\mathbb{C}/2\mathbb{Z}$, we say that Ω_α has degree $\overline{\alpha}$. We can "color" a knot K with a Kirby color Ω_α: let $K(\Omega_\alpha)$ be the formal linear combination of knots $\sum_{k \in H_r} \mathsf{d}(\alpha+k) K_{\alpha+k}$, where $K_{\alpha+k}$ is the knot K colored with $V_{\alpha+k}$. If $\overline{\alpha} \in \mathbb{C}/2\mathbb{Z} \setminus \mathbb{Z}/2\mathbb{Z}$, by $\Omega_{\overline{\alpha}}$ we mean any Kirby color of degree $\overline{\alpha}$. Let Δ_- and Δ_+ be the scalars given by

$$\Delta_- = \overline{\Delta_+} = \begin{cases} i(rq)^{3/2} & \text{if } r \equiv 1 \bmod 4, \\ (i-1)(rq)^{3/2} & \text{if } r \equiv 2 \bmod 4, \\ -(rq)^{3/2} & \text{if } r \equiv 3 \bmod 4. \end{cases}$$

Next we recall the main theorems of [6].

Theorem 8 [6] *If L is a framed oriented link which gives rise to a computable surgery presentation of a compatible triple (M, T, ω), then*

$$\mathsf{N}_r(M, T, \omega) = \frac{F'(L \cup T)}{\Delta_+^p \, \Delta_-^s}$$

is a well-defined topological invariant (ie depends only on the homeomorphism class of the triple (M, T, ω)), where (p, s) is the signature of the linking matrix of the surgery link L and for each i the component L_i is colored by a Kirby color of degree $g_\omega(L_i)$.

The notion of connected sum is easily extended to compatible triples. In particular, given two compatible triples (M_1, T_1, ω_1) and (M_2, T_2, ω_2), let

$$(M_1, T_1, \omega_1) \# (M_2, T_2, \omega_2) = (M_1 \# M_2, T_1 \sqcup T_2, \omega),$$

where $M_1 \# M_2$ is the usual connected sum and ω is the unique cohomology class which restricts to both ω_1 and ω_2 via an isomorphism coming from the Mayer–Vietoris sequence and excision; see [6] for details.

For $\alpha \in \mathbb{C} \setminus X_r$, let u_α be the unknot in S^3 colored by V_α. Let ω_α be the unique element of $H^1(S^3 \setminus u_\alpha; \mathbb{C}/2\mathbb{Z})$ such that $(S^3, u_\alpha, \omega_\alpha)$ is a compatible triple. If (M, T, ω) is any compatible triple, then Proposition 7 implies that $(M, T, \omega) \# (S^3, u_\alpha, \omega_\alpha)$ has a computable surgery presentation.

Theorem 9 [6] *Let (M,T,ω) be a compatible triple. Define*

$$\mathsf{N}^0_r(M,T,\omega)=\frac{\mathsf{N}_r\big((M,T,\omega)\,\#\,(S^3,u_\alpha,\omega_\alpha)\big)}{\mathsf{d}(\alpha)}.$$

Then $\mathsf{N}^0_r(M,T,\omega)$ is a well-defined topological invariant (ie depends only on the homeomorphism class of the compatible triple (M,T,ω)). Moreover, if (M,T,ω) has a computable surgery presentation then $\mathsf{N}^0_r(M,T,\omega)=0$.

Let us also give a definition of the refined Witten–Reshetikhin–Turaev invariants $\mathrm{WRT}_r(M,T,\omega)$. The definition is based on the fact that the Kauffman bracket version of the colored Jones polynomial can be computed through F; see Proposition 5.

We define the Kirby colors of degree $\overline{0}$ and $\overline{1}$ respectively by

$$\Omega^{\mathrm{RT}}_0:=\sum_{j=0,\,j\text{ even}}^{r-2}\frac{\{j+1\}}{\{1\}}S_j\quad\text{and}\quad\Omega^{\mathrm{RT}}_1:=\sum_{j=0,\,j\text{ odd}}^{r-2}-\frac{\{j+1\}}{\{1\}}S_j.$$

Lemma 10 *Let $\Delta^{\mathsf{SO(3)}}_\pm=F(u_{\pm1})$ where $u_{\pm1}$ is the unknot with framing ±1 colored by Ω^{RT}_0. Then*

$$\Delta^{\mathsf{SO(3)}}_+=\frac{\Delta_+}{\{1\}r}\quad\textit{and}\quad\Delta^{\mathsf{SO(3)}}_-=\overline{\Delta^{\mathsf{SO(3)}}_+}=-\frac{\Delta_-}{\{1\}r}.$$

In particular, in both cases, $\Delta^{\mathsf{SO(3)}}_\pm\neq0$.

Proof The proof is a direct computation using the values of the quantum dimension and of the twist for the simple modules S_n. In particular, $\mathrm{qdim}(S_i)=(-1)^i\{i+1\}/\{1\}$ and the twist on S_i acts by the scalar $(-1)^iq^{(i^2+2i)/2}$. Thus

$$\Delta^{\mathsf{SO(3)}}_+=\{1\}^{-2}\sum_{j=0,\,j\text{ even}}^{r-2}\{j+1\}^2q^{(j^2+2j)/2}=\{1\}^{-2}(q^2\Sigma_3+q^{-2}\Sigma_{-1}-2\Sigma_1),$$

where

$$\Sigma_a=\sum_{n=0}^{\lfloor(r-2)/2\rfloor}q^{2(n^2+an)}$$

is part of a quadratic Gauss sum. These terms can be computed using standard results on quadratic Gauss sum. □

Kirby and Melvin [13] and Blanchet [2] consider invariants of $(M,\varnothing,\omega)$ where $\omega\in H^1(M,\mathbb{Z}/2\mathbb{Z})$. The following theorem is a slight generalization of these invariants. (Here we use the conventions of this paper and not those of [2; 13].)

