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We introduce T-adic exponential sums associated to a Laurent polynomial f. They interpolate all classical p^m-power order exponential sums associated to f. We establish the Hodge bound for the Newton polygon of L-functions of T-adic exponential sums. This bound enables us to determine, for all m, the Newton polygons of L-functions of p^m-power order exponential sums associated to an f that is ordinary for $m = 1$. We also study deeper properties of L-functions of T-adic exponential sums. Along the way, we discuss new open problems about the T-adic exponential sum itself.

1. Introduction

Classical exponential sums. We first recall the definition of classical exponential sums over finite fields of characteristic p with values in a p-adic field.

Let p be a fixed prime number, $\mathbb{Z}_p$ the ring of p-adic integers, $\mathbb{Q}_p$ the field of p-adic numbers, and $\overline{\mathbb{Q}}_p$ a fixed algebraic closure of $\mathbb{Q}_p$. Let $q = p^a$ be a power of p, $\mathbb{F}_q$ the finite field of q elements, $\mathbb{Q}_q$ the unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$, and $\mathbb{Z}_q$ the ring of integers of $\mathbb{Q}_q$.

Fix a positive integer n. Let $f(x) \in \mathbb{Z}_q[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent polynomial in n variables of the form

$$f(x) = \sum_u a_u x^u, \quad \text{where } a_u \in \mu_{q-1} \text{ and } x^u = x_1^{u_1} \cdots x_n^{u_n};$$

here μ_k denotes the group of k-th roots of unity in $\overline{\mathbb{Q}}_p$.

Definition 1.1. Let ψ be a locally constant character of $\mathbb{Z}_p$ of order p^m with values in $\overline{\mathbb{Q}}_p$, and let $\pi_\psi = \psi(1) - 1$. The sum

$$S_{f,\psi}(k) = \sum_{x \in \mu_{q^k-1}^n} \psi(\mathrm{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(f(x)))$$

is called a p^m-power order exponential sum on the n-torus $\mathbb{G}_m^n$ over $\mathbb{F}_{q^k}$. The

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generating function

$$L_{f,\psi}(s) = L_{f,\psi}(s; \mathbb{F}_q) = \exp\left(\sum_{k=1}^{\infty} S_{f,\psi}(k)\frac{s^k}{k}\right) \in 1 + s\mathbb{Z}_p[\pi_\psi][[s]]$$

is the L-function of p^m-power order exponential sums over $\mathbb{F}_q$ associated to $f(x)$.

For $m \geq 1$ this is still an exponential sum over a finite field, since we are just summing over the subset of roots of unity (corresponding to the elements of a finite field via the Teichmüller lifting), not over the whole finite residue ring $\mathbb{Z}_q/p^m\mathbb{Z}_q$. The exponential sum over the whole finite ring $\mathbb{Z}_q/p^m\mathbb{Z}_q$ and its generating function as m varies is the subject of Igusa's zeta function [1978].

In general, $L_{f,\psi}(s)$ is rational in s. However, $L_{f,\psi}(s)^{(-1)^{n-1}}$ is a polynomial if f is nondegenerate, as shown in [Adolphson and Sperber 1989; 1987] for ψ of order p, and in [Liu and Wei 2007] for all ψ. By a result of [Gel'fand et al. 1994], if p is large enough, then f is generically nondegenerate. For nondegenerate f, the location of the zeros of $L_{f,\psi}(s)^{(-1)^{n-1}}$ becomes an important issue. The p-adic theory of such L-functions was developed by Dwork [1960], Bombieri [1966], Adolphson and Sperber [1989; 1987], the second author [Wan 1993; 2004], and Blache [2008] for ψ of order p. Recently, the initial part of the theory was extended to all ψ by Liu and Wei [2007] and Liu [2007].

The p-adic theory of the above exponential sum for $n = 1$ and ψ of order p has a long history and has been studied extensively in the literature. For instance, in the simplest case that $f(x) = x^d$, the exponential sum was studied by Gauss; see [Berndt and Evans 1981] for a comprehensive survey. By the Hasse–Davenport relation for Gauss sums, the L-function is a polynomial whose zeros are given by roots of Gauss sums. Thus, the slopes of the L-function are completely determined by the Stickelberger theorem for Gauss sums. The roots of the L-function have explicit p-adic formulas in terms of p-adic Γ-function via the Gross–Koblitz formula [1979]. These ideas can be extended to treat the so-called diagonal f case for general n; see [Wan 2004]. These elementary cases have been used as building blocks to study the deeper nondiagonal $f(x)$ via various decomposition theorems, which are the main ideas of Wan [1993; 2004]. In the case $n = 1$ and ψ of order p, more facts about the slopes of the L-function were found in [Zhu 2003; 2004a; Blache and Férard 2007; Liu 2008].

T-adic exponential sums. We now define the T-adic exponential sum, state our main results, and put forward some new questions.

Definition 1.2. For a positive integer k, the T-adic exponential sum of f over $\mathbb{F}_{q^k}$ is the sum

$$S_f(k, T) = \sum_{x \in \mu_{q^k-1}^n} (1+T)^{\mathrm{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(f(x))} \in \mathbb{Z}_p[[T]].$$

The T-adic L-function of f over $\mathbb{F}_q$ is the generating function

$$L_f(s,T) = L_f(s,T;\mathbb{F}_q) = \exp\left(\sum_{k=1}^{\infty} S_f(k,T)\frac{s^k}{k}\right) \in 1 + s\mathbb{Z}_p[\![T]\!][\![s]\!].$$

The T-adic exponential sum interpolates classical exponential sums of p^m-order over finite fields for all positive integers m. In fact, we have

$$S_f(k,\pi_\psi) = S_{f,\psi}(k).$$

Similarly, one can recover the classical L-function of the p^m-order exponential sum from the T-adic L-function by the formula

$$L_f(s,\pi_\psi) = L_{f,\psi}(s).$$

We view $L_f(s,T)$ as a power series in the single variable s with coefficients in the complete discrete valuation ring $\mathbb{Q}_p[\![T]\!]$ with uniformizer T.

Definition 1.3. The T-adic characteristic function of f over $\mathbb{F}_q$, or C-function of f for short, is the generating function

$$C_f(s,T) = \exp\left(\sum_{k=1}^{\infty} -(q^k-1)^{-n} S_f(k,T)\frac{s^k}{k}\right) \in 1 + s\mathbb{Z}_p[\![T]\!][\![s]\!].$$

The C-function $C_f(s,T)$ and the L-function $L_f(s,T)$ determine each other. They are related by

$$L_f(s,T) = \prod_{i=0}^{n} C_f(q^i s,T)^{(-1)^{n-i-1}\binom{n}{i}}, \quad C_f(s,T)^{(-1)^{n-1}} = \prod_{j=0}^{\infty} L_f(q^j s,T)^{\binom{n+j-1}{j}}.$$

In Section 4, we prove:

Theorem 1.4 (analytic continuation). *The C-function $C_f(s,T)$ is T-adic entire in s. As a consequence, the L-function $L_f(s,T)$ is T-adic meromorphic in s.*

This theorem tells us that the C-function behaves T-adically better than the L-function. In fact, in the T-adic setting, the C-function is a more natural object than the L-function. Thus, we shall focus more on the C-function.

