Gentle algebras arising from surface triangulations

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We associate an algebra $A(\Gamma)$ to a triangulation Γ of a surface S with a set of boundary marking points. This algebra $A(\Gamma)$ is gentle and Gorenstein of dimension one. We also prove that $A(\Gamma)$ is cluster-tilted if and only if it is cluster-tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$, or if and only if the surface S is a disc or an annulus. Moreover all cluster-tilted algebras of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$ are obtained in this way.

1. Introduction

Among the main recent results in the fast-growing theory of cluster algebras is the paper of Fomin, Shapiro and Thurston [Fomin et al. 2008], relating triangulations of oriented surfaces to cluster algebras. This approach, which existed since the beginning of the theory [Caldero et al. 2006], was followed in [Labardini-Fragoso 2009; Schiffler 2008], among others. In the same spirit, we consider in the present paper an unpunctured oriented surface S and a finite set of points M, lying on the boundary of S and intersecting every boundary component of S. We then associate to a triangulation Γ of the marked surface (S, M) a quiver $Q(\Gamma)$, and a potential on $Q(\Gamma)$ (in the sense of [Derksen et al. 2008]), thus defining an algebra $A(\Gamma)$, namely the (noncompleted) Jacobian algebra defined by $Q(\Gamma)$ and the associated potential.

Such an algebra $A(\Gamma)$ has some very nice properties: it is always Gorenstein of dimension one, and also it is a gentle algebra in the sense of [Assem and Skowroński 1987]. In the unpunctured case studied here, our definition coincides with Labardini's definition of a quiver with potential associated to a (possibly punctured) surface [Labardini-Fragoso 2009]. But in the punctured case, one does not

get gentle algebras, or even string algebras. For instance, a once-punctured disc gives rise to cluster-tilted algebras of type $\mathbb{D}$ [Schiffler 2008].

Gentle algebras form a particularly nice subclass of the class of string algebras of [Butler and Ringel 1987] and are much investigated in the representation theory of algebras. For instance, this subclass contains the tilted algebras of type $\mathbb{A}$ and $\widetilde{\mathbb{A}}$ (see [Assem 1982] and [Assem and Skowroński 1987], respectively) and it is closed under tilting and even under derived equivalence (see [Schröer 1999] and [Schröer and Zimmermann 2003], respectively).

Our objective in this paper is twofold. Firstly, we ask which gentle algebras arise in this way, that is, are induced from triangulations of an unpunctured surface with boundary marked points. We show in Proposition 2.8 that this is the case for every gentle algebra such that every relation lies on what we call a 3-cycle with radical-square zero (see definition before Theorem 2.7 or [Buan and Vatne 2008] for the definition). Secondly, we ask which gentle algebras are cluster-tilted. The class of cluster-tilted algebras, introduced in [Buan et al. 2007], has been much investigated and is by now well-understood (see, for instance, [Assem et al. 2008a; Barot et al. 2008; Buan et al. 2006; Buan and Vatne 2008; Caldero et al. 2006; Keller 2009; Keller and Reiten 2007; Schiffler 2008]). In particular, it was shown in [Assem et al. 2008a] that every cluster-tilted algebra is the relation-extension of a tilted algebra, that is, it is the trivial extension of a tilted algebra C by the C-C-bimodule $\mathrm{Ext}^2_C(DC, C)$. We may now state the main result of this paper.

Theorem 1.1. *Let $A(\Gamma)$ be the algebra associated to the triangulation Γ of an unpunctured marked surface (S, M). Then the following statements are equivalent*:

(1) $A(\Gamma)$ *is cluster-tilted.*

(2) $A(\Gamma)$ *is cluster-tilted of type $\mathbb{A}$ and $\widetilde{\mathbb{A}}$.*

(3) $A(\Gamma)$ *is the relation-extension of a tilted algebra of type $\mathbb{A}$ and $\widetilde{\mathbb{A}}$.*

(4) *The surface S is a disc or an annulus.*

Moreover, *all cluster-tilted algebras of type $\mathbb{A}$ (or $\widetilde{\mathbb{A}}$) are of the form $A(\Gamma)$ for some triangulation of a disc S (or an annulus S, respectively).*

Actually, we prove in Theorem 3.3 that a cluster-tilted algebra is gentle if and only if it is of type $\mathbb{A}$ and $\widetilde{\mathbb{A}}$, or if and only if it is the relation-extension of a gentle tilted algebra, and the latter coincide with the tilted algebras of type $\mathbb{A}$ or $\widetilde{\mathbb{A}}$, respectively.

The case where S is a disc has already been studied in [Caldero et al. 2006], and it is known that the bound quivers of all cluster-tilted algebras of type $\mathbb{A}$ arise from triangulations of the (unpunctured) disc. These algebras have also been described explicitly in [Buan and Vatne 2008]. Also, the potential we use for defining the cluster-tilted algebras of type $\widetilde{\mathbb{A}}$ is a particular case of the potential recently defined

by Keller [2009]. However, we do not use this fact, but rather present another proof (predating Keller's result), which uses [Assem et al. 2008a] and properties of the second extension group.

The paper is organised as follows: in Section 2, we define our algebras $A(\Gamma)$ and prove their main properties in Theorem 2.7. Section 3 is devoted to the classification of the gentle cluster-titled algebras and Section 4 to the proof of our main theorem and some of its consequences. We conclude with an example of an algebra $A(\Gamma)$ that is not of polynomial growth in the sense of [Skowroński 1990].

2. Algebras arising from surface triangulations

Throughout this paper, the algebras we consider are basic connected algebras over a fixed algebraically closed field k. Unless otherwise stated, all algebras are finite-dimensional. Consequently, they are given in the form $A = kQ/I$ where Q is a quiver and I is an admissible ideal of the path algebra kQ [Assem et al. 2006]. The pair (Q, I) is called a *bound quiver*, and the algebra $A = kQ/I$ is referred to as a *bound quiver algebra*.

Given a bound quiver algebra $A = kQ/I$, for every vertex x of Q we denote by e_x the idempotent of A associated to x. Also, P_x, I_x and S_x will be the corresponding indecomposable projective module, indecomposable injective module and simple module, respectively.

We study in this section the algebra associated with a surface triangulation. For background material on oriented surfaces we refer to [Massey 1991].

The medial quiver $Q(\Gamma)$. We first recall from [Fomin et al. 2008] the construction of a quiver for every triangulation of a marked surface. Let S be an oriented surface with boundary ∂S, and let M be a nonempty finite set of points on ∂S intersecting each connected component of the boundary ∂S. In this paper, we only consider the case where there are no punctures, that is, we request that the set of marked points M be contained in the boundary ∂S. The pair (S, M) is referred to as an *unpunctured bordered surface with marked points*.

An *arc* in (S, M) is a curve γ in S such that:

- The endpoints of γ are marked points in M.
- γ does not intersect itself, except that its endpoints may coincide.
- γ intersects the boundary of ∂S only in its endpoints.
- γ does not cut out a monogon (that is, γ is not contractible into a point of M).

We call an arc γ a *boundary arc* if it cuts out a digon (that is, γ is homotopic to a curve δ on the boundary ∂S that intersects M only in its endpoints). Otherwise, γ is said to be an *internal arc*. Each arc γ is considered up to homotopy in the

class of such curves. A *triangulation* of (S, M) is a maximal collection Γ of arcs that do not intersect in the interior of S (more precisely, there are curves in their respective homotopy classes that do not intersect in the interior of S).

Proposition 2.1 [Fomin et al. 2008, (2.10)]. *In each triangulation of (S, M), the number of internal arcs is*

$$n = 6g + 3b + c - 6,$$

where g is the genus of S, b is the number of boundary components, and $c = |M|$ is the number of marked points.

This proposition also indicates that in some cases a triangulation does not exist (for instance a disc with one marked point would give $n = -2$). We consider from now on only marked surfaces (S, M) that admit a triangulation. Given a triangulation Γ, we also refer to M as the set of vertices of Γ. The triangles are the components of $S\backslash\Gamma$ with the arcs of Γ as edges.

We denote by $Q(\Gamma)$ the *medial quiver* of internal arcs of Γ. That is, $Q(\Gamma)$ is the quiver whose set of points is the set of internal arcs of Γ, and the arrows are defined as follows: whenever there is a triangle T in Γ containing two internal arcs a and b, then $Q(\Gamma)$ contains an arrow $a \rightarrow b$ if a is a predecessor of b with respect to clockwise orientation at the joint vertex of a and b in T (we can talk about clockwise orientation around each marked point because S is an oriented surface).

Example. We illustrate the construction of $Q(\Gamma)$ when Γ is a triangulation of an octagon:
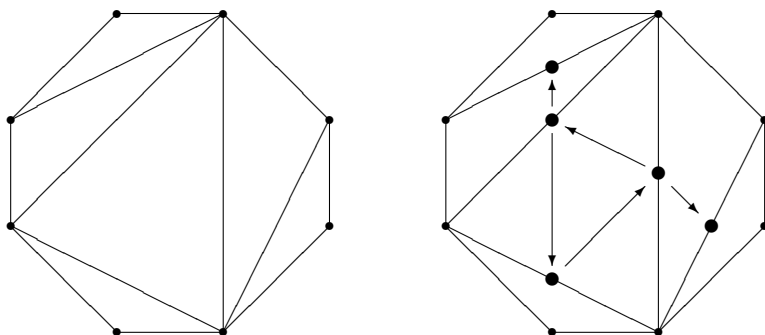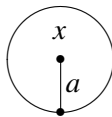

Lemma 2.2. *The quiver $Q(\Gamma)$ contains no oriented cycles of length ≤ 2.*

Proof. We first show that $Q(\Gamma)$ contains no loops. A loop α at the point a of $Q(\Gamma)$ would arise from a triangle T in Γ in the following way:
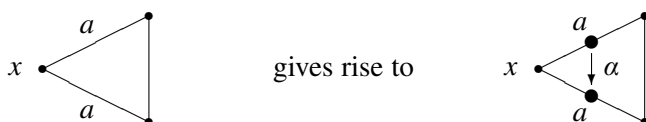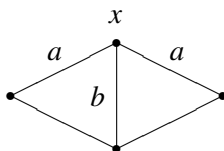
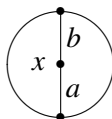But in this case the triangle T is homeomorphic to

which means that x is an internal vertex, contradicting our assumption that M is contained in the boundary of S.

