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On common values of $\phi(n)$ and $\sigma(n)$, II

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For each positive-integer valued arithmetic function f, let $\mathscr{V}_f \subset \mathbb{N}$ denote the image of f, and put $\mathscr{V}_f(x) := \mathscr{V}_f \cap [1, x]$ and $\mathscr{V}_f(x) := \#\mathscr{V}_f(x)$. Recently Ford, Luca, and Pomerance showed that $\mathscr{V}_\phi \cap \mathscr{V}_\sigma$ is infinite, where ϕ denotes Euler's totient function and σ is the usual sum-of-divisors function. Work of Ford shows that $V_\phi(x) \asymp V_\sigma(x)$ as $x \to \infty$. Here we prove a result complementary to that of Ford et al. by showing that most ϕ-values are not σ-values, and vice versa. More precisely, we prove that, as $x \to \infty$,

$$\#\{n \leqslant x : n \in \mathscr{V}_\phi \cap \mathscr{V}_\sigma\} \leqslant \frac{V_\phi(x) + V_\sigma(x)}{(\log\log x)^{1/2+o(1)}}.$$

1. Introduction

1A. ***Summary of results.*** For each positive-integer valued arithmetic function f, let $\mathscr{V}_f$ denote the image of f, and put $\mathscr{V}_f(x) := \mathscr{V}_f \cap [1, x]$ and $V_f(x) := \#\mathscr{V}_f(x)$. In this paper we are primarily concerned with the cases when $f = \phi$, the Euler totient function, and when $f = \sigma$, the usual sum-of-divisors function. When $f = \phi$, the study of the counting function V_f goes back to Pillai [1929], and was subsequently taken up in [Erdős 1935; 1945; Erdős and Hall 1973; 1976; Pomerance 1986; Maier and Pomerance 1988; Ford 1998a] (with an announcement in [Ford 1998b]). From the sequence of results obtained in these papers, we mention Erdős's asymptotic formula [1935] for $\log(V_f(x)/x)$, namely

$$V_f(x) = \frac{x}{(\log x)^{1+o(1)}} \quad (x \to \infty),$$

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and the much more intricate determination of the precise order of magnitude by Ford,

$$V_f(x) \asymp \frac{x}{\log x} \exp\bigl(C(\log_3 x - \log_4 x)^2 + D\log_3 x - (D + \tfrac{1}{2} - 2C)\log_4 x\bigr). \quad (1\text{-}1)$$

Here $\log_k$ denotes the k-th iterate of the natural logarithm, and the constants C and D are defined as follows: Let

$$F(z) := \sum_{n=1}^{\infty} a_n z^n, \quad \text{where } a_n = (n+1)\log(n+1) - n\log n - 1. \quad (1\text{-}2)$$

Since each $a_n > 0$ and $a_n \sim \log n$ as $n \to \infty$, it follows that $F(z)$ converges to a continuous, strictly increasing function on (0, 1), and $F(z) \to \infty$ as $z \uparrow 1$. Thus, there is a unique real number ϱ for which

$$F(\varrho) = 1 \quad (\varrho = 0.542598586098471021959\ldots). \quad (1\text{-}3)$$

In addition, F' is strictly increasing, and $F'(\varrho) = 5.697758\ldots$. Then

$$C = \frac{1}{2\,|\log \varrho|} = 0.817814\ldots$$
$$D = 2C(1 + \log F'(\varrho) - \log(2C)) - \tfrac{3}{2} = 2.176968\ldots.$$

In [Ford 1998a], it is also shown that (1-1) holds for a wide class of ϕ-like functions, including $f = \sigma$. Consequently, $V_\phi(x) \asymp V_\sigma(x)$.

Erdős [1959, p. 172] asked if it could be proved that infinitely many natural numbers appear in both $\mathscr{V}_\phi$ and $\mathscr{V}_\sigma$ (see also [Erdős and Graham 1980]). This question was recently answered by Ford, Luca, and Pomerance [Ford et al. 2010]. Writing $V_{\phi,\sigma}(x)$ for the number of common elements of $\mathscr{V}_\phi$ and $\mathscr{V}_\sigma$ up to x, they proved that

$$V_{\phi,\sigma}(x) \geqslant \exp((\log\log x)^c)$$

for some positive constant $c > 0$ and all large x (in [Garaev 2011] this is shown for *all* constants $c > 0$). This lower bound is probably very far from the truth. For example, if p and $p+2$ form a twin prime pair, then $\phi(p+2) = p+1 = \sigma(p)$; a quantitative form of the twin prime conjecture then implies that $V_{\phi,\sigma}(x) \gg x/(\log x)^2$. In Part I of this article [Ford and Pollack 2011], we showed that a stronger conjecture of the same type allows for an improvement. Roughly, our result is as follows:

Theorem A. *Assume a strong uniform version of Dickson's prime k-tuples conjecture. Then as $x \to \infty$,*

$$V_{\phi,\sigma}(x) = \frac{x}{(\log x)^{1+o(1)}}.$$

Theorem A suggests that $V_{\phi,\sigma}(x)$ is much larger than we might naively expect. This naturally leads one to inquire about what can be proved in the opposite direction; for instance, could it be that a positive proportion of ϕ-values are also σ-values?

N	$V_\phi(N)$	$V_\sigma(N)$	$V_{\phi,\sigma}(N)$	$\frac{V_{\phi,\sigma}(N)}{V_\phi(N)}$	$\frac{V_{\phi,\sigma}(N)}{V_\sigma(N)}$
10000	2374	2503	1368	0.5762426	0.5465441
100000	20254	21399	11116	0.5488299	0.5194635
1000000	180184	189511	95145	0.5280436	0.5020553
10000000	1634372	1717659	841541	0.5149017	0.4899348
100000000	15037909	15784779	7570480	0.5034264	0.4796063
1000000000	139847903	146622886	69091721	0.4940490	0.4712206

Table 1. Data on ϕ-values, σ-values, and common values up to $N = 10^k$, from $k = 5$ to $k = 9$.

The numerical data up to 10^9, exhibited in Table 1, suggests that the proportion of common values is decreasing, but the observed rate of decrease is rather slow.

Our principal result is the following estimate, which implies in particular that almost all ϕ-values are not σ-values, and vice versa.

Theorem 1.1. *As* $x \to \infty$,

$$V_{\phi,\sigma}(x) \leqslant \frac{V_\phi(x) + V_\sigma(x)}{(\log\log x)^{1/2+o(1)}}.$$

The proof of Theorem 1.1 relies on the detailed structure theory of totients as developed in [Ford 1998a]. It would be interesting to know the true rate of decay of $V_{\phi,\sigma}(x)/V_\phi(x)$.

1B. ***Sketch.*** Since the proof of Theorem 1.1 is rather intricate and involves a number of technical estimates, we present a brief outline of the argument in this section.

We start by discarding a sparse set of undesirable ϕ and σ-values. More precisely, we identify (in Lemma 3.2) convenient sets $\mathscr{A}_\phi$ and $\mathscr{A}_\sigma$ with the property that almost all ϕ-values less than or equal to x have all their preimages in $\mathscr{A}_\phi$ and almost all σ-values less than or equal to x have all their preimages in $\mathscr{A}_\sigma$. This reduces us to studying how many ϕ and σ-values arise as solutions to the equation

$$\phi(a) = \sigma(a'), \quad \text{where} \quad a \in \mathscr{A}_\phi, a' \in \mathscr{A}_\sigma.$$

Note that to show that $V_{\phi,\sigma}(x)/V_\phi(x) \to 0$, we need only count the number of common ϕ-σ-values of this kind, and not the (conceivably much larger) number of pairs $(a, a') \in \mathscr{A}_\phi \times \mathscr{A}_\sigma$ corresponding to these values.

What makes the sets $\mathscr{A}_\phi$ and $\mathscr{A}_\sigma$ convenient for us? The properties imposed in the definitions of these sets are of two types, *anatomical* and *structural*. By anatomical considerations, we mean general considerations of multiplicative structure as commonly appear in elementary number theory (for example, consideration of the number and size of prime factors). By structural considerations, we mean those depending for their motivation on the fine structure theory of totients developed by Ford [1998a].

Central to our more anatomical considerations is the notion of a *normal prime*. Hardy and Ramanujan [1917] showed that almost all natural numbers $\leqslant x$ have $\sim \log\log x$ prime factors, and Erdős [1935] showed that the same holds for almost all shifted primes $p-1 \leqslant x$. Moreover, sieve methods imply that if we list the prime factors of $p-1$ on a double-logarithmic scale, then these are typically close to uniformly distributed in $[0, \log\log p]$. Of course, all of this remains true with $p+1$ in place of $p-1$. We assume that the numbers belonging to $\mathscr{A}_\phi$ and $\mathscr{A}_\sigma$ have all their prime factors among this set of normal primes.

If we assume that numbers n all of whose prime factors are normal generate "most" f-values (for $f \in \{\phi, \sigma\}$), we are led to a series of linear inequalities among the (double-logarithmically renormalized) prime factors of n. These inequalities are at the heart of the structure theory of totients as developed in [Ford 1998a]. As one illustration of the power of this approach, mapping the L largest prime factors of n (excluding the largest) to a point in $\mathbb{R}^L$, the problem of estimating $V_f(x)$ reduces to the problem of finding the volume of a certain region of $\mathbb{R}^L$, called the fundamental simplex. In broad strokes, this is how one establishes Ford's bound (1-1). We incorporate these linear inequalities into our definitions of $\mathscr{A}_\phi$ and $\mathscr{A}_\sigma$. One particular linear combination of renormalized prime factors appearing in the definition of the fundamental simplex is of particular interest to us (see condition (8) in the definition of $\mathscr{A}_f$ in Section 3 below); that we can assume this quantity is less than 1 is responsible for the success of our argument.

Suppose now that we have a solution to $\phi(a) = \sigma(a')$, where $(a, a') \in \mathscr{A}_\phi \times \mathscr{A}_\sigma$. We write $a = p_0 p_1 p_2 \cdots$ and $a' = q_0 q_1 q_2 \cdots$, where the sequences of p_i and q_j are nonincreasing. We cut the first of these lists in two places; at the k-th prime p_k and at the L-th prime p_L. The precise choice of k and L is somewhat technical; one should think of the primes p_i larger than p_k as the "large" prime divisors of a, those smaller than p_k but larger than p_L as "small", and those smaller than p_L as "tiny". The equation $\phi(a) = \sigma(a')$ can be rewritten in the form

$$(p_0-1)(p_1-1)(p_2-1)\cdots(p_{k-1}-1)fd \\ = (q_0+1)(q_1+1)(q_2+1)\cdots(q_{k-1}+1)e, \quad (1\text{-}4)$$

where

$$f := \phi(p_k \cdots p_{L-1}), \quad d := \phi(p_L p_{L+1} \cdots), \quad \text{and} \quad e := \sigma(q_k q_{k+1} \cdots). \quad (1\text{-}5)$$

To see that (1-4) correctly expresses the relation $\phi(a) = \sigma(a')$, we recall that the primes $p_1, \ldots, p_k$ are all large, so that by the "anatomical" constraints imposed in the definition of $\mathscr{A}_\phi$, each appears to the first power in the prime factorization of a. An analogous statement holds for the primes $q_1, \ldots, q_k$; this follows from the general principle, established below, that $p_i \approx q_i$ provided that either side is not too small. There is one respect in which (1-5) may not be quite right: Since p_L is tiny, we cannot assume a priori that $p_L \neq p_{L-1}$, and so it may be necessary to amend the definition of d somewhat; we ignore this (ultimately minor) difficulty for now.