Knowing the analytic continuation of $C_f(s,T)$, we are then interested in the location of its zeros. More precisely, we would like to determine the T-adic Newton polygon of this entire function $C_f(s,T)$. This is expected to be a complicated problem in general. It is open even in the simplest case $n=1$, and $f(x)=x^d$ is a monomial if $p \not\equiv 1 \pmod{d}$. What we can do is to give an explicit combinatorial lower bound depending only on q and Δ, called the q-Hodge bound $\mathrm{HP}_q(\Delta)$. This polygon will be described in detail in Section 3.

Let $\mathrm{NP}_T(f)$ denote the T-adic Newton polygon of the C-function $C_f(s, T)$. In Section 5, we prove this:

Theorem 1.5 (Hodge bound). $\mathrm{NP}_T(f) \geq \mathrm{HP}_q(\Delta)$.

This theorem shall give several new results on classical exponential sums, as we shall see in Section 2. In particular, this extends in one stroke all known ordinariness results for ψ of order p to all ψ of any p-power order. It demonstrates the significance of the T-adic L-function. It also gives rise to a definition:

Definition 1.6. A Laurent polynomial f that satisfies $\mathrm{NP}_T(f) = \mathrm{HP}_q(\Delta)$ is called T-adically ordinary.

We shall show that a classically ordinary f is T-adically ordinary, but it is possible that a nonordinary f is T-adically ordinary. Thus, it remains interesting to study exactly when f is T-adically ordinary. For this reason, in Section 6, we extend the facial decomposition theorem in [Wan 1993] to the T-adic case. Let Δ be the convex closure in $\mathbb{R}^n$ of the origin and the exponents of the nonzero monomials in the Laurent polynomial $f(x)$. For any closed face σ of Δ, we let f_σ denote the sum of monomials of f whose exponent vectors lie in σ.

Theorem 1.7 (T-adic facial decomposition). *A Laurent polynomial f is T-adically ordinary if and only if for every closed face σ of Δ of codimension* 1 *not containing the origin, the restriction f_σ is T-adically ordinary.*

In Section 7, we briefly discuss the variation of the C-function $C_f(s, T)$ and its Newton polygon when the reduction of f moves in an algebraic family over a finite field. The main questions concern generic ordinariness, the generic Newton polygon, the analogue of the Adolphson–Sperber conjecture [1989], Wan's limiting conjecture [2004], and Dwork's unit root conjecture [1973] in the T-adic and π_ψ-adic case. We shall give an overview about what can be proved and what is unknown, including a number of conjectures. In summary, a lot can be proved in the ordinary case, and a lot remain to be proved in the nonordinary case.

2. Applications

In this section, we give several applications of the T-adic exponential sum to classical exponential sums.

Theorem 2.1 (integrality theorem). *We have*

$$L_f(s, T) \in 1 + s\mathbb{Z}_p[\![T]\!][\![s]\!] \quad \text{and} \quad C_f(s, T) \in 1 + s\mathbb{Z}_p[\![T]\!][\![s]\!].$$

Proof. Let $|\mathbb{G}_m^n|$ be the set of closed points of $\mathbb{G}_m^n$ over $\mathbb{F}_q$, and let $a \mapsto \hat{a}$ be the Teichmüller lifting. It is easy to check that the T-adic L-function has the Euler product expansion

$$L_f(s,T)=\prod_{x\in|\mathbb{G}_m^n|}\frac{1}{(1-(1+T)^{\mathrm{Tr}_{\mathbb{Q}_{q^{\deg x}}/\mathbb{Q}_p}(f(\hat{x}))}s^{\deg x})}\in 1+s\mathbb{Z}_p[\![T]\!][\![s]\!],$$

where $\hat{x}=(\hat{x}_1,\ldots,\hat{x}_n)$. The theorem now follows. $\square$

This proof shows that the L-function $L_f(s,T)$ is the L-function $L(s,\rho_f)$ of the continuous (p,T)-adic representation of the arithmetic fundamental group given by

$$\rho_f:\pi_1^{\mathrm{arith}}(\mathbb{G}_m^n/\mathbb{F}_q)\to \mathrm{GL}_1(\mathbb{Z}_p[\![T]\!]),\quad \mathrm{Frob}_x\mapsto(1+T)^{\mathrm{Tr}_{\mathbb{Q}_{q^{\deg x}}/\mathbb{Q}_p}(f(\hat{x}))}.$$

The rank one representation ρ_f is transcendental in nature. Its L-function $L(s,\rho_f)$ seems to be beyond the reach of ℓ-adic cohomology, where ℓ is a prime different from p. However, the specialization of ρ_f at the special point $T=\pi_\psi$ is a character of finite order. Thus, the specialization

$$L(s,\rho_f)|_{T=\pi_\psi}=L_{f,\psi}(s)$$

can indeed be studied using Grothendieck's ℓ-adic trace formula [1965]. This gives another proof that the L-function $L_{f,\psi}(s)$ is a rational function in s. But the T-adic L-function $L_f(s,T)$ itself is certainly out of the reach of ℓ-adic cohomology as it is truly transcendental.

Let $\mathrm{NP}_{\pi_\psi}(f)$ denote the π_ψ-adic Newton polygon of the C-function $C_f(s,\pi_\psi)$. The integrality of $C_f(s,T)$ immediately gives the following theorem, whose proof is obvious.

Theorem 2.2 (rigidity bound). *If ψ is nontrivial, then* $\mathrm{NP}_{\pi_\psi}(f)\geq\mathrm{NP}_T(f)$.

A natural question is to ask when $\mathrm{NP}_{\pi_\psi}(f)$ coincides with its rigidity bound.

Theorem 2.3 (transfer theorem). *If* $\mathrm{NP}_{\pi_\psi}(f)=\mathrm{NP}_T(f)$ *holds for one nontrivial ψ, then it holds for all nontrivial ψ.*

Proof. By the integrality of $C_f(s,T)$, the T-adic Newton polygon of $C_f(s,T)$ coincides with the π_ψ-adic Newton polygon of $C_f(s,\pi_\psi)$ if and only if for every vertex (i,e) of the T-adic Newton polygon of $C_f(s,T)$, the coefficients of s^i in $C_f(s,T)$ differs from T^e by a unit in $\mathbb{Z}_p[\![T]\!]^\times$. It follows that if the coincidence happens for one nontrivial ψ, it happens for all nontrivial ψ. $\square$

Definition 2.4. We call f rigid if $\mathrm{NP}_{\pi_\psi}(f)=\mathrm{NP}_T(f)$ for one (and hence for all) nontrivial ψ.

In [Liu et al. 2008], the first author showed in cooperation with his students that f is generically rigid if $n=1$ and p is sufficiently large. So the rigid bound is the best possible bound. In contrast, the weaker Hodge bound $\mathrm{HP}_q(\Delta)$ is only the best possible if $p \equiv 1 \pmod{d}$, where d is the degree of f.

We now pause to describe the relationship between the Newton polygons of $C_f(s, \pi_\psi)$ and $L_{f,\psi}(s)^{(-1)^{n-1}}$. We need the following definitions.

Definition 2.5. A convex polygon with initial point $(0, 0)$ is called algebraic if it is the graph of a $\mathbb{Q}$-valued function defined on $\mathbb{N}$ or on an interval of $\mathbb{N}$, and its slopes are of finite multiplicity and of bounded denominator.

Definition 2.6. For an algebraic polygon with slopes $\{\lambda_i\}$, we define its slope series to be $\sum_i t^{\lambda_i}$.

