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ℓ-modular representations of unramified p-adic U(2,1)

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We construct all irreducible cuspidal ℓ-modular representations of a unitary group in three variables attached to an unramified extension of local fields of odd residual characteristic p with $\ell \neq p$. We describe the ℓ-modular principal series and show that the supercuspidal support of an irreducible ℓ-modular representation is unique up to conjugacy.

1. Introduction

The abelian category $\mathfrak{R}_R(G)$ of smooth representations of a reductive p-adic group G over an algebraically closed field R has been well studied when R has characteristic zero. The same cannot be said when R has positive characteristic ℓ; here many questions remain unanswered. In this paper, we are concerned only with the case $\ell \neq p$. We study the set $\mathrm{Irr}_R(G)$ of isomorphism classes of irreducible R-representations, eventually specialising to $G = \mathrm{U}(2, 1)$, a unitary group in three variables attached to an unramified extension F/F_0 of nonarchimedean local fields of odd residual characteristic. All R-representations henceforth considered will be smooth.

A classical strategy for the classification of irreducible R-representations is to split the problem into two steps: firstly, for any parabolic subgroup P of G with Levi decomposition $P = M \ltimes N$ and any $\sigma \in \mathrm{Irr}_R(M)$, decompose the (normalised) parabolically induced R-representation $i_P^G(\sigma)$; and, secondly, construct the irreducible R-representations which do not appear as a subquotient of an R-representation appearing in the first step, the *supercuspidal* R-representations. For any parabolic subgroup P, a supercuspidal irreducible R-representation π will have trivial Jacquet module $r_P^G(\pi) = 0$, by Frobenius reciprocity (i_P^G is right-adjoint to r_P^G). When R has characteristic zero the irreducible *cuspidal* R-representations, those whose Jacquet modules are all trivial, are all supercuspidal. However, in positive characteristic ℓ, there can exist irreducible cuspidal nonsupercuspidal R-representations.

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By transitivity of the Jacquet module and the geometric lemma — see [Vignéras 1996, II 2.19] — the *cuspidal support* of $\pi \in \mathrm{Irr}_R(G)$, that is, the set of pairs (M, σ) with M a Levi factor of a parabolic subgroup P of G and σ an irreducible cuspidal R-representation of M such that π is a subrepresentation of $i_P^G(\sigma)$, is a nonempty set consisting of a single G-conjugacy class; we say that the cuspidal support is unique up to conjugacy. By transitivity of parabolic induction, the *supercuspidal support* of $\pi \in \mathrm{Irr}_R(G)$, that is, the set of pairs (M, σ) with M a Levi factor of a parabolic subgroup P of G and σ an irreducible supercuspidal R-representation of M such that π is a subquotient of $i_P^G(\sigma)$, is nonempty. However, in general, it is not known if the supercuspidal support of an irreducible R-representation is unique up to conjugacy.

For GL_n and its inner forms, Vignéras [1996] and Mínguez and Sécherre [2014b; 2014a] showed that the supercuspidal support of an irreducible R-representation is unique up to conjugacy. The unicity of supercuspidal support is of great importance. Firstly, the unicity of supercuspidal support (up to inertia) for GL_n leads to the block decomposition of $\mathfrak{R}_R(G)$ into indecomposable summands; see [Vignéras 1998]. Secondly, it is important in Vignéras' ℓ-modular local Langlands correspondence for GL_n, which is first defined on supercuspidal elements by compatibility with the characteristic zero local Langlands correspondence and then extended to all irreducible ℓ-modular representations of GL_n. In this paper, we prove unicity of supercuspidal support for $\mathrm{U}(2, 1)$. We hope this is the first step in establishing similar results for $\mathrm{U}(2, 1)$ and in extending these to classical groups in general.

Our strategy is first to construct all irreducible cuspidal R-representations by compact induction from irreducible R-representations of compact open subgroups. The type of construction we employ has been used to great effect to construct all irreducible cuspidal R-representations in a large class of reductive p-adic groups when R has characteristic zero: [Morris 1999] for level zero R-representations of any reductive p-adic group, [Bushnell and Kutzko 1993a; 1993b] for GL_n and SL_n, [Sécherre and Stevens 2008] for inner forms of GL_n, [Yu 2001] and [Kim 2007] for arbitrary connected reductive groups under "tame" conditions, and [Stevens 2008] for classical p-adic groups with p odd. Vignéras [1996] and Mínguez and Sécherre [2014b; 2014a] adapted the characteristic zero constructions for GL_n and its inner forms to ℓ-modular representations. We perform similar adaptations to Stevens' construction to exhaust all irreducible cuspidal ℓ-modular representations of $\mathrm{U}(2, 1)$.

Theorem 5.3. *Let $G = \mathrm{U}(2, 1)$ and let π be an irreducible cuspidal R-representation of G. There exist a compact open subgroup J of G with pro-unipotent radical J^1 such that J/J^1 is a finite reductive group, an irreducible R-representation κ of J and an irreducible cuspidal R-representation σ of J/J^1 such that $\pi \simeq \mathrm{ind}_J^G(\kappa \otimes \sigma)$.*

The construction is explicit and, furthermore, all R-representations

$$\mathrm{I}_\kappa(\sigma) = \mathrm{ind}_J^G(\kappa \otimes \sigma)$$

constructed in this way are cuspidal. Moreover, we show that $\mathrm{I}_\kappa(\sigma)$ is supercuspidal if and only if σ is supercuspidal (Remark 8.2). In work in progress, joint with Stevens, we extend Stevens' construction for arbitrary classical groups to the ℓ-modular setting.

In the split case, for general linear groups all irreducible cuspidal ℓ-modular representations lift to integral ℓ-adic representations. For inner forms of GL_n, this is no longer true; some cuspidal nonsupercuspidal ℓ-modular representations do not lift. For U(2, 1) we also find cuspidal nonsupercuspidal ℓ-modular representations which do not lift (Remark 5.5). These nonlifting phenomena appear quite different. For U(2, 1) this nonlifting occurs because, in certain cases, there are ℓ-modular representations of the finite group J/J^1 which do not lift. For inner forms of GL_n, the nonlifting occurs when the normaliser of the reduction modulo ℓ of the inflation of a cuspidal ℓ-adic representation of an analogous group to J/J^1 is larger than the normaliser of all of its cuspidal lifts. We find that all supercuspidal ℓ-modular representations of U(2, 1) lift (Remark 8.2), as is the case for GL_n and its inner forms.

Secondly, by studying the corresponding Hecke algebras, we find the characters χ of the maximal diagonal torus T of U(2, 1) such that the principal series R-representation $i_B^{\mathrm{U}(2,1)}(\chi)$ is reducible. We let χ_1 denote the character of $F^\times$ given by $\chi_1(x) = \chi(\mathrm{diag}(x, \bar{x}x^{-1}, \bar{x}^{-1}))$, where $\bar{x}$ is the $\mathrm{Gal}(F/F_0)$-conjugate of x.

Theorem 6.2. *Let $G = \mathrm{U}(2, 1)$. Then $i_B^G(\chi)$ is reducible exactly in the following cases*:

(1) $\chi_1 = \nu^{\pm 2}$, *where ν is the absolute value on F*;

(2) $\chi_1 = \eta\nu^{\pm 1}$, *where η is any extension of the quadratic class field character ω_{F/F_0} to $F^\times$*;

(3) χ_1 *is nontrivial*, *but $\chi_1 |_{F_0^\times}$ is trivial.*

When R is of characteristic zero this is due to Keys [1984]. In our proof we need to apply his results to determine a sign. It should be possible to remove this dependency by computation using the theory of covers (cf. [Blondel 2012, Remark 3.13]). An alternative proof, when F_0 is of characteristic zero, would be to use the computations of [Keys 1984] with [Dat 2005, Proposition 8.4].

Finally, by studying the interaction of the right adjoints R_κ of the functors I_κ with parabolic induction we find cuspidal subquotients of the principal series. When cuspidal subquotients appear in the principal series we show exactly which ones from our exhaustive list do, finding that the supercuspidal support of an irreducible R-representation is unique up to conjugacy.

Theorem 8.1. *Let π be an irreducible R-representation of* $\mathrm{U}(2,1)$. *Then the supercuspidal support of π is unique up to conjugacy.*

In fact, in many cases, we obtain extra information on the irreducible quotients and subrepresentations which appear. If $\ell \neq 2$ and $\ell \mid q-1$, we show that all the principal series R-representations $i_B^{\mathrm{U}(2,1)}(\chi)$ are semisimple (Lemma 6.8). If $\ell \mid q+1$, we show that $i_B^{\mathrm{U}(2,1)}(\chi)$ has a unique irreducible subrepresentation and a unique irreducible quotient, and these are isomorphic (Lemma 6.10). A striking example of the reducibilities that occur is when $\chi = \nu^{-2}$.

Theorem (see Theorem 6.12 for more details). *Let $G = \mathrm{U}(2,1)$.*

(1) *If $\ell \nmid (q-1)(q+1)(q^2-q+1)$, then $i_B^G(\nu^{-2})$ has length two with unique irreducible subrepresentation 1_G and unique irreducible quotient St_G.*

(2) *If $\ell \neq 2$ and $\ell \mid q-1$, then $i_B^G(\nu^{-2}) = 1_G \oplus \mathrm{St}_G$ is semisimple of length two.*

(3) *If $\ell \neq 3$ and $\ell \mid q^2-q+1$, then $i_B^G(\nu^{-2})$ has length three with unique cuspidal subquotient. The unique irreducible subrepresentation is not isomorphic to the unique irreducible quotient.*

(4) *If $\ell \neq 2$ and $\ell \mid q+1$, or if $\ell = 2$ and $4 \mid q+1$, then $i_B^G(\nu^{-2})$ has length six with 1_G appearing as the unique subrepresentation and the unique quotient, and four cuspidal subquotients, one of which appears with multiplicity two. A maximal cuspidal subquotient of $i_B^G(\nu^{-2})$ is not semisimple.*

(5) *If $\ell = 2$ and $4 \mid q-1$, then $i_B^G(\nu^{-2})$ has length five with 1_G appearing as the unique subrepresentation and the unique quotient. All cuspidal subquotients of $i_B^G(\nu^{-2})$ are semisimple and the irreducible cuspidal subquotients are pairwise nonisomorphic.*

2. Notation

2A. *Unramified unitary groups.* Let F_0 be a nonarchimedean local field of odd residual characteristic p. Let F be an unramified quadratic extension of F_0 and $\overline{}$ a generator of $\mathrm{Gal}(F/F_0)$. If D is a nonarchimedean local field, we let $\mathfrak{o}_D$ denote the ring of integers of D, $\mathfrak{p}_D$ denote the unique maximal ideal of $\mathfrak{o}_D$, and $k_D = \mathfrak{o}_D/\mathfrak{p}_D$ denote the residue field. We let $\mathfrak{o}_0 = \mathfrak{o}_{F_0}$, $\mathfrak{p}_0 = \mathfrak{p}_{F_0}$, $k_0 = k_{F_0}$, and $q = q_0 = |k_{F_0}|$. We fix a choice of uniformiser ϖ_F of F_0.

Let V be a finite-dimensional F-vector space and $h : V \times V \to F$ a hermitian form on V, that is, a nondegenerate form which is sesquilinear (linear in the first variable and $\overline{}$-linear in the second variable) and such that $h(v_1, v_2) = \overline{h(v_2, v_1)}$ for all $v_1, v_2 \in V$. The *unitary group* $\mathrm{U}(V, h)$ is the subgroup of isometries of $\mathrm{GL}(V)$, i.e., $\mathrm{U}(V,h) = \{g \in \mathrm{GL}(V) : h(gv_1, gv_2) = h(v_1, v_2),\ v_1, v_2 \in V\}$. The form h induces an anti-involution on $\mathrm{End}_F(V)$ which we denote by $\overline{}$. Let σ denote the involution $g \mapsto \bar{g}^{-1}$ for $g \in \mathrm{GL}(V)$. We also let σ act on $\mathrm{End}_F(V)$ by $a \mapsto -\bar{a}$ for $a \in \mathrm{End}_F(V)$.

2B. ***Parahoric subgroups.*** An $\mathfrak{o}_F$*-lattice* in V is a compact open $\mathfrak{o}_F$-submodule of V. Let L be an $\mathfrak{o}_F$-lattice in V and let $\operatorname{Lat} V$ denote the set of all $\mathfrak{o}_F$-lattices in V. The $\mathfrak{o}_F$-lattice $L^\sharp = \{v \in V : h(v, L) \subseteq \mathfrak{p}_F\}$, defined relative to h, is called the *dual lattice* of L. Let $A = \operatorname{End}_F(V)$ and $\mathfrak{g} = \{X \in A : X + X^\sigma = 0\}$. An $\mathfrak{o}_F$*-lattice sequence* is a function $\Lambda : \mathbb{Z} \to \operatorname{Lat} V$ which is decreasing and periodic. Let Λ be an $\mathfrak{o}_F$-lattice sequence. The *dual* $\mathfrak{o}_F$-lattice sequence $\Lambda^\sharp$ of Λ is the $\mathfrak{o}_F$-lattice sequence defined by $\Lambda^\sharp(n) = (\Lambda(-n))^\sharp$ for all $n \in \mathbb{Z}$. We call Λ *self-dual* if there exists $k \in \mathbb{Z}$ such that $\Lambda(n) = \Lambda^\sharp(n+k)$ for all $n \in \mathbb{Z}$. If Λ is self-dual then we can always consider a translate Λ_k of Λ such that either $\Lambda_k(0) = \Lambda_k^\sharp(0)$ or $\Lambda_k(1) = \Lambda_k^\sharp(0)$.

Let Λ be an $\mathfrak{o}_F$-lattice sequence in V. For $n \in \mathbb{Z}$ define

$$\mathfrak{P}_n(\Lambda) = \{x \in A : x\Lambda(m) \subset \Lambda(m+n) \text{ for all } m \in \mathbb{Z}\},$$

which is an $\mathfrak{o}_F$-lattice in A. We let $\mathfrak{P}_n^-(\Lambda) = \mathfrak{P}_n(\Lambda) \cap \mathfrak{g}$.

If Λ is self-dual then the groups $\mathfrak{P}_n(\Lambda)$ are stable under the involution which h induces on A. In this case, define compact open subgroups of G, called *parahoric subgroups*, by

$$\mathrm{P}(\Lambda) = \mathfrak{P}_0(\Lambda)^\times \cap G,$$
$$\mathrm{P}_m(\Lambda) = (1 + \mathfrak{P}_m(\Lambda)) \cap G, \quad m \in \mathbb{N}.$$

The pro-unipotent radical of $\mathrm{P}(\Lambda)$ is isomorphic to $\mathrm{P}_1(\Lambda)$. The sequence $(\mathrm{P}_m(\Lambda))_{m\in\mathbb{N}}$ is a fundamental system of neighbourhoods of the identity in G and forms a decreasing filtration of $\mathrm{P}(\Lambda)$ by normal compact open subgroups. The quotient $\mathrm{M}(\Lambda) = \mathrm{P}(\Lambda)/\mathrm{P}_1(\Lambda)$ is the k_0-points of a connected reductive group defined over k_0.

Let $\mathrm{P}_1 = \mathrm{P}(\Lambda_1)$ and $\mathrm{P}_2 = \mathrm{P}(\Lambda_2)$ be parahoric subgroups of G. Fix a set of *distinguished* double coset representatives $D_{2,1}$ for $\mathrm{P}_2\backslash G/\mathrm{P}_1$, as in [Morris 1993, §3.10]. Let $n \in D_{2,1}$; then

$$P_{\Lambda_1, n\Lambda_2} = \mathrm{P}_1^1(\mathrm{P}_1 \cap \mathrm{P}_2^n)/\mathrm{P}_1^1$$

is a parabolic subgroup of $\mathrm{M}_1 = \mathrm{P}_1/\mathrm{P}_1^1$, by [Morris 1993, Corollary 3.20]. Furthermore, the pro-p unipotent radical of $\mathrm{P}_1^1(\mathrm{P}_1 \cap \mathrm{P}_2^n)$ is $\mathrm{P}_1^1(\mathrm{P}_1 \cap (\mathrm{P}_2^n)^1)$, by [Morris 1993, Lemma 3.21]. If $D_{2,1}$ is a set of distinguished double coset representatives for $\mathrm{P}_2\backslash G/\mathrm{P}_1$, then $D_{2,1}^{-1}$ is a set of distinguished double coset representatives for $\mathrm{P}_1\backslash G/\mathrm{P}_2$. Hence

$$P_{\Lambda_2, n^{-1}\Lambda_1} = \mathrm{P}_2^1(\mathrm{P}_2 \cap {}^n\mathrm{P}_1)/\mathrm{P}_2^1$$

is a parabolic subgroup of $\mathrm{M}_2 = \mathrm{P}_2/\mathrm{P}_2^1$. Furthermore, the pro-$p$ unipotent radical of $\mathrm{P}_2^1(\mathrm{P}_2 \cap {}^n\mathrm{P}_1)$ is $\mathrm{P}_2^1(\mathrm{P}_2 \cap {}^n\mathrm{P}_1^1)$.

2C. $\mathrm{U}(2,1)(F/F_0)$. Let $x_i \in F$ for $i = 1, 2, \ldots, n$. Denote by $\mathrm{diag}(x_1, \ldots, x_n)$ the n-by-n diagonal matrix with entries x_i on the diagonal and by $\mathrm{adiag}(x_1, \ldots, x_n)$ the n-by-n matrix $(a_{i,j})$ such that $a_{m,n+1-m} = x_{n+1-m}$ and all other entries are zero.

Let V be a three-dimensional F-vector space with standard basis $\{e_{-1}, e_0, e_1\}$ and $h : V \times V \to F$ be the nondegenerate hermitian form on V defined by, for $v, w \in V$,

$$h(v, w) = v_{-1}\overline{w_1} + v_0\overline{w_0} + v_1\overline{w_{-1}}$$

if $v = (v_{-1}, v_0, v_1)$ and $w = (w_{-1}, w_0, w_1)$ with respect to the standard basis $\{e_{-1}, e_0, e_1\}$. Let $\mathrm{U}(2,1)(F/F_0)$ denote the unitary group attached to the hermitian space (V, h), i.e.,

$$\mathrm{U}(2,1)(F/F_0) = \{g \in \mathrm{GL}_3(F) : gJ\bar{g}^T J = 1\},$$

where $J = \mathrm{adiag}(1, 1, 1)$ is the matrix of the form h. We let $\mathrm{U}(1,1)(F/F_0)$ and $\mathrm{U}(2)(F/F_0)$ denote the two-dimensional unitary groups defined by the forms whose associated matrices are $\mathrm{adiag}(1, 1)$ and $\mathrm{diag}(1, \varpi_F)$ respectively. Let

$$\mathrm{U}(1)(F/F_0) = \{g \in F^\times : g\bar{g} = 1\}$$

and occasionally, for brevity, let $F^1 = \mathrm{U}(1)(F/F_0)$. We use analogous notation for unitary groups defined over extensions of F_0 and defined over finite fields.

Let B be the standard Borel subgroup of $\mathrm{U}(2,1)(F/F_0)$ with Levi decomposition $B = T \ltimes N$, where $T = \{\mathrm{diag}(x, y, \bar{x}^{-1}) : x \in F^\times, y \in F^1\}$ and

$$N = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & \bar{x} \\ 0 & 0 & 1 \end{pmatrix} : x, y \in F, y + \bar{y} = x\bar{x} \right\}.$$

The maximal F_0-split torus contained in T is $T_0 = \{\mathrm{diag}(x, 1, x^{-1}) : x \in F_0^\times\}$. The subgroup of T generated by its compact subgroups is

$$T^0 = \{\mathrm{diag}(x, y, \bar{x}^{-1}) : x \in \mathfrak{o}_F^\times, y \in F^1\}.$$

Let $T^1 = T^0 \cap \mathrm{diag}(1 + \mathfrak{p}_F, 1 + \mathfrak{p}_F, 1 + \mathfrak{p}_F)$.

