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Corey Irving and Hal Schenck

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Geometry of Wachspress surfaces

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Let P_d be a convex polygon with d vertices. The associated Wachspress surface W_d is a fundamental object in approximation theory, defined as the image of the rational map

$$\mathbb{P}^2 \xrightarrow{w_d} \mathbb{P}^{d-1},$$

determined by the Wachspress barycentric coordinates for P_d. We show w_d is a regular map on a blowup X_d of $\mathbb{P}^2$ and, if $d > 4$, is given by a very ample divisor on X_d so has a smooth image W_d. We determine generators for the ideal of W_d and prove that, in graded lex order, the initial ideal of I_{W_d} is given by a Stanley–Reisner ideal. As a consequence, we show that the associated surface is arithmetically Cohen–Macaulay and of Castelnuovo–Mumford regularity 2 and determine all the graded Betti numbers of I_{W_d}.

1. Introduction

Introduced by Möbius [1827], barycentric coordinates for triangles appear in a host of applications. Recent work in approximation theory has shown that it is also useful to define barycentric coordinates for a convex polygon P_d with $d \geq 4$ vertices (a d-gon). The idea is as follows. To deform a planar shape, first place the shape inside a control polygon. Then move the vertices of the control polygon, and use barycentric coordinates to extend this motion to the entire shape.

For a d-gon with $d \geq 4$, barycentric coordinates were defined by Wachspress [1975] in his work on finite elements; these coordinates are rational functions depending on the vertices $\nu(P_d)$ of P_d. Warren [2003] shows that Wachspress' coordinates are the unique rational barycentric coordinates of minimal degree. The Wachspress coordinates define a rational map w_d on $\mathbb{P}^2$, whose value at a point $p \in P_d$ is the d-tuple of barycentric coordinates of p. The closure of the image of w_d is the Wachspress surface W_d, first defined and studied by Garcia-Puente and Sottile [2010] in their work on linear precision.

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In Definition 1.3, we fix linear forms ℓ_i that are positive inside P_d and vanish on an edge. Let $A = \ell_1 \cdots \ell_d$, Z be the $\binom{d}{2}$ singular points of $\mathbb{V}(A)$, and $Y = Z \setminus \nu(P_d)$. We call Y the *external vertices* of P_d and show that w_d has basepoints only at Y. Let X_d be the blowup of $\mathbb{P}^2$ at Y. In Section 2, we prove that W_d is the image of X_d, embedded by a certain divisor D_{d-2} on X_d. The global sections of D_{d-2} have a simple interpretation in terms of the edges $\mathbb{V}(\ell_i)$ of P_d: we prove that

$$H^0(\mathbb{O}_{X_d}(D_{d-2})) \quad \text{has basis} \quad \{\ell_3 \cdots \ell_d, \ell_1\ell_4 \cdots \ell_d, \ldots, \ell_2 \cdots \ell_{d-1}\}.$$

We show that D_{d-2} is very ample if $d > 4$; hence, $W_d \subseteq \mathbb{P}^{d-1}$ is a smooth surface.

1A. ***Statement of main results.*** For a d-gon P_d with $d \geq 4$:

(1) We give explicit generators for $I_{W_d} \subseteq S = \mathbb{K}[x_1, \ldots, x_d]$.

(2) We determine $\text{in}_{\prec}(I_{W_d})$, where $\prec$ is graded lex order.

(3) We prove $\text{in}_{\prec}(I_{W_d})$ is the Stanley–Reisner ideal of a graph Γ.

(4) We prove that S/I_{W_d} is Cohen–Macaulay, and $\text{reg}(S/I_{W_d}) = 2$.

(5) We determine the graded Betti numbers of S/I_{W_d}.

In Section 1B, we give some quick background on geometric modeling, and in Section 1C, we do the same for algebraic geometry (in particular, we define all the terms above). Our strategy runs as follows. In Section 2, we study I_{W_d} by blowing up $\mathbb{P}^2$ at the external vertices. Define a divisor

$$D_{d-2} = (d-2)E_0 - \sum_{p \in Y} E_p$$

on X_d, where E_0 is the pullback of a line and E_p is the exceptional fiber over p. We show that D_{d-2} is very ample and that I_{W_d} is the ideal of the image of

$$X_d \to \mathbb{P}(H^0(D_{d-2})).$$

Riemann–Roch then yields the Hilbert polynomial of S/I_{W_d}.

In Sections 3 and 4, we find distinguished sets of quadrics and cubics vanishing on W_d and use them to generate a subideal $I(d) \subseteq I_{W_d}$. In Section 5, we tie everything together, showing that, in graded lex order, $I_\Gamma(d) \subseteq \text{in}_\prec I(d)$, where $I_\Gamma(d)$ is the Stanley–Reisner ideal of a certain graph. Using results on flat deformations and an analysis of associated primes, we prove

$$I_\Gamma(d) = \text{in}_\prec(I(d)).$$

The description in terms of the Stanley–Reisner ring yields the Hilbert series for $S/I_\Gamma(d)$. We prove that $S/I_\Gamma(d)$ is Cohen–Macaulay and has Castelnuovo–Mumford regularity 2, and it follows from uppersemicontinuity that the same is true for $S/I(d)$. The differentials on the quadratic generators of $I_\Gamma(d)$ turn out to be easy to describe, and combining this with the regularity bound and knowledge of the Hilbert series yields the graded Betti numbers for $\mathrm{in}_{\prec}(I(d))$.

Finally, we show that $I(d)$ has no linear syzygies on its quadratic generators, which allows us to prune the resolution of $\mathrm{in}_{\prec}(I(d))$ to obtain the graded Betti numbers of $I(d)$. Comparing Hilbert polynomials shows that up to saturation

$$S/I(d) = S/I_{W_d}.$$

Since I_{W_d} is prime, it is saturated, and a short-exact-sequence argument shows that $S/I(d)$ is also saturated, concluding the proof.

1B. ***Geometric modeling background.*** Let P_d be a d-gon with vertices $v_1, \dots, v_d$ and indices taken modulo d.

Definition 1.1. Functions $\{\beta_i : P_d \to \mathbb{R} \mid 1 \le i \le d\}$ are *barycentric coordinates* if, for all $p \in P_d$,

$$\beta_i(p) \ge 0, \qquad p = \sum_{i=1}^{d} \beta_i(p) v_i, \qquad \sum_{i=1}^{d} \beta_i(p) = 1.$$

Wachspress coordinates have a geometric description in terms of areas of subtriangles of the polygon. Let $A(a, b, c)$ denote the area of the triangle with vertices a, b, and c. For $1 \le j \le d$, set $\alpha_j := A(v_{j-1}, v_j, v_{j+1})$ and $A_j := A(p, v_j, v_{j+1})$.

Definition 1.2. For $1 \le i \le d$, the functions

$$\beta_i = \frac{b_i}{\sum_{j=1}^{d} b_j}, \quad \text{where } b_i = \alpha_i \prod_{j \ne i-1, i} A_j$$

are Wachspress barycentric coordinates for the d-gon P_d; see Figure 1.

We embed P_d in the plane $z = 1 \subseteq \mathbb{R}^3$ and form the cone with $\mathbf{0} \in \mathbb{R}^3$. Explicitly, to each vertex $v_i \in \nu(P_d)$, we associate the ray $\boldsymbol{v}_i := (v_i, 1) \in \mathbb{R}^3$. Let $\boldsymbol{P}_d$ denote the cone generated by the rays $\boldsymbol{v}_i$, and $\nu(\boldsymbol{P}_d) := \{\boldsymbol{v}_i \mid v_i \in \nu(P_d)\}$. The cone over

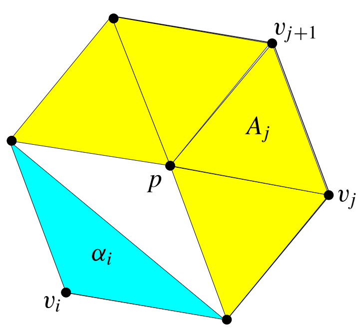


Figure 1. Wachspress coordinates for a polygon.

the edge $[v_i, v_{i+1}]$ corresponds to a facet of $\boldsymbol{P}_d$ with normal vector $\boldsymbol{n}_i := \boldsymbol{v}_i \times \boldsymbol{v}_{i+1}$. We redefine α_j and A_j to be the determinants $|\boldsymbol{v}_{j-1}\boldsymbol{v}_j\boldsymbol{v}_{j+1}|$ and $|\boldsymbol{v}_j\boldsymbol{v}_{j+1}\boldsymbol{p}|$, where $\boldsymbol{p} = (x, y, z)$. This scales the b_i by a factor of 2 so leaves the β_i unchanged, save for homogenizing the A_j with respect to z, and allows us to define Wachspress coordinates for nonconvex polygons, although Property 1 of barycentric coordinates fails when P_d is nonconvex.

Definition 1.3. $$\ell_j := A_j = \boldsymbol{n}_j \cdot \boldsymbol{p} = |\boldsymbol{v}_j\boldsymbol{v}_{j+1}\boldsymbol{p}|.$$

The ℓ_j are homogeneous linear forms in (x, y, z) and vanish on the cone over the edge $[v_j, v_{j+1}]$. We use Theorem 1.6 below, but Warren's proof does not require convexity. Our results hold over an arbitrary field $\mathbb{K}$ as long as no three of the lines $\mathbb{V}(\ell_i) \subseteq \mathbb{P}^2$ meet at a point. For the first condition of Definition 1.1 to make sense, $\mathbb{K}$ should be an ordered field.

Definition 1.4. The *dual cone* to $\boldsymbol{P}_d$ is the cone spanned by the normals $\boldsymbol{n}_1, \dots, \boldsymbol{n}_d$ and is denoted $\boldsymbol{P}_d^*$.

Triangulating P_d yields a triangulation of $\boldsymbol{P}_d$, and the volume of the parallelepiped S spanned by vertices $\{\boldsymbol{v}_i, \boldsymbol{v}_j, \boldsymbol{v}_k, \boldsymbol{0}\}$ is $a_S = |\boldsymbol{v}_i\boldsymbol{v}_j\boldsymbol{v}_k|$.

Definition 1.5. Let C be a cone defined by a polygon P_d and $T(C)$ a triangulation of C obtained from a triangulation of P_d as above. The *adjoint* of C is

$$\mathcal{A}_{T(C)}(\boldsymbol{p}) = \sum_{S \in T(C)} a_S \prod_{\boldsymbol{v} \in \nu(\boldsymbol{P}_d) \setminus \nu(S)} (\boldsymbol{v} \cdot \boldsymbol{p}) \in \mathbb{K}[x, y, z]_{d-3}.$$

Theorem 1.6 [Warren 1996]. *$\mathcal{A}_{T(C)}(\boldsymbol{p})$ is independent of the triangulation $T(C)$.*

1C. ***Algebraic geometry background.*** Next, we review some background in algebraic geometry, referring to [Eisenbud 1995; Hartshorne 1977; Schenck 2003] for more detail. Homogenizing the numerators of Wachspress coordinates yields our main object of study:

Definition 1.7. The Wachspress map defined by a polygon P_d is the rational map $\mathbb{P}^2 \overset{w_d}{\dashrightarrow} \mathbb{P}^{d-1}$ given on the open set $U_{z \neq 0} \subseteq \mathbb{P}^2$ by $(x, y) \mapsto (b_1(x, y), \dots, b_d(x, y))$. The Wachspress variety W_d is the closure of the image of w_d.

The polynomial ring $S = \mathbb{K}[x_1, \dots, x_d]$ is a graded ring: it has a direct-sum decomposition into homogeneous pieces. A finitely generated graded S-module N admits a similar decomposition; if $s \in S_p$ and $n \in N_q$, then $s \cdot n \in N_{p+q}$. In particular, each N_q is a $(S_0 = \mathbb{K})$-vector space.

Definition 1.8. For a finitely generated graded S-module N, the Hilbert series $\mathrm{HS}(N, t) = \sum \dim_{\mathbb{K}} N_q t^q$.

Definition 1.9. A free resolution for an S-module N is an exact sequence

$$\mathbb{F} : \cdots \to F_i \xrightarrow{d_i} F_{i-1} \to \cdots \to F_0 \to N \to 0,$$

where the F_i are free S-modules.