It is clear that an algebraic polygon is uniquely determined by its slope series. So the slope series embeds the set of algebraic polygons into the ring $\lim_{\vec{d}} \mathbb{Z}[[t^{1/d}]]$. The image is $\lim_{\vec{d}} \mathbb{N}[[t^{1/d}]]$ and is closed under addition and multiplication. Therefore one can define addition and multiplication on the set of algebraic polygons.

Lemma 2.7. *Suppose that f is nondegenerate. Then the q-adic Newton polygon of $C_f(s, \pi_\psi; \mathbb{F}_q)$ is the product of the q-adic Newton polygon of $L_{f,\psi}(s; \mathbb{F}_q)^{(-1)^{n-1}}$ and the algebraic polygon $1/(1-t)^n$.*

Proof. The C-value $C_f(s, \pi_\psi)$ and the L-function $L_{f,\psi}(s)$ determine each other. They are related by

$$L_{f,\psi}(s) = \prod_{i=0}^{n} C_f(q^i s, \pi_\psi)^{(-1)^{n-i-1}\binom{n}{i}}, \quad C_f(s, \pi_\psi)^{(-1)^{n-1}} = \prod_{j=0}^{\infty} L_{f,\psi}(q^j s)^{\binom{n+j-1}{j}}.$$

Suppose that $L_{f,\psi}(s)^{(-1)^{n-1}} = \prod_{i=1}^{d}(1 - \alpha_i s)$. Then

$$C_f(s, \pi_\psi) = \prod_{j=0}^{\infty} \prod_{i=1}^{d} (1 - \alpha_i q^j s)^{\binom{n+j-1}{j}}.$$

Let λ_i be the q-adic order of α_i. Then the q-adic order of $\alpha_i q^j$ is $\lambda_i + j$. So the slope series of the q-adic Newton polygon of $L_{f,\psi}(s)^{(-1)^{n-1}}$ is $S(t) = \sum_{i=1}^{d} t^{\lambda_i}$, and the slope series of the q-adic Newton polygon of $C_f(s, \pi_\psi)$ is

$$\sum_{j=0}^{+\infty} \sum_{i=0}^{d} \binom{n+j-1}{j} t^{\lambda_i + j} = \frac{1}{(1-t)^n} S(t). \qquad \square$$

The next theorem, whose proof is obvious, combines the rigidity bound and the Hodge bound.

Theorem 2.8. *If ψ is nontrivial, then* $\mathrm{NP}_{\pi_\psi}(f) \geq \mathrm{NP}_T(f) \geq \mathrm{HP}_q(\Delta)$.

If we drop the middle term, we arrive at the Hodge bound

$$\mathrm{NP}_{\pi_\psi}(f) \ge \mathrm{HP}_q(\Delta)$$

of [Adolphson and Sperber 1987] and [Liu and Wei 2007].

Theorem 2.9. *If* $\mathrm{NP}_{\pi_\psi}(f) = \mathrm{HP}_q(\Delta)$ *holds for one nontrivial* ψ, *then* f *is rigid*, T*-adically ordinary*, *and the equality holds for all nontrivial* ψ.

Proof. Suppose that $\mathrm{NP}_{\pi_{\psi_0}}(f) = \mathrm{HP}_q(\Delta)$ for a nontrivial ψ_0. Then, by the last theorem, we have

$$\mathrm{NP}_{\pi_{\psi_0}}(f) = \mathrm{NP}_T(f) = \mathrm{HP}_q(\Delta).$$

So f is rigid and T-adically ordinary, and $\mathrm{NP}_{\pi_\psi}(f) = \mathrm{NP}_T(f) = \mathrm{HP}_q(\Delta)$ holds for all nontrivial ψ. □

Definition 2.10. We call f ordinary if $\mathrm{NP}_{\pi_\psi}(f) = \mathrm{HP}_q(\Delta)$ holds for one (and hence for all) nontrivial ψ.

The notion of ordinariness now carries much more information than we had known before. From this, we see that the T-adic exponential sum provides a new framework to study all p^m-power order exponential sums simultaneously. Instead of the usual way of extending the methods for ψ of order p to cases of higher order, the T-adic exponential sum has the novel feature that it can sometimes transfer a known result for one nontrivial ψ to all nontrivial ψ. This philosophy is carried out further in [Liu et al. 2008].

Example 2.11. Let

$$f(x) = x_1 + x_2 + \cdots + x_n + \frac{\alpha}{x_1 x_2 \cdots x_n} \quad \text{for } \alpha \in \mu_{q-1}.$$

Then, by the result of [Sperber 1980] and our new information on ordinariness, we have $\mathrm{NP}_{\pi_\psi}(f) = \mathrm{HP}_q(\Delta)$ for all nontrivial ψ.

3. The q-Hodge polygon

Here, we describe explicitly the q-Hodge polygon mentioned in the introduction. Recall that $f(x) \in \mathbb{Z}_q[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ is a Laurent polynomial in n variables of the form

$$f(x) = \sum_{u \in \mathbb{Z}^n} a_u x^u, \quad \text{where } a_u \in \mathbb{Z}_q \text{ and } a_u^q = a_u.$$

We stress that the nonzero coefficients of $f(x)$ are roots of unity in $\mathbb{Z}_q$, and thus correspond uniquely to Teichmüller liftings of elements of the finite field $\mathbb{F}_q$. If the coefficients of $f(x)$ are arbitrary elements in $\mathbb{Z}_q$, much of the theory still holds, but it is more complicated to describe the results. In this paper, we make the simplifying assumption that the nonzero coefficients are always roots of unity.

Let Δ be the convex polyhedron in $\mathbb{R}^n$ associated to f, which is generated by the origin and the exponent vectors of the nonzero monomials of f. Let $C(\Delta)$ be the cone in $\mathbb{R}^n$ generated by Δ. Define the degree function $u \mapsto \deg u$ on $C(\Delta)$ so that $\deg u = 1$ when u lies on a codimensional 1 face of Δ that does not contain the origin, and so that $\deg(ru) = r \deg u$ for $r \in \mathbb{R}_{\geq 0}$ and $u \in C(\Delta)$; we call it the degree function associated to Δ. We have $\deg(u+v) \leq \deg u + \deg v$ for $u, v \in C(\Delta)$, and the equality holds if and only if u and v are cofacial. In other words, the number

$$c(u, v) := \deg u + \deg v - \deg(u + v)$$

is 0 if $u, v \in C(\Delta)$ are cofacial, and is positive otherwise. We call $c(u, v)$ the cofacial defect of u and v. Let

$$M(\Delta) := C(\Delta) \cap \mathbb{Z}^n$$

be the set of lattice points in the cone $C(\Delta)$. Let D be the denominator of the degree function, which is the smallest positive integer such that $\deg M(\Delta) \subset (1/D)\mathbb{Z}$. For every natural number k, we define

$$W(k) := W_\Delta(k) = \#\{u \in M(\Delta) \mid \deg u = k/D\}$$

to be the number of lattice points of degree k/D in $M(\Delta)$. For prime power $q = p^a$, the q-Hodge polygon of f is the polygon with vertices $(0, 0)$ and

$$\left(\sum_{j=0}^{i} W(j), a(p-1) \sum_{j=0}^{i} \frac{j}{D} W(j) \right) \quad \text{for } i = 0, 1, \ldots .$$

It is also called the q-Hodge polygon of Δ and denoted by $\mathrm{HP}_q(\Delta)$. It depends only on q and Δ. It has a side of slope $a(p-1)(j/D)$ with horizontal length $W(j)$ for each nonnegative integer j.