Theorem 11 (Refined Witten–Reshetikhin–Turaev invariants) *Let (M,T,ω) be a compatible triple with T a $\mathscr{C}_{\bar 0}\cup\mathscr{C}_{\bar 1}$–colored ribbon graph and $\omega\in H^1(M\setminus T,\mathbb{Z}/2\mathbb{Z})$. If L is a framed oriented link which gives rise to a surgery presentation of the pair (M,T), then*

$$\mathrm{WRT}_r(M,T,\omega)=\frac{F(L\cup T)}{(\Delta_+^{\mathsf{SO}(3)})^p\,(\Delta_-^{\mathsf{SO}(3)})^s}$$

is a well-defined topological invariant (ie depends only on the homeomorphism class of the triple (M,T,ω)), where (p,s) is the signature of the linking matrix of the surgery link L and for each i the component L_i is colored by a Kirby color of degree $g_\omega(L_i)$.

Proof In [13], for $T=\varnothing$ and r even, this invariant is considered in a slightly different form. Also, in [2, Remark II.4.3], for $T=\varnothing$, the existence of this invariant is discussed. Indeed, the Reshetikhin–Turaev functor applied to graphs colored by the module $S_1\in\mathscr{C}_{\bar 1}$ satisfies the Kauffman skein relation for $A=q^{1/2}=\mathrm{e}^{i\pi/2r}$. It follows that if $L\subset S^3$ is a framed oriented link whose components are colored by elements of $\{S_0,\dots,S_{r-2}\}$, then $F(L)$ is the metabracket [2; 4] evaluated at the element corresponding to the coloring of L at $A=q^{1/2}$. It follows that $\mathrm{WRT}_r(M,\varnothing,\omega)$ is the invariant denoted $\theta_{q^{1/2}}(M_{L,g_\omega})$ in [2, Remark II.4.3].

For a complete proof of the theorem, one can also apply [6, Theorem 3.7] to the modular category obtained as the quotient of the subcategory of $\mathscr{C}$ generated by S_1 by its ideal of projective modules. Indeed, this category is obviously a $\mathbb{Z}/2\mathbb{Z}$–modular category relative to $\varnothing$ with modified dimension qdim and trivial periodicity group. □

In particular, when $\omega=0$ one gets an invariant of manifolds also known as the SO(3) version of the Reshetikhin–Turaev invariants:

Definition 12 Let T be a $\mathscr{C}_{\bar 0}$–colored ribbon graph in a closed 3–manifold M. Then

$$\mathrm{WRT}_r^{\mathsf{SO}(3)}(M,T)=\mathrm{WRT}_r(M,T,0).$$

Remark 13 Let us denote by $\mathrm{WRT}_r^{\mathsf{SU}(2)}(M,T)$ the original WRT-invariant which is obtained as in Theorem 11 except that all components of L are colored by

$$\Omega^{\mathrm{RT}}=\Omega_0^{\mathrm{RT}}+\Omega_1^{\mathrm{RT}}$$

(and the elements $\Delta_\pm^{\mathsf{SU}(2)}$ are also defined with Ω^{RT}). For odd r, it can be shown that if (M,T,ω) is a compatible triple and $\varphi\in H^1(M,\mathbb{Z}/2\mathbb{Z})$, then $(M,T,\omega+\varphi)$ is also a compatible triple. Then $\mathrm{WRT}_r(M,T,\omega+\varphi)=\mathrm{e}^{i\pi\lambda/2}\,\mathrm{WRT}_r(M,T,\omega)$, where $\lambda\in\mathbb{Z}$ only depends on ω, φ and on the linking matrix of a link presentation L of M. Loosely speaking, WRT_r does not have a strong dependance on the cohomology class.

Similarly, $\mathrm{WRT}_r^{\mathsf{SU}(2)}(M,T)$ is proportional to $\mathrm{WRT}_r^{\mathsf{SO}(3)}(M,T)$; see [2, Section III]. A similar property holds for N_r^0 and, more generally, for admissible triples: if $\varphi \in H^1(M,\mathbb{Z}/2\mathbb{Z})$ then $\mathsf{N}_r(M,T,\omega+\varphi) = \mathrm{e}^{i\pi\lambda/2}\mathsf{N}_r(M,T,\omega)$, where $\lambda \in \mathbb{C}$ only depends on ω, φ and on the linking matrix of a link presentation L of M. Thus up to this combinatorial invariant λ, $\mathsf{N}_r(M,T,\omega)$ only depends on the reduction modulo $\mathbb{Z}$ of the compatible cohomology class $\omega \in H^1(M\setminus T,\mathbb{C}/2\mathbb{Z})$.