We now show that $Q(\Gamma)$ contains no oriented cycles of length two. Indeed, such a cycle corresponds to the following situation in Γ:


Then a neighbourhood of x is homeomorphic to

which again contradicts the assumption that Γ contains no internal vertices. □

Remark. Fomin et al. [2008] associate a skew-symmetric matrix $B(\Gamma)$ to a triangulation Γ of (S, M). This construction is equivalent to the construction of the quiver $Q(\Gamma)$ we consider here. Since $Q(\Gamma)$ contains no oriented cycles of length ≤ 2, it is uniquely determined by a skew-symmetric matrix B (where the number of arrows between two vertices is given by the entries of B, and the direction of the arrows is determined by the sign of the matrix entries). It is easy to see that B coincides with $B(\Gamma)$. Thus all the results from [Fomin et al. 2008] apply; in particular, mutations of the quiver $Q(\Gamma)$ correspond to flips of the triangulation Γ.

Let b be an internal arc of Γ. Thus b is one diagonal of the quadrilateral formed by the two triangles of Γ that contain b. The flip of b replaces the edge b by the other diagonal b^* of the same quadrilateral. Keeping all other edges unchanged, one obtains a new triangulation $\mu_b(\Gamma)$.
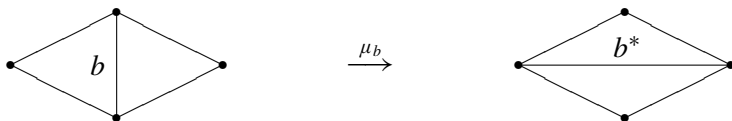
An essential ingredient in the definition of cluster algebras by Fomin and Zelevinsky [2002] is the mutation of skew-symmetric matrices. Reformulated in the language of quivers, one obtains a mutation of quivers $Q \mapsto \mu_b(Q)$. The following proposition shows that flips of the triangulation commute with quiver mutations.

Proposition 2.3 [Fomin et al. 2008, Proposition 4.8]. *Suppose that the triangula-tion* $\mu_b(\Gamma)$ *is obtained from* Γ *by a flip replacing the diagonal labelled b. Then*

$$Q(\mu_b(\Gamma)) = \mu_b(Q(\Gamma))$$

The algebra $A(\Gamma)$. We define in this section an algebra $A(\Gamma)$ for each triangula-tion Γ of the unpunctured marked surface (S, M). Our construction generalizes the one given in [Caldero et al. 2006] for polygons. An even more general case is considered in [Labardini-Fragoso 2009], where such an algebra $A(\Gamma)$ is defined for a general marked surface (allowing punctures). If there are no punctures, the definitions coincide (although Labardini works in the equivalent framework of op-posite medial quivers).

A triangle T in Γ is called an *internal triangle* if all edges of T are internal arcs. Every internal triangle T in Γ gives rise to an oriented cycle $\alpha_T \beta_T \gamma_T$ in $Q(\Gamma)$, unique up to cyclic permutation of the factors $\alpha_T, \beta_T, \gamma_T$. We define

$$W = \sum_T \alpha_T \beta_T \gamma_T,$$

where the sum runs over all internal triangles T of Γ. Then W is a potential on $Q(\Gamma)$ and we define $A(\Gamma)$ to be the (noncompleted) Jacobian algebra of (Q, W) [Derksen et al. 2008; Keller 2007]. Thus $A(\Gamma)$ can be described as a quotient $A(\Gamma) = kQ(\Gamma)/I(\Gamma)$ of the path algebra $kQ(\Gamma)$ by the ideal $I(\Gamma)$ generated by all paths $\alpha_T \beta_T$, $\beta_T \gamma_T$ and $\gamma_T \alpha_T$ whenever T is an internal triangle of Γ. Labardini [2009] showed that flips in the triangulation correspond to mutations of the quiver with potential $(Q(\Gamma), W)$ as defined in [Derksen et al. 2008].

The following result is shown in [Labardini-Fragoso 2009, Theorem 36] for the more general case of punctured marked surfaces.

Lemma 2.4. *Let Γ be a triangulation of an unpunctured marked surface (S, M). Then the algebra $A(\Gamma)$ is finite-dimensional.*

We show in Lemma 2.5 that the algebras $A(\Gamma)$ belong to a class of algebras called *gentle* algebras. Recall from [Assem and Skowroński 1987] that a finite-dimensional algebra is *gentle* if it admits a presentation $A = kQ/I$ satisfying the following conditions:

(G1) At each point of Q start at most two arrows and stop at most two arrows.

(G2) The ideal I is generated by paths of length 2.

(G3) For each arrow β there is at most one arrow α and at most one arrow γ such that $\alpha\beta \in I$ and $\beta\gamma \in I$.

(G4) For each arrow β there is at most one arrow α and at most one arrow γ such that $\alpha\beta \notin I$ and $\beta\gamma \notin I$.

If the pair (Q, I) satisfies conditions (G1) through (G4), we call it a *gentle bound quiver*, or a *gentle presentation* of $A = kQ/I$. Note that in contrast to [Assem and Skowroński 1987], we do not assume that $A = kQ/I$ is triangular. An algebra $A = kQ/I$ where I is generated by paths and (Q, I) satisfies the two conditions (G1) and (G4) is called a *string algebra* [Butler and Ringel 1987], and thus every gentle algebra is a string algebra. The gentle algebras can be characterized by the fact that their repetitive categories are special biserial [Assem and Skowroński 1987; Pogorzały and Skowroński 1991].

We recall here the classification of indecomposable modules over a string algebra $A = kQ/I$ which is given in [Butler and Ringel 1987] in terms of reduced walks in the quiver Q. A *string* in A is by definition a reduced walk w in Q avoiding the zero-relations, and thus w is a sequence

$$ w = x_1 \xleftrightarrow{\alpha_1} x_2 \xleftrightarrow{\alpha_1} \cdots \xleftrightarrow{\alpha_{n-1}} x_n, $$

where the x_i are vertices of Q and each α_i is an arrow between the vertices x_i and x_{i+1} in either direction such that w does not contain a sequence of the form

$$ \xleftarrow{\beta} \xrightarrow{\beta} \quad \text{or} \quad \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_s} $$

with $\beta_1 \cdots \beta_s \in I$, or their duals. A string is *cyclic* if the first and the last vertex coincide. A *band* is defined to be a cyclic string b such that each power b^n is a string, but b itself is not a proper power of some string c.

The *string module $M(w)$* is obtained from the string w by replacing each x_i in w by a copy of the field k. The action of an arrow α on $M(w)$ is induced by the relevant identity morphisms if α lies on w, and is zero otherwise. The dimension vector $\underline{\dim} M(w)$ of $M(w)$ is obtained by counting how often the string w passes through each vertex x of the quiver Q:

$$ \underline{\dim} M(w) = \left(\sum_{1 \le i \le n} \delta_{x,x_i} \right)_{x \in Q_0}, $$

where $\delta_{x,x_i} = 1$ for $x = x_i$ and $\delta_{x,x_i} = 0$ otherwise. Similarly, each band b in A gives rise to a family of *band modules $M(b, \lambda, n)$* where $\lambda \in k$ and $n \in \mathbb{N}$ (we refer to [Butler and Ringel 1987] for the precise definition). All string and band modules are indecomposable, and in fact every indecomposable A-module is either a string module $M(w)$ or a band module $M(b, \lambda, n)$ [Butler and Ringel 1987].

We now return to the study of algebras stemming from surface triangulations:

Lemma 2.5. *Let Γ be a triangulation of an unpunctured marked surface (S, M). Then $A(\Gamma)$ is a gentle algebra.*

Proof. By Lemma 2.4, the algebra $A(\Gamma)$ is finite-dimensional, so we only need to verify conditions (G1) to (G4) for the bound quiver $(Q(\Gamma), I(\Gamma))$ of $A(\Gamma)$.
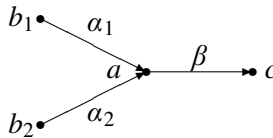
(G2): By definition, the ideal $I(\Gamma)$ is generated by paths of length two.

(G1): Let a be a point of $Q(\Gamma)$ corresponding to an internal arc a of Γ. Since Γ is a triangulation of a surface, the arc a is contained in at most two triangles:
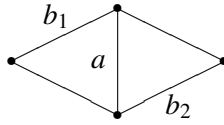
Hence there are at most two arrows $\alpha_1 : b_1 \to a$ and $\alpha_2 : b_2 \to a$ of $Q(\Gamma)$ ending in a. The same holds for arrows starting in a point a.

(G3), (G4): Suppose now that $Q(\Gamma)$ contains $\alpha_1, \alpha_2, \beta$ as follows:

We have to show that precisely one of $\alpha_1\beta$, $\alpha_2\beta$ belongs to $I(\Gamma)$. In Γ, the internal arcs a, b_1, b_2 belong to two triangles as considered in the proof of (G1). The arrow β encodes that the arc c is a successor of a in one of these triangles, say the one formed by a, b_1, c. This gives rise to the relation $\alpha_1\beta$, and $\alpha_2\beta$ does not belong to $I(\Gamma)$ since α_2 and β arise from different triangles.   □

From the construction of $A(\Gamma)$ it is clear that for each $\alpha\beta \in I(\Gamma)$ there is an arrow γ in $Q(\Gamma)$ such that $\beta\gamma \in I(\Gamma)$ and $\gamma\alpha \in I(\Gamma)$. In the following lemma we study a homological property of all gentle algebras satisfying this condition: an algebra A is *Gorenstein of dimension one* if the injective dimension of the (finitely generated) projective A-modules is at most one, and the projective dimension of the (finitely generated) injective A-modules is at most one. Note that all cluster-tilted algebras are Gorenstein of dimension one, and that an algebra of Gorenstein dimension one is either hereditary or has infinite global dimension; see [Keller and Reiten 2007].