4. Analytic continuation

Here we prove the T-adic analytic continuation of the C-function $C_f(s, T)$. The idea is to employ Dwork's trace formula in the T-adic case.

Note that the Galois group $\mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$ is cyclic of order $a = \log_p q$. There is an element σ in the Galois group whose restriction to μ_{q-1} is the p-power morphism. It is of order a, and is called the Frobenius element.

We define a new variable π by the relation $E(\pi) = 1 + T$, where

$$E(\pi) = \exp\left(\sum_{i=0}^{\infty} \frac{\pi^{p^i}}{p^i} \right) \in 1 + \pi \mathbb{Z}_p[[\pi]]$$

is the Artin–Hasse exponential series. Thus, π and T are two different uniformizers of the T-adic local ring $\mathbb{Q}_p[\![T]\!]$. It is clear that

$$E(\pi\alpha) \in 1 + \pi\mathbb{Z}_q[\![\pi]\!] \quad \text{for } \alpha \in \mathbb{Z}_q,$$
$$E(\pi)^\beta \in 1 + \pi\mathbb{Z}_p[\![\pi]\!] \quad \text{for } \beta \in \mathbb{Z}_p.$$

The Galois group $\mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$ can act on $\mathbb{Z}_q[\![\pi]\!]$ but keep π fixed. The Artin–Hasse exponential series has a kind of commutativity, which we express through the following lemma.

Lemma 4.1 (commutativity). *We have the following commutative diagram*:

$$\begin{array}{ccc} \mu_{q-1} & \xrightarrow{E(\pi\cdot)} & \mathbb{Z}_q[\![\pi]\!] \\ {\scriptstyle \mathrm{Tr}}\downarrow & & \downarrow{\scriptstyle \mathrm{Norm}} \\ \mu_{p-1} & \xrightarrow{E(\pi)^{\bullet}} & \mathbb{Z}_p[\![\pi]\!] \end{array}$$

That is, if $x \in \mu_{q-1}$, *then* $E(\pi)^{x+x^p+\cdots+x^{p^{(a-1)}}} = E(\pi x)E(\pi x^p)\cdots E(\pi x^{p^{a-1}})$.

Proof. Since $\sum_{j=0}^{a-1} x^{p^j} = \sum_{j=0}^{a-1} x^{p^{j+i}}$ for $x \in \mu_{q-1}$, we have

$$\begin{aligned} E(\pi)^{x+x^p+\cdots+x^{p^{(a-1)}}} &= \exp\left(\sum_{i=0}^{\infty} \frac{\pi^{p^i}}{p^i} \sum_{j=0}^{a-1} x^{p^{j+i}}\right) \\ &= E(\pi x)E(\pi x^p)\cdots E(\pi x^{p^{a-1}}). \end{aligned}$$ □

Definition 4.2. Let $\pi^{1/D}$ be a fixed D-th root of π. Define

$$L(\Delta) = \left\{ \sum_{u \in M(\Delta)} b_u \pi^{\deg u} x^u : b_u \in \mathbb{Z}_q[\![\pi^{1/D}]\!] \right\},$$
$$B = \left\{ \sum_{u \in M(\Delta)} b_u \pi^{\deg u} x^u \in L(\Delta),\ \mathrm{ord}_T(b_u) \to +\infty \text{ if } \deg u \to +\infty \right\}.$$

The spaces $L(\Delta)$ and B are T-adic Banach algebras over the ring $\mathbb{Z}_q[\![\pi^{1/D}]\!]$. The monomials $\pi^{\deg u} x^u$ for $u \in M(\Delta)$ form an orthonormal basis of B and a formal basis $L(\Delta)$. The algebra B is contained in the larger Banach algebra $L(\Delta)$. If $u \in \Delta$, it is clear that $E(\pi x^u) \in L(\Delta)$. Write

$$E_f(x) := \prod_{a_u \neq 0} E(\pi a_u x^u) \quad \text{if } f(x) = \sum_{u \in \mathbb{Z}^n} a_u x^u.$$

This is an element of $L(\Delta)$ since $L(\Delta)$ is a ring.

The Galois group $\mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$ can act on $L(\Delta)$, while keeping $\pi^{1/D}$ as well as the x_i fixed. From the commutativity of the Artin–Hasse exponential series, one can infer the following lemma.

Lemma 4.3 (Dwork's splitting lemma). *If $x \in \mu_{q^k-1}$, then*

$$E(\pi)^{\mathrm{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(f(x))} = \prod_{i=0}^{ak-1} E_f^{\sigma^i}(x^{p^i}),$$

where a is the order of $\mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$.

Proof. We have

$$\begin{aligned} E(\pi)^{\mathrm{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(f(x))} &= \prod_{a_u \neq 0} E(\pi)^{\mathrm{Tr}_{\mathbb{Q}_{q^k}/\mathbb{Q}_p}(a_u x^u)} \\ &= \prod_{a_u \neq 0} \prod_{i=0}^{ak-1} E(\pi(a_u x^u)^{p^i}) = \prod_{i=0}^{ak-1} E_f^{\sigma^i}(x^{p^i}). \end{aligned}$$

□

Definition 4.4. We define a map

$$\psi_p : L(\Delta) \to L(\Delta), \qquad \sum_{u \in M(\Delta)} b_u x^u \mapsto \sum_{u \in M(\Delta)} b_{pu} x^u.$$

It is clear that the composition map $\psi_p \circ E_f$ sends B to B.

Lemma 4.5. *Write $E_f(x) = \sum_{u \in M(\Delta)} \alpha_u(f) \pi^{\deg u} x^u$. Then*

$$\psi_p \circ E_f(\pi^{\deg u} x^u) = \sum_{w \in M(\Delta)} \alpha_{pw-u}(f) \pi^{c(pw-u,u)} \pi^{(p-1)\deg w} \pi^{\deg w} x^w$$

for $u \in M(\Delta)$, where $c(pw-u, u)$ is the cofacial defect of $pw-u$ and u.

Proof. This follows directly from the definition of ψ_p and $E_f(x)$. □

Definition 4.6. Define $\psi := \sigma^{-1} \circ \psi_p \circ E_f : B \to B$, and its a-th iterate

$$\psi^a = \psi_p^a \circ \prod_{i=0}^{a-1} E_f^{\sigma^i}(x^{p^i}).$$

Note that ψ is linear over $\mathbb{Z}_p[\![\pi^{1/D}]\!]$, but semilinear over $\mathbb{Z}_q[\![\pi^{1/D}]\!]$. On the other hand, ψ^a is linear over $\mathbb{Z}_q[\![\pi^{1/D}]\!]$. By the last lemma, ψ^a is completely continuous in the sense of [Serre 1962].

