The radius of a subcategory of modules

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Dedicated to Professor Craig Huneke on the occasion of his sixtieth birthday

We introduce a new invariant for subcategories $\mathcal{X}$ of finitely generated modules over a local ring R which we call the radius of $\mathcal{X}$. We show that if R is a complete intersection and $\mathcal{X}$ is resolving, then finiteness of the radius forces $\mathcal{X}$ to contain only maximal Cohen–Macaulay modules. We also show that the category of maximal Cohen–Macaulay modules has finite radius when R is a Cohen–Macaulay complete local ring with perfect coefficient field. We link the radius to many well-studied notions such as the dimension of the stable category of maximal Cohen–Macaulay modules, finite/countable Cohen–Macaulay representation type and the uniform Auslander condition.

Introduction

Let R be a commutative Noetherian local ring and mod R the category of finitely generated modules over R. In this paper we introduce and study a new invariant for subcategories $\mathcal{X}$ of mod R which we call the radius of $\mathcal{X}$. Roughly speaking, it is defined as the least number of extensions necessary to build the whole objects in $\mathcal{X}$ out of a single object in mod R. (For the precise definition, see Definition 2.3 in this paper.) Our definition is inspired by the notion of dimension of triangulated categories that was introduced by Rouquier [2008].

We obtain strong evidences that the concept of radius is intimately linked to both the representation theory and the singularity of R. For example, over a Gorenstein complete local ring R, the category of maximal Cohen–Macaulay modules has radius zero if and only if R has finite Cohen–Macaulay representation type, in other words, R is a simple hypersurface singularity (when R has an algebraically closed coefficient field of characteristic zero). In addition, the category of maximal Cohen–Macaulay modules over a complete local hypersurface (over an algebraically closed

field of characteristic not two) of countable Cohen–Macaulay representation type
has radius one. We also observe a tantalizing connection to the uniform Auslander
condition, which has attracted researchers over the years.

Perhaps most surprisingly, one corollary of our first main result (Theorem 3.3)
states:

Theorem I. *Let R be a local complete intersection, and let $\mathscr{X}$ be a resolving
subcategory of* mod *R. If the radius of $\mathscr{X}$ is finite, then $\mathscr{X}$ contains only maximal
Cohen–Macaulay modules.*

We conjecture that the above result holds for all Cohen–Macaulay local rings.
Our second main result below supports this conjecture, which follows from more
general results (Theorems 5.7 and 5.11).

Theorem II. *Let R be a Cohen–Macaulay complete local ring with perfect coef-
ficient field. Then the category of maximal Cohen–Macaulay modules over R has
finite radius.*

The structure of the paper is as follows. In Section 1 we set the basic notations
and definitions. Section 2 contains our key definition (Definition 2.1) of the radius
of a subcategory of mod *R*, as well as some detailed comparisons to similar notions.
We also give several results connecting the radius to the singularities of finite and
countable Cohen–Macaulay representation type. Sections 3 and 4 consist of the
statement and proof of our Theorem I, respectively. We also discuss here thickness
of resolving subcategories of maximal Cohen–Macaulay modules over a complete
intersection. Section 5 contains the proof of (generalizations of) our Theorem II.
Section 6 connects the main results to the uniform Auslander condition and discusses
some open questions.

1. Preliminaries

In this section, we recall the definitions of a resolving subcategory, totally reflexive
modules and a thick subcategory.

Convention 1.1. Throughout this paper, we assume all rings are commutative
Noetherian rings with identity. All modules are finitely generated. All subcategories
are full and strict. (Recall that a subcategory $\mathscr{X}$ of a category $\mathscr{C}$ is called *strict*
provided that for objects M, $N \in \mathscr{C}$ with $M \cong N$, if M is in $\mathscr{X}$, then so is N.) Hence,
the *subcategory* of a category $\mathscr{C}$ consisting of objects $\{M_\lambda\}_{\lambda \in \Lambda}$ always means the
smallest strict full subcategory of $\mathscr{C}$ to which M_λ belongs for all $\lambda \in \Lambda$. Note that
this coincides with the full subcategory of $\mathscr{C}$ consisting of all objects $X \in \mathscr{C}$ such
that $X \cong M_\lambda$ for some $\lambda \in \Lambda$. Let R be a (commutative Noetherian) ring. Denote by
mod *R* the category of (finitely generated) *R*-modules and *R*-homomorphisms. For
a Cohen–Macaulay local ring R, we call a maximal Cohen–Macaulay R-module

just a Cohen–Macaulay R-module. We denote by $\mathrm{CM}(R)$ the subcategory of mod R consisting of Cohen–Macaulay R-modules.

The following notation is used throughout this paper.

Notation 1.2. For a subcategory $\mathscr{X}$ of mod R, we denote by add $\mathscr{X}$ (or $\mathrm{add}_R \mathscr{X}$) the *additive closure* of $\mathscr{X}$, namely, the subcategory of mod R consisting of direct summands of finite direct sums of modules in $\mathscr{X}$. When $\mathscr{X}$ consists of a single module M, we simply denote it by add M (or $\mathrm{add}_R M$). For an R-module M, we denote by M^* the R-dual module $\mathrm{Hom}_R(M, R)$. For a homomorphism $f : M \to N$ of R-modules, f^* denotes the R-dual homomophism $N^* \to M^*$ sending $\sigma \in N^*$ to the composition $\sigma \cdot f \in M^*$.

The notion of a resolving subcategory has been introduced by Auslander and Bridger [1969]. It can actually be defined for an arbitrary abelian category with enough projective object. The only resolving subcategories we deal with in this paper are ones of mod R.

Definition 1.3. A subcategory $\mathscr{X}$ of mod R is called *resolving* if the following hold.

(R1) $\mathscr{X}$ contains the projective R-modules.

(R2) $\mathscr{X}$ is closed under direct summands: if M is an R-module in $\mathscr{X}$ and N is an R-module that is a direct summand of M, then N is also in $\mathscr{X}$.

(R3) $\mathscr{X}$ is closed under extensions: for an exact sequence $0 \to L \to M \to N \to 0$ of R-modules, if L, N are in $\mathscr{X}$, then so is M.

(R4) $\mathscr{X}$ is closed under kernels of epimorphisms: for an exact sequence $0 \to L \to M \to N \to 0$ of R-modules, if M, N are in $\mathscr{X}$, then so is L.

A resolving subcategory is a subcategory such that any two minimal resolutions of a module by modules in it have the same length; see [Auslander and Bridger 1969, Lemma (3.12)]. Note that one can replace the condition (R1) with:

(R1′) $\mathscr{X}$ contains R.

Next we recall the notion of a totally reflexive module.

Definition 1.4. An R-module M is called *totally reflexive* if the natural homomorphism $M \to M^{**}$ is an isomorphism and $\mathrm{Ext}^i_R(M, R) = 0 = \mathrm{Ext}^i_R(M^*, R)$ for all $i > 0$. We denote by $\mathscr{G}(R)$ the subcategory of mod R consisting of totally reflexive modules.

A totally reflexive module was defined by Auslander [1967], and deeply studied by Auslander and Bridger [1969]. The R-dual of a totally reflexive R-module is also totally reflexive. Every projective module is totally reflexive, i.e., add $R \subseteq \mathscr{G}(R)$. If R is a Cohen–Macaulay local ring, then every totally reflexive R-module is Cohen–Macaulay, i.e., $\mathscr{G}(R) \subseteq \mathrm{CM}(R)$. When R is a Gorenstein local ring, an R-module

is totally reflexive if and only if it is Cohen–Macaulay, i.e., $\mathcal{G}(R) = \mathrm{CM}(R)$. For more details, see [Auslander and Bridger 1969] and [Christensen 2000].

Syzygies, cosyzygies and transposes are key tools in this paper. We recall here their precise definitions.

Definition 1.5. Let $(R, \mathfrak{m})$ be a local ring, and let M be an R-module.

(1) Take a minimal free resolution $\cdots \xrightarrow{\delta_{n+1}} F_n \xrightarrow{\delta_n} F_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_1} F_0 \to M \to 0$ of M. Then, for each $n \geq 1$, the image of δ_n is called the *n-th syzygy* of M and denoted by $\Omega^n M$ (or $\Omega_R^n M$). For convention, we set $\Omega^0 M = M$.

(2) The cokernel of the R-dual map $\delta_1^* : F_0^* \to F_1^*$ is called the (*Auslander*) *transpose* of M and denoted by $\mathrm{Tr}\, M$ (or $\mathrm{Tr}_R\, M$).

(3) Let $0 \to M \to F_{-1} \xrightarrow{\delta_{-1}} \cdots \xrightarrow{\delta_{-(n-1)}} F_{-n} \xrightarrow{\delta_{-n}} F_{-(n+1)} \xrightarrow{\delta_{-(n+1)}} \cdots$ be a *minimal free coresolution* of M, that is, an exact sequence with F_{-n} free and $\mathrm{Im}\,\delta_{-n} \subseteq \mathfrak{m} F_{-(n+1)}$ for all $n \geq 1$. Then we call the image of δ_{-n} the *n-th cosyzygy* of M and denote it by $\Omega^{-n} M$ (or $\Omega_R^{-n} M$).

Let R be a local ring. Then by [Yoshino 2005, Lemma 3.2] one can replace (R4) with:

(R4$'$) $\mathscr{X}$ is closed under syzygies: if M is in $\mathscr{X}$, then so is ΩM.

Totally reflexive modules behave well under taking their syzygies, cosyzygies and transposes. Let R be a local ring. Let M be a totally reflexive R-module. The R-dual of a minimal free resolution (respectively, coresolution) of M is a minimal free coresolution (respectively, resolution) of M^*. In particular, a minimal free coresolution of M always exists, and it is uniquely determined up to isomorphism. The n-th syzygy $\Omega^n M$ and cosyzygy $\Omega^{-n} M$ are again totally reflexive for all n. This is an easy consequence of [Christensen 2000, (1.2.9) and (1.4.8)]. The transpose $\mathrm{Tr}\, M$ is also totally reflexive; see [Auslander and Bridger 1969, Proposition (3.8)]. For an R-module M, the n-th syzygy $\Omega^n M$ for any $n \geq 1$ and the transpose $\mathrm{Tr}\, M$ are uniquely determined up to isomorphism, since so is a minimal free resolution of M. If M is totally reflexive, then the n-th cosyzygy $\Omega^{-n} M$ for any $n \geq 1$ is also uniquely determined up to isomorphism, since so is a minimal free coresolution of M.

A lot of subcategories of $\mathrm{mod}\, R$ are known to be resolving. For example, $\mathrm{CM}(R)$ is a resolving subcategory of $\mathrm{mod}\, R$ if R is Cohen–Macaulay. The subcategory of $\mathrm{mod}\, R$ consisting of totally reflexive R-modules is resolving by [Auslander and Bridger 1969, (3.11)]. One can construct a resolving subcategory easily by using the vanishing of Tor or Ext. Also, the modules of complexity less than a fixed integer form a resolving subcategory of $\mathrm{mod}\, R$. For the details, we refer to [Takahashi 2009, Example 2.4].

Now we define a thick subcategory of totally reflexive modules.

Definition 1.6. A subcategory $\mathscr{X}$ of $\mathscr{G}(R)$ is called *thick* if it is closed under direct summands and short exact sequences: for an exact sequence $0 \to L \to M \to N \to 0$ of totally reflexive R-modules, if two of L, M, N are in $\mathscr{X}$, then so is the third.

A typical example of a thick subcategory is obtained by restricting a resolving subcategory to $\mathscr{G}(R)$.

The following proposition is shown by an argument dual to [Yoshino 2005, Lemma 3.2].

Proposition 1.7. *Let R be a local ring. Let $\mathscr{X}$ be a subcategory of $\mathscr{G}(R)$ containing R. Then $\mathscr{X}$ is a thick subcategory of $\mathscr{G}(R)$ if and only if $\mathscr{X}$ is a resolving subcategory of* mod R *and is closed under cosyzygies: if M is in $\mathscr{X}$, then so is $\Omega^{-1}M$.*

Let $(R, \mathfrak{m})$ be a local ring. We call R a *hypersurface* if the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is a residue ring of a complete regular local ring by a principal ideal. We say that R is a *complete intersection* if $\widehat{R}$ is a residue ring of a complete regular local ring by an ideal generated by a regular sequence.

We recall the definitions of *Gorenstein dimension* and *complete intersection dimension*, which are abbreviated to G-dimension and CI-dimension. These notions have been introduced by Auslander and Bridger [1969] and Avramov, Gasharov and Peeva [Avramov et al. 1997], respectively.

Definition 1.8. Let R be a local ring, and let M be an R-module. The *G-dimension* of M, denoted $\mathsf{Gdim}_R M$, is defined as the infimum of the lengths of totally reflexive resolutions of M, namely, exact sequences of the form $0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to M \to 0$ with each X_i being totally reflexive. The *CI-dimension* of M is defined as the infimum of $\mathsf{pd}_S(M \otimes_R R') - \mathsf{pd}_S R'$, where $R \to R' \leftarrow S$ runs over the quasi-deformations of R. Here, a diagram $R \xrightarrow{f} R' \xleftarrow{g} S$ of homomorphisms of local rings is called a quasi-deformation of R if f is faithfully flat and g is a surjection whose kernel is generated by an S-sequence.

Recall that M is said to have *complexity c*, denoted by $\mathsf{cx}_R M = c$, if c is the least nonnegative integer n such that there exists a real number r satisfying the inequality $\beta_i^R(M) \leq ri^{n-1}$ for all $i \gg 0$.

Remark 1.9. For a local ring $(R, \mathfrak{m}, k)$ and a module M over R, the following are known to hold. For the proofs, we refer to [Christensen 2000] and [Avramov et al. 1997].