To complete the argument, we fix d and estimate from above the number of solutions (consisting of $p_0, \ldots, p_{k-1}, q_0, \ldots, q_{k-1}, e, f$) to the relevant equations of the form (1-4); then we sum over d. The machinery facilitating these estimates is encoded in Lemma 4.1, which is proved by a delicate, iterative sieve argument of a kind first introduced in [Maier and Pomerance 1988] and developed in [Ford 1998a, §5]. The hypotheses of that lemma include several assumptions about the p_i and q_j, and about e, f, and d. All of these rather technical hypotheses are, in our situation, consequences of our definitions of $\mathscr{A}_\phi$ and $\mathscr{A}_\sigma$; we say more about some of them in a remark following Lemma 4.1.

Notation. Let $P^+(n)$ denote the largest prime factor of n, understood so that $P^+(1) = 1$, and let $\Omega(n, U, T)$ denote the total number of prime factors p of n such that $U < p \leqslant T$, counted according to multiplicity. Constants implied by the Landau O and the Vinogradov $\ll$ and $\gg$ symbols are absolute unless otherwise specified. Symbols in boldface type indicate vector quantities.

2. Preliminaries

2A. ***Anatomical tools.*** We begin with two tools from the standard chest. The first is a form of the upper bound sieve and the second concerns the distribution of smooth numbers.

Lemma 2.1 (see, e.g., [Halberstam and Richert 1974, Theorem 4.2]). *Suppose $A_1, \ldots, A_h$ are positive integers and $B_1, \ldots, B_h$ are integers such that*

$$E = \prod_{i=1}^{h} A_i \prod_{1 \leqslant i < j \leqslant h} (A_i B_j - A_j B_i) \neq 0.$$

Then

$$\#\{n \leqslant x : A_i n + B_i \text{ prime } (1 \leqslant i \leqslant h)\} \ll \frac{x}{(\log x)^h} \prod_{p \mid E} \frac{1 - \frac{\nu(p)}{p}}{(1 - \frac{1}{p})^h} \ll \frac{x (\log_2(|E| + 2))^h}{(\log x)^h},$$

where $\nu(p)$ is the number of solutions of the congruence $\prod (A_i n + B_i) \equiv 0 \pmod p$, and the implied constant may depend on h.

Let $\Psi(x, y)$ denote the number of $n \leqslant x$ for which $P^+(n) \leqslant y$. The following estimate is due to Canfield, Erdős, and Pomerance:

Lemma 2.2 [Canfield et al. 1983]. *Fix $\epsilon > 0$. If $2 \leqslant y \leqslant x$ and $u = \dfrac{\log x}{\log y}$, then*

$$\Psi(x, y) = x/u^{u+o(u)}$$

for $u \leqslant y^{1-\epsilon}$, as $u \to \infty$.

The next lemma supplies an estimate for how often $\Omega(n)$ is unusually large; this may be deduced from the theorems in Chapter 0 of [Hall and Tenenbaum 1988].

Lemma 2.3. *The number of integers $n \leqslant x$ for which $\Omega(n) \geqslant \alpha \log_2 x$ is*

$$\ll_\alpha \begin{cases} x(\log x)^{-Q(\alpha)} & \text{if } 1 < \alpha < 2, \\ x(\log x)^{1-\alpha \log 2} \log_2 x & \text{if } \alpha \geqslant 2, \end{cases}$$

where $Q(\lambda) = \int_1^\lambda \log t \, dt = \lambda \log(\lambda) - \lambda + 1$.

In the remainder of this section, we give a precise meaning to the term "normal prime" alluded to in the introduction and draw out some simple consequences. For $S \geqslant 2$, a prime p is said to be *S-normal* if the following two conditions hold for each $f \in \{\phi, \sigma\}$:

$$\Omega(f(p), 1, S) \leqslant 2 \log_2 S,$$

and, for every pair of real numbers (U, T) with $S \leqslant U < T \leqslant f(p)$, we have

$$\bigl|\Omega(f(p), U, T) - (\log_2 T - \log_2 U)\bigr| < \sqrt{\log_2 S \log_2 T}. \tag{2-1}$$

This definition is slightly weaker than the corresponding definition on [Ford 1998a, p. 13], and so the results from that paper remain valid in our context. As a straightforward consequence of the definition, if p is S-normal, $f \in \{\phi, \sigma\}$, and $f(p) \geqslant S$, then

$$\Omega(f(p)) \leqslant 3 \log_2 f(p). \tag{2-2}$$

The following lemma is a simple consequence of [Ford 1998a, Lemma 2.10] and (1-1):

Lemma 2.4. *For each $f \in \{\phi, \sigma\}$, the number of f-values less than or equal to x which have a preimage divisible by a prime that is not S-normal is*

$$\ll V_f(x)(\log_2 x)^5 (\log S)^{-1/6}.$$

We also record the observation that if p is S-normal, then $P^+(f(p))$ cannot be too much smaller than p, on a double-logarithmic scale.

Lemma 2.5. *If $5 \leqslant p \leqslant x$ is an S-normal prime and $f(p) \geqslant S$, then*

$$\frac{\log_2 P^+(f(p))}{\log_2 x} \geqslant \frac{\log_2 p}{\log_2 x} - \frac{\log_3 x + \log 4}{\log_2 x}.$$

Proof. We have

$$P^+(f(p)) \geqslant f(p)^{\frac{1}{\Omega(f(p))}} \geqslant f(p)^{\frac{1}{3\log_2 f(p)}} \geqslant p^{\frac{1}{4\log_2 x}}.$$

The result follows upon taking the double logarithm of both sides. □

2B. *Structural tools.* In this section, we describe more fully some components of the structure theory of totients alluded to in the introduction. Given a natural number n, write $n = p_0(n)p_1(n)p_2(n)\cdots$, where $p_0(n) \geqslant p_1(n) \geqslant p_2(n) \geqslant \cdots$ are the primes dividing n (with multiplicity). For a fixed x, we put

$$x_i(n; x) = \begin{cases} (\log_2 p_i(n))/(\log_2 x) & \text{if } i < \Omega(n) \text{ and } p_i(n) > 2, \\ 0 & \text{if } i \geqslant \Omega(n) \text{ or } p_i(n) = 2. \end{cases}$$

Suppose $L \geqslant 2$ is fixed and that $\xi_i \geqslant 0$ for $0 \leqslant i \leqslant L-1$. Recall the definition of the a_i from (1-2) and let $\mathscr{S}_L(\boldsymbol{\xi})$ be the set of $(x_1, \ldots, x_L) \in \mathbb{R}^L$ with $0 \leqslant x_L \leqslant x_{L-1} \leqslant \cdots \leqslant x_1 \leqslant 1$ and

$$\begin{aligned} &(I_0) && a_1x_1 + a_2x_2 + \cdots + a_Lx_L \leqslant \xi_0, \\ &(I_1) && a_1x_2 + a_2x_3 + \cdots + a_{L-1}x_L \leqslant \xi_1 x_1, \\ &\ \ \vdots && \qquad\qquad \vdots \\ &(I_{L-2}) && a_1x_{L-1} + a_2x_L \leqslant \xi_{L-2}x_{L-2}. \end{aligned}$$

Define $T_L(\boldsymbol{\xi})$ as the volume (L-dimensional Lebesgue measure) of $\mathscr{S}_L(\boldsymbol{\xi})$. For convenience, let $\mathbf{1} = (1, 1, \ldots, 1)$, $\mathscr{S}_L = \mathscr{S}_L(\mathbf{1})$ (the "fundamental simplex"), and let T_L be the volume of $\mathscr{S}_L$. Let

$$L_0(x) := \lfloor 2C(\log_3 x - \log_4 x) \rfloor,$$

where C is defined as in the introduction. The next lemma allows us to locate the preimages of almost all f-values within suitable sets of the form $\mathscr{S}_L(\boldsymbol{\xi})$.

Lemma 2.6 [Ford 1998a, Theorem 15]. *Write $L_0 = L_0(x)$. Suppose $0 \leqslant \Psi < L_0$, $L = L_0 - \Psi$, and let*

$$\xi_i = \xi_i(x) = 1 + \frac{1}{10(L_0 - i)^3} \qquad (0 \leqslant i \leqslant L-2).$$

The number of f-values $v \leqslant x$ with a preimage n for which

$$(x_1(n; x), \ldots, x_L(n; x)) \notin \mathscr{S}_L(\boldsymbol{\xi}) \quad \textit{is} \quad \ll V_f(x)\exp(-\Psi^2/4C).$$

For future use, we collect here some further structural lemmas from [Ford 1998a]. The next result, which follows immediately from our (1-1) and Lemma 4.2 of that article, concerns the size of sums of the shape appearing in the definition of inequality (I_0) above.

Lemma 2.7. *Suppose that $L \geqslant 2$, $0 < \omega < \frac{1}{10}$, and x is sufficiently large. The number of f-values $v \leqslant x$ with a preimage satisfying*

$$a_1 x_1(n; x) + \cdots + a_L x_L(n; x) \geqslant 1 + \omega$$

is $\ll V_f(x)(\log_2 x)^5 (\log x)^{-\omega^2/(150 L^3 \log L)}$.

We will make heavy use of the following (purely geometric) statement about the simplices $\mathscr{S}_L(\boldsymbol{\xi})$, which appears as [Ford 1998a, Lemma 3.10]. Recall from (1-3) that $\varrho = 0.542598\ldots$ denotes the unique real number with $\sum_{n=1}^{\infty} a_n \varrho^n = 1$.

Lemma 2.8. *If $\mathbf{x} \in \mathscr{S}_L(\boldsymbol{\xi})$ and $\xi_0^L \xi_1^{L-1} \cdots \xi_{L-2}^2 \leqslant 1.1$, then $x_j \leqslant 3\varrho^{j-i} x_i$ when $i < j$, and $x_j < 3\varrho^j$ for $1 \leqslant j \leqslant L$.*

Define $\mathscr{R}_L(\boldsymbol{\xi}; x)$ as the set of integers n with $\Omega(n) \leqslant L$ and

$$(x_0(n; x), x_1(n; x), \ldots, x_{L-1}(n; x)) \in \mathscr{S}_L(\boldsymbol{\xi}).$$

For $f \in \{\phi, \sigma\}$, put

$$R_L^{(f)}(\boldsymbol{\xi}; x) = \sum_{n \in \mathscr{R}_L(\boldsymbol{\xi}, x)} \frac{1}{f(n)}.$$

The next lemma, extracted from [Ford 1998a, Lemma 3.12], relates the magnitude of $R_L^{(f)}(\boldsymbol{\xi}; x)$ to the volume of the fundamental simplex T_L, whenever $\boldsymbol{\xi}$ is suitably close to $\mathbf{1}$. In that article, it plays a crucial role in the proof of the upper-bound aspect of (1-1).

Lemma 2.9. *If $1/(1000k^3) \leqslant \omega_{L_0-k} \leqslant 1/(10k^3)$ for $1 \leqslant k \leqslant L_0$, $\xi_i = 1 + \omega_i$ for each i, and $L \leqslant L_0$, then*

$$R_L^{(f)}(\boldsymbol{\xi}; x) \ll (\log_2 x)^L T_L$$

for both $f = \phi$ and $f = \sigma$.

