Algebra & Number Theory

Volume 13

2019

No. 9

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ALGEBRA AND NUMBER THEORY 13:9 (2019)
dx.doi.org/10.2140/ant.2019.13.2005

Frobenius–Perron theory of endofunctors

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We introduce the Frobenius–Perron dimension of an endofunctor of a $\Bbbk$-linear category and provide some applications.

0. Introduction

The spectral radius (also called the Frobenius–Perron dimension) of a matrix is an elementary and extremely useful invariant in linear algebra, combinatorics, topology, probability and statistics. The Frobenius–Perron dimension has become a crucial concept in the study of fusion categories and representations of semisimple weak Hopf algebras since it was introduced by Etingof, Nikshych and Ostrik [Etingof et al. 2005] (also see [Etingof et al. 2004; 2015; Nikshych 2004]). In this paper several Frobenius–Perron type invariants are proposed to study derived categories, representations of finite dimensional algebras, and complexity of algebras and categories.

Throughout let $\Bbbk$ be an algebraically closed field, and let everything be over $\Bbbk$.

Definitions. The first goal is to introduce the Frobenius–Perron dimension of an endofunctor of a category. Here we only sketch the definition of $\operatorname{fpd}(\sigma)$ for an endofunctor σ of an abelian category $\mathcal{C}$ and the precise definition is given in Definition 2.3(2). Let $\phi := \{X_1, \dots, X_n\}$ be a finite subset of nonzero objects in $\mathcal{C}$ such that

$$\operatorname{Hom}_{\mathcal{C}}(X_i, X_j) = \begin{cases} \Bbbk & i = j, \\ 0 & i \neq j. \end{cases}$$

Let $\rho(A(\phi, \sigma))$ denote the spectral radius of the $n \times n$-matrix $[\dim \operatorname{Hom}_{\mathcal{C}}(X_i, \sigma(X_j))]_{n\times n}$. The *Frobenius–Perron dimension* of σ is defined to be

$$\operatorname{fpd}(\sigma) = \sup_{\phi}\{\rho(A(\phi, \sigma))\}$$

where ϕ ranges over all finite subsets of $\mathcal{C}$ satisfying the condition mentioned above. If an object V in a fusion category $\mathcal{C}$ is considered as the associated tensor endofunctor $V \otimes_{\mathcal{C}} -$, then our definition of the Frobenius–Perron dimension agrees with the definition given in [Etingof et al. 2005], see Example 2.11 for details. Our definition applies to the derived category of projective schemes and finite dimensional algebras, as well as other abelian and additive categories (Definitions 2.3 and 2.4). We also refer the reader to Section 2 for the following invariants of an endofunctor:

MSC2010: primary 16E10,18E30,16G60; secondary 16E35.
Keywords: Frobenius–Perron dimension, derived categories, embedding of categories, tame and wild dichotomy, complexity.

Frobenius–Perron growth (denoted by fpg).

Frobenius–Perron curvature (denoted by fpv).

Frobenius–Perron series (denoted by FP).

One can further define the above invariants for an abelian or a triangulated category. Note that the Frobenius–Perron dimension/growth/curvature of a category can be a noninteger, see Proposition 5.12(1), Example 8.7, and Remarks 5.13(5) for nonintegral values of fpd, fpg, and fpv respectively.

If $\mathfrak{A}$ is an abelian category, let $D^b(\mathfrak{A})$ denote the bounded derived category of $\mathfrak{A}$. On the one hand it is reasonable to call fpd a dimension function since

$$\operatorname{fpd}(D^b(\operatorname{Mod}-\Bbbk[x_1, \dots, x_n])) = n$$

(Proposition 4.3(1)), but on the other hand, one might argue that fpd should not be called a dimension function since

$$\operatorname{fpd}(D^b(\operatorname{coh}(\mathbb{P}^n))) = \begin{cases} 1 & n = 1, \\ \infty & n \geq 2, \end{cases}$$

(Propositions 6.5 and 6.7). In the latter case, fpd is an indicator of representation type of the category of $\operatorname{coh}(\mathbb{P}^n)$, namely, $\operatorname{coh}(\mathbb{P}^n)$ is tame if $n = 1$, and is of wild representation type for all $n \geq 2$. A similar statement holds for projective curves in terms of genus (Proposition 6.5).

We can define the Frobenius–Perron ("fp") version of several other classical invariants:

fp global dimension (denoted by fpgldim, Definition 2.7(1)).

fp Kodaira dimension (denoted by fp κ) [Chen et al. 2019].

The first one is defined for all triangulated categories and the second one is defined for triangulated categories with Serre functor. In general, the fpgldim A does not agree with the classical global dimension of A (Theorem 7.8). The fp version of the Kodaira dimension agrees with the classical definition for smooth projective schemes [Chen et al. 2019].

Our second goal is to provide several applications.

Embeddings. In addition to the fact that the Frobenius–Perron dimension is an effective and sensible invariant of many categories, this invariant increases when the "size" of the endofunctors and categories increase.

Theorem 0.1. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are $\Bbbk$-linear categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor. Let $\sigma_{\mathcal{C}}$ and $\sigma_{\mathcal{D}}$ be endofunctors of $\mathcal{C}$ and $\mathcal{D}$ respectively. Suppose that $F \circ \sigma_{\mathcal{C}}$ is naturally isomorphic to $\sigma_{\mathcal{D}} \circ F$. Then* $\operatorname{FP}(u, t, \sigma_{\mathcal{C}}) \leq \operatorname{FP}(u, t, \sigma_{\mathcal{D}})$.