(1) $\mathsf{Gdim}_R M = \infty$ if and only if M does not admit a totally reflexive resolution of finite length.

(2) $\mathsf{CIdim}_R M = \infty$ if and only if $\mathsf{pd}_S(M \otimes_R R') = \infty$ for every quasi-deformation $R \to R' \leftarrow S$.

(3) One has $M = 0 \Leftrightarrow \mathsf{Gdim}_R M = -\infty \Leftrightarrow \mathsf{CIdim}_R M = -\infty$.

(4) $\mathrm{Gdim}_R M \le 0$ if and only if M is totally reflexive.

(5) If $\mathrm{Gdim}_R M$ (respectively, $\mathrm{Cldim}_R M$) is finite, it is equal to depth $R - \mathrm{depth}_R M$.

(6) The inequalities $\mathrm{Gdim}_R M \le \mathrm{Cldim}_R M \le \mathrm{pd}_R M$ hold, and equalities hold to the left of any finite dimension.

(7) If $M \ne 0$, then $\mathrm{Gdim}_R(\Omega^n M) = \sup\{\mathrm{Gdim}_R M - n, 0\}$ and $\mathrm{Cldim}_R(\Omega^n M) = \sup\{\mathrm{Cldim}_R M - n, 0\}$ hold for all $n \ge 0$.

(8) If R is a Gorenstein ring (respectively, a complete intersection), then $\mathrm{Gdim}_R M$ (respectively, $\mathrm{Cldim}_R M$) is finite. If $\mathrm{Gdim}_R k$ (respectively, $\mathrm{Cldim}_R k$) is finite, then R is a Gorenstein ring (respectively, a complete intersection).

(9) If $\mathrm{Cldim}_R M < \infty$, then $\mathrm{cx}_R M < \infty$.

2. Definition of the radius of a subcategory

This section contains the key definition and establishes several results. More precisely, we will give the definition of the radius of a subcategory of mod R for a local ring R, and compare it with other notions, such as the dimension of a triangulated category defined by Rouquier. We will also explore its relationships with representation types of a Cohen–Macaulay local ring.

Definition 2.1. Let R be a local ring.

(1) For a subcategory $\mathscr{X}$ of mod R we denote by $[\mathscr{X}]$ the additive closure of the subcategory of mod R consisting of R and all modules of the form $\Omega^i X$, where $i \ge 0$ and $X \in \mathscr{X}$. When $\mathscr{X}$ consists of a single module X, we simply denote it by $[X]$.

(2) For subcategories $\mathscr{X}, \mathscr{Y}$ of mod R we denote by $\mathscr{X} \circ \mathscr{Y}$ the subcategory of mod R consisting of the R-modules M which fits into an exact sequence $0 \to X \to M \to Y \to 0$ with $X \in \mathscr{X}$ and $Y \in \mathscr{Y}$. We set $\mathscr{X} \bullet \mathscr{Y} = [[\mathscr{X}] \circ [\mathscr{Y}]]$.

(3) Let $\mathscr{C}$ be a subcategory of mod R. We define the *ball of radius r centered at $\mathscr{C}$* as

$$[\mathscr{C}]_r = \begin{cases} [\mathscr{C}] & \text{if } r = 1, \\ [\mathscr{C}]_{r-1} \bullet \mathscr{C} = [[\mathscr{C}]_{r-1} \circ [\mathscr{C}]] & \text{if } r \ge 2. \end{cases}$$

If $\mathscr{C}$ consists of a single module C, then we simply denote $[\mathscr{C}]_r$ by $[C]_r$, and call it the ball of radius r centered at C. We write $[\mathscr{C}]_r^R$ when we should specify that mod R is the ground category where the ball is defined.

Some similar notions have already been introduced. Takahashi [2009, Definition 3.1] defines the subcategory $\mathrm{res}^n \mathscr{X}$ of the resolving closure $\mathrm{res}\, \mathscr{X}$ of a given subcategory $\mathscr{X}$ of mod R. This is different from ours in that $\mathrm{res}^n \mathscr{X}$ is not closed under syzygies. In [Avramov et al. 2010] the *thickening* $\mathrm{thick}^n \mathscr{X}$ of a given subcategory $\mathscr{X}$ of a triangulated category is defined. This cannot be applied directly to a module category.

Proposition 2.2. *Let R be a local ring.*

(1) *Let $\mathcal{X}, \mathcal{Y}$ be subcategories of* $\mathrm{mod}\, R$. *The following are equivalent for an R-module M:*

    (a) *M belongs to $\mathcal{X} \bullet \mathcal{Y}$.*

    (b) *There exists an exact sequence $0 \to X \to Z \to Y \to 0$ of R-modules with $X \in [\mathcal{X}]$ and $Y \in [\mathcal{Y}]$ such that M is a direct summand of Z.*

(2) *For subcategories $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ of* $\mathrm{mod}\, R$, *one has $(\mathcal{X} \bullet \mathcal{Y}) \bullet \mathcal{Z} = \mathcal{X} \bullet (\mathcal{Y} \bullet \mathcal{Z})$.*

(3) *Let $\mathcal{C}$ be a subcategory of* $\mathrm{mod}\, R$, *and let a, b be positive integers. Then one has $[\mathcal{C}]_a \bullet [\mathcal{C}]_b = [\mathcal{C}]_{a+b} = [\mathcal{C}]_b \bullet [\mathcal{C}]_a$.*

Proof. (1) The implication (b) $\Rightarrow$ (a) is obvious. To prove the opposite implication (a) $\Rightarrow$ (b), let M be an R-module in $\mathcal{X} \bullet \mathcal{Y} = [[\mathcal{X}] \circ [\mathcal{Y}]]$. By definition, M is isomorphic to a direct summand of $R^{\oplus p} \oplus \bigoplus_{i=0}^{n}(\Omega^i Z_i)^{\oplus q_i}$, where $p, q_i \geq 0$ and $Z_i \in [\mathcal{X}] \circ [\mathcal{Y}]$. For each $0 \leq i \leq n$ there is an exact sequence $0 \to X_i \to Z_i \to Y_i \to 0$ with $X_i \in [\mathcal{X}]$ and $Y_i \in [\mathcal{Y}]$. Taking syzygies and direct sums, we have an exact sequence

$$0 \to R^{\oplus p} \oplus \bigoplus_{i=0}^{n}(\Omega^i X_i)^{\oplus q_i} \to R^{\oplus p} \oplus \bigoplus_{i=0}^{n}(\Omega^i Z_i)^{\oplus q_i} \oplus R^{\oplus r} \to \bigoplus_{i=0}^{n}(\Omega^i Y_i)^{\oplus q_i} \to 0.$$

The left and right terms are in $[\mathcal{X}]$ and $[\mathcal{Y}]$, respectively. The middle term contains an R-module isomorphic to M. Thus the statement (b) follows.

(2) First, let M be an R-module in $(\mathcal{X} \bullet \mathcal{Y}) \bullet \mathcal{Z}$. By the assertion (1) there is an exact sequence $0 \to W \xrightarrow{f} V \to Z \to 0$ with $W \in \mathcal{X} \bullet \mathcal{Y}$ and $Z \in [\mathcal{Z}]$ such that M is a direct summand of V. By (1) again, we have an exact sequence $0 \to X \to U \to Y \to 0$ with $X \in [\mathcal{X}]$ and $Y \in [\mathcal{Y}]$ such that W is a direct summand of U. Writing $U = W \oplus W'$, we make the following pushout diagram.

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
X & = & X \\
\downarrow & & \downarrow \\
0 \longrightarrow W \oplus W' \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & 1 \end{smallmatrix}\right)} V \oplus W' \longrightarrow Z \longrightarrow 0 \\
\downarrow \qquad\qquad \downarrow \qquad\quad \| \\
0 \longrightarrow Y \longrightarrow T \longrightarrow Z \longrightarrow 0 \\
\downarrow \qquad\qquad \downarrow \\
0 \qquad\qquad 0
\end{array}
$$

The bottom row implies that T is in $\mathcal{Y} \bullet \mathcal{Z}$, and it follows from the middle column that M belongs to $\mathcal{X} \bullet (\mathcal{Y} \bullet \mathcal{Z})$. Hence we have $(\mathcal{X} \bullet \mathcal{Y}) \bullet \mathcal{Z} \subseteq \mathcal{X} \bullet (\mathcal{Y} \bullet \mathcal{Z})$.

Next, let M be an R-module in $\mathcal{X} \bullet (\mathcal{Y} \bullet \mathcal{Z})$. Then it follows from (1) that there is an exact sequence $0 \to X \to V \xrightarrow{f} W \to 0$ with $X \in [\mathcal{X}]$ and $W \in \mathcal{Y} \bullet \mathcal{Z}$ such that M is a direct summand of V. Applying (1) again, we have an exact sequence $0 \to Y \to U \to Z \to 0$ with $Y \in [\mathcal{Y}]$ and $Z \in [\mathcal{Z}]$ such that W is a direct summand of U. Write $U = W \oplus W'$, and we have a pullback diagram:

$$
\begin{array}{ccccccccc}
& & & & 0 & & 0 & & \\
& & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & T & \longrightarrow & Y & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & V \oplus W' & \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & 1 \end{smallmatrix}\right)} & W \oplus W' & \longrightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & Z & = & Z & & \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}
$$

We see from the first row that T is in $\mathcal{X} \bullet \mathcal{Y}$, and from the middle column that M is in $(\mathcal{X} \bullet \mathcal{Y}) \bullet \mathcal{Z}$. Therefore $\mathcal{X} \bullet (\mathcal{Y} \bullet \mathcal{Z}) \subseteq (\mathcal{X} \bullet \mathcal{Y}) \bullet \mathcal{Z}$ holds.

(3) It is enough to show the equality $[\mathcal{C}]_a \bullet [\mathcal{C}]_b = [\mathcal{C}]_{a+b}$. We prove this by induction on b. It holds by definition when $b = 1$. Let $b \geq 2$. Then we have

$$[\mathcal{C}]_a \bullet [\mathcal{C}]_b = [\mathcal{C}]_a \bullet ([\mathcal{C}]_{b-1} \bullet \mathcal{C}) = ([\mathcal{C}]_a \bullet [\mathcal{C}]_{b-1}) \bullet \mathcal{C} = [\mathcal{C}]_{a+b-1} \bullet \mathcal{C} = [\mathcal{C}]_{a+b},$$

where the second equality follows from (2), and the induction hypothesis implies the third equality. $\qquad \square$

Let $\mathcal{C}$ be a subcategory of $\bmod R$ and $r \geq 0$ an integer. By the second and third assertions of Proposition 2.2, without danger of confusion we can write

$$[\mathcal{C}]_r = \overbrace{\mathcal{C} \bullet \cdots \bullet \mathcal{C}}^{r}.$$

Now we can make the definition of the radius of a subcategory.

Definition 2.3. Let R be a local ring, and let $\mathcal{X}$ be a subcategory of $\bmod R$. We define the *radius* of $\mathcal{X}$, denoted by radius $\mathcal{X}$, as the infimum of the integers $n \geq 0$ such that there exists a ball of radius $n + 1$ centered at a module containing $\mathcal{X}$. By definition, radius $\mathcal{X} \in \mathbb{N} \cup \{\infty\}$.

The definition of the radius of a resolving subcategory looks similar to that of the *dimension* of a triangulated category which has been introduced by Rouquier [2008, Definition 3.2]. The stable category $\underline{\mathrm{CM}}(R)$ of Cohen–Macaulay modules over a Gorenstein local ring R is triangulated by [Buchweitz 1986; Happel 1988], and the dimension of $\underline{\mathrm{CM}}(R)$ in the sense of Rouquier is defined. It might look the same as the radius of $\mathrm{CM}(R)$ in our sense.

However there are (at least) two differences in the definitions:

(1) A defining object for $\dim \underline{\mathrm{CM}}(R)$ is required to be inside the category $\underline{\mathrm{CM}}(R)$, but a defining object for $\mathrm{radius}\,\mathrm{CM}(R)$ is not, i.e., it is enough to be an object of $\mathrm{mod}\,R$. More precisely, $\dim \underline{\mathrm{CM}}(R)$ (respectively, $\mathrm{radius}\,\mathrm{CM}(R)$) is defined as the infimum of the integers $n \geq 0$ such that $\underline{\mathrm{CM}}(R) = \langle G \rangle_{n+1}$ for some object G (respectively, $\mathrm{CM}(R) \subseteq [C]_{n+1}$ for some object C). Then G must be an object of $\underline{\mathrm{CM}}(R)$, while C may not be an object of $\mathrm{CM}(R)$, just being an object of $\mathrm{mod}\,R$.

(2) Let $\mathcal{X}$ and $\mathcal{Y}$ be subcategories of $\mathrm{CM}(R)$ and $\underline{\mathrm{CM}}(R)$, respectively. Then the subcategory $\langle \mathcal{Y} \rangle$ of $\underline{\mathrm{CM}}(R)$ is closed under taking cosyzygies of Cohen–Macaulay modules in it, but the subcategory $[\mathcal{X}]$ of $\mathrm{CM}(R)$ is not in general. (In fact, this difference is a reason why we can prove Proposition 2.5 below but do not know whether the analogue for dimension holds or not; see Question 2.6 below.)

Thus these two notions are different, but they are still related to each other. In fact, we can show that the following relationship exists between them.

Proposition 2.4. *Let R be a Gorenstein local ring.*

(1) *One has the inequality* $\dim \underline{\mathrm{CM}}(R) \leq \mathrm{radius}\,\mathrm{CM}(R)$.