While only the case $f = \phi$ of Lemma 2.9 appears in the statement of [ibid., Lemma 3.12], the $f = \sigma$ case follows trivially, since $\sigma(n) \geqslant \phi(n)$. In order to apply Lemma 2.9, we need estimates for the volume T_L; this is handled by the next lemma, extracted from [ibid., Corollary 3.4].

Lemma 2.10. *Assume $1 \leqslant \xi_i \leqslant 1.1$ for $0 \leqslant i \leqslant L-2$ and that $\xi_0^L \xi_1^{L-1} \cdots \xi_{L-2}^2 \asymp 1$. If $L = L_0 - \Psi > 0$, then*

$$(\log_2 x)^L T_L(\boldsymbol{\xi}) \ll Y(x) \exp(-\Psi^2/4C).$$

Here

$$Y(x) := \exp\bigl(C(\log_3 x - \log_4 x)^2 + D \log_3 x - (D + \tfrac{1}{2} - 2C) \log_4 x\bigr). \tag{2-3}$$

We conclude this section with the following technical lemma, which will be needed when we select the sets $\mathscr{A}_\phi$ and $\mathscr{A}_\sigma$ in Section 3.

Lemma 2.11. *For $f \in \{\phi, \sigma\}$ and $y \geqslant 20$,*

$$\sum_{\substack{v \in \mathscr{V}_f \\ P^+(v) \leqslant y}} \frac{1}{v} \ll \frac{\log_2 y}{\log_3 y} Y(y), \tag{2-4}$$

where Y is as defined in (2-3). Moreover, for any $b > 0$,

$$Y(\exp((\log x)^b)) \ll_b Y(x) \left(\frac{\log_3 x}{\log_2 x}\right)^{-2C \log b}. \tag{2-5}$$

Proof. We split the left-hand sum in (2-4) according to whether or not $v \leqslant y^{\log_2 y}$. The contribution of the large v is $O(1)$ and so is negligible: Indeed, for $t > y^{\log_2 y}$, we have $\log t/\log y > \log_2 y$. Thus, by Lemma 2.2, we have $\Psi(t, y) \ll t/(\log t)^2$ (say), and the $O(1)$ bound follows by partial summation. We estimate the sum over small v by ignoring the smoothness condition. Put $X = y^{\log_2 y}$. Since $V_f(t) \asymp (t/\log t) Y(t)$, partial summation gives that

$$\sum_{\substack{v \in \mathscr{V}_f \\ v \leqslant X}} \frac{1}{v} \ll 1 + \int_3^X \frac{Y(t)}{t \log t}\, dt = (1 + o(1)) Y(X) \frac{\log_2 X}{\log_3 X},$$

as $y \to \infty$. (The last equality follows, for instance, from L'Hôpital's rule.) Since $\log_2 X/\log_3 X \sim \log_2 y/\log_3 y$ and $Y(X) \sim Y(y)$, we have (2-4). Estimate (2-5) follows from the definition of Y and a direct computation; here it is helpful to note that if we redefine $X := \exp((\log x)^b)$, then $\log_3 X = \log_3 x + \log b$ and $\log_4 X = \log_4 x + O_b(1/\log_3 x)$. □

3. Definition of the sets $\mathscr{A}_\phi$ and $\mathscr{A}_\sigma$

We continue fleshing out the introductory sketch, giving precise definitions to the preimage sets $\mathscr{A}_\phi$ and $\mathscr{A}_\sigma$. Put

$$L := \left\lfloor L_0(x) - 2\sqrt{\log_3 x} \right\rfloor, \quad \xi_i := 1 + \frac{1}{10(L_0 - i)^3} \quad (1 \leqslant i \leqslant L). \tag{3-1}$$

The next lemma is a final technical preliminary.

Lemma 3.1. *Let $f \in \{\phi, \sigma\}$. The number of f-values $v \leqslant x$ with a preimage n for which*

(i) $(x_1(n; x), \ldots, x_L(n; x)) \in \mathscr{S}_L(\boldsymbol{\xi})$ *and*

(ii) *n has fewer than $L + 1$ odd prime divisors (counted with multiplicity)*

is $\ll V_f(x)/\log_2 x$.

Proof. We treat the case when $f=\phi$; the case when $f=\sigma$ requires only small modifications. We can assume that $x/\log x \leqslant n \leqslant 2x\log_2 x$, where the last inequality follows from known results on the minimal order of the Euler function. By Lemma 2.3, we can also assume that $\Omega(n) \leqslant 10\log_2 x$. Put $p_i := p_i(n)$, as defined in Section 2B. Since $(x_1(n;x),\dots,x_L(n;x)) \in \mathscr{S}_L(\boldsymbol{\xi})$ by hypothesis, Lemma 2.8 gives that $x_2 < 3\varrho^2 < 0.9$, and so $p_2 \leqslant \exp((\log x)^{0.9})$. Thus,

$$n/(p_0p_1) = p_2p_3\cdots \leqslant \exp(10(\log_2 x)(\log x)^{0.9}) = x^{o(1)},$$

and so $p_0 \geqslant x^{2/5}$ (say) for large x. In particular, we can assume that $p_0^2 \nmid n$.

Suppose now that n has exactly L_0-k+1 odd prime factors, where we fix $k > L_0 - L$. Then

$$v = (p_0-1)\phi(p_1p_2\cdots p_{L_0-k})2^s$$

for some integer $s \geqslant 0$. Using the prime number theorem to estimate the number of choices for p_0 given $p_1\cdots p_{L_0-k}$ and 2^s, we obtain that the number of v of this form is

$$\ll \frac{x}{\log x}\sum_{p_1\cdots p_{L_0-k}} \frac{1}{\phi(p_1\cdots p_{L_0-k})}\sum_{s\geqslant 0}\frac{1}{2^s}.$$

(We use here that $x/(\phi(p_1\cdots p_{L_0-k})2^s) \gg p_0 \geqslant x^{2/5}$.) The sum over s is $\ll 1$. To handle the remaining sum, we observe that $p_1\cdots p_{L_0-k}$ belongs to the set $\mathscr{R}_{L_0-k}(\boldsymbol{\xi}_k, x)$, where $\boldsymbol{\xi}_k := (\xi_0,\dots,\xi_{L_0-k-2})$. Thus, the remaining sum is bounded by

$$R^{(\phi)}_{L_0-k}(\boldsymbol{\xi}_k; x) = \sum_{m\in\mathscr{R}_{L_0-k}(\boldsymbol{\xi}_k,x)} \frac{1}{\phi(m)}.$$

So by Lemmas 2.9 and 2.10, both of whose hypotheses are straightforward to verify,

$$R^{(\phi)}_{L_0-k}(\boldsymbol{\xi}_k;x) \ll (\log_2 x)^{L_0-k}T_{L_0-k} \leqslant (\log_2 x)^{L_0-k}T_{L_0-k}(\boldsymbol{\xi}_k) \ll Y(x)\exp(-k^2/4C).$$

Collecting our estimates, we obtain a bound of

$$\ll \frac{x}{\log x}Y(x)\exp(-k^2/4C) \ll V_\phi(x)\exp(-k^2/4C).$$

Now since $L_0 - L > 2\sqrt{\log_3 x}$, summing over $k > L_0 - L$ gives a final bound which is

$$\ll V_\phi(x)\exp(-(\log_3 x)/C) \ll V_\phi(x)/\log_2 x,$$

as desired. □

For the rest of this paper, we fix $\epsilon > 0$ and assume that $x \geqslant x_0(\epsilon)$. Put

$$S := \exp((\log_2 x)^{36}), \qquad \delta := \sqrt{\frac{\log_2 S}{\log_2 x}}, \qquad \omega := (\log_2 x)^{-1/2+\epsilon/2}. \tag{3-2}$$

For $f \in \{\phi, \sigma\}$, let $\mathscr{A}_f$ be the set of $n = p_0(n)p_1(n)\cdots$ satisfying $f(n) \leqslant x$ and

(0) $n \geqslant x/\log x$,

(1) every squarefull divisor m of n or $f(n)$ satisfies $m \leqslant \log^2 x$,

(2) all of the primes $p_j(n)$ are S-normal,

(3) $\Omega(f(n)) \leqslant 10\log_2 x$ and $\Omega(n) \leqslant 10\log_2 x$,

(4) if $d \parallel n$ and $d \geqslant \exp((\log_2 x)^{1/2})$, then $\Omega(f(d)) \leqslant 10\log_2 f(d)$,

(5) $(x_1(n; x), \ldots, x_L(n; x)) \in \mathscr{S}_L(\boldsymbol{\xi})$,

(6) n has at least $L+1$ odd prime divisors,

(7) $P^+(f(p_0)) \geqslant x^{1/(\log_2 x)}$, $p_1(n) < x^{1/(100\log_2 x)}$,

(8) $a_1x_1(n; x) + \cdots + a_Lx_L(n; x) \leqslant 1 - \omega$.

The following lemma asserts that a generic f-value has all of its preimage in $\mathscr{A}_f$.

Lemma 3.2. *For each $f \in \{\phi, \sigma\}$, the number of f-values $\leqslant x$ with a preimage $n \notin \mathscr{A}_f$ is*

$$\ll V_f(x)(\log_2 x)^{-1/2+\epsilon}.$$

Remarks. (i) The $\mathscr{A}_f$ not only satisfy Lemma 3.2 but do so economically. In fact, from condition (5) and the work of [Ford 1998a, §4], we have that $\#\mathscr{A}_f \ll V_f(x)$. Thus, on average, an element of $\mathscr{V}_f(x)$ has only a bounded number of preimages from $\mathscr{A}_f$. So when we turn in Sections 4 and 5 to counting ϕ-values arising from solutions to $\phi(a) = \sigma(a')$, with $(a, a') \in \mathscr{A}_\phi \times \mathscr{A}_\sigma$, we expect not to be (excessively) overcounting.

(ii) Of the nine conditions defining $\mathscr{A}_f$, conditions (0)–(4) are, in the nomenclature of the introduction, purely *anatomical*, while conditions (5)–(8) depend to some degree on the fine structure theory of [Ford 1998a]. Condition (8) is particularly critical. It is (8) which ensures that the sieve bounds developed in Section 4 result in a nontrivial estimate for $V_{\phi,\sigma}(x)$. Our inability to replace the exponent $\frac{1}{2}$ on $\log_2 x$ in Lemma 3.2 (or in Theorem 1.1) by a larger number is also rooted in (8).

Proof. It is clear that the number of values of $f(n)$ corresponding to n failing (0) or (1) is $\ll x\log_2 x/\log x$, which (recalling (1-1)) is permissible for us. By Lemma 2.4 and our choice of S, the number of values of $f(n)$ coming from n failing (2) is $\ll V_f(x)/\log_2 x$. The same holds for values coming from n failing (3), by Lemma 2.3.