The behavior for r even is different; in this case results of [13; 2] suggest the following conjecture:

$$\mathrm{WRT}_r^{\mathsf{SU}(2)}(M,T) = \sum_{\substack{\text{compatible}\\ \omega\in H^1(M\setminus T,\mathbb{Z}/2\mathbb{Z})}} \mathrm{WRT}_r(M,T,\omega).$$

2 Relations between F' and the colored Jones polynomial

Recall the r–dimensional modules V_α, $\alpha \in \mathbb{C}$, given in Section 1.2. Using the basis given in (6) and its dual basis, we identify V_α and V_α^* with $\mathbb{C}^r$. With these identifications we can identify certain Hom-spaces with spaces of matrices. For example, we can make the following identifications: $\mathrm{End}_{\mathscr{C}}(V_\alpha) = \mathrm{Mat}_{r\times r}(\mathbb{C})$ and $\mathrm{Hom}(\mathbb{C}, V_\alpha\otimes V_\alpha^*) = \mathrm{Mat}_{1\times r^2}(\mathbb{C})$.

We say a function $g\colon \mathbb{C}\to\mathbb{C}$ is a *Laurent polynomial in* q^α if there exists a Laurent polynomial $f\in\mathbb{C}[x,x^{-1}]$ such that $g(\alpha)=f(q^\alpha)$. The action of the basis given in (6) implies that all the entries in the matrices $\rho_{V_\alpha}(E)$, $\rho_{V_\alpha}(F)$, $\rho_{V_\alpha}(H)$ and $\rho_{V_\alpha}(K)$ are Laurent polynomials in q^α.

Lemma 14 *All the entries in the images of the maps*

$$\begin{aligned} g_b&\colon \mathbb{C}\to\mathrm{Mat}_{1\times r^2}(\mathbb{C}) \quad \textit{given by } \alpha\mapsto b_{V_\alpha},\\ g_d&\colon \mathbb{C}\to\mathrm{Mat}_{r^2\times 1}(\mathbb{C}) \quad \textit{given by } \alpha\mapsto d_{V_\alpha},\\ g_{b'}&\colon \mathbb{C}\to\mathrm{Mat}_{1\times r^2}(\mathbb{C}) \quad \textit{given by } \alpha\mapsto b'_{V_\alpha},\\ g_{d'}&\colon \mathbb{C}\to\mathrm{Mat}_{r^2\times 1}(\mathbb{C}) \quad \textit{given by } \alpha\mapsto d'_{V_\alpha}\end{aligned}$$

are Laurent polynomials in q^α*. Also, for each entry* f_{ij} *in the image of the map*

$$f\colon \mathbb{C}\times\mathbb{C}\to\mathrm{Mat}_{r^2\times r^2}(\mathbb{C}),\quad (\alpha,\beta)\mapsto q^{-\alpha\beta/2}q^{-(r-1)(\alpha+\beta)/2}c_{V_\alpha,V_\beta}$$

there is a two-variable Laurent polynomial $g_{ij}(x,y)$ *such that* $f_{ij}(\alpha,\beta)=g_{ij}(q^\alpha,q^\beta)$*.*

Proof The first statement follows from the formulas for b, d, b' and d' given in Section 1.2. For example, the entry in the image of $g_{d'}$ corresponding to $v_i\otimes v_j^*$

is $v_j^*(K^{1-r}v_i)=\delta_{ij}q^{(1-r)(\alpha+r-1-2i)}$. The second statement follows from the form of the R–matrix given in (5). In particular, if v_i and v_j are any basis vectors of V_α and V_β, respectively, then

$$q^{-\alpha\beta/2}q^{-(r-1)(\alpha+\beta)/2}q^{H\otimes H/2}E^n\otimes F^n.v_i\otimes v_j \\ = q^{-\alpha(j+n)-\beta(i-n)}q^{c/2}E^n\otimes F^n.v_i\otimes v_j,$$

where c is an integer which does not depend on α or β. Also,

$$E^n\otimes F^n(v_i\otimes v_j)=\frac{\{i\}!}{\{i-n\}!\{1\}^{2n}}\{i-\alpha\}\{i-1-\alpha\}\cdots\{i-(n-1)-\alpha\}v_{i-n}\otimes v_{j+n}.$$

Since the coefficients in the last two equalities are Laurent polynomials in q^α and q^β, the desired result about the function f follows. □

The above lemma has the following corollaries.

Corollary 15 *Let $T_{(V_{\alpha_1},\dots,V_{\alpha_n})}$ be a $(1,1)$–tangle with n components whose i^{th} component is colored by V_{α_i}, $\alpha_i\in\mathbb{C}$. Then the function $g_T\colon\mathbb{C}^n\to\mathbb{C}$ given by $(\alpha_1,\dots,\alpha_n)\mapsto\langle T_{(V_{\alpha_1},\dots,V_{\alpha_n})}\rangle$ is a holomorphic function in $\mathbb{C}^n$. In particular g_T is continuous.*