(2) *The equality holds if R is a hypersurface.*

Proof. (1) We may assume that $n := \mathrm{radius}\,\mathrm{CM}(R) < \infty$. Then there exists an R-module C such that $\mathrm{CM}(R)$ is contained in the ball $[C]_{n+1}$.

We claim that $\mathrm{CM}(R) = [\Omega^{-d}\Omega^d C]_{n+1}$ holds, where $d = \dim R$. This claim implies $\underline{\mathrm{CM}}(R) = \langle \Omega^d C \rangle_{n+1}$, which shows $\dim \underline{\mathrm{CM}}(R) \leq n$.

In the following, we show this claim. Since $\Omega^{-d}\Omega^d C$ is a Cohen–Macaulay R-module, and $\mathrm{CM}(R)$ is a resolving subcategory of $\mathrm{mod}\,R$, the inclusion $\mathrm{CM}(R) \supseteq [\Omega^{-d}\Omega^d C]_{n+1}$ holds. To get the opposite inclusion, it is enough to prove that for every $m \geq 1$ and $M \in [C]_m$ we have $\Omega^{-d}\Omega^d M \in [\Omega^{-d}\Omega^d C]_m$. Let us prove this by induction on m. The case $m = 1$ is obvious, so let $m \geq 2$. According to Proposition 2.2(1), there is an exact sequence $0 \to X \to Y \to Z \to 0$ of R-modules with $X \in [C]_{m-1}$ and $Z \in [C]$ such that M is a direct summand of Y. Taking the d-th syzygies, we have an exact sequence $0 \to \Omega^d X \to \Omega^d Y \oplus R^{\oplus l} \to \Omega^d Z \to 0$

of Cohen–Macaulay R-modules. Since R is Gorenstein, taking the d-th cosyzygies makes an exact sequence

$$0 \to \Omega^{-d}\Omega^d X \to \Omega^{-d}\Omega^d Y \oplus R^{\oplus k} \to \Omega^{-d}\Omega^d Z \to 0$$

of Cohen–Macaulay modules. The induction hypothesis implies $\Omega^{-d}\Omega^d Z \in [\Omega^{-d}\Omega^d C]$ and $\Omega^{-d}\Omega^d X \in [\Omega^{-d}\Omega^d C]_{m-1}$. Since $\Omega^{-d}\Omega^d M$ is a direct summand of $\Omega^{-d}\Omega^d Y$, it belongs to $[\Omega^{-d}\Omega^d C]_m$.

(2) Let $n := \dim \underline{CM}(R) < \infty$. We find a Cohen–Macaulay R-module G such that $\underline{CM}(R) = \langle G \rangle_{n+1}$. We want to prove that $CM(R) = [G]_{n+1}$ holds, and it suffices to show that for every $m \geq 1$ and $M \in \langle G \rangle_m$ we have $M \in [G]_m$. Let us use induction on m. The case $m = 1$ follows from the fact that $\Omega N \cong \Omega^{-1}N$ up to free summand for each Cohen–Macaulay R-module N, since R is a hypersurface. When $m \geq 2$, there exists an exact triangle $X \to Y \to Z \to \Sigma X$ in $\underline{CM}(R)$ with $X \in \langle G \rangle_{m-1}$ and $Z \in \langle G \rangle$ such that M is a direct summand of Y. Then we have an exact sequence $0 \to X \to Y \oplus R^{\oplus h} \to Z \to 0$ of R-modules, and we are done by applying the induction hypothesis. $\qquad\square$

In the rest of this section, we will study the relationships between the representation types of a Cohen–Macaulay local ring and the radius of the category of Cohen–Macaulay modules. Recall that a Cohen–Macaulay local ring R is said to be of *finite* (respectively, *countable*) *Cohen–Macaulay representation type* if $CM(R)$ has only finitely (respectively, countably but not finitely) many indecomposable modules up to isomorphism.

We can describe the property of finite Cohen–Macaulay representation type in terms of a radius.

Proposition 2.5. *Let R be a Gorenstein Henselian local ring. The following are equivalent*:

(1) *One has* $\operatorname{radius} CM(R) = 0$;

(2) *The ring R has finite Cohen–Macaulay representation type.*

Proof. (2) $\Rightarrow$ (1): If $M_1, \ldots, M_r$ are the nonisomorphic indecomposable Cohen–Macaulay R-modules, then we have $CM(R) = [M_1 \oplus \cdots \oplus M_r]$.

(1) $\Rightarrow$ (2): There is an R-module C satisfying $CM(R) \subseteq [C]$. Setting $d = \dim R$, we have $CM(R) = [\Omega^{-d}\Omega^d C]$. Replacing C with $\Omega^{-d}\Omega^d C$, we may assume that $CM(R) = [C]$ with C being Cohen–Macaulay.

Note that since R is Henselian, the Krull–Schmidt theorem holds, i.e., each R-module uniquely decomposes into indecomposable R-modules up to isomorphism. Let $C_1, \ldots, C_n$ be the indecomposable direct summands of C. We may assume that $C = C_1 \oplus \cdots \oplus C_n$. Since R is Gorenstein, taking syzygies preserves

indecomposability of nonfree Cohen–Macaulay R-modules. We see that the set of nonisomorphic indecomposable Cohen–Macaulay R-modules is

$$\{R\} \cup \{\, \Omega^i C_j \mid i \geq 0, \ 1 \leq j \leq n \,\}.$$

We may assume that for all $i \geq 0$ and $1 \leq j \neq j' \leq n$ we have $\Omega^i C_j \not\cong C_{j'}$, because if $\Omega^i C_j \cong C_{j'}$ for some such i, j, j', then we can exclude $C_{j'}$ from C. Now fix an integer j with $1 \leq j \leq n$. As taking cosyzygies preserves indecomposability of nonfree Cohen–Macaulay R-modules, $\Omega^{-1} C_j$ is isomorphic to $\Omega^a C_b$ for some $a \geq 0$ and $1 \leq b \leq n$. Taking the a-th cosyzygies, we have $\Omega^{-1-a} C_j \cong C_b$, hence $C_j \cong \Omega^{1+a} C_b$. This forces us to have $b = j$, which says that C_j is periodic. Hence there are only finitely many indecomposable Cohen–Macaulay R-modules. $\square$

Question 2.6. Does the equality in Proposition 2.4(1) hold true? If it does, then Proposition 2.5 will say that *a Gorenstein Henselian local ring R has finite Cohen–Macaulay representation type if and only if* $\dim \underline{\mathrm{CM}}(R) \leq 0$. This statement is a partial generalization of Minamoto's theorem [2013, Theorem 0.2], which asserts that the same statement holds for a finite-dimensional self-injective algebra over a perfect field, extending Yoshiwaki's recent theorem [2011, Corollary 3.10].

The next result hints at further relationship between finite radius of $\mathrm{CM}(R)$ and more well-known classification of singularities.

Proposition 2.7. *Let R be a complete local hypersurface over an algebraically closed field of characteristic not two. Assume that R is of countable Cohen–Macaulay representation type. Then* $\mathrm{radius}\,\mathrm{CM}(R) = 1$.

Proof. It follows from [Araya et al. 2012, Theorem 1.1] that there exists an R-module X such that for every indecomposable module $M \in \mathrm{CM}(R)$ there is an exact sequence $0 \to L \to M \oplus R^n \to N \to 0$ with $L, N \in \{0, X, \Omega X\}$. This shows that $\mathrm{CM}(R) = [X]_2$. Now we see from Proposition 2.5 that the radius of $\mathrm{CM}(R)$ is equal to one. $\square$

3. Finiteness of the radius of a resolving subcategory

In this section we state our guiding conjecture and first main result.

Conjecture 3.1. Let R be a Cohen–Macaulay local ring. Let $\mathscr{X}$ be a resolving subcategory of $\mathrm{mod}\,R$ with finite radius. Then every R-module in $\mathscr{X}$ is Cohen–Macaulay.

Remark 3.2. The converse of Conjecture 3.1 also seems to be true. We consider this in Section 5.

Let $\mathscr{X}$ be a subcategory of $\mathrm{mod}\,R$. We denote by $\mathrm{res}\,\mathscr{X}$ (or $\mathrm{res}_R \mathscr{X}$) the *resolving closure* of $\mathscr{X}$, namely, the smallest resolving subcategory of $\mathrm{mod}\,R$ containing $\mathscr{X}$. If

$\mathscr{X}$ consists of a single module M, then we simply denote it by res M (or $\mathrm{res}_R M$). For a prime ideal $\mathfrak{p}$ of R, we denote by $\mathscr{X}_{\mathfrak{p}}$ the subcategory of $\mathrm{mod}\, R_{\mathfrak{p}}$ consisting of all modules of the form $X_{\mathfrak{p}}$, where $X \in \mathscr{X}$. The first main result of this paper is the following theorem.

Theorem 3.3. *Let R be a commutative Noetherian ring. Let $\mathscr{X}$ be a resolving subcategory of $\mathrm{mod}\, R$. Suppose that there exist a prime ideal $\mathfrak{p}$ of R with $\mathrm{ht}\,\mathfrak{p} > 0$ and an $R_{\mathfrak{p}}$-module M with $0 \neq M \in \mathrm{add}_{R_{\mathfrak{p}}} \mathscr{X}_{\mathfrak{p}}$ which satisfy one of the following conditions.*

(1) *$\mathfrak{p}M = 0$.*
(2) *$0 < \mathrm{Gdim}_{R_{\mathfrak{p}}} M = n < \infty$ and $\Omega_{R_{\mathfrak{p}}}^{-2}\Omega_{R_{\mathfrak{p}}}^{n} M \in \mathrm{add}_{R_{\mathfrak{p}}} \mathscr{X}_{\mathfrak{p}}$.*
(3) *$0 < \mathrm{Gdim}_{R_{\mathfrak{p}}} M < \infty$ and $\mathrm{res}_{R_{\mathfrak{p}}}(\Omega_{R_{\mathfrak{p}}}^{n} M)$ is a thick subcategory of $\mathscr{G}(R_{\mathfrak{p}})$ for some $n \geq 0$.*
(4) *$0 < \mathrm{Cldim}_{R_{\mathfrak{p}}} M < \infty$.*

Then $\mathscr{X}$ has infinite radius.

The proof of this theorem will be given in the next section. As a direct consequence of the above theorem, we obtain two cases in which our conjecture holds true.

Corollary 3.4. *Conjecture 3.1 is true if*

(1) *R is a complete intersection, or*
(2) *R is Gorenstein, and every resolving subcategory of $\mathrm{mod}\, R$ contained in $\mathrm{CM}(R)$ is a thick subcategory of $\mathrm{CM}(R)$.*

Proof. Conjecture 3.1 trivially holds in the case where R is Artinian, so let $(R, \mathfrak{m}, k)$ be a Cohen–Macaulay local ring of positive dimension. Then we have $\mathrm{ht}\,\mathfrak{m} > 0$. Let $\mathscr{X}$ be a resolving subcategory of $\mathrm{mod}\, R$, and suppose that $\mathscr{X}$ contains a non-Cohen–Macaulay R-module M.

(1) We have $0 < \dim R - \mathrm{depth}_R M = \mathrm{depth}\, R - \mathrm{depth}_R M = \mathrm{Cldim}_R M < \infty$, and Theorem 3.3(4) implies that $\mathscr{X}$ has infinite radius.

(2) We have $0 < n := \dim R - \mathrm{depth}_R M = \mathrm{depth}\, R - \mathrm{depth}_R M = \mathrm{Gdim}_R M < \infty$. The module $\Omega_R^n M$ is Cohen–Macaulay, and by assumption $\mathrm{res}_R(\Omega_R^n M)$ is a thick subcategory of $\mathrm{CM}(R) = \mathscr{G}(R)$. Theorem 3.3(3) implies that $\mathscr{X}$ has infinite radius. $\square$

4. Proof of Theorem I

This section is devoted to give the proof of Theorem 3.3 (hence of Theorem I from the introduction), which we break up into several parts. Most of them also reveal properties of subcategories of $\mathrm{mod}\, R$ which are interesting in their own right.

First of all, we make a remark to reduce our theorem to the local case.

Remark 4.1. Let C be an R-module, $n \geq 0$ an integer and $\mathfrak{p}$ a prime ideal of R. Then for a subcategory $\mathcal{X}$ of mod R the implication

$$\mathcal{X} \subseteq [C]_n^R \quad \Rightarrow \quad \mathrm{add}_{R_\mathfrak{p}} \mathcal{X}_\mathfrak{p} \subseteq [C_\mathfrak{p}]_n^{R_\mathfrak{p}}$$

holds. It follows from [Takahashi 2010, Lemma 4.8] that $\mathrm{add}_{R_\mathfrak{p}} \mathcal{X}_\mathfrak{p}$ is a resolving subcategory of mod $R_\mathfrak{p}$. (The ring R in (loc. cit.) is assumed to be local, but its proof does not use this assumption, so it holds for an arbitrary commutative Noetherian ring.) Hence, to prove Theorem 3.3, without loss of generality we can assume $(R, \mathfrak{p})$ is a local ring with $\dim R > 0$ and M is an R-module with $0 \neq M \in \mathcal{X}$.

***Proof of Theorem 3.3**(1).* First, we investigate the annihilators of torsion submodules. For an ideal I of R and an R-module M, we denote by $\Gamma_I(M)$ the *I-torsion submodule* of M. Recall that $\Gamma_I(M)$ is by definition the subset of M consisting of all elements that are annihilated by some power of I, and the assignment $M \mapsto \Gamma_I(M)$ defines a left exact additive covariant functor $\Gamma_I : \mathrm{mod}\, R \to \mathrm{mod}\, R$.