If N is graded, then the F_i are also graded, so letting $S(-m)$ denote a rank-1 free module generated in degree m, we may write $F_i = \bigoplus_j S(-j)^{a_{i,j}}$. By the Hilbert syzygy theorem [Eisenbud 1995], a finitely generated, graded S-module N has a free resolution of length at most d with all the F_i of finite rank.

Definition 1.10. For a finitely generated graded S-module N, a free resolution is minimal if, for each i, $\mathrm{Im}(d_i) \subseteq \mathfrak{m}F_{i-1}$, where $\mathfrak{m} = \langle x_1, \ldots, x_d \rangle$. The graded Betti numbers of N are the $a_{i,j}$ that appear in a minimal free resolution, and the Castelnuovo–Mumford regularity of N is $\max_{i,j}\{a_{i,j} - i\}$.

While the differentials d_i that appear in a minimal free resolution of N are not unique, the ranks and degrees of the free modules that appear are unique. The graded Betti numbers are displayed in a *Betti table*. Reading this table right and down, starting at (0, 0), the entry $b_{ij} := a_{i,i+j}$, and the regularity of N is the index of the bottommost nonzero row in the Betti table for N.

Example 1.11. In Examples 2.9 and 3.11 of [Garcia-Puente and Sottile 2010], it is shown that I_{W_6} is generated by three quadrics and one cubic. The variety $\mathbb{V}(\ell_1 \cdots \ell_6)$ of the edges of P_6 has $\binom{6}{2} = 15$ singular points, of which six are vertices of P_6, and S/I_{W_6} has Betti table

total	1	4	6	3
0	1	–	–	–
1	–	3	–	–
2	–	1	6	3

For example, $b_{1,2} = a_{1,3} = 1$ reflects that I_{W_6} has a cubic generator, and S/I_{W_6} has regularity 2. The Hilbert series can be read off the Betti table:

$$\mathrm{HS}(S/I_{W_6}, t) = \frac{1 - 3t^2 - t^3 + 6t^4 - 3t^5}{(1-t)^6} = \frac{1 + 3t + 3t^2}{(1-t)^3}.$$

Theorem 5.11 gives a complete description of the Betti table of S/I_{W_d}.

2. $H^0(D_{d-2})$ and the Wachspress surface

2A. ***Background on blowups of $\mathbb{P}^2$.*** Fix points $p_1, \ldots, p_k \in \mathbb{P}^2$, and let

$$X \xrightarrow{\pi} \mathbb{P}^2 \tag{1}$$

be the blowup of $\mathbb{P}^2$ at these points. Then $\mathrm{Pic}(X)$ is generated by the exceptional curves E_i over the points p_i and the proper transform E_0 of a line in $\mathbb{P}^2$. A classical geometric problem asks for a relationship between numerical properties of a divisor $D_m = mE_0 - \sum a_i E_i$ on X and the geometry of

$$X \xrightarrow{\phi} \mathbb{P}(H^0(D_m)^\vee).$$

First, we discuss some basics. Let m and a_i be nonnegative, let I_{p_i} denote the ideal of a point p_i, and define

$$J = \bigcap_{i=1}^{k} I_{p_i}^{a_i} \subseteq \mathbb{K}[x, y, z] = R. \tag{2}$$

Then $H^0(D_m)$ is isomorphic to the m-th graded piece J_m of J (see [Harbourne 2002]). Davis and Geramita [1988] show that, if $\gamma(J)$ denotes the smallest degree t such that J_t defines J scheme theoretically, then D_m is very ample if $m > \gamma(J)$, and if $m = \gamma(J)$, then D_m is very ample if and only if J does not contain m collinear points, counted with multiplicity. Note that $\gamma(J) \le \operatorname{reg}(J)$.

2B. *Wachspress surfaces.* For a polygon P_d, fix defining linear forms ℓ_i as in Definition 1.3 and let $A := \ell_1 \cdots \ell_d$; the edges of P_d are defined by the $\mathbb{V}(\ell_i)$. Let Z denote the $\binom{d}{2}$ singular points of $\mathbb{V}(A)$ and $Y = Z \setminus \nu(P_d)$. Finally, X_d will be the blowup of $\mathbb{P}^2$ at Y. We study the divisor

$$D_{d-2} = (d-2)E_0 - \sum_{p \in Y} E_p$$

on X_d. First, we present some preliminaries.

Definition 2.1. Let L be the ideal in $R = \mathbb{K}[x, y, z]$ given by

$$L = \langle \ell_3 \cdots \ell_d, \ell_1\ell_4 \cdots \ell_d, \ldots, \ell_2 \cdots \ell_{d-1} \rangle = \langle A/\ell_1\ell_2, A/\ell_2\ell_3, \ldots, A/\ell_d\ell_1 \rangle,$$

where $A = \prod_{i=1}^{d} \ell_i$.

For any variety V, we use I_V to denote the ideal of polynomials vanishing on V.

Lemma 2.2. *The ideals L and I_Y are equal up to saturation at $\langle x, y, z \rangle$.*

Proof. Being equal up to saturation at $\langle x, y, z \rangle$ means that the localizations at any associated prime except $\langle x, y, z \rangle$ are equal. The ideal I_p of a point p is a prime ideal. Recall that the localization of a ring T at a prime ideal $\mathfrak{p}$ is a new ring $T_{\boldsymbol{p}}$ whose elements are of the form f/g with $f, g \in T$ and $g \notin \mathfrak{p}$. Localize R at I_p, where $p \in Y$. Then in R_{I_p}, ℓ_i is a unit if $p \notin \mathbb{V}(\ell_i)$. Without loss of generality, suppose forms ℓ_1 and ℓ_2 vanish on p (note that all points of Y are intersections of exactly two lines) and the remaining forms do not. Thus, $L_{I_p} = \langle \ell_1, \ell_2 \rangle = (I_Y)_{I_p}$. $\square$

The ideal L is not saturated.

Lemma 2.3. *I_Y is generated by one form F of degree $d-3$ and $d-3$ forms of degree $d-2$. Hence, a basis for L_{d-2} consists of $F \cdot x$, $F \cdot y$, $F \cdot z$, and the $d-3$ forms.*

Proof. First, note that I_Y cannot contain any form of degree $d-4$ since Y contains d sets of $d-3$ collinear points. So the smallest degree of a minimal generator for I_Y is $d-3$. Since Y consists of $\binom{d-1}{2}-1$ distinct points and the space of forms of degree $d-3$ has dimension $\binom{d-1}{2}$, there is at least one form F of degree $d-3$ in I_Y. We claim that it is unique. To see this, first note that no ℓ_i can divide F: by symmetry, if one ℓ_i divides F, they all must, which is impossible for degree reasons. Now suppose G is a second form of degree $d-3$ in I_Y. Let $p \in \nu(P_d)$ and $\mathbb{V}(\ell_i)$ be a line corresponding to an edge containing p. $F(p)$ must be nonzero since if not $\mathbb{V}(F)$ would contain $d-2$ collinear points of $\mathbb{V}(\ell_i)$, forcing $\mathbb{V}(F)$ to contain $\mathbb{V}(\ell_i)$, a contradiction. This also holds for G. But in this case, $F(p)G - G(p)F$ is a polynomial of degree $d-3$ vanishing at $d-2$ collinear points, again a contradiction. So F is unique (up to scaling), which shows that the Hilbert function satisfies

$$\mathrm{HF}(R/L, d-3) = |Y|,$$

so $\mathrm{HF}(R/L, t) = |Y|$ for all $t \geq d-3$ (see [Schenck 2003]). As the polynomials $A/\ell_i\ell_{i+1}$ are linearly independent and there are the correct number, L_{d-2} must be the degree-$(d-2)$ component of I_Y. □

Theorem 2.4. *The minimal free resolution of R/L is*

$$0 \to R(-d) \xrightarrow{d_3} R(-d+1)^d \xrightarrow{d_2} R(-d+2)^d \xrightarrow{\left[\frac{A}{\ell_1\ell_2}\ \frac{A}{\ell_2\ell_3}\ \cdots\ \frac{A}{\ell_d\ell_1}\right]} R \to R/L \to 0,$$

$$\text{where } d_2 = \begin{bmatrix} \ell_1 & 0 & \cdots & \cdots & 0 & 0 & m_1 \\ -\ell_3 & \ell_2 & 0 & \cdots & \vdots & \vdots & m_2 \\ 0 & -\ell_4 & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ell_{d-2} & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & -\ell_d & \ell_{d-1} & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & -\ell_1 & m_d \end{bmatrix}$$

and the m_i are linear forms.

Proof. By Lemma 2.3, the generators of I_Y are known. Since I_Y is saturated, the Hilbert–Burch theorem implies that the free resolution of R/I_Y has the form

$$0 \to R(-d+1)^{d-3} \to R(-d+3) \oplus R(-d+2)^{d-3} \to R \to R/I_Y \to 0.$$

Writing I_Y as $\langle f_1, \ldots, f_{d-3}, F\rangle$ and L as $\langle f_1, \ldots, f_{d-3}, xF, yF, zF\rangle$, the task is to understand the syzygies on L given the description above of the syzygies on I_Y. From the Hilbert–Burch resolution, any minimal syzygy on I_Y is of the form

$$\sum g_i f_i + qF = 0,$$

where g_i are linear and q is a quadric (or zero). Since

$$qF = g_1 xF + g_2 yF + g_3 zF \quad \text{with } g_i \text{ linear,}$$

all $d-3$ syzygies on I_Y lift to give linear syzygies on L. Furthermore, we obtain three linear syzygies on $\{xF, yF, zF\}$ from the three Koszul syzygies on $\{x, y, z\}$. It is clear from the construction that these d linear syzygies are linearly independent. Since $\mathrm{HF}(R/L, d-1) = |Y|$, this means we have determined all the linear first syzygies. Furthermore, the three Koszul first syzygies on $\{xF, yF, zF\}$ generate a linear second syzygy, so the complex given above is a subcomplex of the minimal free resolution. A check shows that the Buchsbaum–Eisenbud criterion [1973] holds, so the complex above is actually exact and hence a free resolution. The differential d_2 above involves the canonical generators $A/\ell_i\ell_{i+1}$ rather than a set involving $\{xF, yF, zF\}$. Since the $d-1$ linear syzygies appearing in the first $d-1$ columns of d_2 are linearly independent, they agree up to a change of basis; the last column of d_2 is a vector of linear forms determined by the change of basis. □

Theorem 2.5.

(i) $H^0(D_{d-2}) \simeq \mathrm{Span}_{\mathbb{K}}\{A/\ell_1\ell_2, A/\ell_2\ell_3, \dots\}$.

(ii) $H^1(D_{d-2}) = 0 = H^2(D_{d-2})$.