Theorem 4.7 (Dwork's trace formula). *For every positive integer k,*

$$(q^k-1)^{-n} S_f(k,T) = \mathrm{Tr}_{B/\mathbb{Z}_q[\![\pi^{1/D}]\!]}(\psi^{ak}).$$

Proof. Let $g(x) \in B$. We have

$$\psi^{ak}(g) = \psi_p^{ak}(g \prod_{i=0}^{ak-1} E_f^{\sigma^i}(x^{p^i})).$$

Write $\prod_{i=0}^{ak-1} E_f^{\sigma^i}(x^{p^i}) = \sum_{u \in M(\Delta)} \beta_u x^u$. One computes that

$$\psi^{ak}(\pi^{\deg v} x^v) = \sum_{u \in M(\Delta)} \beta_{q^k u - v} \pi^{\deg v} x^u.$$

Thus, $\mathrm{Tr}(\psi^{ak} \mid B/\mathbb{Z}_q[\![\pi^{1/D}]\!]) = \sum_{u \in M(\Delta)} \beta_{(q^k-1)u}$. But, by Dwork's splitting lemma, we have

$$(q^k-1)^{-n} S_f(k,T) = (q^k-1)^{-n} \sum_{x \in \mu^n_{q^k-1}} \prod_{i=0}^{ak-1} E_f^{\sigma^i}(x^{p^i}) = \sum_{u \in M(\Delta)} \beta_{(q^k-1)u}. \qquad \square$$

Theorem 4.8 (analytic trace formula). *We have*

$$C_f(s,T) = \det(1 - \psi^a s \mid B/\mathbb{Z}_q[\![\pi^{1/D}]\!]).$$

In particular, the T-adic C-function $C_f(s,T)$ is T-adic analytic in s.

Proof. It follows from the last theorem and the well-known identity

$$\det(1 - \psi^a s) = \exp\left(-\sum_{k=1}^{\infty} \mathrm{Tr}(\psi^{ak}) \frac{s^k}{k}\right). \qquad \square$$

This theorem gives another proof that the coefficients of $C_f(s,T)$ and $L_f(s,T)$ as power series in s are T-adically integral.

Corollary 4.9. *For each nontrivial ψ, the C-value $C_f(s,\pi_\psi)$ is p-adic entire in s and the L-function $L_{f,\psi}(s)$ is rational in s.*

5. The Hodge bound

The analytic trace formula in the previous section reduces the study of $C_f(s,T)$ to the study of the operator ψ^a. We consider ψ first. Note that ψ operates on B and is linear over $\mathbb{Z}_p[\![\pi^{1/D}]\!]$.

Theorem 5.1. *The T-adic Newton polygon of* $\det(1-\psi s \mid B/\mathbb{Z}_p[\![\pi^{1/D}]\!])$ *lies above the polygon with vertices* $(0,0)$ *and*

$$\left(a \sum_{k=0}^{i} W(k),\, a(p-1) \sum_{k=0}^{i} \frac{k}{D} W(k)\right) \quad \textit{for } i = 0, 1, \ldots.$$

Proof. Let $\xi_1, \xi_2, \ldots, \xi_a$ be a normal basis of $\mathbb{Q}_q$ over $\mathbb{Q}_p$. Write

$$(\xi_j \alpha_{pw-u}(f))^{\sigma^{-1}} = \sum_{i=0}^{a-1} \alpha_{(i,w),(j,u)}(f) \xi_i \quad \text{for } \alpha_{(i,w),(j,u)}(f) \in \mathbb{Z}_p[\![\pi^{1/D}]\!].$$

Then

$$\psi(\xi_j \pi^{\deg u} x^u) = \sum_{i=0}^{a-1} \sum_{w \in M(\Delta)} \alpha_{(i,w),(j,u)}(f) \pi^{c(pw-u,u)} \pi^{(p-1)\deg w} \xi_i \pi^{\deg w} x^w.$$

That is, the matrix of ψ over $\mathbb{Z}_p[\![\pi^{1/D}]\!]$ with respect to the orthonormal basis $\{\xi_j \pi^{\deg u} x^u\}_{0 \le j < a, u \in M(\Delta)}$ is

$$A = \left(\alpha_{(i,w),(j,u)}(f) \pi^{c(pw-u,u)} \pi^{(p-1)\deg w}\right)_{(i,w),(j,u)}.$$

The claim follows. □

We are now ready to prove the Hodge bound for the Newton polygon.

Theorem 5.2. $\mathrm{NP}_T(f) \ge \mathrm{HP}_q(\Delta)$.

Proof. By the theorem above, it suffices to prove that the T-adic Newton polygon of $\det\left(1 - \psi^a s^a \mid B/\mathbb{Z}_q[\![\pi^{1/D}]\!]\right)$ coincides with that of $\det\left(1 - \psi s \mid B/\mathbb{Z}_p[\![\pi^{1/D}]\!]\right)$. Note that

$$\det\left(1 - \psi^a s \mid B/\mathbb{Z}_p[\![\pi^{1/D}]\!]\right) = \mathrm{Norm}\left(\det\left(1 - \psi^a s \mid B/\mathbb{Z}_q[\![\pi^{1/D}]\!]\right)\right),$$

where the norm map is the norm from $\mathbb{Z}_q[\![\pi^{1/D}]\!]$ to $\mathbb{Z}_p[\![\pi^{1/D}]\!]$. The theorem now follows from the equality

$$\prod_{\zeta^a=1} \det\left(1 - \psi\zeta s \mid B/\mathbb{Z}_p[\![\pi^{1/D}]\!]\right) = \det\left(1 - \psi^a s^a \mid B/\mathbb{Z}_p[\![\pi^{1/D}]\!]\right). \qquad \square$$

6. Facial decomposition

In this section, we extend the facial decomposition theorem in [Wan 1993]. Recall that the operator $\psi = \sigma^{-1} \circ (\psi_p \circ E_f)$ is only semilinear over $\mathbb{Z}_q[\![\pi^{1/D}]\!]$. But its second factor $\psi_p \circ E_f$ is clearly linear, and so $\det\left(1 - (\psi_p \circ E_f)s \mid B/\mathbb{Z}_q[\![\pi^{1/D}]\!]\right)$ is well defined. We begin with the following theorem.

Theorem 6.1. *The T-adic Newton polygon of $C_f(s,T)$ coincides with* $\mathrm{HP}_q(\Delta)$ *if and only if the T-adic Newton polygon of* $\det\left(1 - (\psi_p \circ E_f)s \mid B/\mathbb{Z}_q[\![\pi^{1/D}]\!]\right)$ *coincides with the polygon with vertices* $(0,0)$ *and*

$$\left(\sum_{k=0}^{i} W(k),\ (p-1)\sum_{k=0}^{i} \frac{k}{D} W(k)\right) \quad \textit{for } i = 0, 1, \ldots.$$

Proof. In the proof of Theorem 5.2, we showed that the T-adic Newton polygon of $C_f(s^a, T)$ coincides with that of $\det\left(1 - \psi s \mid B/\mathbb{Z}_p[\![\pi^{1/D}]\!]\right)$. Note that

$$\det\left(1 - (\psi_p \circ E_f)s \mid B/\mathbb{Z}_p[\![\pi^{1/D}]\!]\right) = \mathrm{Norm}\left(\det\left(1 - (\psi_p \circ E_f)s \mid B/\mathbb{Z}_q[\![\pi^{1/D}]\!]\right)\right),$$

where the norm map is the norm from $\mathbb{Z}_q[\![\pi^{1/D}]\!]$ to $\mathbb{Z}_p[\![\pi^{1/D}]\!]$. The theorem is equivalently stated that the T-adic Newton polygon of $\det\bigl(1-\psi s \mid B/\mathbb{Z}_p[\![\pi^{1/D}]\!]\bigr)$ coincides with the polygon with vertices (0, 0) and

$$\left(\sum_{k=0}^{i} aW(k),\, a(p-1)\sum_{k=0}^{i}\frac{k}{D}W(k)\right) \quad \text{for } i=0,1,\dots$$

if and only if the T-adic Newton polygon of $\det\bigl(1-(\psi_p\circ E_f)s \mid B/\mathbb{Z}_p[\![\pi^{1/D}]\!]\bigr)$ does. Therefore it suffices to show that the determinant of the matrix

$$(\alpha_{(i,w),(j,u)}(f)\pi^{c(pw-u,u)})_{0\le i,j<a,\,\deg w,\deg u\le k/D}$$

is not divisible by T in $\mathbb{Z}_p[\![\pi^{1/D}]\!]$ if and only if the determinant of the matrix

$$(\alpha_{pw-u}(f)\pi^{c(pw-u,u)})_{\deg w,\deg u\le k/D}$$

is not divisible by T in $\mathbb{Z}_q[\![\pi^{1/D}]\!]$. The theorem now follows from the fact that the former determinant is the norm of the latter from $\mathbb{Q}_q[\![\pi^{1/D}]\!]$ to $\mathbb{Q}_p[\![\pi^{1/D}]\!]$ up to a sign. □

We now define the open facial decomposition $F(\Delta)$. It is the decomposition of $C(\Delta)$ into a disjoint union of relatively open cones generated by the relatively open faces of Δ whose closure does not contain the origin. Note that every relatively open cone generated by cofacial vectors in $C(\Delta)$ is contained in a unique element of $F(\Delta)$.