Lemma 4.2. *Let I be an ideal of R. Let C, M be R-modules and $n \geq 1$ an integer. If M belongs to $[C]_n$, then one has $\mathrm{Ann}_R \Gamma_I(M) \supseteq (\mathrm{Ann}_R \Gamma_I(R) \cdot \mathrm{Ann}_R \Gamma_I(C))^n$.*

Proof. Let us prove the lemma by induction on n.

If $n = 1$, the module M is isomorphic to a direct summand of $\left(R \oplus \bigoplus_{i=0}^a \Omega^i C\right)^{\oplus b}$ for some $a, b \geq 0$. Hence $\Gamma_I(M)$ is isomorphic to a direct summand of

$$\left(\Gamma_I(R) \oplus \bigoplus_{i=0}^a \Gamma_I(\Omega^i C)\right)^{\oplus b}.$$

For $i \geq 1$, the syzygy $\Omega^i C$ is a submodule of some free module $R^{\oplus c_i}$, and $\Gamma_I(\Omega^i C)$ is a submodule of $\Gamma_I(R)^{\oplus c_i}$, which implies that $\mathrm{Ann}_R \Gamma_I(\Omega^i C)$ contains $\mathrm{Ann}_R \Gamma_I(R)$. Hence we obtain

$$\mathrm{Ann}_R \Gamma_I(M) \supseteq \mathrm{Ann}_R \Gamma_I(R) \cap \left(\bigcap_{i=0}^a \mathrm{Ann}_R \Gamma_I(\Omega^i C)\right)$$

$$= \mathrm{Ann}_R \Gamma_I(R) \cap \mathrm{Ann}_R \Gamma_I(C) \supseteq \mathrm{Ann}_R \Gamma_I(R) \cdot \mathrm{Ann}_R \Gamma_I(C).$$

Let $n \geq 2$. Then M is in $[C]_n = [C]_{n-1} \bullet [C]$, and Proposition 2.2(1) says that there is an exact sequence $0 \to X \to Y \to Z \to 0$ with $X \in [C]_{n-1}$ and $Z \in [C]$ such that M is a direct summand of Y. We have an exact sequence $0 \to \Gamma_I(X) \to \Gamma_I(Y) \to \Gamma_I(Z)$, and therefore we obtain

$$\mathrm{Ann}_R \Gamma_I(M) \supseteq \mathrm{Ann}_R \Gamma_I(Y) \supseteq \mathrm{Ann}_R \Gamma_I(X) \cdot \mathrm{Ann}_R \Gamma_I(Z)$$

$$\supseteq (\mathrm{Ann}_R \Gamma_I(R) \cdot \mathrm{Ann}_R \Gamma_I(C))^{n-1} \cdot (\mathrm{Ann}_R \Gamma_I(R) \cdot \mathrm{Ann}_R \Gamma_I(C))$$

$$= (\mathrm{Ann}_R \Gamma_I(R) \cdot \mathrm{Ann}_R \Gamma_I(C))^n,$$

which is what we want.   □

Let $(R, \mathfrak{m})$ be a local ring, and let M be an R-module. We denote by $\ell\ell(M)$ the *Loewy length* of M, which is by definition the infimum of the integers $n \geq 0$ such that $\mathfrak{m}^n M = 0$. Obviously, $\ell\ell(M)$ is finite if and only if M has finite length. There is a relationship between finite radius and Loewy length:

Proposition 4.3. *Let $(R, \mathfrak{m})$ be a local ring, and let $\mathscr{X}$ be a resolving subcategory of* mod R. *If* radius $\mathscr{X} < \infty$, *then* $\sup_{X \in \mathscr{X}}\{\ell\ell(\Gamma_{\mathfrak{m}}(X))\} < \infty$.

Proof. Put $r = \mathrm{radius}\,\mathscr{X}$. By definition, there exists an R-module C such that $[C]_{r+1}$ contains $\mathscr{X}$. Let X be a module in $\mathscr{X}$. It follows from Lemma 4.2 that the annihilator Ann $\Gamma_{\mathfrak{m}}(X)$ contains the ideal $(\mathrm{Ann}\,\Gamma_{\mathfrak{m}}(R) \cdot \mathrm{Ann}\,\Gamma_{\mathfrak{m}}(C))^{r+1}$. As $\Gamma_{\mathfrak{m}}(R)$ and $\Gamma_{\mathfrak{m}}(C)$ have finite length, they have finite Loewy length. Set $a = \ell\ell(\Gamma_{\mathfrak{m}}(R))$ and $b = \ell\ell(\Gamma_{\mathfrak{m}}(C))$. Then Ann $\Gamma_{\mathfrak{m}}(X)$ contains $\mathfrak{m}^{(a+b)(r+1)}$, which means that $\ell\ell(\Gamma_{\mathfrak{m}}(X))$ is at most $(a+b)(r+1)$. Since the number $(a+b)(r+1)$ is independent of the choice of X, we have $\sup_{X \in \mathscr{X}}\{\ell\ell(\Gamma_{\mathfrak{m}}(X))\} \leq (a+b)(r+1) < \infty$.   □

The following is the essential part of Theorem 3.3(1).

Theorem 4.4. *Let $(R, \mathfrak{m}, k)$ be a local ring of positive dimension. Let $\mathscr{X}$ be a resolving subcategory of* mod R. *If $\mathscr{X}$ contains k, then $\mathscr{X}$ has infinite radius.*

Proof. We claim that $R/\mathfrak{m}^i$ belongs to $\mathscr{X}$ for all integers $i > 0$. Indeed, there is an exact sequence $0 \to \mathfrak{m}^{i-1}/\mathfrak{m}^i \to R/\mathfrak{m}^i \to R/\mathfrak{m}^{i-1} \to 0$, and the left term belongs to $\mathscr{X}$ as it is a k-vector space. Induction on i shows the claim.

We have $\ell\ell(\Gamma_{\mathfrak{m}}(R/\mathfrak{m}^i)) = \ell\ell(R/\mathfrak{m}^i) = i$, where the second equality follows from the assumption that $\dim R > 0$. Therefore it holds that

$$\sup_{X \in \mathscr{X}}\{\ell\ell(\Gamma_{\mathfrak{m}}(X))\} \geq \sup_{i > 0}\{\ell\ell(\Gamma_{\mathfrak{m}}(R/\mathfrak{m}^i))\} = \sup_{i > 0}\{i\} = \infty,$$

which implies radius $\mathscr{X} = \infty$ by Proposition 4.3.   □

Now, let us prove Theorem 3.3(1). By Remark 4.1, we may assume that $(R, \mathfrak{p})$ is a local ring with $\dim R > 0$ and that M is a nonzero R-module in $\mathscr{X}$. The assumption $\mathfrak{p}M = 0$ says that M is a nonzero k-vector space, where $k = R/\mathfrak{p}$ is the residue field of R. Since $\mathscr{X}$ is closed under direct summands, k belongs to $\mathscr{X}$. Theorem 4.4 yields radius $\mathscr{X} = \infty$.

***Proof of Theorem 3.3*(2)(3).** Establishing several preliminary lemmas and propositions is necessary, which will also be used in the proof of Theorem 3.3(4).

We begin with stating an elementary lemma, whose proof we omit.

Lemma 4.5. *Let $(R, \mathfrak{m}, k)$ be a local ring. Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules. Then* $\inf\{\mathrm{depth}\,L, \mathrm{depth}\,N\} = \inf\{\mathrm{depth}\,M, \mathrm{depth}\,N\}$.

For an ideal I of R, we denote by $V(I)$ (respectively, $D(I)$) the closed (respectively, open) subset of Spec R defined by I in the Zariski topology, namely, $V(I)$ is the set of prime ideals containing I and $D(I) = \operatorname{Spec} R \setminus V(I)$. For an R-module M we denote by $\operatorname{NF}(M)$ the *nonfree locus* of M, that is, the set of prime ideals $\mathfrak{p}$ of R such that the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is nonfree. It is well-known that $\operatorname{NF}(M)$ is a closed subset of Spec R.

The next result builds, out of each module in a resolving subcategory and each point in its nonfree locus, another module in the same resolving subcategory whose nonfree locus coincides with the closure of the point. Such a construction has already been given in [Takahashi 2009, Theorem 4.3], but we need in this paper a more detailed version. Indeed, the following lemma yields a generalization of (loc. cit.).

Lemma 4.6. *Let M be an R-module. For every $\mathfrak{p} \in \operatorname{NF}(M)$ there exists $X \in \operatorname{res} M$ satisfying $\operatorname{NF}(X) = V(\mathfrak{p})$ and $\operatorname{depth} X_{\mathfrak{q}} = \inf\{\operatorname{depth} M_{\mathfrak{q}}, \operatorname{depth} R_{\mathfrak{q}}\}$ for all $\mathfrak{q} \in V(\mathfrak{p})$.*

Proof. Note that $V(\mathfrak{p})$ is contained in $\operatorname{NF}(M)$. If $V(\mathfrak{p}) = \operatorname{NF}(M)$, then we can take $X := M \oplus R$. Suppose $V(\mathfrak{p})$ is strictly contained in $\operatorname{NF}(M)$. Then there is a prime ideal $\mathfrak{r}$ in $\operatorname{NF}(M)$ that is not in $V(\mathfrak{p})$. Choose an element $x \in \mathfrak{p} \setminus \mathfrak{r}$. By [ibid., Proposition 4.2], we have a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega M & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle x} & & \Big\downarrow & & \Big\| & & \\
0 & \longrightarrow & \Omega M & \longrightarrow & N & \longrightarrow & M & \longrightarrow & 0
\end{array}
$$

where F is free, $V(\mathfrak{p}) \subseteq \operatorname{NF}(N) \subseteq \operatorname{NF}(M)$ and $D((x)) \cap \operatorname{NF}(N) = \varnothing$. The second row shows that N belongs to $\operatorname{res} M$. Since $\mathfrak{r}$ is in $D((x))$, it is not in $\operatorname{NF}(N)$, and we have $V(\mathfrak{p}) \subseteq \operatorname{NF}(N) \subsetneq \operatorname{NF}(M)$.

Now we claim that $\operatorname{depth} N_{\mathfrak{q}} = \inf\{\operatorname{depth} M_{\mathfrak{q}}, \operatorname{depth} R_{\mathfrak{q}}\}$ for all $\mathfrak{q} \in V(\mathfrak{p})$. Indeed, localizing the above diagram at $\mathfrak{q}$ and taking long exact sequences with respect to Ext, we get a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
E^{i-1}(M_{\mathfrak{q}}) & \longrightarrow & E^i((\Omega M)_{\mathfrak{q}}) & \longrightarrow & E^i(F_{\mathfrak{q}}) & \longrightarrow & E^i(M_{\mathfrak{q}}) & \longrightarrow & E^{i+1}((\Omega M)_{\mathfrak{q}}) \\
\Big\| & & {\scriptstyle x}\Big\downarrow{\scriptstyle (1)} & & \Big\downarrow & & \Big\| & & {\scriptstyle x}\Big\downarrow{\scriptstyle (2)} \\
E^{i-1}(M_{\mathfrak{q}}) & \xrightarrow{\;(3)\;} & E^i((\Omega M)_{\mathfrak{q}}) & \longrightarrow & E^i(N_{\mathfrak{q}}) & \longrightarrow & E^i(M_{\mathfrak{q}}) & \xrightarrow{\;(4)\;} & E^{i+1}((\Omega M)_{\mathfrak{q}})
\end{array}
$$

for $i \in \mathbb{Z}$, where $E^i(-) = \operatorname{Ext}^i_{R_{\mathfrak{q}}}(\kappa(\mathfrak{q}), -)$. As x is an element of $\mathfrak{q}$, the maps (1), (2) are zero maps, and so are (3), (4). Thus we have a short exact sequence

$$0 \to \operatorname{Ext}^i(\kappa(\mathfrak{q}), (\Omega M)_{\mathfrak{q}}) \to \operatorname{Ext}^i(\kappa(\mathfrak{q}), N_{\mathfrak{q}}) \to \operatorname{Ext}^i(\kappa(\mathfrak{q}), M_{\mathfrak{q}}) \to 0$$

for each integer i. It is easy to see from this that the first equality in the following holds, while the second equality is obtained by applying Lemma 4.5 to the exact

sequence $0 \to (\Omega M)_{\mathfrak{q}} \to F_{\mathfrak{q}} \to M_{\mathfrak{q}} \to 0$:

$$\operatorname{depth} N_{\mathfrak{q}} = \inf\{\operatorname{depth}(\Omega M)_{\mathfrak{q}}, \operatorname{depth} M_{\mathfrak{q}}\} = \inf\{\operatorname{depth} R_{\mathfrak{q}}, \operatorname{depth} M_{\mathfrak{q}}\}.$$

Thus the claim follows.

If $V(\mathfrak{p}) = \mathrm{NF}(N)$, then we can take $X := N$. If $V(\mathfrak{p})$ is strictly contained in $\mathrm{NF}(N)$, then the above procedure gives rise to an R-module $L \in \operatorname{res} N$ (so $L \in \operatorname{res} M$) with $V(\mathfrak{p}) \subseteq \mathrm{NF}(L) \subsetneq \mathrm{NF}(N) \subsetneq \mathrm{NF}(M)$ such that

$$\operatorname{depth} L_{\mathfrak{q}} = \inf\{\operatorname{depth} N_{\mathfrak{q}}, \operatorname{depth} R_{\mathfrak{q}}\} = \inf\{\operatorname{depth} M_{\mathfrak{q}}, \operatorname{depth} R_{\mathfrak{q}}\}$$

for all $\mathfrak{q} \in V(\mathfrak{p})$. Since $\operatorname{Spec} R$ is a Noetherian space, iteration of this procedure must stop in finitely many steps, and we eventually obtain such a module X as in the lemma. $\qquad\square$

The next lemma will play a crucial role in the proofs of our theorems. The main idea of the proof is similar to that of Lemma 4.2, but a much closer examination is necessary.