Let Λ_I be the $\mathfrak{o}_F$-lattice sequence of period three given by $\Lambda_I(0) = \mathfrak{o}_F \oplus \mathfrak{o}_F \oplus \mathfrak{o}_F$, $\Lambda_I(1) = \mathfrak{o}_F \oplus \mathfrak{o}_F \oplus \mathfrak{p}_F$ and $\Lambda_I(2) = \mathfrak{o}_F \oplus \mathfrak{p}_F \oplus \mathfrak{p}_F$ with respect to the standard basis. The (standard) Iwahori subgroup of G is the parahoric subgroup

$$\mathrm{P}(\Lambda_I) = \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{p}_F \end{pmatrix} \cap G.$$

There are two parahoric subgroups of G which contain $\mathrm{P}(\Lambda_I)$, both of which are maximal. These correspond to the lattice sequences Λ_x of period one and Λ_y of period two with $\Lambda_x(0) = \mathfrak{o}_F \oplus \mathfrak{o}_F \oplus \mathfrak{o}_F$, $\Lambda_y(0) = \mathfrak{o}_F \oplus \mathfrak{o}_F \oplus \mathfrak{p}_F$ and $\Lambda_y(1) = \mathfrak{o}_F \oplus \mathfrak{p}_F \oplus \mathfrak{p}_F$.

Note that we have $\mathrm{M}(\Lambda_x) \simeq \mathrm{U}(2,1)(k_F/k_0)$, $\mathrm{M}(\Lambda_y) \simeq \mathrm{U}(1,1)(k_F/k_0) \times k_F^1$ and $\mathrm{M}(\Lambda_I) \simeq k_F^\times \times k_F^1$. Furthermore, $\mathrm{M}(\Lambda_I)$ is a maximal torus in $\mathrm{M}(\Lambda_x)$ and $\mathrm{P}(\Lambda_I)$ is equal to the preimage in $\mathrm{P}(\Lambda_x)$ of a Borel subgroup B_x, which we call *standard*, under the projection map $\mathrm{P}(\Lambda_x) \to \mathrm{M}(\Lambda_x)$; the same holds with y in place of x throughout.

The *affine Weyl group* $\widetilde{W} = N_G(T)/T^0$ of $\mathrm{U}(2,1)(F/F_0)$ is an infinite dihedral group generated by the cosets represented by the elements $w_x = \mathrm{adiag}(1,1,1)$ and $w_y = \mathrm{adiag}(\varpi_F, 1, \varpi_F^{-1})$. Furthermore, we have $\mathrm{P}(\Lambda_x) = \mathrm{P}(\Lambda_I) \cup \mathrm{P}(\Lambda_I) w_x \mathrm{P}(\Lambda_I)$ and $\mathrm{P}(\Lambda_y) = \mathrm{P}(\Lambda_I) \cup \mathrm{P}(\Lambda_I) w_y \mathrm{P}(\Lambda_I)$.

2D. ***Reduction modulo ℓ.*** Let $\overline{\mathbb{Q}}_\ell$ be an algebraic closure of the ℓ-adic numbers, $\overline{\mathbb{Z}}_\ell$ be the ring of integers of $\overline{\mathbb{Q}}_\ell$, Γ be the unique maximal ideal of $\overline{\mathbb{Z}}_\ell$, and $\overline{\mathbb{F}}_\ell = \overline{\mathbb{Z}}_\ell/\Gamma$ be the residue field of $\overline{\mathbb{Q}}_\ell$, which is an algebraic closure of the finite field with ℓ elements. Let $\mathfrak{Gr}_R(G)$ denote the *Grothendieck group* of R-representations, i.e., the free abelian group with $\mathbb{Z}$-basis $\mathrm{Irr}_R(G)$. A representation in $\mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(G)$ will be called *ℓ-adic* and a representation in $\mathfrak{R}_{\overline{\mathbb{F}}_\ell}(G)$ will be called *ℓ-modular*. We say ℓ is *banal* for G if it does not divide the pro-order of any compact open subgroup of G.

Let $(\pi, \mathcal{V})$ be a finite-length ℓ-adic representation of G. We call π *integral* if π stabilises a $\overline{\mathbb{Z}}_\ell$-lattice $\mathcal{L}$ in $\mathcal{V}$. In this case π stabilises $\Gamma\mathcal{L}$ and π induces a finite-length ℓ-modular representation on the space $\mathcal{L}/\Gamma\mathcal{L}$. In general, this depends on the choice of the lattice $\mathcal{L}$. However, due to [Vignéras 2004, Theorem 1], the semisimplification of $\mathcal{L}/\Gamma\mathcal{L}$ is independent of the lattice chosen and we define $r_\ell(\pi)$, the *reduction modulo ℓ* of π, to be this semisimple ℓ-modular representation. If π is a finite-length R-representation of G we write $[\pi]$ for the semisimplification of π in $\mathfrak{Gr}_R(G)$.

We fix choices of square roots of p in $\overline{\mathbb{Q}}_\ell^\times$ and $\overline{\mathbb{F}}_\ell^\times$ such that our chosen square root of p in $\overline{\mathbb{F}}_\ell^\times$ is the reduction modulo ℓ of our chosen square root of p in $\overline{\mathbb{Q}}_\ell^\times$, and make use of these choices in our definitions of normalised parabolic induction and the Jacquet module.

Parabolic induction preserves integrality and commutes with reduction modulo ℓ: if $P = M \ltimes N$ is a parabolic subgroup of G and σ is a finite-length integral ℓ-adic representation of M, then $r_\ell(i_P^G(\sigma)) \simeq \left[i_P^G(r_\ell(\sigma))\right]$. Furthermore, compact induction commutes with reduction modulo ℓ: if H is a closed subgroup of G and σ an integral finite-length representation of H such that $\mathrm{ind}_H^G(\sigma)$ is of finite length, then $r_\ell(\mathrm{ind}_H^G(\sigma)) = [\mathrm{ind}_H^G(r_\ell(\sigma))]$. For classical groups, due to [Dat 2005], the Jacquet module preserves integrality and commutes with reduction modulo ℓ: if $P = M \ltimes N$ is a parabolic subgroup of G and π is a finite-length integral ℓ-adic representation of G, then $r_\ell(r_P^G(\pi)) \simeq \left[r_P^G(r_\ell(\pi))\right]$. This implies that the reduction modulo ℓ of a finite-length integral cuspidal ℓ-adic representation is cuspidal.

An irreducible R-representation is admissible, due to [Vignéras 1996, II 2.8]. If π is an R-representation, we let $\tilde{\pi}$ or $\pi^\sim$ denote the contragredient representation of π.

The abelian category $\mathfrak{R}_R(G)$ has a decomposition as a direct product of full subcategories $\mathfrak{R}_R^x(G)$, consisting of all representations all of whose irreducible subquotients have level x for $x \in \mathbb{Q}_{\geqslant 0}$, which is preserved by parabolic induction and the Jacquet functor, by [Vignéras 1996, II 5.8 and 5.12].

3. Cuspidal representations of $\mathrm{U}(1,1)(k_F/k_0)$ and $\mathrm{U}(2,1)(k_F/k_0)$

Our description of the supercuspidal ℓ-adic representations of $\mathrm{U}(1,1)(k_F/k_0)$ and $\mathrm{U}(2,1)(k_F/k_0)$ and the decomposition of the ℓ-adic principal series follow from similar arguments made for $\mathrm{GL}_2(k_F)$ and $\mathrm{SL}_2(k_F)$ by Digne and Michel [1991, §15.9]. The character tables of both groups were first computed by Ennola [1963] and the ℓ-modular representations of $\mathrm{U}(2,1)(k_F/k_0)$ were first studied by Geck [1990]. In this section, let $H = \mathrm{U}(1,1)(k_F/k_0)$ and $G = \mathrm{U}(2,1)(k_F/k_0)$. We can realise H and G as the fixed points of $\mathrm{GL}_2(\bar{k})$ and $\mathrm{GL}_3(\bar{k})$ under twisted Frobenius morphisms $\widetilde{F} : (a_{ij}) \mapsto (a_{ji}^q)^{-1}$, where $\bar{k}$ is an algebraic closure of k_0 containing k_F. A torus T of $\mathrm{GL}_2(\bar{k})$ (resp. $\mathrm{GL}_3(\bar{k})$) is called *minisotropic* if it is stable under the twisted Frobenius morphism $\widetilde{F}$ and is not contained in any $\widetilde{F}$-stable parabolic subgroup of $\mathrm{GL}_2(\bar{k})$ (resp. $\mathrm{GL}_3(\bar{k})$). We call a torus in H or G *minisotropic* if it is equal to the $\widetilde{F}$-fixed points of a minisotropic torus of the corresponding algebraic group.

3A. *Cuspidals of* $\mathrm{U}(1,1)(k_F/k_0)$.

3A1. *Cuspidals.* There are $\frac{1}{2}(q^2+q)$ irreducible ℓ-adic supercuspidal representations of H. These can be parametrised by the regular irreducible characters of the minisotropic tori of H. There is only one conjugacy class of minisotropic tori in G, which is isomorphic to $k_F^1 \times k_F^1$; hence a character of this torus corresponds to two characters of k_F^1. Furthermore, this character is regular if and only if it corresponds to two distinct characters of k_F^1. Thus the ℓ-adic supercuspidals can be parametrised by unordered pairs of distinct irreducible characters of k_F^1. Let χ_1, χ_2 be distinct ℓ-adic characters of k_F^1. Let $\sigma(\chi_1, \chi_2)$ denote the ℓ-adic supercuspidal representation parametrised by the set $\{\chi_1, \chi_2\}$.

Using Clifford Theory, the decomposition numbers for H follow from the well-known decomposition numbers of $\mathrm{SU}(1,1)(k_F/k_0) \simeq \mathrm{SL}_2(k_0)$. We have $|H| = q(q-1)(q+1)$; hence, because q is odd, there are four cases to consider: $\ell \mid q-1$, $\ell \mid q+1$, $\ell = 2$, and ℓ is prime to (q^2-1).

All irreducible ℓ-modular cuspidal representations of H are isomorphic to the reduction modulo ℓ of an irreducible ℓ-adic supercuspidal representation. If χ is an ℓ-adic character we let $\overline{\chi}$ denote its reduction modulo ℓ. If χ_1', χ_2' are ℓ-adic characters of k_F^1, we have $r_\ell(\sigma(\chi_1, \chi_2)) = r_\ell(\sigma(\chi_1', \chi_2'))$ if and only if $\{\overline{\chi}_1, \overline{\chi}_2\} = \{\overline{\chi}_1', \overline{\chi}_2'\}$. We let $\overline{\sigma}(\overline{\chi}_1, \overline{\chi}_2) = r_\ell(\sigma(\chi_1, \chi_2))$. Furthermore, $\overline{\sigma}(\overline{\chi}_1, \overline{\chi}_2)$ is supercuspidal if and only if $|\{\overline{\chi}_1, \overline{\chi}_2\}| = 2$ and we have $\overline{\sigma}(\overline{\chi}_1, \overline{\chi}_2) = \overline{\sigma}(\overline{\chi}_2, \overline{\chi}_1)$. Hence the irreducible cuspidal nonsupercuspidal ℓ-modular representations of H are parametrised by the

ℓ-modular characters of k_F^1 and, if $\bar\chi$ is an ℓ-modular character of k_F^1 equal to the reduction modulo ℓ of two distinct ℓ-adic characters of k_F^1, we let $\bar\sigma(\bar\chi)=\bar\sigma(\bar\chi,\bar\chi)$. When $\ell\nmid q+1$, all irreducible cuspidal ℓ-modular representations are supercuspidal.

3A2. *Cuspidal nonsupercuspidals when $\ell \mid q+1$.* Let $\ell^a \| q+1$, so that there are $(q+1)/\ell^a$ cuspidal nonsupercuspidal ℓ-modular representations denoted by $\bar\sigma(\bar\chi)$; these occur as the reduction modulo ℓ of $\sigma(\chi_1,\chi_2)$ when $\bar\chi=\bar\chi_1=\bar\chi_2$. Let $T=\{\mathrm{diag}(x,\bar x^{-1}) : x\in k_F^\times\}$ be the maximal diagonal torus of H and B_H be the standard Borel subgroup containing T. The principal series representations $i_{B_H}^H(\bar\chi\circ\xi)\simeq i_{B_H}^H(\bar 1)(\bar\chi\circ\det)$ are uniserial of length three with $(\bar\chi\circ\det)$ appearing as the unique irreducible subrepresentation and the unique irreducible quotient, and unique irreducible cuspidal subquotient $\bar\sigma(\bar\chi)$.

3B. *Cuspidals of* $\mathrm{U}(2,1)(k_F/k_0)$.

3B1. *ℓ-adic supercuspidals.* There are two conjugacy classes of minisotropic tori in G, which give rise to two classes of irreducible supercuspidal ℓ-adic representations coming from regular irreducible characters of these tori. Let E be an unramified cubic extension of F. One conjugacy class of the minisotropic tori has representatives isomorphic to $k_F^1\times k_F^1\times k_F^1$; the other conjugacy class has representatives isomorphic to k_E^1. However, in contrast to H, the irreducible representations parametrised by the irreducible regular characters of these tori do not constitute all the irreducible supercuspidal representations of G: additionally there exist unipotent supercuspidal representations of G. Thus we have three classes of ℓ-adic supercuspidals:

(1) There are $\frac{1}{6}(q+1)q(q-1)$ ℓ-adic supercuspidals of dimension $(q-1)(q^2-q+1)$ parametrised by the irreducible regular characters of $k_F^1\times k_F^1\times k_F^1$. An irreducible ℓ-adic character of $k_F^1\times k_F^1\times k_F^1$ is of the form $\chi_1\otimes\chi_2\otimes\chi_3$, with χ_1,χ_2,χ_3 irreducible ℓ-adic characters of k_F^1, and is regular if and only if $|\{\chi_1,\chi_2,\chi_3\}|=3$. We let $\sigma(\chi_1,\chi_2,\chi_3)$ denote the ℓ-adic supercuspidal corresponding to the set $\{\chi_1,\chi_2,\chi_3\}$.

(2) There are $\frac{1}{3}(q+1)q(q-1)$ ℓ-adic supercuspidals of dimension $(q-1)(q+1)^2$ parametrised by the irreducible regular characters of k_E^1. An irreducible ℓ-adic character ψ of k_E^1 is regular if and only if $\psi^{q+1}\neq 1$. We let $\tau(\psi)$ denote the ℓ-adic supercuspidal representation corresponding to ψ.

(3) There are $(q+1)$ unipotent ℓ-adic supercuspidals of dimension $q(q-1)$. These can be parametrised by the irreducible characters of k_F^1. We write $\nu(\chi)$ for the unipotent ℓ-adic supercuspidal representation corresponding to the irreducible ℓ-adic character χ of k_F^1.

3B2. *ℓ-modular cuspidals.* We have $|G|=q^3(q-1)(q+1)^3(q^2-q+1)$; hence there are six cases to consider: $\ell=2$, $\ell=3$ and $\ell\mid q+1$, $\ell\mid q-1$, $\ell\mid q+1$, $\ell\mid q^2-q+1$, and ℓ is prime to $(q-1)(q+1)(q^2-q+1)$. When $\ell\neq 2$, the decomposition numbers can be obtained from [Geck 1990] and [Okuyama and Waki 2002] using Clifford theory. Parabolic induction of the trivial character is completely described in [Hiss 2004, Theorem 4.1]. When $\ell \mid q-1$ or $\ell \mid q+1$, all irreducible cuspidal ℓ-modular representations lift to irreducible cuspidal ℓ-adic representations. Analogously to the two-dimensional case, we write $\overline{\nu}(\overline{\chi}) = r_\ell(\nu(\chi))$, $\overline{\tau}(\overline{\psi}) = r_\ell(\tau(\psi))$ and $\overline{\sigma}(\overline{\chi}_1, \overline{\chi}_2, \overline{\chi}_3) = r_\ell(\sigma(\chi_1, \chi_2, \chi_3))$.

When $\ell\neq 3$ and $\ell \mid q^2-q+1$, we have irreducible ℓ-modular cuspidal representations which do not lift: if ψ is an ℓ-adic character of k_E^1 such that $\psi^{q+1}\neq 1$ but $\overline{\psi}^{q+1}=\overline{1}$, then $r_\ell(\tau(\psi)) = \overline{\nu}(\overline{\chi})\oplus\overline{\tau}^+(\overline{\chi})$, where $\overline{\chi}$ is the character of k_F^1 such that $\overline{\psi}=\overline{\chi}\circ\xi$, where $\xi(x)=x^{q-1}$, and $\overline{\tau}^+(\overline{\chi})$ does not lift. When $\ell=2$ and $4\mid q-1$, we also have cuspidal representations which do not lift: if ψ is an ℓ-adic character of k_E^1 such that $\psi^{q+1}\neq 1$ but $\overline{\psi}^{q+1}=\overline{1}$, then $r_\ell(\tau(\psi)) = \overline{\nu}(\overline{\chi})\oplus\overline{\nu}(\overline{\chi})\oplus\overline{\tau}^+(\overline{\chi})$, where $\overline{\chi}$ is the character of k_F^1 such that $\overline{\psi}=\overline{\chi}\circ\xi$, where $\xi(x)=x^{q-1}$, and $\overline{\tau}^+(\overline{\chi})$ does not lift. All other irreducible cuspidal ℓ-modular representations of G lift to ℓ-adic representations and we use the same notation as before.

3B3. *ℓ-adic principal series.* Let $T=\{\mathrm{diag}(x,y,\bar{x}^{-1}) : x\in k_F^\times,\ y\in k_F^1\}$ be the maximal diagonal torus in G and B be the standard Borel subgroup of G containing T.

Let χ_1 be an ℓ-adic character of $k_F^\times$ and χ_2 an ℓ-adic character of k_F^1. Let χ be the irreducible character of T defined by $\chi(\mathrm{diag}(x,y,x^{-q}))=\chi_1(x)\chi_2(xyx^{-q})$. The character χ is regular if and only if $\chi_1^{q+1}\neq 1$, and in this case the principal series representation $i_B^G(\chi)$ is irreducible.

If $\chi_1^{q+1}=1$ then $\chi_1=\chi_1'\circ\xi$, where $\xi(x)=x^{q-1}$ and χ_1' is an ℓ-adic character of k_F^1. If $\chi_1'=1$, or equivalently $\chi_1=1$, then

$$i_B^G(\chi)=1_G(\chi_2\circ\det)\oplus\mathrm{St}_G(\chi_2\circ\det),$$

where St_G is an irreducible q^3-dimensional representation of G. If $\chi_1'\neq 1$ then

$$i_B^G(\chi)=R_{1_{H(\chi_1')}}(\chi_2\circ\det)\oplus R_{\mathrm{St}_{H(\chi_1')}}(\chi_2\circ\det),$$

where $R_{1_{H(\chi_1')}}$ and $R_{\mathrm{St}_{H(\chi_1')}}$ are irreducible representations of G of dimensions q^2-q+1 and $q(q^2-q+1)$ respectively. The reducibility here comes from inducing first to the Levi subgroup $L^*=\mathrm{U}(1,1)(k_F/k_0)\times\mathrm{U}(1)(k_F/k_0)$, which is not contained in any proper rational parabolic subgroup of G. Here 1_H and St_H denote the trivial and Steinberg representations of $\mathrm{U}(1,1)(k_F/k_0)$, and R is a generalised induction from L^* to G.

3B4. *Cuspidal subquotients of ℓ-modular principal series.* If $\ell\neq 2$ and $\ell\mid q-1$, or ℓ is prime to $(q-1)(q+1)(q^2-q+1)$, then all irreducible cuspidal ℓ-modular representations are supercuspidal and the principal series representations are all semisimple.

Let $\overline{\chi}_2$ be an ℓ-modular character of k_F^1. We first describe the ℓ-modular principal series representations $i_B^G(\overline{1})(\overline{\chi}_2\circ\det)$ in all the cases where cuspidal subquotients appear.