Suppose now that n fails condition (4). Then n has a unitary divisor $d \geqslant \exp((\log_2 x)^{1/2})$ with $\Omega(f(d)) \geqslant 10\log_2 f(d)$. Put $w := f(d)$. Then $w \mid f(n)$, and

$f(n) \ll x\log_2 x$. So if $w \geqslant x^{1/2}$, then the number of possibilities for $f(n)$ is

$$\ll x\log_2 x \sum_{\substack{w\geqslant x^{1/2}\\ \Omega(w)\geqslant 10\log_2 w}} \frac{1}{w} \ll \frac{x\log_2 x}{\log x},$$

using Lemma 2.3 to estimate sum over w. If $w \leqslant x^{1/2}$, we observe that $f(n)/w = f(n/d) \in \mathscr{V}_f$; hence, with $Y(x)$ defined as in (2-3), the number of corresponding values of $f(n)$ is

$$\ll \sum_{\substack{\exp((\log_2 x)^{1/3})\leqslant w\leqslant x^{1/2}\\ \Omega(w)\geqslant 10\log_2 w}} V_f(x/w) \ll \frac{x}{\log x}\, Y(x) \sum_{\substack{w\geqslant \exp((\log_2 x)^{1/3})\\ \Omega(w)\geqslant 10\log_2 w}} \frac{1}{w} \ll \frac{V_f(x)}{\log_2 x}.$$

By Lemma 2.6, the number of f-values with a preimage failing (5) is

$$\ll \frac{V_f(x)}{\log_2 x}.$$

According to Lemma 3.1, the number of f-values with a preimage satisfying (5) but not (6) is also $\ll V_f(x)/\log_2 x$.

Suppose now that n satisfies (0)–(6). In what follows, we write $x_i = x_i(n; x)$. From (5), we have $\xi_0 \geqslant a_1x_1 + a_2x_2 \geqslant (a_1 + a_2)x_2$, and so $x_2 \leqslant 0.8$. So from (3),

$$\frac{n}{p_0(n)p_1(n)} = p_2(n)p_3(n)\cdots \leqslant \exp(10(\log_2 x)(\log x)^{0.8}) < x^{1/100}. \tag{3-3}$$

In particular, $p_0 > x^{1/3} + 1$ and $f(p_0) > x^{1/3}$, so that $v := f(p_1p_2\cdots) \leqslant x^{2/3}$. The prime p_0 satisfies $f(p_0) \leqslant x/v$. For z with $x^{1/3} < z \leqslant x$, the number of primes p_0 with $f(p_0) \leqslant z$ and $P^+(f(p_0)) \leqslant x^{1/\log_2 x}$ is (crudely) bounded by $\Psi(z, x^{1/\log_2 x}) \ll z/(\log x)^2$, by Lemma 2.2. So the number of values of $f(n)$ coming from n with $P^+(f(p_0)) \leqslant x^{1/\log_2 x}$ is

$$\ll \sum_{\substack{v\leqslant x^{2/3}\\ v\in\mathscr{V}_f}} \sum_{\substack{p:\ f(p)\leqslant x/v\\ P^+(f(p))\leqslant x^{1/\log_2 x}}} 1 \ll \frac{x}{(\log x)^2} \sum_{\substack{v\leqslant x^{2/3}\\ v\in\mathscr{V}_f}} \frac{1}{v} \ll \frac{x}{(\log x)^{2-\epsilon}}.$$

To handle the second condition in (7), observe that since $f(p_0) \leqslant x/v$, the prime number theorem (and the bound $v \leqslant x^{2/3}$) shows that given v, the number of possibilities for p_0 is $\ll x/(v\log x)$. Suppose that $p_1(n) > x^{1/(100\log_2 x)}$. Then $x_1 = x_1(n; x) \geqslant 0.999$, and we conclude from $\sum_{i\geqslant 1} a_ix_i \leqslant \xi_0$ that either $x_2 \leqslant \varrho^{3/2}$ or $x_3 \leqslant \varrho^{5/2}$. Writing v_2 for $f(p_2p_3\cdots)$ and v_3 for $f(p_3p_4\cdots)$, we see that the number of such f-values is

$$\ll \frac{x}{\log x}\sum_{p_1}\frac{1}{p_1}\Bigg(\sum_{\substack{P^+(v_2)\\ \leqslant \exp((\log x)^{\varrho^{3/2}})}}\frac{1}{v_2}+\sum_{p_2}\frac{1}{p_2}\sum_{\substack{P^+(v_3)\\ \leqslant \exp((\log x)^{\varrho^{5/2}})}}\frac{1}{v_3}\Bigg)$$

$$\ll \frac{x}{\log x}\log_3 x\bigg(Y(x)\Big(\frac{\log_3 x}{\log_2 x}\Big)^{1/2}+(\log_2 x)Y(x)\Big(\frac{\log_3 x}{\log_2 x}\Big)^{3/2}\bigg)\ll \frac{V_f(x)}{(\log_2 x)^{1/2-\epsilon}},$$

using Lemma 2.11 to estimate the sums over v_2 and v_3.

Finally, we consider n for which (0)–(7) hold but where condition (8) fails. By Lemma 2.7, we can assume that $a_1x_1+\cdots+a_Lx_L<1+\omega$, since the number of exceptional f-values is

$$\ll V(x)\exp(-(\log_2 x)^{\epsilon/2})\ll \frac{V(x)}{\log_2 x}.$$

Thus,

$$1-\omega<a_1x_1+\cdots+a_Lx_L<1+\omega, \tag{3-4}$$

while by condition (I_1) in the definition of $\mathscr{S}_L(\boldsymbol{\xi})$, $a_1x_2+\cdots+a_{L-1}x_L\leqslant \xi_1x_1$. We claim that if J is fixed large enough depending on ϵ, then there is some $2\leqslant j\leqslant J$ with $x_j\leqslant \varrho^{j-\epsilon/3}$. If not, then for large enough J,

$$\xi_1x_1\geqslant\sum_{j=1}^{J-1}a_jx_{j+1}\geqslant\varrho^{1-\epsilon/3}(a_1\varrho+a_2\varrho^2+\cdots+a_J\varrho^J)>\varrho^{1-\epsilon/4}.$$

Thus, $x_1\geqslant\varrho^{1-\epsilon/4}\xi_1^{-1}\geqslant\varrho^{1-\epsilon/5}$, and so $\xi_0\geqslant\varrho^{-\epsilon/5}(a_1\varrho+a_2\varrho^2+\cdots+a_J\varrho^J)\geqslant\varrho^{-\epsilon/6}$, which is false. This proves the claim. We assume below that $j\in[2,J]$ is chosen as the smallest index with $x_j\leqslant\varrho^{j-\epsilon/3}$; by condition (1), this implies that all of $p_1,\dots,p_{j-1}$ appear to the first power in the prime factorization of n.

Now given $x_2,\dots,x_L$, we have from (3-4) that $x_1\in[\alpha,\alpha+2\omega]$ for a certain α. Thus,

$$\sum_{p_1}\frac{1}{p_1}\ll\omega\log_2 x=(\log_2 x)^{1/2+\epsilon/2}.$$

So the number of f-values that arise from n satisfying (0)–(7) but failing (8) is

$$\ll\frac{x}{\log x}\sum_{j=2}^{J}\sum_{p_1,\dots,p_{j-1}}\frac{1}{p_1\cdots p_{j-1}}\sum_{\substack{P^+(v)\leqslant\exp((\log x)^{\varrho^{j-\epsilon/3}})\\ v\in\mathscr{V}_f}}\frac{1}{v}$$

$$\ll\frac{x}{\log x}\sum_{j=2}^{J}(\log_2 x)^{j-3/2+\epsilon/2}Y(x)\Big(\frac{\log_3 x}{\log_2 x}\Big)^{-1+j-\epsilon/3}\ll V_f(x)(\log_2 x)^{-1/2+\epsilon}.$$

This completes the proof of Lemma 3.2. $\square$

As a corollary of Lemma 3.2, we have that $V_{\phi,\sigma}(x)$ is bounded, up to an additive error of $\ll (V_\phi(x)+V_\sigma(x))/(\log_2 x)^{1/2-\epsilon}$, by the number of values $\phi(a)$ that appear in solutions to the equation

$$\phi(a)=\sigma(a'), \quad \text{where} \quad (a,a')\in\mathscr{A}_\phi\times\mathscr{A}_\sigma.$$

In Sections 4 and 5, we develop the machinery required to estimate the number of such values. Ultimately, we find that it is smaller than $(V_\phi(x)+V_\sigma(x))/(\log_2 x)^A$ for any fixed A, which immediately gives Theorem 1.1.

4. The fundamental sieve estimate

Lemma 4.1. *Let y be large, $k\geqslant 1$, $l\geqslant 0$, $30\leqslant S\leqslant v_k\leqslant v_{k-1}\leqslant\cdots\leqslant v_0=y$, and $u_j\leqslant v_j$ for $0\leqslant j\leqslant k-1$. Put $\delta=\sqrt{\log_2 S/\log_2 y}$, $\nu_j=\log_2 v_j/\log_2 y$, $\mu_j=\log_2 u_j/\log_2 y$. Suppose that d is a natural number for which $P^+(d)\leqslant v_k$. Moreover, suppose that both of the following hold*:

(a) *For $2\leqslant j\leqslant k-1$, either $(\mu_j,\nu_j)=(\mu_{j-1},\nu_{j-1})$ or $\nu_j\leqslant\mu_{j-1}-2\delta$. Also, $\nu_k\leqslant\mu_{k-1}-2\delta$.*

(b) *For $1\leqslant j\leqslant k-2$, we have $\nu_j>\nu_{j+2}$.*

The number of solutions of

$$(p_0-1)\cdots(p_{k-1}-1)fd=(q_0+1)\cdots(q_{k-1}+1)e\leqslant y \tag{4-1}$$

in $p_0,\dots,p_{k-1},q_0,\dots,q_{k-1},e,f$ satisfying

(i) *p_i and q_i are S-normal primes,*

(ii) *$u_i\leqslant P^+(p_i-1)$, $P^+(q_i+1)\leqslant v_i$ for $0\leqslant i\leqslant k-1$,*

(iii) *neither $\phi(\prod_{i=0}^{k-1}p_i)$ nor $\sigma(\prod_{i=0}^{k-1}q_i)$ is divisible by r^2 for a prime $r\geqslant v_k$,*

(iv) *$P^+(ef)\leqslant v_k$; $\Omega(f)\leqslant 4l\log_2 v_k$,*

(v) *p_0-1 has a divisor $\geqslant y^{1/2}$ which is composed of primes $>v_1$*

is

$$\ll\frac{y}{d}(c\log_2 y)^{6k}(k+1)^{\Omega(d)}(\log v_k)^{8(k+l)\log(k+1)+1}(\log y)^{-2+\sum_{i=1}^{k-1}a_i\nu_i+E},$$

where $E=\delta\sum_{i=2}^{k}(i\log i+i)+2\sum_{i=1}^{k-1}(\nu_i-\mu_i)$. Here c is an absolute positive constant.

Remarks. Since the lemma statement is very complicated, it may be helpful to elaborate on how it will be applied in Section 5 below. Given $(a,a')\in\mathscr{A}_\phi\times\mathscr{A}_\sigma$ satisfying $\phi(a)=\sigma(a')$, rewrite the corresponding equation in the form (1-4), with d, e, and f as in (1-5). (Here L is as in (3-1), and k, given more precisely in the next section, satisfies $k\approx L/2$.) We are concerned with counting the number of values $\phi(a)$ which arise from such solutions. We partition the solutions according to the value of d, which describes the contribution of the "tiny" primes to $\phi(a)$, and by the rough location of the primes p_i and q_i, which we encode in the selection of intervals $[u_i, v_i]$ (cf. Lemma 2.5). Finally, we apply Lemma 4.1 and sum over both d and the possible selections of intervals; this gives an estimate for the number of $\phi(a)$ which is smaller than $(V_\phi(x)+V_\sigma(x))/(\log_2 x)^A$, for any fixed A.