Proof. The remark following Equation (2) shows that $H^0(D_{d-2}) \simeq L_{d-2}$. Since $K = -3E_0 + \sum_{p\in Y} E_p$ (see [Hartshorne 1977]), by Serre duality,

$$H^2(D_{d-2}) \simeq H^0\Big((-d-1)E_0 + \sum_{p\in Y} E_p\Big),$$

which is clearly zero. Using that X_d is rational, it follows from Riemann–Roch that

$$h^0(D_{d-2}) - h^1(D_{d-2}) = \frac{D_{d-2}^2 - D_{d-2}\cdot K}{2} + 1.$$

The intersection pairing on X_d is given by $E_i^2 = 1$ if $i = 0$ and -1 if $i \neq 0$, and

$$E_i \cdot E_j = 0 \quad \text{if } i \neq j.$$

Thus,

$$D_{d-2}^2 = (d-2)^2 - |Y| \quad \text{and} \quad -D_{d-2}K = 3(d-2) - |Y|, \tag{3}$$

yielding

$$h^0(D_{d-2}) - h^1(D_{d-2}) = \frac{d^2 - d - 2 - 2|Y|}{2} + 1 = d. \tag{4}$$

Thus, $h^0(D_{d-2}) - h^1(D_{d-2}) = d$. Now apply the remark following Equation (2). □

Corollary 2.6. *If $d > 4$, D_{d-2} is very ample, so the image of X_d in $\mathbb{P}^{d-1}$ is smooth.*

Proof. By Theorem 2.4, the ideal L is $d-2$ regular. Furthermore, the set Y contains d sets of $d-3$ collinear points but no set of $d-2$ collinear points if $d > 4$. The result follows from the Davis–Geramita criterion. □

Theorem 2.7. *$W_4 \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and $X_4 \to W_4$ is an isomorphism away from the (-1) curve $E_0 - E_1 - E_2$, which is contracted to a smooth point.*

Proof. The surface X_4 is $\mathbb{P}^2$ blown up at two points, which is toric, and isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at a point. By Proposition 6.12 of [Cox et al. 2011], D_2 is basepoint free. Since $D_2^2 = 2$, W_4 is an irreducible quadric surface in $\mathbb{P}^3$. As $D_2 \cdot (E_0 - E_1 - E_2) = 0$, the result follows. □

Replacing D_{d-2} with $t D_{d-2}$, a computation as in Equations (3) and (4) and Serre vanishing shows that the Hilbert polynomial $\mathrm{HP}(S/I_{W_d}, t)$ is equal to

$$\frac{((d-2)^2-|Y|)t^2+(3(d-2)-|Y|)t}{2}+1=\frac{d^2-5d+8}{4}t^2-\frac{d^2-9d+12}{4}t+1. \quad (5)$$

3. The Wachspress quadrics

In this section, we construct a set of quadrics that vanish on W_d. These quadrics are polynomials that are expressed as a scalar product with a fixed vector τ. The vector τ defines a linear projection $\mathbb{P}^{d-1} \dashrightarrow \mathbb{P}^2$, also denoted by τ, given by

$$\boldsymbol{x} \mapsto \sum_{i=1}^{d} x_i \boldsymbol{v}_i,$$

where $\boldsymbol{x} = [x_1 : \cdots : x_d] \in \mathbb{P}^{d-1}$. By the second property of barycentric coordinates, the composition $\tau \circ w_d : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is the identity map on $\mathbb{P}^2$. Since $\boldsymbol{v}_i \in \mathbb{K}^3$, the vector τ is a triple of linear forms $(\tau_1, \tau_2, \tau_3) \in S^3$. The linear subspace $\mathscr{C}$ of $\mathbb{P}^{d-1}$ where the projection is undefined is the *center of projection*, and $I_{\mathscr{C}} = \langle \tau_1, \tau_2, \tau_3 \rangle$.

3A. *Diagonal monomials.* A *diagonal monomial* is a monomial $x_i x_j \in S_2$ such that $j \notin \{i-1, i, i+1\}$. We write $\mathscr{D}$ for the subspace of S_2 spanned by the diagonal monomials; identifying x_i with the vertex v_i, a diagonal monomial is a diagonal in P_d; see Figure 2.

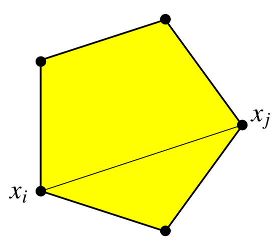


Figure 2. A diagonal monomial.

Lemma 3.1. *Any quadric that vanishes on W_d is a linear combination of elements of $\mathfrak{D}$.*

Proof. Let Q be a polynomial in $(I_{W_d})_2$. Then $Q(w_d) = Q(b_1, \dots, b_d) = 0$. On the edge $[v_k, v_{k+1}]$, all the b_i vanish except b_k and b_{k+1}. Thus, on this edge, the expression $Q(w_d) = 0$ is

$$c_1 b_k^2 + c_2 b_k b_{k+1} + c_3 b_{k+1}^2 = 0 \tag{6}$$

for some constants c_1, c_2, and c_3 in $\mathbb{K}$. Recall that $b_i(v_j) = 0$ if $i \neq j$ and $b_i(v_i) \neq 0$ for each i. Evaluating (6) at v_k and v_{k+1}, we conclude $c_1 = c_3 = 0$. At an interior point of edge $[v_k, v_{k+1}]$, neither b_k nor b_{k+1} vanishes. This implies that $c_2 = 0$. A similar calculation on each edge shows that all coefficients of nondiagonal terms in Q are zero. □

3B. *The map to* $(I_{\mathscr{C}})_2$. We define a surjective map onto $(I_{\mathscr{C}})_2$ and use the map to calculate the dimension of the vector space of polynomials in $(I_{\mathscr{C}})_2$ that are supported on diagonal monomials. Let S_1^3 denote the space of triples of linear forms on $\mathbb{P}^{d-1}$. Define the map $\Psi : S_1^3 \to (I_{\mathscr{C}})_2$ by $F \mapsto F \cdot \tau$, where $\cdot$ is the scalar product.

Lemma 3.2. *The kernel of Ψ is three-dimensional.*

Proof. Since $I_{\mathscr{C}}$ is a complete intersection, the kernel is generated by the three Koszul syzygies on the τ_i. □

Next we determine conditions on F so that $\Psi(F) \in \mathfrak{D}$. If $\boldsymbol{u}_i \in \mathbb{K}^3$ for $i = 1, \dots, d$, then

$$F = \sum_{i=1}^{d} x_i \boldsymbol{u}_i$$

is an element of S_1^3. Viewing the projection τ as an element of S_1^3, we have

$$\Psi(F) = F \cdot \tau = \left(\sum_{i=1}^{d} x_i \boldsymbol{u}_i\right) \cdot \left(\sum_{i=1}^{d} x_i \boldsymbol{v}_i\right) = \sum_{i,j=1}^{d} (\boldsymbol{u}_i \cdot \boldsymbol{v}_j + \boldsymbol{u}_j \cdot \boldsymbol{v}_i) x_i x_j. \tag{7}$$

If $\Psi(F) \in \mathfrak{D}$, then the coefficients of nondiagonal monomials must vanish:

$$\boldsymbol{u}_i \cdot \boldsymbol{v}_i = 0 \quad \text{and} \quad \boldsymbol{u}_i \cdot \boldsymbol{v}_{i+1} + \boldsymbol{u}_{i+1} \cdot \boldsymbol{v}_i = 0 \quad \text{for all } i. \tag{8}$$

Lemma 3.3. *The dimension of the vector space $\mathfrak{D} \cap (I_{\mathscr{C}})_2$ is $d - 3$.*

Proof. We show the conditions in (8) give $2d$ independent conditions on the $3d$-dimensional vector space S_1^3, and the solution space is $\Psi^{-1}(\mathfrak{D} \cap (I_{\mathscr{C}})_2)$; thus,

$\dim(\Psi^{-1}(\mathscr{D} \cap (I_{\mathscr{C}})_2)) = d$. The conditions are represented by the matrix equation

$$\begin{pmatrix} \boldsymbol{v}_1 \cdot \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{v}_d \cdot \boldsymbol{u}_d \\ \boldsymbol{v}_1 \cdot \boldsymbol{u}_2 + \boldsymbol{v}_2 \cdot \boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{v}_d \cdot \boldsymbol{u}_1 + \boldsymbol{v}_1 \cdot \boldsymbol{u}_d \end{pmatrix} = \overbrace{\begin{pmatrix} \boldsymbol{v}_1^T & 0 & \cdots & 0 \\ 0 & \boldsymbol{v}_2^T & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \boldsymbol{v}_d^T \\ \boldsymbol{v}_2^T & \boldsymbol{v}_1^T & & 0 \\ 0 & & \ddots & \\ \boldsymbol{v}_d^T & & & \boldsymbol{v}_1^T \end{pmatrix}}^{M} \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \boldsymbol{u}_d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix},$$

where the $\boldsymbol{v}_i$ and $\boldsymbol{u}_i$ are column vectors and the superscript T indicates transpose. The matrix M in the middle is a $2d \times 3d$ matrix, and the proof will be complete if the rows are shown to be independent. Denote the rows of M by $r_1, \ldots, r_d, r_{d+1}, \ldots, r_{2d}$, and let $c_1 r_1 + \cdots + c_d r_d + c_{d+1} r_{d+1} + \cdots + c_{2d} r_{2d}$ be a dependence relation among them. The first three columns of M give the dependence relation $c_1 \boldsymbol{v}_1 + c_{d+1} \boldsymbol{v}_2 + c_{2d} \boldsymbol{v}_d = 0$. Since $\boldsymbol{v}_d$, $\boldsymbol{v}_1$, and $\boldsymbol{v}_2$ define adjacent rays of a polyhedral cone, they must be independent, so c_1, c_{d+1}, and c_{2d} must be zero. Repeating the process at each triple $\boldsymbol{v}_{i-1}$, $\boldsymbol{v}_i$, and $\boldsymbol{v}_{i+1}$ shows the rest of the c_i's vanish. Since the restriction $\Psi : \Psi^{-1}(\mathscr{D} \cap (I_{\mathscr{C}})_2) \to \mathscr{D} \cap (I_{\mathscr{C}})_2$ remains surjective, we find $\dim(\mathscr{D} \cap (I_{\mathscr{C}})_2) = \dim(\Psi^{-1}(\mathscr{D} \cap (I_{\mathscr{C}})_2)) - \dim(\ker(\Psi)) = d - 3$. □

3C. *Wachspress quadrics.* We now compute the dimension and a generating set for $(I_{W_d})_2$.

Definition 3.4. Let $\gamma(i)$ denote the set $\{1, \ldots, d\} \setminus \{i-1, i\}$, $\gamma(i, j) = \gamma(i) \cap \gamma(j)$, and $\gamma(i, j, k) = \gamma(i) \cap \gamma(j) \cap \gamma(k)$.

The image of a diagonal monomial $x_i x_j$ under the pullback map $w_d^* : S \to R$ is

$$b_i b_j = \alpha_i \alpha_j \prod_{k \in \gamma(i)} \ell_k \prod_{m \in \gamma(j)} \ell_m = \alpha_i \alpha_j \prod_{k=1}^{d} \ell_k \prod_{m \in \gamma(i,j)} \ell_m,$$

and each diagonal monomial has a common factor $A = \prod_{k=1}^{d} \ell_k$. To find the quadratic relations among Wachspress coordinates, it suffices to find linear relations among products $\prod_{m \in \gamma(i,j)} \ell_m \in R_{d-4}$ for diagonal pairs i and j. Define the map $\phi : \mathscr{D} \to R_{d-4}$ by $x_i x_j \mapsto b_i b_j / A$, and extend by linearity; this is w_d^* restricted to $\mathscr{D}$ and divided by A. By Lemma 3.1, it follows that $(I_{W_d})_2 = \ker(\phi) \subseteq \mathscr{D}$.

Lemma 3.5. *The dimension of* $(I_{W_d})_2$ *is* $d - 3$.

Proof. We will show $\phi : \mathscr{D} \to R_{d-4}$ is surjective with $\dim(\ker \phi) = d - 3$. To see this, note that there are $d - 3$ diagonal monomials that have x_1 as a factor. We show

	$x_{2,4}$	$\cdots$	$x_{2,d}$	$x_{3,5}$	$\cdots$	$x_{3,d}$	$\cdots$	$x_{d-3,d-1}$	$x_{d-3,d}$	$x_{d-2,d}$
$p_{1,3}$	$*$									
$\vdots$		$\ddots$								
$p_{1,d-1}$			$*$							
$p_{2,4}$	$*$			$*$						
$\vdots$					$\ddots$					
$p_{2,d-1}$			$*$			$*$				
$\vdots$							$\ddots$			
$p_{(d-4)(d-2)}$								$*$		
$p_{(d-4)(d-1)}$								$*$	$*$	
$p_{(d-3)(d-1)}$								$*$	$*$	$*$

Table 1. Values of images of diagonal monomials at external vertices.

that the images of the remaining

$$d(d-3)/2-(d-3)=(d-3)(d-2)/2=\dim(R_{d-4})$$

diagonal monomials are independent. Let $p_{s,t}=\ell_s\cap\ell_t$ and $x_{p,q}=x_px_q$. In Table 1, a star, $*$, represents a nonzero number and a blank space is zero. The $(i,\,j)$ entry in the table represents the value of the image of the diagonal monomial in column j at the external vertex in row i. The external vertices not lying on ℓ_d are arranged down the rows with their indices in lexicographic order.