Lemma 2.6. *Let $A = kQ/I$ be a gentle algebra such that for each $\alpha\beta \in I$ there is an arrow γ in Q such that $\beta\gamma \in I$ and $\gamma\alpha \in I$. Then A is Gorenstein of dimension one.*

Proof. We only compute the projective dimension of the injective modules here; the proof of the other part in the definition of Gorenstein of dimension one is dual. It is sufficient to show that for every vertex x of Q the corresponding indecomposable injective A-module I_x has projective dimension at most one. To do so, we construct explicitly a projective resolution of I_x. We write the string module I_x as $I_x = M(u_1\alpha_1\alpha_2^{-1}u_2^{-1})$, where u_1 and u_2 are oriented paths. Both paths might have length

zero, and in this case, also the arrows α_1 and α_2 might not be present. The following figure is used throughout the proof:
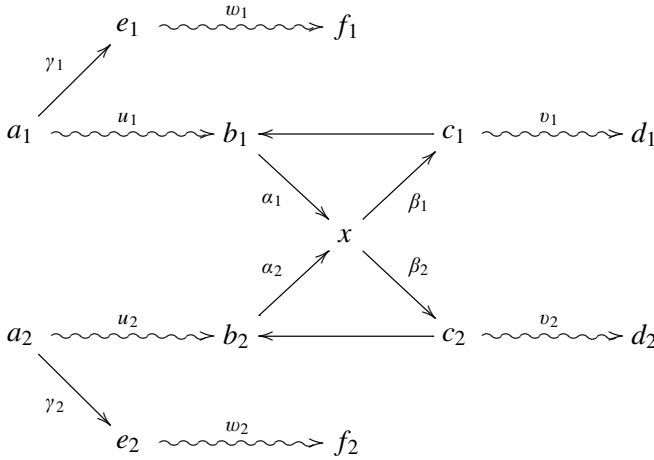
Note that $\{x, c_1, b_1\}$ and $\{x, c_2, b_2\}$ form oriented cycles in Q such that the composition of any two consecutive arrows is zero. Let

$$p_0 : P(0) \to I_x$$

be a projective cover; then

$$P(0) = M(w_1^{-1}\gamma_1^{-1}u_1\alpha_1\beta_2 v_2) \oplus M(w_2^{-1}\gamma_2^{-1}u_2\alpha_2\beta_1 v_1)$$

and

$$\operatorname{Ker} p_0 = M(w_1) \oplus M(w_2) \oplus M(v_1^{-1}\beta_1^{-1}\beta_2 v_2)$$

(note that some summands of the terms of this sequence can be zero). We show that $\operatorname{Ker} p_0$ is projective, thus obtaining the desired projective resolution

$$0 \longrightarrow \operatorname{Ker} p_0 \longrightarrow P(0) \xrightarrow{\ p_0\ } I_x \longrightarrow 0.$$

In order to see that the first two summands of $\operatorname{Ker} p_0$ are projective (namely the indecomposable projectives P_{e_1} and P_{e_2}), one has to show that there are no other arrows starting at the vertices e_1, e_2. Suppose there is an arrow $\delta_1 : e_1 \to y$ in Q. Since the algebra A is gentle, the composition $\gamma_1\delta_1$ lies in the ideal I. The assumption of the lemma guarantees the existence of a cycle $\gamma_1\delta_1\epsilon_1$ such that $\gamma_1\delta_1, \delta_1\epsilon_1, \epsilon_1\gamma_1 \in I$. But then the simple A-module S_y would be a composition factor of I_x, contradicting the assumption $I_x = M(u_1\alpha_1\alpha_2^{-1}u_2^{-1})$. This shows that $M(w_1) = P_{e_1}$, and a similar argument shows that $M(w_2) = P_{e_2}$. Since $M(v_1^{-1}\beta_1^{-1}\beta_2 v_2) = P_x$, we conclude that $\operatorname{Ker} p_0$ is projective. $\qquad\square$

Example. We illustrate the projective resolution constructed in Lemma 2.6 when Γ is the following triangulation of a polygon with 11 vertices (where the midpoints

of internal arcs are labeled):
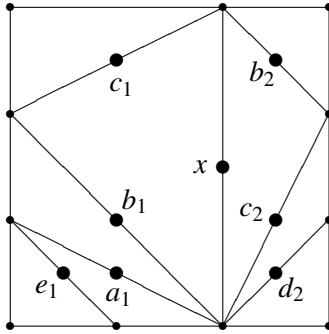
The corresponding algebra $A(\Gamma)$ is given by the quiver
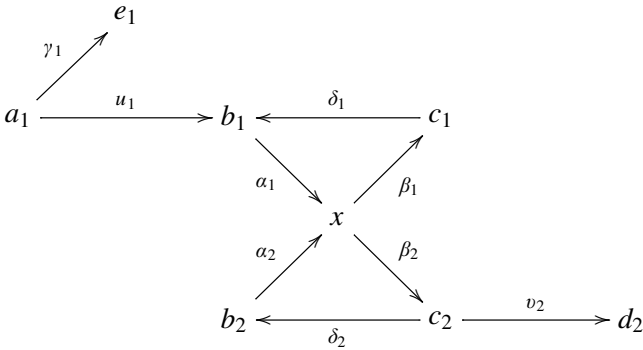
with relations $\alpha_1\beta_1 = \beta_1\delta_1 = \delta_1\alpha_1 = 0$ and $\alpha_2\beta_2 = \beta_2\delta_2 = \delta_2\alpha_2 = 0$. The projective resolution of the injective module I_x is then

$$0 \longrightarrow P_{e_1} \oplus P_x \longrightarrow P_{a_1} \oplus P_{b_2} \longrightarrow I_x \longrightarrow 0,$$

where P_{e_1} is simple and $P_x = M(\beta_1^{-1}\beta_2 v_2)$, $P_{a_1} = M(u_1\alpha_1\beta_2 v_2)$ and $P_{b_2} = M(\alpha_2\beta_1)$.

We recall from [Gabriel 1981] the concept of Galois coverings of bound quiver algebras: Let $\Lambda = k\tilde{Q}/\tilde{I}$ be a bound quiver algebra (where the quiver $\tilde{Q}$ is not necessarily finite). A group G of k-linear automorphisms of Λ is acting freely on Λ if $ge_x \neq e_x$ for each vertex x of $\tilde{Q}$ and each $g \neq 1$ in G. In this case the multiplication in Λ induces a multiplication on the set Λ/G of G-orbits which turns Λ/G into an algebra. The canonical projection $\Lambda \to \Lambda/G$ is called the *Galois covering* of Λ/G with group G.

In the following theorem we call (as in [Buan and Vatne 2008]) a *3-cycle* an oriented cycle $\alpha\beta\gamma$ where α, β, γ are three distinct arrows; and by a *3-cycle with radical square zero* we mean a 3-cycle $\alpha\beta\gamma$ in an algebra kQ/I such that $\alpha\beta, \beta\gamma, \gamma\alpha$

all lie in I. By a *simple cycle* we refer to a subquiver C of Q with n distinct vertices $\{x_0, x_1, \ldots, x_{n-1}, x_n = x_0\}$ and n arrows $\alpha_i : x_i \to x_{i+1}$, for $i = 1, \ldots, n-1$.

Theorem 2.7. *Let Γ be a triangulation of an unpunctured marked surface (S, M).*

(1) *The algebra $A(\Gamma)$ is a gentle algebra.*

(2) *The algebra $A(\Gamma)$ is Gorenstein of dimension one.*

(3) *There is a relation in $A(\Gamma)$ from x to y only if there is an arrow $y \to x$.*

(4) *$A(\Gamma)$ admits a Galois covering by a bound quiver algebra $k\tilde{Q}/\tilde{I}$ satisfying:*

   (T1) *Every simple cycle in $\tilde{Q}$ is a 3-cycle with radical square zero.*

   (T2) *The only relations in $\tilde{I}$ are those in the 3-cycles.*

Proof. Part (1) is shown in Lemma 2.5, and part (2) is shown in Lemma 2.6 since the condition imposed on the gentle algebra A there clearly holds for the algebra $A(\Gamma)$. Part (3) follows directly from the definition of $A(\Gamma)$. Maybe the most intuitive way to obtain the Galois covering required in part (4) is the following. By construction, the only relations in the algebra $A(\Gamma)$ are those in the 3-cycles. In a first step, we identify all 3-cycles to points, replacing each 3-cycle C with vertices $\{x_1, x_2, x_3\}$ by one single vertex x and replacing each arrow $y \to x_i$ (or $x_i \to y$, respectively) by an arrow $y \to x$ (or $x \to y$, respectively). The quiver $\bar{Q}$ thus obtained contains no relations, and we let $\bar{\bar{Q}}$ be its universal Galois covering, a (maybe infinite) tree. The bound quiver $(\tilde{Q}, \tilde{I})$ is then obtained by placing back the 3-cycles $C = \{x_1, x_2, x_3\}$ for all contracted vertices x of $\bar{\bar{Q}}$. $\qquad\square$

Note that the finite quivers satisfying conditions (T1) and (T2) from the previous theorem form precisely the class of quivers $\mathcal{Q}_n$ considered in [Buan and Vatne 2008], where also the same relations are imposed. It would be interesting to relate the Galois covering $(\tilde{Q}, \tilde{I})$ constructed above with the universal cover of the bordered surface (S, M).

Recovering topological data from $A(\Gamma)$. The condition (4) in Theorem 2.7 is very strong. Combined with the fact that the algebra is gentle, it implies the remaining conditions (2) and (3). We show in this section that a gentle algebra satisfying condition (4) is given by an unpunctured marked surface.

