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Berger and Colmez (2008) formulated a theory of families of overconvergent étale (φ, Γ)-modules associated to families of p-adic Galois representations over p-adic Banach algebras. In contrast with the classical theory of (φ, Γ)-modules, the functor they obtain is not an equivalence of categories. In this paper, we prove that when the base is an affinoid space, every family of (overconvergent) étale (φ, Γ)-modules can locally be converted into a family of p-adic representations in a unique manner, providing the "local" equivalence. There is a global mod p obstruction related to the moduli of residual representations.

Introduction

Berger and Colmez [2008] introduced a theory of families of overconvergent étale (φ, Γ)-modules associated to families of p-adic Galois representations over p-adic Banach algebras. The p-adic families of local Galois representations emerging from number theory are usually over rigid analytic spaces. So we are mainly interested in the case where the bases are reduced affinoid spaces. However, even in this case, the functor of Berger and Colmez is far from an equivalence of categories, in contrast with the classical theory of (φ, Γ)-modules. This was first noticed by Chenevier [Berger and Colmez 2008, remarque 4.2.10]: if the base is the p-adic unit circle $M(\mathbb{Q}\langle X, Y\rangle/(XY-1))$, then it is easy to see that the free rank-1 overconvergent étale (φ, Γ)-module D with a basis e such that $\varphi(e) = Ye$ and $\gamma(e) = e$ for $\gamma \in \Gamma$ does not come from a family of p-adic representations over the same base.

On the other hand, in his proof of the density of crystalline representations, Colmez [2008, proposition 5.2] proved that for certain families of rank-2 triangular étale (φ, Γ)-modules, one can locally convert such a family into a family of p-adic representations using his theory of *Espaces Vectoriels de dimension finie* (it is clear that we can also convert Chenevier's example locally). Moreover, Colmez

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remarked [2008, remarque 5.3(2)]: *On aurait pu aussi utiliser une version «en famille» des théorèmes à la Dieudonné–Manin de Kedlaya. Il y a d'ailleurs une concordance assez frappante entre ce que permettent de démontrer ces théorèmes de Kedlaya et la théorie des Espaces Vectoriels de dimension finie.*

Unfortunately, as noted in [Liu 2008], there is no family version of Kedlaya's slope filtrations theorem in general, because the slope polygons of families of Frobenius modules are not necessarily locally constant. Nonetheless, one may still ask to what extent one can convert a globally étale family of (φ, Γ)-modules back into a Galois representation. As Chenevier's example shows, this cannot be done in general over an affinoid base. The best one can hope for in general is the following theorem, which extends a result of Dee [2001]. (In the statement, the distinction between a (φ, Γ)-module and a family of (φ, Γ)-modules is that the former is defined as a module over a ring, whereas the latter is defined as a coherent sheaf over a rigid analytic space.)

Theorem 0.1. *Let S be a Banach algebra over $\mathbb{Q}_p$ of the form $R \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, where R is a complete noetherian local domain of characteristic 0 whose residue field is finite over $\mathbb{F}_p$. Then for any finite extension K of $\mathbb{Q}_p$, the categories of S-linear representations of G_K, of étale (φ, Γ)-modules over $\mathbf{B}^{\dagger}_{K} \widehat{\otimes}_{\mathbb{Q}_p} S$, and of families of étale (φ, Γ)-modules over $\mathbf{B}^{\dagger}_{\mathrm{rig},K} \widehat{\otimes}_{\mathbb{Q}_p} S$ are all equivalent.*

For instance, if S is an affinoid algebra and we are given an étale (φ, Γ)-module over $\mathbf{B}^{\dagger}_{K} \widehat{\otimes}_{\mathbb{Q}_p} S$, we recover a linear representation over each residue disc of S (and every affinoid subdomain of such a disc), but these representations may not glue. This is what happens in Chenevier's example, because the mod p representations cannot be uniformly trivialized. In fact, the obstruction to converting a (φ, Γ)-module back into a representation exists purely at the residual level; it suggests a concrete realization of the somewhat murky notion of "moduli of residual (local) representations".

By combining Theorem 0.1 with the results of [Liu 2008], we obtain a result that applies when only one fiber of the (φ, Γ)-module is known to be étale. (Beware that the natural analogue of this statement in which the rigid analytic point x is replaced by a Berkovich point is trivially false.)

Theorem 0.2. *Let S be an affinoid algebra over $\mathbb{Q}$, and let M_S be a family of (φ, Γ)-modules over $\mathbf{B}^{\dagger}_{\mathrm{rig},K} \widehat{\otimes}_{\mathbb{Q}_p} S$. If M_x is étale for some $x \in M(S)$, then there exists an affinoid neighborhood $M(B)$ of x and a B-linear representation V_B of G_K whose associated (φ, Γ)-module is isomorphic to $M_S \widehat{\otimes}_S B$. Moreover, V_B is unique for this property.*

To prove the Fontaine–Colmez theorem, Berger [2008] constructed a morphism from the category of filtered (φ, N)-modules to the category of (φ, Γ)-modules. It should be possible to generalize Berger's construction to families of filtered (φ, N)-modules; upon doing so, one would get a family version of the Fontaine–Colmez theorem by Theorem 0.2. That is, one would know that a weakly admissible family of filtered (φ, N)-modules over an affinoid base (with trivial φ-action on the base) becomes admissible in a neighborhood of each rigid analytic point.

1. Rings of p-adic Hodge theory

We begin by introducing some of the rings used in p-adic Hodge theory. This is solely to fix notation; we do not attempt to expose the constructions in any detail. For that, see for instance [Berger 2004]. Here, whenever a ring is defined whose notation includes a boldface $\mathbf{A}$, the same notation with $\mathbf{A}$ replaced by $\mathbf{B}$ will indicate the result of inverting p.

Let $\mathbb{C}_p$ be a completed algebraic closure of $\mathbb{Q}$, with valuation subring $\mathbb{O}_{\mathbb{C}_p}$ and p-adic valuation v_p normalized with $v_p(p) = 1$. Let $\overline{v}_p : \mathbb{O}_{\mathbb{C}_p}/(p) \to [0, 1) \cup \{+\infty\}$ be the semivaluation obtained by truncation. Define $\widetilde{\mathbf{E}}^+$ to be the ring of sequences $(x_n)_{n=0}^{\infty}$ in $\mathbb{O}_{\mathbb{C}_p}/(p)$ such that $x_{n+1}^p = x_n$ for all n. Define a function

$$v_{\mathbf{E}} : \widetilde{\mathbf{E}}^+ \to [0, +\infty]$$

by sending the zero sequence to $+\infty$, and sending each nonzero sequence (x_n) to the common value of $p^n \overline{v}_p(x_n)$ for all n with $x_n \neq 0$. This gives a valuation under which $\widetilde{\mathbf{E}}^+$ is complete. Moreover, if we put $\widetilde{\mathbf{E}} = \mathrm{Frac}(\widetilde{\mathbf{E}}^+)$, and let $\epsilon = (\epsilon_n)$ be an element of $\widetilde{\mathbf{E}}^+$ with $\epsilon_0 = 1$ and $\epsilon_1 \neq 1$, then $\widetilde{\mathbf{E}}$ is a completed algebraic closure of $\mathbb{F}_p((\epsilon - 1))$.

Let $\widetilde{\mathbf{A}}$ be the p-typical Witt ring $W(\widetilde{\mathbf{E}})$, which is the unique complete discrete valuation ring with maximal ideal (p) and residue field $\widetilde{\mathbf{E}}$. For each positive integer n, $W(\widetilde{\mathbf{E}})/p^n W(\widetilde{\mathbf{E}})$ inherits a topology from the valuation topology on $\widetilde{\mathbf{E}}$, under which it is complete. We call the inverse limit of these the *weak topology* on $\widetilde{\mathbf{A}}$. We similarly obtain a weak topology on $\widetilde{\mathbf{B}}$.

For any $n \geq 0$, we let μ_{p^n} denote the set of p^n-th roots of unity in $\overline{\mathbb{Q}}_p$, and let $\mu_{p^\infty} = \bigcup_{n \geq 0} \mu_{p^n}$. For K a finite extension of $\mathbb{Q}$, let

$$K_\infty = K(\mu_{p^\infty}), \quad H_K = \mathrm{Gal}(\overline{K}/K_\infty), \quad \Gamma = \Gamma_K = \mathrm{Gal}(K_\infty/K), \quad K_0' = \mathbb{Q}_p^{\mathrm{ur}} \cap K_\infty.$$

Put $\pi = [\epsilon] - 1$, where brackets denote the Teichmüller lift. Using the completeness of $\widetilde{\mathbf{A}}$ for the weak topology, we may embed $\mathbb{Z}_p((\pi))$ into $\widetilde{\mathbf{A}}$. Let $\mathbf{A}$ be the p-adic completion of the integral closure of $\mathbb{Z}_p((\pi))$ in $\widetilde{\mathbf{A}}$, and put $\mathbf{A}_K = \mathbf{A}^{H_K}$. These rings carry actions of G_K that are continuous for the weak topology on the rings and the profinite topology on G_K. They also carry endomorphisms φ (which are weakly and p-adically continuous) induced by the Witt vector Frobenius on $\widetilde{\mathbf{A}}$.

For $s > 0$, the subset

$$\widetilde{\mathbf{A}}^{\dagger,s} = \left\{ x \in \widetilde{\mathbf{A}} \,\middle|\, x = \sum_{k\in\mathbb{Z}} p^k[x_k],\, v_{\widetilde{\mathbf{E}}}(x_k) + \frac{psk}{p-1} \ge 0,\, \lim_{k\to+\infty} v_{\widetilde{\mathbf{E}}}(x_k) + \frac{psk}{p-1} = +\infty \right\}$$

is a subring of $\widetilde{\mathbf{A}}$ that is complete for the valuation

$$w_s(x) = \inf_k \left\{ v_{\widetilde{\mathbf{E}}}(x_k) + \frac{psk}{p-1} \right\}.$$

Put

$$\widetilde{\mathbf{B}}^{\dagger} = \bigcup_{s>0} \widetilde{\mathbf{B}}^{\dagger,s}, \quad \mathbf{B}_K^{\dagger,s} = \mathbf{B}_K \cap \widetilde{\mathbf{B}}^{\dagger,s}, \quad \mathbf{B}_K^{\dagger} = \bigcup_{s>0} \mathbf{B}_K^{\dagger,s},$$
$$\mathbf{A}_K^{\dagger,s} = \mathbf{A}_K \cap \widetilde{\mathbf{A}}^{\dagger,s}, \quad \mathbf{A}_K^{\dagger} = \mathbf{A} \cap \mathbf{B}_K^{\dagger}.$$

(This last ring is strictly larger than $\bigcup_{s>0} \mathbf{A}_K^{\dagger,s}$.) These rings carry an action of φ. with the proviso that φ gtakes a ring with a superscript of s to the corresponding ring with s replaced by ps. For n a positive integer, write

$$\mathbf{A}_{K,n}^{\dagger,s} = \varphi^{-n}(\mathbf{A}_K^{\dagger,p^n s}).$$

Let $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,s}$ be the Fréchet completion of $\widetilde{\mathbf{B}}^{\dagger,s'}$ under the valuations $w_{s'}$ for all $s' \ge s$, and put $\widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger} = \bigcup_{s>0} \widetilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger,s}$. Similarly, let $\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$ be the Fréchet completion of $\mathbf{B}_K^{\dagger,s'}$ under the valuations $w_{s'}$ for all $s' \ge s$, and put $\mathbf{B}_{\mathrm{rig},K}^{\dagger} = \bigcup_{s>0} \mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$. It turns out that $(\mathbf{B}_{\mathrm{rig}}^{\dagger,s})^{H_K} = \mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$.