Since Table 1 is lower triangular, the images are independent. We have found $\dim(R_{d-4})$ independent images, and hence, ϕ is surjective. This is a map from a vector space of dimension $d(d-3)/2$ to one of dimension $(d-2)(d-3)/2$. The map is surjective, so the kernel has dimension $d(d-3)/2-(d-2)(d-3)/2=d-3$. $\square$

There is a generating set for $(I_{W_d})_2$ where each generator is a scalar product with the vector τ. The other vectors in these scalar products are

$$\Lambda_k=\frac{x_{k+1}}{\alpha_{k+1}}\boldsymbol{n}_{k+1}-\frac{x_k}{\alpha_k}\boldsymbol{n}_{k-1}\in S^3_1.$$

Lemma 3.6. *The vectors* $\{\Lambda_1,\ldots,\Lambda_d\}$ *form a basis for the space* $\Psi^{-1}(\mathscr{D}\cap(I_{\mathscr{C}})_2)$.

Proof. Suppose that $\sum_{k=1}^d c_k\Lambda_k=0$ is a linear dependence relation among the Λ_k. The coefficient of a variable x_k is

$$\frac{1}{\alpha_k}(c_{k-1}\boldsymbol{n}_k-c_k\boldsymbol{n}_{k-1}).$$

By the dependence relation, this must be zero, which implies that $\boldsymbol{n}_{k-1}$ and $\boldsymbol{n}_k$ are scalar multiples. This is impossible since they are normal vectors of adjacent facets of a polyhedral cone. Hence, $c_{k-1} = c_k = 0$ for all k, which shows that the Λ_k are independent.

In the proof of Lemma 3.3, we showed that $\dim(\Psi^{-1}(\mathscr{D} \cap (I_{\mathscr{C}})_2)) = d$, and we have just shown $\dim(\langle \Lambda_k \mid k = 1, \ldots, d \rangle) = d$. To prove the result, it suffices to show $\langle \Lambda_k \mid k = 1, \ldots, d \rangle \subseteq \Psi^{-1}(\mathscr{D} \cap (I_{\mathscr{C}})_2)$. The conditions of (8) are required for $\Lambda_k \in S_1^3$ to lie in $\Psi^{-1}(\mathscr{D} \cap (I_{\mathscr{C}})_2)$. We show these conditions are satisfied for each Λ_k.

Let $\boldsymbol{u}_i = 0$ if $i \neq k, k+1$, $\boldsymbol{u}_k = -\boldsymbol{n}_{k-1}/\alpha_k$, and $\boldsymbol{u}_{k+1} = \boldsymbol{n}_{k+1}/\alpha_{k+1}$ for each fixed k. Then

$$\Lambda_k = \frac{x_{k+1}}{\alpha_{k+1}} \boldsymbol{n}_{k+1} - \frac{x_k}{\alpha_k} \boldsymbol{n}_{k-1} = \sum_{i=1}^{d} \boldsymbol{u}_i x_i.$$

Since $\boldsymbol{n}_{k-1} \cdot \boldsymbol{v}_k = 0$, $\boldsymbol{n}_{k+1} \cdot \boldsymbol{v}_{k+1} = 0$, and $\boldsymbol{u}_i = 0$ for $i \neq k, k+1$, we have that $\boldsymbol{u}_i \cdot \boldsymbol{v}_i = 0$ for each $i = 1, \ldots d$. The expression $\boldsymbol{u}_i \cdot \boldsymbol{v}_{i+1} + \boldsymbol{u}_{i+1} \cdot \boldsymbol{v}_i$ is zero for all $i \neq k-1, k, k+1$ simply because $\boldsymbol{u}_i = 0$ for $i \neq k, k+1$. We have

$$\begin{aligned}
\boldsymbol{u}_k \cdot \boldsymbol{v}_{k+1} + \boldsymbol{u}_{k+1} \cdot \boldsymbol{v}_k &= -\frac{\boldsymbol{n}_{k-1}}{\alpha_k} \cdot \boldsymbol{v}_{k+1} + \frac{\boldsymbol{n}_{k+1}}{\alpha_{k+1}} \cdot \boldsymbol{v}_k \\
&= -\frac{\boldsymbol{v}_{k-1} \times \boldsymbol{v}_k \cdot \boldsymbol{v}_{k+1}}{\alpha_k} + \frac{\boldsymbol{v}_{k+1} \times \boldsymbol{v}_{k+2} \cdot \boldsymbol{v}_k}{\alpha_{k+1}} \\
&= -\frac{|\boldsymbol{v}_{k-1} \boldsymbol{v}_k \boldsymbol{v}_{k+1}|}{\alpha_k} + \frac{|\boldsymbol{v}_{k+1} \boldsymbol{v}_{k+2} \boldsymbol{v}_k|}{\alpha_{k+1}} = 0
\end{aligned}$$

as $\alpha_j = |\boldsymbol{v}_{j-1} \boldsymbol{v}_j \boldsymbol{v}_{j+1}|$. It is easy to show that the expression $\boldsymbol{u}_i \cdot \boldsymbol{v}_{i+1} + \boldsymbol{u}_{i+1} \cdot \boldsymbol{v}_i$ is zero for $i = k \pm 1$. Thus, the $\boldsymbol{u}_i$ satisfy the conditions in (8), so $\Lambda_k \in \Psi^{-1}(\mathscr{D} \cap (I_{\mathscr{C}})_2)$. □

Theorem 3.7 (Wachspress quadrics). *The Wachspress quadrics $(I_{W_d})_2$ are those elements of S_2 that are diagonally supported and vanish on $\mathscr{C}$. The quadrics $Q_k = \Lambda_k \cdot \tau$ for $k = 1, \ldots, d$ span $(I_{W_d})_2$.*

Proof. Let $\boldsymbol{p}$ be the vector (x, y, z). By definition of Wachspress coordinates,

$$\tau(w_d(\boldsymbol{p})) = \sum_{i=1}^{d} b_i(\boldsymbol{p}) \boldsymbol{v}_i = \boldsymbol{p} \sum_{i=1}^{d} b_i(\boldsymbol{p}).$$

We have

$$\begin{aligned}
\Lambda_k(w_d(\boldsymbol{p})) &= \frac{b_{k+1}(\boldsymbol{p})}{\alpha_{k+1}} \boldsymbol{n}_{k+1} - \frac{b_k(\boldsymbol{p})}{\alpha_k} \boldsymbol{n}_{k-1} \\
&= \Big(\prod_{j \neq k, k+1} \ell_j \Big) \boldsymbol{n}_{k+1} - \Big(\prod_{j \neq k-1, k} \ell_j \Big) \boldsymbol{n}_{k-1} \\
&= \Big(\prod_{j \neq k-1, k, k+1} \ell_j \Big) (\ell_{k-1} \boldsymbol{n}_{k+1} - \ell_{k+1} \boldsymbol{n}_{k-1}) \\
&= H[\boldsymbol{n}_{k+1}(\boldsymbol{n}_{k-1} \cdot \boldsymbol{p}) - \boldsymbol{n}_{k-1}(\boldsymbol{n}_{k+1} \cdot \boldsymbol{p})],
\end{aligned}$$

where $H = \prod_{j \neq k-1,k,k+1} \ell_j$. Set $\overline{H} := H \sum_{i=1}^{d} b_i(\boldsymbol{p})$. Then we have

$$\begin{aligned} Q_k(w_d(\boldsymbol{p})) &= \tau(w_d(\boldsymbol{p})) \cdot \Lambda_k(w_d(\boldsymbol{p})) \\ &= \overline{H}\boldsymbol{p} \cdot [\boldsymbol{n}_{k+1}(\boldsymbol{n}_{k-1} \cdot \boldsymbol{p}) - \boldsymbol{n}_{k-1}(\boldsymbol{n}_{k+1} \cdot \boldsymbol{p})] \\ &= \overline{H}[(\boldsymbol{n}_{k+1} \cdot \boldsymbol{p})(\boldsymbol{n}_{k-1} \cdot \boldsymbol{p}) - (\boldsymbol{n}_{k-1} \cdot \boldsymbol{p})(\boldsymbol{n}_{k+1} \cdot \boldsymbol{p})] = 0. \end{aligned}$$

We have just shown that $Q_k \in (I_{W_d})_2$. By Lemma 3.6, $\Psi^{-1}(\mathfrak{D} \cap (I_{\mathscr{C}})_2)$ is spanned by the Λ_k. Observe that $\langle Q_1, \ldots, Q_d \rangle = \Psi(\langle \Lambda_k \rangle) = \mathfrak{D} \cap (I_{\mathscr{C}})_2$. Thus, $\dim(\langle Q_1, \ldots, Q_d \rangle) = d - 3$, and by Lemma 3.5, $\dim((I_{W_d})_2) = d - 3$. Therefore, since $\langle Q_1, \ldots, Q_d \rangle \subseteq (I_{W_d})_2$, we have $\langle Q_1, \ldots, Q_d \rangle = (I_{W_d})_2 = \mathfrak{D} \cap (I_{\mathscr{C}})_2$. □

Corollary 3.8. *The quadrics* $\{\Lambda_2 \cdot \tau, \ldots, \Lambda_{d-2} \cdot \tau\}$ *are a basis for the quadrics in* I_{W_d}, *and in graded lex order*, $\{x_1x_3, \ldots, x_1x_{d-1}\}$ *is a basis for* $\mathrm{in}_{\prec}(I_{W_d})_2$.

Proof. Expanding the expression for $\Lambda_i \cdot \tau$ yields

$$\Lambda_i \cdot \tau = x_1x_{i+1}\left(\frac{\boldsymbol{v}_1 \cdot \boldsymbol{n}_{i+1}}{\alpha_{i+1}}\right) - x_1x_i\left(\frac{\boldsymbol{v}_1 \cdot \boldsymbol{n}_{i-1}}{\alpha_i}\right) + \zeta_i,$$

where $\zeta_i \in \mathbb{K}[x_2, \ldots, x_d]$. Since $\boldsymbol{n}_i = \boldsymbol{v}_i \times \boldsymbol{v}_{i+1}$,

$$\Lambda_2 \cdot \tau = x_1x_3\left(\frac{\boldsymbol{v}_1 \cdot \boldsymbol{n}_3}{\alpha_3}\right) + \zeta_2.$$

Since no three of the lines $\mathbb{V}(l_i)$ are concurrent, $\boldsymbol{v}_i \cdot \boldsymbol{n}_j$ is nonzero unless $j \in \{i, i+1\}$, so we may use the lead term of $\Lambda_2 \cdot \tau$ to reduce $\Lambda_3 \cdot \tau$ to $x_1x_4 + f(x_2, \ldots, x_d)$. Repeating the process proves that

$$\{x_1x_3, \ldots, x_1x_{d-1}\} \subseteq \mathrm{in}_{\prec}(I_{W_d})_2.$$

By Lemma 3.5, $(I_{W_d})_2$ has dimension $d - 3$, which concludes the proof. □

Corollary 3.9. *There are no linear first syzygies on* $(I_{W_d})_2$.