In our application, conditions (i)–(v) of Lemma 4.1 are either immediate from the definitions, or are readily deduced from the defining properties of $\mathscr{A}_\phi$ and $\mathscr{A}_\sigma$. Conditions (a) and (b) are rooted in the observation that while neighboring primes in the prime factorization of a (or a') may be close together (requiring us to allow $[u_{i+1}, v_{i+1}] = [u_i, v_i]$), the primes $p_i(a)$ and $p_{i+2}(a)$ are forced to be far apart on a double-logarithmic scale. Indeed, since $(x_1(a;x), \dots, x_L(a;x)) \in \mathscr{S}_L(\boldsymbol{\xi})$, Lemma 2.8 shows that $x_{i+2} < 3\varrho^2 x_i < 0.9x_i$.

Proof. We consider separately the prime factors of each shifted prime lying in each interval $(v_{i+1}, v_i]$. For $0 \leqslant j \leqslant k-1$ and $0 \leqslant i \leqslant k$, let

$$s_{i,j}(n) = \prod_{\substack{p^a \| (p_j-1) \\ p \leqslant v_i}} p^a, \qquad s'_{i,j}(n) = \prod_{\substack{p^a \| (q_j+1) \\ p \leqslant v_i}} p^a, \qquad s_i = df \prod_{j=0}^{k-1} s_{i,j} = e \prod_{j=0}^{k-1} s'_{i,j}.$$

Also, for $0 \leqslant j \leqslant k-1$ and $1 \leqslant i \leqslant k$, let

$$t_{i,j} = \frac{s_{i-1,j}}{s_{i,j}}, \qquad t'_{i,j} = \frac{s'_{i-1,j}}{s'_{i,j}}, \qquad t_i = \prod_{j=0}^{k-1} t_{i,j} = \prod_{j=0}^{k-1} t'_{i,j}.$$

For each solution $\mathscr{A} = (p_0, \dots, p_{k-1}, f, q_0, \dots, q_{k-1}, e)$ of (4-1), let

$$\sigma_i(\mathscr{A}) = \{s_i; s_{i,0}, \dots, s_{i,k-1}, f; s'_{i,0}, \dots, s'_{i,k-1}, e\},$$
$$\tau_i(\mathscr{A}) = \{t_i; t_{i,0}, \dots, t_{i,k-1}, 1; t'_{i,0}, \dots, t'_{i,k-1}, 1\}.$$

Defining multiplication of $(2k+l+2)$-tuples component-wise, we have

$$\sigma_{i-1}(\mathscr{A}) = \sigma_i(\mathscr{A})\tau_i(\mathscr{A}). \tag{4-2}$$

Let $\mathfrak{S}_i$ denote the set of $\sigma_i(\mathscr{A})$ arising from solutions $\mathscr{A}$ of (4-1) and $\mathfrak{T}_i$ the corresponding set of $\tau_i(\mathscr{A})$. By (4-2), the number of solutions of (4-1) satisfying the required conditions is

$$|\mathfrak{S}_0| = \sum_{\sigma_1 \in \mathfrak{S}_1} \sum_{\substack{\tau_1 \in \mathfrak{T}_1 \\ \sigma_1\tau_1 \in \mathfrak{S}_0}} 1. \tag{4-3}$$

First, fix $\sigma_1 \in \mathfrak{S}_1$. By assumption (v) in the lemma, $t_{1,0} \geqslant y^{1/2}$. Also,

$$t_1 = t_{1,0} = t'_{1,0} \leqslant y/s_1,$$

t_1 is composed of primes $> v_1$, and $s_{1,0}t_1+1$ and $s'_{1,0}t_1-1$ are prime. Write $t_1=t'_1 Q$, where $Q=P^+(t_1)$. Since p_0 is an S-normal prime, (2-2) gives that

$$Q \geqslant t_1^{1/\Omega(t_1)} \geqslant t_1^{1/\Omega(p_0-1)} \geqslant y^{1/(2\Omega(p_0-1))} \geqslant y^{1/(6\log_2 y)},$$

Given t'_1, Lemma 2.1 implies that the number of Q is $O(y(\log_2 y)^6/(s_1 t'_1 \log^3 y))$. Moreover,

$$\sum \frac{1}{t'_1} \leqslant \prod_{v_1<p\leqslant y}\Big(1+\frac{1}{p}+\frac{1}{p^2}+\cdots\Big) \ll \frac{\log y}{\log v_1} = (\log y)^{1-\nu_1}.$$

Consequently, for each $\sigma_1\in\mathfrak{S}_1$,

$$\sum_{\substack{\tau_1\in\mathfrak{T}_1\\ \sigma_1\tau_1\in\mathfrak{S}_0}} 1 \ll \frac{y(\log_2 y)^6}{s_1(\log y)^{2+\nu_1}}. \tag{4-4}$$

Next, suppose $2\leqslant i\leqslant k$. We now apply an iterative procedure: If $v_i<v_{i-1}$, we use the identity

$$\sum_{\sigma_{i-1}\in\mathfrak{S}_{i-1}} \frac{1}{s_{i-1}} = \sum_{\sigma_i\in\mathfrak{S}_i}\frac{1}{s_i}\sum_{\substack{\tau_i\in\mathfrak{T}_i\\ \sigma_i\tau_i\in\mathfrak{S}_{i-1}}}\frac{1}{t_i}. \tag{4-5}$$

If $v_i=v_{i-1}$, then (4-5) remains true but contains no information, and in this case we use the alternative identity

$$\sum_{\sigma_{i-1}\in\mathfrak{S}_{i-1}} \frac{1}{s_{i-1}} = \sum_{\sigma_{i+1}\in\mathfrak{S}_{i+1}}\frac{1}{s_{i+1}}\sum_{\substack{\tau_{i+1}\in\mathfrak{T}_{i+1}\\ \sigma_{i+1}\tau_{i+1}\in\mathfrak{S}_{i-1}}}\frac{1}{t_{i+1}}. \tag{4-6}$$

We consider first the simpler case when $v_i<v_{i-1}$. Suppose $\sigma_i\in\mathfrak{S}_i$, $\tau_i\in\mathfrak{T}_i$ and $\sigma_i\tau_i\in\mathfrak{S}_{i-1}$. By assumption (ii), $t_i=t_{i,0}\cdots t_{i,i-1}=t'_{i,0}\cdots t'_{i,i-1}$. In addition, $s_{i,i-1}t_{i,i-1}+1=p_{i-1}$ and $s'_{i,i-1}t'_{i,i-1}-1=q_{i-1}$ are prime. Let $Q:=P^+(t_{i,i-1})$, $Q':=P^+(t'_{i,i-1})$, $b:=t_{i,i-1}/Q$ and $b':=t'_{i,i-1}/Q'$.

We consider separately $\mathfrak{T}_{i,1}$, the set of τ_i with $Q=Q'$ and $\mathfrak{T}_{i,2}$, the set of τ_i with $Q\neq Q'$. First,

$$\Sigma_1 := \sum_{\substack{\tau_i\in\mathfrak{T}_{i,1}\\ \sigma_i\tau_i\in\mathfrak{S}_{i-1}}}\frac{1}{t_i} \leqslant \sum_t \frac{h(t)}{t}\max_{b,b'}\sum_Q\frac{1}{Q},$$

where $h(t)$ denotes the number of solutions of $t_{i,0}\cdots t_{i,i-2}b=t=t'_{i,0}\cdots t'_{i,i-2}b'$, and in the sum on Q, $s_{i,i-1}bQ+1$ and $s'_{i,i-1}b'Q-1$ are prime. By Lemma 2.1, the number of $Q\leqslant z$ is $\ll z(\log z)^{-3}(\log_2 y)^3$ uniformly in b,b'. By partial summation,

$$\sum_{Q\geqslant u_{i-1}}\frac{1}{Q} \ll (\log_2 y)^3(\log y)^{-2\mu_{i-1}}.$$

Also, $h(t)$ is at most the number of dual factorizations of t into i factors each, that is, $h(t) \leqslant i^{2\Omega(t)}$. By (2-1), $\Omega(t) \leqslant i(\nu_{i-1}-\nu_i+\delta)\log_2 y =: I$. Also, by assumption (iii), t is squarefree. Thus,

$$\sum_t \frac{h(t)}{t} \leqslant \sum_{j\leqslant I} \frac{i^{2j}H^j}{j!},$$

where

$$\sum_{\nu_i<p\leqslant\nu_{i-1}} \frac{1}{p} \leqslant (\nu_{i-1}-\nu_i)\log_2 y + 1 =: H.$$

By assumption (a), $\nu_{i-1}-\nu_i \geqslant 2\delta$, hence $I \leqslant \frac{3}{2}iH \leqslant \frac{3}{4}i^2H$. Hence,

$$\sum_t \frac{h(t)}{t} \leqslant \left(\frac{i^2H}{I}\right)^I \sum_{j\leqslant I}\frac{I^j}{j!} < i^I \exp(I) = (\log y)^{(i+i\log i)(\nu_{i-1}-\nu_i+\delta)}. \tag{4-7}$$

This gives

$$\Sigma_1 \ll (\log_2 y)^3 (\log y)^{-2\mu_{i-1}+(i+i\log i)(\nu_{i-1}-\nu_i+\delta)}.$$

For the sum over $\mathfrak{T}_{i,2}$, set $t_i = tQQ'$. Note that $tQ' = t_{i,0}\cdots t_{i,i-2}b$ and $tQ = t'_{i,0}\cdots t'_{i,i-2}b'$, so $Q \mid t'_{i,0}\cdots t'_{i,i-2}b'$ and $Q' \mid t_{i,0}\cdots t_{i,i-2}b$. If we fix the factors divisible by Q and by Q', then the number of possible ways to form t is $\leqslant i^{2\Omega(t)}$ as before. Then

$$\Sigma_2 := \sum_{\substack{\tau_i\in\mathfrak{T}_{i,2}\\ \sigma_i\tau_i\in\mathfrak{S}_{i-1}}} \frac{1}{t_i} \leqslant \sum_t \frac{i^{2\Omega(t)+2}}{t} \max_{b,b'} \sum_{Q,Q'} \frac{1}{QQ'},$$

where $s_{i,i-1}bQ+1$ and $s'_{i,i-1}b'Q'-1$ are prime. By Lemma 2.1, the number of $Q \leqslant z$ (respectively $Q' \leqslant z$) is $\ll z(\log z)^{-2}(\log_2 y)^2$. Hence,

$$\sum_{Q,Q'} \frac{1}{QQ'} \ll (\log_2 y)^4 (\log y)^{-2\mu_{i-1}}.$$

Combined with (4-7), this gives $\Sigma_2 \ll i^2(\log_2 y)^4(\log y)^{-2\mu_{i-1}+(i+i\log i)(\nu_{i-1}-\nu_i+\delta)}$. From (a) and (b), $i^2 \leqslant k^2 \leqslant (\log_2 y)^2$. Adding Σ_1 and Σ_2 shows that for each σ_i,

$$\sum_{\substack{\tau_i\in\mathfrak{T}_i\\ \sigma_i\tau_i\in\mathfrak{S}_{i-1}}} \frac{1}{t_i} \ll (\log_2 y)^6 (\log y)^{-2\mu_{i-1}+(i\log i+i)(\nu_{i-1}-\nu_i+\delta)}. \tag{4-8}$$

We consider now the case when $v_i = v_{i-1}$. Set $Q_1 := P^+(t_{i+1,i-1})$, $Q_2 := P^+(t_{i+1,i})$, $Q_3 := P^+(t'_{i+1,i-1})$, and $Q_4 := P^+(t'_{i+1,i})$. From (iii), we have that $Q_1 \neq Q_2$ and $Q_3 \neq Q_4$. Moreover, letting b_i denote the cofactor of Q_i in each case, we have that

$$\begin{aligned} s_{i+1,i-1}b_1Q_1+1&=p_{i-1}, & s'_{i+1,i-1}b_3Q_3-1&=q_{i-1},\\ s_{i+1,i}b_2Q_2+1&=p_i, & s'_{i+1,i}b_4Q_4-1&=q_i. \end{aligned} \tag{4-9}$$

Since there are now several ways in which the various Q_i may coincide, the combinatorics is more complicated than in the case when $v_i < v_{i-1}$. We index the cases by fixing the incidence matrix (δ_{ij}) with $\delta_{ij}=1$ if $Q_i=Q_j$ and $\delta_{ij}=0$ otherwise.