First we give a different combinatorial description of the algebras studied here. Consider the following two bound quivers, where type I is a quiver of type $\mathbb{A}_2$, and type II is a 3-cycle with radical square zero:

Using these bound quivers one can construct algebras in the following way. Suppose we start with a collection C of disjoint blocks of types I and II. Choose

a partial matching (that is to say a partial bijection) π of the vertices in C, where matching a vertex to another vertex of the same block is not allowed. Identifying (or "gluing") the vertices within each pair of the matching we obtain an algebra $A(C, \pi)$. Note that the arrows are not identified by this procedure, so one might obtain parallel arrows or two-cycles. We consider only matchings where the algebra $A(C, \pi)$ is connected.

The procedure of gluing blocks is considered in a more general situation (using plenty of building blocks) in [Brüstle 2001], where the resulting algebras are called *kit algebras*. A similar construction to glue blocks of type I, II and four more types is described in [Fomin et al. 2008].

We show below that the gentle algebras that admit a Galois covering satisfying conditions (T1) and (T2) from Theorem 2.7 are algebras of the form $A(C, \pi)$, and thus results from [Fomin et al. 2008] concerning these algebras can be applied.

Proposition 2.8. *Let $A = kQ/I$ be a gentle algebra where every relation lies on a 3-cycle with radical square zero. Then there exists an unpunctured marked surface (S, M) with a triangulation Γ such that $A(\Gamma) = A$.*

Proof. The statement follows from [Fomin et al. 2008, (14.1)] once we show that the algebra A admits a unique block decomposition $A = A(C, \pi)$ using blocks of types I and II. We therefore define C to be the disjoint union of all 3-cycles with radical square zero of A together with the disjoint union of all remaining arrows from A. Denote by f the quiver morphism $f : C \to Q$ that identifies the blocks of C with their images in Q.

We first show that $|f^{-1}(x)| \leq 2$ for each vertex $x \in Q$. Indeed, if $f^{-1}(x)$ contains three different vertices, then there are three different arrows in Q adjacent to the vertex x. But since the algebra A is gentle, there has to be one relation between these three arrows. However, the set C is constructed in such a way that all relations of A belong to one of the components in C, so there are no relations between arrows corresponding to different components of C, and so the fiber $f^{-1}(x)$ contains at most two vertices.

We now define a matching π on C relating x_1 to x_2 whenever $f^{-1}(x) = \{x_1, x_2\}$. As required in the definition of $A(C, \pi)$, we do not match a vertex to itself or to some vertex in the same block. It is clear from the construction that $A = A(C, \pi)$. Moreover, the choice of blocks of type I or II is unique since all relations have to correspond to a block of type II. $\square$

We would like to point out that all algebras $A(\Gamma)$ given by a triangulation Γ of an unpunctured marked surface are of the form $A(C, \pi)$ for some C and π, but the converse is not true: One can easily produce two-cycles in an algebra $A(C, \pi)$, but this never occurs for the algebras $A(\Gamma)$ as we have shown in Lemma 2.2.

3. Gentle cluster-tilted algebras

Cluster-tilted algebras. Let Δ be an acyclic quiver. In [Buan et al. 2006] the cluster category $\mathscr{C}_\Delta$ is studied in order to obtain a categorical interpretation of the cluster variables of the cluster algebra associated with Δ. It is shown there that clusters correspond bijectively to tilting objects T in $\mathscr{C}_\Delta$. Their endomorphism rings $\operatorname{End}_{\mathscr{C}_\Delta}(T)$ are called *cluster-tilted algebras of type* Δ. They were introduced and studied in [Buan et al. 2007].

Here we use a different description that has been given in [Assem et al. 2008a]. Let A be the hereditary algebra $k\Delta$. An A-module T is called a *tilting module* if $\operatorname{Ext}^1_A(T, T) = 0$ and the number of isomorphism classes of indecomposable summands of T equals the number of isomorphism classes of simple A-modules. In this case, the endomorphism ring $\operatorname{End}_A(T)$ is called a *tilted algebra of type* Δ.

Let C be an algebra of global dimension two. The trivial extension

$$\tilde{C} = C \ltimes \operatorname{Ext}^2_C(DC, C)$$

of C by the C-C-bimodule $\operatorname{Ext}^2_C(DC, C)$ is called the *relation-extension* of C. It is useful to describe explicitly the operations on $\tilde{C}$. As an abelian group, $\tilde{C} = C \oplus \operatorname{Ext}^2_C(DC, C)$. Therefore, let (c, e) and (c', e') be two elements of $\tilde{C}$, where e and e' are respectively represented by the exact sequences of C-modules

$$e : \qquad 0 \longrightarrow P \longrightarrow M \longrightarrow N \longrightarrow I \longrightarrow 0,$$
$$e' : \qquad 0 \longrightarrow P' \longrightarrow M' \longrightarrow N' \longrightarrow I' \longrightarrow 0,$$

with P, P' projective and I, I' injective. The addition is given by

$$(c, e) + (c', e') = (c + c', e + e'),$$

where the sum $c + c'$ is the ordinary sum inside C, while $e + e'$ is the Baer sum in $\operatorname{Ext}^2_C(DC, C)$ (for which we refer to any textbook of homological algebra). The product in $\tilde{C}$ is given by the formula

$$(c, e)(c', e') = (cc', ce' + ec'),$$

where the product cc' is the ordinary product inside C, while ce' and $e'c$ are defined as follows. Viewing $c \in C$ as an element of $\operatorname{End} C_C \cong C$, then $e_1 = ce'$ is represented by the sequence obtained by pulling down the sequence e':

$$
\begin{array}{ccccccccccc}
e' : & \quad & 0 \longrightarrow & P' & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & I' & \longrightarrow 0 \\
& & & \downarrow c & & \downarrow & & \| & & \| & \\
e_1 : & \quad & 0 \longrightarrow & P_1 & \longrightarrow & M_1 & \longrightarrow & N' & \longrightarrow & I' & \longrightarrow 0
\end{array}
$$

Similarly, if we view $c' \in C$ as an element of $\operatorname{End} DC_C \cong C$, then $e_2 = ec'$ is represented by the sequence obtained by lifting the sequence e:

$$e_2: \quad 0 \longrightarrow P \longrightarrow M \longrightarrow N_2 \longrightarrow I_2 \longrightarrow 0$$

$$e: \quad 0 \longrightarrow P \longrightarrow M \longrightarrow N \longrightarrow I \longrightarrow 0$$

The following theorem allows us to view cluster-tilted algebras as relation-extensions of tilted algebras.

Theorem 3.1 [Assem et al. 2008a]. *An algebra Λ is cluster-tilted of type Δ if and only if there exists a tilted algebra C of type Δ such that Λ is isomorphic to the relation-extension $\tilde{C}$ of C.*

Every cluster-tilted algebra satisfies conditions (2) and (3) from Theorem 2.7 [Keller and Reiten 2007]. The bound quivers of cluster-tilted algebras of type $\mathbb{A}$ are explicitly described in [Buan and Vatne 2008, Proposition 3.1]. In fact they were already described in [Caldero et al. 2006] as the algebras $A(\Gamma)$ arising from a triangulation of an unpunctured polygon. The following proposition is contained in [Buan and Vatne 2008], but it also follows from Theorem 3.3 below.

Proposition 3.2 [Buan and Vatne 2008, (3.1)]. *An algebra A is cluster-tilted of type $\mathbb{A}$ precisely when A is gentle and there is a presentation $A = kQ/I$ which satisfies conditions (T1) and (T2) from Theorem 2.7.*

In particular, the cluster-tilted algebras of type $\mathbb{A}$ are gentle. We describe in the following theorem, whose proof occupies the rest of the section, which of the gentle algebras are cluster-tilted:

Theorem 3.3. *Let $C = kQ_C/I_C$ be a tilted algebra, and $\tilde{C}$ be its relation-extension. The following are equivalent.*

(1) *C is gentle.*

(2) *C is tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$.*

(3) *$\tilde{C}$ is gentle.*

(4) *$\tilde{C}$ is cluster-tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$.*

A preliminary part of the proof follows from a result in [Schröer 1999], which says that the class of gentle algebras is stable under tilting.

Lemma 3.4. *If a tilted algebra is gentle, then it is tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$.*

Proof. Let Δ be a quiver such that C is tilted of type Δ. Then there exists a tilting C-module T such that $\operatorname{End} T = k\Delta$. According to [Schröer 1999], $k\Delta$ is a gentle algebra. This implies that the quiver Δ is of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$. $\square$

Lemma 3.5. *If $\tilde{C}$ is gentle, then so is C.*

Proof. This follows from the fact that $\tilde{C}$ is a split extension of C and from [Assem et al. 2008b, (2.7)]. $\square$

Lemma 3.6. *The algebra C is tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$ if and only if $\tilde{C}$ is cluster-tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$.*

Proof. Clearly, if C is tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$, then $\tilde{C}$ is cluster-tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$. Conversely, suppose $\tilde{C}$ is cluster-tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$. By [Assem et al. 2008a] there exists a local slice Σ' in mod $\tilde{C}$ such that $C' = \tilde{C}/\text{Ann}\,\Sigma'$ is tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$. On the other hand, since $\tilde{C} = C \ltimes \text{Ext}_C^2(DC, C)$, then there exists a local slice Σ in mod $\tilde{C}$ such that $C = \tilde{C}/\text{Ann}\,\Sigma$. Since both Σ and Σ' have the same underlying graph, C and C' have the same type, so C is tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$. $\square$

The main part of the proof of Theorem 3.3 is concerned with the problem of showing that $\tilde{C}$ is gentle if C is tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$. This will be done next.

Relation-extensions of tilted algebras of types $\mathbb{A}$ and $\tilde{\mathbb{A}}$. Suppose $C = kQ_C/I_C$ is tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$. In particular, C is gentle because of [Assem 1982] and [Assem and Skowroński 1987]. Moreover, the quiver of $\tilde{C}$ is known, as are some of its relations, namely those already in C [Assem et al. 2008a; Assem et al. 2008b]. The aim here is to study the remaining relations of $\tilde{C}$.