(1) If $\ell\neq 3$ and $\ell\mid q^2-q+1$, $i_B^G(\overline{1})(\overline{\chi}_2\circ\det)$ are uniserial of length three with $(\overline{\chi}\circ\det)$ appearing as the unique irreducible subrepresentation and the unique irreducible quotient and $\overline{\tau}^+(\overline{\chi})$ as the unique irreducible cuspidal subquotient.

(2) If $\ell\neq 2$ and $\ell\mid q+1$, or $\ell=2$ and $4\mid q+1$, then $i_B^G(\overline{1})(\overline{\chi}\circ\det)$ have irreducible cuspidal subquotients $\overline{\nu}(\overline{\chi})$ and $\overline{\sigma}(\overline{\chi})=\overline{\sigma}(\overline{\chi},\overline{\chi},\overline{\chi})$. The principal series representations $i_B^G(\overline{1})(\overline{\chi}\circ\det)$ are uniserial of length five with $(\overline{\chi}\circ\det)$ appearing as the unique irreducible subrepresentation and the unique irreducible quotient. A maximal cuspidal subquotient of $i_B^G(\overline{1})(\overline{\chi}\circ\det)$ is uniserial of length three with $\overline{\nu}(\overline{\chi})$ appearing as the unique irreducible quotient and the unique irreducible subrepresentation, and remaining subquotient $\overline{\sigma}(\overline{\chi})$.

(3) If $\ell=2$ and $4\mid q-1$ then $i_B^G(\overline{1})(\overline{\chi}\circ\det)$ has length four with $(\overline{\chi}\circ\det)$ appearing as the unique irreducible subrepresentation and the unique irreducible quotient, and cuspidal subquotient $\overline{\nu}(\overline{\chi})\oplus\overline{\tau}^+(\overline{\chi})$.

Now let $\overline{\chi}_1'$ and $\overline{\chi}_2$ be ℓ-modular characters of k_F^1 with $\overline{\chi}_1'$ nontrivial and let $\overline{\chi}_1=\overline{\chi}_1'\circ\xi$. Let $\overline{\chi}$ be the ℓ-modular character of T defined by

$$\overline{\chi}(\operatorname{diag}(x,y,x^{-q}))=\overline{\chi}_1(x)\overline{\chi}_2(xyx^{-q}).$$

If $\ell\nmid q+1$ then $i_B^G(\overline{\chi})$ does not possess any cuspidal subquotients. If $\ell\mid q+1$ then $i_B^G(\overline{\chi})$ is uniserial of length three with $\overline{R}_{\overline{1}_H(\overline{\chi}_1')}(\overline{\chi}_2\circ\det)$ appearing as the unique irreducible subrepresentation and the unique irreducible quotient and cuspidal subquotient $\sigma(\overline{\chi}_1',\overline{\chi}_1',\overline{\chi}_2)$. This follows from [Bonnafé and Rouquier 2003, Theorem 11.8] and the principal block of H as χ corresponds to a semisimple element with centraliser $H\times k_F^1$ in the dual group.

4. Irreducible cuspidal R-representations of $\mathrm{U}(2,1)(F/F_0)$

Let $G=\mathrm{U}(2,1)(F/F_0)$. We construct all irreducible cuspidal representations of G by compact induction from certain irreducible representations of compact open subgroups. We review some general theory first and recall results of Vignéras on level zero representations. Our construction of all irreducible cuspidal representations of G then follows the outline of Stevens' construction [2008] of all irreducible cuspidal representations of classical p-adic groups in the complex case. While his construction is carried out when $R=\mathbb{C}$ the first part remains equally valid when R is any algebraically closed field of characteristic unequal to p, essentially as all groups involved are pro-p . However, when we move to defining β-extensions and beyond the subgroups we are dealing with no longer have pro-order necessarily invertible in $\overline{\mathbb{F}}_\ell$. It is here, and after, where we need to be careful and have to make nontrivial changes to the proofs of the statements of [Stevens 2008]. It turns out that, even though we have to change the proofs, the definitions and properties of β-extensions in the ℓ-modular case are completely analogous to those of complex β-extensions. We note that as we are in the special case of unramified $\mathrm{U}(2,1)(F/F_0)$, using the framework of Stevens, we can show that our β-extensions satisfy closer compatibility properties than are available in the general case of classical groups.

4A. ***Types and Hecke algebras.*** By an R-*type*, we mean a pair (K,σ) consisting of a compact open subgroup K of G and an irreducible R-representation σ of K. Given an R-type we consider the compactly induced representation $\mathrm{ind}_K^G(\sigma)$ of G, the goal being to find pairs (K,σ) such that $\mathrm{ind}_K^G(\sigma)$ is irreducible and cuspidal. Let $\pi\in\mathrm{Irr}_R(G)$; we say that π *contains* the R-type (K,σ) if π is a quotient of $\mathrm{ind}_K^G(\sigma)$.

Let (K,σ) be an R-type in G and $\mathcal{W}$ be the space of σ. The *spherical Hecke algebra* $\mathcal{H}(G,\sigma)$ of σ is the R-module consisting of the set of all functions $f: G\to\mathrm{End}_R(\mathcal{W})$ such that the support of f is a finite union of double cosets in $K\backslash G/K$ and f transforms by σ on the left and the right, i.e., for all $k_1,k_2\in K$ and all $g\in G$, $f(k_1gk_2)=\sigma(k_1)f(g)\sigma(k_2)$. The product in $\mathcal{H}(G,\sigma)$ is given by convolution: if $f_1,f_2\in\mathcal{H}(G,\sigma)$ then

$$f_1\star f_2(h)=\sum_{G/K} f_1(g)f_2(g^{-1}h).$$

The spherical Hecke algebra $\mathcal{H}(G,\sigma)$ is isomorphic to $\mathrm{End}_G(\mathrm{ind}_K^G(\sigma))$, where multiplication in $\mathrm{End}_G(\mathrm{ind}_K^G(\sigma))$ is defined by composition. For $g\in G$, let $I_g(\sigma)=\mathrm{Hom}_K(\sigma,\mathrm{ind}_{K\cap K^g}^K\sigma^g)$ and let $I_G(\sigma)=\{g\in G: I_g(\sigma)\neq 0\}$.

Let $\mathcal{M}(G,\sigma)$ denote the category of right $\mathcal{H}(G,\sigma)$-modules. Define

$$M_\sigma:\mathfrak{R}_R(G)\to\mathcal{M}(G,\sigma)$$

by $\pi\mapsto\mathrm{Hom}_G(\mathrm{ind}_K^G(\sigma),\pi)$; this is a (right) $\mathrm{End}_G(\mathrm{ind}_K^G(\sigma))$-module by precomposition. In the ℓ-adic case, if (K,σ) is a type in the sense of [Bushnell and Kutzko 1998, p. 584], M_σ induces an equivalence of categories between $\mathcal{M}(G,\sigma)$ and the full subcategory of $\mathfrak{R}_R(G)$ of representations all of whose irreducible subquotients contain (K,σ).

An R-representation $(\pi,\mathcal{V})$ of G is *quasiprojective* if, for all R-representations $(\sigma,\mathcal{W})$ of G, all surjective $\Phi\in\mathrm{Hom}_G(\mathcal{V},\mathcal{W})$ and all $\Psi\in\mathrm{Hom}_G(\mathcal{V},\mathcal{W})$, there exists $\Xi\in\mathrm{End}_G(\mathcal{V})$ such that $\Psi=\Phi\circ\Xi$.

Theorem 4.1 [Vignéras 1998, Appendix, Theorem 10]. *Let π be a quasiprojective, finitely generated R-representation of G. The map $\rho\mapsto\mathrm{Hom}_G(\pi,\rho)$ induces a bijection between the irreducible quotients of π and the simple right $\mathrm{End}_G(\pi)$-modules.*

Let P be a parabolic subgroup of G with Levi decomposition $P = M \ltimes N$. Let P^{op} be the opposite parabolic subgroup of P with Levi decomposition $P^{\mathrm{op}} = M \ltimes N^{\mathrm{op}}$. Let $K^+ = K \cap N$ and $K^- = K \cap N^{\mathrm{op}}$. An element z of the centre of M is called *strongly* (P, K)*-positive* if:

(1) $zK^+z^{-1} \subset K^+$ and $zK^-z^{-1} \supset K^-$.

(2) For all compact subgroups H_1, H_2 of N (resp. N^{op}), there exists a positive (resp. negative) integer m such that $z^m H_1 z^{-m} \subset H_2$.

Let (K_M, σ_M) be an R-type of M. An R-type (K, σ) is called a G*-cover* of (K_M, σ_M) relative to P if we have:

(1) $K \cap M = K_M$ and we have an Iwahori decomposition $K = K^- K_M K^+$.

(2) $\mathrm{Res}^K_{K_M}(\sigma) = \sigma_M$, $\mathrm{Res}^K_{K^+}(\sigma)$ and $\mathrm{Res}^K_{K^-}(\sigma)$ are both multiples of the trivial representation.

(3) There exists a strongly (P, K)-positive element z of the centre of M such that the double coset $Kz^{-1}K$ supports an invertible element of $\mathscr{H}_R(G, \sigma)$.

The point is that the properties of a G-cover allow one to define an injective homomorphism of algebras $j_P : \mathscr{H}(M, \sigma_M) \to \mathscr{H}(G, \sigma)$ and hence a (normalised) restriction functor $(j_P)^* : \mathscr{M}(G, \sigma) \to \mathscr{M}(M, \sigma_M)$; see [Bushnell and Kutzko 1998, p. 585] and [Vignéras 1998, II §10].

Theorem 4.2 [Vignéras 1998, II §10.1]. *Let π be a finitely generated ℓ-modular representation of G. We have an isomorphism $(j_P)^*(M_\sigma(\pi)) \simeq M_{\sigma_M}(r^G_P(\pi))$ of representations of M.*

4B. ***Level zero ℓ-modular representations.*** An irreducible representation π of G is of level zero if it has nontrivial invariants under the pro-p unipotent radical of some maximal parahoric subgroup of G.

Let Λ be a self-dual $\mathfrak{o}_F$-lattice sequence in V and $\mathrm{P}(\Lambda)$ the associated parahoric subgroup in G. We define *parahoric induction* $\mathrm{I}_\Lambda : \mathfrak{R}_R(\mathrm{M}(\Lambda)) \to \mathfrak{R}_R(G)$ on the objects of $\mathfrak{R}_R(\mathrm{M}(\Lambda))$ by

$$\mathrm{I}_\Lambda(\sigma) = \mathrm{ind}^G_{\mathrm{P}(\Lambda)}(\sigma)$$

for σ an R-representation of $\mathrm{M}(\Lambda)$, where, by abuse of notation, we also write σ for the inflation of σ to $\mathrm{P}(\Lambda)$ by defining $\mathrm{P}_1(\Lambda)$ to act trivially. This functor has a right-adjoint, *parahoric restriction* $\mathrm{R}_\Lambda : \mathfrak{R}_R(G) \to \mathfrak{R}_R(\mathrm{M}(\Lambda))$, defined on the objects of $\mathfrak{R}_R(G)$ by

$$\mathrm{R}_\Lambda(\pi) = \pi^{\mathrm{P}_1(\Lambda)}$$

for π an R-representation of G. Parahoric induction and restriction are exact functors.

We have the following important lemma, due to Vignéras [2001]. In her paper, the statement is for a general p-adic reductive group G.

Lemma 4.3 [Vignéras 2001]. *Let* $\mathrm{P}_1=\mathrm{P}(\Lambda_1)$ *and* $\mathrm{P}_2=\mathrm{P}(\Lambda_2)$ *be parahoric subgroups of* G. *Let* σ *be a representation of* $\mathrm{M}(\Lambda_2)$ *and fix a set* $D_{1,2}$ *of distinguished double coset representatives of* $\mathrm{P}_1\backslash G/\mathrm{P}_2$. *We have an isomorphism*

$$\mathrm{R}_{\Lambda_1}\circ \mathrm{I}_{\Lambda_2}(\sigma)\simeq \bigoplus_{n\in D_{1,2}} i^{\mathrm{M}(\Lambda_1)}_{P_{\Lambda_1,n\Lambda_2}}\big(r^{\mathrm{M}(\Lambda_2)}_{P_{\Lambda_2,n^{-1}\Lambda_1}}(\sigma)\big)^n.$$

Lemma 4.4. *Let* $\mathrm{P}(\Lambda_1)$ *and* $\mathrm{P}(\Lambda_2)$ *be parahoric subgroups of* G *associated to the* $\mathfrak{o}_F$*-lattice sequences* Λ_1 *and* Λ_2 *in* V. *Suppose that* $\mathrm{P}(\Lambda_2)$ *is maximal and let* σ *be an irreducible cuspidal representation of* $\mathrm{M}(\Lambda_2)$. *We have*

$$\mathrm{R}_{\Lambda_1}\circ \mathrm{I}_{\Lambda_2}(\sigma)\simeq \begin{cases}\sigma & \textit{if } \mathrm{P}(\Lambda_1) \textit{ is conjugate to } \mathrm{P}(\Lambda_2) \textit{ in } G,\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. By cuspidality of σ, if $r^{\mathrm{M}(\Lambda_2)}_{P_{\Lambda_2,n^{-1}\Lambda_1}}(\sigma)\neq 0$, then $P_{\Lambda_2,n^{-1}\Lambda_1}=M(\Lambda_2)$. If $\mathrm{P}(\Lambda_2)$ is not conjugate to $\mathrm{P}(\Lambda_1)$ then, for all $n\in D_{1,2}$, the parabolic subgroup $P_{\Lambda_2,n^{-1}\Lambda_1}$ is a proper parabolic subgroup of $\mathrm{M}(\Lambda_2)$. Hence $\mathrm{R}_{\Lambda_2}\circ \mathrm{I}_{\Lambda_1}(\sigma)\simeq 0$ by Lemma 4.3. As $N_G(\mathrm{P}(\Lambda_2))=\mathrm{P}(\Lambda_2)$, if there exists $n\in D_{1,2}$ such that $\mathrm{P}(n^{-1}\Lambda_1)=\mathrm{P}(\Lambda_2)$, there can be only one such n. In this case, $\mathrm{R}_{\Lambda_1}\circ \mathrm{I}_{\Lambda_2}(\sigma)\simeq\sigma$, by Lemma 4.3. □

4C. *Positive level cuspidal ℓ-modular representations.*

4C1. *Semisimple strata and characters.* Let $[\Lambda,n,r,\beta]$ be a *skew semisimple stratum* in A; see [Stevens 2008, Definition 2.8]. Associated to $[\Lambda,n,r,\beta]$ and a fixed level one character of $F_0^\times$ are:

(1) A decomposition $V=\bigoplus_{i=1}^l V_i$, orthogonal with respect to h, and a sum of field extensions $E=\bigoplus_{i=1}^l E_i$ of E such that $\Lambda=\bigoplus_{i=1}^l\Lambda_i$ with Λ_i an $\mathfrak{o}_{E_i}$-lattice sequence in V_i; we say that Λ is an $\mathfrak{o}_E$*-lattice sequence* and write Λ_E when we are considering Λ as such.

(2) The F_0-points of a product of unramified unitary groups defined over F_0, $G_E=\prod_{i=1}^l G_{E_i}$.

(3) Compact open subgroups $H(\Lambda,\beta)\subseteq J(\Lambda,\beta)$ of G with decreasing filtrations by pro-p normal compact open subgroups $H^n(\Lambda,\beta)=H(\Lambda,\beta)\cap \mathrm{P}_n(\Lambda)$ and $J^n(\Lambda,\beta)=J(\Lambda,\beta)\cap\mathrm{P}_n(\Lambda)$, $n\geqslant 1$. When Λ is fixed we write $J=J(\Lambda,\beta)$, $H=H(\Lambda,\beta)$, and use similar notation for their filtration subgroups. We have $J=P(\Lambda_E)J^1$, where $\mathrm{P}(\Lambda_E)$ is the parahoric subgroup of G_E obtained by considering Λ as an $\mathfrak{o}_E$-lattice sequence.

(4) A set of *semisimple characters* $\mathscr{C}_-(\Lambda,r,\beta)$ of $H^{r+1}(\Lambda,\beta)$. For $r=0$, we write $\mathscr{C}_-(\Lambda,\beta)=\mathscr{C}_-(\Lambda,0,\beta)$.

Let $[\Lambda_i, n, 0, \beta]$, $i = 1, 2$, be skew semisimple strata in A. For all $\theta_1 \in \mathscr{C}_-(\Lambda_1, \beta)$, there is a unique $\theta_2 \in \mathscr{C}_-(\Lambda_2, \beta)$ such that $1 \in I_G(\theta_1, \theta_2)$, by [Stevens 2005, Proposition 3.32]. This defines a bijection

$$\tau_{\Lambda_1,\Lambda_2,\beta} : \mathscr{C}_-(\Lambda_1, \beta) \to \mathscr{C}_-(\Lambda_2, \beta)$$

and we call $\theta_2 = \tau_{\Lambda_1,\Lambda_2,\beta}(\theta_1)$ the *transfer* of θ_1.

The skew semisimple strata in A fall into three classes:

(1) Skew simple strata $[\Lambda, n, 0, \beta]$, where E is a field.

(a) If $E = F$ we say that $[\Lambda, n, 0, \beta]$ is a scalar skew simple stratum. In this case, $J/J^1 = \mathrm{P}(\Lambda)/\mathrm{P}_1(\Lambda)$ is isomorphic to one of $\mathrm{GL}_1(k_F) \times \mathrm{U}(1)(k_F/k_0)$, $\mathrm{U}(1, 1)(k_F/k_0) \times \mathrm{U}(1)(k_F/k_0)$ or $\mathrm{U}(2, 1)(k_F/k_0)$.

(b) Otherwise, E/F is cubic and $J/J^1 \simeq \mathrm{P}(\Lambda_E)/\mathrm{P}_1(\Lambda_E) \simeq \mathrm{U}(1)(k_E/k_{E^0})$ is a finite unitary group of order $q_{E^0} + 1$, where

$$q_{E^0} = \begin{cases} q_0^3 & \text{if } E/F \text{ is unramified,} \\ q_0 & \text{if } E/F \text{ is ramified.} \end{cases}$$

(2) Skew semisimple strata $[\Lambda, n, 0, \beta] = [\Lambda_1, n_1, 0, \beta_1] \oplus [\Lambda_2, n_2, 0, \beta_2]$, not equivalent to a skew simple stratum, with $[\Lambda_i, n_i, 0, \beta_i]$ skew simple strata in $\mathrm{End}_{F_0}(V_i)$. Without loss of generality, suppose that V_1 is one-dimensional and V_2 is two-dimensional. We have $J/J^1 \simeq \prod_{i=1}^{2} \mathrm{P}(\Lambda_{i,E})/\mathrm{P}_1(\Lambda_{i,E})$. If $\beta_2 \in F$ and V_2 is hyperbolic, then $G_E \simeq U(1, 1)(F/F_0) \times \mathrm{U}(1)(F/F_0)$ and $\mathrm{P}(\Lambda_{2,E})$ is a parahoric subgroup of $\mathrm{U}(1, 1)(F/F_0)$ and need not be maximal. If $\beta_2 \in F$ and V_2 is anisotropic, then $G_E \simeq \mathrm{U}(2)(F/F_0) \times \mathrm{U}(1)(F/F_0)$ is compact. If E_2/F is quadratic then it is ramified, because there is a unique unramified extension of F_0 in each degree and E_2^0/F_0 is quadratic and also fixed by the involution. Thus, if E_2/F is quadratic then $J/J^1 \simeq \mathrm{U}(1)(k_F/k_0) \times \mathrm{U}(1)(k_F/k_0)$.

(3) Skew semisimple strata $[\Lambda, n, 0, \beta] = \bigoplus_{i=1}^{3} [\Lambda_i, n_i, 0, \beta_i]$, not equivalent to a skew semisimple stratum of the first two classes, with $[\Lambda_i, n_i, 0, \beta_i]$ skew simple strata in $\mathrm{End}_{F_0}(V_i)$. In this case, $J/J^1 \simeq \mathrm{U}(1)(k_F/k_0) \times \mathrm{U}(1)(k_F/k_0) \times \mathrm{U}(1)(k_F/k_0)$.