Lemma 6.2. *Let $\sigma\in F(\Delta)$, and $u\in\sigma$. Then $\alpha_u(f_{\bar\sigma})\equiv\alpha_u(f) \pmod{\pi^{1/D}}$, where $f_{\bar\sigma}$ is the sum of monomials of f whose exponent vectors lie in the closure $\bar\sigma$ of σ.*

Proof. Let $v_1,\dots,v_j$ be exponent vectors of monomials of f such that $a_1v_1+\cdots+a_jv_j=u$, with $a_1>0,\dots,a_j>0$. It suffices to show that either $v_1,\dots v_j$ lie in the closure of σ, or their contribution to $\alpha_u(f)$ is $\equiv 0 \pmod{\pi^{1/D}}$. Suppose their contribution to $\alpha_u(f)$ is $\not\equiv 0 \pmod{\pi^{1/D}}$. Then $v_1,\dots,v_j$ must be cofacial. So the interior of the cone generated by those vectors is contained in a unique element of $F(\Delta)$. Since that interior has a common point u with σ, it must be σ. It follows that $v_1,\dots v_j$ lie in the closure of σ. □

Lemma 6.3. *Let $\sigma,\tau\in F(\Delta)$ be distinct. Let $w\in\sigma$ and $u\in\tau$. Suppose that the dimension of σ is no greater than that of τ. Then $pw-u$ and u are not cofacial, that is, $c(pw-u,u)>0$.*

Proof. Suppose that $pw-u$ and u are cofacial. Then the interior of the cone generated by $pw-u$ and u is contained in a unique element of $F(\Delta)$. Since that interior has a common point w with σ, it must be σ. It follows that u lies in the closure of σ. Since σ and τ are distinct, u lies in the boundary of σ. This implies that the dimension of τ is less than that of σ, which is a contradiction. Therefore $pw-u$ and u are not cofacial. $\square$

For $\sigma \in F(\Delta)$, we define $M(\sigma) = M(\Delta) \cap \sigma = \mathbb{Z}^n \cap \sigma$ to be the set of lattice points in the cone σ.

Theorem 6.4 (open facial decomposition). *The T-adic Newton polygon of $C_f(s,T)$ coincides with* $\mathrm{HP}_q(\Delta)$ *if and only if for every $\sigma \in F(\Delta)$, the determinants of the matrices*

$$\{\alpha_{pw-u}(f_{\bar{\sigma}})\pi^{c(pw-u,u)}\}_{w,u\in M(\sigma),\,\deg w,\deg u\le k/D} \quad \textit{for } k = 0, 1, \dots$$

are not divisible by T in $\mathbb{Z}_q[\![\pi^{1/D}]\!]$, where $\bar{\sigma}$ is the closure of σ.

Proof. By Theorem 6.1, the T-adic Newton polygon of $C_f(s,T)$ coincides with the q-Hodge polygon of f if and only if the determinants of the matrices

$$A^{(k)} = \{\alpha_{pw-u}(f)\pi^{c(pw-u,u)}\}_{w,u\in M(\Delta),\,\deg w,\deg u\le k/D} \quad \text{for } k = 0, 1, \dots$$

are not divisible by T in $\mathbb{Z}_q[\![\pi^{1/D}]\!]$. Write

$$A^{(k)}_{\sigma,\tau} = \{\alpha_{pw-u}(f)\pi^{c(pw-u,u)}\}_{w\in M(\sigma),\,u\in M(\tau),\,\deg w,\deg u\le k/D}.$$

The facial decomposition shows that $A^{(k)}$ has the block form $(A^{(k)}_{\sigma,\tau})_{\sigma,\tau\in F(\Delta)}$. The last lemma shows that the block form modulo $\pi^{1/D}$ is triangular if we order the cones in $F(\Delta)$ by increasing dimension. It follows that $\det A^{(k)}$ is not divisible by T in $\mathbb{Z}_q[\![\pi^{1/D}]\!]$ if and only if for all $\sigma \in F(\Delta)$, $\det A^{(k)}_{\sigma,\sigma}$ is not divisible by T in $\mathbb{Z}_q[\![\pi^{1/D}]\!]$. By Lemma 6.2, modulo $\pi^{1/D}$, $A^{(k)}_{\sigma,\sigma}$ is congruent to the matrix

$$\{\alpha_{pw-u}(f_{\bar{\sigma}})\pi^{c(pw-u,u)}\}_{w,u\in M(\sigma),\,\deg w,\deg u\le k/D}.$$

So $\det A^{(k)}_{\sigma,\sigma}$ is not divisible by T in $\mathbb{Z}_q[\![\pi^{1/D}]\!]$ if and only if the determinant of the matrix

$$\{\alpha_{pw-u}(f_{\bar{\sigma}})\pi^{c(pw-u,u)}\}_{w,u\in M(\sigma),\,\deg w,\deg u\le k/D}$$

is not divisible by T in $\mathbb{Z}_q[\![\pi^{1/D}]\!]$. $\square$

The closed facial decomposition Theorem 1.7 follows from the open decomposition theorem and the fact that

$$F(\Delta) = \bigcup_{\substack{\sigma\in F(\Delta)\\ \dim\sigma=\dim\Delta}} F(\bar{\sigma}).$$

A similar π_ψ-adic facial decomposition theorem for $C_f(s,\pi_\psi)$ can be proved in a similar way. Alternatively, it follows from the transfer theorem together with the π_ψ-adic facial decomposition in [Wan 1993] for ψ of order p.

7. Variation of C-functions in a family

Fix an n-dimensional integral convex polytope $\triangle$ in $\mathbb{R}^n$ containing the origin. For each prime p, let $P(\Delta, \mathbb{F}_p)$ denote the parameter space of all Laurent polynomials $f(x)$ over $\overline{\mathbb{F}}_p$ such that $\Delta(f) = \Delta$. This is a connected rational variety defined over $\mathbb{F}_p$. For each $f \in P(\Delta, \mathbb{F}_p)(\mathbb{F}_q)$, the Teichmüller lifting gives a Laurent polynomial $\tilde{f}$ whose nonzero coefficients are roots of unity in $\mathbb{Z}_q$. The C-function $C_{\tilde{f}}(s, T)$ is then defined and T-adically entire. For simplicity of notation, we shall just write $C_f(s, T)$ for $C_{\tilde{f}}(s, T)$ and similarly $L_f(s, T)$ for $L_{\tilde{f}}(s, T)$. Thus, our C-function and L-function are now defined for Laurent polynomials over finite fields via the Teichmüller lifting. We would like to study how $C_f(s, T)$ varies when f varies in the algebraic variety $P(\Delta, \mathbb{F}_p)$.