Proof Assume the first component is the open component. By definition we have

$$F(T_{(V_{\alpha_1},\dots,V_{\alpha_n})})=\langle T_{(V_{\alpha_1},\dots,V_{\alpha_n})}\rangle\,\mathrm{Id}_{V_{\alpha_1}},$$

so it is enough to consider $F(T_{(V_{\alpha_1},\dots,V_{\alpha_n})})$. The value of $F(T_{(V_{\alpha_1},\dots,V_{\alpha_n})})$ is computed by decomposing a projection of $T_{(V_{\alpha_1},\dots,V_{\alpha_n})}$ into building blocks made of cups, caps, vertical edges and crossings. Then the building blocks are associated with the duality morphisms, the identity and the positive and negative braidings, respectively. These morphisms are tensored and composed according to the projection of $T_{(V_{\alpha_1},\dots,V_{\alpha_n})}$. Lemma 14 implies that the contribution from a duality morphism corresponding to a cup or cap on the i^{th} component is a Laurent polynomial in q^{α_i}. Lemma 14 also implies that all contributions of a crossing between the i^{th} and j^{th} components are Laurent polynomials in q^{α_i} and q^{α_j} times a factor of $q^{-\alpha_i\alpha_j/2}q^{-(r-1)(\alpha_i+\alpha_j)/2}$. Thus, the map $g_T(\alpha_1,\dots,\alpha_n)=\langle T_{(V_{\alpha_1},\dots,V_{\alpha_n})}\rangle$ is a Laurent polynomial in the variables $q^{\alpha_1},\dots,q^{\alpha_n}$ times an integral power of $q^{-\alpha_i\alpha_j/2}q^{-(r-1)(\alpha_i+\alpha_j)/2}$ and so g_T is holomorphic. □

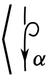
Corollary 16 *Let K be a knot. Let $K^f_{V_\alpha}$ be K colored by V_α with framing $f\in\mathbb{Z}$. Let $T^0_{V_\alpha}$ be a $(1,1)$–tangle with zero framing whose closure is $K^0_{V_\alpha}$. Then there is a Laurent polynomial $\widetilde{K}(X)\in\mathbb{C}[X,X^{-1}]$ such that $\langle T^0_{V_\alpha}\rangle=\widetilde{K}(q^\alpha)$ and*

$$F'(K^f_{V_\alpha})=\theta^f_\alpha\,\mathsf{d}(\alpha)\,\widetilde{K}(q^\alpha), \tag{11}$$

where $\theta_\alpha = q^{(\alpha^2-(r-1)^2)/2}$ is the twist on V_α. Moreover, we have

$$\widetilde{K}(q^{\alpha+r}) = \widetilde{K}(q^\alpha), \quad F'(K^f_{V_{\alpha+2r}}) = q^{2r\alpha f} F'(K^f_{V_\alpha}),$$
$$F'(K^f_{V_{\alpha+r}}) = (-1)^{r+1}(iq^\alpha)^{rf} F'(K^f_{V_\alpha}).$$

Proof As in the proof of Corollary 15, the function $g_T(\alpha) = \langle T^0_{V_\alpha}\rangle$ is a Laurent polynomial in q^α times an integral power of $q^{\alpha^2/2}$. From the form of the map c_{V_α,V_α} in Lemma 14, the integral power of $q^{\alpha^2/2}$ is equal to the number of positive crossings minus the number of negative crossings in the projection of $T^0_{V_\alpha}$. Since the framing of K_α is zero, this power is zero. Thus, $g_T(\alpha)$ is a Laurent polynomial in q^α and so there exists a $\widetilde{K}(X) \in \mathbb{C}[X, X^{-1}]$ such that $\langle T^0_\alpha\rangle = \widetilde{K}(q^\alpha)$. Now we can use the duality and the braiding to compute the value of the twist:

$$\theta_\alpha = \Big\langle$$  $$\Big\rangle = q^{(\alpha^2-(r-1)^2)/2}.$$

Then (11) follows from the above discussion and the definition of F',

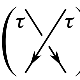
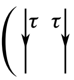
$$F'(K^f_{V_\alpha}) = \theta^f_\alpha F'(K^0_{V_\alpha}) = \theta^f_\alpha \mathsf{d}(\alpha)\langle T^0_{V_\alpha}\rangle = \theta^f_\alpha \mathsf{d}(\alpha)\widetilde{K}(q^\alpha).$$

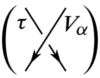
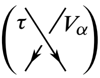
Next we show that $\widetilde{K}(q^{\alpha+r}) = \widetilde{K}(q^\alpha)$. Consider the one-dimensional space $\tau = \mathbb{C}$ with the $\overline{U}^H_q\mathfrak{sl}(2)$–module structure given by

$$Ev = Fv = 0, \quad Hv = rv$$

for any $v \in \tau$. The quantum dimension of τ is $(-1)^{r+1}$. From the form of the R–matrix we have

(12)

$$\Big\langle$$  $$\Big\rangle = -i^{-r}, \quad F\Big($$  $$\Big) = i^r F\Big($$  $$\Big),$$

$$F\Big($$  $$\Big) = q^{(\alpha+r-1)r} F\Big($$  $$\Big).$$

Hence for a 0–framed knot K colored with τ, one has $F(K) = F(\text{unknot}) = (-1)^{r+1}$. Let T^0 be the 0–framed tangle underlying $T^0_{V_\alpha}$. Let T^0_τ be T^0 colored with τ. Since T^0_τ has zero framing then $\langle T^0_\tau\rangle = 1$. Now $F(T^0_{V_{\alpha+r}})$ is equal to the endomorphism associated to T^0 labeled with $V_\alpha \otimes \tau$ or equivalently the 2–cabling of T^0 where the two components are labeled by V_α and τ, respectively. We can use the third equality in (12) to unlink the component labeled with τ from the component labeled with V_α. Therefore, since $T^0_{V_{\alpha+r}}$ has zero framing we have

$$\langle T^0_{V_{\alpha+r}}\rangle = \langle T^0_{V_\alpha}\rangle\langle T^0_\tau\rangle = \langle T^0_{V_\alpha}\rangle.$$