Lemma 4.7. *Let $S \to R$ be a homomorphism of rings. Let C, M be R-modules with $M \in [C]_n^R$, and let N be an S-module of injective dimension $m < \infty$. Then*

$$\bigcap_{i>0} \operatorname{Ann}_S \operatorname{Ext}_S^i(M, N) = \bigcap_{i=1}^{m} \operatorname{Ann}_S \operatorname{Ext}_S^i(M, N)$$

$$\supseteq \left(\prod_{i=1}^{m} \operatorname{Ann}_S \operatorname{Ext}_S^i(R, N) \cdot \operatorname{Ann}_S \operatorname{Ext}_S^i(C, N) \right)^n.$$

Proof. For each integer $i \le 1$ and R-module L, set $\mathfrak{a}_L^i = \operatorname{Ann}_S \operatorname{Ext}_S^i(L, N)$. Note that $\mathfrak{a}_L^h = S$ for all $h > m$ since $\operatorname{Ext}_S^h(L, N) = 0$. It suffices to prove that $\mathfrak{a}_M^i \supseteq (\prod_{j=i}^{m} \mathfrak{a}_R^j \mathfrak{a}_C^j)^n$. Let us proceed by induction on n.

When $n = 0$, we have $M = 0$, and the above two ideals coincide with S.

Let $n = 1$. Then M is isomorphic to a direct summand of a finite direct sum of copies of $R \oplus (\bigoplus_{j=0}^{l} \Omega^j C)$. Hence $\operatorname{Ext}_S^i(M, N)$ is isomorphic to a direct summand of a finite direct sum of copies of $\operatorname{Ext}_S^i(R, N) \oplus (\bigoplus_{j=0}^{l} \operatorname{Ext}_S^i(\Omega^j C, N))$. Thus we have $\mathfrak{a}_M^i \supseteq \mathfrak{a}_R^i \cap (\bigcap_{j=0}^{l} \mathfrak{a}_{\Omega^j C}^i)$. For each $j \ge 1$ there is an exact sequence $0 \to \Omega^j C \to R^{\oplus k_j} \to \Omega^{j-1} C \to 0$, which induces an exact sequence

$$\operatorname{Ext}_S^i(R, N)^{\oplus k_j} \to \operatorname{Ext}_S^i(\Omega^j C, N) \to \operatorname{Ext}_S^{i+1}(\Omega^{j-1} C, N).$$

This gives

$$\mathfrak{a}_{\Omega^j C}^i \supseteq \mathfrak{a}_R^i \mathfrak{a}_{\Omega^{j-1} C}^{i+1} \supseteq \mathfrak{a}_R^i \mathfrak{a}_R^{i+1} \mathfrak{a}_{\Omega^{j-2} C}^{i+2} \supseteq \cdots \supseteq \mathfrak{a}_R^i \mathfrak{a}_R^{i+1} \cdots \mathfrak{a}_R^{i+j-1} \mathfrak{a}_C^{i+j}.$$

Regarding $\mathfrak{a}_R^i \mathfrak{a}_R^{i+1} \cdots \mathfrak{a}_R^{i+j-1} \mathfrak{a}_C^{i+j}$ as $\mathfrak{a}_C^i$ when $j = 0$, we have

$$\mathfrak{a}_{\Omega^j C}^i \supseteq \mathfrak{a}_R^i \mathfrak{a}_R^{i+1} \cdots \mathfrak{a}_R^{i+j-1} \mathfrak{a}_C^{i+j}$$

for all $j \geq 0$. Thus we obtain

$$\mathfrak{a}_M^i \supseteq \mathfrak{a}_R^i \cap \left(\bigcap_{j=0}^{l} \mathfrak{a}_R^i \mathfrak{a}_R^{i+1} \cdots \mathfrak{a}_R^{i+j-1} \mathfrak{a}_C^{i+j} \right)$$

$$\supseteq (\mathfrak{a}_R^i \mathfrak{a}_R^{i+1} \cdots \mathfrak{a}_R^m \mathfrak{a}_R^{m+1} \cdots)(\mathfrak{a}_C^i \mathfrak{a}_C^{i+1} \cdots \mathfrak{a}_C^m \mathfrak{a}_C^{m+1} \cdots)$$

$$= (\mathfrak{a}_R^i \mathfrak{a}_R^{i+1} \cdots \mathfrak{a}_R^m)(\mathfrak{a}_C^i \mathfrak{a}_C^{i+1} \cdots \mathfrak{a}_C^m) \supseteq \prod_{j=i}^{m} \mathfrak{a}_R^j \mathfrak{a}_C^j.$$

Now let us consider the case where $n \geq 2$. We have $M \in [C]_n = [C]_{n-1} \bullet [C]$, and by [Proposition 2.2](1), there exists an exact sequence $0 \to X \to Y \to Z \to 0$ with $X \in [C]_{n-1}$ and $Z \in [C]$ such that M is a direct summand of Y. Using the induction hypothesis, we have

$$\mathfrak{a}_M^i \supseteq \mathfrak{a}_Y^i \supseteq \mathfrak{a}_X^i \cdot \mathfrak{a}_Z^i \supseteq \left(\prod_{j=i}^{m} \mathfrak{a}_R^j \mathfrak{a}_C^j \right)^{n-1} \cdot \left(\prod_{j=i}^{m} \mathfrak{a}_R^j \mathfrak{a}_C^j \right) = \left(\prod_{j=i}^{m} \mathfrak{a}_R^j \mathfrak{a}_C^j \right)^n,$$

which completes the proof of the lemma. $\qquad\square$

Here we prepare a lemma, which is an easy consequence of Krull's intersection theorem.

Lemma 4.8. *Let* $(R, \mathfrak{m})$ *be a local ring and M an R-module. Then* $\operatorname{Ann}_R M = \bigcap_{i>0} \operatorname{Ann}_R(M/\mathfrak{m}^i M)$.

Now we can prove the following proposition, which will be the base of the proofs of our theorems. Actually, all of them will be proved by making use of this proposition.

Proposition 4.9. *Let* $(R, \mathfrak{m}, k)$ *be a local ring. Let $\mathscr{X}$ be a resolving subcategory of* mod R. *If $\mathscr{X}$ contains a module M such that* $0 < \operatorname{pd}_R M < \infty$, *then* radius $\mathscr{X} = \infty$.

Proof. Applying Lemma 4.6 to $\mathfrak{m} \in \operatorname{NF}(M)$, we find a module $X \in \operatorname{res} M \subseteq \mathscr{X}$ satisfying $\operatorname{NF}(X) = \{\mathfrak{m}\}$ and depth $X = \inf\{\operatorname{depth} M, \operatorname{depth} R\}$. Since M has finite projective dimension, the depth of M is at most that of R. Hence, we have depth $X = \operatorname{depth} M$. Note that the subcategory of mod R consisting of R-modules of finite projective dimension is resolving. Since it contains M, it also contains res M. This implies that X has finite projective dimension, and we have

$$\operatorname{pd}_R X = \operatorname{depth} R - \operatorname{depth} X = \operatorname{depth} R - \operatorname{depth} M = \operatorname{pd}_R M.$$

Thus, replacing M with X, we may assume that M is locally free on the punctured spectrum of R. Taking the n-th syzygy of M where $n = \operatorname{pd}_R M - 1 \geq 0$, we may also assume that the projective dimension of M is equal to 1.

Now $\operatorname{Ext}^1_R(M, R)$ is a nonzero R-module of finite length, and we can choose a socle element $0 \neq \sigma \in \operatorname{Ext}^1_R(M, R)$. It can be represented as a short exact sequence

$$\sigma : 0 \to R \to N \to M \to 0.$$

The module N belongs to $\mathscr{X}$, is locally free on the punctured spectrum of R and has projective dimension at most 1. Hence $\operatorname{pd}_R N = 1$ if and only if $\operatorname{Ext}^1_R(N, R) \neq 0$. Applying the functor $\operatorname{Hom}_R(-, R)$, we get an exact sequence

$$R \xrightarrow{f} \operatorname{Ext}^1_R(M, R) \to \operatorname{Ext}^1_R(N, R) \to 0,$$

where f sends $1 \in R$ to $\sigma \in \operatorname{Ext}^1_R(M, R)$. Hence we obtain an exact sequence

$$0 \to k \to \operatorname{Ext}^1_R(M, R) \to \operatorname{Ext}^1_R(N, R) \to 0.$$

This implies $\operatorname{length}(\operatorname{Ext}^1_R(N, R)) = \operatorname{length}(\operatorname{Ext}^1_R(M, R)) - 1$. Replacing M by N and repeating this process if $\operatorname{length}(\operatorname{Ext}^1_R(N, R)) > 0$, we can assume that $\operatorname{Ext}^1_R(N, R) = 0$. Therefore $\operatorname{Ext}^1_R(M, R) \cong k$. Since $\operatorname{pd}_R M = 1$, we easily get an isomorphism $\operatorname{Tr} M \cong k$. Taking the transpose of this isomorphism, we see that $\operatorname{Tr} k$ is isomorphic to M up to free summand (see [Auslander and Bridger 1969, Proposition (2.6)(d)]). It follows that $\operatorname{Tr} k$ belongs to $\mathscr{X}$.

We claim that $\operatorname{Tr} L$ is in $\mathscr{X}$ for any R-module L of finite length. This is shown by induction on $\operatorname{length} L$. If $\operatorname{length} L > 0$, then there is an exact sequence $0 \to L' \to L \to k \to 0$, and applying [ibid., Lemma (3.9)] (see also [Takahashi 2013, Proposition 3.3(3)]), we have an exact sequence

$$0 = (L')^* \to \operatorname{Tr} k \to \operatorname{Tr} L \oplus R^{\oplus n} \to \operatorname{Tr} L' \to 0,$$

where the equality follows from the fact that R has positive depth. (As R possesses a module of finite positive projective dimension, the depth of R is positive.) The induction hypothesis implies $\operatorname{Tr} L' \in \mathscr{X}$, and the above exact sequence shows $\operatorname{Tr} L \in \mathscr{X}$, as desired.

Now, assume that we have radius $\mathscr{X} = r < \infty$. We want to deduce a contradiction. There is a ball $[C]^R_{r+1}$ that contains $\mathscr{X}$. Since $\mathscr{X}$ contains $\operatorname{Tr}_R(R/\mathfrak{m}^i)$ for all $i > 0$, the ball $[C]^R_{r+1}$ also contains it. Taking the completions, we have $\operatorname{Tr}_{\widehat{R}}(\widehat{R}/\mathfrak{m}^i \widehat{R}) \in [\widehat{C}]^{\widehat{R}}_{r+1}$ for all $i > 0$. By virtue of Cohen's structure theorem, there exists a surjective homomorphism $S \to \widehat{R}$ such that S is a Gorenstein local ring with $\dim S = \dim R =: d$. Let $\mathfrak{n}$ denote the maximal ideal of S and note that we have $\widehat{R}/\mathfrak{m}^i \widehat{R} = \widehat{R}/\mathfrak{n}^i \widehat{R}$ for

any $i > 0$. Lemma 4.7 gives an inclusion relation

$$\bigcap_{j>0} \mathrm{Ann}_S \, \mathrm{Ext}_S^j(\mathrm{Tr}_{\widehat{R}}(\widehat{R}/\mathfrak{n}^i \widehat{R}), S) \supseteq \left(\prod_{j=1}^d \mathrm{Ann}_S \, \mathrm{Ext}_S^j(\widehat{R}, S) \cdot \mathrm{Ann}_S \, \mathrm{Ext}_S^j(\widehat{C}, S) \right)^{r+1}.$$

Fix an integer $i > 0$ and let $a_1, \ldots, a_m$ be a system of generators of the ideal $\mathfrak{n}^i$ of S. There is an exact sequence $\widehat{R}^{\oplus m} \xrightarrow{(a_1,\ldots,a_m)} \widehat{R} \to \widehat{R}/\mathfrak{n}^i \widehat{R} \to 0$ of $\widehat{R}$-modules. Dualizing this by $\widehat{R}$ induces an exact sequence

$$0 = \mathrm{Hom}_{\widehat{R}}(\widehat{R}/\mathfrak{n}^i \widehat{R}, \widehat{R}) \to \widehat{R} \xrightarrow{\left(\begin{smallmatrix} a_1 \\ \vdots \\ a_m \end{smallmatrix}\right)} \widehat{R}^{\oplus m} \to \mathrm{Tr}_{\widehat{R}}(\widehat{R}/\mathfrak{n}^i \widehat{R}) \to 0,$$

where the equality follows since depth $\widehat{R} = $ depth $R > 0$. This makes an exact sequence

$$\mathrm{Hom}_S(\widehat{R}, S)^{\oplus m} \xrightarrow{(a_1,\ldots,a_m)} \mathrm{Hom}_S(\widehat{R}, S) \to \mathrm{Ext}_S^1(\mathrm{Tr}_{\widehat{R}}(\widehat{R}/\mathfrak{n}^i \widehat{R}), S),$$

which yields an injection $\mathrm{Hom}_S(\widehat{R}, S)/\mathfrak{n}^i \, \mathrm{Hom}_S(\widehat{R}, S) \to \mathrm{Ext}_S^1(\mathrm{Tr}_{\widehat{R}}(\widehat{R}/\mathfrak{n}^i \widehat{R}), S)$. Thus

$$\mathrm{Ann}_S(\mathrm{Hom}_S(\widehat{R}, S)/\mathfrak{n}^i \, \mathrm{Hom}_S(\widehat{R}, S))$$
$$\supseteq \mathrm{Ann}_S(\mathrm{Ext}_S^1(\mathrm{Tr}_{\widehat{R}}(\widehat{R}/\mathfrak{n}^i \widehat{R}), S))$$
$$\supseteq \bigcap_{j>0} \mathrm{Ann}_S \, \mathrm{Ext}_S^j(\mathrm{Tr}_{\widehat{R}}(\widehat{R}/\mathfrak{n}^i \widehat{R}), S) \supseteq \left(\prod_{j=1}^d \mathrm{Ann}_S \, \mathrm{Ext}_S^j(\widehat{R}, S) \cdot \mathrm{Ann}_S \, \mathrm{Ext}_S^j(\widehat{C}, S) \right)^{r+1},$$

and we obtain

$$\left(\prod_{j=1}^d \mathrm{Ann}_S \, \mathrm{Ext}_S^j(\widehat{R}, S) \cdot \mathrm{Ann}_S \, \mathrm{Ext}_S^j(\widehat{C}, S) \right)^{r+1}$$
$$\subseteq \bigcap_{i>0} \mathrm{Ann}_S\big(\mathrm{Hom}_S(\widehat{R}, S)/\mathfrak{n}^i \, \mathrm{Hom}_S(\widehat{R}, S)\big) = \mathrm{Ann}_S \, \mathrm{Hom}_S(\widehat{R}, S),$$

where the equality follows from Lemma 4.8.