Recall that for a closed face $\sigma \in \Delta$, f_σ denotes the restriction of f to σ. That is, f_σ is the sum of those nonzero monomials in f whose exponents are in σ.

Definition 7.1. A Laurent polynomial $f \in P(\Delta, \mathbb{F}_p)$ is called nondegenerate if for every closed face σ of Δ of arbitrary dimension that does not contain the origin, the system

$$\frac{\partial f_\sigma}{\partial x_1} = \cdots = \frac{\partial f_\sigma}{\partial x_n} = 0$$

has no common zeros with $x_1 \cdots x_n \neq 0$ over the algebraic closure of $\mathbb{F}_p$.

The nondegeneracy condition is a geometric condition that insures the associated Dwork cohomology can be calculated. In particular, it implies that if ψ is of order p^m, then the L-function $L_{f,\psi}(s)^{(-1)^{n-1}}$ is a polynomial in s whose degree is precisely $n!\,\mathrm{Vol}(\Delta)p^{n(m-1)}$; see [Liu and Wei 2007]. Consequently:

Theorem 7.2. *Let $f \in P(\Delta, \mathbb{F}_p)(\mathbb{F}_q)$. Write*

$$L_f(s, T)^{(-1)^{n-1}} = \sum_{k=0}^{\infty} L_{f,k}(T)s^k \quad \textit{for } L_{f,k}(T) \in \mathbb{Z}_p[\![T]\!].$$

Assume that f is nondegenerate. Then for every positive integer m and all positive integers $k > n!\,\mathrm{Vol}(\Delta)p^{n(m-1)}$, we have the congruence

$$L_{f,k}(T) \equiv 0 \pmod{((1+T)^{p^m} - 1)/T} \quad \textit{in } \mathbb{Z}_p[\![T]\!].$$

Proof. Write $((1+T)^{p^m} - 1)/T = \prod(T - \xi)$. The nondegeneracy assumption implies that

$$L_f(s, \xi)^{(-1)^{n-1}} = \sum_{j=0}^{\infty} L_{f,j}(\xi)s^j,$$

is a polynomial in s of degree $\leq n!\,\mathrm{Vol}(\Delta)p^{n(m-1)} < k$. It follows that $L_{f,k}(\xi) = 0$ for all ξ. That is, $L_{f,k}(T)$ is divisible by $(T - \xi)$ for ξ. □

Definition 7.3. Let $N(\Delta, \mathbb{F}_p)$ denote the subset of all nondegenerate Laurent polynomials $f \in P(\Delta, \mathbb{F}_p)$.

The subset $N(\Delta, \mathbb{F}_p)$ is Zariski open in $P(\Delta, \mathbb{F}_p)$. It can be empty for some pair $(\Delta, \mathbb{F}_p)$. But, $N(\Delta, \mathbb{F}_p)$ for a given Δ is Zariski open dense in $P(\Delta, \mathbb{F}_p)$ for all primes p except for possibly finitely many primes depending on Δ. It is an interesting and independent question to classify the primes p for which $N(\Delta, \mathbb{F}_p)$ is nonempty. This is related to the GKZ discriminant [Gel'fand et al. 1994]. For simplicity, we shall only consider nondegenerate f in the following.

Generic ordinariness. The first question is, How often f is T-adically ordinary when f varies in the nondegenerate locus $N(\Delta, \mathbb{F}_p)$? Let $U_p(\Delta, T)$ be the subset of $f \in N(\Delta, \mathbb{F}_p)$ such that f is T-adically ordinary, and $U_p(\Delta)$ the subset of $f \in N(\Delta, \mathbb{F}_p)$ such that f is ordinary. One can prove this:

Lemma 7.4. *The set $U_p(\Delta)$ is Zariski open in $N(\Delta, \mathbb{F}_p)$.*

Is $U_p(\Delta, T)$ also Zariski open in $N(\Delta, \mathbb{F}_p)$? We do not know the answer.

For which p are $U_p(\Delta)$ and $U_p(\Delta, T)$ Zariski dense in $N(\Delta, \mathbb{F}_p)$? The rigidity bound as well as the Hodge bound imply that $U_p(\Delta) \subseteq U_p(\Delta, T)$. It follows that if $U_p(\Delta)$ is Zariski dense in $N(\Delta, \mathbb{F}_p)$, then $U_p(\Delta, T)$ is also Zariski dense in $N(\Delta, \mathbb{F}_p)$.

The Adolphson–Sperber conjecture [1989] says that if $p \equiv 1 \pmod{D}$, then $U_p(\Delta)$ is Zariski dense in $N(\Delta, \mathbb{F}_p)$. This conjecture was proved to be true in [Wan 1993; 2004] if $n \leq 3$, which implies this:

Theorem 7.5. *If $p \equiv 1 \pmod{D}$ and $n \leq 3$, then $U_p(\Delta, T)$ is Zariski dense in $N(\Delta, \mathbb{F}_p)$.*

For $n \geq 4$, it was shown in [Wan 1993; 2004] that there is an effectively computable positive integer $D^*(\Delta)$ depending only on Δ such that $U_p(\Delta)$ is Zariski dense in $N(\Delta, \mathbb{F}_p)$ if $p \equiv 1 \pmod{D^*(\Delta)}$.

Theorem 7.6. *For each Δ, there exists an effectively computable positive integer $D^*(\Delta)$ such that $U_p(\Delta, T)$ is Zariski dense in $N(\Delta, \mathbb{F}_p)$ if $p \equiv 1 \pmod{D^*(\Delta)}$.*

The smallest possible $D^*(\Delta)$ is rather subtle to compute in general, and it can be much larger than D. We now state a conjecture giving reasonably precise estimates of $D^*(\Delta)$.

Definition 7.7. Let $S(\triangle)$ be the monoid generated by the degree 1 lattice points in $M(\Delta)$, that is, those lattice points on the codimension 1 faces of Δ not containing the origin. Define the exponent of $\triangle$ by

$$I(\triangle) = \inf\{d \in \mathbb{Z}_{>0} \mid dM(\Delta) \subseteq S(\triangle)\}.$$

If $u \in M(\Delta)$, then the degree of Du will be integral, but Du may not be a nonnegative integral combination of degree 1 elements in $M(\Delta)$; thus $DM(\Delta)$ may not be a subset of $S(\Delta)$. It is not hard to show that $I(\Delta) \geq D$. In general they are different but they are equal if $n \leq 3$. This explains why the Adolphson–Sperber conjecture is true if $n \leq 3$ but may be false if $n \geq 4$. The following conjecture is a modified form, and it is a consequence of [Wan 1993, Conjecture 9.1].

Conjecture 7.8. *If $p \equiv 1 \mod I(\triangle)$, then $U_p(\Delta)$ is Zariski dense in $N(\Delta, \mathbb{F}_p)$. In particular, $U_p(\Delta, T)$ is Zariski dense in $N(\Delta, \mathbb{F}_p)$ for such p.*

By the facial decomposition theorem, it suffices in proving this conjecture to assume that Δ has only one codimension 1 face not containing the origin.