Finally, (11) and the above formulas for θ_α and $\mathsf{d}(\alpha)$ imply

$$\begin{aligned} F'(K^f_{V_{\alpha+2r}}) &= \theta^f_{\alpha+2r}\mathsf{d}(\alpha+2r)\widetilde{K}(q^{\alpha+2r}) \\ &= (q^{(2r\alpha+2r^2)}\theta_\alpha)^f\mathsf{d}(\alpha)\widetilde{K}(q^\alpha) = q^{2r\alpha f}F'(K^f_{V_\alpha}), \\ F'(K^f_{V_{\alpha+r}}) &= \theta^f_{\alpha+r}\mathsf{d}(\alpha+r)\widetilde{K}(q^{\alpha+r}) = (-1)^{r+1}(iq^\alpha)^{rf}F'(K^f_{V_\alpha}), \end{aligned}$$

which concludes the proof. □

Remark 17 Corollary 16 with Proposition 4 implies the well-known symmetry principle relating the colored Jones polynomial associated to S_{k-1} with the one associated to S_{r-1-k} for $k \in \{1,\dots,r-2\}$.

Corollary 18 *Let K be a knot and let K_{V_α} be K colored by V_α. The function $g_K\colon \mathbb{C}\setminus X_r \to \mathbb{C}$ defined by $\alpha \mapsto F'(K_{V_\alpha})$ is a meromorphic function on the whole plane $\mathbb{C}$. Moreover, the residue at each pole is determined by the colored Jones polynomial.*

Proof Recall that

$$F'(K_{V_\alpha}) = \mathsf{d}(\alpha)\langle T_{V_\alpha}\rangle = (-1)^{r-1}\prod_{j=1}^{r-1}\frac{\{j\}}{\{\alpha+r-j\}}\langle T_{V_\alpha}\rangle,$$

where T_{V_α} is the $(1,1)$–tangle obtained from cutting K_{V_α}. From Corollary 15 it follows that $\alpha \mapsto (-1)^{r-1}\prod_{j=1}^{r-1}\{j\}\langle T_{V_\alpha}\rangle$ is a holomorphic function in the entire plane $\mathbb{C}$. And it is clear as well that $\alpha \mapsto \prod_{j=1}^{r-1}\{\alpha+r-j\}$ is a holomorphic function in the entire plane $\mathbb{C}$ which is zero when $\alpha\in\mathbb{Z}\setminus r\mathbb{Z}$. Therefore, the quotient of these two functions is a meromorphic function whose set of poles is $\mathbb{Z}\setminus r\mathbb{Z}$.

All of these poles are simple and so the residue can be computed as follows. Let $n\in\mathbb{Z}\setminus r\mathbb{Z}$. The residue at n of the $2r$–periodic meromorphic function d is given by

$$\begin{aligned} \mathrm{Res}(\mathsf{d},n) &= \lim_{\alpha\to n}(\alpha-n)(-1)^{r-1}\frac{r\{\alpha\}}{\{r\alpha\}} \\ &= \lim_{x\to 0}(-1)^{r-1}\frac{xr\sin(\frac{\pi(n+x)}{r})}{\sin(\pi(n+x))} = (-1)^{r-1+n}\frac{r}{\pi}\sin\Big(\frac{n\pi}{r}\Big). \end{aligned}$$

So the residue of g_K at n is equal to

$$\mathrm{Res}(g_K,n) = \mathrm{Res}(\mathsf{d},n)\langle T_{V_n}\rangle = (-1)^{r-1+n}\frac{r}{\pi}\sin\Big(\frac{n\pi}{r}\Big)\langle T_{V_n}\rangle.$$

To finish the proof we show that the above formula for $\mathrm{Res}(g_K,n)$ can be rewritten in terms of the colored Jones polynomial. To do this we have two cases. First, suppose

$n = k + 2mr$ with $k \in \{1, \dots, r-1\}$ and $m \in \mathbb{Z}$. By Corollary 16 and Proposition 4 we have

$$\langle T_{V_n} \rangle = \langle T_{V_k} \rangle = \langle T_{S_{r-1-k}} \rangle.$$

Combining the fact that

$$\operatorname{qdim}(S_{r-1-k}) = (-1)^{r-1-k} \frac{\{r-k\}}{\{1\}} = (-1)^{r-1-k} \frac{\sin\left(\frac{k\pi}{r}\right)}{\sin\left(\frac{\pi}{r}\right)}$$

and Proposition 5 we have

$$J_{r-1-k}(K)|_{q=e^{i\pi/r}} = (-1)^{r-1-k} \frac{\sin\left(\frac{k\pi}{r}\right)}{\sin\left(\frac{\pi}{r}\right)} \langle T_{S_{r-1-k}} \rangle.$$

Thus,

$$\operatorname{Res}(g_K, n) = \frac{r}{\pi} \sin\left(\frac{\pi}{r}\right) J_{r-1-k}(K)\big|_{q=e^{i\pi/r}}.$$

Similarly, if $n = k + 2mr$ with $k \in \{1-r, \dots, -1\}$ and $m \in \mathbb{Z}$ then one can show that

$$\operatorname{Res}(g_K, n) = -\frac{r}{\pi} \sin\left(\frac{\pi}{r}\right) J_{r-1+k}(K)\big|_{q=e^{i\pi/r}}. \qquad \square$$

3 Surgery on a knot in the 3–sphere S^3

In this section we prove Conjecture 1 when M is an empty closed manifold obtained by surgery on a nonzero framed knot in S^3.