We say that π *contains* the skew semisimple stratum $[\Lambda, n, 0, \beta]$ if it contains a character $\theta \in \mathscr{C}_-(\Lambda, \beta)$.

Theorem 4.5 [Stevens 2005, Theorem 5.1]. *Let π be an irreducible cuspidal ℓ-modular representation of G. Then π contains a skew semisimple stratum* $[\Lambda, n, 0, \beta]$.

4C2. *Heisenberg representations.* Let $\theta \in \mathscr{C}_-(\Lambda, \beta)$. By [Stevens 2008, Corollary 3.29], there exists a unique irreducible representation η of $J^1(\Lambda, \beta)$ which contains θ. We call such an η a *Heisenberg representation*. Furthermore, by [Stevens

2008, Proposition 3.31],

$$\dim_R(I_g(\eta)) = \begin{cases} 1 & \text{if } g \in J^1 G_E J^1, \\ 0 & \text{otherwise.} \end{cases}$$

4C3. *β-extensions.* Assume $P(\Lambda_E)$ is maximal. A *β-extension* of a Heisenberg representation η to $J = J(\Lambda, \beta)$ is an extension κ with maximal intertwining, $I_G(\kappa) = I_G(\eta)$. By [Blasco 2002, Lemma 5.8], for all maximal skew semisimple strata which are not skew scalar simple strata, β-extensions exist in the ℓ-adic case for G, and for ℓ-modular representations we obtain β-extensions by reduction modulo ℓ from the ℓ-adic extensions. Note that the reduction modulo ℓ of an ℓ-adic β-extension $\tilde{\kappa}$ of J is irreducible: its restriction to J^1 is the reduction modulo ℓ of $\tilde{\eta} = \mathrm{Res}^J_{J^1}(\tilde{\kappa})$, reduction modulo ℓ commutes with restriction, and, as J^1 is pro-p , the reduction modulo ℓ of $\tilde{\eta}$ is irreducible. Let $[\Lambda, n, 0, \beta]$ be a scalar skew simple stratum and $\theta \in \mathscr{C}_-(\Lambda, \beta)$. Then $J^1 = H^1 = \mathrm{P}_1(\Lambda)$, $J = \mathrm{P}(\Lambda)$, and $\theta = \chi \circ \det$ for some character χ of $\mathrm{P}_1(\Lambda)$ (cf. [Bushnell and Kutzko 1993a, Definition 3.23]). The character χ extends to a character $\tilde{\chi}$ of F^1 and we define $\kappa : J \to R^\times$ by $\kappa = \tilde{\chi} \circ \det$. Then κ extends θ and is intertwined by all of G, hence is a β-extension. Hence, in the maximal case, β-extensions exist.

Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum. Suppose $P(\Lambda_E)$ is not maximal and choose a maximal parahoric subgroup $P(\Lambda_E^m)$ of G_E associated to the $\mathfrak{o}_E$-lattice sequence Λ_E^m in V such that $P(\Lambda_E) \subset P(\Lambda_E^m)$. This implies that $P(\Lambda) \subset P(\Lambda^m)$. Note that this is the case for unramified $\mathrm{U}(2, 1)(E/F)$, but not for classical groups in general. Let $\theta \in \mathscr{C}_-(\Lambda, \beta)$ and let η be the irreducible representation of $J^1_m = J^1(\beta, \Lambda)$ which contains θ. Let $\theta_m = \tau_{\Lambda, \Lambda^m, \beta}(\theta)$ and let η_m be the irreducible representation of $J^1(\beta, \Lambda^m)$ which contains θ_m. Let κ_m be a β-extension of η_m.

Lemma 4.6. *There exists a unique extension κ of η to J such that $\mathrm{Res}^{J_m}_{P(\Lambda_E)J^1_m}(\kappa_m)$ and κ induce equivalent irreducible representations of $P(\Lambda_E)P_1(\Lambda)$.*

Proof. If $\mathrm{P}(\Lambda_E)$ is maximal then $\kappa_m = \kappa$ and there is nothing to prove. Let $\tilde{\kappa}_m$ be a lift of κ. By [Stevens 2008, Lemma 4.3], there exists a unique irreducible ℓ-adic representation $\tilde{\kappa}$ of J such that $\mathrm{Res}^{J_m}_{P(\Lambda_E)J^1_m}(\tilde{\kappa}_m)$ and κ induce equivalent irreducible representations of $P(\Lambda_E)P_1(\Lambda)$. By reduction modulo ℓ, we have an irreducible ℓ-modular representation $\kappa = r_\ell(\tilde{\kappa})$ which extends η such that

$$\big[\mathrm{ind}_J^{\mathrm{P}(\Lambda_E)\,\mathrm{P}_1(\Lambda)} \kappa\big] = \big[\mathrm{ind}_{\mathrm{P}(\Lambda_E)J^1_m}^{\mathrm{P}(\Lambda_E)\,\mathrm{P}_1(\Lambda)} \mathrm{Res}^{J_m}_{\mathrm{P}(\Lambda_E)J^1_m}(\kappa_m)\big].$$

By Mackey Theory,

$$\mathrm{Res}^{P(\Lambda_E)P_1(\Lambda)}_{P_1(\Lambda)}(\mathrm{ind}_J^{P(\Lambda_E)P_1(\Lambda)} \kappa) \simeq \mathrm{ind}_{J^1}^{P_1(\Lambda)} \kappa.$$

Furthermore, $J^1 \subseteq I_{P_1(\Lambda)}(\kappa) \subseteq I_{P_1(\Lambda)}(\eta) = J^1$, so $\mathrm{ind}_{J^1}^{P_1(\Lambda)}\kappa$ and hence $\mathrm{ind}_J^{\mathrm{P}(\Lambda_E)\,\mathrm{P}_1(\Lambda)}\kappa$ are irreducible. □

A *β-extension* of η is an extension κ of η to J constructed in this way. We call two β-extensions which induce equivalent representations, as in Lemma 4.6, *compatible*. With the next lemma we show we can "go backwards" and from a β-extension defined in the minimal case we define two unique compatible β-extensions in the maximal case. In this way we get a triple of *compatible* β-extensions. Let $\mathrm{P}(\Lambda_E^r)$ be a maximal parahoric subgroup of G_E containing $\mathrm{P}(\Lambda_E)$ associated to the $\mathfrak{o}_E$-lattice sequence Λ_E^r in V. Let $\theta_r = \tau_{\Lambda,\Lambda^r,\beta}(\theta)$, η_r be the irreducible representation of $J^1(\beta,\Lambda^r)$ which contains θ_r, and κ be a β-extension of η.

Lemma 4.7. *There exists a unique β-extension κ_r of η_r which is compatible with κ.*

Proof. By [Stevens 2008, Lemma 4.3], there exists a representation $\hat{\kappa}$ of $P(\Lambda_E)J_r^1$ such that κ and $\hat{\kappa}$ induce equivalent representations of $\mathrm{P}(\Lambda_E)\,\mathrm{P}_1(\Lambda)$. Let κ' be a β-extension of η_r. The restriction to $\mathrm{P}(\Lambda_E)J_r^1$ of κ' and $\hat{\kappa}$ differ by a character χ of $B_r = \mathrm{P}(\Lambda_E)/\,\mathrm{P}_1(\Lambda_E^r)$ which is trivial on the unipotent part of B_r and intertwined by the nontrivial Weyl group element w. By the Bruhat decomposition, $\mathrm{M}_r = \mathrm{M}(\Lambda_E^r) = B_r \cup B_r w B_r$; hence χ is intertwined by the whole of M_r and extends to a character of M_r. Hence $\kappa_r = \kappa\otimes\chi^{-1}$ is a β-extension of η_r which is compatible with κ. By reduction modulo ℓ, as in the proof of Lemma 4.6, we have the corresponding statement in the ℓ-modular setting. □

4C4. *κ-induction and restriction.* Fix $[\Lambda, n, 0, \beta]$ a skew semisimple stratum in A, $\theta \in \mathscr{C}_-(\Lambda,\beta)$, η the unique Heisenberg representation containing θ and κ a β-extension of η.

Let σ be an R-representation of $\mathrm{M}(\Lambda_E)$ and, by abuse of notation, we also write σ for the inflation of σ to J obtained by defining J^1 to act trivially. The functor $\mathfrak{R}_R(\mathrm{M}(\Lambda_E)) \to \mathfrak{R}_R(J)$ given by $\sigma\mapsto\kappa\otimes\sigma$ identifies $\mathfrak{R}_R(\mathrm{M}(\Lambda_E))$ with the full subcategory of η-isotypic representations of J; see [Vignéras 2001, Definition 8.1]. Define *κ-induction*, $\mathrm{I}_\kappa : \mathfrak{R}_R(\mathrm{M}(\Lambda_E)) \to \mathfrak{R}_R(G)$, by

$$\mathrm{I}_\kappa(\sigma) = \mathrm{ind}_J^G(\kappa\otimes\sigma)$$

for σ an R-representation of $\mathrm{M}(\Lambda_E)$ and defined analogously on the morphisms of $\mathfrak{R}_R(\mathrm{M}(\Lambda_E))$. This functor has a right adjoint, $\mathrm{R}_\kappa : \mathfrak{R}_R(G) \to \mathfrak{R}_R(\mathrm{M}(\Lambda_E))$, called *$\kappa$-restriction*, defined by

$$\mathrm{R}_\kappa(\pi) = \mathrm{Hom}_{J^1}(\kappa,\pi),$$

where the action of $\mathrm{M}(\Lambda_E)$ is given by: for $f\in\mathrm{Hom}_{J^1}(\kappa,\pi)$ and $m\in\mathrm{M}(\Lambda_E)$, let $j\in J$ represent the coset $m\in J/J^1$, then $m\cdot f = \pi(j)\circ f\circ\kappa(j^{-1})$.

In the level zero case, we have $J = \mathrm{P}(\Lambda)$ and we can choose κ to be trivial, thus we have $\mathrm{I}_\kappa = \mathrm{I}_\Lambda$ and $\mathrm{R}_\kappa = \mathrm{R}_\Lambda$. Hence κ-restriction and induction generalise parahoric restriction and induction. Related to $[\Lambda, n, 0, \beta]$, we also have functors of parahoric induction $\mathrm{I}^E_\Lambda : \mathfrak{R}_R(M(\Lambda_E)) \to \mathfrak{R}_R(G_E)$ and parahoric restriction $\mathrm{R}^E_\Lambda : \mathfrak{R}_R(G_E) \to \mathfrak{R}_R(M(\Lambda_E))$ obtained by considering Λ as an $\mathfrak{o}_E$-lattice sequence.

Theorem 4.8 [Kurinczuk and Stevens 2014]. *Let $[\Lambda^i, n, 0, \beta]$, $i = 1, 2$, be skew semisimple strata. Let $\theta_1 \in \mathscr{C}_-(\Lambda^1, \beta)$ and $\theta_2 = \tau_{\Lambda^1,\Lambda^2,\beta}(\theta_1)$. For $i = 1, 2$, let η_i be a Heisenberg extension of θ_i, κ_i be compatible β-extensions of η_i, and let σ be an R-representation of $\mathrm{M}(\Lambda^1_E)$. Then*

$$\mathrm{R}_{\kappa_2} \circ \mathrm{I}_{\kappa_1}(\sigma) \simeq \mathrm{R}^E_{\Lambda^2} \circ \mathrm{I}^E_{\Lambda^1}(\sigma).$$

The proof of Theorem 4.8 in [Kurinczuk and Stevens 2014] follows from a combination of Mackey theory, isomorphisms defined as in [Bushnell and Kutzko 1993a, Proposition 5.3.2], and the computation of the intertwining spaces $\mathrm{I}_g(\eta_1, \eta_2)$ for $g \in G$, which are one-dimensional if $g \in G_E$ and zero otherwise.

Lemma 4.9. *In the setting of Lemma 4.6, let κ and κ_m be compatible β-extensions. Then, for all $\sigma \in \mathfrak{R}_R(\mathrm{M}(\Lambda_E))$, we have*

$$\mathrm{I}_\kappa(\sigma) \simeq \mathrm{ind}^G_{J^1_m\,\mathrm{P}(\Lambda_E)}(\kappa_m \otimes \sigma)$$

and, for all R-representations π of G, we have $\mathrm{R}_\kappa(\pi) \simeq \mathrm{Hom}_{J^1_m\,\mathrm{P}_1(\Lambda_E)}(\kappa_m, \pi)$.

Proof. By transitivity of induction and Lemma 4.6, $\mathrm{I}_\kappa(\sigma) \simeq \mathrm{ind}^G_{J^1_m\,\mathrm{P}(\Lambda_E)}(\kappa_m \otimes \sigma)$. By reciprocity, for π an R-representation of G, $\mathrm{R}_\kappa(\pi) \simeq \mathrm{Hom}_{J^1_m\,\mathrm{P}_1(\Lambda_E)}(\kappa_m, \pi)$. □

Define $\tilde{\kappa}$*-induction* $\mathrm{I}_{\tilde{\kappa}} : \mathfrak{R}_R(\mathrm{M}(\Lambda_E)) \to \mathfrak{R}_R(G)$ by $\mathrm{I}_{\tilde{\kappa}}(\sigma) = \mathrm{ind}^G_J(\tilde{\kappa} \otimes \sigma)$ for σ an R-representation of $\mathrm{M}(\Lambda_E)$. This functor has a right adjoint, $\tilde{\kappa}$*-restriction*, $\mathrm{R}_{\tilde{\kappa}} : \mathfrak{R}_R(G) \to \mathfrak{R}_R(\mathrm{M}(\Lambda_E))$ defined by $\mathrm{R}_{\tilde{\kappa}}(\pi) = \mathrm{Hom}_{J^1}(\tilde{\kappa}, \pi)$, where the action of $\mathrm{M}(\Lambda_E)$ on $\mathrm{R}_{\tilde{\kappa}}(\pi)$ is defined analogously to κ-restriction. In fact, $\tilde{\kappa}$ is a $-\beta$-extension for the semisimple character θ^{-1} for the semisimple stratum $[\Lambda, n, 0, -\beta]$.

Lemma 4.10. *Let π be an R-representation of G and σ be an irreducible representation of $\mathrm{M}(\Lambda_E)$. Then $(\mathrm{R}_\kappa(\pi))^\sim \simeq \mathrm{R}_{\tilde{\kappa}}(\tilde{\pi})$ and, if $\mathrm{I}_\kappa(\sigma)$ is irreducible, then $\mathrm{I}_\kappa(\sigma)^\sim \simeq \mathrm{I}_{\tilde{\kappa}}(\tilde{\sigma})$.*

Proof. We have an isomorphism of vector spaces

$$\mathrm{Hom}_{J^1}(\kappa, \pi)^\sim \simeq \mathrm{Hom}_{J^1}(\pi, \kappa) \simeq \mathrm{Hom}_{J^1}(\tilde{\kappa}, \tilde{\pi})$$

by [Henniart and Sécherre 2014, Proposition 2.6], and checking the action of J/J^1 we have $(\mathrm{R}_\kappa(\pi))^\sim \simeq \mathrm{R}_{\tilde{\kappa}}(\tilde{\pi})$. If $\mathrm{I}_\kappa(\sigma)$ is irreducible, then it is admissible and we have $\mathrm{I}_\kappa(\sigma)^\sim \simeq \mathrm{I}_{\tilde{\kappa}}(\tilde{\sigma})$ by [Vignéras 1996, I 8.4]. □

If $\mathrm{P}(\Lambda_E)$ is not maximal, let $\kappa_T = \mathrm{Res}^J_{T^0}(\kappa)$. Define $\mathrm{R}_{\kappa_T,\Lambda} : \mathfrak{R}_R(T) \to \mathfrak{R}_R(\overline{T})$ by $\mathrm{R}_{\kappa_T,\Lambda}(\pi) = \mathrm{Hom}_{T^1}(\kappa_T, \pi)$.

5. Exhaustion of cuspidal representations

In this section, we exhaust all irreducible cuspidal ℓ-modular representations of unramified $\mathrm{U}(2,1)(F/F_0)$. To do this we construct covers. The construction we give here is a vast simplification of that of [Stevens 2008], available as we are in the special case of unramified $\mathrm{U}(2,1)$. As the covers are constructed on compact open subgroups with pro-order not necessarily invertible in $\overline{\mathbb{F}}_\ell$ it is not clear whether or not the construction will follow *mutatis mutandis* the complex construction. In fact, for the relatively simple proof we give here for $\mathrm{U}(2,1)(F/F_0)$, it does. It is only when we come to computing the parameters of associated Hecke algebras later that we have to change the complex proof, and these changes occur in computing the parameters of Hecke algebras of certain associated finite reductive groups.

5A. ***Covers.*** In the ℓ-adic case our construction of G-covers is a special case of the general results of [Stevens 2008, Propositions 7.10 and 7.13]. Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum in A such that $\mathrm{P}(\Lambda_E)$ is not a maximal parahoric subgroup of G_E. For unramified $\mathrm{U}(2,1)(F/F_0)$, this implies $H^1(\Lambda,\beta)=J^1(\Lambda,\beta)$, which, in the notation of [ibid.], implies that $J=J_B$. Moreover, $\mathrm{P}(\Lambda_E)/\mathrm{P}_1(\Lambda_E)$ is abelian and isomorphic to $k_E^1\times k_E^\times$. Let $\theta\in\mathscr{C}_-(\Lambda,\beta)$; then $\eta=\theta$ is the unique Heisenberg representation containing θ. Let κ be a β-extension of η and $\sigma\in\mathrm{Irr}_R(J/J^1)$. Then $\lambda=\kappa\otimes\sigma$ is a character of J. Let $\kappa_T=\mathrm{Res}^J_{T^0}(\kappa)$ and set $\lambda_T=\kappa_T\otimes\sigma$. Let $J=(J\cap\overline{N})(J\cap T)(J\cap N)$ be the Iwahori decomposition of J with respect to B. We have $\lambda(j^-j_Tj^+)=\lambda_T(j_T)$ for $j^-\in(J\cap\overline{N})$, $j_T\in J\cap T$, and $j^+\in(J\cap N)$.

Lemma 5.1. *The element w_x intertwines λ if and only if w_y intertwines λ.*

Proof. Suppose $w_x\in I_G(\lambda)$. Then, as w_x normalises T^0, w_x normalises $\mathrm{Res}^J_{T^0}(\lambda)$. For all $t\in T^0$ we have $w_xtw_x=w_ytw_y$; hence w_y normalises $\mathrm{Res}^J_{T^0}(\lambda)$. Let $j\in J\cap w_yJw_y$ be such that $j=w_yj'w_y$ with $j'\in J$. Using the Iwahori decomposition of J we have $j=j_{\overline{N}}j_Tj_N$ and $j'=j'_Nj'_Tj'_{\overline{N}}$ with j_N, j'_N upper triangular unipotent, $j_{\overline{N}}$, $j'_{\overline{N}}$ lower triangular unipotent and j_T, j'_T in T. Thus

$$j=w_yj'w_y^{-1}=(w_yj'_Nw_y)(w_yj'_Tw_y)(w_yj'_{\overline{N}}w_y)$$

and, by unicity of the Iwahori decomposition, $j_{\overline{N}}=w_yj'_Nw_y$, $j_T=w_yj'_Tw_y$ and $j_N=w_yj'_{\overline{N}}w_y$. Therefore $w_y\in I_G(\lambda)$. □

Lemma 5.2. *Let $\lambda_T=\kappa_T\otimes\sigma$. Then (J,λ) is a G-cover of (T^0,λ_T).*

Proof. In the ℓ-modular case, it remains to show that there exists a strongly (B,J)-positive element z of the centre of T such that $Jz^{-1}J$ supports an invertible element of $\mathscr{H}(G,\lambda)$. Let $\zeta=w_xw_y$. Then ζ is strongly (B,J)-positive. For $g\in I_G(\lambda)$, because λ is a character, $I_g(\lambda)\simeq R$ and there is a unique function in

$f_g \in \mathscr{H}(G,\lambda)$ with support JgJ such that $f_g(g)=1$. We have $\zeta, \zeta^{-1} \in I_G(\lambda)$; hence $f_\zeta, f_{\zeta^{-1}} \in \mathscr{H}(G,\lambda)$.