First, the bound quiver of a tilted algebra of type $\mathbb{A}$ has been described in [Assem 1982], and that of a tilted algebra of type $\tilde{\mathbb{A}}$ in [Roldán 1983]. The criterion given here is derived from [Huard and Liu 2000].

We recall that a *double-zero* in a gentle algebra is a reduced walk of the form $\alpha\beta\omega\gamma\delta$, where α, β, γ and δ are arrows such that $\alpha\beta$ and $\gamma\delta$ are relations, while ω is a nonzero reduced walk (that is, a walk which does not contain any relation). Note that ω may be trivial and that in this case β and γ may coincide.

Example. The algebra
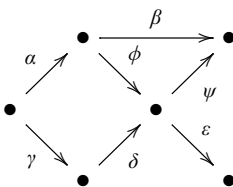
where $\alpha\beta = \phi\psi = \delta\varepsilon = 0$, is gentle with a double-zero (namely $\phi\psi\beta^{-1}\phi\psi$).

Proposition 3.7 [Assem 1982; Assem and Skowroński 1987]. (1) *An algebra is tilted of type $\mathbb{A}$ if and only if it admits a bound quiver presentation kQ/I, with (Q, I) a gentle tree with no double-zero.*

(2) *An algebra is tilted of type $\tilde{\mathbb{A}}$ if and only if it admits a bound quiver presentation kQ/I, with (Q, I) a gentle presentation with no double-zero and a unique (nonoriented) cycle such that, if the cycle is a band, then all arrows attached to the cycle either enter it or leave it.*

Example. Consider the algebras given by the bound quivers

$$\alpha\beta = 0 \qquad\qquad \alpha\beta = \gamma\delta = 0$$
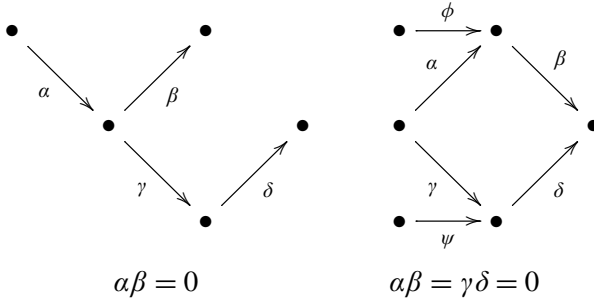
Using [Proposition 3.7](#), we see that the first one is tilted of type $\mathbb{A}$, while the second one is tilted of type $\tilde{\mathbb{A}}$.

A vanishing criterion. We need a criterion to verify whether a given exact sequence represents the zero element in the second extension group:

Lemma 3.8 [Happel et al. 1996, (II.1.3)]. *Given a morphism $f : M \longrightarrow N$, the exact sequence*

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow M \xrightarrow{\ f\ } N \longrightarrow \operatorname{Coker} f \longrightarrow 0$$

represents the zero element of $\operatorname{Ext}^2(\operatorname{Coker} f, \operatorname{Ker} f)$ *if and only if there exist a module X and morphisms g, h such that the sequence*

$$0 \longrightarrow M \xrightarrow{(p,g)^t} \operatorname{Im} f \oplus X \xrightarrow{(j,h)} N \longrightarrow 0$$

is exact, where p and j are the natural morphisms arising from f.

The following lemma will be used frequently.

Lemma 3.9. *Let (Q, I) be a gentle presentation of an algebra C, and let $\alpha : c \longrightarrow b$ and $\beta : b \longrightarrow a$ be arrows in Q. Let σ and η be strings, not passing through b, such that $\beta\sigma$ and $\eta\alpha$ are strings. Let $f : M(\beta\sigma) \longrightarrow M(\eta\alpha)$ be a morphism such that $\operatorname{Im} f = S_b$.*

Then the exact sequence

$$e : \qquad 0 \longrightarrow \operatorname{Ker} f \longrightarrow M(\beta\sigma) \xrightarrow{\ f\ } M(\eta\alpha) \longrightarrow \operatorname{Coker} f \longrightarrow 0$$

represents a nonzero element of $\operatorname{Ext}^2_C(\operatorname{Coker} f, \operatorname{Ker} f)$ *if and only if $\alpha\beta$ lies in I.*

Proof. In view of [Lemma 3.8](#), the sequence *e* represents a nonzero element of $\operatorname{Ext}^2_C(\operatorname{Coker} f, \operatorname{Ker} f)$ if and only if there exists no short exact sequence of the form

$$0 \longrightarrow M(\beta\sigma) \xrightarrow{(p,g)^t} \operatorname{Im} f \oplus X \xrightarrow{(j,h)} M(\eta\alpha) \longrightarrow 0,$$

Assume such a sequence exists. Since S_b appears exactly once as a composition factor of $M(\beta\sigma)$ and $M(\eta\alpha)$, then it also appears exactly once as a composition factor of X. Therefore, there exists a unique indecomposable summand Y of X admitting S_b as a composition factor.

We claim that $g : M(\beta\sigma) \longrightarrow X$ is a monomorphism: let $x \in Q_0$ and take a vector $v \in M(\beta\sigma)_x$ such that $g_x(v) = 0$. If $x \neq b$, then $p_x(v) = 0$ and so $(p, g)_x^t(v) = 0$ which implies $v = 0$. If $x = b$, then $(p_a(\beta v), g_a(M(\beta\sigma)_\beta(v)))^t = (p, g)_a^t(M(\beta\sigma)_\beta(v)) = (\operatorname{Im} f \oplus X)_\beta(p, g)_b^t(v) = 0$ which implies $M(\beta\sigma)_\beta(v) = 0$. Since $(M(\beta\sigma)_\beta$ is injective, then $v = 0$. This completes the proof of our claim.

Since the evaluation $M(\beta\sigma)_\beta$ of the module $M(\beta\sigma)$ on the arrow β is nonzero, we must have $X_\beta \neq 0$. Now, S_b is a composition factor of Y, hence $Y_\beta \neq 0$ as well. Similarly, h is an epimorphism and it follows that $Y_\alpha \neq 0$. On the other hand, Y must be a string or a band module. The above reasoning implies that $\alpha\beta$ must then be a subpath of a string or a band, which implies that $\alpha\beta \notin I$, as required.

Conversely, if $\alpha\beta \notin I$, then we have a short exact sequence

$$0 \longrightarrow M(\beta\sigma) \longrightarrow S_b \oplus M(\eta\alpha\beta\sigma) \longrightarrow M(\eta\alpha) \longrightarrow 0,$$

and hence e represents the zero element in $\operatorname{Ext}^2_C(\operatorname{Coker} f, \operatorname{Ker} f)$. $\qquad\square$

Arrows. From now on, let C be a tilted algebra of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$. We give a description of the elements of $\tilde{C} = C \ltimes \operatorname{Ext}^2_C(DC, C)$ corresponding to the arrows of its ordinary quiver. In [Assem et al. 2008a, (2.4)], it is proved that the quiver of $\tilde{C}$ is obtained from that of C by adding an arrow from x to y for each relation from y to x. The elements of $\tilde{C}$ corresponding to the arrows of C are of the form $(\alpha, 0)$, where α is an arrow of C.

The other arrows correspond to relations in C. Let $\alpha\beta$ be a relation from c to a in C, and let $\xi_{\alpha\beta}$ be the corresponding new arrow in $\tilde{C}$.

Lemma 3.10. *The new arrow $\xi_{\alpha\beta}$ lies in $0 \oplus \operatorname{Ext}^2_C(I_c, P_a)$.*

Proof. This new arrow lies in $e_a \tilde{C} e_c$, which can be written as the direct sum of $e_a C e_c$ and $e_a \operatorname{Ext}^2_C(DC, C) e_c$. We know from Proposition 3.7 that the quiver of C contains no double-zero. Consequently, there are no paths from a to c, and hence $e_a C e_c = 0$. Moreover, $e_a \operatorname{Ext}^2_C(DC, C) e_c = \operatorname{Ext}^2_C(I_c, P_a)$. The element $\xi_{\alpha\beta}$ thus lies in $0 \oplus \operatorname{Ext}^2_C(I_c, P_a)$. $\qquad\square$

The following lemma gives the dimension and a basis of the extension space involved in the last expression.

Lemma 3.11. *Let $\alpha : c \longrightarrow b$ and $\beta : b \longrightarrow a$ be two arrows of C such that $\alpha\beta \in I_C$.*

(a) *The dimension of the vector space* $\mathrm{Ext}_C^2(I_c, P_a)$ *is* 1 *or* 2. *Its dimension is* 2 *if and only if the following situation occurs in the bound quiver of* C:
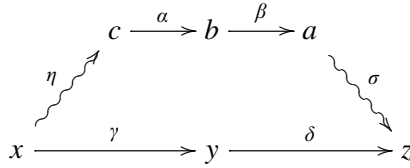
where γ *and* δ *are arrows,* η *and* σ *are paths, possibly stationary, without relations, and* $\alpha\beta$, $\gamma\delta$ *are relations.*

(b) *If the dimension of the space is* 1, *then a basis is given by the sequence*

$$\mathbf{e}_1: \qquad 0 \longrightarrow P_a \longrightarrow M(\beta\sigma) \longrightarrow M(\eta\alpha) \longrightarrow I_c \longrightarrow 0,$$

where η *and* σ *are paths such that* $I_c = M(\eta)$ *and* $P_a = M(\sigma)$.

(c) *If the dimension of the space is* 2, *then a basis is given by the sequences*

$$\mathbf{e}_1: \qquad 0 \longrightarrow P_a \longrightarrow M(\beta\sigma) \longrightarrow M(\eta\alpha) \longrightarrow I_c \longrightarrow 0,$$

$$\mathbf{e}_2: \qquad 0 \longrightarrow P_a \longrightarrow M(\sigma\delta^{-1}) \longrightarrow M(\gamma^{-1}\eta) \longrightarrow I_c \longrightarrow 0,$$

where γ, δ, η *and* σ *are as in the figure in part* (a).