Proof. By Corollary 3.8, we may assume that a basis for $(I_{W_d})_2$ has the form

$$\begin{gathered} x_1x_3 + \zeta_3(x_2, \ldots, x_d), \\ x_1x_4 + \zeta_4(x_2, \ldots, x_d), \\ x_1x_5 + \zeta_5(x_2, \ldots, x_d), \\ \vdots \\ x_1x_{d-1} + \zeta_{d-1}(x_2, \ldots, x_d). \end{gathered}$$

Since the ζ_i do not involve x_1, this implies that any linear first syzygy on $(I_{W_d})_2$ must be a linear combination of the Koszul syzygies on $\{x_3, \ldots, x_{d-1}\}$. Now change the term order to graded lex with $x_i > x_{i+1} > \cdots > x_d > x_1 > x_2 > \cdots > x_{i-1}$. In this order, arguing as in the proof of Corollary 3.8 shows that we may assume a basis for $(I_{W_d})_2$ has the form

$$\begin{aligned} &x_i x_{i+2} + \zeta_{i+2}(x_1, \ldots, \widehat{x_i}, \ldots, x_d),\\ &x_i x_{i+3} + \zeta_{i+3}(x_1, \ldots, \widehat{x_i}, \ldots, x_d),\\ &x_i x_{i+4} + \zeta_{i+4}(x_1, \ldots, \widehat{x_i}, \ldots, x_d),\\ &\qquad\vdots\\ &x_i x_{i-2} + \zeta_{i-2}(x_1, \ldots, \widehat{x_i}, \ldots, x_d). \end{aligned}$$

Hence, any linear first syzygy on $(I_{W_d})_2$ must be a combination of Koszul syzygies on $x_{i+2}, x_{i+3}, \ldots, x_{i-2}$. Iterating this process for the term orders above shows there can be no linear first syzygies on $(I_{W_d})_2$. □

3D. *Decomposition of* $\mathbb{V}(\langle(I_{W_d})_2\rangle)$. We now prove that $\mathbb{V}(\langle(I_{W_d})_2\rangle) = \mathscr{C} \cup W_d$. The results in Sections 4 and 5 are independent of this fact.

Lemma 3.10. *For any i, j, and k, we have*

$$|\boldsymbol{n}_i \boldsymbol{n}_j \boldsymbol{n}_k| = |\boldsymbol{v}_j \boldsymbol{v}_k \boldsymbol{v}_{k+1}| \cdot |\boldsymbol{v}_i \boldsymbol{v}_{i+1} \boldsymbol{v}_{j+1}| - |\boldsymbol{v}_{j+1} \boldsymbol{v}_k \boldsymbol{v}_{k+1}| \cdot |\boldsymbol{v}_i \boldsymbol{v}_{i+1} \boldsymbol{v}_j|.$$

Proof. Apply the formulas $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$ and $|\boldsymbol{abc}| = \boldsymbol{a} \times \boldsymbol{b} \cdot \boldsymbol{c}$:

$$\begin{aligned} |\boldsymbol{n}_i \boldsymbol{n}_j \boldsymbol{n}_k| &= \boldsymbol{n}_i \times \boldsymbol{n}_j \cdot \boldsymbol{n}_k = (\boldsymbol{n}_i \times (\boldsymbol{v}_j \times \boldsymbol{v}_{j+1})) \cdot \boldsymbol{n}_k\\ &= [\boldsymbol{v}_j(\boldsymbol{n}_i \cdot \boldsymbol{v}_{j+1}) - \boldsymbol{v}_{j+1}(\boldsymbol{n}_i \cdot \boldsymbol{v}_j)] \cdot \boldsymbol{n}_k\\ &= (\boldsymbol{v}_j \cdot \boldsymbol{n}_k)(\boldsymbol{n}_i \cdot \boldsymbol{v}_{j+1}) - (\boldsymbol{v}_{j+1} \cdot \boldsymbol{n}_k)(\boldsymbol{n}_i \cdot \boldsymbol{v}_j)\\ &= |\boldsymbol{v}_j \boldsymbol{v}_k \boldsymbol{v}_{k+1}| \cdot |\boldsymbol{v}_i \boldsymbol{v}_{i+1} \boldsymbol{v}_{j+1}| - |\boldsymbol{v}_{j+1} \boldsymbol{v}_k \boldsymbol{v}_{k+1}| \cdot |\boldsymbol{v}_i \boldsymbol{v}_{i+1} \boldsymbol{v}_j|. \end{aligned}$$ □

Corollary 3.11. *We have $|\boldsymbol{n}_i \boldsymbol{n}_j \boldsymbol{n}_{j+1}| = \alpha_{j+1} |\boldsymbol{v}_i \boldsymbol{v}_{i+1} \boldsymbol{v}_{j+1}|$.*

Proof. This follows from Lemma 3.10 and the definition of α_{j+1}. □

Corollary 3.12. *We have $|\boldsymbol{n}_{i-1} \boldsymbol{n}_i \boldsymbol{n}_{i+1}| = \alpha_i \alpha_{i+1}$.*

Proof. This follows from Lemma 3.10 and the definition of α_i and α_{i+1}. □

Lemma 3.13. *Let $\boldsymbol{x} = [x_1 : \cdots : x_d] \in \mathbb{V}(\langle(I_{W_d})_2\rangle) \setminus \mathscr{C}$. If $\tau(\boldsymbol{x})$ is a base point $p_{ij} = \boldsymbol{n}_i \times \boldsymbol{n}_j$, then $\boldsymbol{x}$ lies on the exceptional line $\hat{p}_{ij}$ over p_{ij}.*

Proof. Since indices are cyclic, we assume that $i = 1$. Thus, $\tau(\boldsymbol{x}) = p_{1,j} = \boldsymbol{n}_1 \times \boldsymbol{n}_j$ for some $j \notin \{d, 1, 2\}$. The relation $Q_1(\boldsymbol{x}) = \Lambda_1 \cdot \tau(\boldsymbol{x}) = \Lambda_1 \cdot (\boldsymbol{n}_1 \times \boldsymbol{n}_j) = 0$ yields

$$L(1) := x_2 \boldsymbol{n}_2 \cdot p_{1,j} - x_1 \boldsymbol{n}_d \cdot p_{1,j} = 0. \tag{9}$$

The relation $Q_j(\boldsymbol{x}) = 0$ implies

$$L(j) := x_{j+1} |\boldsymbol{n}_{j+1} \boldsymbol{n}_1 \boldsymbol{n}_j| - x_j |\boldsymbol{n}_2 \boldsymbol{n}_1 \boldsymbol{n}_j| = 0. \tag{10}$$

Also,

$$Q_2(\boldsymbol{x}) = (x_3\boldsymbol{n}_3 - x_2\boldsymbol{n}_1)\cdot\boldsymbol{n}_1\times\boldsymbol{n}_j = x_3|\boldsymbol{n}_3\boldsymbol{n}_1\boldsymbol{n}_j| = 0,$$

implying $x_3 = 0$ since $|\boldsymbol{n}_3\boldsymbol{n}_1\boldsymbol{n}_j| \neq 0$ if $j \neq 3$. Assume $x_k = 0$ for $3 \le k < j-1$. Note that

$$Q_k(\boldsymbol{x}) = (x_{k+1}\boldsymbol{n}_{k+1} - x_k\boldsymbol{n}_{k-1})\cdot\boldsymbol{n}_1\times\boldsymbol{n}_j = x_{k+1}|\boldsymbol{n}_{k+1}\boldsymbol{n}_1\boldsymbol{n}_j| = 0;$$

hence, $x_{k+1} = 0$ since $|\boldsymbol{n}_{k+1}\boldsymbol{n}_1\boldsymbol{n}_j| \neq 0$ and by induction $x_k = 0$ for $3 \le k \le j-1$. An analogous argument shows that $x_k = 0$ for $j+2 \le k \le d$. Hence, $\boldsymbol{x}$ lies on the line $\mathbb{V}(L(1), L(j), x_k \mid k \notin \{1, 2, j, j+1\})$, which is the exceptional line $\hat{p}_{1,j}$. □

Theorem 3.14. *The subset $\mathbb{V}(\langle (I_{W_d})_2\rangle) \setminus \mathscr{C}$ is contained in W_d. It follows that the variety $\mathbb{V}(\langle (I_{W_d})_2\rangle)$ has irreducible decomposition $W_d \cup \mathscr{C}$.*

Proof. Let $\boldsymbol{x} = [x_1 : \cdots : x_d] \in \mathbb{V}(\langle (I_{W_d})_2\rangle) \setminus \mathscr{C}$. The Wachspress quadrics give the relations

$$x_{r+1}\boldsymbol{n}_{r+1}\cdot\tau = x_r\boldsymbol{n}_{r-1}\cdot\tau \tag{11}$$

for each $r = 1, \ldots, d$. By Theorem 1.6, the adjoint is independent of triangulation,

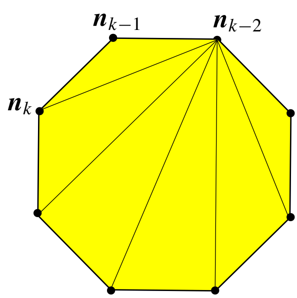


Figure 3. Triangulation used for adjoint.

so we use $\mathscr{A}$ to denote the adjoint, specifying the triangulation if necessary. We now show, for each $k \in \{1, \ldots, d\}$, $b_k(\tau(\boldsymbol{x})) = \mathscr{A}(\tau(\boldsymbol{x}))x_k$, where the triangulation above is used for the adjoint $\mathscr{A}$. It follows from the uniqueness of Wachspress coordinates that the denominator $\sum_{i=1}^d b_i$ of β_i is the adjoint of $\boldsymbol{P}_d^*$, so it follows that

$$w_d(\tau(\boldsymbol{x})) = \mathscr{A}(\tau(\boldsymbol{x}))\boldsymbol{x}. \tag{12}$$

Provided $\mathscr{A}(\tau(\boldsymbol{x})) \neq 0$, the result follows since $w_d(\tau(\boldsymbol{x})) \in \mathbb{P}^{d-1}$ is a nonzero scalar multiple of $\boldsymbol{x}$; hence, $\boldsymbol{x}$ is in the image of the Wachspress map and thus lies on W_d. If $\boldsymbol{x} \in \mathbb{V}(\langle (I_{W_d})_2\rangle)\setminus\mathscr{C}$ and $\mathscr{A}(\tau(\boldsymbol{x})) = 0$, then by (12) $w_d(\tau(\boldsymbol{x})) = 0$, and hence, $\tau(\boldsymbol{x})$ is a basepoint of w_d. Thus, $\tau(\boldsymbol{x}) = \boldsymbol{n}_i \times \boldsymbol{n}_j$ for some diagonal pair (i, j). By Lemma 3.13,

$\boldsymbol{x}$ lies on an exceptional line and hence lies on W_d. To prove the claim, note that since all indices are cyclic it suffices to assume $k = 3$. Let $|\boldsymbol{n}_i\boldsymbol{n}_j\boldsymbol{n}_k| = |\boldsymbol{n}_{ijk}|$ and

$$\boldsymbol{n}_{i_1,\ldots,i_m} \cdot \tau := \prod_{j=1}^{m} (\boldsymbol{n}_{i_j} \cdot \tau).$$

This is the product of m linear forms in S, and with this notation,

$$b_3(\tau) = \boldsymbol{n}_{1,4,5,\ldots,d} \cdot \tau.$$

For each $r \in \{3, \ldots, d\}$, define

$$\sigma_r := (\boldsymbol{n}_{4,\ldots,r} \cdot \tau)\boldsymbol{n}_1 \cdot \left[\sum_{i=3}^{r} \boldsymbol{v}_i (\boldsymbol{n}_{r+1,\ldots,d} \cdot \tau) x_i + \sum_{i=r+1}^{d} \boldsymbol{v}_i (\boldsymbol{n}_{r-1,\ldots,i-2} \cdot \tau)(\boldsymbol{n}_{i+1,\ldots,d} \cdot \tau) x_r\right],$$

where we set $\boldsymbol{n}_{i,\ldots,j} \cdot \tau = 1$ if $j < i$. We show $x_3\mathscr{A}(\tau(\boldsymbol{x})) = \sigma_3 = \sigma_d = b_3(\tau(\boldsymbol{x}))$. First, we show $\sigma_3 = x_3\mathscr{A}(\tau)$: to see this, note that

$$x_3\mathscr{A}(\tau) = |\boldsymbol{n}_{123}|(\boldsymbol{n}_{4,\ldots,d} \cdot \tau)x_3 + \sum_{i=4}^{d} |\boldsymbol{n}_{1,i-1,i}|(\boldsymbol{n}_{2,\ldots,i-2} \cdot \tau)(\boldsymbol{n}_{i+1,\ldots,d} \cdot \tau)x_3, \quad (13)$$

where we express the adjoint $\mathscr{A}$ using the triangulation in Figure 3. Applying the scalar triple product to $|\boldsymbol{n}_{123}|$ and $|\boldsymbol{n}_{1,i-1,i}|$ in the expression (13) yields

$$\boldsymbol{n}_1 \cdot (\boldsymbol{n}_2 \times \boldsymbol{n}_3)(\boldsymbol{n}_{4,\ldots,d} \cdot \tau)x_3 + \sum_{i=4}^{d} \boldsymbol{n}_1 \cdot (\boldsymbol{n}_{i-1} \times \boldsymbol{n}_i)(\boldsymbol{n}_{2,\ldots,i-2} \cdot \tau)(\boldsymbol{n}_{i+1,\ldots,d} \cdot \tau)x_3. \quad (14)$$