Some of these rings admit more explicit descriptions, as follows. It turns out that $\mathbf{B}_K$ is isomorphic to the p-adic local field

$$\mathscr{E}_{K_0'} = \left\{ f = \sum_{i=-\infty}^{+\infty} a_i T^i \,\middle|\, a_i \in K_0',\, \inf_i \{v_p(a_i)\} > -\infty,\, \lim_{i\to-\infty} v_p(a_i) = +\infty \right\}$$

with valuation $w(f) = \min_{i\in\mathbb{Z}} v_p(a_i)$ and imperfect residue field $k'((T))$, where k' is the residue field of K_0'. There is no distinguished such isomorphism in general (except for $K = \mathbb{Q}_p$, where one may take $T = \pi$), but suppose we fix a choice. Then $\mathbf{B}$ corresponds to the completion of the maximal unramified extension of $\mathbf{B}_K$. For $s \gg 0$ (depending on K and the choice of the isomorphism $\mathbf{B}_K \cong \mathscr{E}_{K_0'}$), $\mathbf{B}_K^{\dagger,s}$ corresponds to the subring $\mathscr{E}_{K_0'}^s$ of $\mathscr{E}_{K_0'}$ defined as

$$\mathscr{E}_{K_0'}^s = \left\{ f = \sum_{i=-\infty}^{+\infty} a_i T^i \,\middle|\, a_i \in K_0',\, \inf_i \{v_p(a_i)\} > -\infty,\, \lim_{i\to-\infty} i + \frac{ps}{p-1} v_p(a_i) = +\infty \right\},$$

that is, the bounded Laurent series in T convergent on the annulus $0 < v_p(T) \le 1/s$. Meanwhile, $\mathbf{B}_{\mathrm{rig},K}^{\dagger,s}$ corresponds to the ring

$$\mathscr{R}_{K_0'}^s = \left\{ f = \sum_{i=-\infty}^{+\infty} a_i T^i \,\middle|\, a_i \in K_0',\, \lim_{i\to+\infty} i + r v_p(a_i) = +\infty \text{ for all } r > 0, \right.$$
$$\left. \lim_{i\to-\infty} i + \frac{ps}{p-1} v_p(a_i) = +\infty \right\},$$

that is, the unbounded Laurent series in T convergent on $0 < v_p(T) \leq 1/s$. The union $\mathcal{R}_{K_0'} = \bigcup_{s>0} \mathcal{R}^s_{K_0'}$ is commonly called the *Robba ring* over K_0'.

2. p-adic representations and (φ, Γ)-modules

We next introduce p-adic representations and the objects of semilinear algebra used to describe them. Fix a finite extension K of $\mathbb{Q}_p$. For R a topological ring, we will mean by an *R-linear representation* a finite R-module equipped with a continuous linear action of G_K. (We will apply additional adjectives like "free", which are to be passed through to the underlying R-module.) Fontaine [1990] constructed a functor giving an equivalence of categories between $\mathbb{Q}$-linear representations and certain linear (or rather semilinear) algebraic data, as follows. (We may extend to L-linear representations for finite extensions L of $\mathbb{Q}$ by restricting the coefficient field to $\mathbb{Q}$ and then keeping track of the L-action separately.)

An *étale φ-module* over $\mathbf{A}_K$ is a finite module N over $\mathbf{A}_K$ equipped with a semilinear action of φ such that the $\mathbf{A}_K$-linear map $\varphi^* N \to N$ induced by the φ-action is an isomorphism. An *étale φ-module* over $\mathbf{B}_K$ is a finite module M over $\mathbf{B}_K$, equipped with a semilinear action of φ, that contains an $\mathbf{A}_K$-lattice N (that is, a finite $\mathbf{A}_K$-submodule such that the induced map $N \otimes_{\mathbf{A}_K} \mathbf{B}_K \to M$ is an isomorphism) that forms an étale φ-module over $\mathbf{A}_K$. An *étale (φ, Γ)-module* over $\mathbf{A}_K$ or $\mathbf{B}_K$ is an étale φ-module equipped with a semilinear action of Γ which commutes with the φ-action and is continuous for the profinite topology on Γ and the weak topology on $\mathbf{A}_K$. Note that an étale (φ, Γ)-module over $\mathbf{B}_K$ may contain an $\mathbf{A}_K$-lattice that forms an étale φ-module over $\mathbf{A}_K$ but is not stable under Γ; on the other hand, the images of such a lattice under Γ span another lattice which forms an étale φ-module over $\mathbf{A}_K$.

For T a $\mathbb{Z}$-linear representation, define $\mathrm{D}(T) = (\mathbf{A} \otimes_{\mathbb{Z}_p} T)^{H_K}$; this gives an $\mathbf{A}_K$-module equipped with commuting semilinear actions of φ and Γ. Similarly, for V a $\mathbb{Q}$-linear representation, define $\mathrm{D}(V) = (\mathbf{B} \otimes_{\mathbb{Q}_p} V)^{H_K}$.

Theorem 2.1 [Fontaine 1990]. *The functor $T \mapsto \mathrm{D}(T)$ (resp. $V \mapsto \mathrm{D}(V)$) is an equivalence from the category of $\mathbb{Z}$-linear representations (resp. $\mathbb{Q}$-linear representations) of G_K to the category of étale (φ, Γ)-modules over $\mathbf{A}_K$ (resp. $\mathbf{B}_K$); a quasiinverse functor is given by $D \mapsto (\mathbf{A} \otimes_{\mathbf{A}_K} D)^{\varphi=1}$ (resp. $D \mapsto (\mathbf{B} \otimes_{\mathbf{B}_K} D)^{\varphi=1}$).*

Dee [2001] extended Fontaine's results to families of $\mathbb{Z}$-representations, as follows. Let R be a complete noetherian local ring whose residue field k_R is finite over $\mathbb{F}_p$, equipped with the topology defined by its maximal ideal $\mathfrak{m}_R$; we may then view R as a topological $\mathbb{Z}$-algebra. We form the completed tensor product $R \widehat{\otimes}_{\mathbb{Z}} \mathbf{A}$ by completing the ordinary tensor product for the ideal $p\mathbf{A} + \mathfrak{m}_R$, and similarly with $\mathbf{A}$ replaced by $\mathbf{A}_K$.

We define (φ, Γ)-modules and étale (φ, Γ)-modules over $R \mathbin{\widehat{\otimes}}_{\mathbb{Z}} \mathbf{A}_K$ by analogy with the definitions over $\mathbf{A}_K$. For T_R an R-representation, define $\mathrm{D}(T_R) = ((R \mathbin{\widehat{\otimes}}_{\mathbb{Z}} \mathbf{A}) \otimes_R T_R)^{H_K}$.

Theorem 2.2 [Dee 2001]. *The functor*

$$T_R \mapsto \mathrm{D}(T_R)$$

is an equivalence from the category of R-representations to the category of étale (φ, Γ)-modules over $R \mathbin{\widehat{\otimes}}_{\mathbb{Z}} \mathbf{A}_K$; a quasiinverse functor is given by

$$D \mapsto ((R \mathbin{\widehat{\otimes}}_{\mathbb{Z}} \mathbf{A}) \otimes_{R \mathbin{\widehat{\otimes}}_{\mathbb{Z}} \mathbf{A}_K} D)^{\varphi=1}.$$

We next cite a refinement of Fontaine's result. We define (φ, Γ)-modules and étale (φ, Γ)-modules over $\mathbf{A}_K^{\dagger}$ and $\mathbf{B}_K^{\dagger}$ by analogy with the definitions over $\mathbf{A}_K$ and $\mathbf{B}_K$. For V a $\mathbb{Q}_p$-linear representation, define $\mathrm{D}_K^{\dagger,r}(V) = (\mathbf{B}^{\dagger,r} \otimes_{\mathbb{Q}_p} V)^{H_K}$ (where $\mathbf{B}^{\dagger,r} = \mathbf{B} \cap \widetilde{\mathbf{B}}^{\dagger,r}$) and $\mathrm{D}_K^{\dagger}(V) = \bigcup_{r>0} \mathrm{D}_K^{\dagger,r}(V) = (\mathbf{B}^{\dagger} \otimes_{\mathbb{Q}_p} V)^{H_K}$.

Theorem 2.3 [Cherbonnier and Colmez 1998]. *For any $\mathbb{Q}_p$-linear representation V, there exists $r(V) > 0$ such that*

$$\mathrm{D}_K(V) = \mathbf{B}_K \otimes_{\mathbf{B}_K^{\dagger,r}} \mathrm{D}_K^{\dagger,r}(V) \quad \textit{for all} \quad r \geqslant r(V).$$

Equivalently, $\mathrm{D}_K^{\dagger}(V)$ is an étale (φ, Γ)-module over $\mathbf{B}_K^{\dagger}$ of dimension $\dim_{\mathbb{Q}} V$. Therefore $V \mapsto \mathrm{D}_K^{\dagger}(V)$ is an equivalence from the category of p-adic representations of G_K to the category of étale (φ, Γ)-modules over $\mathbf{B}_K^{\dagger}$. Furthermore, $\mathrm{D}_K^{\dagger}(V)$ is the unique maximal étale (φ, Γ)-submodule of $\mathrm{D}_K(V)$ over $\mathbf{B}_K^{\dagger}$.

Berger and Colmez [2008] extended these results to families of p-adic representations. However, unlike Dee's families, the families considered by Berger and Colmez are over Banach algebras over $\mathbb{Q}$. (Berger and Colmez were forced to make a freeness hypothesis on the representation space; we relax this hypothesis later in the case of an affinoid algebra. See Definition 3.12.)

For S a commutative Banach algebra over $\mathbb{Q}_p$, let $\mathbb{O}_S$ be the ring of elements of S of norm at most 1, and let I_S be the ideal of elements of $\mathbb{O}_S$ of norm strictly less than 1. Note that it makes sense to form a completed tensor product with S or $\mathbb{O}_S$ when the other tensorand carries a norm under which it is complete — for example, for the rings $\widetilde{\mathbf{A}}^{\dagger,s}, \mathbf{A}_{L,n}^{\dagger,s}, \widetilde{\mathbf{B}}^{\dagger,s}, \mathbf{B}_L^{\dagger,s}$ using the norm corresponding to the valuation w_s.