Write $D=\gcd(Q_1Q_2,\, Q_3Q_4)$, and let $Q := Q_1Q_2/D$ and $Q' := Q_3Q_4/D$, so that D, Q, and Q' are formally determined by (δ_{ij}). Then $QQ' \mid t_{i+1}$, and writing $t_{i+1}/D = tQQ'$, we have

$$tQ = t_{i+1,0}t_{i+1,1}\cdots t_{i+1,i-2}b_3b_4, \tag{4-10}$$

$$tQ' = t'_{i+1,0}t'_{i+1,1}\cdots t'_{i+1,i-2}b_1b_2. \tag{4-11}$$

We now choose which terms on the right-hand sides of (4-10) and (4-11) contain the prime factors of Q and Q', respectively; since $\Omega(Q)\leqslant 2$ and $\Omega(Q')\leqslant 2$, this can be done in at most $(i+1)^4$ ways. Having made this choice, the number of ways to form t is bounded by $(i+1)^{2\Omega(t)}$, and so

$$\sum_{\substack{\tau_{i+1}\in\mathfrak{T}_{i+1}\\ \sigma_{i+1}\tau_{i+1}\in\mathfrak{S}_{i-1}}} \frac{1}{t_{i+1}} \leqslant \sum_t \frac{(i+1)^{2\Omega(t)+4}}{t} \max_{b_1,b_2,b_3,b_4} \sum \frac{1}{DQQ'}. \tag{4-12}$$

It is easy to check that $DQQ' = \prod_{j\in\mathscr{J}} Q_j$, where $\mathscr{J}$ indexes the distinct Q_j. For each $j\in\mathscr{J}$, let n_j be the number of linear forms appearing in (4-9) involving Q_j. Since each of these n_j linear forms in Q_j is prime, as is Q_j itself, Lemma 2.1 implies that the number of possibilities for $Q_j\leqslant z$ is $\ll z(\log z)^{-n_j-1}(\log_2 y)^{n_j+1}$, and so

$$\sum_{Q_j\geqslant u_{i-1}} \frac{1}{Q_j} \ll (\log_2 y)^{n_j+1}(\log u_{i-1})^{-n_j} \ll (\log_2 y)^{n_j+1}(\log y)^{-n_j\mu_{i-1}},$$

uniformly in the choice of the b's. Since $\sum_{j\in\mathscr{J}} n_j = 4$ and $\sum_{j\in\mathscr{J}} 1\leqslant 4$,

$$\sum \frac{1}{DQQ'} \leqslant \prod_{j\in\mathscr{J}}\Biggl(\sum_{Q_j\geqslant u_{i-1}} \frac{1}{Q_j}\Biggr) \ll (\log_2 y)^8(\log y)^{-4\mu_{i-1}}. \tag{4-13}$$

The calculation (4-7), with i replaced by $i+1$, shows that

$$\sum_t \frac{(i+1)^{2\Omega(t)}}{t} \leqslant (\log y)^{((i+1)+(i+1)\log(i+1))(\nu_i-\nu_{i+1}+\delta)}. \tag{4-14}$$

Combining (4-12), (4-13), and (4-14) shows that

$$\sum_{\substack{\tau_{i+1}\in\mathfrak{T}_{i+1}\\ \sigma_{i+1}\tau_{i+1}\in\mathfrak{S}_{i-1}}} \frac{1}{t_{i+1}} \leqslant (i+1)^4(\log_2 y)^8(\log y)^{-4\mu_{i-1}+((i+1)+(i+1)\log(i+1))(\nu_i-\nu_{i+1}+\delta)}$$
$$\leqslant (\log_2 y)^{12}(\log y)^{-2\mu_{i-1}+(i\log i+i)(\nu_i-\nu_{i-1})-2\mu_i+((i+1)\log(i+1)+(i+1))(\nu_{i+1}-\nu_i+\delta)}, \tag{4-15}$$

where in the last line we use that $v_{i-1}=v_i$ and $(i+1)^4\leqslant k^4\leqslant(\log_2 y)^4$.

Using (4-3), (4-5), and (4-6) together with the inequalities (4-4), (4-8), and (4-15), we find that the number of solutions of (4-1) is

$$\ll y(c\log_2 y)^{6k}(\log y)^{-2-\nu_1+\sum_{i=2}^{k}(\nu_{i-1}-\nu_i+\delta)(i\log i+i)-2\mu_{i-1}}\sum_{\sigma_k\in\mathfrak{S}_k}\frac{1}{s_k},$$

where c is some positive constant. Note that the exponent of $\log y$ is

$$\leqslant -2+\sum_{i=1}^{k-1}a_i\nu_i+E.$$

It remains to treat the sum on σ_k. Given $s_k'=s_k/d$, the number of possible σ_k is at most the number of factorizations of s_k' into $k+1$ factors times the number of factorizations of ds_k' into $k+1$ factors, which is at most $(k+1)^{\Omega(ds_k')}(k+1)^{\Omega(s_k')}$. By assumptions (i) and (iv), $\Omega(s_k')\leqslant 4(k+l)\log_2 v_k$. Thus,

$$\sum_{\sigma_k\in\mathfrak{S}_k}\frac{1}{s_k}\leqslant\frac{(k+1)^{\Omega(d)}(k+1)^{8(k+l)\log_2 v_k}}{d}\sum_{P^+(s_k')\leqslant v_k}\frac{1}{s_k'}$$
$$\ll\frac{(k+1)^{\Omega(d)}(\log v_k)^{8(k+l)\log(k+1)+1}}{d}. \qquad \square$$

5. Counting common values: Application of Lemma 4.1

In this section we prove the following proposition, which combined with Lemma 3.2 immediately yields Theorem 1.1. Throughout the rest of this paper, we adopt the definitions of L, the ξ_i, S, δ, and ω from (3-1) and (3-2).

Proposition 5.1. *Fix $A>0$. For large x, the number of distinct values of $\phi(a)$ that arise from solutions to the equation*

$$\phi(a)=\sigma(a'), \quad \text{with} \quad (a,a')\in\mathscr{A}_\phi\times\mathscr{A}_\sigma,$$

is smaller than $(V_\phi(x)+V_\sigma(x))/(\log_2 x)^A$.

Let us once again recall the strategy outlined in the introduction and in the remarks following Lemma 4.1. Let $(a,a')\in\mathscr{A}_\phi\times\mathscr{A}_\sigma$ be a solution to $\phi(a)=\sigma(a')$. Let $p_i:=p_i(a)$ and $q_i:=p_i(a')$, in the notation of Section 2B. We choose a cutoff

k so that all of $p_0, \dots, p_{k-1}$ and $q_0, \dots, q_{k-1}$ are "large". Then by condition (1) in the definition of the sets $\mathscr{A}_f$, neither $p_i^2 \mid a$ nor $q_i^2 \mid a'$, for $0 \leqslant i \leqslant k-1$. Fixing a notion of "small" and "tiny", we rewrite the equation $\phi(a) = \sigma(a')$ in the form

$$(p_0 - 1) \cdots (p_{k-1} - 1) f d = (q_0 + 1) \cdots (q_{k-1} + 1) e, \tag{5-1}$$

where f is the contribution to $\phi(a)$ from the "small" primes, d is the contribution from the "tiny" primes, and e is the contribution of both the "small" and "tiny" primes to $\sigma(a')$.

We then fix d and numbers u_i and v_i, chosen so that $u_i \leqslant P^+(p_i - 1)$, $P^+(q_i + 1) \leqslant v_i$ for each $0 \leqslant i \leqslant k-1$. With these fixed, Lemma 4.1 provides us with an upper bound on the number of corresponding solutions to (5-1). Such a solution determines the common value $\phi(a) = \sigma(a') \in \mathscr{V}_\phi \cap \mathscr{V}_\sigma$. We complete the proof of Proposition 5.1 by summing the upper bound estimates over all choices of d and all selections of the u_i and v_i.

We carry out this plan in four stages, each of which is treated in more detail below:

- Finalize the notions of "small" and "tiny", and so also the choices of d, e, and f.
- Describe how to choose the u_i and v_i so that the intervals $[u_i, v_i]$ capture $P^+(p_i - 1)$ and $P^+(q_i + 1)$ for all $0 \leqslant i \leqslant k-1$.
- Check that the hypotheses of Lemma 4.1 are satisfied.
- Take the estimate of Lemma 4.1 and sum over d and the choices of u_i and v_i.

5A. *"Small" and "tiny".* Suppose we are given a solution $(a, a') \in \mathscr{A}_\phi \times \mathscr{A}_\sigma$ to $\phi(a) = \sigma(a')$. Set $x_j = x_j(a; x)$ and $y_j = x_j(a'; x)$, in the notation of Section 2B, so that (from the definition of $\mathscr{A}_f$) the sequences $\mathbf{x} = (x_1, \dots, x_L)$ and $\mathbf{y} = (y_1, \dots, y_L)$ belong to $\mathscr{S}_L(\boldsymbol{\xi})$.

Lemma 5.2. *With $\{z_j\}_{j=1}^L$ denoting either of the sequences $\{x_j\}$ or $\{y_j\}$, we have*

(i) $z_j < 3\varrho^j$ *for* $1 \leqslant j \leqslant L$,

(ii) $z_{L-j} \geqslant \frac{3}{100}\varrho^{-j} / \log_2 x$ *for* $0 \leqslant j < L$.

(iii) $z_{j+2} \leqslant 0.9 z_j$ *for* $1 \leqslant j \leqslant L-2$.

Proof. Claim (i) is repeated verbatim from Lemma 2.8. By the same lemma, $z_j \leqslant 3\varrho^{j-i} z_i$ for $1 \leqslant i < j \leqslant L$. This immediately implies (iii), since $\varrho^2 < 0.9$. Moreover, fixing $j = L$, condition (6) in the definition of $\mathscr{A}_f$ gives that

$$z_i \geqslant \tfrac{1}{3}\varrho^{i-L} z_L \geqslant \frac{\log_2 3}{3} \varrho^{i-L} / \log_2 x > \tfrac{3}{100}\varrho^{i-L} / \log_2 x,$$

which is (ii) up to a change of variables. $\square$

Lemma 5.3. *The minimal index $k_0 \leqslant L$ for which*

$$\log_2 P^+(p_{k_0}-1) < (\log_2 x)^{1/2+\epsilon/10}$$

satisfies $k_0 \sim (1/2-\epsilon/10)L$ as $x \to \infty$.