Factoring an $\boldsymbol{n}_1$ and noting that $\boldsymbol{n}_i \times \boldsymbol{n}_{i+1} = \boldsymbol{v}_{i+1}$, (14) becomes

$$\boldsymbol{n}_1 \cdot \left[\boldsymbol{v}_3(\boldsymbol{n}_{4,\ldots,d} \cdot \tau)x_3 + \sum_{i=4}^{d} \boldsymbol{v}_i (\boldsymbol{n}_{2,\ldots,i-2} \cdot \tau)(\boldsymbol{n}_{i+1,\ldots,d} \cdot \tau)x_3\right] = \sigma_3.$$

Now we show $\sigma_d = b_3(\tau)$. Since $\boldsymbol{n}_{d+1,\ldots,d} \cdot \tau = 1$,

$$\sigma_d = (\boldsymbol{n}_{4,\ldots,d} \cdot \tau)\boldsymbol{n}_1 \cdot \left(\sum_{i+3}^{d} \boldsymbol{v}_i (\boldsymbol{n}_{d+1,\ldots,d} \cdot \tau)x_i\right) = (\boldsymbol{n}_{4,\ldots,d} \cdot \tau)\boldsymbol{n}_1 \cdot \left(\sum_{i+3}^{d} \boldsymbol{v}_i x_i\right). \quad (15)$$

Observing that $\boldsymbol{n}_1 \cdot \sum_{i=1}^{2} x_i \boldsymbol{v}_i = 0$, we see that (15) is

$$(\boldsymbol{n}_{4,\ldots,d} \cdot \tau)(\boldsymbol{n}_1 \cdot \tau) = \boldsymbol{n}_{1,4,\ldots,d} \cdot \tau = b_3(\tau).$$

We now claim that for $r \in \{3, \dots, d-1\}$ we have $\sigma_r = \sigma_{r+1}$. Indeed,

$$\begin{aligned}\sigma_r &= (\boldsymbol{n}_{4,\dots,r}\cdot\tau)\boldsymbol{n}_1\cdot\Bigg[\sum_{i=3}^{r}\boldsymbol{v}_i(\boldsymbol{n}_{r+1,\dots,d}\cdot\tau)x_i \\ &\qquad\qquad + \sum_{i=r+1}^{d}\boldsymbol{v}_i(\boldsymbol{n}_{r,\dots,i-2}\cdot\tau)(\boldsymbol{n}_{i+1,\dots,d}\cdot\tau)(\boldsymbol{n}_{r-1}\cdot\tau)x_r\Bigg] \\ &= (\boldsymbol{n}_{4,\dots,r}\cdot\tau)\boldsymbol{n}_1\cdot\Bigg[\sum_{i=3}^{r}\boldsymbol{v}_i(\boldsymbol{n}_{r+1,\dots,d}\cdot\tau)x_i \\ &\qquad\qquad + \sum_{i=r+1}^{d}\boldsymbol{v}_i(\boldsymbol{n}_{r,\dots,i-2}\cdot\tau)(\boldsymbol{n}_{i+1,\dots,d}\cdot\tau)(\boldsymbol{n}_{r+1}\cdot\tau)x_{r+1}\Bigg],\end{aligned}$$

where we have applied (11) to the last term. Factoring out $\boldsymbol{n}_{r+1}\cdot\tau$ yields

$$(\boldsymbol{n}_{4,\dots,r+1}\cdot\tau)\boldsymbol{n}_1\cdot\Bigg[\sum_{i=3}^{r}\boldsymbol{v}_i(\boldsymbol{n}_{r+2,\dots,d}\cdot\tau)x_i + \sum_{i=r+1}^{d}\boldsymbol{v}_i(\boldsymbol{n}_{r,\dots,i-2}\cdot\tau)(\boldsymbol{n}_{i+1,\dots,d}\cdot\tau)x_{r+1}\Bigg].$$

Lastly, since the expressions in both summations agree at the index $i = r+1$, we can shift the indices of summation,

$$(\boldsymbol{n}_{4,\dots,r+1}\cdot\tau)\boldsymbol{n}_1\cdot\Bigg[\sum_{i=3}^{r+1}\boldsymbol{v}_i(\boldsymbol{n}_{r+2,\dots,d}\cdot\tau)x_i + \sum_{i=r+2}^{d}\boldsymbol{v}_i(\boldsymbol{n}_{r,\dots,i-2}\cdot\tau)(\boldsymbol{n}_{i+1,\dots,d}\cdot\tau)x_{r+1}\Bigg],$$

which is precisely σ_{r+1}, proving the claim. The claim shows that $\sigma_3 = \sigma_d$; hence, (12) holds, and so $\boldsymbol{x}$ lies in W_d if $\mathscr{A}(\tau(\boldsymbol{x})) \neq 0$. □

4. The Wachspress cubics

Theorem 3.14 shows that the Wachspress quadrics do not suffice to cut out the Wachspress variety W_d. We now construct cubics, the *Wachspress cubics*, that lie in I_{W_d} and do not arise from the Wachspress quadrics. These cubics are determinants of 3×3 matrices of linear forms. The key to showing that they are in I_{W_d} is to write them as a difference of adjoints $\mathscr{A}_{T_1(C)} - \mathscr{A}_{T_2(C)}$, where $T_1(C)$ and $T_2(C)$ are two different triangulations of a subcone C of the dual cone $\boldsymbol{P}_d^*$. By Theorem 1.6, the difference is zero, so the cubic is in I_{W_d}.

4A. ***Construction of Wachspress cubics.*** As in Lemma 3.6, let

$$\Lambda_r = \frac{x_{r+1}}{\alpha_{r+1}}\boldsymbol{n}_{r+1} - \frac{x_r}{\alpha_r}\boldsymbol{n}_{r-1}.$$

Theorem 4.1. *If $i \neq j \neq k \neq i$, then $w_{i,j,k} := |\Lambda_i, \Lambda_j, \Lambda_k| \in I_{W_d}$.*

Proof. We break the proof into two parts. First, suppose no pair of (i, j, k) corresponds to an edge of P_d. We call such an (i, j, k) a T-triple. A direct calculation shows that, if (i, j, k) is a T-triple, then evaluating the monomial $x_i x_j x_k$ at Wachspress coordinates yields

$$x_i x_j x_k(w_d) = b_i b_j b_k = A^2 \prod_{m \in \gamma(i,j,k)} \ell_m, \tag{16}$$

where $\gamma(i, j, k)$ is as in Definition 3.4. Since there are no T-triples if $d < 6$, we may assume $d \geq 6$. Changing variables by replacing x_i with x_i/α_i, we may ignore the constants α_i. Using the definition of the Λ's, observe that

$$\begin{aligned} w_{i,j,k} = |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{k+1}|x_{i+1}x_{j+1}x_{k+1} - |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{k-1}|x_{i+1}x_{j+1}x_k \\ - |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{k+1}|x_{i+1}x_j x_{k+1} + |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{k-1}|x_{i+1}x_j x_k \\ - |\boldsymbol{n}_{i-1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{k+1}|x_i x_{j+1}x_{k+1} + |\boldsymbol{n}_{i-1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{k-1}|x_i x_{j+1}x_k \\ + |\boldsymbol{n}_{i-1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{k+1}|x_i x_j x_{k+1} - |\boldsymbol{n}_{i-1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{k-1}|x_i x_j x_k. \end{aligned} \tag{17}$$

There are several situations to consider, depending on various possibilities for interactions among the indices. Interactions may occur if $i+1 = j-1$ or $j+1 = k-1$ or $k+1 = i-1$, so there are four cases:

1. All three hold. 2. Two hold. 3. One holds. 4. None hold.

Case 1. The indices (i, j, k) satisfy Case 1 if and only if $d = 6$. For $d = 6$, there are only two T-triples: $(1, 3, 5)$ and $(2, 4, 6)$. We show that $w_{1,3,5}$ vanishes on Wachspress coordinates; the case of $w_{2,4,6}$ is similar. All but two of the determinants in Equation (17) vanish, leaving

$$w_{1,3,5} = |\Lambda_1, \Lambda_3, \Lambda_5| = |\boldsymbol{n}_2\boldsymbol{n}_4\boldsymbol{n}_6|x_2x_4x_6 - |\boldsymbol{n}_6\boldsymbol{n}_2\boldsymbol{n}_4|x_1x_3x_5. \tag{18}$$

Notice that the coefficients are equal, and we conclude by showing that

$$x_1x_3x_5 - x_2x_4x_6$$

vanishes on Wachspress coordinates. The monomials $x_1x_3x_5$ and $x_2x_4x_6$ evaluated at Wachspress coordinates are $b_1b_3b_5$ and $b_2b_4b_6$, respectively. Both of these are equal to A^2, so $x_1x_3x_5 - x_2x_4x_6$ vanishes on Wachspress coordinates.

Case 2. We can assume without loss of generality $i+1 \neq j-1$, $j+1 = k-1$, and $k+1 = i-1$. Four coefficients vanish in (17), yielding

$$\begin{aligned} w_{i,j,k} = |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{i-1}|x_{i+1}x_{j+1}x_{i-1} \\ - |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{i-1}|x_{i+1}x_j x_{i-1} \\ + |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{j+1}|x_{i+1}x_j x_{i-2} \\ - |\boldsymbol{n}_{i-1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{j+1}|x_i x_j x_{i-2}. \end{aligned}$$

Evaluating this at Wachspress coordinates yields

$$
\begin{aligned}
w_{i,j,k}\circ w_d &= |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{i-1}| \prod_{m\in\gamma(i+1,j+1,i-1)} \ell_m + |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{i-1}| \prod_{m\in\gamma(i+1,j,i-1)} \ell_m \\
&\quad - |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{j+1}| \prod_{m\in\gamma(i+1,j,i-1)} \ell_m - |\boldsymbol{n}_{i-1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{j+1}| \prod_{m\in\gamma(i,j,i-1)} \ell_m \\
&= A^2\Bigl(\prod_{m\in\gamma(i-1,i+1,j+1,j)} \ell_m\Bigr)\bigl(|\boldsymbol{n}_{i+1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{i-1}|\ell_{j-1} - |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{i-1}|\ell_{j+1} \\
&\qquad + |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{j+1}|\ell_{i-1} - |\boldsymbol{n}_{i-1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{j+1}|\ell_{i+1}\bigr) \\
&= A^2\Bigl(\prod_{m\in\gamma(i-1,i+1,j+1,j)} \ell_m\Bigr)\bigl[(|\boldsymbol{n}_{i+1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{i-1}|\ell_{j-1} + |\boldsymbol{n}_{i-1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{j-1}|\ell_{i+1}) \\
&\qquad - (|\boldsymbol{n}_{i+1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{i-1}|\ell_{j+1} + |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{j-1}|\ell_{i-1})\bigr],
\end{aligned}
$$

where

$$A = \prod_{i=1}^{d} \ell_i.$$

The last factor is the difference of two adjoints with respect to the triangulations of the quadrilateral in Figure 4. The vanishing can be seen directly: write $\boldsymbol{n}_1, \ldots, \boldsymbol{n}_4$ for $\boldsymbol{n}_{i-1}$, $\boldsymbol{n}_{i+1}$, $\boldsymbol{n}_{j-1}$, and $\boldsymbol{n}_{j+1}$. Then the last factor is

$$|\boldsymbol{n}_2\boldsymbol{n}_3\boldsymbol{n}_4|\ell_1 - |\boldsymbol{n}_1\boldsymbol{n}_3\boldsymbol{n}_4|\ell_2 + |\boldsymbol{n}_1\boldsymbol{n}_2\boldsymbol{n}_4|\ell_3 - |\boldsymbol{n}_1\boldsymbol{n}_2\boldsymbol{n}_3|\ell_4.$$

Applying $\dfrac{d}{dx}$ to this shows the x coefficient is

$$|\boldsymbol{n}_2\boldsymbol{n}_3\boldsymbol{n}_4|\boldsymbol{n}_{11} - |\boldsymbol{n}_1\boldsymbol{n}_3\boldsymbol{n}_4|\boldsymbol{n}_{21} + |\boldsymbol{n}_1\boldsymbol{n}_2\boldsymbol{n}_4|\boldsymbol{n}_{31} - |\boldsymbol{n}_1\boldsymbol{n}_2\boldsymbol{n}_3|\boldsymbol{n}_{41}.$$

This is the determinant of the matrix of the $\boldsymbol{n}_i$ with a repeat row for the x coordinates $\boldsymbol{n}_{i1}$, so it vanishes. Reason similarly for the y and z coefficients.