Proposition 2.4 [Berger and Colmez 2008, proposition 4.2.8]. *Let S be a commutative Banach algebra over $\mathbb{Q}_p$. Let T_S be a free $\mathbb{O}_S$-linear representation of rank d. Let L be a finite Galois extension of K such that G_L acts trivially on $T_S/12pT_S$. Then there exists $n(L, T_S) \geq 0$ such that for $n \geq n(L, T_S)$,*

$$(\mathbb{O}_S \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} \tilde{\mathbf{A}}^{\dagger,(p-1)/p}) \otimes_{\mathbb{O}_S} T_S$$

has a unique sub-$(\mathbb{O}_S \widehat{\otimes}_{\mathbb{Z}_p} \mathbf{A}_{L,n}^{\dagger,(p-1)/p})$*-module* $\mathrm{D}_{L,n}^{\dagger,(p-1)/p}(T_S)$ *that is free of rank* d*, is fixed by* H_L*, has a basis almost invariant under* Γ_L *(that is, for each* $\gamma \in \Gamma_L$*, the matrix of action of* $\gamma - 1$ *on the basis has positive valuation), and satisfies*

$$(\mathbb{O}_S \widehat{\otimes}_{\mathbb{Z}_p} \tilde{\mathbf{A}}^{\dagger,(p-1)/p}) \otimes_{\mathbb{O}_S \widehat{\otimes}_{\mathbb{Z}_p} \mathbf{A}_{L,n}^{\dagger,(p-1)/p}} \mathrm{D}_{L,n}^{\dagger,(p-1)/p}(T_S) = (\mathbb{O}_S \widehat{\otimes} \tilde{\mathbf{A}}^{\dagger,(p-1)/p}) \otimes_{\mathbb{O}_S} T_S.$$

Theorem 2.5 [Berger and Colmez 2008, théorème 4.2.9]. *Let* S *be a commutative Banach algebra over* $\mathbb{Q}_p$*. Let* V_S *be an* S*-linear representation admitting a free Galois-stable* $\mathbb{O}_S$*-lattice* T_S*. There exists an* $s(V_S) \geq 0$ *such that for any* $s \geq s(V_S)$*, we may define*

$$\mathrm{D}_K^{\dagger,s}(V_S) = ((S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_L^{\dagger,s}) \otimes_{\mathbb{O}_S \widehat{\otimes}_{\mathbb{Z}_p} \mathbf{A}_L^{\dagger,s(V_S)}} \varphi^n(\mathrm{D}_{L,n}^{\dagger,p-1/p}(T_S)))^{H_K}$$

for some L *and* n*, so that the construction does not depend on the choices of* T_S*,* L*, and* n*, and the following statements hold.*

(a) *The* $(S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_K^{\dagger,s})$*-module* $\mathrm{D}_K^{\dagger,s}(V_S)$ *is locally free of rank* d*.*

(b) *The natural map* $\mathrm{D}_K^{\dagger,s}(V_S) \otimes_{S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_K^{\dagger,s}} (S \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^{\dagger,s}) \to V_S \otimes_S (S \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^{\dagger,s})$ *is an isomorphism.*

(c) *For any maximal ideal* $\mathfrak{m}_x$ *of* S*, for* $V_x = V_S \otimes_S (S/\mathfrak{m}_x)$*, the natural map* $\mathrm{D}_K^{\dagger,s}(V_S) \otimes_S (S/\mathfrak{m}_x) \to \mathrm{D}_K^{\dagger,s}(V_x)$ *is an isomorphism.*

We write $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_K^{\dagger} = \bigcup_{s>0}(S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_K^{\dagger,s})$ and $S \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^{\dagger} = \bigcup_{s>0}(S \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^{\dagger,s})$. (Note that $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_K^{\dagger}$ does not necessarily embed into $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_K$, due to the incompatibility between the topologies used for the completed tensor products.) We then put

$$\mathrm{D}_K^{\dagger}(V_S) = \mathrm{D}_K^{\dagger,s}(V_S) \otimes_{S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_K^{\dagger,s}} (S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_K^{\dagger}).$$

We may recover V_S from $\mathrm{D}_K^{\dagger}(V_S)$ as follows.

Lemma 2.6. $(S \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^{\dagger})^{\varphi=1} = S$.

Proof. We reduce at once to the case where S is countably topologically generated over $\mathbb{Q}_p$. In this case, by [Bosch et al. 1984, Proposition 2.7.2/3], we can find a *Schauder basis* of S over $\mathbb{Q}$; in other words, there exists an index set I such that S is isomorphic as a topological $\mathbb{Q}$-vector space to the Banach space

$$l_0^{\infty}(I, \mathbb{Q}) = \{(a_i)_{i \in I} \mid a_i \in \mathbb{Q}, a_i \to 0\}.$$

(The supremum norm need only be equivalent to the Banach norm on S; the two need not be equal.) We can then write $S \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^{\dagger}$, as a topological $\mathbb{Q}$-vector space, as

$$l_0^{\infty}(I, \widetilde{\mathbf{B}}^{\dagger}) = \{(a_i)_{i \in I} \mid a_i \in \widetilde{\mathbf{B}}^{\dagger}, a_i \to 0\}.$$

In this presentation, the φ-action carries $(a_i)_{i\in I}$ to $(\varphi(a_i))_{i\in I}$. It is then clear that $(S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^\dagger)^{\varphi=1} = (l_0^\infty(I, \widetilde{\mathbf{B}}^\dagger))^{\varphi=1} = l_0^\infty(I, \mathbb{Q}) = S$. □

Proposition 2.7. $(\mathrm{D}_K^\dagger(V_S) \otimes_{S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}_K^\dagger} (S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^\dagger))^{\varphi=1} = V_S$.

Proof. From Theorem 2.5(b) we get

$$\mathrm{D}_K^\dagger(V_S) \otimes_{S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}_K^\dagger} (S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^\dagger) = V_S \otimes_S (S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^\dagger).$$

By Lemma 2.6, it follows that

$$(\mathrm{D}_K^\dagger(V_S) \otimes_{S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}_K^\dagger} (S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^\dagger))^{\varphi=1} = V_S \otimes_S (S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^\dagger)^{\varphi=1} = V_S. \quad \square$$

This suggests that the object $\mathrm{D}_K^\dagger(V_S)$ merits the following definition.

Definition 2.8. Define a (φ, Γ)-*module* over $S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}_K^\dagger$ to be a finite locally free module over $S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}_K^\dagger$, equipped with commuting continuous (φ, Γ)-actions such that $\varphi^* D_S \to D_S$ is an isomorphism. We say a (φ, Γ)-module M_S over $S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}_K^\dagger$ is *étale* if it admits a finite (φ, Γ)-stable $(\mathbb{O}_S \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} \mathbf{A}_K^\dagger)$-submodule N_S such that $\varphi^* N_S \to N_S$ is an isomorphism and the induced map

$$N_S \otimes_{\mathbb{O}_S \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} \mathbf{A}_K^\dagger} S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}_K^\dagger \to M_S$$

is an isomorphism. In this language, Theorem 2.5 implies that $\mathrm{D}_K^\dagger(V_S)$ is an étale (φ, Γ)-module over $S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}_K^\dagger$.

3. Gluing on affinoid spaces

Throughout this section, let S denote an affinoid algebra over $\mathbb{Q}_p$. We explain how to perform gluing for finite modules over $S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}_K^\dagger$. We start with some basic notions from [Bosch et al. 1984].

Definition 3.1. Let $M(S)$ be the set of maximal ideals of S, that is, the affinoid space associated to S. For X a subset of $M(S)$, an *affinoid subdomain* of X is a subset U of X for which there exists a morphism $S \to S'$ of affinoid algebras such that the induced map $M(S') \to M(S)$ is universal for maps from an affinoid space to $M(S)$ landing in U. The algebra S' is then unique up to unique isomorphism, and the resulting map $M(S) \to U$ is a bijection.

The set $M(S)$ carries two canonical G-topologies, defined as follows. In the *weak G-topology*, the admissible open sets are the affinoid subdomains, and the admissible coverings are the finite coverings. In the *strong G-topology*, the admissible open sets are the subsets U of $M(S)$ admitting a covering by affinoid subdomains such that the induced covering of any affinoid subdomain of U can be refined to a finite cover by affinoid subdomains, and the admissible coverings are the ones whose restriction to any affinoid subdomain can be refined to a finite cover by affinoid subdomains. The categories of sheaves on these two topologies are equivalent, because the strong G-topology is *slightly finer* than the weak one [Bosch et al. 1984, §9.1].

We need a generalization of the Tate and Kiehl theorems on coherent sheaves on affinoid spaces.

Definition 3.2. For A a commutative Banach algebra over $\mathbb{Q}_p$, define the presheaf $\mathscr{A}$ on the weak G-topology of $M(S)$ by declaring that

$$\mathscr{A}(M(S')) = S' \widehat{\otimes}_{\mathbb{Q}_p} A.$$

Lemma 3.3. *For A a commutative Banach algebra over $\mathbb{Q}_p$, the presheaf $\mathscr{A}$ is a sheaf for the weak G-topology of $M(S)$, and hence extends uniquely to the strong G-topology.*

Proof. Since every finite covering of an affinoid space by affinoid subdomains can be refined to a Laurent covering, it is enough to check the sheaf condition for Laurent coverings [Bosch et al. 1984, Proposition 8.2.2/5]. This reduces to checking for coverings of the form

$$M(S) = M(S\langle f\rangle) \cup M(S\langle f^{-1}\rangle)$$

for $f \in S$. The claim then is that the sequence

$$0 \to S \widehat{\otimes}_{\mathbb{Q}_p} A \to (S\langle f\rangle \widehat{\otimes}_{\mathbb{Q}_p} A) \times (S\langle f^{-1}\rangle \widehat{\otimes}_{\mathbb{Q}_p} A) \xrightarrow{d^0} S\langle f, f^{-1}\rangle \widehat{\otimes}_{\mathbb{Q}_p} A \to 0$$

is exact; this follows from the corresponding assertion for $A = \mathbb{Q}_p$, for which see [Bosch et al. 1984, §8.2.3]. □

From now on, we consider only the strong G-topology on $M(S)$.

Definition 3.4. For A a commutative Banach algebra over $\mathbb{Q}_p$, an $\mathscr{A}$-module N on $M(S)$ is *coherent* if there exists an admissible covering $\{M(S_i)\}_{i\in I}$ of $M(S)$ by affinoid subdomains such that for each $i \in I$, we have

$$N|_{M(S_i)} = \operatorname{coker}(\varphi : \mathscr{A}^m|_{M(S_i)} \to \mathscr{A}^n|_{M(S_i)})$$

for some morphism φ of $\mathscr{A}$-modules. By Lemma 3.3, this is equivalent to requiring $N|_{M(S_i)}$ to be the sheaf associated to some finitely presented $(S_i \widehat{\otimes}_{\mathbb{Q}_p} A)$-module.