Suppose that $w_x \notin I_G(\lambda)$, i.e., $I_G(\lambda) = JTJ$. As ζ is strongly positive,

$$J\zeta J\zeta^{-1}J = J\zeta J^-\zeta^{-1}J.$$

Suppose $y \in J\zeta J\zeta^{-1}J \cap JTJ$. Then we can write $y = j_1tj_2$ and $y = j_3\zeta j^-\zeta^{-1}j_4$ with $j_1, j_2, j_3, j_4 \in J$, $t \in T$ and $j^- \in J^-$. Thus, we can write

$$\zeta j^-\zeta^{-1} = jtj'$$

with $j, j' \in J$. By the Iwahori decomposition of J applied to the elements j and $(j')^{-1}$, we have

$$\zeta j^-\zeta^{-1} = j_{\overline{N}}j_Tj_Ntj'_Nj'_Tj'_{\overline{N}}$$

with $j_N, j'_N \in J\cap N$, $j_{\overline{N}}, j'_{\overline{N}} \in J\cap\overline{N}$ and $j_T, j'_T \in J\cap T$. Then $j_{\overline{N}}^{-1}\zeta j^-\zeta^{-1}(j'_{\overline{N}})^{-1} \in N$ and $j_{\overline{N}}^{-1}\zeta j^-\zeta^{-1}(j'_{\overline{N}})^{-1} = j_Tj_Ntj'_Nj'_T \in B$; hence $j_{\overline{N}}^{-1}\zeta j^-\zeta^{-1}(j'_{\overline{N}})^{-1} = 1$ and $\zeta j^-\zeta^{-1} \in J$. Therefore, $y \in J$ and $J\zeta J\zeta^{-1}J \cap JTJ = J$. Hence $f_\zeta \star f_{\zeta^{-1}}$ is supported on the single double coset J. We have $f_\zeta \star f_{\zeta^{-1}}(1_G) = q^4$. Hence $f_{\zeta^{-1}}$ is an invertible element of $\mathscr{H}(G,\lambda)$ supported on the single double coset $J\zeta^{-1}J$.

Now, suppose that $w_x \in I_G(\lambda)$, then $w_y \in I_G(\lambda)$ by Lemma 5.1. Hence $f_{w_x}, f_{w_y} \in \mathscr{H}(G,\lambda)$. Let $s \in \{x,y\}$. The maximal parahoric subgroup $P(\Lambda_s)$ of G contains J and w_s and $P(\Lambda_s)\cap G_E$ is a maximal parahoric subgroup of G_E. Moreover, $(J\cap G_E)/(P_1(\Lambda_s)\cap G_E)$ is a Borel subgroup of $(P(\Lambda_s)\cap G_E)/(P_1(\Lambda_s)\cap G_E)$. By [Stevens 2008, Lemma 5.12], $I_G(\eta) = JG_EJ$, thus the support of $\mathscr{H}(G,\lambda)$ is contained in JG_EJ. Hence,

$$\begin{aligned}\operatorname{supp}(f_{w_s}\star f_{w_s}) &\subseteq (Jw_sJw_sJ)\cap JG_EJ\\ &\subseteq P(\Lambda_s)\cap JG_EJ = J(P(\Lambda_s)\cap G_E)J\\ &= J((J\cap G_E)\cup(J\cap G_E)w_s(J\cap G_E))J,\end{aligned}$$

by the Bruhat decomposition of $(P(\Lambda_s)\cap G_E)/(P_1(\Lambda_s)\cap G_E)$, which is a finite reductive group. Thus, $\operatorname{supp}(f_{w_s}\star f_{w_s}) \subseteq J\cup Jw_sJ$. We have that $f_{w_s}\star f_{w_s}(1_G) = [J : J\cap w_sJw_s]$ is a power of q. Let $a_s = f_{w_s}\star f_{w_s}(1_G)$ and $b_s = f_{w_s}\star f_{w_s}(w_s)$. Therefore, for $s\in\{x,y\}$, f_{w_s} is an invertible element of $\mathscr{H}(G,\lambda)$ with inverse $(1/a_s)(f_{w_s} - b_sf_1)$. By [Stevens 2008, Lemma 7.11], we have $(J\cap N)^{w_x} \subseteq J\cap N$ and $(J\cap\overline{N})^{w_y} \subseteq J\cap\overline{N}$. By the Iwahori decomposition of J,

$$\begin{aligned}Jw_yJw_xJ &= J(w_y(J\cap\overline{N})w_y)w_yw_x(w_x(J\cap T)w_x)(w_x(J\cap N)w_x)J\\ &= J(J\cap\overline{N})^{w_y}w_yw_x(J\cap T)^{w_x}(J\cap N)^{w_x}J\\ &\subseteq J(J\cap\overline{N})w_yw_x(J\cap T)(J\cap N)J = Jw_yw_xJ.\end{aligned}$$

Moreover, we clearly have $Jw_yw_xJ \subseteq Jw_yJw_xJ$. Hence $Jw_yw_xJ = Jw_yJw_xJ$.

Therefore, $f_{w_y} \star f_{w_x}$ is an invertible element of $\mathscr{H}(G, \lambda)$ supported on the single double coset $J\zeta^{-1}J$. □

5B. *Cuspidal representations.* The following theorem addresses the construction of all irreducible cuspidal ℓ-modular and ℓ-adic representations of G.

Theorem 5.3. (1) *Let* $[\Lambda, n, 0, \beta]$ *be a skew semisimple stratum in* A, $\theta \in \mathscr{C}_-(\beta, \Lambda)$, η *the unique Heisenberg representation containing* θ, κ *a* β*-extension of* η *and* σ *an irreducible cuspidal representation of* $\mathrm{M}(\Lambda_E)$. *Then* $\mathrm{I}_\kappa(\sigma)$ *is quasiprojective. Furthermore, if* $\mathrm{P}(\Lambda_E)$ *is a maximal parahoric subgroup of* G_E, *then* $\mathrm{I}_\kappa(\sigma)$ *is irreducible and cuspidal.*

(2) *Let* π *be an irreducible cuspidal representation of* G. *Then there exist a skew semisimple stratum* $[\Lambda, n, 0, \beta]$ *with* $\mathrm{P}(\Lambda_E)$ *a maximal parahoric subgroup of* G_E, $\theta \in \mathscr{C}_-(\beta, \Lambda)$, *a* β*-extension* κ *of the unique Heisenberg representation* η *which contains* θ *and an irreducible cuspidal representation* σ *of* $\mathrm{M}(\Lambda_E)$ *such that* $\pi \simeq \mathrm{I}_\kappa(\sigma)$.

Proof. (1) Quasiprojectivity follows *mutatis mutandis* the proof given in [Vignéras 2001, Proposition 6.1]. So suppose $\mathrm{P}(\Lambda_E)$ is a maximal parahoric subgroup of G_E. By Theorem 4.8 and Lemma 4.4 we have

$$\mathrm{R}_\kappa \circ \mathrm{I}_\kappa(\sigma) \simeq \mathrm{R}^E_\Lambda \circ \mathrm{I}^E_\Lambda(\sigma) \simeq \sigma.$$

The proof of irreducibility follows *mutatis mutandis* the proof given in [Vignéras 2001, Proposition 7.1].

(2) By Theorem 4.5, π contains a skew semisimple stratum $[\Lambda, n, 0, \beta]$. Suppose $\theta \in \mathscr{C}_-(\Lambda, \beta)$ is a skew semisimple character which π contains. Let κ be a β-extension of the unique Heisenberg representation η which contains θ. Then π contains $\kappa \otimes \sigma$ for some $\sigma \in \mathrm{Irr}_R(\mathrm{M}(\Lambda_E))$. We show that we may assume that σ is cuspidal. If $\mathrm{P}(\Lambda_E)$ is not maximal then σ is cuspidal, so we can suppose that $\mathrm{P}(\Lambda_E)$ is maximal. Let $B(\Lambda_E)$ be the standard Borel subgroup of $\mathrm{M}(\Lambda_E)$ and $\mathrm{P}(\Lambda'_E)$ the preimage of $B(\Lambda_E)$ under the projection map. Suppose that $r^{\mathrm{M}(\Lambda_E)}_{B(\Lambda_E)}(\sigma) \neq 0$. Then, as π contains σ, $r^{\mathrm{M}(\Lambda_E)}_{B(\Lambda_E)}(\mathrm{R}_\kappa(\pi)) \neq 0$. We have

$$r^{\mathrm{M}(\Lambda_E)}_{B(\Lambda_E)}(\mathrm{R}_\kappa(\pi)) \simeq \mathrm{Hom}_{J^1}(\kappa, \pi)^{\mathrm{P}_1(\Lambda'_E)J^1/J^1} \simeq \mathrm{Hom}_{\mathrm{P}_1(\Lambda'_E)J^1}(\kappa, \pi),$$

which, by Lemma 4.9, implies that $\mathrm{R}_{\kappa', \Lambda'}(\pi) \neq 0$ where κ' is the unique β-extension containing $\tau_{\Lambda, \Lambda', \beta}(\theta)$ compatible with κ. Hence π contains a skew semisimple stratum $[\Lambda', n, 0, \beta]$ such that $\mathrm{P}(\Lambda'_E)$ is not maximal and thus contains $\kappa' \otimes \sigma'$, with σ' a cuspidal representation of $\mathrm{M}(\Lambda'_E)$. By Theorem 4.2 and Lemma 5.2, if π contains a skew semisimple stratum $[\Lambda, n, 0, \beta]$ such that $\mathrm{P}(\Lambda_E)$ is not a maximal parahoric subgroup of G_E, then π is not cuspidal. Therefore $\mathrm{P}(\Lambda_E)$ is maximal and σ is cuspidal. □

For level zero representations we can refine the exhaustive list of irreducible cuspidal representations given in Theorem 5.3 into a classification.

Theorem 5.4. *For $i = 1, 2$, let $\mathrm{P}(\Lambda_i)$ be a maximal parahoric subgroup of G and σ_i an irreducible cuspidal representation of $\mathrm{M}(\Lambda_i)$. If $\mathrm{Hom}_G(\mathrm{I}_{\Lambda_1}(\sigma_1), \mathrm{I}_{\Lambda_2}(\sigma_2)) \neq 0$ then $(\mathrm{P}(\Lambda_1), \sigma_1)$ and $(\mathrm{P}(\Lambda_2), \sigma_2)$ are conjugate.*

Proof. By reciprocity and Lemma 4.3,

$$\mathrm{Hom}_G(\mathrm{I}_{\Lambda_1}(\sigma_1), \mathrm{I}_{\Lambda_2}(\sigma_2)) \simeq \bigoplus_{n\in D_{1,2}} \mathrm{Hom}_{\mathrm{M}(\Lambda_1)}\big(\sigma_1, i^{\mathrm{M}(\Lambda_1)}_{P_{\Lambda_1,n\Lambda_2}}\big(r^{\mathrm{M}(\Lambda_2)}_{P_{\Lambda_2,n^{-1}\Lambda_1}}(\sigma_2)\big)^n\big).$$

Hence

$$\mathrm{Hom}_G(\mathrm{I}_{\Lambda_1}(\sigma_1), \mathrm{I}_{\Lambda_2}(\sigma_2)) \neq 0$$

if and only if there exists $n \in D_{1,2}$ such that

$$\mathrm{Hom}_{\mathrm{M}(\Lambda_1)}\big(\sigma_1, i^{\mathrm{M}(\Lambda_1)}_{P_{\Lambda_1,n\Lambda_2}}\big(r^{\mathrm{M}(\Lambda_2)}_{P_{\Lambda_2,n^{-1}\Lambda_1}}(\sigma_2)\big)^n\big) \neq 0.$$

Assume there exists such an element n. By cuspidality of σ_2, $P_{\Lambda_2,n^{-1}\Lambda_1} = \mathrm{M}(\Lambda_2)$, so $\mathrm{P}_1(\Lambda_2)(\mathrm{P}(\Lambda_2)\cap\mathrm{P}(n^{-1}\Lambda_1))/\mathrm{P}_1(\Lambda_2) = M(\Lambda_2)$. By cuspidality of σ_1, $P_{\Lambda_1,n\Lambda_2} = \mathrm{M}(\Lambda_1)$, so $\mathrm{P}_1(\Lambda_1)(\mathrm{P}(\Lambda_1)\cap\mathrm{P}(n\Lambda_2))/\mathrm{P}_1(\Lambda_1) = \mathrm{M}(\Lambda_1)$. If $\mathrm{P}(\Lambda_1)$ and $\mathrm{P}(\Lambda_2)$ are not conjugate then for all $g \in G$, in particular $n \in D_{1,2}$, the group $\mathrm{P}(\Lambda_1)\cap\mathrm{P}(g\Lambda_2)$ must stabilise an edge in the building and hence is not maximal. Thus it cannot surject onto either $\mathrm{M}(\Lambda_1)$ or $\mathrm{M}(\Lambda_2)$. Hence there exists $n \in D_{1,2}$ such that $\mathrm{P}(\Lambda_1) = \mathrm{P}(n\Lambda_2)$ and $\mathrm{Hom}_{\mathrm{M}(\Lambda_1)}(\sigma_1, \sigma_2^n) \neq \{0\}$; i.e., $(\mathrm{P}(\Lambda_1), \sigma_1)$ and $(\mathrm{P}(\Lambda_2), \sigma_2)$ are conjugate. □

Remark 5.5. Let $\ell \mid (q^2 - q + 1)$. The irreducible cuspidal ℓ-modular representations $\mathrm{I}_{\Lambda_x}(\bar{\tau}^+(\bar{\chi}))$ do not lift. A lift must necessarily be cuspidal as the Jacquet functor commutes with reduction modulo ℓ. However, by Theorem 5.3, all ℓ-adic level zero irreducible cuspidal representations are of the form $\mathrm{I}_{\Lambda_x}(\sigma_x)$ or $\mathrm{I}_{\Lambda_y}(\sigma_y)$ with σ_x (resp. σ_y) an irreducible cuspidal ℓ-adic representation of $\mathrm{M}(\Lambda_x)$ (resp. $\mathrm{M}(\Lambda_y)$). Furthermore, $r_\ell(\mathrm{I}_{\Lambda_w}(\sigma_w)) = \mathrm{I}_{\Lambda_w}(r_\ell(\sigma_w))$ as compact induction commutes with reduction modulo ℓ, for $w \in \{x, y\}$. Hence, by Section 3B2, $\mathrm{I}_{\Lambda_x}(\bar{\tau}^+(\bar{\chi}))$ does not lift, but does appear in the reduction modulo ℓ of $\mathrm{I}_{\Lambda_x}(\tau(\psi))$, where $r_\ell(\mathrm{I}_{\Lambda_x}(\tau(\psi)) = \mathrm{I}_{\Lambda_x}(\bar{\nu}(\bar{\chi})) \oplus \mathrm{I}_{\Lambda_x}(\bar{\tau}^+(\bar{\chi}))$.

6. Parabolically induced representations

Let ω_{F/F_0} be the unique character of $F_0^\times$ associated to F/F_0 by local class field theory. That is, ω_{F/F_0} is defined by $\omega_{F/F_0}|_{\mathfrak{o}_{F_0}^\times} = 1$ and $\omega_{F/F_0}(\varpi_F) = -1$. All extensions of ω_{F/F_0} to $F^\times$ take values in $\overline{\mathbb{Z}}_\ell^\times$, hence are integral. Let χ_1 be a character of $F^\times$ and χ_2 be a character of F^1. Let χ be the character of T defined by

$$\chi(\mathrm{diag}(x, y, \bar{x}^{-1})) = \chi_1(x)\chi_2(x\bar{x}^{-1}y),$$

which is well-defined because $x \mapsto x\bar{x}^{-1}$ is a surjective map $F^\times \to F^1$. Every character of T appears in this way: we can recover χ_1 and χ_2 from χ by

$$\chi_1(x) = \chi(\operatorname{diag}(x, \bar{x}/x, \bar{x}^{-1})), \quad \chi_2(y) = \chi(\operatorname{diag}(1, y, 1)).$$

The character χ_2 factors through the determinant and

$$i_B^G(\chi) \simeq i_B^G(\chi_1)(\chi_2 \circ \det),$$

where χ_1 is the character $\chi_1(\operatorname{diag}(x, y, \bar{x}^{-1})) = \chi_1(x)$ of T. Hence the reducibility of $i_B^G(\chi)$ is completely determined by that of $i_B^G(\chi_1)$. The character χ is not regular if $\chi_1(x) = \chi_1(\bar{x})^{-1}$, which occurs if and only if χ_1 is an extension of 1 or ω_{F/F_0} to $F^\times$. An irreducible character χ has level zero if and only if both χ_1 and χ_2 have level zero.

Let ν be the character of T given by $\nu(\operatorname{diag}(x, y, \bar{x}^{-1})) = |x|_F$, i.e., the character with $\chi_1(x) = |x|_F$ and χ_2 trivial, where we normalise $|\cdot|_F$ so that $|\varpi|_F = 1/q$. The modulus character δ_B of B is given on T by $\delta_B = \nu^{-4}$. Because the image of ν is contained in $\overline{\mathbb{Z}}_\ell^\times$, ν and δ_B are integral. If $q^4 \equiv 1 \bmod \ell$ then δ_B is trivial.

6A. *Hecke Algebras.* To find the characters χ such that the induced representation $i_B^G(\chi)$ is reducible we study the algebras $\mathcal{H}(G, \lambda)$.

Theorem 6.1. *Suppose λ_T is a character of T^0. Let (J, λ) be a G-cover of (T^0, λ_T) as constructed in Lemma 5.2.*

(1) *If λ_T is regular then $\mathcal{H}(G, \lambda) \simeq R[X^{\pm 1}]$.*

(2) *If λ_T is not regular then $\mathcal{H}(G, \lambda)$ is a two-dimensional algebra generated as an R-algebra by f_{w_x} and f_{w_y} and the relations*

$$f_{w_x} \star f_{w_x} = (q^a - 1) f_{w_x} + q^a,$$
$$f_{w_y} \star f_{w_y} = (q - 1) f_{w_y} + q,$$

where $a = 3$ and $f_{w_x}(1) = f_{w_y}(1) = 1$ if λ_T is trivial on T^1 and factors through the determinant, and $a = 1$, $f_{w_x}(1) = 1/q$ and $f_{w_y}(1) = 1$ if not.

Proof. If $g \in I_G(\lambda)$ then $\mathrm{I}_g(\lambda) \simeq R$, because χ is a character. For $g \in I_G(\lambda)$, $r \in R$, we let $f_{g,r}$ denote the unique function supported on JgJ with $f_{g,r}(g) = r$. If λ_T is regular then the support of $\mathcal{H}(G, \lambda)$ is $JTJ = \bigcup_{n \in \mathbb{Z}} J\zeta^n J$ and, since each intertwining space is one-dimensional and $f_{\zeta^n,1}$ has support $J\zeta^n J$, we have an isomorphism $\mathcal{H}(G, \lambda) \simeq R[X^{\pm 1}]$ defined by $f_{\zeta,1} \mapsto X$.