Let I be the kernel of the surjection $S \to \widehat{R}$. Since $\dim S = d = \dim \widehat{R}$, the ideal I of S has height zero. Hence there exists a minimal prime ideal $\mathfrak{p}$ of S which contains I. Since we have a ring epimorphism from the Artinian Gorenstein local ring $S_{\mathfrak{p}}$ to $\widehat{R}_{\mathfrak{p}}$, the $\widehat{R}_{\mathfrak{p}}$-module $\mathrm{Hom}_S(\widehat{R}, S)_{\mathfrak{p}} = \mathrm{Hom}_{S_{\mathfrak{p}}}(\widehat{R}_{\mathfrak{p}}, S_{\mathfrak{p}})$ is isomorphic to the injective hull of the residue field of $\widehat{R}_{\mathfrak{p}}$, which is in particular nonzero. This implies that $\mathfrak{p}$ contains the ideal $\mathrm{Ann}_S \, \mathrm{Hom}_S(\widehat{R}, S)$. Therefore, for some integer $1 \leq l \leq d$ the ideal $\mathfrak{p}$ contains either $\mathrm{Ann}_S \, \mathrm{Ext}_S^l(\widehat{R}, S)$ or $\mathrm{Ann}_S \, \mathrm{Ext}_S^l(\widehat{C}, S)$. If $\mathfrak{p}$ contains $\mathrm{Ann}_S \, \mathrm{Ext}_S^l(\widehat{R}, S)$, then we have $\mathrm{Ext}_{S_{\mathfrak{p}}}^l(\widehat{R}_{\mathfrak{p}}, S_{\mathfrak{p}}) \neq 0$, which contradicts the

fact that $S_{\mathfrak{p}}$ is injective as an $S_{\mathfrak{p}}$-module. Similarly, we have a contradiction when $\mathfrak{p}$ contains $\mathrm{Ann}_S \, \mathrm{Ext}^l_S(\widehat{C}, S)$. This contradiction proves that radius $\mathscr{X} = \infty$. $\qquad \square$

Now we can show the essential part of Theorem 3.3(2)(3).

Theorem 4.10. *Let R be a local ring. Let $\mathscr{X}$ be a resolving subcategory of* mod *R. One has* radius $\mathscr{X} = \infty$ *if there exists a module $M \in \mathscr{X}$ with $0 < \mathrm{Gdim}_R M < \infty$ that satisfies either of the following conditions.*

(1) $\Omega^{-2}\Omega^g M \in \mathscr{X}$, *where $g = \mathrm{Gdim}_R M$.*

(2) $\mathrm{res}(\Omega^n M)$ *is a thick subcategory of $\mathcal{G}(R)$ for some $n \geq 0$.*

Proof. According to Proposition 4.9, it suffices to show that $\mathscr{X}$ contains a module of projective dimension one.

(1) We consider a construction whose idea essentially comes from the Auslander–Buchweitz approximation theorem [1989, Theorem 1.1]. There are exact sequences $0 \to \Omega^g M \to R^{\oplus a} \to \Omega^{g-1}M \to 0$ and $0 \to \Omega^g M \to R^{\oplus b} \to \Omega^{-1}\Omega^g M \to 0$, where the latter is possible as $\Omega^g M$ is a totally reflexive module. We make the following pushout diagram.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega^g M & \longrightarrow & R^{\oplus a} & \longrightarrow & \Omega^{g-1}M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & R^{\oplus b} & \longrightarrow & N & \longrightarrow & \Omega^{g-1}M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega^{-1}\Omega^g M & = \!\!=\!\!= & \Omega^{-1}\Omega^g M & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

As $\Omega^{-1}\Omega^g M$ is totally reflexive, we have $\mathrm{Ext}^1_R(\Omega^{-1}\Omega^g M, R) = 0$. Hence the second column in the above diagram splits, and we get an exact sequence

$$0 \to R^{\oplus b} \to R^{\oplus a} \oplus \Omega^{-1}\Omega^g M \to \Omega^{g-1}M \to 0.$$

There is an exact sequence $0 \to \Omega^{-1}\Omega^g M \to R^{\oplus c} \to \Omega^{-2}\Omega^g M \to 0$, and taking the direct sum with $0 \to R^{\oplus a} \overset{=}{\to} R^{\oplus a} \to 0 \to 0$, we have an exact sequence $0 \to R^{\oplus a} \oplus \Omega^{-1}\Omega^g M \to R^{\oplus(a+c)} \to \Omega^{-2}\Omega^g M \to 0$. Thus the following pushout

diagram is obtained.

$$
\begin{array}{ccccccccc}
& & & & 0 & & 0 & & \\
& & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^{\oplus b} & \longrightarrow & R^{\oplus a} \oplus \Omega^{-1}\Omega^g M & \longrightarrow & \Omega^{g-1} M & \longrightarrow & 0 \\
& & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^{\oplus b} & \longrightarrow & R^{\oplus(a+c)} & \longrightarrow & L & \longrightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & \Omega^{-2}\Omega^g M & =\!=\!= & \Omega^{-2}\Omega^g M & & \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}
$$

As $\Omega^{g-1} M$ and $\Omega^{-2}\Omega^g M$ are in $\mathscr{X}$, the module L is also in $\mathscr{X}$. The second row shows that L has projective dimension at most 1. Since $\Omega^{-2}\Omega^g M$ is totally reflexive but $\Omega^{g-1} M$ is not, it follows from the second column that L is nonfree. Therefore the projective dimension of L is equal to 1.

(2) Since by assumption $\operatorname{res}\Omega^n M$ is a subcategory of $\mathscr{G}(R)$, the module $\Omega^n M$ is totally reflexive. Hence $n \geq g := \operatorname{\mathsf{Gdim}}_R M$.

We claim that $\operatorname{res}\Omega^n M = \operatorname{res}\Omega^g M$. In fact, since $n - g \geq 0$ and $\Omega^n M = \Omega^{n-g}(\Omega^g M)$, we observe that $\operatorname{res}\Omega^n M$ is contained in $\operatorname{res}\Omega^g M$. There is an exact sequence

$$
0 \to \Omega^n M \to F_{n-1} \to F_{n-2} \to \cdots \to F_{g+1} \to F_g \to \Omega^g M \to 0
$$

of totally reflexive R-modules with each F_i being free. Since $\operatorname{res}\Omega^n M$ is assumed to be thick in $\mathscr{G}(R)$, decomposing the above exact sequence into short exact sequences of totally reflexive modules, we see that $\Omega^g M$ belongs to $\operatorname{res}\Omega^n M$. Therefore $\operatorname{res}\Omega^g M$ is contained in $\operatorname{res}\Omega^n M$. Thus the claim follows.

Set $\mathscr{X} = \operatorname{res}\Omega^n M = \operatorname{res}\Omega^g M$. Our assumption implies that $\mathscr{X}$ is closed under cosyzygies, whence $\Omega^{-2}\Omega^g M \in \mathscr{X}$. By (1), we conclude that $\mathscr{X}$ contains a module of projective dimension 1. $\qquad\square$

Now, using Remark 4.1 and Theorem 4.10, we deduce Theorem 3.3(2)(3).

***Proof of Theorem 3.3*(4).** We use the notion of a module of reducible complexity, which has been introduced by Bergh [2007]. Let us recall the definition.

Definition 4.11. The subcategory $\mathscr{C}^r_R$ of $\operatorname{mod} R$ is defined inductively as follows.

(1) Every module of finite projective dimension belongs to $\mathscr{C}^r_R$.

(2) A module M with $0 < \mathrm{cx}_R M < \infty$ belongs to $\mathscr{C}_R^r$ if there exists a homogeneous element $\eta \in \mathrm{Ext}_R^*(M, M)$ with $|\eta| > 0$ which is represented by a short exact sequence $0 \to M \to K \to \Omega^{|\eta|-1} M \to 0$ with $K \in \mathscr{C}_R^r$, $\mathrm{cx}\, K < \mathrm{cx}\, M$ and depth $K = $ depth M.

An R-module is said to have *reducible complexity* if it is in $\mathscr{C}_R^r$.

The result below is shown in [Bergh 2007, Proposition 2.2(i)], which is implicitly stated in [Avramov et al. 1997].

Proposition 4.12. *Let R be a local ring. Every R-module of finite CI-dimension has reducible complexity.*

In a resolving subcategory, for any fixed integer $n \geq 0$, existence of modules of CI-dimension n is equivalent to existence of modules of projective dimension n.

Lemma 4.13. *Let R be a local ring. Let $\mathscr{X}$ be a resolving subcategory of* mod R. *Suppose that there is a module $M \in \mathscr{X}$ such that* $\mathrm{CIdim}_R M < \infty$. *Then $\mathscr{X}$ contains a module N with* $\mathrm{pd}_R N = \mathrm{CIdim}_R M$.

Proof. Since M has finite CI-dimension, it has finite complexity. It follows from Proposition 4.12 that M has reducible complexity. If $\mathrm{cx}\, M = 0$, then $\mathrm{pd}\, M < \infty$, and we can take $N := M$. Hence we may assume $\mathrm{cx}\, M > 0$. There exists an exact sequence $0 \to M \to K \to \Omega_R^{|\eta|-1} M \to 0$ with $\mathrm{cx}\, K < \mathrm{cx}\, M$ and depth $K = $ depth M, where η is a homogeneous element of $\mathrm{Ext}_R^*(M, M)$. We have $K \in \mathscr{X}$ and

$$\mathrm{CIdim}\, K = \text{depth } R - \text{depth } K = \text{depth } R - \text{depth } M = \mathrm{CIdim}\, M.$$

Replacing M with K and iterating this procedure, we can eventually arrive at a module $N \in \mathscr{X}$ with $\mathrm{CIdim}\, N = \mathrm{CIdim}\, M$ and $\mathrm{cx}\, N = 0$. The module N has finite projective dimension, and we have $\mathrm{pd}\, N = \mathrm{CIdim}\, N = \mathrm{CIdim}\, M$. $\square$

Lemma 4.13 and Proposition 4.9 immediately yield the following theorem. This is not only the essential part of Theorem 3.3(4) but also a generalization of Proposition 4.9.

Theorem 4.14. *Let R be a local ring. Let $\mathscr{X}$ be a resolving subcategory of* mod R. *Suppose that there exists a module $M \in \mathscr{X}$ with* $0 < \mathrm{CIdim}_R M < \infty$. *Then the radius of $\mathscr{X}$ is infinite.*

Theorem 3.3(4) now follows from Theorem 4.14 and Remark 4.1.

Another proof of Theorem 4.14. In the next theorem, we study the thickness of resolving subcategories of modules of CI-dimension at most zero. This will give another proof of Theorem 4.14.

Theorem 4.15. *Let R be a local ring.*

(1) *Let M be an R-module of CI-dimension at most zero. Then $\Omega^{-1}M$ belongs to res M.*

(2) *Let $\mathscr{X}$ be a resolving subcategory of* mod R. *Suppose that every module in $\mathscr{X}$ has CI-dimension at most zero. Then $\mathscr{X}$ is a thick subcategory of $\mathscr{G}(R)$.*

Proof. (1) Proposition 4.12 implies that M has reducible complexity. Let $K_0 = M$ and let K_{i+1} be a reduction in complexity of K_i for each $i \geq 0$. Then we have a short exact sequence $0 \to K_i \xrightarrow{f_i} K_{i+1} \to \Omega^{t_i-1}K_i \to 0$ with $t_i > 0$ (see also [Avramov et al. 1997, Proposition 7.2]), and eventually we must have $\operatorname{cx}_R K_e = 0$ for some $e \geq 0$. Then K_e has finite projective dimension. As $\operatorname{CIdim}_R M = 0$, we have depth $K_e = $ depth $M = $ depth R. Therefore K_e is a free module. Note that the above exact sequence also shows that

$$K_i \in \operatorname{res} M \quad \text{for all } i \geq 0. \tag{4.15.1}$$

If $e = 0$, then M is free and we have $\Omega^{-1}M = 0 \in \operatorname{res} M$. So we may assume $e \geq 1$.