Lemma 3.5. *Let A be a commutative Banach algebra over $\mathbb{Q}_p$ such that for each Tate algebra T_n over $\mathbb{Q}_p$, $T_n \widehat{\otimes}_{\mathbb{Q}_p} A$ is noetherian. Then for any coherent $\mathscr{A}$-module N on $M(S)$, the first Čech cohomology $\check{H}^1(N)$ vanishes.*

Proof. As in Lemma 3.3, it suffices to check vanishing of the first Čech cohomology computed on a cover of $M(S)$ of the form

$$M(S) = M(S\langle f\rangle) \cup M(S\langle f^{-1}\rangle)$$

for some $f \in S$, such that N is represented on each of the two covering subsets by a finite module. For this, we may follow the proof of [Bosch et al. 1984, Lemma 9.4.3/5] verbatim. (The noetherian condition is needed so that the invocation of [Bosch et al. 1984, Proposition 3.7.3/3] within the proof of [Bosch et al. 1984, Lemma 9.4.3/5] remains valid.) □

To recover an analogue of Kiehl's theorem, however, we need an extra condition.

Proposition 3.6. *Let A be a commutative Banach algebra over $\mathbb{Q}_p$ such that for each Tate algebra T_n over $\mathbb{Q}_p$, $T_n \widehat{\otimes}_{\mathbb{Q}_p} A$ is noetherian and the map*

$$\operatorname{Spec}(T_n \widehat{\otimes}_{\mathbb{Q}_p} A) \to \operatorname{Spec}(T_n)$$

carries $M(T_n \widehat{\otimes}_{\mathbb{Q}_p} A)$ to $M(T_n)$. Then any coherent $\mathscr{A}$-module N on $M(S)$ is associated to a finite $(S \widehat{\otimes}_{\mathbb{Q}_p} A)$-module.

Proof. There must exist a finite covering of $M(S)$ by affinoid subdomains $M(S_1), \dots, M(S_n)$ such that $N|_{M(S_i)}$ is associated to a finite $(S_i \widehat{\otimes}_{\mathbb{Q}_p} A)$-module N_i. As in [Bosch et al. 1984, Lemma 9.4.3/6], we may deduce from Lemma 3.5 that for each $\mathfrak{m} \in M(S_i)$, the map $N(M(S)) \to (N/\mathfrak{m}N)(M(S_i))$ is surjective. By the hypothesis on A, each maximal ideal of $S_i \widehat{\otimes}_{\mathbb{Q}_p} A$ lies over a maximal ideal of S_i; we may thus deduce that $N(M(S)) \otimes_S S_i$ surjects onto $N(M(S_i))$. Since the latter is a finite $S_i \widehat{\otimes}_{\mathbb{Q}} A$-module, we can choose finitely many elements of $N(M(S))$ that generate all of the $N(M(S_i))$. That is, we have a surjection $\mathscr{A}^n \to N$ for some n; repeating the argument for the kernel of this map yields the claim. □

To use the above argument, we need to prove a variant of the Nullstellensatz; for simplicity, we restrict to the case where K is discretely valued (the case of interest in this paper). We first prove a finite generation result using ideas from the theory of Gröbner bases.

Lemma 3.7. *Let K be a complete discretely valued field extension of $\mathbb{Q}_p$. Let A be a commutative Banach algebra over K such that A has the same set of nonzero norms as K, and the ring $\mathbb{O}_A/I_A$ is noetherian. Then $T_n \widehat{\otimes}_{\mathbb{Q}_p} A$ is also noetherian.*

Proof. Equip the monoid $\mathbb{Z}^n_{\geq 0}$ with the componentwise partial ordering $\leq$ and the lexicographic total ordering $\preceq$. That is, $(x_1, \dots, x_n) \leq (y_1, \dots, y_n)$ if $x_i \leq y_i$ for all i, whereas $(x_1, \dots, x_n) \preceq (y_1, \dots, y_n)$ if there exists an index $i \in \{1, \dots, n+1\}$ such that $x_j = y_j$ for $j < i$, and either $i = n+1$ or $x_i \leq y_i$. Recall that $\leq$ is a well partial ordering and that $\preceq$ is a well total ordering; in particular, any sequence in $\mathbb{Z}^n_{\geq 0}$ has a subsequence that is weakly increasing under both orderings.

For $I = (i_1, \dots, i_n) \in \mathbb{Z}^n_{\geq 0}$, write t^I for $t_1^{i_1} \cdots t_n^{i_n}$. We represent each element $x \in T_n \widehat{\otimes}_{\mathbb{Q}_p} A$ as a formal sum $\sum_I x_I t^I$ with $x_I \in A$, such that for each $\epsilon > 0$, there exist only finitely many indices I with $|x_I| \leq \epsilon$. For x nonzero, define the *degree* of x, denoted $\deg(x)$, to be the maximal index I under $\preceq$ among those indices maximizing $|x_I|$. Define the *leading coefficient* of x to be the coefficient $x_{\deg(x)}$.

Let J be any ideal of $T_n \widehat{\otimes}_{\mathbb{Q}_p} A$. We apply a Buchberger-type algorithm to construct a generating set $m_1, \dots, m_k$ for J, as follows. Start with the empty list (that is, $k = 0$). As long as possible, given $m_1, \dots, m_k$, choose an element m_{k+1} of $J \cap \mathbb{O}_A$ with leading coefficient a_{k+1}, for which we *cannot* choose $I_1, \dots, I_k \in \mathbb{Z}^n_{\geq 0}$ and $b_1, \dots, b_k \in \mathbb{O}_A$ satisfying both

(a) $\deg(m_{k+1}) = \deg(m_i t^{I_i})$ whenever $b_i \neq 0$ and

(b) $a_{k+1} - a_1 b_1 - \cdots - a_k b_k \in I_A$.

In particular, we must have $|a_{k+1}| = 1$.

We claim this process must terminate. Suppose the contrary; then there must exist a sequence of indices $i_1 < i_2 < \cdots$ such that $\deg(m_{i_1}) \leq \deg(m_{i_2}) \leq \cdots$. Then the sequence of ideals $(a_{i_1}), (a_{i_1}, a_{i_2}), \dots$ in $\mathbb{O}_A / I_A$ must be strictly increasing, but this violates the hypothesis that $\mathbb{O}_A / I_A$ is noetherian. Hence the process terminates.

Let $|\cdot|_1$ denote the 1-Gauss norm on $T_n \widehat{\otimes}_{\mathbb{Q}_p} A$. We now write each element of J as a linear combination of $m_1, \dots, m_k$ using a form of the Buchberger division algorithm. Start with some nonzero $x \in J$ and put $x_0 = x$. Given $x_l \in J$, if $x_l = 0$, put $y_{l,1} = \cdots = y_{l,k} = 0$ and $x_{l+1} = 0$. Otherwise, choose $\lambda \in A^\times$ with $|\lambda x_l|_1 = 1$. By the construction of $m_1, \dots, m_k$, there must exist $I_1, \dots, I_k \in \mathbb{Z}^n_{\geq 0}$ and $b_1, \dots, b_k \in \mathbb{O}_A$ satisfying conditions (a) and (b) with m_{k+1} replaced by λx_l. Put $y_{l,i} = \lambda^{-1} b_i t^{I_i}$ and $x_{l+1} = x_l - y_{l,1} m_1 - \cdots - y_{l,k} m_k$.

If $|x_{l+1}|_1 = |x_l|_1$, we must have $\deg(x_{l+1}) \prec \deg(x_l)$. Since $\preceq$ is a well ordering, we must have $|x_{l'}|_1 < |x_l|_1$ for some $l' > l$. Since K is discretely valued and A has the same group of nonzero norms as K, we conclude that $|x_l|_1 \to 0$ as $l \to \infty$.

Since $|y_{l,i}|_1 \leq |x_l|_1$, we may set $y_i = \sum_{l=0}^{\infty} y_{l,i}$ to get elements of $T_n \widehat{\otimes}_{\mathbb{Q}_p} A$ such that $x = y_1 m_1 + \cdots + y_k m_k$. This proves that J is always finitely generated, so $T_n \widehat{\otimes}_{\mathbb{Q}_p} A$ is noetherian. □

Next we establish an analogue of the Nullstellensatz by combining the previous argument with an idea of Munshi [May 2003].

Lemma 3.8. *Take K and A as in Lemma 3.7, but suppose also that the intersection of the nonzero prime ideals of A is zero. Then for any maximal ideal $\mathfrak{m}$ of $T_n \widehat{\otimes}_{\mathbb{Q}_p} A$, the intersection $\mathfrak{m} \cap A$ is nonzero.*

Proof. Suppose on the contrary that $\mathfrak{m}$ is a maximal ideal of $T_n \widehat{\otimes}_{\mathbb{Q}_p} A$ such that $\mathfrak{m} \cap A = 0$. Since $T_n \widehat{\otimes}_{\mathbb{Q}_p} A$ is noetherian by Lemma 3.7, $\mathfrak{m}$ is closed by [Bosch et al. 1984, Proposition 3.7.2/2]. Hence $\mathfrak{m} + A$ is also a closed subspace of $T_n \widehat{\otimes}_{\mathbb{Q}_p} A$. Let $\psi : T_n \widehat{\otimes}_{\mathbb{Q}_p} A \to (T_n \widehat{\otimes}_{\mathbb{Q}_p} A)/A$ be the canonical projection; it is a bounded surjective morphism of Banach spaces with kernel A. Put $V = \psi(\mathfrak{m} + A)$; since

$\mathfrak{m} + A = \psi^{-1}(V)$, the open mapping theorem [Bosch et al. 1984, §2.8.1] implies that V is closed. Hence ψ induces a bounded bijective map $\mathfrak{m} \to V$ between two Banach spaces; by the open mapping theorem again, the inverse of ψ is also bounded.

Using the power series representation of elements of $T_n \widehat{\otimes}_{\mathbb{Q}_p} A$, let us represent $(T_n \widehat{\otimes}_{\mathbb{Q}_p} A)/A$ as the set of series in $T_n \widehat{\otimes}_{\mathbb{Q}_p} A$ with zero constant term. We may then represent ψ as the map that subtracts off the constant term. Define the *nonconstant degree* of $x \in T_n \widehat{\otimes}_{\mathbb{Q}_p} A$ as $\deg'(x) = \deg(\psi(x))$, and define the *leading nonconstant coefficient* of x to be the coefficient $x_{\deg'(x)}$.

We construct $m_1, \dots, m_k \in \mathfrak{m}$ using the following modified Buchberger algorithm. As long as possible, choose an element m_{k+1} of $\mathfrak{m} \cap \mathbb{O}_A$ with nonconstant leading coefficient a_{k+1}, for which we cannot choose $I_1, \dots, I_k \in \mathbb{Z}^n_{\geq 0}$ and $b_1, \dots, b_k \in \mathbb{O}_A$ satisfying both

(a) $\deg'(m_{k+1}) = \deg'(m_i t^{I_i})$ whenever $b_i \neq 0$ and

(b) $a_{k+1} - a_1 b_1 - \cdots - a_k b_k \in I_A$.