Suppose $w_x \in I_G(\lambda)$. By Lemma 5.1, w_x intertwines λ if and only if w_y intertwines λ. The support of the Hecke algebra is contained in the intertwining of $\eta = \operatorname{Res}^J_{J^1}(\kappa)$, which is JG_EJ. By the semisimple intersection property [Stevens 2008, Lemma 2.6] and the Bruhat decomposition we have $JG_EJ = \bigcup_{w \in \widetilde{W}} JwJ$. As in the proof of Lemma 5.2 we have $Jw_xJw_yJ = Jw_xw_yJ$ and, similarly,

$Jw_yJw_xJ = Jw_yw_xJ$. Hence, as the intertwining spaces are one-dimensional, the support of $f_{w_x} \star f_{w_y} \star f_{w_x} \star \cdots \star f_{w_i}$ is $Jw_xw_yw_x \cdots w_iJ$. Thus, as $\widetilde{W}$ is an infinite dihedral group generated by w_x and w_y, $\mathcal{H}(G, \lambda)$ is generated by $f_{w_x,1}$ and $f_{w_y,1}$ and the quadratic relations $f_{w_x,1} \star f_{w_x,1}$ and $f_{w_y,1} \star f_{w_y,1}$. Let Λ^x and Λ^y be $\mathfrak{o}_E$-lattice sequences such that the parahoric subgroups $\mathrm{P}(\Lambda_E^x) = \mathrm{P}(\Lambda_E) \cup \mathrm{P}(\Lambda_E) w_x \mathrm{P}(\Lambda_E)$ and $\mathrm{P}(\Lambda_E^y) = \mathrm{P}(\Lambda_E) \cup \mathrm{P}(\Lambda_E) w_y \mathrm{P}(\Lambda_E)$. The parahoric subgroups $\mathrm{P}(\Lambda_E^x)$ and $\mathrm{P}(\Lambda_E^y)$ are nonconjugate, maximal and contain $\mathrm{P}(\Lambda_E)$. Let κ_x and κ_y be the β-extensions, compatible with κ, defined by Lemma 4.6 related to the skew semisimple strata $[\Lambda^x, n, 0, \beta]$ and $[\Lambda^y, n, 0, \beta]$.

For $z \in \{x, y\}$, let $\hat{\kappa}_z = \mathrm{Res}^J_{J^1(\beta,\Lambda^i)\mathrm{P}(\Lambda_E)}(\kappa_z)$. We have a support-preserving isomorphism

$$\mathcal{H}(G, \kappa \otimes \sigma) \simeq \mathcal{H}(G, \hat{\kappa}_z \otimes \sigma)$$

by Lemma 4.6 and transitivity of compact induction. We have a support-preserving injection of algebras

$$\mathcal{H}(\mathrm{P}(\Lambda_E^z), \sigma) \to \mathcal{H}(\mathrm{P}(\Lambda^z), \hat{\kappa}_z \otimes \sigma)$$

defined by $\Phi \mapsto \hat{\kappa}_z \otimes \Phi$, where σ is considered as a character of $\mathrm{P}(\Lambda_E)$ trivial on $\mathrm{P}_1(\Lambda_E)$.

Let B_z be the standard Borel subgroup of $\mathrm{M}(\Lambda_E^z)$. In the ℓ-adic case, by [Howlett and Lehrer 1980, Theorem 4.14], if $i_{B_z}^{\mathrm{M}(\Lambda_E^z)}(\bar{\sigma}) = \rho_1^z \oplus \rho_2^z$ with $\dim(\rho_1^z) \geqslant \dim(\rho_2^z)$ then $\mathcal{H}(\mathrm{M}(\Lambda_E^z), \bar{\sigma})$ is generated by T_w^z, which is supported on the double coset $B_z w_x B_z$ and satisfies the quadratic relation

$$T_w^z \star T_w^z = (d_z - 1) T_w^z + d_z T_1^z,$$

where $d_z = \dim(\rho_1^z)/\dim(\rho_2^z)$ and T_1^z is the identity of $\mathcal{H}(\mathrm{M}(\Lambda_E^z), \bar{\sigma})$. By Section 3, $d_y = q$ and

$$d_x = \begin{cases} q^3 & \text{if } \lambda_T \text{ is trivial on } T^1 \text{ and factors through the determinant,} \\ q & \text{otherwise.} \end{cases}$$

In the ℓ-modular case, we choose a lift $\hat{\bar{\sigma}}$ of $\bar{\sigma}$ such that $\hat{\bar{\sigma}}^{w_x} = \hat{\bar{\sigma}}$. Let L be a lattice in $\hat{\bar{\sigma}}$. Recall that $\hat{\bar{\sigma}}$ is called a reduction stable of $\bar{\sigma}$ if $\mathcal{H}(\mathrm{M}(\Lambda_E^z), \bar{\sigma}) = \overline{\mathbb{Z}}_\ell \otimes_{\mathbb{F}_\ell} \mathcal{H}(\mathrm{M}(\Lambda_E^z), L)$ and $\mathcal{H}(\mathrm{M}(\Lambda_E^z), \hat{\bar{\sigma}}) = \overline{\mathbb{Q}}_\ell \otimes_{\mathbb{F}_\ell} \mathcal{H}(\mathrm{M}(\Lambda_E^z), L)$. A basis of $\mathcal{H}(\mathrm{M}(\Lambda_E^z), \hat{\bar{\sigma}})$ is called reduction stable if it is a basis of $\mathcal{H}(\mathrm{M}(\Lambda_E^z), L)$ and $\hat{\bar{\sigma}}$ is reduction stable. By [Geck et al. 1996, Section 3.1], $\hat{\bar{\sigma}}$ is reduction stable and a basis of $\mathcal{H}(\mathrm{M}(\Lambda_E^z), \hat{\bar{\sigma}})$ is reduction stable. Hence we obtain a basis of $\mathcal{H}(\mathrm{M}(\Lambda_E^z), \bar{\sigma})$ satisfying the quadratic relations required by reduction modulo ℓ.

By inflation, T_w^z determines an element $f_{w_z,r_z} \in \mathcal{H}(\mathrm{P}(\Lambda_E^z), \sigma)$ supported on Jw_zJ. Furthermore, $f_{w_x,1} \star f_{w_x,1}(1_G) = [J : J \cap w_x J w_x] = q^3$ and $f_{w_y,1} \star f_{w_y,1}(1_G) =$

$[J : J \cap w_y J w_y] = q$ in all cases; hence $r_x = r_y = 1$ if λ_T is trivial on T^1 and factors through the determinant, and $r_x = 1/q$ and $r_y = 1$ otherwise. $\square$

6B. ***Reducibility points.*** Suppose $i_B^G(\chi)$ is reducible and let $\lambda_T = \mathrm{Res}_{T^0}^T(\chi)$. By Theorem 6.1, λ_T is not regular. Let (J, λ) be a G-cover of (T^0, λ_T) as constructed in Lemma 5.2 with $\lambda = \kappa \otimes \sigma$. If π is an irreducible quotient of $\mathrm{I}_\kappa(\sigma)$ and an irreducible quotient of $i_B^G(\chi)$ then, by exactness of the Jacquet functor, $r_B^G(\pi)$ is one-dimensional. Hence, as $(j_B)^*(M_\lambda(\pi)) \simeq M_{\lambda_T}(r_B^G(\pi))$ by Theorem 4.2, π must correspond to a character of $\mathscr{H}(G, \lambda)$ under the bijection of Theorem 4.1. The characters of $\mathscr{H}(G, \lambda)$ are determined by their values on the generators $f_{w_x,b}$ and $f_{w_y,1}$, where we let $b = 1$ if λ_T is trivial on T^1 and factors through the determinant and $b = 1/q$ otherwise. Let a be given by Theorem 6.1. The characters of $\mathscr{H}(G, \lambda)$ are summarised as follows:

Character of $\mathscr{H}_R(G, \lambda)$	Value on $f_{w_x,b}$	Value on $f_{w_y,1}$
Ξ_{sgn}	-1	-1
Ξ_{ind}	q^a	q
Ξ_1	q^a	-1
Ξ_2	-1	q

If $q^a \neq -1 \bmod \ell$, these characters are distinct; if $q^a = -1 \bmod \ell$ but $q \neq -1 \bmod \ell$, there are two characters, $\Xi_{\mathrm{sgn}} = \Xi_1$ and $\Xi_{\mathrm{ind}} = \Xi_2$; if $q = -1 \bmod \ell$, there is a unique character $\Xi_{\mathrm{sgn}} = \Xi_1 = \Xi_{\mathrm{ind}} = \Xi_2$.

To calculate the values of χ where this reducibility occurs we study the restriction of the characters of $\mathscr{H}_R(G, \lambda)$ to $\mathscr{H}(T, \lambda_T)$ under $(j_B)^*$. The injection $j_B : \mathscr{H}(T, \lambda_T) \to \mathscr{H}(G, \lambda)$ is induced by taking the unique function $f_{\zeta,1}^T \in \mathscr{H}(T, \lambda_T)$ with support $J_T \zeta$ and $f_{\zeta,1}^T(\zeta) = 1$ to $f_{\zeta,1}$, the unique function in $\mathscr{H}(G, \lambda)$ with support $J \zeta J$ and $f_{\zeta,1}(\zeta) = 1$. Moreover, we know that $f_{\zeta,1} = \varepsilon f_{w_x,1} \star f_{w_y,1}$ for some scalar $\varepsilon \in R$. It is determining the sign of this scalar which requires work. The normalised restriction map $(j_B)^*$ is then induced by this injection and twisting by ν^{-2}. To find ε we compare the value of the characters of $\mathscr{H}(G, \lambda)$ on $f_{w_x,1} \star f_{w_y,1}$ twisted by $\varepsilon \nu^{-2}(\zeta)$ to known reducibility points.

Character χ of $\mathscr{H}_R(G, \lambda)$	$\varepsilon \nu^{-2}(\zeta) \chi(f_{w_x,1} \star f_{w_y,1})$ $a = 3,\ b = 1$	$\varepsilon \nu^{-2}(\zeta) \chi(f_{w_x,1} \star f_{w_y,1})$ $a = 1,\ b = 1/q$
Ξ_{sgn}	$q^{-2}\varepsilon$	$q^{-1}\varepsilon$
Ξ_{ind}	$q^2\varepsilon$	$q\varepsilon$
Ξ_1	$-q\varepsilon$	$-\varepsilon$
Ξ_2	$-q^{-1}\varepsilon$	$-\varepsilon$

First consider the case when $a = 3$ and $b = 1/q$. As the trivial representation is an irreducible subquotient of $i_B^G(\nu^{\pm 2})$, the induced representations are reducible. Thus $\nu^{\pm 2}(\zeta) = q^{\pm 2} \in \{q^{-2}\varepsilon, q^2\varepsilon, -q\varepsilon, -q^{-1}\varepsilon\}$ and this multiset of values of the characters must be $\{q^{-2}, q^2, -q, -q^{-1}\}$. Moreover, by compatibility with the ℓ-adic case by reduction modulo ℓ, we must have $\varepsilon = 1$ with $\nu^{\pm 2}$ corresponding to Ξ_{sgn} and Ξ_{ind}. The other reducibility points, corresponding to the characters Ξ_1 and Ξ_2 of $\mathscr{H}(G, \lambda)$, are the characters χ of T of the form $\chi = \eta\nu^{\pm 1}$, where η is any extension of ω_{F/F_0} to $F^\times$ which is trivial on F^1 such that $\chi\mid_{T^0}$ factors through the determinant.

Now consider the case when $a = 1$ and $b = 1/q$. Using an alternative method, Keys [1984] computed the ℓ-adic reducibility points. Comparing the value of a pair of these on ζ — see [Keys 1984, Section 7] — with our values in the table we must have $\varepsilon = -1$ in all other cases. This gives reducibility points the characters χ of T of the form $\chi = \eta\nu^{\pm 1}$, where η is any extension of ω_{F/F_0} to $F^\times$ not trivial on F^1, corresponding to the characters Ξ_{sgn} and Ξ_{ind} of $\mathscr{H}(G, \lambda)$, and the characters χ of T of the form χ_1 is nontrivial, but $\chi_1\mid_{F_0^\times}$ is trivial, corresponding to the characters Ξ_1 and Ξ_2 of $\mathscr{H}(G, \lambda)$.

Theorem 6.2. *Let χ be an irreducible ℓ-modular character of T. Then $i_B^G(\chi)$ is reducible exactly in the following cases*:

(1) $\chi = \nu^{\pm 2}$;

(2) $\chi = \eta\nu^{\pm 1}$, *where η is any extension of ω_{F/F_0} to $F^\times$*;

(3) χ_1 *is nontrivial, but $\chi_1\mid_{F_0^\times}$ is trivial.*

6C. ***Parahoric restriction and parabolic induction.*** As the parabolic functors respect the decomposition of $\mathfrak{R}_R(G)$ by level, by [Vignéras 1996, II 5.12], if χ is a level zero character of T (i.e., a character of T trivial on T^1) then all irreducible subquotients of $i_B^G(\chi)$ have level zero.

Lemma 6.3. *Let $w \in \{x, y\}$ and let χ be a level zero character of T. Then* $\mathrm{R}_{\Lambda_w}(i_B^G(\chi)) \simeq i_{B_w}^{\mathrm{M}(\Lambda_w)}(\chi)$.

Proof. The proof follows by Mackey theory, as the maximal parahoric subgroups of G satisfy the Iwasawa decomposition. □

Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum in A. Let $\theta \in \mathscr{C}_-(\Lambda, \beta)$ and κ be a β-extension of the unique Heisenberg representation containing θ. Let χ be an irreducible ℓ-modular character of T which contains the R-type $(J_T, \kappa_T \otimes \sigma)$. Furthermore, suppose that $(J, \kappa \otimes \sigma)$ is a G-cover of $(J_T, \kappa_T \otimes \sigma)$ relative to B, as in Lemma 5.2. Let Λ^m be an $\mathfrak{o}_E$-lattice sequence in V such that $\mathrm{P}(\Lambda_E^m)$ is maximal and $\mathrm{P}(\Lambda_E) \subset \mathrm{P}(\Lambda_E^m)$. Let $\theta_m = \tau_{\Lambda, \Lambda^m, \beta}(\theta)$ and κ_m be the unique β-extension of the unique Heisenberg representation containing θ_m which is compatible with κ, as in Lemma 4.6. Let $B(\Lambda_E^m)$ be the Borel subgroup of $\mathrm{M}(\Lambda_E^m)$ whose preimage under the projection map $\mathrm{P}(\Lambda_E^m) \to \mathrm{M}(\Lambda_E^m)$ is equal to J. Suppose $B(\Lambda_E^m)$ has Levi decomposition $B(\Lambda_E^m) = T(\Lambda_E^m) \ltimes N(\Lambda_E^m)$.

The next theorem is a generalisation of a weakening of Lemma 6.3; precisely, it generalises the isomorphism Lemma 6.3 induces in the Grothendieck group $\mathfrak{Gr}_R(\mathrm{M}(\Lambda_w))$.

Theorem 6.4. *With the notation as above, there is an isomorphism*

$$\big[\mathrm{R}_{\kappa_m}(i_B^G(\chi))\big] \simeq \big[i_{B(\Lambda_E^m)}^{M(\Lambda_E^m)}(\mathrm{R}_{\kappa_T}(\chi))\big].$$

Proof. We prove the corresponding result in the ℓ-adic case first and deduce the ℓ-modular result by reduction modulo ℓ. The proof in the ℓ-adic case follows a similar argument made for $\mathrm{GL}_n(F)$ in [Schneider and Zink 1999]. Let $\Omega_T = [T, \rho]_T$ and $\Omega = [T, \rho]_G$ be inertial equivalence classes. Let $\mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(\Omega)$ denote the full subcategory of $\mathrm{R}_{\overline{\mathbb{Q}}_\ell}(G)$ of representations all of whose irreducible subquotients have inertial support in Ω, and $\mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(\Omega_T)$ denote the full subcategory of $\mathrm{R}_{\overline{\mathbb{Q}}_\ell}(T)$ of representations all of whose irreducible subquotients have inertial support in Ω_T. Let ω denote the $\mathrm{M}(\Lambda_E^m)$-conjugacy class of σ and ω_T the $T(\Lambda_E^m)$-conjugacy class of σ. Let $\mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(\omega)$ be the full subcategory of $\mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(\mathrm{M}(\Lambda_E^m))$ of representations all of whose irreducible subquotients have supercuspidal support in ω and $\mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(\omega_T)$ be the full subcategory of $\mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(T(\Lambda_E^m))$ of representations all of whose irreducible subquotients have lie in ω_T. Let $M_\omega : \mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(\omega) \to \mathcal{M}(\mathrm{M}(\Lambda_E^m), \sigma)$ be defined by $\rho \mapsto \mathrm{Hom}_{B(\Lambda_E^m)}(\sigma, \rho)$ for $\rho \in \mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(\omega)$. Similarly, let $M_{\omega_T} : \mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(\omega_T) \to \mathcal{M}(T(\Lambda_E^m), \sigma)$ be defined by $\rho \mapsto \mathrm{Hom}_{T(\Lambda_E^m)}(\sigma, \rho)$ for $\rho \in \mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(\omega_T)$. We prove that the following diagram commutes.

$$\begin{tikzcd}
 & \mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(\omega) \arrow[rr, "\simeq", "M_\omega"'] & & \mathcal{M}(\mathrm{M}(\Lambda_E^m), \sigma) & & \mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(\omega) \arrow[ll, "\simeq"', "M_\omega"] & \\
\mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(\Omega) \arrow[ur, "\mathrm{R}_{\kappa_m}"] \arrow[rr, "\simeq", "M_{\kappa\otimes\sigma}"'] & & \mathcal{M}(G, \kappa\otimes\sigma) \arrow[ur, "\mathrm{Res}"'] & & \mathcal{M}(T(\Lambda_E^m), \sigma) \arrow[ul, "(j_{B(\Lambda_E^m)})_*"'] & & \mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(\omega_T) \arrow[ll, "\simeq"', "M_{\omega_T}"] \arrow[ul, "i_{B(\Lambda_E^m)}^{\mathrm{M}(\Lambda_E^m)}"'] \\
 & \mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(\Omega_T) \arrow[ul, "i_B^G"] \arrow[rr, "\simeq", "M_{\kappa_T\otimes\sigma}"'] & & \mathcal{M}(T, \kappa_T\otimes\sigma) \arrow[ul, "(j_B)_*"'] \arrow[ur, "\mathrm{Res}"] & & \mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(\Omega_T) \arrow[ll, "\simeq"', "M_{\kappa_T\otimes\sigma}"] \arrow[ur, "\mathrm{R}_{\kappa_T}"'] &
\end{tikzcd}$$

We have $M_\omega \circ i_{B(\Lambda_E^m)}^{\mathrm{M}(\Lambda_E^m)} \simeq (j_{B(\Lambda_E^m)})_* \circ M_{\omega_T}$ and $M_{\kappa\otimes\sigma} \circ i_B^G \simeq (j_B)_* \circ M_{\kappa_T\otimes\sigma}$ by [Bushnell and Kutzko 1998, Corollary 8.4], and $M_{\kappa\otimes\sigma}$ is an equivalences of categories by [Bushnell and Kutzko 1998, Theorems 4.3 and 8.3].

We have support-preserving injections $\alpha_1 : \mathcal{H}(\mathrm{M}(\Lambda_E^m), \sigma) \to \mathcal{H}(G, \kappa\otimes\sigma)$ and $\alpha_2 : \mathcal{H}(T(\Lambda_E^m), \sigma) \to \mathcal{H}(T, \kappa_T \otimes \sigma)$, as in the proof of Theorem 6.1, hence restriction functors $\mathcal{M}(G, \kappa \otimes \sigma) \to \mathcal{M}(\mathrm{M}(\Lambda_E^m), \sigma)$ and $\mathcal{M}(T, \kappa_T \otimes \sigma) \to \mathcal{M}(T(\Lambda_E^m), \sigma)$, denoted in the diagram by Res. Because $\mathcal{H}(T(\Lambda_E^m), \sigma)$ is one-dimensional and the injections defined are homomorphisms of algebras, we must have $j_B \circ \alpha_1 \simeq j_{B(\Lambda_E^m)} \circ \alpha_2$, hence also $\mathrm{Res} \circ (j_B)_* \simeq (j_{B(\Lambda_E^m)})_* \circ \mathrm{Res}$.