Generic Newton polygon. In the case that $U_p(\Delta, T)$ is empty, we expect the existence of a generic T-adic Newton polygon. For this purpose, we need to rescale the uniformizer. For $f \in N(\Delta, \mathbb{F}_p)(\mathbb{F}_{p^a})$, the $T^{a(p-1)}$-adic Newton polygon of $C_f(s, T; \mathbb{F}_{p^a})$ is independent of the choice of a for which f is defined over $\mathbb{F}_{p^a}$. We call this the absolute T-adic Newton polygon of f.

Conjecture 7.9. *There is a Zariski open dense subset $G_p(\Delta, T)$ of $N(\Delta, \mathbb{F}_p)$ such that the absolute T-adic Newton polygon of f is constant for all $f \in G_p(\Delta, T)$. Denote this common polygon by* $\mathrm{GNP}_T(\Delta, p)$, *and call it the generic Newton polygon of* $(\triangle, T)$.

More generally, one expects that much of the classical theory for finite rank F-crystals extends to a certain nuclear infinite rank setting. This includes the classical Dieudonne–Manin isogeny theorem, the Grothendieck specialization theorem, the Katz isogeny theorem [1979]. All these are essentially understood in the ordinary infinite rank case, but open in the nonordinary infinite rank case.

Similarly, for each nontrivial ψ there is a Zariski open dense subset $G_p(\Delta, \psi)$ of $N(\Delta, \mathbb{F}_p)$ with the property that the $\pi_\psi^{a(p-1)}$-adic Newton polygon of the C-value $C_f(s, \pi_\psi; \mathbb{F}_{p^a})$ is constant for all $f \in G_p(\Delta, \psi)$. Denote this common polygon by $\mathrm{GNP}_p(\Delta, \psi)$, and call it the generic Newton polygon of $(\triangle, \psi)$. The existence of $G_p(\Delta, \psi)$ can be proved, since the nondegeneracy assumption implies that the C-function $C_f(s, \pi_\psi)$ is determined by a single finite rank F-crystal via a Dwork type cohomological formula for $L_{f,\psi}(s)$. In the T-adic case, we are not aware of any such finite rank reduction.

Clearly, we have the relation

$$\mathrm{GNP}_p(\Delta, \psi) \geq \mathrm{GNP}_T(\Delta, p).$$

Conjecture 7.10. *If p is sufficiently large, then*

$$\mathrm{GNP}_p(\Delta, \psi) = \mathrm{GNP}_T(\Delta, p).$$

This conjecture is proved in the case $n = 1$ in [Liu et al. 2008].

Let $\mathrm{HP}(\Delta)$ denote the absolute Hodge polygon with vertices $(0, 0)$ and

$$\left(\sum_{k=0}^{i} W(k), \sum_{k=0}^{i} \frac{k}{D} W(k)\right) \quad \text{for } i = 0, 1, \ldots.$$

Note that $\mathrm{HP}(\Delta)$ depends only on Δ, and no longer on q. It is rescaled from the q-Hodge polygon $\mathrm{HP}_q(\Delta)$. Clearly, we have

$$\mathrm{GNP}_p(\Delta, \psi) \geq \mathrm{GNP}_T(\Delta, p) \geq \mathrm{HP}(\Delta).$$

Conjecture 7.8 says that if $p \equiv 1 \pmod{I(\triangle)}$, then both $\mathrm{GNP}_p(\Delta, \psi)$ and $\mathrm{GNP}_T(\Delta, p)$ are equal to $\mathrm{HP}(\Delta)$. In general, the generic Newton polygon lies above $\mathrm{HP}(\Delta)$, but for many Δ it should get closer and closer to $\mathrm{HP}(\Delta)$ as p goes to infinity. We now make this more precise. Let $E(\Delta)$ be the monoid generated by the lattice points in Δ. This is a subset of $M(\Delta)$. We may generalize the limiting [Wan 2004, Conjecture 1.11] for ψ of order p:

Conjecture 7.11. *If the difference $M(\Delta) - E(\Delta)$ is a finite set, then for each nontrivial ψ, we have*

$$\lim_{p \to \infty} \mathrm{GNP}_p(\Delta, \psi) = \mathrm{HP}(\Delta).$$

In particular, $\lim_{p\to\infty} \mathrm{GNP}_T(\Delta, p) = \mathrm{HP}(\Delta)$.

This conjecture is equivalent to the existence of the limit. This is because for all primes $p \equiv 1 \pmod{D^*(\Delta)}$, we already have by Theorem 7.6 the equality $\mathrm{GNP}_p(\Delta, \psi) = \mathrm{HP}(\Delta)$. A stronger version of this conjecture (namely, [Wan 2004, Conjecture 1.12]) has been proved by Zhu [2003; 2004a; 2004b] in the case $m = 1$ and $n = 1$; see also [Blache and Férard 2007; Blache et al. 2008] and [Liu 2008] for related further work in the case $m = 1$ and $n = 1$; see [Hong 2001; 2002] and [Yang 2003] for more specialized one variable results. For $n \geq 2$, the conjecture is clearly true for any Δ for which both $D \leq 2$ and the Adolphson–Sperber conjecture holds, because then $\mathrm{GNP}_p(\Delta, \psi) = \mathrm{HP}(\Delta)$ for every $p > 2$. There are many such higher-dimensional examples [Wan 2004]. Using free products of polytopes and the examples above, one can construct further examples [Blache 2008].

T-adic Dwork conjecture. In this final subsection, we describe the T-adic version of Dwork's conjecture [1973] on pure slope zeta functions.

Let Λ be a quasiprojective subvariety of $N(\Delta, \mathbb{F}_p)$ defined over $\mathbb{F}_p$. Let f_λ be a family of Laurent polynomials parameterized by $\lambda \in \Lambda$. For each closed point $\lambda \in \Lambda$, the Laurent polynomial f_λ is defined over the finite field $\mathbb{F}_{p^{\deg \lambda}}$. The T-adic entire function $C_{f_\lambda}(s, T)$ has the pure slope factorization

$$C_{f_\lambda}(s, T) = \prod_{\alpha \in \mathbb{Q}_{\geq 0}} P_\alpha(f_\lambda, s),$$

where each $P_\alpha(f_\lambda, s) \in 1 + s\mathbb{Z}_p[\![T]\!][s]$ is a polynomial in s whose reciprocal roots all have $T^{\deg \lambda (p-1)}$-slope equal to α.

Definition 7.12. For $\alpha \in \mathbb{Q}_{\geq 0}$, the T-adic pure slope L-function of the family f_Λ is defined to be the infinite Euler product

$$L_\alpha(f_\Lambda, s) = \prod_{\lambda \in |\Lambda|} \frac{1}{P_\alpha(f_\lambda, s^{\deg \lambda})} \in 1 + s\mathbb{Z}_p[\![T]\!][\![s]\!],$$

where $|\Lambda|$ denotes the set of closed points of Λ over $\mathbb{F}_p$.

Dwork's conjecture then has a T-adic version:

Conjecture 7.13. *For $\alpha \in \mathbb{Q}_{\geq 0}$, the T-adic pure slope L-function $L_\alpha(f_\Lambda, s)$ is T-adic meromorphic in s.*

In the ordinary case, this conjecture can be proved using the methods from [Wan 2000a; 2000b; 1999]. It would be interesting to prove this conjecture in the general case. The π_ψ-adic version of this conjecture is essentially Dwork's original conjecture, which can be proved as it reduces to finite rank F-crystals. The difficulty of the T-adic version is that we have to work with infinite rank objects, where much less is known in the nonordinary case.

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