Again, this algorithm must terminate.

By the hypothesis on A, we can choose a nonzero prime ideal $\mathfrak{p}$ of A not containing the product $a_1 \cdots a_k$. By our earlier hypothesis that $\mathfrak{m} \cap A = 0$, we have $\mathfrak{m} \cap \mathfrak{p} = 0$. Hence $\mathfrak{m} + \mathfrak{p}(T_n \widehat{\otimes}_{\mathbb{Q}_p} A)$ is the unit ideal, so we can find $x_0 \in \mathfrak{p}(T_n \widehat{\otimes}_{\mathbb{Q}_p} A)$ such that $1 + x_0 \in \mathfrak{m}$.

We now perform a modified division algorithm. Given $x_l \in \mathfrak{p}(T_n \widehat{\otimes}_{\mathbb{Q}_p} A)$ such that $1 + x_l \in \mathfrak{m}$, we cannot have $x_l \in A$. We may thus choose $\lambda \in A^\times$ with $|\psi(\lambda x_l)|_1 = 1$. By the construction of $m_1, \dots, m_k$, there must exist $I_1, \dots, I_k \in \mathbb{Z}^n_{\geq 0}$ and $b_1, \dots, b_k \in A$ satisfying conditions (a) and (b) with m_{k+1} replaced by λx_l. Put $y_{l,i} = \lambda^{-1} b_i t^{I_i}$ and $x_{l+1} = x_l - y_{l,1} m_1 - \cdots - y_{l,k} m_k$.

As in the proof of Lemma 3.7, we see that $|\psi(x_l)|_1 \to 0$ as $l \to \infty$. Since ψ has bounded inverse, we also conclude that $|x_l|_1 \to 0$ as $l \to \infty$. However, since $\mathfrak{m}$ is closed, this yields the contradiction $1 \in \mathfrak{m}$. We conclude that $\mathfrak{m} \cap A \neq 0$, as desired. □

Lemma 3.9. *For any Tate algebra T_n over $\mathbb{Q}_p$, any rational $s > 0$, and any complete discretely valued field extension K of $\mathbb{Q}_p$, $T_n \widehat{\otimes}_{\mathbb{Q}_p} \mathscr{E}^s_K$ is noetherian and each of its maximal ideals has residue field finite over K. In particular, every maximal ideal of $T_n \widehat{\otimes}_{\mathbb{Q}_p} \mathscr{E}^s_K$ lies over a maximal ideal of T_n.*

Proof. The Banach norm on $\mathscr{E}^s_K$ is the maximum of the p-adic norm and the norm induced by w_s. By enlarging K, we may assume that the nonzero values of this norm are all achieved by elements of K. In this case, we check that $A = \mathscr{E}^s_K$ satisfies the hypotheses of Lemma 3.8. First, the nonzero norms of elements of A are all realized by units of the form λt^i with $\lambda \in K^\times$ and $i \in \mathbb{Z}$. Second, the residue ring $\mathbb{O}_A/I_A$ is isomorphic to a Laurent polynomial ring over a field, which is noetherian. Third, for each nonzero element x of A, we can construct $y \in A$ whose Newton polygon has no slopes in common with that of x; this implies that x and y generate the unit ideal (see, for example, [Kedlaya 2005b, §2.6]), so any maximal ideal containing y fails to contain x. Hence the intersection of the nonzero prime ideals of A is zero; moreover, the quotient of A by any nonzero ideal is finite over K. We may thus apply Lemma 3.8 to deduce the claim. □

By combining Proposition 3.6 with Lemma 3.9, we deduce the following. (The second assertion follows from the first because for a coherent module, local freeness can be checked at each maximal ideal.)

Proposition 3.10. *For any $s > 0$ and any finite extension K of $\mathbb{Q}_p$, for $A = \mathscr{E}^s_K$, any coherent $\mathscr{A}$-module $\mathscr{V}$ on $M(S)$ is associated to a finite $(S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} A)$-module V. Moreover, $\mathscr{V}$ is locally free if and only if V is.*

Using this, we may extend Theorem 2.5 for affinoid algebras, to eliminate the hypothesis requiring a free Galois-stable lattice. We first handle the case where V_S is itself free.

Theorem 3.11. *Let S be an affinoid algebra over $\mathbb{Q}_p$. Let V_S be a free S-linear representation. There exists $s(V_S) \geq 0$ such that for $s \geq s(V_S)$, we may construct a $(S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}^{\dagger,s}_K)$-module $\mathrm{D}^{\dagger,s}_K(V_S)$ satisfying the following conditions.*

- *The $(S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}^{\dagger,s}_K)$-module $\mathrm{D}^{\dagger,s}_K(V_S)$ is locally free of rank d.*
- *The natural map $\mathrm{D}^{\dagger,s}_K(V_S) \otimes_{S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}^{\dagger,s}_K} (S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^{\dagger,s}) \to V_S \otimes_S (S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^{\dagger,s})$ is an isomorphism.*
- *For any maximal ideal $\mathfrak{m}_x$ of S, for $V_x = V_S \otimes_S (S/\mathfrak{m}_x)$, the natural map $\mathrm{D}^{\dagger,s}_K(V_S) \otimes_S (S/\mathfrak{m}_x) \to \mathrm{D}^{\dagger,s}_K(V_x)$ is an isomorphism.*
- *The construction is functorial in V_S, compatible with passage from K to a finite extension, and compatible with Theorem 2.5 in case V_S admits a Galois-stable free lattice.*

Proof. Let T_S be any free $\mathbb{O}_S$-lattice in V_S. Since the Galois action is continuous, there exists a finite Galois extension L of K such that G_L carries T_S into itself. For such L, for s sufficiently large, $\mathrm{D}^{\dagger,s}_L(V_S)$ is locally free of rank d by Theorem 2.5; moreover, it carries an action of $\mathrm{Gal}(L/K)$. If we restrict scalars on this module back to $S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}^{\dagger,s}_K$, then $\mathrm{D}^{\dagger,s}_K(V_S)$ appears as a direct summand; this summand is then finite projective, and hence locally free (since $T_n \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathscr{E}^s_K$ is noetherian by Lemma 3.9). Moreover, the $\mathrm{Gal}(L/K)$-action on $\mathrm{D}^{\dagger,s}_L(V_S)$ allows us to extend the Γ_L-action on $\mathrm{D}^{\dagger,s}_K(V_S)$ to a Γ_K-action. This yields the desired assertions. □

Definition 3.12. Let S be an affinoid algebra over $\mathbb{Q}_p$. Let V_S be a locally free S-linear representation; we can then choose a finite covering of $M(S)$ by affinoid subdomains $M(S_1), \dots, M(S_n)$ such that $V_i = V_S \otimes_S S_i$ is free over S_i for each i. We may then apply Theorem 3.11 to V_i to produce $\mathrm{D}_K^{\dagger,s}(V_i)$ for s sufficiently large.

By Proposition 3.10, these glue to form a finite $(S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}_K^{\dagger,s})$-module $\mathrm{D}_K^{\dagger,s}(V_S)$, which satisfies the analogues of the assertions of Theorem 3.11. We may then define

$$\mathrm{D}_K^{\dagger}(V_S) = \mathrm{D}^{\dagger,s}(V_S) \otimes_{S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}_K^{\dagger,s}} (S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}_K^{\dagger}),$$

and this will be an étale (φ, Γ)-module over $S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}_K^{\dagger}$. The analogue of Proposition 2.7 will also carry over.

Remark 3.13. Chenevier has pointed out that Theorem 3.11 is also an easy consequence of [Chenevier 2009, Lemme 3.18]. That lemma implies that for S an affinoid algebra over $\mathbb{Q}_p$ and V_S a locally free S-linear representation, there exist an affine formal scheme $\mathrm{Spf}(R)$ of finite type over $\mathbb{Z}$, equipped with an isomorphism $R \otimes_{\mathbb{Z}} \mathbb{Q}_p \cong S$, and a locally free R-linear representation T_R admitting an isomorphism $T_R \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong V_S$. This makes it possible to glue the Berger–Colmez theorem by doing so on a suitable formal model of S.

4. Local coefficient algebras

Here we show that in a restricted setting, it is possible to invert the (φ, Γ)-module functor $D_K^{\dagger}$.

Definition 4.1. By a *coefficient algebra*, we mean a commutative Banach algebra S over $\mathbb{Q}$ satisfying the following conditions.

- The norm on S restricts to the norm on $\mathbb{Q}$.
- For each maximal ideal $\mathfrak{m}$ of S, the residue field of $\mathfrak{m}$ is finite over $\mathbb{Q}$.
- The Jacobson radical of S is zero; in particular, S is reduced.

For instance, any reduced affinoid algebra over $\mathbb{Q}$ is a coefficient algebra.

By a *local coefficient algebra*, we mean a coefficient algebra S of the form $R \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, where R is a complete noetherian local domain of characteristic 0 with residue field finite over $\mathbb{F}_p$. For instance, if S is a reduced affinoid algebra over $\mathbb{Q}_p$ equipped with the spectral norm, and R is the completion of $\mathbb{O}_S$ at a maximal ideal, then $R \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a local coefficient algebra.

One special property of local coefficient algebras is the following. (Compare the discussion preceding Lemma 2.6.)

Proposition 4.2. *Let R be a complete noetherian local domain of characteristic* 0 *with residue field finite over $\mathbb{F}_p$, and let S be the local coefficient algebra $R \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.*

(1) *We may naturally identify $(R \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} \mathbf{A}_K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with the p-adic completion of $S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}_K^{\dagger}$.*

(2) *We may naturally identify* $(R \widehat{\otimes}_{\mathbb{Z}_p} \widetilde{\mathbf{A}}_K) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ *with a subring of the p-adic completion of* $S \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^{\dagger}_K$.

Proof. Let $P_{1,n,s}, P_{2,n,s}, P_{3,n,s}$ denote the completed tensor products

$$(R/p^n R) \widehat{\otimes}_{\mathbb{Z}_p} (\mathbf{A}^{\dagger,s}/p^n \mathbf{A}^{\dagger,s})$$

formed using the following choices for the topologies on the two sides.

- For $P_{1,n,s}$, use on the left side the discrete topology, and on the right side the topology induced by w_s.
- For $P_{2,n,s}$, use on the left side the topology induced by $\mathfrak{m}_R$, and on the right side the topology induced by w_s.
- For $P_{3,n,s}$, use on the left side the topology induced by $\mathfrak{m}_R$, and on the right side the discrete topology.