Theorem 19 *Suppose that K is a knot in S^3 with nonzero framing f. Let M be the manifold obtained by surgery on the knot K and $\omega \in H^1(M, \mathbb{Z}/2\mathbb{Z})$. Then*

$$\mathsf{N}_r^0(M, \varnothing, \omega) = |f| \operatorname{WRT}_r(M, \varnothing, \omega) = \operatorname{ord}(H_1(M;\mathbb{Z})) \operatorname{WRT}_r(M, \varnothing, \omega).$$

Corollary 20 *Let M be a rational homology sphere obtained by surgery on a knot in S^3. Then*

$$\operatorname{WRT}_r^{\mathsf{SO}(3)}(M, \varnothing) = \frac{1}{\operatorname{ord}(H_1(M;\mathbb{Z}))} \mathsf{N}_r^0(M, \varnothing, 0).$$

Remark 21 The three invariants $\operatorname{WRT}_r^{\mathsf{SO}(3)}$, N_r^0 and $M \mapsto \operatorname{ord}(H_1(M, \mathbb{Z}))$ are multiplicative with respect to the connected sum of 3–manifolds. Hence Theorem 19 implies that Conjecture 1 is also true for a connected sum of manifolds each obtained by surgery on a knot in S^3.

The rest of this section is devoted to the proof of Theorem 19.

Proof of Theorem 19. First we improve the results of [6, Section 2.4] and derive a formula for $\mathsf{N}_r^0(M,\varnothing,\omega)$. We still denote by ω the integer in $\{0,1\}$ whose class modulo 2 is the value $g_\omega(K)$ of the cohomology class on the meridian of K and let $e\in\{0,1\}$ be such that $\overline{e}=\overline{r-1+\omega}\in\mathbb{Z}/2\mathbb{Z}$.

For $\alpha\in\mathbb{C}\setminus X_r$, recall the function $\mathsf{P}(\alpha)=\sum_{k\in H_r}F'(K_{V_{\alpha+k}})$ of [6, Section 2.4]. (As above, K_V means K colored by V.) The function P is continuous and so can be naturally extended to all of $\mathbb{C}$. Indeed, let $DK_{(V_\alpha,V_\beta)}$ be the 2–cable of K whose components are colored with V_α and V_β such that α or β is in $\mathbb{C}\setminus X_r$. From Lemma 14 we have that the map $(\alpha,\beta)\mapsto q^{-f(\alpha^2+\beta^2+2\alpha\beta)/2}F'(DK_{(V_\alpha,V_\beta)})$ is a rational function in

$$\frac{1}{q^{r\alpha}-q^{-r\alpha}}\mathbb{C}[q^{\pm\alpha},q^{\pm\beta}]\cap\frac{1}{q^{r\beta}-q^{-r\beta}}\mathbb{C}[q^{\pm\alpha},q^{\pm\beta}].$$

(Also see the proof of Corollary 15.) Thus, this function is a Laurent polynomial in $\mathbb{C}[q^{\pm\alpha},q^{\pm\beta}]$. In addition, if $\alpha+\beta\in\mathbb{C}\setminus X_r$ then $F'(DK_{(V_\alpha,V_\beta)})$ can be computed by coloring K with $V_\alpha\otimes V_\beta\simeq\bigoplus_{k\in H_r}V_{\alpha+\beta+k}$. Combining the statements of this paragraph we have that

$$\mathsf{P}(\alpha+\beta)=\sum_{k\in H_r}F'(K_{V_{\alpha+\beta+k}})=F'(DK_{(V_\alpha,V_\beta)})$$

is a continuous function of (α,β), which we extend to all of $\mathbb{C}\times\mathbb{C}$.

Next we give a formula for N_r^0 in terms of P. By sliding the unknot o_α over K we obtain a computable presentation of $(M,\varnothing,\omega)\,\#\,(S^3,o_\alpha,\omega_\alpha)$ as in Theorem 9. This produces the $\mathscr{C}$–colored framed oriented link $DK_{(\Omega_{e-\alpha},V_\alpha)}$, where $\Omega_{e-\alpha}=\sum_{h\in H_r}\mathsf{d}(e-\alpha+h)V_{e-\alpha+h}$ is a Kirby color of degree $\overline{\omega-\alpha}$. By definition of N_r^0,

$$\mathsf{N}_r^0(M,\varnothing,\omega)=\frac{1}{\Delta_{\mathrm{sign}(f)}\mathsf{d}(\alpha)}\sum_{h\in H_r}\mathsf{d}(e-\alpha+h)F'(DK_{(V_\alpha,V_{e-\alpha+h})}).$$

Since $\{r(e-\alpha+h)\}=\{-r(e-\alpha+h)\}=(-1)^\omega\{r\alpha\}$ we have

$$\begin{aligned}\Delta_{\mathrm{sign}(f)}\mathsf{N}_r^0(M,\varnothing,\omega)&=\frac{(-1)^\omega}{\{\alpha\}}\sum_{h\in H_r}\{\alpha-h-e\}\mathsf{P}(h+e)\\&=\frac{(-1)^\omega q^\alpha}{q^\alpha-q^{-\alpha}}\sum_{h\in H_r}q^{-h-e}\mathsf{P}(h+e)-\frac{(-1)^\omega q^{-\alpha}}{q^\alpha-q^{-\alpha}}\sum_{h\in H_r}q^{h+e}\mathsf{P}(h+e).\end{aligned}$$