We claim that for each $0 \leq i \leq e-1$ the cokernel C_i of the composite map $f_i \cdots f_1 f_0 : M \to K_{i+1}$ belongs to res M. Let us show this claim by induction on i. When $i = 0$, we have $C_i = \Omega^{t_0-1}M \in \operatorname{res} M$. Let $i \geq 1$. We have the following commutative diagram with exact rows and columns.

$$
\begin{array}{ccccccccc}
& & & & 0 & & 0 & & \\
& & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \xrightarrow{f_{i-1}\cdots f_0} & K_i & \longrightarrow & C_{i-1} & \longrightarrow & 0 \\
& & \| & & \downarrow{\scriptstyle f_i} & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & K_{i+1} & \longrightarrow & C_i & \longrightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & \Omega^{t_i-1}K_i & \!\!=\!\!\!=\!\!\!=\!\! & \Omega^{t_i-1}K_i & & \\
& & & & \downarrow & & \downarrow & & \\
& & & & 0 & & 0 & &
\end{array}
$$

The induction hypothesis implies that C_{i-1} belongs to res M. Since $\Omega^{t_i-1}K_i$ is in res M by (4.15.1), the right column shows that C_i is also in res M. Thus the claim follows.

Now we have a short exact sequence $0 \to M \xrightarrow{f_{e-1}\cdots f_1 f_0} K_e \to C_{e-1} \to 0$, where K_e is free and C_{e-1} is in res M by the claim. Since M is totally reflexive and $\mathscr{G}(R)$ is a resolving subcategory of mod R, all the modules in res M are totally reflexive. Hence all modules appearing in the above exact sequence belong to $\mathscr{G}(R)$. It is

easy to verify that there exists an isomorphism $C_{e-1} \cong \Omega^{-1}M \oplus F$ with F being free. Consequently, $\Omega^{-1}M$ belongs to res M.

(2) By assumption, $\mathscr{X}$ is a subcategory of $\mathscr{G}(R)$. Thanks to Proposition 1.7, it is enough to show that $\mathscr{X}$ is closed under cosyzygies. Let M be an R-module in $\mathscr{X}$. Then M is of CI-dimension at most zero, and $\Omega^{-1}M$ belongs to res M by (1). Since $\mathscr{X}$ is resolving and contains M, it also contains res M. Thus $\Omega^{-1}M$ is in $\mathscr{X}$.   □

Theorem 4.15(2) immediately implies:

Corollary 4.16. *Let R be a local complete intersection. The following two are the same.*

- *A resolving subcategory of* mod R *contained in* $\mathrm{CM}(R)$.
- *A thick subcategory of* $\mathrm{CM}(R)$ *containing R.*

Now let us give another proof of Theorem 4.14. Let R be a local ring, and let $M \in \mathscr{X}$ be an R-module with $0 < \mathrm{CIdim}_R M < \infty$. Then we have $c := \mathrm{CIdim}_R M = \mathrm{Gdim}_R M$, and $\Omega^c M$ has CI-dimension zero. In particular, $\Omega^c M$ is totally reflexive, and hence so is $\Omega^{-1}\Omega^c M$. We have

$$0 = \mathrm{CIdim}_R(\Omega^c M) = \sup\{\mathrm{CIdim}_R(\Omega^{-1}\Omega^c M) - 1, 0\},$$

which especially says that $\Omega^{-1}\Omega^c M$ has finite CI-dimension. Therefore

$$\mathrm{CIdim}_R(\Omega^{-1}\Omega^c M) = \mathrm{Gdim}_R(\Omega^{-1}\Omega^c M) = 0,$$

and Theorem 4.15(1) yields $\Omega^{-2}\Omega^c M \in \mathrm{res}(\Omega^c M) \subseteq \mathscr{X}$. Now Theorem 4.10(1) implies that the radius of $\mathscr{X}$ is infinite.

5. Proof of Theorem II

In this section we prove the main Theorem II from the introduction. In fact, we can prove significantly more general statements (Theorems 5.7 and 5.11). In order to state and prove such results we need to first introduce a couple of definitions related to the concept of radius. In this section, an *n-th syzygy* $\Omega^n M$ of an R-module M means the image of the n-th differential map in a projective resolution of M in mod R. (So it is not unique up to isomorphism but unique up to projective summands.)

Definition 5.1. Let $\mathscr{X}, \mathscr{Y}$ be subcategories of mod R. We put $|\mathscr{X}| = \mathrm{add}\,\mathscr{X}$, and set $\mathscr{X} * \mathscr{Y} = \big||\mathscr{X}| \circ |\mathscr{Y}|\big|$. (The notation "$\circ$" was introduced in Definition 2.1.) For an integer $r > 0$, set

$$|\mathscr{X}|_r = \begin{cases} |\mathscr{X}| & \text{if } r = 1, \\ |\mathscr{X}|_{r-1} * \mathscr{X} & \text{if } r \geq 2. \end{cases}$$

Let $\mathscr{X}, \mathscr{Y}, \mathscr{Z}$ be subcategories of mod R. We observe that an object $M \in \text{mod } R$ is in $\mathscr{X} * \mathscr{Y}$ if and only if there is an exact sequence $0 \to X \to E \to Y \to 0$ with $X \in |\mathscr{X}|$ and $Y \in |\mathscr{Y}|$ such that M is a direct summand of E. Also, one has $(\mathscr{X} * \mathscr{Y}) * \mathscr{Z} = \mathscr{X} * (\mathscr{Y} * \mathscr{Z})$ and $|\mathscr{X}|_a * |\mathscr{X}|_b = |\mathscr{X}|_{a+b}$ for all $a, b > 0$.

Definition 5.2. For a subcategory $\mathscr{X}$ of mod R we define size $\mathscr{X}$ (respectively, rank $\mathscr{X}$) to be the infimum of integers $n \geq 0$ such that $\mathscr{X} \subseteq |G|_{n+1}$ (respectively, $\mathscr{X} = |G|_{n+1}$) for some $G \in \text{mod } R$.

It always holds that size $\mathscr{X} \leq \text{rank } \mathscr{X}$. Since $|\mathscr{X}|_n \subseteq [\mathscr{X}]_n$ for all $n > 0$, one has dim $\mathscr{X} \geq$ radius $\mathscr{X} \leq$ size $\mathscr{X}$. If $\mathscr{X}$ is resolving, then dim $\mathscr{X} \leq$ rank $\mathscr{X}$.

For an R-module M, we denote by $M^{\oplus}$ an object in $\text{add}_R M$.

Proposition 5.3. *Let I be an ideal of R and let M be an R/I-module.*

(1) *There is an exact sequence $0 \to I^{\oplus} \to \Omega_R M \to \Omega_{R/I} M \to 0$.*

(2) *One has $\Omega_R^n M \in \left| \Omega_{R/I}^n M \oplus \left(\bigoplus_{i=0}^{n-1} \Omega_R^i I \right) \right|_{n+1}$ for all $n \geq 0$.*

Proof. (1) Take a surjection from a free R-module F to M. Then this factors through a surjection $F/IF \to M$. The assertion follows from this.

(2) We induce on n. Let $n > 0$. The induction hypothesis shows

$$\Omega_R^{n-1} \Omega_{R/I} M \in \left| \Omega_{R/I}^n M \oplus \left(\bigoplus_{i=0}^{n-2} \Omega_R^i I \right) \right|_n.$$

By (1) we have an exact sequence $0 \to \Omega_R^{n-1} I^{\oplus} \to \Omega_R^n M \to \Omega_R^{n-1} \Omega_{R/I} M \to 0$. Now the assertion follows. $\qquad \square$

For $n \geq 0$ we denote by $\Omega_R^n(\text{mod } R)$ the subcategory of mod R consisting of n-th syzygies of R-modules. For an ideal I of R, let $\Omega_R^n(\text{mod } R/I)$ be the subcategory of mod R consisting of n-th syzygies of R-modules annihilated by I.

Corollary 5.4. *Let $d = \dim R < \infty$. Suppose that $R/\mathfrak{p}$ is regular for all $\mathfrak{p} \in \text{Min } R$. Then size $\Omega^d(\text{mod } R) < \infty$.*

Proof. There is a filtration $R = I_0 \supsetneq I_1 \supsetneq \cdots \supsetneq I_n = 0$ of ideals of R such that for each i one has $I_i/I_{i+1} \cong R/\mathfrak{p}_i$ with $\mathfrak{p}_i \in \text{Spec } R$. Choose a minimal prime $\mathfrak{q}_i$ contained in $\mathfrak{p}_i$. Let M be an R-module. Setting $M_i = I_i M/I_{i+1} M$, we have an exact sequence $0 \to \Omega_R^d(I_{i+1} M) \to \Omega_R^d(I_i M) \to \Omega_R^d M_i \to 0$. Note that each d-th syzygy $R/\mathfrak{q}_i$-module is free. Hence $\Omega_R^d M_i \in |R/\mathfrak{q}_i \oplus L_i|_{d+1}$ by Proposition 5.3(2), where $L_i := \bigoplus_{j=0}^{d-1} \Omega_R^j \mathfrak{q}_i$. Thus $\Omega_R^d M \in \left| \bigoplus_{i=1}^n (R/\mathfrak{q}_i \oplus L_i) \right|_{n(d+1)}$, which implies size $\Omega^d(\text{mod } R) < n(d+1) < \infty$. $\qquad \square$

Corollary 5.5. *Let I be an ideal of R and $n \geq 0$ an integer. Then*

$$\text{size } \Omega_R^n(\text{mod } R/I) < (n+1)(\text{size } \Omega_{R/I}^n(\text{mod } R/I) + 1).$$

In particular, if $\operatorname{size} \Omega_{R/I}^n (\operatorname{mod} R/I)$ *is finite, then so is* $\operatorname{size} \Omega_R^n (\operatorname{mod} R/I)$.

Proof. This is a consequence of Proposition 5.3(2).   □

Lemma 5.6. *Let M be an R-module. Let $x \in R$ be R-regular. Then*

$$\Omega_{R/xR}^n (\Omega_R M / x \Omega_R M) \cong \Omega_R^{n+1} M / x \Omega_R^{n+1} M$$

for any $n \geq 0$.

Proof. We use induction on n. Let $n > 0$. We have

$$\Omega_{R/xR}^{n-1} (\Omega M / x \Omega M) \cong \Omega^n M / x \Omega^n M$$

by the induction hypothesis, and hence

$$\Omega_{R/xR}^n (\Omega M / x \Omega M) \cong \Omega_{R/xR} (\Omega^n M / x \Omega^n M).$$

Note that x is $\Omega_R^n M$-regular. There is an exact sequence

$$0 \to \Omega^{n+1} M \to P \to \Omega^n M \to 0$$

of R-modules with P projective, which gives an exact sequence

$$0 \to \Omega^{n+1} M / x \Omega^{n+1} M \to P/xP \to \Omega^n M / x \Omega^n M \to 0.$$

Hence $\Omega^{n+1} M / x \Omega^{n+1} M \cong \Omega_{R/xR} (\Omega^n M / x \Omega^n M)$.   □

Theorem 5.7. *Let $(R, \mathfrak{m})$ be a d-dimensional complete local ring with perfect coefficient field. Then one has $\operatorname{size} \Omega^d (\operatorname{mod} R) < \infty$. Hence $\operatorname{radius} \Omega^d (\operatorname{mod} R) < \infty$.*

Proof. We use induction on d. When $d = 0$, we have $\operatorname{mod} R = |k|_{\ell\ell(R)}$, hence $\operatorname{size}(\operatorname{mod} R) < \ell\ell(R) < \infty$. Let $d \geq 1$. Take a filtration $R = I_0 \supsetneq \cdots \supsetneq I_n = 0$ of ideals such that for each i one has $I_i / I_{i+1} \cong R/\mathfrak{p}_i$ with $\mathfrak{p}_i \in \operatorname{Spec} R$. Suppose that $\operatorname{size} \Omega_{R/\mathfrak{p}_i}^{d_i} (\operatorname{mod} R/\mathfrak{p}_i) < \infty$ for all i, where $d_i = \dim R/\mathfrak{p}_i$. Then we have

$$\operatorname{size} \Omega_{R/\mathfrak{p}_i}^d (\operatorname{mod} R/\mathfrak{p}_i) < \infty,$$

since $\Omega_{R/\mathfrak{p}_i}^d (\operatorname{mod} R/\mathfrak{p}_i)$ is contained in $\Omega_{R/\mathfrak{p}_i}^{d_i} (\operatorname{mod} R/\mathfrak{p}_i)$. Corollary 5.5 implies $\operatorname{size} \Omega_R^d (\operatorname{mod} R/\mathfrak{p}_i) < \infty$. For each R-module M there is an exact sequence $0 \to I_{i+1} M \to I_i M \to I_i M / I_{i+1} M \to 0$, which gives an exact sequence

$$0 \to \Omega_R^d (I_{i+1} M) \to \Omega_R^d (I_i M) \to \Omega_R^d (I_i M / I_{i+1} M) \to 0.$$

As $\Omega_R^d (I_i M / I_{i+1} M)$ is in $\Omega_R^d (\operatorname{mod} R/\mathfrak{p}_i)$, we have $\operatorname{size} \Omega_R^d (\operatorname{mod} R) < \infty$. Thus, we may assume R is a domain.