These constructions relate to our original question as follows. If we take the inverse limit of the $P_{1,n,s}$ as $n \to \infty$, then invert p, then take the union over all choices of s, we recover $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}^{\dagger}$. If we take the union of the $P_{3,n,s}$ over all choices of s, then take the inverse limit as $n \to \infty$, and finally invert p, we recover $(R \widehat{\otimes}_{\mathbb{Z}_p} \mathbf{A}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

To establish (1), it thus suffices to check that the natural maps $P_{1,n,s} \to P_{2,n,s}$ and $P_{3,n,s} \to P_{2,n,s}$ are both bijections. Put $A = \mathfrak{m}_R(R/p^n R)$ and $I = \mathfrak{m}_R A$. Put $B = \mathbf{A}^{\dagger,s}/p^n \mathbf{A}^{\dagger,s}$ and choose an ideal of definition $J \subseteq B$ for the topology induced by w_s. In this notation, A is I-adically complete and separated, B is J-adically complete and separated, and both A and B are flat over $\mathbb{Z}/p^n\mathbb{Z}$. Put $C = A \otimes_{\mathbb{Z}/p^n\mathbb{Z}} B$. The IC-adic completion of C is then the inverse limit over m of the quotients $C/I^m C = (A/I^m A) \otimes_{\mathbb{Z}/p^n\mathbb{Z}} B$. Since B is flat over $\mathbb{Z}/p^n\mathbb{Z}$ and $A/I^m A$ has finite cardinality, the completeness of B with respect to J implies the completeness of $C/I^m C$ with respect to $J(C/I^m C)$. It follows that C is complete with respect to $IC + JC$, which means that $P_{1,n,s} \to P_{2,n,s}$ is a bijection. Similarly, we may argue that $P_{3,n,s} \to P_{2,n,s}$ is bijective using the fact that $B/J^m B$ is of finite cardinality.

This yields (1). The whole argument carries over in the case of (2) except for the finiteness of $B/J^m B$; hence in this case, we only have that $P_{1,n,s} \to P_{2,n,s}$ is a bijection and $P_{3,n,s} \to P_{2,n,s}$ is injective. □

Theorem 4.3. *Let S be a local coefficient algebra. Let M_S be an étale (φ, Γ)-module over $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}^{\dagger}_K$, and put*

$$V_S = (M_S \otimes_{S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}^{\dagger}_K} (S \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^{\dagger}))^{\varphi=1}.$$

Then V_S is an S-linear representation for which the natural map $\mathrm{D}^{\dagger}_K(V_S) \to M_S$ is an isomorphism.

Proof. By Proposition 4.2(1), we may identify the p-adic completion of $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_K^\dagger$ with $(R \widehat{\otimes}_{\mathbb{Z}_p} \mathbf{A}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. This allows us to define

$$V'_S = \big(M_S \otimes_{S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_K^\dagger} ((R \widehat{\otimes}_{\mathbb{Z}_p} \widetilde{\mathbf{A}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)\big)^{\varphi=1}.$$

By Theorem 2.2, the natural map

$$V'_S \otimes_{(R \widehat{\otimes}_{\mathbb{Z}_p} \mathbf{A}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p} ((R \widehat{\otimes}_{\mathbb{Z}_p} \widetilde{\mathbf{A}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \to M_S \otimes_{S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_K^\dagger} ((R \widehat{\otimes}_{\mathbb{Z}_p} \widetilde{\mathbf{A}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$$

is an isomorphism.

By Proposition 4.2(2), we may identify $(R \widehat{\otimes}_{\mathbb{Z}_p} \widetilde{\mathbf{A}}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with a subring of the p-adic completion of $S \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^\dagger$. Using this identity, we may argue as in [Kedlaya 2008, Proposition 1.2.7] to show that $V'_S \subseteq V_S$, which is enough to establish the desired result. □

5. A lifting argument

While one cannot invert the functor $\mathrm{D}_K^\dagger$ for an arbitrary S, one can give a partial result.

Lemma 5.1. *For any commutative Banach algebra S over $\mathbb{Q}_p$, any $s > 0$, and any $x \in S \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^{\dagger,s}$, the equation*

$$y - \varphi^{-1}(y) = x$$

has a solution $y \in S \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^{\dagger,s}$. More precisely, we may choose y such that $v_p(y) \geq v_p(x)$ and $w_s(y) \geq w_s(x)$.

Proof. For $S = \mathbb{Q}_p$, the existence of a solution $y \in \widetilde{\mathbf{B}}$ follows from the fact that $\widetilde{\mathbf{B}}$ is a complete discretely valued field with algebraically closed residue field. Write $x = \sum_k p^k[x_k]$ and $y = \sum_k p^k[y_k]$. We claim that y can be chosen such that for each k,

$$\inf\{v_{\widetilde{\mathbf{E}}}(y_\ell) : \ell \leq k\} \geq \inf\{v_{\widetilde{\mathbf{E}}}(x_\ell) : \ell \leq k\},$$

which yields the desired results. This choice can be made because for any $\bar{x} \in \widetilde{\mathbf{E}}$, the equation $\bar{y} - \bar{y}^{1/p} = \bar{x}$ always has a solution $\bar{y} \in \widetilde{\mathbf{E}}$ with

$$v_{\widetilde{\mathbf{E}}}(\bar{y}) \geq \begin{cases} v_{\widetilde{\mathbf{E}}}(\bar{x}) & \text{if } v_{\widetilde{\mathbf{E}}}(\bar{x}) \leq 0, \\ p v_{\widetilde{\mathbf{E}}}(\bar{x}) & \text{if } v_{\widetilde{\mathbf{E}}}(\bar{x}) \geq 0. \end{cases}$$

For general S, write x as a convergent sum $\sum_i u_i \otimes x_i$ with $u_i \in S$ and $x_i \in \widetilde{\mathbf{B}}^{\dagger,s}$. For each i, let $y_i \in \widetilde{\mathbf{B}}^{\dagger,s}$ be a solution of $y_i - \varphi^{-1}(y_i) = x_i$ with $w_s(y_i) \geq w_s(x_i)$. Then the sum $y = \sum_i u_i \otimes y_i$ converges with the desired effect. □

Theorem 5.2. *Let S be a commutative Banach algebra over $\mathbb{Q}_p$. Let M_S be a free étale (φ, Γ)-module over $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_K^\dagger$. Suppose that there exists a basis of M_S on which $\varphi - 1$ acts via a matrix whose entries have positive p-adic valuation. Then*

$$V_S = (M_S \otimes_{S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_K^\dagger} (S \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^\dagger))^{\varphi=1}$$

is a free S-linear representation for which the natural map $\mathrm{D}_K^\dagger(V_S) \to M_S$ is an isomorphism.

Proof. Choose a basis of

$$M'_S = M_S \otimes_{S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_K^\dagger} (S \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^\dagger)$$

on which $\varphi - 1$ acts via a matrix A whose entries belong to $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_K^{\dagger,s}$ for some $s > 0$ and have p-adic valuation bounded below by $c > 0$. We may apply Lemma 5.1 to choose a matrix X such that X has entries in $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_K^{\dagger,s}$ with p-adic valuation bounded below by c, $\min_{i,j}\{w_s(X_{i,j})\} \geq \min_{i,j}\{w_s(A_{i,j})\}$, and $X - \varphi^{-1}(X) = A$. We can thus change basis to get a new basis of M'_S on which $\varphi - 1$ acts via the matrix

$$(I_n - \varphi^{-1}(X))^{-1}(I_n + A)(I_n - X) - I_n,$$

whose entries have valuation bounded below by $2c$. If we repeat this process, we get a sequence of matrices $X_1, X_2, \ldots$ such that $w_s(X_i)$ is bounded below, and the p-adic valuation of X_i is at least ci. It follows that $w_{s'}(X_i)$ tends to infinity for any $s' > s$, so the product $(I_n + X_1)(I_n + X_2) \cdots$ converges in $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_K^{\dagger,s'}$ and defines a basis of M'_S fixed by φ. This proves the claim. □

Remark 5.3. The hypothesis about the basis of M_S is needed in Theorem 5.2 for the following reason. For R an arbitrary $\mathbb{F}_p$-algebra, if φ acts as the identity on R and as the p-power Frobenius on $\widetilde{\mathbf{E}}$, given an invertible square matrix A over $R \otimes_{\mathbb{F}_p} \widetilde{\mathbf{E}}$, we cannot necessarily solve the matrix equation $U^{-1} A \varphi(U) = A$ for an invertible matrix U over $R \otimes_{\mathbb{F}_p} \widetilde{\mathbf{E}}$. For instance, in Chenevier's example, there is no solution of the equation $\varphi(z) = Yz$.

One may wish to view the collection of isomorphism classes of (φ, Γ)-modules over $R \otimes_{\mathbb{F}_p} \mathbb{F}_p((\epsilon - 1))$, for R an $\mathbb{F}_p$-algebra, as the "R-valued points of the moduli space of mod p representations of $G_{\mathbb{Q}_p}$." To replace $\mathbb{Q}_p$ with K, one should replace $\mathbb{F}_p((\epsilon - 1))$ with the H_K-invariants of its separable closure.

6. Families of (φ, Γ)-modules and étale models

We turn from (φ, Γ)-modules over $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_K^\dagger$ to those over $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{rig},K}^\dagger$. In the absolute case, these have important applications to the study of de Rham representations, as shown by Berger; see for instance [2004]. In the relative case, however, they do not form a robust enough category to be useful; it is better to pass to a more geometric notion. For this, we must restrict to the case where S is an affinoid algebra.

Definition 6.1. Let K be a finite extension of $\mathbb{Q}_p$, and let S be an affinoid algebra over K. Recall that $\mathcal{R}^s_K$ denotes the ring of Laurent series with coefficients in K in a variable T convergent on the annulus $0 < v_p(T) \leq 1/s$, and that $\mathcal{R}_K = \bigcup_{s>0} \mathcal{R}^s_K$. By a *vector bundle* over $S \mathbin{\widehat{\otimes}}_K \mathcal{R}^s_K$, we will mean a coherent locally free sheaf over the product of this annulus with $M(S \otimes_K K)$ in the category of rigid analytic spaces over K. (In case S is disconnected, we insist that the rank be constant, not just locally constant.) By a vector bundle over $S \mathbin{\widehat{\otimes}}_K \mathcal{R}_K$, we will mean an object in the direct limit as $s \to \infty$ of the categories of vector bundles over $S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathcal{R}^s_K$.

Recall that for s sufficiently large, we can produce an isomorphism

$$\mathbf{B}^{\dagger,s}_{\mathrm{rig},K} \cong \mathcal{R}^s_{K'_0}.$$

We thus obtain the notion of a vector bundle over $S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}^{\dagger,s}_{\mathrm{rig},K}$, dependent on the choice of the isomorphism. However, the notion of a vector bundle over $S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}^{\dagger}_{\mathrm{rig},K}$ does not depend on any choices.