We show that $M_\omega \circ \mathrm{R}_{\kappa_m} \simeq \mathrm{Res} \circ M_{\kappa\otimes\sigma}$; a similar argument shows that $M_{\omega_T} \circ \mathrm{R}_{\kappa_T} \simeq \mathrm{Res} \circ M_{\kappa_T \otimes \sigma}$. Let $\pi \in \mathfrak{R}_{\overline{\mathbb{Q}}_\ell}(\Omega)$. By Lemma 4.9 and adjointness, we have

$$\begin{aligned} M_\omega(\mathrm{R}_{\kappa_m}(\pi)) = \mathrm{Hom}_{B(\Lambda_E^m)}(\sigma, \mathrm{R}_{\kappa_m}(\pi)) &= \mathrm{Hom}_{B(\Lambda_E^m)}(\sigma, \mathrm{R}_{\kappa_m}(\pi)) \\ &\simeq \mathrm{Hom}_J(\sigma, (\mathrm{R}_{\kappa_m}(\pi))^{J_m^1 \mathrm{P}_1(\Lambda_E)/J_m^1}) \\ &\simeq \mathrm{Hom}_J(\sigma, \mathrm{R}_\kappa(\pi)) \\ &\simeq \mathrm{Hom}_{J^1}(\kappa \otimes \sigma, \pi) = M_{\kappa\otimes\sigma}(\pi). \end{aligned}$$

In the ℓ-modular case, we choose lifts of κ and χ and then by the ℓ-adic isomorphism and reduction modulo ℓ we have $\left[\mathrm{R}_{\kappa_m}(i_B^G(\chi))\right] \simeq \left[i_{B(\Lambda_E^m)}^{M(\Lambda_E^m)}(\mathrm{R}_{\kappa_T}(\chi))\right]$. □

6D. *Parabolic induction, κ-restriction, and covers.* Let χ be an irreducible character of T. Let (T^0, λ_T) be an R-type contained in χ such that (J, λ) is a G-cover of (T^0, λ_T) relative to B as constructed in Lemma 5.2 with $\lambda = \kappa \otimes \sigma$ and $\lambda_T = \kappa_T \otimes \sigma$, where $\kappa_T = \mathrm{Res}_{T^0}^J(\kappa)$. Hence $J = \mathrm{P}(\Lambda_E)J^1$ with $\mathrm{P}(\Lambda_E)$ a nonmaximal parahoric subgroup of G_E corresponding to the $\mathfrak{o}_E$-lattice sequence Λ_E. In all cases, there are two nonconjugate maximal parahoric which contain $\mathrm{P}(\Lambda_E)$; we denote the $\mathfrak{o}_E$-lattice sequences that correspond to these by Λ_E^x and Λ_E^y. Let $m \in \{x, y\}$ and let (κ_m, Λ_E^m) be the unique pair compatible with (κ, Λ_E) as in Lemma 4.6.

Lemma 6.5. *Let π be an irreducible subrepresentation or quotient of $i_B^G(\chi)$ and $m \in \{x, y\}$. Then $\mathrm{R}_{\kappa_m}(\pi) \neq 0$.*

Proof. By the geometric lemma, $r_B^G(i_B^G(\chi))$ is filtered by χ and $\chi^{w_x} = \psi\chi$ for some unramified character ψ. Hence, by exactness of the Jacquet functor, $r_B^G(\pi) = \psi\chi$. By Theorem 4.2, π contains (J, λ) if and only if $r_B^G(\pi)$ contains (T^0, λ_T). Thus π contains (J, λ); hence $\mathrm{R}_\kappa(\pi) \neq 0$. Therefore $\mathrm{R}_{\kappa_m}(\pi) \neq 0$. □

The next lemma is crucial in our proof of unicity of supercuspidal support. It shows that parabolic induction preserves the semisimple character up to transfer.

Lemma 6.6. *Suppose that $i_B^G(\chi)$ has an irreducible cuspidal subquotient π. Then there exists $m \in \{x, y\}$ such that $\mathrm{R}_{\kappa_m}(\pi) \neq 0$.*

Proof. By Theorem 5.3 there exist a skew semisimple stratum $[\Lambda', n', 0, \beta']$ such that $\mathrm{P}(\Lambda'_{E'})$ is a maximal parahoric subgroup of $G_{E'}$, where $G_{E'}$ denotes the G-centraliser of β', a semisimple character $\theta' \in \mathscr{C}_-(\Lambda', \beta')$, a β'-extension κ' to $J' = J(\Lambda', \beta')$ of the unique Heisenberg representation η' containing θ' and a cuspidal representation $\sigma' \in \mathrm{Irr}(J'/(J')^1)$ such that $\pi \simeq \mathrm{I}_{\kappa'}(\sigma')$.

As π contains $\kappa'\otimes\sigma'$, the restriction of $i_B^G(\chi)$ to J' has $\kappa'\otimes\sigma'$ as a subquotient. We choose $\hat{\chi}$ an ℓ-adic character lifting χ such that $i_B^G(\hat{\chi})$ is reducible. Then, because restriction and parabolic induction commute with reduction modulo ℓ, the restriction of $i_B^G(\hat{\chi})$ to J' has an irreducible subquotient δ such that $r_\ell(\delta)$ contains $\kappa'\otimes\sigma'$. On restricting to $(J')^1$ we see that δ contains the unique lift $\hat{\eta}'$ of η' and, since δ is irreducible and J' normalises $\hat{\eta}'$, $\mathrm{Res}^{J'}_{(J')^1}(\delta)$ is a multiple of η'. Thus $\delta=\hat{\kappa}'\otimes\xi$, with $\hat{\kappa}'$ a lift of κ' and ξ an irreducible representation of $J'/(J')^1$ whose reduction modulo ℓ contains σ. However, ξ cannot be cuspidal, otherwise $i_B^G(\hat{\chi})$ would have a cuspidal subquotient $\mathrm{I}_{\hat{\kappa}'}(\xi)$. Hence $G_{E'}$ is not compact. Therefore $[\Lambda',n',0,\beta']$ is either a scalar skew simple stratum or a skew semisimple stratum with splitting $V=V_1'\oplus V_2'$, with V_1' one-dimensional and V_2' two-dimensional hyperbolic. (Note that, as σ is cuspidal nonsupercuspidal, we must have $\ell\mid q+1$ or $\ell\mid q^2-q+1$ by Section 3.)

We continue by induction on the level $l(\pi)$ of π.

The base step is when π has level zero. If π has level zero then, as all subquotients of $i_B^G(\chi)$ have the same level as χ by [Vignéras 1996, 5.12], χ and $i_B^G(\chi)$ have level zero. Thus we can choose, and assume that we have chosen, κ', κ and κ_T to be trivial. By conjugating, we may assume $\Lambda'=\Lambda_m$ for some $m\in\{x,y\}$ and then $\kappa_m=\kappa'$ is trivial and $\mathrm{R}_{\kappa_m}(\pi)=\mathrm{R}_{\Lambda_m}(\pi)\neq 0$.

Suppose first that $[\Lambda,n,n-1,\beta]$ is equivalent to a scalar stratum $[\Lambda,n,n-1,\gamma]$. The stratum $[\Lambda,n,n-1,\gamma]$ corresponds to a character ψ_γ of $P_n(\Lambda)$ which extends to a character $\phi\circ\det$ of G. Twisting by $\phi^{-1}\circ\det$ we reduce the level of π and the level of $i_B^G(\chi)$. The stratum $[\Lambda,n,n-1,\beta-\gamma]$ is equivalent to a semisimple stratum $[\Lambda,n,n-1,\alpha]$ and the representations $\kappa(\phi^{-1}\circ\det)$, $\kappa_T(\phi^{-1}\circ\det)$ and $\kappa_m(\phi^{-1}\circ\det)$ for $m\in\{x,y\}$ are α-extensions defined on the relevant groups. Similarly, the stratum $[\Lambda',n',n'-1,\beta'-\gamma]$ is equivalent to a semisimple stratum $[\Lambda,n',n'-1,\alpha']$ and $\kappa'(\phi^{-1}\circ\det)$ is an α'-extension. Moreover, $\kappa_m(\phi^{-1}\circ\det)$ is compatible with $\kappa(\phi^{-1}\circ\det)$ for $m\in\{x,y\}$, $(\kappa\otimes\sigma)(\phi^{-1}\circ\det)$ is a G-cover of $(\kappa_T\otimes\sigma)(\phi^{-1}\circ\det)$ relative to B, $(\kappa_T\otimes\sigma)(\phi^{-1}\circ\det)$ is contained in $\chi(\phi^{-1}\circ\det)$, and $(\kappa'\otimes\sigma')(\phi^{-1}\circ\det)$ is contained in $\pi(\phi^{-1}\circ\det)$. Thus, by induction, we have

$$\mathrm{R}_{\kappa_m}(\pi)\simeq\mathrm{R}_{\kappa_m(\phi^{-1}\circ\det)}(\pi(\phi^{-1}\circ\det))$$

is nonzero for some $m\in\{x,y\}$.

Secondly, suppose that $[\Lambda',n',n'-1,\beta']$ is equivalent to a scalar stratum $[\Lambda',n',n'-1,\gamma']$. As in the last case, we can twist by a character to reduce the level.

Hence we may assume that both $[\Lambda,n,n-1,\beta]$ and $[\Lambda',n',n'-1,\beta']$ are not equivalent to scalar simple strata. This forces $[\Lambda,n,0,\beta]$ (resp. $[\Lambda',n',0,\beta']$) to be semisimple — and nonsimple — with splitting $V=V_1\oplus V_2$ (resp. $V=V_1'\oplus V_2'$) with V_1 (resp. V_1') one-dimensional and V_2 (resp. V_2') two-dimensional hyperbolic.

Thus, by conjugation we may assume that the splitting of $[\Lambda', n', 0, \beta']$ is the same as the splitting of $[\Lambda, n, 0, \beta]$, i.e., $V_1' = V_1$ and $V_2' = V_2$. We have $E = E'$ and $G_E = G_{E'}$, and conjugating further we may assume that Λ_E' and Λ_E lie in the closure of the same chamber of the building of G_E. Moreover, Λ_E' is a vertex and Λ_E is the barycentre of the chamber.

Let

$$w = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We have

$$J(\beta', \Lambda) = w \left(\begin{array}{c|c} \mathfrak{A}_0(\Lambda)^{11} & \mathfrak{A}_{\lfloor \frac{r'+1}{2} \rfloor}(\Lambda)^{12} \\ \hline \mathfrak{A}_{\lfloor \frac{r'+1}{2} \rfloor}(\Lambda)^{12} & \mathfrak{A}_0(\Lambda)^{22} \end{array} \right) w \cap G,$$

and

$$J(\beta, \Lambda) = w \left(\begin{array}{c|c} \mathfrak{A}_0(\Lambda)^{11} & \mathfrak{A}_{\lfloor \frac{r+1}{2} \rfloor}(\Lambda)^{12} \\ \hline \mathfrak{A}_{\lfloor \frac{r+1}{2} \rfloor}(\Lambda)^{12} & \mathfrak{A}_0(\Lambda)^{22} \end{array} \right) w \cap G,$$

where r' (resp. r) is minimal such that $[\Lambda, n', r', \beta]$ (resp. $[\Lambda, n, r, \beta]$) is equivalent to a scalar stratum. Thus, as we are now assuming that $[\Lambda, n, n-1, \beta]$ and $[\Lambda', n', n'-1, \beta']$ are not equivalent to scalar simple strata, we have $r' = n'$ and $r = n$. Furthermore, we have $l(\chi) = l(\pi)$, i.e., $n'/e(\Lambda') = n/e(\Lambda)$. We let κ'' be the unique β-extension to $J(\beta', \Lambda)$ compatible with κ' relative to a semisimple stratum $[\Lambda, n, 0, \beta']$. Therefore, $J(\beta, \Lambda) = J(\beta', \Lambda)$. Similar considerations yield $H(\beta, \Lambda) = H(\beta', \Lambda)$ and $J(\beta, \Lambda') = J(\beta', \Lambda')$.

As ξ is not cuspidal, it is a direct factor of $i_{B(\Lambda_E')}^{M(\Lambda_E')}(\hat{\tau}')$, where we choose $B(\Lambda_E')$ to be the image of $P(\Lambda_E)$ in $M(\Lambda_E')$, for some representation $\hat{\tau}'$ of $T(\Lambda_E')$. Furthermore, $i_B^G(\hat{\chi})$ contains $\hat{\kappa}'' \otimes \hat{\tau}'$ with $\hat{\kappa}''$ a lift of κ'', by Lemma 4.6 and transitivity of induction. By Lemma 5.2, $(J, \hat{\kappa}'' \otimes \hat{\tau}')$ is a G-cover of $(T^0, \hat{\kappa}_T'' \otimes \hat{\tau}')$ relative to B, where $\hat{\kappa}_T'' = \mathrm{Res}^J_{T^0}(\hat{\kappa}'')$. By [Blondel 2005, Theorem 2], $\mathrm{ind}_J^G(\hat{\kappa}'' \otimes \hat{\tau}') \simeq \mathrm{Ind}^G_{B^{\mathrm{op}}}(\mathrm{ind}^T_{T_0}(\hat{\kappa}_T'' \otimes \hat{\tau}'))$. By second adjunction of parabolic induction and parabolic restriction for ℓ-adic representations, and right adjunction of restriction with compact induction we have

$$\mathrm{Hom}_{T^0}(\hat{\kappa}_T'' \otimes \hat{\tau}', r_B^G \circ i_B^G(\hat{\chi})) \simeq \mathrm{Hom}_G(\mathrm{ind}_J^G(\hat{\kappa}'' \otimes \hat{\tau}'), i_B^G(\hat{\chi})) \neq 0.$$

We have $[r_B^G \circ i_B^G(\hat{\chi}) \mid_{T^0}] = \hat{\chi} \oplus \hat{\chi}^{w_x} \mid_{T^0} = \hat{\chi} \oplus \hat{\chi} \mid_{T^0}$. Hence $\hat{\kappa}_T'' \otimes \hat{\tau}' = \mathrm{Res}^T_{T^0}(\hat{\chi})$. Similarly if we let $\hat{\kappa}$ be a lift of κ, $\hat{\sigma}$ be a lift of σ, and $\hat{\kappa}_T = \mathrm{Res}^T_{T^0}(\hat{\kappa})$, then we have $\hat{\kappa}_T \otimes \hat{\sigma} = \mathrm{Res}^J_{T^0}(\hat{\chi})$. This implies that we have an equality of semisimple characters $\tau_{\Lambda', \Lambda, \beta'}(\hat{\theta}') = \hat{\theta}$, where $\hat{\theta}' \in \mathscr{C}_-(\beta', \Lambda')$ is contained in $\hat{\kappa}'$ and $\hat{\theta} \in \mathscr{C}(\beta, \Lambda)$ is contained in $\hat{\kappa}$.

We let $\widetilde{H}(\beta, \Lambda)$ (resp. $\widetilde{H}(\beta', \Lambda')$) denote the compact open subgroup of $\mathrm{GL}_3(F)$ defined in [Stevens 2008], which defines $H(\beta, \Lambda)$ (resp. $H(\beta', \Lambda')$) by intersecting with $\mathrm{U}(2,1)(F/F_0)$. The Iwahori decomposition for $\widetilde{H}^1(\beta', \Lambda')$ gives $\widetilde{H}^1(\beta', \Lambda') = \widetilde{H}^1(\beta', \Lambda')^-(\widetilde{H}^1(\beta', \Lambda') \cap \widetilde{M})\widetilde{H}^1(\beta', \Lambda')^+$ where $\widetilde{H}^1(\beta', \Lambda')^-$ denotes the lower triangular unipotent matrices in $\widetilde{H}^1(\beta', \Lambda')$, $\widetilde{H}^1(\beta', \Lambda')^+$ denotes the upper triangular unipotent matrices in $\widetilde{H}^1(\beta', \Lambda')$, and $\widetilde{M}$ the subgroup of diagonal matrices. As $\widetilde{H}^1(\beta', \Lambda)$ contains $(\widetilde{H}^1(\beta', \Lambda') \cap \widetilde{M})$ and is contained in $\widetilde{H}^1(\beta', \Lambda')$, we have

$$\widetilde{H}^1(\beta', \Lambda') = \widetilde{H}^1(\beta', \Lambda')^-\big(\widetilde{H}^1(\beta', \Lambda') \cap \widetilde{H}^1(\beta', \Lambda)\big)\widetilde{H}^1(\beta', \Lambda')^+.$$

Thus a character of $\widetilde{H}^1(\beta', \Lambda')$ is determined by its values on $\widetilde{H}^1(\beta', \Lambda')^-$, $\widetilde{H}^1(\beta', \Lambda') \cap \widetilde{H}^1(\beta', \Lambda)$, and $\widetilde{H}^1(\beta', \Lambda')^+$.

The semisimple characters $\hat{\theta}$ and $\hat{\theta}'$ are equal to the restriction of semisimple characters $\tilde{\theta}$ and $\tilde{\theta}'$ of $\mathrm{GL}_3(F)$. Moreover, $\tau_{\Lambda',\Lambda,\beta'}(\tilde{\theta}') = \tilde{\theta}$ as $\tau_{\Lambda',\Lambda,\beta'}(\hat{\theta}') = \hat{\theta}$. It follows from the decomposition of $\widetilde{H}^1(\beta', \Lambda')$ given above that $\tau_{\Lambda,\Lambda',\beta}(\tilde{\theta}) = \tilde{\theta}'$; they are both trivial on $\widetilde{H}^1(\beta', \Lambda')^-$ and $\widetilde{H}^1(\beta', \Lambda')^+$, and as $\theta' = \tau_{\Lambda,\Lambda',\beta'}(\theta)$ they both agree with $\tilde{\theta}$ on $\widetilde{H}^1(\beta, \Lambda') \cap \widetilde{H}^1(\beta, \Lambda) = \widetilde{H}^1(\beta', \Lambda') \cap \widetilde{H}^1(\beta', \Lambda)$. Hence, $\tau_{\Lambda,\Lambda',\beta}(\theta) = \theta'$ by restriction and reduction modulo ℓ. As there is a unique Heisenberg representation containing θ', we have $\mathrm{R}_{\kappa_m}(\pi) \neq 0$ for some $m \in \{x, y\}$. □

Lemma 6.7. *Suppose that $i_B^G(\chi)$ is reducible with irreducible subrepresentation π_1 and quotient $\pi_2 = i_B^G(\chi)/\pi_1$. If Σ is a maximal cuspidal subquotient of $\mathrm{R}_{\kappa_m}(i_B^G(\chi))$, i.e., all subquotients of $\mathrm{R}_{\kappa_m}(i_B^G(\chi))$ not contained in Σ are not cuspidal, then $\mathrm{I}_{\kappa_m}(\Sigma)$ is a subrepresentation of π_2.*

Proof. Let Σ be a maximal cuspidal subquotient of $\mathrm{R}_{\kappa_m}(i_B^G(\chi))$. By Lemma 6.5, $\mathrm{R}_{\kappa_m}(\pi_1)$ and $\mathrm{R}_{\kappa_m}(\pi_2)$ are nonzero and must contain noncuspidal subquotients as π_1 and π_2 are not cuspidal. However, by Theorem 6.4 and Section 3, there are only two noncuspidal subquotients of $\mathrm{R}_{\kappa_m}(i_B^G(\chi))$. Thus each of $\mathrm{R}_{\kappa_m}(\pi_1)$ and $\mathrm{R}_{\kappa_m}(\pi_2)$ must have a single noncuspidal irreducible subquotient, say ρ_1 and ρ_2 respectively.