Finally, as $\mathsf{N}_r^0(M,\varnothing,\omega)$ does not depend on α we have

$$\mathsf{N}_r^0(M,\varnothing,\omega)=\frac{(-1)^\omega}{\Delta_{\mathrm{sign}(f)}}\sum_{k\in H_r}q^{k+e}\mathsf{P}(k+e)=\frac{(-1)^\omega}{\Delta_{\mathrm{sign}(f)}}\sum_{k\in H_r}q^{-k-e}\mathsf{P}(k+e).$$

Next we use the last formula and the continuity of P to write a multiple of N_r^0. In particular, let S be the limit

$$\begin{aligned} S = (-1)^{\omega}\Delta_{\mathrm{sign}(f)}\mathsf{N}_r^0(M,\varnothing,\omega) &= \lim_{\varepsilon\to 0}\sum_{\ell\in H_r} q^{\ell+e}\mathsf{P}(\varepsilon+\ell+e) \\ &= \lim_{\varepsilon\to 0}\sum_{k,\ell\in H_r} q^{\ell+e}F'(K_{(\varepsilon+k+\ell+e)}) \\ &= \lim_{\varepsilon\to 0}\sum_{n=1-r}^{r-1}\sum_{\substack{k,\ell\in H_r\\ k+\ell=2n}} q^{\ell+e}F'(K_{(\varepsilon+2n+e)}). \end{aligned}$$

In this sum, for fixed n the only part of the interior sum which varies is q^{ℓ} for $k,\ell\in H_r$ with $k+\ell=2n$. Here the possible values of ℓ are integers from $\max(1-r,1-r+2n)$ to $\min(r-1,r-1+2n)$, so the sum of q^{ℓ} over these values is equal to

$$q^n\frac{\{r-|n|\}}{\{1\}} = q^n\frac{\{|n|\}}{\{1\}}.$$

Therefore, we have the following expression for S:

$$\begin{aligned} S &= \lim_{\varepsilon\to 0}\frac{1}{\{1\}}\sum_{n=1-r}^{r-1} q^{n+e}\{|n|\}F'(K_{V_{\varepsilon+2n+e}}) \\ &= \lim_{\varepsilon\to 0}\frac{1}{\{1\}}\sum_{n=1}^{r-1}\big(\{|n|\}q^{n+e}F'(K_{V_{\varepsilon+2n+e}})+\{|n-r|\}q^{n+e-r}F'(K_{V_{\varepsilon+2n+e-2r}})\big) \end{aligned}$$

Now Corollary 16 and a direct computation show that

$$\begin{aligned} S &= \lim_{\varepsilon\to 0}\frac{1}{\{1\}}\sum_{n=1}^{r-1} F'(K_{V_{\varepsilon+2n+e}})\{n\}q^{n+e}(1-q^{-2rf(\varepsilon+2n+e)}) \\ &= \frac{1}{\{1\}}\sum_{n=1}^{r-1}\langle T_{V_{2n+e}}\rangle\{n\}q^{n+e}\lim_{\varepsilon\to 0}\mathsf{d}(\varepsilon+2n+e)(1-q^{-2rf\varepsilon}) \\ &= \frac{(-1)^{r-1}r}{\{1\}}\sum_{n=1}^{r-1}\langle T_{V_{2n+e}}\rangle\{n\}q^{n+e}\{2n+e\}\lim_{\varepsilon\to 0}\frac{\{rf\varepsilon\}}{\{r\varepsilon+re\}} \\ &= \frac{(-1)^{\omega}rf}{\{1\}}\sum_{n=1}^{r-1} q^{n+e}\{n\}\{2n+e\}\langle T_{V_{2n+e}}\rangle. \end{aligned}$$

Coming back to N_r^0, we have

$$\mathsf{N}_r^0(M,\varnothing,\omega)=\frac{rf}{\{1\}\Delta_{\mathrm{sign}(f)}}\sum_{n=1}^{r-1}q^{n+e}\{n\}\{2n+e\}\langle T_{V_{2n+e}}\rangle=c\sum_{n=0}^{r-1}\varphi_e(2n+e),$$

where $c=rf/(\{1\}\Delta_{\mathrm{sign}(f)})$ and $\varphi_e(k)=(q^k-q^e)\{k\}\langle T_{V_k}\rangle$. From Corollary 16, φ_e is $2r$–periodic. Furthermore, Proposition 4 implies that for $k\in\{1,\dots,r-1\}$, one has

$$\varphi_e(k)+\varphi_e(-k)=(q^k-q^e-q^{-k}+q^e)\{k\}\langle T_{S_{r-1-k}}\rangle=\{k\}^2\langle T_{S_{r-1-k}}\rangle.$$