Remark 6.2. For $S = K$ discretely valued, every vector bundle over $\mathcal{R}^s_K$ is freely generated by global sections [Kedlaya 2005a, Theorem 3.4.1]. On the other hand, for S an affinoid algebra over $\mathbb{Q}_p$, we do not know whether any vector bundle over $S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathcal{R}^s_K$ is S-locally free; this does not follow from [Lütkebohmert 1977], which only applies to closed annuli.

Definition 6.3. Let K be a finite extension of $\mathbb{Q}_p$, and let S be an affinoid algebra over $\mathbb{Q}_p$. By a *family of* (φ, Γ)*-modules* over $S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}^{\dagger}_{\mathrm{rig},K}$, we mean a vector bundle V over $S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}^{\dagger}_{\mathrm{rig},K}$ equipped with an isomorphism $\varphi^* V \to V$, viewed as a semilinear φ-action, and a semilinear Γ-action commuting with the φ-action. Call a family of (φ, Γ)-modules over $S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}^{\dagger}_{\mathrm{rig},K}$ *étale* if it arises by base extension from an étale (φ, Γ)-module over $S \mathbin{\widehat{\otimes}}_{\mathbb{Q}_p} \mathbf{B}^{\dagger}_K$, called an *étale model* of the family.

It turns out that étale models are unique when they exist. To check this without any reducedness hypothesis on S, we need a generalization of the fact that a reduced affinoid algebra embeds into a product of complete fields [Berkovich 1990, Proposition 2.4.4].

Lemma 6.4. *Let K be a finite extension of $\mathbb{Q}_p$, and let S be an affinoid algebra over $\mathbb{Q}_p$. Then there exists a strict inclusion $S \to \prod_{i=1}^n A_i$ of topological rings, in which each A_i is a finite connected algebra over a complete discretely valued field.*

Proof. Let T be the multiplicative subset of $\mathcal{O}_S$ consisting of elements whose images in $\mathcal{O}_S/I_S$ are not zero divisors. For any $s \in S$ and $t \in T$, we have $|st| = |s||t|$, so the norm on S extends uniquely to the localization $S[T^{-1}]$. The completion of this localization has the desired form. □

Proposition 6.5. *Let K be a finite extension of $\mathbb{Q}_p$, and let S be an affinoid algebra over $\mathbb{Q}_p$. Then the natural base change functor from étale (φ, Γ)-modules over $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}^{\dagger}_K$ to families of (φ, Γ)-modules over $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}^{\dagger}_{\mathrm{rig},K}$ is fully faithful. In fact, this holds even without the Γ-action.*

Proof. Note that if we replace S by a complete discretely valued field L, we may deduce the analogous claim by [Kedlaya 2005b, Theorem 6.3.3] after translating notations. (Families of (φ, Γ)-modules over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ are finite free over $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$, by Remark 6.2.) In fact, if we replace S by a finite algebra over L, we may make the same deduction by restricting scalars to L. We may thus deduce the original claim by embedding S into a product of finite algebras over complete discretely valued fields using Lemma 6.4. □

Corollary 6.6. *Let K be a finite extension of $\mathbb{Q}_p$, and let S be an affinoid algebra over $\mathbb{Q}_p$. Then an étale model of a family of (φ, Γ)-modules over $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}^{\dagger}_{\mathrm{rig},K}$ is unique if it exists.*

Definition 6.7. Let S be an affinoid algebra over $\mathbb{Q}_p$. Let V_S be a locally free S-linear representation. We define $\mathrm{D}^{\dagger}_K(V_S)$ as in Definition 3.12, then put

$$\mathrm{D}^{\dagger}_{\mathrm{rig},K}(V_S) = \mathrm{D}^{\dagger}_K(V_S) \otimes_{S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}^{\dagger}_K} (S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}^{\dagger}_{\mathrm{rig},K}).$$

This is an étale (φ, Γ)-module over $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}^{\dagger}_{\mathrm{rig},K}$, from which we may recover V_S by taking

$$V_S = (\mathrm{D}^{\dagger}_{\mathrm{rig},K}(V_S) \otimes_{S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}^{\dagger}_{\mathrm{rig},K}} (S \widehat{\otimes}_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}))^{\varphi=1}.$$

We may now obtain Theorem 0.1 by combining Theorem 4.3 (via Definition 3.12) with Proposition 6.5.

7. Local étaleness

We now turn to Theorem 0.2 of the introduction. Given what we already have proven, this can be obtained by invoking some results from [Liu 2008]. For the convenience of the reader, we recall these results in detail.

Lemma 7.1. *Let K be a finite extension of $\mathbb{Q}_p$, and let S be an affinoid algebra over K. For any $x \in M(S)$ and $\lambda > 0$, there exists an affinoid subdomain $M(B)$ of $M(S)$ containing x such that if $f \in S$ vanishes at x, then $|f(y)| \leq \lambda |f|_S$ for any $y \in M(B)$.*

Proof. We first prove the lemma for $S = T_n = K\langle x_1, \dots, x_n \rangle$, the n-dimensional Tate algebra over K. It is harmless to enlarge K, so we may suppose without loss of generality that x is the origin $x_1 = \cdots = x_n = 0$. Choosing a rational number

$\lambda' < \lambda$, the affinoid subdomain $\{(x_1, \dots, x_n) \in M(S) \,\big|\, |x_1| \le \lambda', \dots, |x_n| \le \lambda'\}$ satisfies the required property.

For general S, the reduction $\overline{S} = \mathcal{O}_S/\mathfrak{m}_K\mathcal{O}_S$ is a finite type scheme over the residue field k of K. For n sufficiently large, we take a surjective k-algebra homomorphism $\overline{\alpha}: k[\overline{x}_1, \dots, \overline{x}_n] \twoheadrightarrow \overline{S}$. We lift $\overline{\alpha}$ to a K-affinoid algebra homomorphism $\alpha : K\langle x_1, \dots, x_n\rangle \to S$ by mapping x_i to a lift of $\overline{\alpha}(\overline{x}_i)$ in $\mathcal{O}_S$. Then it follows from Nakayama's lemma that α maps $\mathcal{O}_K\langle x_1, \dots, x_n\rangle$ onto $\mathcal{O}_S$. Let α also denote the induced map from $M(S)$ to $M(K\langle x_1, \dots, x_n\rangle)$. By the case of $K\langle x_1, \dots, x_n\rangle$, we can find an affinoid neighborhood $M(B)$ of $\alpha(x)$ satisfying the required property for λ/p. Now for any nonzero $f \in S$ vanishing at x, we choose $c \in \mathbb{Q}$ such that $|c| \le |f|_S \le p|c|$, yielding $pf/c \in \mathcal{O}_S$. Pick $f' \in \mathcal{O}_K\langle x_1, \dots, x_n\rangle$ such that $\alpha(f') = pf/c$. Then $f'(\alpha(x)) = (pf/c)(x) = 0$ implies that $|f'(y)| \le (\lambda/p)|f'|_{T_n} \le \lambda/p$ for any $y \in M(B)$. Then for any $y \in \alpha^{-1}(M(B))$, we have $|pf(y)|/|c| = |f'(\alpha(y))| \le \lambda/p$, yielding $|f(y)| \le \lambda|c| \le \lambda|f|_S$. Hence $\alpha^{-1}(M(B))$ is an affinoid neighborhood of x satisfying the property we need. □

Definition 7.2. For S a commutative Banach algebra over $\mathbb{Q}_p$ and I a subinterval of $\mathbb{R}$, let $\mathcal{R}_S^I$ be the ring of Laurent series over S in the variable T convergent for $v(T)^{-1} \in I$. Let v_S be the valuation on S, and for $s \in I$ and $x = \sum_i x_i T^i \in \mathcal{R}_S^I$ put

$$w_s(x) = \inf_i \{i + s v_S(x_i)\}.$$

Put $\mathcal{R}_S^s = \mathcal{R}_S^{[s,+\infty)}$, which we may identify with the completed tensor product $S \,\widehat{\otimes}_{\mathbb{Q}_p} \mathcal{R}_{\mathbb{Q}_p}^s$ for the Fréchet topology on the right, and put $\mathcal{R}_S = \bigcup_{s>0} \mathcal{R}_S^s$. Let $\mathcal{R}_S^{\mathrm{int},s}$ be the subring of $\mathcal{R}_S^s$ consisting of series with coefficients in $\mathcal{O}_S$.

Lemma 7.3 (based on [Kedlaya 2005b, Lemma 6.1.1]). *Let K be a finite extension of $\mathbb{Q}_p$, and let S be an affinoid algebra over K. Pick $s_0 > 0$. Let $\varphi : \mathcal{R}_S^{s_0/p} \to \mathcal{R}_S^{s_0}$ be a map of the form $\sum_i c_i T^i \mapsto \sum_i \varphi_S(c_i) W^i$, where $\varphi_S : S \to S$ is an isometry and $W \in \mathcal{R}_S^{s_0}$ satisfies $w_{s_0}(W - T^p) > w_{s_0}(T^p)$. For some $s \ge s_0$, suppose D is an invertible $n \times n$ matrix over $\mathcal{R}_S^{[s,s]}$, and put $h = -w_s(D) - w_s(D^{-1})$; it is clear that $h \ge 0$. Let F be an $n \times n$ matrix over $\mathcal{R}_S^{[s,s]}$ such that $w_s(FD^{-1} - I_n) \ge c + h/(p-1)$ for a positive number c. Then for any positive integer k satisfying $2(p-1)sk \le c$, there exists an invertible $n \times n$ matrix U over $\mathcal{R}_S^{[s/p,s]}$ such that $U^{-1}F\varphi(U)D^{-1} - I_n$ has entries in $p^k\mathcal{R}_S^{\mathrm{int},s}$ and $w_s(U^{-1}F\varphi(U)D^{-1} - I_n) \ge c + h/(p-1)$.*

Proof. For $i \in \mathbb{R}$, $s > 0$, $f = \sum_{j=-\infty}^{+\infty} a_j T^j \in \mathcal{R}_S$, we set

$$v_i(f) = \min\{j : v_S(a_j) \le i\} \quad \text{and} \quad v_{i,s}(f) = v_i(f) + si.$$

It is clear that $v_{i,s}(f) \ge w_s(f)$. (If S is a field, these quantities are similar to v_i^{naive}, $v_{i,r}^{\mathrm{naive}}$ in [Kedlaya 2005b, p. 458], albeit with a slightly different normalization.)