If $\mathrm{R}_{\kappa_m}(\pi_1) \neq \rho_1$ then $\mathrm{R}_{\kappa_m}(\pi_1)$ has an irreducible cuspidal subrepresentation or an irreducible cuspidal quotient. If $\mathrm{R}_{\kappa_m}(\pi_1)$ has an irreducible cuspidal subrepresentation σ then, by adjointness of R_{κ_m} and I_{κ_m}, $\mathrm{I}_{\kappa_m}(\sigma)$ is an irreducible cuspidal subrepresentation of π_1, contradicting the irreducibility and noncuspidality of π_1. If $\mathrm{R}_{\kappa_m}(\pi_1)$ has an irreducible cuspidal quotient σ then $\mathrm{R}_{\tilde{\kappa}_m}(\tilde{\pi}_1)$ has a cuspidal subrepresentation $\tilde{\sigma}$ by Lemma 4.10. Thus $\mathrm{I}_{\tilde{\kappa}_m}(\tilde{\sigma})$ is an irreducible cuspidal subrepresentation of $\tilde{\pi}_1$ by adjointness. Hence $\mathrm{I}_{\kappa_m}(\sigma)$ is an irreducible cuspidal quotient of π_1 by Lemma 4.10, contradicting the irreducibility and noncuspidality of π_1. Thus $\mathrm{R}_{\kappa_m}(\pi_1) = \rho_1$.

Similarly, if $\mathrm{R}_{\kappa_m}(\pi_2)$ has an irreducible cuspidal quotient σ, then $\mathrm{I}_{\kappa_m}(\sigma)$ is an irreducible cuspidal quotient of π_2. Hence $\mathrm{I}_{\kappa_m}(\sigma)$ is a quotient of $i_B^G(\chi)$, contradicting the cuspidality of $\mathrm{I}_{\kappa_m}(\sigma)$. Hence $\mathrm{R}_{\kappa_m}(\pi_2)$ can have no cuspidal quotients. Hence, by Section 3, Lemma 6.3 and Theorem 6.4, Σ is a subrepresentation of $\mathrm{R}_{\kappa_m}(\pi_2)$. Note that, as Theorem 6.4 only gives us an isomorphism in the Grothendieck group of finite-length representations of $\mathrm{M}(\Lambda_E)$, we have used that Σ is irreducible by Section 3 in the skew semisimple nonscalar case to imply it is a subrepresentation of $\mathrm{R}_{\kappa_m}(\pi_2)$; in all other cases we twist by a character (if necessary) and use Lemma 6.3. By reciprocity, $\mathrm{I}_{\kappa_m}(\Sigma)$ is a subrepresentation of π_2. □

By [Blondel 2005, Theorem 2] and Lemma 5.2, $\mathrm{I}_\kappa(\sigma) \simeq \mathrm{Ind}_{B^{\mathrm{op}}}^G(\mathrm{ind}_{T^0}^T(\kappa_T \otimes \sigma))$. By second adjunction (cf. [Dat 2009, Corollaire 3.9]),

$$\mathrm{Hom}_G(\mathrm{Ind}_{B^{\mathrm{op}}}^G(\mathrm{ind}_{T^0}^T(\kappa_T \otimes \sigma)), \pi) \simeq \mathrm{Hom}_T(\mathrm{ind}_{T^0}^T(\kappa_T \otimes \sigma), r_B^G(\pi)).$$

By Clifford theory, the irreducible quotients of $\mathrm{ind}_{T^0}^T(\kappa_T \otimes \sigma)$ are all the twists of χ by an unramified character. Hence, π is an irreducible quotient of $\mathrm{I}_\kappa(\sigma)$ if and only if it is an irreducible quotient of $i_B^G(\chi\psi)$ for some unramified character ψ of T.

The R-type (J, λ) is quasiprojective by Theorem 5.3; hence a simple module of $\mathcal{H}(G, \lambda)$ corresponds to an irreducible quotient of $i_B^G(\chi\psi)$ for some unramified character ψ, by the bijection of Theorem 4.1. If $i_B^G(\chi\psi)$ is reducible with proper quotient π, then the Jacquet module of π is one-dimensional by the geometric lemma. Hence, by Theorem 4.2, π must correspond to a character of $\mathcal{H}(G, \lambda)$ under the bijection of Theorem 4.1 and all characters of $\mathcal{H}(G, \lambda)$ must correspond to a proper quotient of a reducible principal series representation $i_B^G(\chi\psi)$ with ψ an unramified character of T.

Lemma 6.8. *Suppose $\ell \neq 2$ and $\ell \mid q-1$. Then $i_B^G(\chi)$ is semisimple.*

Proof. If $i_B^G(\chi)$ is irreducible then it is semisimple, so suppose $i_B^G(\chi)$ is reducible. If $i_B^G(\chi)$ has a cuspidal subquotient it is of the form $\mathrm{I}_{\kappa_m}(\sigma)$ for $m \in \{x, y\}$ and σ an irreducible cuspidal representation of $\mathrm{M}(\Lambda_E^x)$ by Theorem 5.3. By Theorem 6.4, $\mathrm{R}_{\kappa_x}(i_B^G(\chi)) = i_{B(\Lambda_E^x)}^{\mathrm{M}(\Lambda_E^x)}(\mathrm{R}_{\kappa_T}(\chi))$, and $\mathrm{R}_{\kappa_x}(\mathrm{I}_{\kappa_x}(\sigma)) = \sigma$, by Theorem 4.8 and Lemma 4.4. Hence, by exactness, σ is a cuspidal subquotient of $i_{B(\Lambda_E^x)}^{\mathrm{M}(\Lambda_E^x)}(\mathrm{R}_{\kappa_T}(\chi))$. However, by Section 3, when $\ell \mid q-1$ no such cuspidal subquotients exist; hence $i_B^G(\chi)$ has no cuspidal subquotients. Thus, by exactness of the Jacquet functor and the geometric lemma, $i_B^G(\chi)$ has length two. When $\ell \neq 2$ and $\ell \mid q-1$ there are four characters of $\mathcal{H}(G, \lambda)$, yet only two reducibility points. Hence these two reducible principal series representations must both have two nonisomorphic irreducible quotients and must be semisimple. □

Remark 6.9. If $\chi^{-1} = \chi$ then $i_B^G(\chi)$ is self-contragredient and there is a simple proof of Lemma 6.8 using the contragredient representation and avoiding the use of covers or second adjunction.

Lemma 6.10. *Let $\ell \mid q+1$. Then the unique irreducible quotient of $i_B^G(\chi)$ is isomorphic to the unique irreducible subrepresentation.*

Proof. Let π denote the unique irreducible quotient of $i_B^G(\chi)$. When $\ell \mid q+1$ there is only one character of $\mathscr{H}(G, \kappa\otimes\sigma)$. Hence π corresponds to the unique character of $\mathscr{H}(G,\lambda)$. Hence, if $\mathscr{V}$ is the space of π, $\mathrm{R}_\kappa(\mathscr{V})$ is one-dimensional and the action of J is given by σ. As δ_B is trivial, the contragredient commutes with parabolic induction: we have $(i_B^G(\chi))^\sim \simeq i_B^G(\tilde{\chi})$. Furthermore, $\tilde{\chi}=\chi^{-1}$, where χ^{-1} is the character defined by, for all $x\in F^\times$, $\chi^{-1}(x)=\chi(x^{-1})$. The character χ^{-1} is not regular and similar arguments, given for $i_B^G(\chi)$, apply to $i_B^G(\chi^{-1})$. We find that $i_B^G(\chi^{-1})$ has a unique irreducible quotient ρ which corresponds to the unique character of $\mathscr{H}(G,\tilde{\lambda})$ under the bijection of Theorem 4.1. As the contragredient is contravariant and exact, $\tilde{\rho}$ is a subrepresentation of $i_B^G(\chi)$. By Lemma 4.10, we have $(\mathrm{R}_{\tilde{\kappa}}(\rho))^\sim \simeq \mathrm{R}_\kappa(\tilde{\rho})$ which is one-dimensional and hence must be isomorphic to σ. Hence $\tilde{\rho}$ is irreducible and isomorphic to π. Thus π appears twice in the composition series of $i_B^G(\chi)$, as the unique irreducible quotient and as the unique irreducible subrepresentation. □

Remark 6.11. If $\ell\neq 3$ and $\ell \mid q^2-q+1$, then similar counting arguments show that the unique irreducible subrepresentation is not isomorphic to the unique irreducible quotient. However, in these cases we find out more information later so this argument is not necessary.

6E. *On the unramified principal series.*

6E1. *Decomposition of $i_B^G(\nu^2)$ and $i_B^G(\nu^{-2})$.* In all cases of coefficient field, the space of constant functions forms an irreducible subrepresentation of $i_B^G(\nu^{-2})$ isomorphic to 1_G. We let St_G denote the quotient of $i_B^G(\nu^{-2})$ by 1_G. Parabolic induction preserves finite-length representations; hence St_G has an irreducible quotient υ_G. By the geometric lemma, $\left[r_B^G\circ i_B^G(\nu^{-2})\right]\simeq \nu^{-2}\oplus(\nu^{-2})^{w_x}$. Considering ν^{-2} as a character of $F^\times$, we have $(\nu^{-2})^{w_x}(x)=\nu^{-2}(\bar{x}^{-1})=\nu^2(x)$, as $\nu^{-2}(x)=\nu^{-2}(\bar{x})$. Thus $\left[r_B^G\circ i_B^G(\nu^{-2})\right]=\nu^{-2}\oplus\nu^2$. We have $r_B^G(1_G)=\nu^{-2}$, thus $r_B^G(\mathrm{St}_G)=\nu^2$ by exactness of the Jacquet functor. A quotient of a parabolically induced representation has nonzero Jacquet module; hence $r_B^G(\upsilon_G)=\nu^2$. Thus any other composition factors which occur in $i_B^G(\nu^{-2})$ must be cuspidal.

Theorem 6.12. (1) *If $\ell\nmid(q-1)(q+1)(q^2-q+1)$ then $i_B^G(\nu^{-2})$ has length two with unique irreducible subrepresentation 1_G and unique irreducible quotient St_G.*

(2) *If $\ell\neq 2$ and $\ell\mid q-1$ then $i_B^G(\nu^{-2})=1_G\oplus\mathrm{St}_G$ is semisimple of length two.*

(3) *If $\ell\neq 3$ and $\ell\mid q^2-q+1$ then $i_B^G(\nu^{-2})$ has length three with unique cuspidal subquotient $\mathrm{I}_{\Lambda_x}(\bar{\tau}^+(\bar{1}))$. The unique irreducible quotient υ_G is not a character.*

(4) *If $\ell\neq 2$ and $\ell\mid q+1$, or if $\ell=2$ and $4\mid q+1$, then $i_B^G(\nu^{-2})$ has length six with 1_G appearing as the unique subrepresentation and the unique quotient,*

and four cuspidal subquotients. Let π be a maximal proper submodule of St_G. *Then* $\pi \simeq \rho \oplus \mathrm{I}_{\Lambda_y}(\bar\sigma(\bar 1) \otimes \bar 1)$, *where ρ is of length three with unique irreducible subrepresentation and unique irreducible quotient, both of which are isomorphic to* $\mathrm{I}_{\Lambda_x}(\bar\upsilon(\bar 1))$, *and remaining subquotient isomorphic to* $\mathrm{I}_{\Lambda_x}(\bar\sigma(\bar 1))$.

(5) *If $\ell = 2$ and $4 \mid q-1$, then $i_B^G(\nu^{-2})$ has length five with unique irreducible subrepresentation and unique irreducible quotient both isomorphic to 1_G. Let π be a maximal proper submodule of* St_G. *Then*

$$\pi \simeq \mathrm{I}_{\Lambda_x}(\bar\upsilon(\bar 1)) \oplus \mathrm{I}_{\Lambda_x}(\bar\tau^+(\bar\chi)) \oplus \mathrm{I}_{\Lambda_y}(\bar\sigma(\bar 1) \otimes \bar 1).$$

Proof. By Theorem 5.3 and Lemma 6.6, if $i_B^G(\nu^{-2})$ has a cuspidal subquotient π then $\pi \simeq \mathrm{I}_{\Lambda_w}(\sigma)$ for $w \in \{x, y\}$ and σ an irreducible cuspidal representation of $\mathrm{P}(\Lambda_w)/\mathrm{P}_1(\Lambda_w)$.

If Σ_w is a maximal cuspidal subquotient of $\mathrm{R}_{\Lambda_w}(i_B^G(\nu^{-2}))$ then $\mathrm{I}_{\Lambda_w}(\Sigma_w)$ is a subrepresentation of St_G, by Lemma 6.7. Thus, we have an exact sequence

$$0 \to \mathrm{I}_{\Lambda_x}(\Sigma_x) \oplus \mathrm{I}_{\Lambda_y}(\Sigma_y) \to \mathrm{St}_G \to \upsilon_G \to 0.$$

By exactness and Section 3, we obtain composition series of $\mathrm{I}_{\Lambda_x}(\Sigma_x)$ and of $\mathrm{I}_{\Lambda_y}(\Sigma_y)$.

If $\ell \nmid (q-1)(q+1)(q^2-q+1)$, or $\ell \neq 2$ and $\ell \mid q-1$, then $\mathrm{R}_{\Lambda_x}(i_B^G(\nu^{-2}))$ and $\mathrm{R}_{\Lambda_y}(i_B^G\nu^{-2}))$ are of length two with no cuspidal subquotients, by Theorem 5.3 and Lemma 6.6. Hence, $i_B^G(\nu^{-2})$ has no cuspidal subquotients as $\mathrm{R}_{\Lambda_w}(\mathrm{I}_{\Lambda_w}(\sigma)) \simeq \sigma$ is cuspidal by Lemma 4.4. By the geometric lemma, $i_B^G(\nu^{-2})$ is of length two with 1_G as an irreducible subrepresentation and St_G as an irreducible quotient. By second adjunction,

$$\mathrm{Hom}_G(i_B^G(\nu^{-2}), 1_G) \simeq \mathrm{Hom}_T(\nu^{-2}, 1_T).$$

The character ν^{-2} is nontrivial when $\ell \nmid (q-1)(q+1)(q^2-q+1)$ and trivial when $\ell \mid q-1$. Hence 1_G is a direct factor when $\ell \neq 2$ and $\ell \mid q-1$ and $i_B^G(\nu^{-2})$ is semisimple, and $i_B^G(\nu^{-2})$ is nonsplit when $\ell \nmid (q-1)(q+1)(q^2-q+1)$.

In all other cases, $i_B^G(\nu^{-2})$ has cuspidal subquotients. Thus 1_G cannot be a direct factor. Therefore $i_B^G(\nu^{-2})$ has a unique irreducible quotient υ_G and a unique irreducible subrepresentation 1_G. When $\ell \mid q+1$ the unique irreducible quotient is isomorphic to the unique irreducible subrepresentation by Lemma 6.10; hence $\upsilon_G \simeq 1_G$. When $\ell \neq 3$ and $\ell \mid q^2-q+1$, the representation $\mathrm{R}_{\Lambda_y}(i_B^G(\nu^{-2}))$ has noncuspidal subquotients 1_{M_y} and St_{M_y}. By exactness, $\mathrm{R}_{\Lambda_y}(\upsilon_G) \simeq \mathrm{St}_{M_y}$; hence 1_G is not isomorphic to υ_G, which is not a character. □

Note that $i_B^G(\nu^2) \simeq i_B^G(\nu^{-2})^\sim$; hence decompositions of $i_B^G(\nu^2)$ can be obtained from Theorem 6.12.

6E2. *Decomposition of unramified $i_B^G(\eta\nu)$ and $i_B^G(\eta\nu^{-1})$.* Let η be the unique unramified character of $F^\times$ extending ω_{F/F_0}. If $\ell \mid q+1$, then $\omega_{F/F_0}\nu^{-1} = \omega_{F/F_0}\nu = 1$;

hence we refer to Theorem 6.12. When $\ell \mid q^2+q+1$ we have $\nu^2 = \eta\nu^{-1}$ and $\nu^{-2} = \eta\nu$; hence once more we refer to Theorem 6.12. When $\ell \mid q-1$, ν is trivial, hence $\eta\nu = \eta\nu^{-1} = \eta$. Thus $i_B^G(\eta)$ is self-contragredient. By Lemma 6.8, $i_B^G(\eta)$ has length two and is semisimple.

6F. ***Cuspidal subquotients of the ramified level zero principal series.*** We describe the reducible principal series $i_B^G(\chi)$ which have length greater than two when χ is a level zero character of T which does not factor through the determinant map. We twist by a character that factors through the determinant map so that we can assume $\chi_2 = 1$. Then $\chi^{q+1} = 1$ and $\chi = \overline{\psi} \circ \xi$ for $\overline{\psi}$ a nontrivial character of k_F^1.

When $\ell \nmid q+1$, because $\mathrm{R}_{\Lambda_x}(i_B^G(\chi))$ and $\mathrm{R}_{\Lambda_y}(i_B^G(\chi))$ have no cuspidal subquotients, $i_B^G(\chi)$ is of length two.

Theorem 6.13. *Let $\ell \mid q+1$. The representation $i_B^G(\chi)$ has length four with a unique irreducible subrepresentation and a unique irreducible quotient, and cuspidal subquotient isomorphic to $\mathrm{I}_{\Lambda_x}(\overline{\sigma}(\overline{\psi}, \overline{\psi}, \overline{1})) \oplus \mathrm{I}_{\Lambda_y}(\overline{\sigma}(\overline{\psi}) \otimes \overline{1})$. Furthermore, the unique irreducible subrepresentation is isomorphic to the unique irreducible quotient.*

Proof. The proof is similar to the proof of Theorem 6.12. □

7. Cuspidal subquotients of positive level principal series

In this section, suppose that χ_1 is a positive level character of $F^\times$ trivial on $F_0^\times$ and χ is the character of T given by χ_1 and $\chi_2 = 1$. We assume we are in the same setting as Section 6D with (T^0, λ_T) an R-type contained in χ, (J, λ) a G-cover of (T^0, λ_T) relative to B with $\lambda = \kappa \otimes \sigma$, and (κ_m, Λ^m) compatible with (κ, Λ) for $m \in \{x, y\}$. We have $\mathrm{M}(\Lambda_E^m) \simeq \mathrm{U}(1,1)(k_F/k_0) \times \mathrm{U}(1)(k_F/k_0)$. When $\ell \nmid q+1$, there are no cuspidal subquotients of $\mathrm{U}(1,1)(k_F/k_0)$, and hence no cuspidal subquotients of $i_B^G(\chi)$, by Lemma 6.6. Thus it remains to look at the case when $\ell \mid q+1$. Let $\overline{\psi} = (\chi\kappa_T^{-1})^{T^1}$ and $\overline{\chi}$ the character of k_F^1 such that $\overline{\psi} = \overline{\chi} \circ \xi$.

Theorem 7.1. *Suppose $\ell \mid q+1$. The representation $i_B^G(\chi)$ has length four with unique irreducible subrepresentation and unique irreducible quotient which are isomorphic, and cuspidal subquotient isomorphic to $\mathrm{I}_{\kappa_x}(\sigma(\overline{\chi}) \otimes \overline{1}) \oplus \mathrm{I}_{\kappa_y}(\sigma(\overline{\chi}) \otimes \overline{1})$.*

Proof. The proof is similar to the proof of Theorem 6.12. □

8. Supercuspidal support

Theorem 8.1. *Let G be an unramified unitary group in three variables and π an irreducible ℓ-modular representation of G. Then the supercuspidal support of π is unique up to conjugacy.*

Proof. Suppose π is not cuspidal. Then the supercuspidal support of π is equal to the cuspidal support of π and is thus unique up to conjugacy. If π is cuspidal nonsupercuspidal then it appears in one of the decompositions given in Theorems 6.12, 6.13, and 7.1, or is a twist of such a representation by a character that factors through the determinant map, and we see that the supercuspidal support of π is unique up to conjugacy. □

Remark 8.2. Let $\mathrm{I}_{\kappa'}(\sigma')$ be an irreducible cuspidal representation of G as constructed in Theorem 5.3. After the decomposition of the parabolically induced representations given in Theorems 6.12, 6.13, and 7.1, we see that $\mathrm{I}_{\kappa'}(\sigma')$ is supercuspidal if and only if σ' is supercuspidal. Hence all supercuspidal representations of G lift, by Section 3.

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