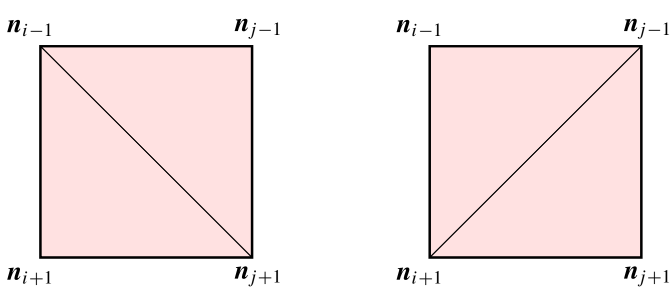


Figure 4. Case 2 triangulation.

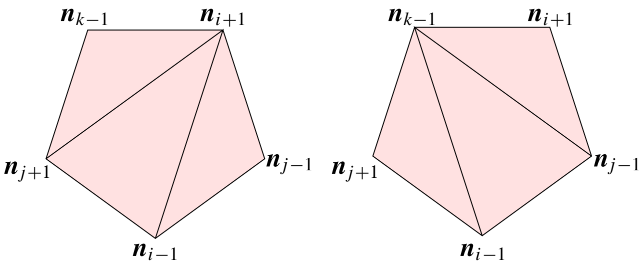


Figure 5. Case 3 triangulation.

Case 3. Assume without loss of generality $i+1 \neq j-1$, $j+1 \neq k-1$, and $k+1 = i-1$. In this case, two coefficients vanish in (17), and after evaluating at Wachspress coordinates, we obtain

$$\begin{aligned} w_{i,j,k} \circ w_d &= |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{i-1}| \prod_{m\in\gamma(i+1,j+1,k+1)} \ell_m - |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{k-1}| \prod_{m\in\gamma(i+1,j+1,k)} \ell_m \\ &\quad - |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{i-1}| \prod_{m\in\gamma(i+1,j,k+1)} \ell_m + |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{k-1}| \prod_{m\in\gamma(i+1,j,k)} \ell_m \\ &\quad + |\boldsymbol{n}_{i-1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{k-1}| \prod_{m\in\gamma(i,j+1,k)} \ell_m - |\boldsymbol{n}_{i-1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{k-1}| \prod_{m\in\gamma(i,j,k)} \ell_m \\ &= A^2\Bigg(\prod_{\substack{m\in\gamma(i,j,k,\\ i+1,j+1,k+1)}} \ell_m\Bigg)\big(|\boldsymbol{n}_{i+1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{i-1}|\ell_{j-1}\ell_{k-1} - |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{k-1}|\ell_{i-1}\ell_{j-1} \\ &\quad - |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{i-1}|\ell_{j+1}\ell_{k-1} + |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{k-1}|\ell_{j+1}\ell_{i-1} \\ &\quad + |\boldsymbol{n}_{i-1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{k-1}|\ell_{i+1}\ell_{j-1} - |\boldsymbol{n}_{i-1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{k-1}|\ell_{i+1}\ell_{j+1}\big). \end{aligned}$$

The last factor is the difference of adjoints with respect to the triangulations of the pentagon in Figure 5.

Case 4. In this case, evaluation at Wachspress coordinates yields

$$\begin{aligned} w_{i,j,k} \circ w_d = |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{k+1}| &\prod_{m\in\gamma(i+1,j+1,k+1)} \ell_m - |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{k-1}| \prod_{m\in\gamma(i+1,j+1,k)} \ell_m \\ - |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{k+1}| &\prod_{m\in\gamma(i+1,j,k+1)} \ell_m + |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{k-1}| \prod_{m\in\gamma(i+1,j,k)} \ell_m \\ - |\boldsymbol{n}_{i-1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{k+1}| &\prod_{m\in\gamma(i,j+1,k+1)} \ell_m + |\boldsymbol{n}_{i-1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{k-1}| \prod_{m\in\gamma(i,j+1,k)} \ell_m \\ + |\boldsymbol{n}_{i-1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{k+1}| &\prod_{m\in\gamma(i,j,k+1)} \ell_m - |\boldsymbol{n}_{i-1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{k-1}| \prod_{m\in\gamma(i,j,k)} \ell_m \end{aligned}$$

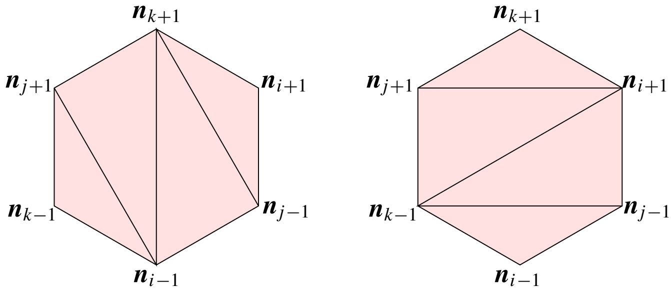


Figure 6. Case 4 triangulation.

$$
\begin{aligned}
= A^2\Big(\prod_{\substack{m\in\gamma(i,j,k,\\ i+1,j+1,k+1)}} \ell_m\Big) \\
\times \big(&|\boldsymbol{n}_{i+1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{k+1}|\ell_{i-1}\ell_{j-1}\ell_{k-1} - |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{k-1}|\ell_{i-1}\ell_{j-1}\ell_{k+1} \\
&- |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{k+1}|\ell_{i-1}\ell_{j+1}\ell_{k-1} + |\boldsymbol{n}_{i+1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{k-1}|\ell_{i-1}\ell_{j+1}\ell_{k+1} \\
&- |\boldsymbol{n}_{i-1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{k+1}|\ell_{i+1}\ell_{j-1}\ell_{k-1} + |\boldsymbol{n}_{i-1}\boldsymbol{n}_{j+1}\boldsymbol{n}_{k-1}|\ell_{j+1}\ell_{i-1}\ell_{k+1} \\
&+ |\boldsymbol{n}_{i-1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{k+1}|\ell_{i+1}\ell_{j+1}\ell_{k-1} - |\boldsymbol{n}_{i-1}\boldsymbol{n}_{j-1}\boldsymbol{n}_{k-1}|\ell_{i+1}\ell_{j+1}\ell_{k+1}\big).
\end{aligned}
$$

The last factor is the difference of adjoints expressed using the triangulations of the hexagon in Figure 6. This completes the analysis when (i, j, k) is a T-triple.

Next, we consider the situation when (i, j, k) contains a pair of consecutive indices. Suppose first that there are exactly two consecutive vertices; without loss of generality, we assume the indices are $(2, 3, i)$ with $i > 4$. We have

$$
\begin{aligned}
w_{2,3,i} := |\Lambda_2\Lambda_3\Lambda_i| = &|\boldsymbol{n}_2\boldsymbol{n}_4\boldsymbol{n}_{i+1}|x_3x_4x_{i+1} - |\boldsymbol{n}_3\boldsymbol{n}_4\boldsymbol{n}_{i-1}|x_3x_4x_i \\
&- |\boldsymbol{n}_3\boldsymbol{n}_2\boldsymbol{n}_{i+1}|x_3x_3x_{i+1} + |\boldsymbol{n}_3\boldsymbol{n}_2\boldsymbol{n}_{i-1}|x_3x_3x_i \\
&- |\boldsymbol{n}_1\boldsymbol{n}_4\boldsymbol{n}_{i+1}|x_2x_4x_{i+1} + |\boldsymbol{n}_1\boldsymbol{n}_4\boldsymbol{n}_{i-1}|x_2x_4x_i \\
&+ |\boldsymbol{n}_1\boldsymbol{n}_2\boldsymbol{n}_{i+1}|x_2x_3x_{i+1} - |\boldsymbol{n}_1\boldsymbol{n}_2\boldsymbol{n}_{i-1}|x_2x_3x_i.
\end{aligned}
$$

We show that $w_{2,3,i} \circ w_d$ is a multiple of the difference between two expressions of the adjoint polynomial of a polygon with respect to two different triangulations. After evaluation at w_d, each monomial has a common factor of $A \prod_{j\neq 2,3} \ell_j$. Thus, we can express

$$
\frac{w_{2,3,i}(w_d)}{A \prod_{j\neq 2,3} \ell_j}
$$

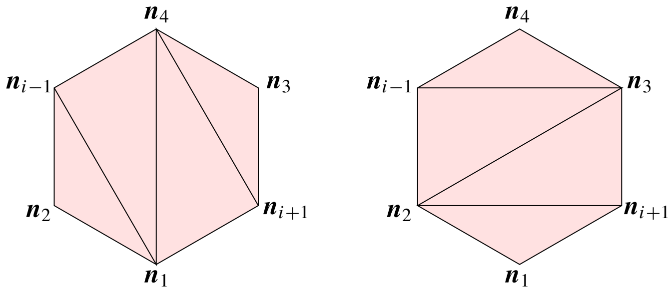


Figure 7. Triangulations for the non-T-triples.

as

$$\begin{aligned}\frac{w_{2,3,i}(w_d)}{A\prod_{j\neq 2,3}\ell_j} = |\boldsymbol{n}_2\boldsymbol{n}_4\boldsymbol{n}_{i+1}|\prod_{j\neq 3,4,i+1}\ell_j - |\boldsymbol{n}_3\boldsymbol{n}_4\boldsymbol{n}_{i-1}|\prod_{j\neq 3,4,i-1}\ell_j \\ - |\boldsymbol{n}_3\boldsymbol{n}_2\boldsymbol{n}_{i+1}|\prod_{j\neq 2,3,i+1}\ell_j + |\boldsymbol{n}_3\boldsymbol{n}_2\boldsymbol{n}_{i-1}|\prod_{j\neq 2,3,i-1}\ell_j \\ - |\boldsymbol{n}_1\boldsymbol{n}_4\boldsymbol{n}_{i+1}|\prod_{j\neq 1,4,i+1}\ell_j + |\boldsymbol{n}_1\boldsymbol{n}_4\boldsymbol{n}_{i-1}|\prod_{j\neq 1,4,i-1}\ell_j \\ + |\boldsymbol{n}_1\boldsymbol{n}_2\boldsymbol{n}_{i+1}|\prod_{j\neq 1,2,i+1}\ell_j - |\boldsymbol{n}_1\boldsymbol{n}_2\boldsymbol{n}_{i-1}|\prod_{j\neq 1,2,i-1}\ell_j \\ = \Bigl(\prod_{j\in\gamma(2,4,i,i+1)}\ell_j\Bigr)\bigl(|\boldsymbol{n}_2\boldsymbol{n}_4\boldsymbol{n}_{i+1}|\ell_1\ell_3\ell_{i-1} - |\boldsymbol{n}_3\boldsymbol{n}_4\boldsymbol{n}_{i-1}|\ell_1\ell_2\ell_{i+1} \\ - |\boldsymbol{n}_3\boldsymbol{n}_2\boldsymbol{n}_{i+1}|\ell_1\ell_4\ell_{i-1} + |\boldsymbol{n}_3\boldsymbol{n}_2\boldsymbol{n}_{i-1}|\ell_1\ell_4\ell_{i+1} \\ - |\boldsymbol{n}_1\boldsymbol{n}_4\boldsymbol{n}_{i+1}|\ell_2\ell_3\ell_{i-1} + |\boldsymbol{n}_1\boldsymbol{n}_4\boldsymbol{n}_{i-1}|\ell_2\ell_3\ell_{i+1} \\ + |\boldsymbol{n}_1\boldsymbol{n}_2\boldsymbol{n}_{i+1}|\ell_3\ell_4\ell_{i-1} - |\boldsymbol{n}_1\boldsymbol{n}_2\boldsymbol{n}_{i-1}|\ell_3\ell_4\ell_{i+1}\bigr).\end{aligned}$$

The factor in parentheses is the difference of the adjoints computed with respect to the triangulations of the polygon in Figure 7.