Proof. Lemma 5.2(i) shows that the least K with $\log_2 p_K < (\log_2 x)^{1/2+\epsilon/10}$ satisfies $K \leqslant (1/2-\epsilon/10+o(1))L$, as $x \to \infty$. Since $\log_2 P^+(p_K-1) \leqslant \log_2 p_K$, this gives the asserted upper bound on k_0. The lower bound follows in a similar fashion from Lemma 5.2(ii) and Lemma 2.5. □

Recall the definition of δ from (3-2), and put

$$\eta := 10L\delta, \quad \text{so that} \quad \eta \asymp (\log_3 x)^{3/2}(\log_2 x)^{-1/2}.$$

We choose our "large"/"small" cutoff point k by taking $k = k_0$ if $x_{k_0-1} - x_{k_0} \geqslant 20\eta$, and taking $k = k_0 - 1$ otherwise. For future use, we note that with this choice of k,

$$x_{k-1} - x_k \geqslant 20\eta. \tag{5-2}$$

This inequality is immediate if $k = k_0$; in the opposite case, by Lemma 5.2(iii),

$$\begin{aligned} x_{k-1} - x_k = x_{k_0-2} - x_{k_0-1} &\geqslant x_{k_0-2} - x_{k_0} - 20\eta \\ &\geqslant 0.1x_{k_0-2} - 20\eta \geqslant 0.1(\log_2 x)^{-1/2+\epsilon/10} - 20\eta > 20\eta. \end{aligned}$$

Note that with this choice of k, we have $\log_2 p_i > (\log_2 x)^{1/2+\epsilon/10}$ for $0 \leqslant i \leqslant k-1$, and so condition (1) in the definition of $\mathscr{A}_\phi$ guarantees that each p_i divides a to the first power only, for $0 \leqslant i \leqslant k-1$. Moreover, from Lemmas 5.2(ii) and 5.3, we have $\log_2 q_i > (\log_2 x)^{1/2+\epsilon/11}$ for $0 \leqslant i \leqslant k-1$. So each q_i divides a' only to the first power, for $0 \leqslant i \leqslant k-1$. Now take

$$f := \phi(p_k p_{k+1} \cdots p_{L-1}), \quad d := \begin{cases} \phi(p_L p_{L+1} \cdots) & \text{if } p_{L-1} \neq p_L, \\ \frac{p_L}{\phi(p_L)}\phi(p_L p_{L+1} \cdots) & \text{if } p_{L-1} = p_L, \end{cases} \tag{5-3}$$

and

$$e := \sigma(q_k q_{k+1} \cdots),$$

and observe that (5-1) holds.

5B. ***Selection of the u_j and v_j.*** Rather than choose the u_j and v_j directly, it is more convenient to work with the μ_j and ν_j; then u_j and v_j are defined by $u_j := \exp((\log x)^{\mu_j})$ and $v_j := \exp((\log x)^{\nu_j})$. Put

$$\zeta_0 := 1 - \frac{\log_3 x + \log 100}{\log_2 x} \quad \text{and} \quad \zeta_j := \zeta_0 - j\eta \quad (j \geqslant 1), \tag{5-4}$$

and note that with $\nu_0 := 1$ and $\mu_0 := \zeta_0$, we have

$$u_0 = x^{1/(100\log_2 x)} < x^{1/\log_2 x} \leqslant P^+(p_0-1),\ P^+(q_0+1) \leqslant x = v_0,$$

by condition (7) in the definitions of $\mathscr{A}_\phi$ and $\mathscr{A}_\sigma$. To choose the remaining μ_j and ν_j, it is helpful to know that p_j and q_j are close together (renormalized on a double logarithmic scale) for $1 \leqslant j \leqslant k$. This is the substance of the following lemma.

Lemma 5.4. *If $p_j \geqslant S$ and $q_j \geqslant S$, then $|x_j - y_j| \leqslant (2j+1)\delta < \eta$. These hypotheses hold if $L - j \geqslant 2C \log_4 x + 12$, and so in particular for $1 \leqslant j \leqslant k$.*

Proof. Suppose for the sake of contradiction that $y_j \geqslant x_j + (2j+1)\delta$; since the p_i and q_i are all S-normal, this would imply that

$$(j+1)(y_j - x_j - \delta) \leqslant \frac{\Omega(\sigma(a'), p_j, q_j)}{\log_2 x} = \frac{\Omega(\phi(a), p_j, q_j)}{\log_2 x} \leqslant j(y_j - x_j + \delta),$$

which is false. We obtain a similar contradiction if we suppose $x_j \geqslant y_j + (2j+1)\delta$. The second half of the lemma follows from Lemma 5.2 and a short calculation, together with the estimate $k \sim (1/2 - \epsilon/10)L$ of Lemma 5.3. $\square$

We choose the intervals $[\mu_j, \nu_j]$ for $1 \leqslant j \leqslant k-1$ successively, starting with $j = 1$. (We select ν_k last, by a different method.) Say that the pair $\{x_j, x_{j+1}\}$ is *well-separated* if $x_j - x_{j+1} \geqslant 10\eta$, and *poorly separated* otherwise.

In the well-separated case, among all ζ_i (with $i \geqslant 0$), choose ζ minimal and ζ' maximal with

$$\zeta' \log_2 x \leqslant \log_2 \min\{P^+(p_j - 1), P^+(q_j + 1)\} \\ \leqslant \log_2 \max\{P^+(p_j - 1), P^+(q_j + 1)\} \leqslant \zeta \log_2 x,$$

and put

$$\mu_j := \zeta, \qquad \nu_j := \zeta'.$$

In the poorly separated case, $j < k-1$, by (5-2). We select $[\mu_j, \nu_j] = [\mu_{j+1}, \nu_{j+1}]$ by a similar recipe: Among all ζ_i (with $i \geqslant 0$), choose ζ minimal and ζ' maximal with

$$\zeta' \log_2 x \leqslant \log_2 \min\{P^+(p_j - 1), P^+(q_j + 1), P^+(p_{j+1} - 1), P^+(q_{j+1} + 1)\} \\ \leqslant \log_2 \max\{P^+(p_j - 1), P^+(q_j + 1), P^+(p_{j+1} - 1), P^+(q_{j+1} + 1)\} \leqslant \zeta \log_2 x,$$

and put

$$\nu_j = \nu_{j+1} = \zeta, \quad \text{and} \quad \mu_j = \mu_{j+1} = \zeta'.$$

To see that these choices are well-defined, note that by (7) in the definition of $\mathscr{A}_f$, we have $x_j, y_j \leqslant \zeta_0$, which implies that a suitable choice of ζ above exists in both cases. Also, for $1 \leqslant i \leqslant k$, we have $x_i, y_i \geqslant (\log_2 x)^{-1/2+\epsilon/11}$ (by Lemma 5.3 and 5.2(ii)). So by Lemma 2.5,

$$\log_2 \min\{P^+(p_i - 1), P^+(q_i + 1)\} / \log_2 x \geqslant (\log_2 x)^{-1/2+\epsilon/12},$$

say. Since neighboring ζ_i are spaced at a distance $\eta \asymp (\log_2 x)^{-1/2}(\log_3 x)^{3/2}$, a suitable choice of ζ' also exists in both cases.

For our application of Lemma 4.1, it is expedient to keep track at each step of the length of the intervals $[\mu_j, \nu_j]$, as well as the distance between the left-endpoint of the last interval chosen and the right-endpoint of the succeeding interval (if any). In the well-separated case, Lemmas 5.4 and 2.5 show that

$$\nu_j \leqslant \max\{x_j, y_j\} + \eta \leqslant x_j + 2\eta,$$

while

$$\mu_j \geqslant \min\{x_j, y_j\} - \frac{\log_3 x + \log 4}{\log_2 x} - \eta \geqslant x_j - 3\eta, \tag{5-5}$$

so that $\nu_j - \mu_j \leqslant 5\eta$. Also, if a succeeding interval exists (so that $j+1 \leqslant k-1$), then $\nu_{j+1} \leqslant \max\{x_{j+1}, y_{j+1}\} + \eta \leqslant x_{j+1} + 2\eta$, and the separation between μ_j and ν_{j+1} satisfies the lower bound

$$\mu_j - \nu_{j+1} \geqslant x_j - x_{j+1} - 5\eta \geqslant 5\eta. \tag{5-6}$$

In the poorly separated case, we have

$$\nu_j \leqslant \max\{x_j, y_j, x_{j+1}, y_{j+1}\} + \eta = \max\{x_j, y_j\} + \eta \leqslant x_j + 2\eta,$$

as before, but the lower bound on μ_j takes a slightly different form;

$$\begin{aligned}\mu_j &\geqslant \min\{x_j, y_j, x_{j+1}, y_{j+1}\} - \frac{\log_3 x + \log 4}{\log_2 x} - \eta \\ &\geqslant (x_{j+1} - \eta) - \frac{\log_3 x + \log 4}{\log_2 x} - \eta \geqslant x_{j+1} - 3\eta \geqslant x_j - 13\eta, \end{aligned} \tag{5-7}$$

so that $\nu_j - \mu_j \leqslant 15\eta$. In this case, since $\nu_j = \nu_{j+1}$ and $\mu_j = \mu_{j+1}$, the succeeding interval (if it exists) is $[\mu_{j+2}, \nu_{j+2}]$. By Lemma 5.2(iii),

$$x_j - x_{j+2} \geqslant 0.1 x_j \geqslant 0.1 (\log_2 x)^{-1/2+\epsilon/10} > 20\eta,$$

say. Thus, $\nu_{j+2} \leqslant \max\{x_{j+2}, y_{j+2}\} + \eta \leqslant x_{j+2} + 2\eta \leqslant x_j - 18\eta$, and so

$$\mu_{j+1} - \nu_{j+2} = \mu_j - \nu_{j+2} \geqslant (x_j - 13\eta) - (x_j - 18\eta) \geqslant 5\eta. \tag{5-8}$$

At this point we have selected intervals $[\mu_j, \nu_j]$, for all $0 \leqslant j \leqslant k-1$. We choose $\nu_k = \zeta$, where ζ is the minimal ζ_i satisfying $\zeta \geqslant x_k + \eta$. Note that

$$\log_2 S / \log_2 x = 36 \log_3 x / \log_2 x < (\log_2 x)^{-1/2+\epsilon/11} \leqslant x_k < \zeta = \nu_k \leqslant x_k + 2\eta.$$

Thus, $\upsilon_k > S$. From (5-5) and (5-7), $\mu_{k-1} \geqslant x_{k-1} - 3\eta$, so that also

$$\mu_{k-1} - \nu_k \geqslant x_{k-1} - x_k - 5\eta \geqslant 15\eta, \tag{5-9}$$

where the last estimate uses (5-2).