So, using that $\varphi_0(0)=\varphi_e(r)=0$, we can write

$$\begin{aligned}\mathsf{N}_r^0(M,\varnothing,\omega)&=c\sum_{\substack{k\in e+2\mathbb{Z}\\0<k<2r}}\varphi_e(k)=c\Bigg(\sum_{\substack{k\in e+2\mathbb{Z}\\0<k<r}}\varphi_e(k)+\sum_{\substack{k\in e+2\mathbb{Z}\\-r<k<0}}\varphi_e(k)\Bigg)\\&=c\sum_{\substack{k\in e+2\mathbb{Z}\\0<k<r}}\{k\}^2\langle T_{S_{r-1-k}}\rangle=\frac{rf}{\{1\}\Delta_{\mathrm{sign}(f)}}\sum_{\substack{k\in e+2\mathbb{Z}\\0<k<r}}\{r-k\}^2\langle T_{S_{r-1-k}}\rangle\\&=\frac{rf}{\{1\}\Delta_{\mathrm{sign}(f)}}\sum_{\substack{n\in\omega+2\mathbb{Z}\\0\le n\le r-2}}\{n+1\}^2\langle T_{S_n}\rangle.\end{aligned}$$

Finally, $\mathrm{qdim}(S_n)=(-1)^n\{n+1\}/\{1\}$ implies

$$\mathsf{N}_r^0(M,\varnothing,\omega)=\frac{|f|}{\Delta^{\mathsf{SO}(3)}_{\mathrm{sign}(f)}}\sum_{\substack{n\in\omega+2\mathbb{Z}\\0\le n\le r-2}}\mathrm{qdim}(S_n)J_n(K)=|f|\,\mathrm{WRT}_r(M,\varnothing,\omega).\qquad\square$$

4 Vanishing of N_r^0 for nonhomology spheres

Theorem 22 *Let (M,T,ω) be any compatible triple. If $b_1(M)>0$, $\mathsf{N}_r^0(M,T,\omega)=0$.*

Proof Since $b_1(M)>0$ there exists a nontrivial $\delta\in H^1(M;\mathbb{Z})\subset H^1(M;\mathbb{C})$. For $\alpha\in\mathbb{C}$, let $\bar\alpha\delta\in H^1(M\setminus T;\mathbb{C}/2\mathbb{Z})$ be the trivial extension of $\bar\alpha\delta\in H^1(M;\mathbb{C}/2\mathbb{Z})$, where $\bar\alpha$ is the image of α in $\mathbb{C}/2\mathbb{Z}$. Then $(M,T,\omega+\bar\alpha\delta)$ is a compatible triple for all $\alpha\in\mathbb{C}$. Moreover, there exists a neighborhood N of $0\in\mathbb{C}$ such that $\omega+\bar\alpha\delta$ is nonintegral for all $\alpha\in N\setminus\{0\}$. Then for a complex number $\alpha\in N\setminus\{0\}$, [6, Propositions 1.5 and 3.14] implies that $\mathsf{N}_r^0(M,T,\omega+\bar\alpha\delta)=0$.

Now, for all $\alpha\in\mathbb{C}$, by definition of N_r^0 we have

$$\mathsf{N}_r^0(M,T,\omega+\bar\alpha\delta)=\frac{\mathsf{N}_r\big((M,T,\omega+\bar\alpha\delta)\,\#\,(S^3,o_\beta,\omega_\beta)\big)}{\mathsf{d}(\beta)},$$

where o_β is the unknot in S^3 colored by V_β, $\beta \in \mathbb{C} \setminus X_r$, and ω_β is the unique element of $H^1(S^3 \setminus o_\beta, \mathbb{C}/2\mathbb{Z})$ such that $(S^3, o_\beta, \omega_\beta)$ is a compatible triple. To compute the right side of this equation, we choose a colored framed oriented link $L^{\omega \# \omega_\beta} \cup T \cup o_\beta$ which is a computable presentation of $(M, T, \omega) \# (S^3, o_\beta, \omega_\beta)$. Then the same link colored by $\omega'_\alpha = (\omega + \overline{\alpha}\delta) \# \omega_\beta$ gives a presentation of $(M, T, \omega + \overline{\alpha}\delta) \# (S^3, o_\beta, \omega_\beta)$. For each component L_i of $L^{\omega'_\alpha}$ the color $g_{\omega'_\alpha}(L_i)$ is an affine function of α. The framed oriented link $L^{\omega'_\alpha}$ is computable if and only if all the colors $g_{\omega'_\alpha}(L_i)$ are in $\mathbb{C}/2\mathbb{Z} \setminus \mathbb{Z}/2\mathbb{Z}$. Let N' be the open set of $\mathbb{C}$ consisting of α such that $L^{\omega'_\alpha}$ is computable. Then N' contains 0 since $L^{\omega'_0}$ is computable.

Now we have

$$\mathsf{N}^0_r(M, T, \omega + \overline{\alpha}\delta) = \frac{F'(L^{\omega'_\alpha} \cup T \cup o_\beta)}{\mathsf{d}(\beta)\Delta_+^p \, \Delta_-^s} = \frac{\langle L^{\omega'_\alpha} \cup T \cup |_{V_\beta} \rangle}{\Delta_+^p \, \Delta_-^s},$$

where $|_{V_\beta}$ is the trivial one-component $(1,1)$–tangle colored with V_β. The function

$$\alpha \mapsto \langle L^{\omega'_\alpha} \cup T \cup |_{V_\beta} \rangle$$

is continuous on N' since it is a weighted sum of continuous functions (by Corollary 15), where the weights are products of functions d evaluated away from their poles. Thus, $\mathsf{N}^0_r(M, T, \omega + \overline{\alpha}\delta)$ is continuous at $\alpha = 0$. Finally, since $\mathsf{N}^0_r(M, T, \omega + \overline{\alpha}\delta)$ vanishes on N, we have $\mathsf{N}^0_r(M, T, \omega) = 0$. □

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