We define a sequence of invertible matrices $U_0, U_1, \dots$ over $\mathcal{R}_S^{[s/p,s]}$ and a sequence of matrices $F_0, F_1, \dots$ over $\mathcal{R}_S^{[s,s]}$ as follows. Set $U_0 = I_n$. Given U_l, put $F_l = U_l^{-1} F\varphi(U_l)$. Suppose

$$F_l D^{-1} - I_n = \sum_{m=-\infty}^{\infty} V_m T^m,$$

where the V_m's are $n\times n$ matrices over S. Let $X_l = \sum_{v_S(V_m)<k} V_m T^m$, and put $U_{l+1} = U_l(I_n + X_l)$. Set

$$c_l = \inf_{i\leq k-1} \{v_{i,s}(F_l D^{-1} - I_n) - h/(p-1)\}.$$

We now prove by induction that $c_l \geq ((l+1)/2)c$, $w_s(F_l D^{-1} - I_n) \geq c + h/(p-1)$ and U_l is invertible over $\mathcal{R}_S^{[s/p,s]}$ for any $l \geq 0$. This is obvious for $l = 0$. Suppose that the claim is true for some $l \geq 0$. Then for any $t \in [s/p, s]$, since

$$c_l \geq \frac{l+1}{2} c \geq (p-1)sk,$$

we have

$$w_t(X_l) \geq w_s(X_l) - (s-t)k \geq (c_l + h/(p-1)) - (s-t)k > 0.$$

Hence U_{l+1} is also invertible over $\mathcal{R}_S^{[s/p,s]}$. Furthermore, we have

$$\begin{aligned} w_s(D\varphi(X_l)D^{-1}) &\geq w_s(D) + w_s(\varphi(X_l)) + w_s(D^{-1}) = p w_{s/p}(X_l) - h \\ &> p\Big(c_l + \frac{h}{p-1}\Big) - h - (p-1)sk = pc_l + \frac{h}{p-1} - (p-1)sk \\ &\geq c_l + \tfrac{1}{2}c + \frac{h}{p-1} + (\tfrac{1}{2}c - (p-1)sk) \geq \frac{l+2}{2}c + \frac{h}{p-1}, \end{aligned}$$

since $c_l \geq \frac{l+1}{2} c$ by the inductive assumption. Note that

$$\begin{aligned} F_{l+1}D^{-1} - I_n &= (I_n + X_l)^{-1} F_l D^{-1}(I_n + D\varphi(X_l)D^{-1}) - I_n \\ &= ((I_n + X_l)^{-1} F_l D^{-1} - I_n) + (I_n + X_l)^{-1}(F_l D^{-1}) D\varphi(X_l) D^{-1}. \end{aligned}$$

Since $w_s(F_l D^{-1}) \geq 0$ and $w_s((I_n + X_l)^{-1}) \geq 0$, we have

$$w_s((I_n + X_l)^{-1}(F_l D^{-1}) D\varphi(X_l) D^{-1}) \geq \frac{l+2}{2} c + \frac{h}{p-1}.$$

Write

$$\begin{aligned} (I_n + X_l)^{-1} F_l D^{-1} - I_n &= (I_n + X_l)^{-1}(F_l D^{-1} - I_n - X_l) \\ &= \sum_{j=0}^{\infty} (-X_l)^j (F_l D^{-1} - I_n - X_l). \end{aligned}$$

For $j \geq 1$, we have

$$w_s((-X_l)^j(F_l D^{-1} - I_n - X_l)) \geq c + c_l + \frac{2h}{p-1} > \frac{l+2}{2}c + \frac{h}{p-1}.$$

By the definition of X_l, we also have $v_i(F_l D^{-1} - I_n - X_l) = \infty$ for $i < k$ and $w_s(F_l D^{-1} - I_n - X_l) \geq c + h/(p-1)$. Putting these together, we get that

$$v_{i,s}(F_{l+1}D^{-1} - I_n) \geq \frac{l+2}{2}c + \frac{h}{p-1}$$

for any $i < k$, that is, $c_{l+1} \geq \dfrac{l+2}{2}c$, and that $w_s(F_{l+1}D^{-1} - I_n) \geq c + \dfrac{h}{p-1}$. The induction step is finished.

Now since $w_t(X_l) \geq c_l + h/(p-1) - (p-1)ps/k$ for $t \in [s/p, s]$, and $c_l \to \infty$ as $l \to \infty$, the sequence U_l converges to a limit U, which is an invertible $n \times n$ matrix over $\mathcal{R}_S^{[s/p,s]}$ satisfying $w_s(U^{-1}F\varphi(U)D^{-1} - I_n) \geq c + h/(p-1)$. Furthermore,

$$\begin{aligned} v_{m,s}(U^{-1}F\varphi(U)D^{-1} - I_n) &= \lim_{l\to\infty} v_{m,s}(U_l^{-1}F\varphi(U_l)D^{-1} - I_n) \\ &= \lim_{l\to\infty} v_{m,s}(F_{l+1}D^{-1} - I_n) = \infty, \end{aligned}$$

for any $m < k$. Therefore $U^{-1}F\varphi(U)D^{-1} - I_n$ has entries in $p^k\mathcal{R}_S^{\mathrm{int},s}$. □

Theorem 7.4. *Let S be an affinoid algebra over $\mathbb{Q}_p$, and let M_S be a family of (φ, Γ)-modules over $S \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}^{\dagger}_{\mathrm{rig},K}$, such that for some $x \in M(S)$ whose residue field is contained in S, the fiber M_x of M_S over x is étale. Then there exist an affinoid neighborhood $M(B)$ of x and a finite extension L of K such that the base extension M_B of M_S to $B \widehat{\otimes}_{\mathbb{Q}_p} \mathbf{B}^{\dagger}_{\mathrm{rig},L}$ has an étale model in which the entries of the matrix of $\varphi - 1$ have positive p-adic valuation.*

Proof. Because Proposition 6.5 does not require the Γ-action, it suffices to construct an étale model just for the φ-action. Choose an isomorphism

$$\mathbf{B}^{\dagger,s_0}_{\mathrm{rig},K} \cong \mathcal{R}^{s_0}_{K_0'}$$

for some $s_0 > 0$, via which φ induces a map from $\mathcal{R}^{s_0/p}_{K_0'}$ to $\mathcal{R}^{s_0}_{K_0'}$ satisfying

$$w_{s_0}(\varphi(T) - T^p) > w_{s_0}(T^p).$$

Choose $s \geq s_0$ such that M_S is represented by a vector bundle V_S over $S \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{R}^{s/p}_{K_0'}$ equipped with an isomorphism $\varphi^* V_S \to V_S$ of vector bundles over $S \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{R}^{s}_{K_0'}$.

By hypothesis, M_x is étale. After increasing s, we may therefore assume that M_x admits a basis e_x on which φ acts via an invertible matrix over $\mathcal{R}^{\mathrm{int},s}_{S/\mathfrak{m}_x}$. Lift this matrix to a matrix D over $\mathcal{R}^{\mathrm{int},s}_S$, using the inclusion $S/\mathfrak{m}_x \hookrightarrow S$ which was assumed to exist. By enlarging K, we can ensure that $D - 1$ has positive p-adic valuation (by first doing so modulo $\mathfrak{m}_x$).

By results of Lütkebohmert [1977, Sätze 1 and 2], the restriction of V_S to $S \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{R}^{[s/p,s]}_{K'_0}$ is S-locally free. By replacing $M(S)$ with an affinoid subdomain containing x, we may reduce to the case where this restriction admits a basis e_S. Let A be the matrix via which φ acts on this basis; it has entries in $S \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{R}^{[s,s]}_{K'_0}$. Let V be a matrix over $S \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{R}^{[s/p,s]}_{K'_0}$ lifting (again using the inclusion $S/\mathfrak{m}_x \hookrightarrow S$) the change-of-basis matrix from the mod-$\mathfrak{m}_x$ reduction of e_S to e_x.

By Lemma 7.1, we can shrink S so as to make D invertible over $\mathcal{R}^{\text{int},s}_S$. We can also force V to become invertible, and we may make $V^{-1}A\varphi(V) - D$ as small as desired. We may thus put ourselves in a position to apply Lemma 7.3 with $F = V^{-1}A\varphi(V)$, to produce an invertible $n \times n$ matrix U over $S \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{R}^{[s/p,s]}_{K'_0}$ such that

$$W = U^{-1}F\varphi(U)D^{-1} - I_n$$

has entries in $p\mathcal{O}_S \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{R}^{\text{int},s}_{K'_0}$ and $w_s(W) > 0$.

Changing basis from e_S via the matrix VU gives another basis e'_S of V_S over $S \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{R}^{[s/p,s]}_{K'_0}$, on which φ acts via the matrix $W + I_n$.

We may change the basis e'_S using $(W + I_n)D$ to get a new basis of V_S over $S \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{R}^{[s,ps]}_{K'_0}$; since the matrix $(W + I_n)D$ is invertible over $\mathcal{O}_S \widehat{\otimes}_{\mathbb{Z}_p} \mathcal{R}^{\text{int},s}_{K'_0}$, it is also the case that the basis e'_S also generates V_S over $S \widehat{\otimes}_{\mathbb{Q}_p} \mathcal{R}^{[s,ps]}_{K'_0}$. Repeating the argument, we can deduce that e'_S is actually a basis of V_S generating an étale model. This proves the claim. □

Combining Theorem 5.2 with Theorem 7.4 yields Theorem 0.2. Note that before applying Theorem 7.4, we must first extend scalars from S to $S \otimes_{\mathbb{Q}_p} L$ for $L = S/\mathfrak{m}_x$; we then use Galois descent for the action of $\text{Gal}(L/\mathbb{Q}_p)$ to recover a statement about S itself.

Remark 7.5. Unfortunately, there is no natural extension of Theorem 7.4 to the Berkovich analytic space $\mathcal{M}(S)$ associated to S. For instance, take $K = \mathbb{Q}_p$, $S = \mathbb{Q}_p\langle y \rangle$, and let M_S be free of rank 2 with the action of φ given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & y/p \end{pmatrix}$$

(in which T does not appear). The locus of $x \in M(S)$ where M_x is étale is precisely the disc $|y| \leq |p|$, which does not correspond to an open subset of $\mathcal{M}$.

On the other hand, it may still be the case that M_S is étale if and only if M_x is étale (in an appropriate sense) for each $x \in \mathcal{M}(S)$.

Remark 7.6. It should be possible to generalize Berger's construction [2008] to families of filtered (φ, N)-modules. With such a generalization, one would deduce immediately from Theorem 7.4 that any family of weakly admissible (φ, N)-modules over an affinoid base (with trivial φ-action on the base) arises from a Galois representation in a neighborhood of any given rigid analytic point. However, in view of Remark 7.5, we cannot make the corresponding assertion for Berkovich points.

Remark 7.7. The families of (φ, Γ)-modules considered here are "arithmetic" in the sense that φ acts trivially on the base S. They correspond to "arithmetic" families of Galois representations, such as the p-adic families arising in the theory of p-adic modular forms. There is also a theory of "geometric" families of (φ, Γ)-modules, in which φ acts as a Frobenius lift on the base S. These correspond to representations of arithmetic fundamental groups via the work of Faltings, Andreatta, Brinon, Iovita, et al. In this theory, one does expect the étale locus to be open, as in [Hartl 2006, Theorem 5.2]. One also expects a family of (φ, Γ)-modules to be globally étale if and only if it is étale over each Berkovich point (but not if it is only étale over each rigid point, as shown by the Rapoport–Zink spaces). We hope to consider this question in subsequent work.

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