5C. ***Verification of hypotheses.*** We now check that Lemma 4.1 may be applied with $y = x$. By construction, $S \leqslant v_k \leqslant v_{k-1} \leqslant \cdots \leqslant v_0 = x$, and $u_i \leqslant v_i$ for all $0 \leqslant i \leqslant k-1$. Moreover, if $[\mu_j, \nu_j] \neq [\mu_{j-1}, \nu_{j-1}]$ (where $2 \leqslant j \leqslant k-1$), then from (5-6) and (5-8), $\mu_{j-1} - \nu_j \geqslant 5\eta = 50L\delta > 2\delta$, and from (5-9), $\mu_{k-1} - \nu_k \geqslant 15\eta > 2\delta$. Thus, condition (a) of Lemma 4.1 is satisfied. It follows from our method of selecting the μ_j and ν_j that if $\nu_j = \nu_{j+1}$, then (again by (5-8)) $\nu_{j+2} \leqslant \mu_{j+1} - 5\eta < \nu_{j+1} = \nu_j$, which shows that condition (b) is also satisfied. Moreover, since $\nu_k > x_k$, we have $P^+(d) \leqslant p_L \leqslant p_k < v_k$. So we may focus our attention on hypotheses (i)–(v) of Lemma 4.1. We claim that these hypotheses are satisfied with our choices of d, e, and f from Section 5A and with

$$l := L - k. \tag{5-10}$$

Property (i) is contained in (2) from the definition of $\mathscr{A}_f$. By construction,

$$u_i \leqslant P^+(p_i - 1),\ P^+(q_i + 1) \leqslant v_i$$

for all $0 \leqslant i \leqslant k-1$, which is (ii). Since $v_k \geqslant S > \log y$, property (iii) holds by (1) in the definition of $\mathscr{A}_f$. The verification of (iv) is somewhat more intricate. Recalling that $\nu_k > x_k$, it is clear from (5-3) that

$$P^+(f) < p_k \leqslant v_k.$$

To prove the same estimate for $P^+(e)$, we can assume $e \neq 1$. Let $r = P^+(e)$, and observe that $r \mid \sigma(R)$, for some prime power R with $R \parallel q_k q_{k+1} \cdots$. If R is a proper prime power, then from (1) in the definition of $\mathscr{A}_f$, we have

$$r \leqslant \sigma(R) \leqslant 2R \leqslant 2(\log x)^2 < v_k.$$

So we can assume that R is prime, and so $R \leqslant q_k$ and $r \leqslant P^+(R+1) \leqslant \max\{3, R\} \leqslant q_k$. But by Lemma 5.4,

$$\log_2 q_k / \log_2 x = y_k \leqslant x_k + (2k+1)\delta < x_k + \eta \leqslant \nu_k.$$

Thus, $P^+(e) = r \leqslant v_k$. Hence, $P^+(ef) \leqslant v_k$. Turning to the second half of (iv), write $p_k \cdots p_{L-1} = AB$, where A is squarefree, B is squarefull and $\gcd(A, B) = 1$. Recalling (2-2), we see that

$$\Omega(\phi(A)) \leqslant 3\Omega(A) \log_2 v_k \leqslant 3l \log_2 v_k,$$

with l as in (5-10). Let B' be the largest divisor of a supported on the primes dividing B, so that B' is squarefull and $B \mid B'$. By (1) in the definition of $\mathscr{A}_f$, we have $B' \leqslant (\log x)^2$. If $B' \leqslant \exp((\log_2 x)^{1/2})$, then (estimating crudely)

$$\Omega(\phi(B)) \leqslant \Omega(\phi(B')) \leqslant 2 \log \phi(B') \leqslant 2 \log B' \leqslant 2(\log_2 x)^{1/2}.$$

On the other hand, if $B' > \exp((\log_2 x)^{1/2})$, then by (4) in the definition of $\mathscr{A}_f$,

$$\Omega(\phi(B)) \leqslant \Omega(\phi(B')) \leqslant 10\log_2 \phi(B') \leqslant 10\log_2 B' \ll \log_3 x.$$

Since $\log_2 v_k = \nu_k \log_2 x > \eta \log_2 x > (\log_2 x)^{1/2}$, we have we have $\Omega(\phi(B)) \leqslant 2\log_2 v_k$ in either case. Hence,

$$\Omega(f) = \Omega(\phi(A)) + \Omega(\phi(B)) \leqslant (3l+2)\log_2 v_k \leqslant 4l \log_2 v_k,$$

which completes the proof of (iv). Finally, we prove (v): Suppose that $b \geqslant x^{1/3}$ is a divisor of $p_0 - 1$. Recalling again (2-2),

$$P^+(b) \geqslant b^{1/\Omega(p_0-1)} \geqslant b^{\frac{1}{3\log_2 x}} \geqslant x^{\frac{1}{9\log_2 x}} > x^{\frac{1}{100\log_2 x}} \geqslant v_1.$$

Thus, setting b to be the largest divisor of $p_0 - 1$ supported on the primes $\leqslant v_1$, we have $b < x^{1/3}$. From (3-3) and conditions (0) and (7) in the definition of $\mathscr{A}_\phi$,

$$p_0 = \frac{a}{p_1 p_2 p_3 \cdots} > \frac{x/\log x}{x^{1/100} p_1} > x^{0.95},$$

say. Thus, $(p_0 - 1)/b$ is a divisor of $p_0 - 1$ composed of primes greater than v_1 and of size at least $(p_0 - 1)x^{-1/3} > x^{9/10}x^{-1/3} > x^{1/2}$.

5D. ***Denouement.*** We are now in a position to establish Proposition 5.1 and so also Theorem 1.1. Suppose that k and the μ_i and ν_i are fixed, as is d; this also fixes $l = L - k$. By Lemma 4.1, whose hypotheses were verified above, the number of values $\phi(a)$ coming from corresponding solutions to $\phi(a) = \sigma(a')$, with $(a, a') \in \mathscr{A}_\phi \times \mathscr{A}_\sigma$, is

$$\begin{aligned} &\ll \frac{x}{d}(c\log_2 x)^{6k}(k+1)^{\Omega(d)}(\log v_k)^{8(k+l)\log(k+1)+1}(\log x)^{-2+\sum_{i=1}^{k-1} a_i \nu_i + E} \\ &\qquad \leqslant \frac{x}{d}\exp(O((\log_3 x)^2))L^{\Omega(d)}(\log v_k)^{L^2}(\log x)^{-2+\sum_{i=1}^{k-1} a_i x_i + E'}, \end{aligned} \tag{5-11}$$

where

$$E' := E + \sum_{i=1}^{k-1} a_i(\nu_i - x_i) = \delta\sum_{i=2}^{k}(i\log i + i) + 2\sum_{i=1}^{k-1}(\nu_i - \mu_i) + \sum_{i=1}^{k-1} a_i(\nu_i - x_i).$$

By our choice of ν_i and μ_i in Section 5B, we have $\nu_i - \mu_i \ll \eta$ and $\nu_i - x_i \ll \eta$. Hence,

$$E' \ll \delta L^2 \log L + \eta\biggl(L + \sum_{i=1}^{k-1} a_i\biggr) \ll \delta L^2\log L + \eta L^2 \log L \ll \delta L^3 \log L.$$

In combination with (8) from the definition of $\mathscr{A}_\phi$, this shows that the exponent of $\log x$ on the right-hand side of (5-11) is at most $-1-\omega+E' \leqslant -1-\omega/2$, and so

$$(\log x)^{-2+\sum_{i=1}^{k-1} a_i x_i + E'} \leqslant (\log x)^{-1} \exp\bigl(-\tfrac{1}{2}(\log_2 x)^{1/2+\epsilon/2}\bigr).$$

Moreover, by Lemma 5.3 and Lemma 5.2(i),

$$\nu_k \leqslant x_k + 2\eta \leqslant (\log_2 x)^{-1/2+\epsilon/9} + 2\eta \leqslant (\log_2 x)^{-1/2+\epsilon/5}, \tag{5-12}$$

and hence

$$(\log v_k)^{L^2} = \exp(L^2(\log_2 x)\nu_k) \leqslant \exp((\log_2 x)^{1/2+\epsilon/4}).$$

Inserting all of this back into (5-11), we obtain an upper bound which is

$$\ll \frac{x}{\log x} \exp\bigl(-\tfrac{1}{3}(\log_2 x)^{1/2+\epsilon/2}\bigr) \frac{L^{\Omega(d)}}{d}. \tag{5-13}$$

Now we sum over the parameters previously held fixed. We have $k < L$; also, for $i > 0$, each μ_i and ν_i has the form ζ_j of (5-4). Thus, the number of possibilities for k and the μ_i and ν_i is

$$\leqslant L(1+\lfloor \eta^{-1} \rfloor)^{2L} \leqslant \exp(O((\log_3 x)^2)). \tag{5-14}$$

Next, we prove that

$$\Omega(d) \ll (\log_2 x)^{1/2} \tag{5-15}$$

uniformly for the d under consideration, so that

$$L^{\Omega(d)} \leqslant \exp(O((\log_2 x)^{1/2} \log_4 x)). \tag{5-16}$$

Put $m := p_L p_{L+1} \cdots$. Suppose first that $p_L \neq p_{L-1}$, so that m is a unitary divisor of a and $d = \phi(m)$. If $m \leqslant \exp((\log_2 x)^{1/2})$, then (5-15) follows from the crude bound $\Omega(d) \ll \log d$. On the other hand, if $m > \exp((\log_2 x)^{1/2})$, then from (4) in the definition of $\mathscr{A}_\phi$, we have $\Omega(d) = \Omega(\phi(m)) \ll \log_2 m$. But by (3) in the definition of $\mathscr{A}_\phi$ and Lemma 5.2(i),

$$\begin{aligned}\log_2 m \leqslant \log_2 p_L^{10\log_2 x} &\ll \log_3 x + \log_2 p_L \ll \log_3 x + \varrho^L \log_2 x \\ &\ll \log_3 x + \varrho^{-2\sqrt{\log_3 x}} \varrho^{L_0} \log_2 x \ll \varrho^{-2\sqrt{\log_3 x}} \log_3 x \ll \exp(O(\sqrt{\log_3 x})),\end{aligned}$$

which again gives (5-15). Suppose now that $p_L = p_{L-1}$. In this case, let m' be the largest divisor of a supported on the primes dividing m. Then $d \mid \phi(m')$, and so $\Omega(d) \leqslant \Omega(\phi(m'))$. Write $m' = p_L^j m''$, where $j \geqslant 2$ and $p_L \nmid m''$; both p_L^j and m'' are unitary divisors of a. We have $\Omega(\phi(m'')) \ll (\log_2 x)^{1/2}$, by mimicking the argument used for m in the case when $p_L \neq p_{L-1}$. Also, $\Omega(\phi(p_L^j)) \ll (\log_2 x)^{1/2}$

except possibly if $p_L^j > \exp((\log_2 x)^{1/2})$, in which case, invoking (1) and (4) in the definition of $\mathscr{A}_\phi$,

$$\Omega(\phi(p_L^j)) \leqslant 10 \log_2 \phi(p_L^j) \leqslant 10 \log_2 p_L^j \leqslant 10 \log_2 (\log^2 x) \ll \log_3 x.$$

So

$$\Omega(d) \leqslant \Omega(\phi(m')) = \Omega(\phi(p_L^j)) + \Omega(\phi(m'')) \ll (\log_2 x)^{1/2},$$

confirming (5-15).

Referring back to (5-13), we see that it remains only to estimate the sum of $1/d$. Since $P^+(d) \leqslant v_k$, (5-12) shows that every prime dividing d belongs to the set $\mathscr{P} := \{p : \log_2 p \leqslant (\log_2 x)^{1/2+\epsilon/5}\}$. Thus,

$$\sum \frac{1}{d} \leqslant \prod_{p \in \mathscr{P}} \Big(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\Big) \ll \exp((\log_2 x)^{1/2+\epsilon/5}). \tag{5-17}$$

Combining the estimates (5-13), (5-14), (5-16), and (5-17), we find that

$$\#\{\phi(a) : a \in \mathscr{A}_\phi, a' \in \mathscr{A}_\sigma, \phi(a) = \sigma(a')\} \ll \frac{x}{\log x} \exp\big(-\tfrac{1}{4}(\log_2 x)^{1/2+\epsilon/2}\big),$$

which completes the proof of Proposition 5.1 and of Theorem 1.1.

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