   If R is regular, then $\Omega^d (\operatorname{mod} R) = |R|_1$ and $\operatorname{size} \Omega^d (\operatorname{mod} R) = 0 < \infty$, so we may assume that R is singular. By [Wang 1994, 5.15] there is an ideal $J \subseteq \mathfrak{m}$ with

Sing $R = V(J)$ and $J \operatorname{Ext}_R^{d+1}(\operatorname{mod} R, \operatorname{mod} R) = 0$. Since R is a domain, we find an element $0 \neq x \in J$. The induction hypothesis guarantees

$$\Omega_{R/xR}^{d-1}(\operatorname{mod} R/xR) \subseteq |G|_n^{R/xR}$$

for some R/xR-module G and an integer $n > 0$. Let M be an R-module and put $N = \Omega_R^d M$. Note that x is N-regular as $d > 0$. Hence N is isomorphic to a direct summand of $\Omega_R(N/xN)$ (see [Takahashi 2010, Lemma 2.1]). In view of Lemma 5.6, we have

$$N/xN \cong \Omega_{R/xR}^{d-1}(\Omega_R M/x\Omega_R M) \in \Omega_{R/xR}^{d-1}(\operatorname{mod} R/xR) \subseteq |G|_n^{R/xR}.$$

Hence N/xN is in $|G|_n^R$, which implies $\Omega_R(N/xN) \in |\Omega_R G|_n^R$. Therefore N belongs to $|\Omega_R G|_n^R$, and we obtain $\Omega^d(\operatorname{mod} R) \subseteq |\Omega_R G|_n^R$. It now follows that size $\Omega^d(\operatorname{mod} R) < \infty$. $\square$

Lemma 5.8. *Let*

$$0 \to M \to C^0 \to C^1 \to \cdots \to C^{n-1} \to N \to 0$$

be an exact sequence in $\operatorname{mod} R$ *with* $n \geq 0$. *Then* M *is in* $\left|\Omega^n N \oplus \left(\bigoplus_{i=0}^{n-1} \Omega^i C^i\right)\right|_{n+1}$.

Proof. We induce on n. The case $n = 0$ is trivial, so let $n \geq 1$. There are two exact sequences $0 \to M \to C^0 \to L \to 0$ and $0 \to L \to C^1 \to \cdots \to C^{n-1} \to N \to 0$. The induction hypothesis shows $L \in \left|\Omega^{n-1} N \oplus \left(\bigoplus_{i=0}^{n-2} \Omega^i C^{i+1}\right)\right|_n$. A pullback diagram makes an exact sequence $0 \to \Omega L \to M \oplus R^\oplus \to C^0 \to 0$. Since ΩL belongs to $\left|\Omega^n N \oplus \left(\bigoplus_{i=0}^{n-2} \Omega^{i+1} C^{i+1}\right)\right|_n$, we see that M is in $\left|\Omega^n N \oplus \left(\bigoplus_{i=0}^{n-1} \Omega^i C^i\right)\right|_{n+1}$. $\square$

Corollary 5.9. *Let R be a Cohen–Macaulay complete local ring with perfect coefficient field. Then* size $\operatorname{CM}(R) < \infty$.

Proof. As R is complete, it admits a canonical module ω. Theorem 5.7 implies size $\Omega^d(\operatorname{mod} R) =: n < \infty$, so we have $\Omega^d(\operatorname{mod} R) \subseteq |G|_{n+1}$ for some R-module G. Let M be a Cohen–Macaulay R-module. Then there exists an exact sequence $0 \to M \to \omega^{\oplus s_0} \to \cdots \to \omega^{\oplus s_{d-1}} \to N \to 0$. It follows from Lemma 5.8 that M is in $|\Omega^d N \oplus W|_{d+1}$, where $W := \bigoplus_{i=0}^{d-1} \Omega^i \omega$. Since $\Omega^d N \in |G|_{n+1}$, we have $M \in |G \oplus W|_{(n+1)(d+1)}$. Thus size $\operatorname{CM}(R) < (n+1)(d+1) < \infty$. $\square$

Proposition 5.10. *Let R be a Cohen–Macaulay local ring with a canonical module ω.*

 (1) rank $\operatorname{CM}(R) < (\dim R + 1)(\text{size } \operatorname{CM}(R) + 1)$.

 (2) dim $\operatorname{CM}(R) < (\dim R + 1)(\text{radius } \operatorname{CM}(R) + 1)$.

In particular, one has

rank $\operatorname{CM}(R) < \infty \Leftrightarrow$ size $\operatorname{CM}(R) < \infty \Rightarrow$ dim $\operatorname{CM}(R) < \infty \Leftrightarrow$ radius $\operatorname{CM}(R) < \infty$.

Proof. (1) Let $n = \operatorname{size} \operatorname{CM}(R)$. We find an R-module G with $\operatorname{CM}(R) \subseteq |G|_{n+1}$. Let $d = \dim R$ and $M \in \operatorname{CM}(R)$. Similarly to the proof of Corollary 5.9, there exists $N \in \operatorname{CM}(R)$ such that M is in $|\Omega^d N \oplus W|_{d+1}$, where $W := \bigoplus_{i=0}^{d-1} \Omega^i \omega \in \operatorname{CM}(R)$. Note that $\Omega^d N \in |\Omega^d G|_{n+1}$. Thus we obtain

$$\operatorname{CM}(R) = |\Omega^d G \oplus W|_{(n+1)(d+1)},$$

and $\operatorname{rank} \operatorname{CM}(R) < (n+1)(d+1)$.

(2) In the proof of (1), replace "size", "rank" and "| |" with "radius", "dim" and "[ ]", respectively. □

Theorem 5.11. *Let R be a Cohen–Macaulay local ring admitting perfect coefficient field. Assume that either of the following holds.*

(1) *R is complete.*

(2) *R is excellent with an isolated singularity.*

Then one has $\operatorname{rank} \operatorname{CM}(R) < \infty$.

Proof. (1) This assertion follows from Corollary 5.9 and Proposition 5.10(1).

(2) It follows by (1) that $\operatorname{CM}(\widehat{R})$ has finite rank, where $\widehat{R}$ denotes the completion of R. One can prove that $\operatorname{CM}(R)$ also has finite rank, making an argument similar to [Dao and Takahashi 2012, Remark 6.5]:

Putting $n = \operatorname{rank} \operatorname{CM}(\widehat{R})$, we have $\operatorname{CM}(\widehat{R}) = |C|_{n+1}$ for some $C \in \operatorname{CM}(\widehat{R})$. Since R is Cohen–Macaulay, excellent and with an isolated singularity, we can apply [Takahashi 2010, Corollary 3.6] to see that C is isomorphic to a direct summand of $\widehat{G}$ for some $G \in \operatorname{CM}(R)$. Hence the equality $\operatorname{CM}(\widehat{R}) = |\widehat{G}|_{n+1}$ holds.

Claim. Let $m > 0$. For any $N \in |\widehat{G}|_m$ there exists $M \in |G|_m$ such that N is isomorphic to a direct summand of $\widehat{M}$.

To show this claim, we use induction on m. As R is an isolated singularity, for all $X, Y \in \operatorname{CM}(R)$ the R-module $\operatorname{Ext}^1_R(X, Y)$ has finite length. Hence there are isomorphisms $\operatorname{Ext}^1_R(X, Y) \cong \operatorname{Ext}^1_R(X, Y)\widehat{\ } \cong \operatorname{Ext}^1_{\widehat{R}}(\widehat{X}, \widehat{Y})$, which imply that every short exact sequence $0 \to \widehat{Y} \to E \to \widehat{X} \to 0$ of $\widehat{R}$-modules is isomorphic to the completion of some short exact sequence $0 \to Y \to E' \to X \to 0$ of R-modules. The claim follows from this. Using this claim and [Aihara and Takahashi 2011, Lemma 5.7], we observe that $\operatorname{CM}(R) = |G|_{n+1}$ holds. Therefore $\operatorname{rank} \operatorname{CM}(R) \leq n < \infty$. □

6. Some discussions and open questions

In this section we relate our results to the uniform Auslander condition and discuss some open questions. For a local ring R, Jorgensen and Şega [2004] introduced the *uniform Auslander condition*:

(UAC): There exists an integer n such that for all R-modules M, N with
$\operatorname{Ext}^i_R(M, N) = 0$ for all $i \gg 0$ one has $\operatorname{Ext}^i_R(M, N) = 0$ for all $i \geq n$.

It is known that this condition is satisfied if the local ring R is a complete intersection, a Golod ring, a Gorenstein ring with $\operatorname{mult} R = \operatorname{codim} R + 2$, or a Gorenstein ring with $\operatorname{codim} R \leq 4$. Here $\operatorname{mult} R$ denotes the multiplicity of R. These are proved in [Jorgensen and Şega 2004, Proposition 1.4], [Avramov and Buchweitz 2000, Theorem 4.7], [Huneke and Jorgensen 2003, Theorem 3.5] and [Şega 2003, Theorem 3.4], respectively. More information can be found in [Christensen and Holm 2010, Appendix A]. On the other hand, there exists an example of a Gorenstein local ring which does not satisfy (UAC); see [Jorgensen and Şega 2004, Theorem in §0].

There result below says that over a Gorenstein local ring the condition (UAC) is closely related to the thickness of resolving subcategories of Cohen–Macaulay modules.

Proposition 6.1. *Let R be a Gorenstein local ring. Assume every resolving subcategory of $\operatorname{mod} R$ contained in $\operatorname{CM}(R)$ is a thick subcategory of $\operatorname{CM}(R)$. Then R satisfies* (UAC).

Proof. Let $t \geq 0$ be an integer, and let M, N be R-modules with $\operatorname{Ext}^i_R(M, N) = 0$ for all $i > t$. We define a subcategory $\mathscr{X}$ of $\operatorname{mod} R$ to consist of all Cohen–Macaulay R-modules X satisfying $\operatorname{Ext}^i_R(X, N) = 0$ for all $i > t$. Then $\mathscr{X}$ is a resolving subcategory of $\operatorname{mod} R$ contained in $\operatorname{CM}(R)$. By assumption, $\mathscr{X}$ is a thick subcategory of $\operatorname{CM}(R)$. Set $d = \dim R$. Since $\Omega^d M$ is in $\mathscr{X}$, so is $\Omega^{-t}\Omega^d M$. We have

$$\operatorname{Ext}^i_R(M, N) \cong \operatorname{Ext}^{i-d}_R(\Omega^d M, N) \cong \operatorname{Ext}^{i-d}_R(\Omega^t(\Omega^{-t}\Omega^d M), N)$$
$$\cong \operatorname{Ext}^{i-d+t}_R(\Omega^{-t}\Omega^d M, N) = 0$$

for all integers $i > d$. $\qquad\square$

There is also a connection between thickness of resolving subcategories of totally reflexive modules and closure under R-duals. Here we say that a subcategory $\mathscr{X}$ of $\operatorname{mod} R$ is *closed under R-duals* if for each module M in $\mathscr{X}$ its R-dual M^* is also in $\mathscr{X}$.

Proposition 6.2. (1) *Let R be local. Let $\mathscr{X}$ be a resolving subcategory of $\operatorname{mod} R$ contained in $\mathcal{G}(R)$. If $\mathscr{X}$ is closed under R-duals, then $\mathscr{X}$ is a thick subcategory of $\mathcal{G}(R)$.*

(2) *Let R be a local hypersurface. Then every resolving subcategory of $\operatorname{mod} R$ contained in $\operatorname{CM}(R)$ is closed under R-duals.*

Proof. (1) According to Proposition 1.7, we have only to show that $\mathscr{X}$ is closed under cosyzygies. Let $X \in \mathscr{X}$. There is an exact sequence $0 \to \Omega(X^*) \to F \to X^* \to 0$,

where F is free. Dualizing this by R, we get an exact sequence

$$0 \to X \to F^* \to (\Omega(X^*))^* \to 0.$$

Note that $(\Omega(X^*))^*$ is totally reflexive. We easily see that $(\Omega(X^*))^*$ is isomorphic to $\Omega^{-1}X$ up to free summand. As $\mathscr{X}$ is a resolving subcategory closed under R-duals, $(\Omega(X^*))^*$ belongs to $\mathscr{X}$, and so does $\Omega^{-1}X$.

(2) It follows from [Takahashi 2010, main theorem] that every resolving subcategory of mod R contained in $CM(R)$ can be described as $NF_{CM}^{-1}(W)$, where W is a specialization-closed subset of Spec R contained in Sing R. If M is an R-module in $NF_{CM}^{-1}(W)$, then we have $NF(M^*) = NF(M) \subseteq W$, which shows that $NF_{CM}^{-1}(W)$ also contains M^*. $\square$

Now we have reached the following question.

Question 6.3. Let R be a Gorenstein local ring. Let us consider the following five conditions.

(1) R is a complete intersection.

(2) Every resolving subcategory of mod R contained in $CM(R)$ is closed under R-duals.

(3) Every resolving subcategory of mod R contained in $CM(R)$ is a thick subcategory of $CM(R)$.

(4) R satisfies (UAC).

(5) Conjecture 3.1 is true for R.

We know that the implications $(2) \Rightarrow (3)$ and $(1) \Rightarrow (3) \Rightarrow (4)$ hold by Propositions 6.1, 6.2(1) and Corollary 4.16. The implication $(1) \Rightarrow (2)$ is also true if R is a hypersurface by Proposition 6.2(2). Very recently, motivated by the first version of the present paper, Stevenson [2013a] proved that the implication $(1) \Rightarrow (2)$ holds in the case where R is a quotient of a regular local ring. Corollary 3.4 says that $(3) \Rightarrow (5)$ holds. How about the other implications among these five conditions?

Remark 6.4. According to a recent preprint by Stevenson [2013b] (see also [Iyengar 2009]), if R is a quotient of a regular local ring by a regular sequence, then one can classify the thick subcategories of $\underline{CM}(R)$ in terms of "support varieties". Thus, one can also classify the resolving subcategories of mod R contained in $CM(R)$ by using Corollary 4.16 and [Takahashi 2010, Proposition 6.2]. In relation to this, the resolving subcategories over a regular ring can be classified completely. This classification theorem is stated and proved in [Dao and Takahashi 2013].

Acknowledgments

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