Finally, for the case where the three vertices are consecutive, assume without loss of generality the triple is $(2, 3, 4)$, and proceed as above. In this case, the triangulations that arise are those that appear in Figure 5. □

Definition 4.2. $I(d)$ is the ideal generated by the Wachspress quadrics appearing in Corollary 3.8 and the Wachspress cubics appearing in Theorem 4.1.

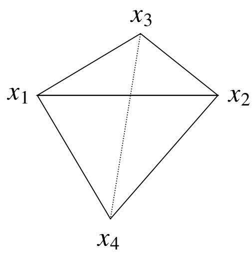
5. Gröbner basis, Stanley–Reisner ring, and free resolution

In this section, we determine the initial ideal of $I(d)$ in graded lex order and prove $I(d) = I_{W_d}$. First, we present some preliminaries.

5A. *Simplicial complexes and combinatorial commutative algebra.* An abstract n-simplex is a set consisting of all subsets of an $(n+1)$-element ground set. Typically a simplex is viewed as a geometric object; for example, a 2-simplex on the set $\{a, b, c\}$ can be visualized as a triangle with the subset $\{a, b, c\}$ corresponding to the whole triangle, $\{a, b\}$ an edge, and $\{a\}$ a vertex. For this reason, elements of the ground set are called the vertices.

Definition 5.1 [Ziegler 1995]. A simplicial complex Δ on a vertex set V is a collection of subsets σ of V such that, if $\sigma \in \Delta$ and $\tau \subset \sigma$, then $\tau \in \Delta$. If $|\sigma| = i + 1$, then σ is called an i-face. Let $f_i(\Delta)$ denote the number of i-faces of Δ, and define $\dim(\Delta) = \max\{i \mid f_i(\Delta) \neq 0\}$. If $\dim(\Delta) = n - 1$, we define $f_\Delta(t) = \sum_{i=0}^{n} f_{i-1}t^{n-i}$. The ordered list of coefficients of $f_\Delta(t)$ is the f-vector of Δ, and the coefficients of $h_\Delta(t) := f_\Delta(t-1)$ are the h-vector of Δ.

Example 5.2. Consider the 1-skeleton of a tetrahedron with vertices x_1, x_2, x_3, x_4, as in the figure.



The corresponding simplicial complex Δ consists of all vertices and edges, so $\Delta = \{\varnothing, \{x_i\}, \{x_i, x_j\} \mid 1 \le i \le 4 \text{ and } i < j \le 4\}$. Thus, $f(\Delta) = (1, 4, 6)$ and $h(\Delta) = (1, 2, 3)$; the empty face gives $f_{-1}(\Delta) = 1$.

A simplicial complex Δ can be used to define a commutative ring, known as the Stanley–Reisner ring. This construction allows us to use tools of commutative algebra to prove results about the topology or combinatorics of Δ.

Definition 5.3. Let Δ be a simplicial complex on vertices $\{x_1, \dots, x_n\}$. The Stanley–Reisner ideal I_Δ is

$$I_\Delta = \langle x_{i_1} \cdots x_{i_j} \mid \{x_{i_1}, \dots, x_{i_j}\} \text{ is not a face of } \Delta \rangle \subseteq \mathbb{K}[x_1, \dots, x_n],$$

and the Stanley–Reisner ring is $\mathbb{K}[x_1, \dots, x_k]/I_\Delta$.

In Example 5.2, since Δ has no 2-faces,

$$I_\Delta = \langle x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4 \rangle = \bigcap_{1 \le i < j \le 4} \langle x_i, x_j \rangle.$$

Definition 5.4. A prime ideal P is associated to a graded S-module N if P is the annihilator of some $n \in N$, and $\operatorname{Ass}(N)$ is the set of all associated primes of N.

Definition 5.5. Let $\operatorname{codim}(N) = \min\{\operatorname{codim}(P) \mid P \in \operatorname{Ass}(N)\}$ for a finitely generated graded S-module N. The projective dimension $\operatorname{pdim}(N)$ is the length of a minimal free resolution of N; N is Cohen–Macaulay if $\operatorname{codim}(N) = \operatorname{pdim}(N)$. S/I is arithmetically Cohen–Macaulay if it is Cohen–Macaulay as an S-module.

5B. ***Application to Wachspress surfaces.***

Definition 5.6. Define $I_\Gamma(d) \subseteq \mathbb{K}[x_1, \dots, x_d]$ as

$$I_\Gamma(d) = \langle x_1x_3, \dots, x_1x_{d-1}\rangle + K_{2,d-1},$$

where $K_{2,d-1}$ consists of all square-free cubic monomials in $x_2, \dots, x_{d-1}$.

Theorem 5.7. *The quotient $S/I_\Gamma(d)$ is arithmetically Cohen–Macaulay, of Castelnuovo–Mumford regularity two, and has Hilbert series*

$$\operatorname{HS}(S/I_\Gamma(d), t) = \frac{1 + (d-3)t + \binom{d-3}{2}t^2}{(1-t)^3}.$$

Proof. The ideal $I_\Gamma(d)$ is the Stanley–Reisner ideal of a one-dimensional simplicial complex Γ consisting of a complete graph on vertices $\{x_2, \dots, x_{d-1}\}$ with a single additional edge $\overline{x_1x_2}$ attached. All connected graphs are shellable, so since shellable implies Cohen–Macaulay (see [Miller and Sturmfels 2005]), $S/I_\Gamma(d)$ is Cohen–Macaulay. Since $I_\Gamma(d)$ contains no terms involving x_d, if $S' = \mathbb{K}[x_1, \dots, x_{d-1}]$, then

$$S/I_\Gamma(d) \simeq S'/I_\Gamma(d) \otimes \mathbb{K}[x_d].$$

The Hilbert series of a Stanley–Reisner ring has numerator equal to the h-vector of the associated simplicial complex (see [Schenck 2003]), which in this case is a graph on $d-1$ vertices with $\binom{d-2}{2}+1$ edges. Converting $f(\Gamma) = (1, d-1, \binom{d-2}{2}+1)$ to $h(\Gamma)$ yields the Hilbert series of $S'/I_\Gamma(d)$. The Hilbert series of a graph has denominator $(1-t)^2$, and tensoring with $\mathbb{K}[x_d]$ contributes a factor of $1/(1-t)$, yielding the result. □

Theorem 5.8. *In graded lex order*, $\operatorname{in}_\prec I(d) = I_\Gamma(d)$.

Proof. First, note that

$$I_\Gamma(d) \subseteq \operatorname{in}_\prec I(d),$$

which follows from Corollary 3.8 and Theorem 4.1, combined with the observation that, in graded lex order, $\operatorname{in}(|\Lambda_i\Lambda_j\Lambda_k|) = x_ix_jx_k$ if $i < j < k$ as long as $k \neq d$. Since $I(d) \subseteq I_{W_d}$, there is a surjection

$$S/I(d) \twoheadrightarrow S/I_{W_d};$$

hence, $\operatorname{HP}(S/I(d), t) \geq \operatorname{HP}(S/I_{W_d}, t)$. Since

$$\operatorname{HP}(S/I(d), t) = \operatorname{HP}(S/\operatorname{in}_\prec I(d), t)$$

and

$$I_\Gamma(d) \subseteq \mathrm{in}_\prec I(d),$$

we have

$$\mathrm{HP}(S/I_\Gamma(d), t) \geq \mathrm{HP}(S/\mathrm{in}_\prec I(d), t) = \mathrm{HP}(S/I(d), t) \geq \mathrm{HP}(S/I_{W_d}, t).$$

The Hilbert polynomial $\mathrm{HP}(S/I_{W_d}, t)$ is given by Equation (5). The Hilbert series of $S/I_\Gamma(d)$ is given by Theorem 5.7, from which we can extract the Hilbert polynomial:

$$\mathrm{HP}(S/I_\Gamma(d), t) = \binom{d-3}{2}\binom{t}{2} + (d-3)\binom{t+1}{2} + \binom{t+2}{2},$$

and a check shows this agrees with Equation (5). Since $I_\Gamma(d) \subseteq \mathrm{in}_\prec I(d)$, equality of the Hilbert polynomials implies that in high degree (i.e., up to saturation)

$$I_\Gamma(d) = \mathrm{in}_\prec I(d) \quad \text{and} \quad I(d) = I_{W_d}.$$

Consider the short exact sequence

$$0 \to \mathrm{in}_\prec I(d)/I_\Gamma(d) \to S/I_\Gamma(d) \to S/\mathrm{in}_\prec I(d) \to 0.$$

By Lemma 3.6 of [Eisenbud 1995],

$$\mathrm{Ass}(\mathrm{in}_\prec I(d)/I_\Gamma(d)) \subseteq \mathrm{Ass}(S/I_\Gamma(d)). \tag{19}$$

Since $\mathrm{HP}(S/I_\Gamma(d), t) = \mathrm{HP}(S/\mathrm{in}_\prec I(d), t)$, the module $\mathrm{in}_\prec I(d)/I_\Gamma(d)$ must vanish in high degree so is supported at $\mathfrak{m}$, which is of codimension d. But $I_\Gamma(d)$ is a radical ideal supported in codimension $d-3$, so it follows from Equation (19) that $\mathrm{in}_\prec I(d)/I_\Gamma(d)$ must vanish. □

Corollary 5.9. *The ideal $I(d)$ is the ideal of the image of*

$$X_d \to \mathbb{P}(H^0(D_{d-2})).$$

In particular, $I(d) = I_{W_d}$, and $S/I(d)$ is arithmetically Cohen–Macaulay.

Proof. By the results of Sections 2 and 3, $I(d) \subseteq I_{W_d}$, and the proof of Theorem 5.8 showed that they are equal up to saturation. Hence, $I_{W_d}/I(d)$ is supported at $\mathfrak{m}$. Consider the short exact sequence

$$0 \to I_{W_d}/I(d) \to S/I(d) \to S/(I_{W_d}) \to 0.$$

Since $S/I_\Gamma(d) = S/\mathrm{in}_\prec I(d)$ is arithmetically Cohen–Macaulay of codimension $d-3$, by uppersemicontinuity [Herzog 2005], so is $S/I(d)$, so $I_{W_d}/I(d) = 0$. □

Corollary 5.10. *The quotient S/I_{W_d} has regularity* 2.

Proof. Since $S/I(d)$ is Cohen–Macaulay, reducing modulo a linear regular sequence of length 3 yields an Artinian ring with the same regularity, which is equal to the socle degree [Eisenbud 2005]. By Theorems 5.7 and 5.8, this is 2, so the regularity of S/I_{W_d} is 2. □

Theorem 5.11. *The nonzero graded Betti numbers of the minimal free resolution of $S/I(d)$ are given by $b_{12} = d - 3$ and for $i \geq 3$ by*

$$b_{i-2,i} = \binom{d-3}{i} - (d-3)\binom{d-3}{i-1} + \binom{d-3}{2}\binom{d-3}{i-2}.$$

Proof. By Corollary 5.10, there are only two rows in the Betti table of $S/I(d)$. By Corollary 3.9, the top row is empty, save for the quadratic generators at the first step. Thus, the entire Betti diagram may be obtained from the Hilbert series, which is given in Theorem 5.7, and the result follows. □

We are at work on generalizing the results here to higher dimensions.

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cfirving@scu.edu *Department of Mathematics and Computer Science, Santa Clara University, 500 El Camino Real, Santa Clara, CA 95053, United States*

schenck@math.uiuc.edu *Department of Mathematics, University of Illinois at Urbana–Champaign, 1409 West Green Street, Urbana, IL 61801, United States*

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