Proof. (a) It is known from [Assem et al. 2008a] that there is a new arrow from a to c; thus the dimension cannot be zero. On the other hand, since C is gentle and without double-zero, the local situation of the relation $\alpha\beta$ can be described by the following figure, where dotted lines represent relations.
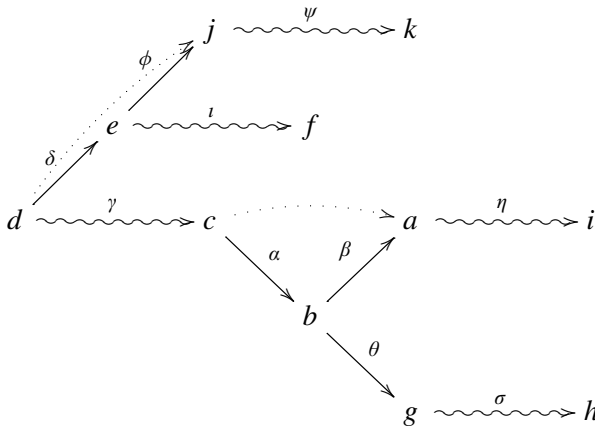
This diagram allows us to compute a projective resolution of I_c in mod C:

$$0 \xrightarrow{p_3} P(2) \xrightarrow{p_2} P(1) \xrightarrow{p_1} P(0) \xrightarrow{p_0} I_c \longrightarrow 0,$$

where $P(2) = M(\psi) \oplus M(\eta)$, $P(1) = M(\iota^{-1}\phi\psi) \oplus M(\sigma^{-1}\theta^{-1}\beta\eta)$ and $P(0) = M(\iota^{-1}\delta^{-1}\gamma\alpha\theta\sigma)$. Note that some direct summands of the terms of this sequence can be zero. Applying $\mathrm{Hom}_C(-, P_a)$, we get a complex

$$0 \longrightarrow \mathrm{Hom}_C(I_c, P_a) \xrightarrow{(p_0, P_a)} \mathrm{Hom}_C(P(0), P_a)$$

$$\xrightarrow{(p_1, P_a)} \mathrm{Hom}_C(P(1), P_a) \xrightarrow{(p_2, P_a)} \mathrm{Hom}_C(P(2), P_a) \xrightarrow{(p_3, P_a)} 0.$$

This yields

$$\mathrm{Ext}_C^2(I_c, P_a) = \frac{\mathrm{Ker}\,\mathrm{Hom}(p_3, P_a)}{\mathrm{Im}\,\mathrm{Hom}(p_2, P_a)} = \frac{\mathrm{Hom}(M(\psi), P_a) \oplus \mathrm{Hom}(M(\eta), P_a)}{\mathrm{Im}\,\mathrm{Hom}(p_2, P_a)}.$$

Since $P_a = M(\eta)$, we have $\dim \mathrm{Hom}(M(\eta), P_a) = 1$, and since

$$\mathrm{Hom}(M(\sigma^{-1}\theta^{-1}\beta\eta), P_a) = 0,$$

no nonzero morphism in $\mathrm{Hom}(M(\eta), P_a)$ factors through p_2.

We claim that $\mathrm{Hom}(M(\psi), P_a)$ is nonzero if and only if $j = i$. Indeed, a nonzero morphism from $M(\psi)$ to P_a can only exist when j coincides with a vertex on the path η. But if j were a vertex different from i, then there would be an arrow $\phi' : j \to j'$ in the path η, forcing the relation $\phi\phi''$ and creating the double-zero $\delta\phi\phi'$. Thus $j = i$. In this case, ψ has no choice but to be the trivial path in i, and $\dim \mathrm{Hom}(M(\psi), P_a) = 1$. Since $\mathrm{Hom}(M(\iota^{-1}\phi\psi, P_a) = 0$, no nonzero morphism in $\mathrm{Hom}(M(\psi), P_a)$ factors through p_2.

Hence no nonzero morphism in $\mathrm{Hom}(M(\psi), P_a) \oplus \mathrm{Hom}(M(\eta), P_a)$ factors through p_2. Thus the dimension of this space is either 1 or 2, and it is 2 exactly when $i = j$. In this case, and in this case only, we have

as desired.

(b) It follows from Lemma 3.9 that e_1 is nonzero. The result follows.

(c) It follows from Lemma 3.9 that e_1 and e_2 are nonzero. It remains to be shown that e_1 and e_2 are linearly independent. Suppose there exists a nonzero scalar λ such that $e_2 + \lambda e_1 = 0$. Computing this sum, we get the sequence

$$0 \longrightarrow P_a \longrightarrow M(\beta\sigma\delta^{-1}) \xrightarrow{f} M(\gamma^{-1}\eta\alpha) \longrightarrow I_c \longrightarrow 0,$$

where all morphisms are multiples of the natural morphisms between string modules.

Here, applying Lemma 3.9 is not possible, since $\mathrm{Im}\, f = S_b \oplus S_y$, but a similar technique of proof can be used.

Suppose there exist a module X and morphisms g and h such that the sequence

$$0 \longrightarrow M(\beta\sigma\delta^{-1}) \xrightarrow{(p,g)^t} (S_b \oplus S_y) \oplus X \xrightarrow{(j,h)} M(\gamma^{-1}\eta\alpha) \longrightarrow 0$$

is exact, where $f = jp$ is the canonical factorisation. Since S_b appears exactly once as a composition factor of $M(\beta\sigma\delta^{-1})$ and $M(\gamma^{-1}\eta\alpha)$, it also appears exactly once as a composition factor of X. Therefore, there exists a unique indecomposable summand Y of X admitting S_b as a composition factor. As in the proof of Lemma 3.9, we show that $Y_\beta \neq 0$ and $Y_\alpha \neq 0$. Therefore, $\alpha\beta$ must be a subpath of a string or a band, which is a contradiction, since it is a relation.

The sequences e_1 and e_2 thus form a basis of the extension space.   □

It remains to determine which of the basis elements are represented by arrows of $\tilde{C}$.

Lemma 3.12. *Let $\alpha : c \longrightarrow b$ and $\beta : b \longrightarrow a$ be two arrows of the quiver of C such that $\alpha\beta$ is a relation. Let $\xi_{\alpha\beta}$ be the corresponding new arrow in $\tilde{C}$. With the notation of Lemma 3.11, the element $(0, e_1)$ can be chosen to represent $\xi_{\alpha\beta}$.*

Proof. The space $0 \oplus \mathrm{Ext}^2_C(I_c, P_a)$ contains at least one arrow.

If its dimension is 1, the result is obvious.

If its dimension is 2, Lemma 3.11 describes the situation of $\alpha\beta$ in the quiver of C. Two cases arise.

First, suppose that η and σ are both trivial paths.
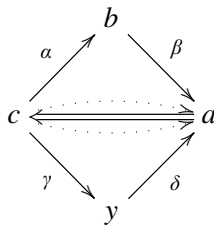
In this case, two arrows from a to c are added to the quiver. Both $(0, e_1)$ and $(0, e_2)$ must thus represent arrows of $\tilde{C}$.

Second, suppose η and σ are not both trivial. In this case, Lemma 3.11 implies that $\mathrm{Ext}^2_C(I_x, P_z)$ is of dimension 1, and that a basis is given by

$$e' : \qquad 0 \longrightarrow P_z \longrightarrow M(\delta) \longrightarrow M(\gamma) \longrightarrow I_x \longrightarrow 0.$$

Reasoning as above, we get that $(0, e')$ represents the new arrow from z to x. Moreover, a straightforward calculation yields $(\sigma, 0)(0, e')(\eta, 0) = (0, e_2)$.

Since one of η and σ is not trivial, one of $(0, \eta)$ and $(0, \sigma)$ must lie in rad $\tilde{C}$. Therefore $(0, e_2) \in \text{rad}^2\tilde{C}$, and $(0, e_1) \in \text{rad } \tilde{C} \setminus \text{rad}^2\tilde{C}$; in other words, $(0, e_1)$ represents an arrow from a to c. □

Relations. Knowing how to write arrows in $\tilde{C}$ allows us to compute the relations.

Lemma 3.13. *Let $C = kQ_C/I_C$ and $\tilde{C} = kQ_{\tilde{C}}/I_{\tilde{C}}$.*

(1) *Let $\omega_1, \omega_2, \ldots, \omega_n$ be paths from x to y in the quiver of C, and let $\lambda_1, \lambda_2, \ldots, \lambda_n \in k$. Then $\sum_{i=1}^n \lambda_i(\omega_i, 0) = 0$ in $\tilde{C}$ if and only if $\sum_{i=1}^n \lambda_i\omega_i = 0$ in C.*

(2) *Let $\alpha : c \longrightarrow b$ and $\beta : b \longrightarrow a$ be two arrows in the quiver of C such that $\alpha\beta$ is a relation. Let $(0, e_1)$ be the element representing the corresponding new arrow, where e_1 is as in Lemma 3.11. Then $(0, e_1)(\alpha, 0) = 0$ and $(\beta, 0)(0, e_1) = 0$.*

(3) *The ideal $I_{\tilde{C}}$ is generated by the relations of C and those described in (2).*

Proof. (1) This is shown in [Assem et al. 2008b].

(2) Viewing α as an element of End DC, or more precisely as a morphism from I_b to I_c, we can compute $e_1\beta$:

$$e_1\beta : \quad 0 \longrightarrow P_a \longrightarrow M(\beta\sigma) \longrightarrow M(\eta\alpha) \oplus M(\varphi\gamma) \longrightarrow I_b \longrightarrow 0,$$

where $I_b = M(\eta\alpha\gamma^{-1}\varphi^{-1})$. This sequence represents the zero element, because of Lemma 3.8 and exactness of the sequence

$$0 \longrightarrow M(\beta\sigma) \longrightarrow S_b \oplus M(\varphi\gamma\beta\sigma) \oplus M(\eta\alpha) \longrightarrow M(\varphi\gamma) \oplus M(\eta\alpha) \longrightarrow 0.$$

Therefore $(0, e_1)(\alpha, 0) = 0$.

In a dual way, we prove that $(\beta, 0)(0, e_1) = 0$.

(3) It is sufficient to show that new arrows in the quiver of $\tilde{C}$ are not involved in other relations than those described in (2).

First suppose that w is a monomial relation involving new arrows and relations other than those described in (2). Then it must contain exactly one new arrow ξ, corresponding to a relation $\alpha\beta$; otherwise the quiver of C would contain a double-zero. Write $w = u\xi v$, where u and v are nonzero paths consisting of arrows of C. Let e_1 be the sequence as in Lemma 3.11 corresponding to ξ. Then $(u, 0)(0, e_1)(v, 0) = (0, ue_1v)$, where ue_1v is the sequence

$$0 \longrightarrow M(u^{-1}u') \longrightarrow M(\beta u^{-1}u') \longrightarrow M(v'v^{-1}\alpha) \longrightarrow M(v'v^{-1}) \longrightarrow 0,$$

where u' and v' are paths in the quiver of C. The figure at the top of the next page illustrates the local situation, where $\alpha\beta = \gamma'\delta' = 0$; the last arrow of u and the first

of t form a relation, as do the last of v and v' and the first of w' and w, respectively.
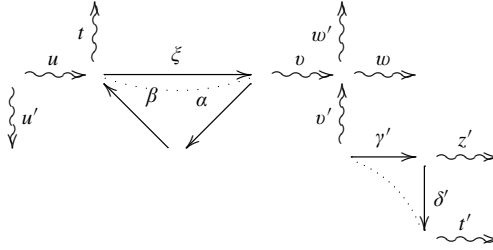
This yields the following commutative diagram, where the first line is a projective resolution of $M(v'v^{-1})$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & P(2) & \xrightarrow{\ p_2\ } & P(1) & \xrightarrow{\ p_1\ } & P(0) & \xrightarrow{\ p_0\ } & M(v'v^{-1}) \longrightarrow 0 \\
 & & \downarrow{\scriptstyle (f,0)} & & \downarrow{\scriptstyle (0,g,0)} & & \downarrow{\scriptstyle (h,\ell)} & & \parallel \\
0 & \longrightarrow & M(u^{-1}u') & \longrightarrow & M(\beta u^{-1}u') & \longrightarrow & M(v'v^{-1}\alpha) & \longrightarrow & M(v'v^{-1}) \longrightarrow 0,
\end{array}
$$

where $P(2) = M(t) \oplus M(t')$, $P(1) = M(w'w) \oplus M(z^{-1}\gamma^{-1}\beta t) \oplus M(z'^{-1}\delta't')$ and $P(0) = M(z^{-1}\gamma^{-1}\alpha^{-1}vw) \oplus M(z'^{-1}\gamma'^{-1}v'w')$, and all nonzero morphisms are the natural morphisms between string modules. It is then seen that $(f,0)$ cannot factor through p_2, and thus the lower exact sequence is nonzero. Hence there are no other monomial relations than those in (2).

Now suppose we have a minimal relation of the form $\sum_{i=1}^{m} \lambda_i w_i$, where each λ_i is a nonzero scalar, each w_i is a path in the quiver of $\tilde{C}$, and $m \geq 2$. At least one of the w_i must pass through a new arrow, and since C contains no double zero, this implies that each w_i must pass through exactly one new arrow, say ξ_i, corresponding to a relation $\alpha_i \beta_i$. Write $w_i = u_i \xi_i v_i$, where u_i and v_i are paths of the quiver of C.

Since the quiver of C contains at most one cycle, we must have $m = 2$. Since k is a field, we may suppose that $\lambda_1 = 1$. Letting e_1 and e_2 be the sequences associated to ξ_1 and ξ_2, respectively, we get that $u_1 e_1 v_1$ and $\lambda_2 u_2 e_2 v_2$ are both sequences of the form above. Their sum is the sequence

$$
0 \longrightarrow M(u_2^{-1}u_1) \longrightarrow M(\beta_2 u_2^{-1}u_1\beta_1^{-1})
$$
$$
\longrightarrow M(\alpha_2^{-1}v_2 v_1^{-1}\alpha_1) \longrightarrow M(v_1 v_2^{-1}) \longrightarrow 0.
$$

By an argument similar to the one given in the proof of Lemma 3.11(c), this element is not zero; a contradiction. Hence no binomial relations exist in $\tilde{C}$. □

The relations described in the preceding lemma make $\tilde{C}$ a gentle algebra.

Lemma 3.14. *If $\tilde{C}$ is cluster-tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$, then $\tilde{C}$ is gentle.*

Proof. The relations of $\tilde{C}$ are known (see Lemma 3.13). Moreover, C is gentle.

Suppose that there are r new arrows. Let us add the new arrows and the corresponding new relations one by one, thus obtaining a sequence $C = C_0, C_1, \ldots,$ $C_r = \tilde{C}$ of algebras. We show that C_i is gentle for all i in $\{0, 1, 2, \ldots, r\}$.

Since C is gentle, then so is C_0. Suppose that C_i is gentle, where i is in $\{0, 1, 2, \ldots, r-1\}$. To get C_{i+1}, we add one new arrow, say γ from x to y. This arrow comes from a relation $\alpha\beta$ from y to x in C. We must add the relations $\beta\gamma$ and $\gamma\alpha$ to obtain C_{i+1}.

Since C_i is gentle, there were already at most two arrows starting from x in C_i. Suppose that there were two, say η_1 and η_2. Since C_i is gentle, then β is involved in a relation with one of the two, say η_1. The arrow η_1 cannot be in C, otherwise there would be a double zero involving $\alpha\beta$ and $\beta\eta_1$. So the arrow η_1 comes from a relation $\sigma\beta$ in C. Since C is gentle, we must have that $\sigma = \alpha$, so that $\eta_1 = \gamma$, which is absurd because γ is not in C_i.

Therefore, in C_i, there is at most one outgoing arrow from x, and this arrow is not involved in a relation with β. This shows that in C_{i+1}, there are at most two arrows starting from x, say η and γ, and that $\beta\eta$ is not a relation while $\beta\gamma$ is. Moreover, there is at most one more arrow ending in x, say δ, and since C_i is gentle, we have that $\delta\eta$ is a relation, while $\delta\gamma$ is not. So the relations at x are those found in a gentle algebra.
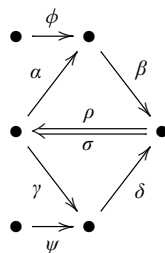
Using a similar argument for the vertex y, we get that C_{i+1} is a gentle algebra. By induction, $\tilde{C}$ is a gentle algebra. $\qquad\square$

Example. Lemma 3.13 allows us to compute the relation-extension of any gentle tilted algebra. As an illustration, consider the two algebras given in the example on page 216. The relation-extension of each is given in the following diagram:
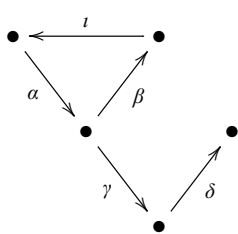
$$\alpha\beta = \iota\alpha = \beta\iota = 0$$

$$\alpha\beta = \gamma\delta = \rho\alpha = 0$$

$$\beta\rho = \sigma\gamma = \delta\sigma = 0$$

Proof of the main theorem. Now the proof of Theorem 3.3, developed in separate parts over the last several pages, can be stated properly.

Proof of Theorem 3.3. That (1) implies (2) is shown in Lemma 3.4, that (2) implies
(4) in Lemma 3.6, that (4) implies (3) in Lemma 3.14, and that (3) implies (1) in
Lemma 3.5.   □

4. Geometry of surfaces and $A(\Gamma)$

We study in this section more connections between geometric properties of the
marked surface (S, M) and properties of the algebra $A(\Gamma)$ given by a triangulation
of (S, M).

Cluster-tilted algebras arising from surfaces. We first address the question of
which of the algebras $A(\Gamma)$ are cluster-tilted. Recall that all algebras $A(\Gamma)$ share the
properties (2) and (3) from Theorem 2.7 with every cluster-tilted algebra. More-
over, it is shown in [Caldero et al. 2006] and [Buan and Vatne 2008] that the
cluster-tilted algebras of type $\mathbb{A}$ are algebras $A(\Gamma)$ arising from a triangulation of
an unpunctured polygon. In this section, we show the following generalization:

Theorem 4.1. *Let $A(\Gamma)$ be the algebra associated to the triangulation Γ of an
unpunctured marked surface (S, M). Then the following statements are equivalent*:

(1) *The algebra $A(\Gamma)$ is cluster-tilted.*

(2) *The algebra $A(\Gamma)$ is cluster-tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$.*

(3) *S is a disc or an annulus.*

Moreover, *all cluster-tilted algebras of type $\mathbb{A}$ (or $\tilde{\mathbb{A}}$) are of the form $A(\Gamma)$ for some
triangulation Γ of a disc S (or an annulus S, respectively*).

Proof. It is clear that (2) implies (1). Let us show the converse: Suppose that the
algebra $A(\Gamma)$ is cluster-tilted. Thus there is a sequence of mutations transforming
the quiver with potential defining $A(\Gamma)$ into some quiver Q with zero potential.
This sequence of mutations corresponds to a sequence of flips, transforming the
triangulation Γ of (S, M) into a triangulation T with $Q(T) = Q$ and zero potential.
Hence $A(T) = kQ$ is hereditary. Since we know from Theorem 2.7 that $A(T)$ is
gentle, this leaves only the possibilities that Q is of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$. Therefore the
algebra $A(\Gamma)$ is cluster-tilted of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$.

We prove now the equivalence of (2) and (3). Since all triangulations on (S, M)
are flip-equivalent [Hatcher 1991] and flips of the triangulation correspond to mu-
tations of the corresponding quiver with potential [Labardini-Fragoso 2009], it is
sufficient to consider one particular triangulation. In the case where S is a disc, we
choose the triangulation to be in the form of a fan, giving rise to a linear oriented
quiver of type $\mathbb{A}$. In the case where S is an annulus, we choose the triangulation
given by two fans in opposite direction as shown in the figure at the top of the next
page (where the left and right vertical edge should be identified).
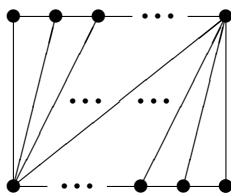
The corresponding quiver is of type $\tilde{\mathbb{A}}$ with zero potential; thus (3) implies (2). Conversely, we know from Proposition 2.8 that the quivers $Q(\Gamma)$ uniquely determine the topology of the unpunctured marked surface (S, M). Therefore S is a disc or an annulus, respectively, and since all triangulations are flip-equivalent, it is clear that all cluster-tilted algebras of the corresponding type occur. □

Curves in (S, M) and string modules. In this section we are comparing strings in $A(\Gamma)$ to curves in (S, M). By a curve in (S, M) we mean a curve γ in S whose endpoints lie in M and where all points except the endpoints lie in the interior of S. We usually consider curves up to homotopy. For instance, for two distinct curves γ and δ in (S, M), the intersection number $I_\Gamma(\gamma, \delta)$ is defined as the minimal number of transversal intersections of two representatives of the homotopy classes of γ and δ. Denote the internal arcs of the triangulation Γ by $\{a_1, \ldots, a_n\}$. Then we define the *intersection vector* $I_\Gamma(\gamma)$ of a curve γ as

$$I_\Gamma(\gamma) = (I_\Gamma(\gamma, a_1) \ldots, I_\Gamma(\gamma, a_n)).$$

Proposition 4.2. *Let Γ be a triangulation of (S, M), an unpunctured marked surface. Then there exists a bijection $\{\gamma\} \mapsto w(\gamma)$ between the homotopy classes of curves in (S, M) not homotopic to an arc in Γ and the strings of $A(\Gamma)$. Under this bijection, the intersection vector corresponds to the dimension vector of the corresponding string module, that is,*

$$I_\Gamma(\gamma) = \underline{\dim} M(w(\gamma)).$$

Proof. Let $w = x_1 \overset{\alpha_1}{\longleftrightarrow} x_2 \overset{\alpha_2}{\longleftrightarrow} \cdots \overset{\alpha_{s-1}}{\longleftrightarrow} x_s$ be a string in $A(\Gamma)$. We define a curve $\gamma(w)$ in (S, M) as follows: The arcs x_1 and x_2 belong to the same triangle T_1 since they are joined by an arrow in $A(\Gamma)$. We connect the midpoints of x_1 and x_2 by a curve γ_1 in the interior of T_1. Proceeding in the same way with the remaining arcs $x_2, \ldots, x_s$ we obtain curves $\gamma_2, \ldots, \gamma_{s-1}$ connecting the midpoints of the respective arcs. The internal arc x_1 belongs to two triangles: the triangle T_1 which we considered above and another triangle T_0. Let $P \in M$ be the marked point in T_0 opposite to the arc x_1. We now connect P with the midpoint of x_1 by a curve γ_0 in the interior of T_0, and proceed in the same way on the other end of the string w, connecting the midpoint of x_s with a marked point Q by some curve γ_s. The curve $\gamma(w)$ is then defined as the concatenation of the curves $\gamma_0, \ldots, \gamma_s$.
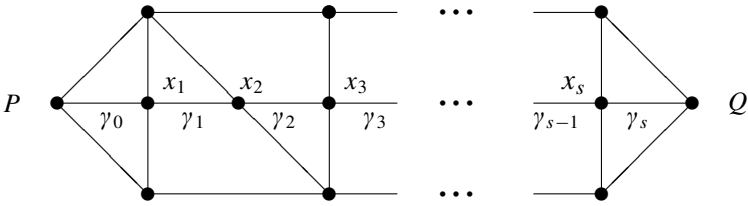
By construction, the points of intersection of the curve $\gamma(w)$ with arcs in Γ are indexed by the vertices of the string w. The curve intersects the arcs of Γ transversally, and since the string w is reduced, none of the γ_i is homotopic to a piece of an arc in Γ. Thus the intersection numbers are minimal, and $I_\Gamma(\gamma(w)) = \dim M(w)$. Since $\gamma(w)$ has nontrivial intersection with arcs of Γ, it is clear that it is not homotopic to an arc in the triangulation Γ.

Conversely, let $\gamma : [0,1] \to S$ be a curve in (S, M) which is not homotopic to an arc in Γ. We assume that the curve γ is chosen (in its homotopy class) such that it intersects the arcs a of Γ transversally (if at all) and such that the intersection numbers $I_\Gamma(\gamma, a)$ are minimal. Orienting γ from $P = \gamma(0) \in M$ to $Q = \gamma(1) \in M$, we denote by x_1 the first internal arc of Γ that intersects γ, by x_2 the second arc, and so on. We thus obtain a sequence $x_1, \ldots, x_s$ of (not necessarily different) internal arcs in Γ. Since the intersection numbers are minimal, we know that $x_i \neq x_{i+1}$. Thus there are arrows, either $\alpha_i : x_i \to x_{i+1}$ or $\alpha_i : x_{i+1} \to x_i$ in $Q(\Gamma)$, and we obtain a walk $w(\gamma) = x_1 \overset{\alpha_1}{\longleftrightarrow} x_2 \overset{\alpha_1}{\longleftrightarrow} \cdots \overset{\alpha_{s-1}}{\longleftrightarrow} x_s$ in $Q(\Gamma)$. The fact that γ intersects the arcs of Γ transversally implies that the walk $w(\gamma)$ is reduced and avoids the zero-relations, and thus $w(\gamma)$ is a string in $A(\Gamma)$.

It follows from their construction that the two maps between strings and homotopy classes of curves defined above are mutually inverse.   □

Remark. Recall that two string modules $M(w)$ and $M(v)$ are isomorphic precisely when $v = w$ or $v = w^{-1}$. The inverse string w^{-1} corresponds to orienting the curve in the opposite direction.

Proposition 4.3. *Let Γ be a triangulation of (S, M), an unpunctured marked surface. Then there exists a bijection between the homotopy classes of closed curves in (S, M) and powers b^n of bands b of $A(\Gamma)$.*

The proof is analogous to that of the previous proposition.

An example where $A(\Gamma)$ is not cluster-tilted. We finally present in this section an example of an algebra $A(\Gamma)$ which is not cluster-tilted. Recall that an algebra A is *tame* if for all $d \in \mathbb{N}$ there is a finite number n_d of one-parameter families of A-modules such that almost every d-dimensional A-module belongs to one of these n_d families. The algebra A is said to be *domestic* if there is a constant c such that $n_d \leq c$ for all $d \in \mathbb{N}$. On the other hand, if the numbers n_d grow faster than any polynomial, then the tame algebra A is said to be of *nonpolynomial growth*.

It is well-known that every string algebra A is tame, and that the one-parameter families are given by the bands in A [Butler and Ringel 1987]. In particular, all the algebras $A(\Gamma)$ are tame, because they are gentle and thus string algebras. Moreover the tame cluster-tilted algebras of the form $A(\Gamma)$ studied here are all domestic; in fact, they are of type $\mathbb{A}$ or $\tilde{\mathbb{A}}$, and thus we may assume above that $c = 0$ or $c = 1$. We construct in this section an example of an algebra $A(\Gamma)$ which is of nonpolynomial growth, and thus cannot be cluster-tilted. To obtain this example, we consider a sphere S with three holes and choose one marked point in each boundary component. We fix the following triangulation Γ of (S, M):
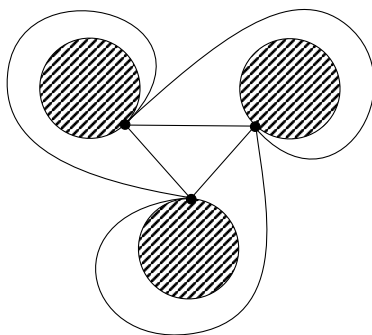
Then the algebra $A(\Gamma)$ is given by the following quiver with relations $\epsilon_i \rho_i = 0$, $\rho_i \sigma_i = 0$ and $\sigma_i \epsilon_i = 0$ for $i = 1$ and $i = 2$.
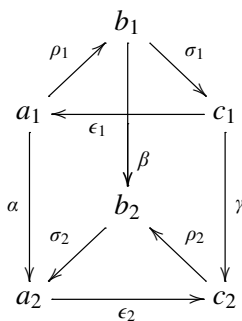
The string algebra $A(\Gamma)$ admits the two bands

$$\xi = b_2 \xrightarrow{\sigma_2} a_2 \xleftarrow{\alpha} a_1 \xrightarrow{\rho_1} b_1 \xrightarrow{\beta} b_2,$$

$$\eta = b_2 \xleftarrow{\rho_2} c_2 \xleftarrow{\gamma} c_1 \xleftarrow{\sigma_1} b_1 \xrightarrow{\beta} b_2.$$

Since ξ and η can be composed arbitrarily, the number of bands of a fixed length l grows exponentially with l, so the algebra $A(\Gamma)$ is of nonpolynomial growth.

We would like to point out that the notion of nonpolynomial growth of tame algebras discussed here does not coincide with the notion of nonpolynomial growth cluster algebras discussed in [Fomin et al. 2008]: There one counts the number of

cluster variables, that is to say, of arcs in (S, M), instead of one-parameter families, that is to say, closed curves in (S, M). In [Fomin et al. 2008] the example we are considering in this section is classified as being of polynomial growth, meaning that, even if the number of curves is growing exponentially, the number of arcs is bounded for the sphere with three holes.

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