See Theorem 3.2 for the proof. By taking σ to be the suspension functor of a pretriangulated category [Neeman 2001, Definition 1.1.2], we have the following immediate consequence. (Note that the fp-dimension of a triangulated category $\mathcal{T}$ is defined to be $\operatorname{fpd}(\Sigma)$, where Σ is the suspension of $\mathcal{T}$.)

Corollary 0.2. *Let $\mathcal{T}_2$ be a pretriangulated category and $\mathcal{T}_1$ a full pretriangulated subcategory of $\mathcal{T}_2$. Then the following hold*:

(1) $\operatorname{fpd} \mathcal{T}_1 \leq \operatorname{fpd} \mathcal{T}_2$.

(2) $\operatorname{fpg} \mathcal{T}_1 \leq \operatorname{fpg} \mathcal{T}_2$.

(3) $\operatorname{fpv} \mathcal{T}_1 \leq \operatorname{fpv} \mathcal{T}_2$.

(4) *If $\mathcal{T}_2$ has* fp-*subexponential growth*, *so does $\mathcal{T}_1$*.

Fully faithful embeddings of derived categories of projective schemes have been investigated in the study of Fourier–Mukai transforms, birational geometry, and noncommutative crepant resolutions (NCCRs) by Bondal and Orlov [2001; 2002], Van den Bergh [2004], Bridgeland [2002], Bridgeland, King and Reid [Bridgeland et al. 2001] and more.

Note that if $\operatorname{fpgldim}(\mathcal{T}) < \infty$, then $\operatorname{fpg}(\mathcal{T}) = 0$. If $\operatorname{fpg}(\mathcal{T}) < \infty$, then $\operatorname{fpv}(\mathcal{T}) \leq 1$. Hence, fpd, fpgldim, fpg and fpv measure the "size", "representation type", or "complexity" of a triangulated category $\mathcal{T}$ at different levels. Corollary 0.2 has many consequences concerning nonexistence of fully faithful embeddings provided that we compute the invariants fpd, fpg and fpv of various categories efficiently.

Tame vs wild. Here we mention a couple of more applications. First we extend the classical trichotomy on the representation types of quivers to the fpd. A proof of the following theorem is given in Section 7.

Theorem 0.3. *Let Q be a finite quiver and let $\mathcal{Q}$ be the bounded derived category of finite dimensional left $\Bbbk Q$-modules*:

(1) *$\Bbbk Q$ is of finite representation type if and only if* $\operatorname{fpd} \mathcal{Q} = 0$.

(2) *$\Bbbk Q$ is of tame representation type if and only if* $\operatorname{fpd} \mathcal{Q} = 1$.

(3) *$\Bbbk Q$ is of wild representation type if and only if* $\operatorname{fpd} \mathcal{Q} = \infty$.

By the classical theorems of Gabriel [1972] and Nazarova [1973], the quivers of finite and tame representation types correspond to the ADE and $\tilde{A}\tilde{D}\tilde{E}$ diagrams respectively.

The above theorem fails for quiver algebras with relations (Proposition 5.12). As we have already seen, fpd is related to the "size" of a triangulated category, as well as, the representation types. We will see soon that fpg is also closely connected with the complexity of representations. When we focus on the representation types, we make some tentative definitions.

Let $\mathcal{T}$ be a triangulated category (such as $D^b(\mathrm{Mod}_{f.d.} - A)$):

(i) We call $\mathcal{T}$ fp-*trivial*, if $\operatorname{fpd} \mathcal{T} = 0$.

(ii) We call $\mathcal{T}$ fp-*tame*, if $\operatorname{fpd} \mathcal{T} = 1$.

(iii) We call $\mathcal{T}$ fp-*potentially wild*, if $\operatorname{fpd} \mathcal{T} > 1$. Further:

  (a) $\mathcal{T}$ is fp-*finitely wild*, if $1 < \operatorname{fpd} \mathcal{T} < \infty$.

  (b) $\mathcal{T}$ is fp-*locally-finitely wild*, if $\operatorname{fpd} \mathcal{T} = \infty$ and $\operatorname{fpd}^n(\mathcal{T}) < \infty$ for all n.

  (c) $\mathcal{T}$ is fp-*wild*, if $\operatorname{fpd}^1 \mathcal{T} = \infty$.

There are other notions of tame/wildness in representation theory, see for example, [Geiss and Krause 2002; Drozd 2004]. Following the above definition, fpd provides a quantification of the tame-wild dichotomy. By Theorem 0.3, finite/tame/wild representation types of the path algebra $\Bbbk Q$ are equivalent to the fp-version of these properties of $\mathcal{Q}$. Let A be a quiver algebra with relations and let $\mathcal{A}$ be the derived category $D^b(\mathrm{Mod}_{f.d.}-A)$. Then, in general, finite/tame/wild representation types of A are NOT equivalent to the fp-version of these properties of $\mathcal{A}$ (Example 5.5). It is natural to ask

Question 0.4. For which classes of algebras A, is the fp-wildness of $\mathcal{A}$ equivalent to the classical and other wildness of A in representation theory literature?

Complexity. The complexity of a module or of an algebra is an important invariant in studying representations of finite dimensional algebras [Alperin and Evens 1981; Carlson 1996; Carlson et al. 1994; Guo et al. 2009]. Let A be the quiver algebra $\Bbbk Q/(R)$ with relations R. The *complexity* of A is defined to be the complexity of the A-module $T := A/\operatorname{Jac}(A)$, namely,

$$\operatorname{cx}(A) = \operatorname{cx}(T) := \limsup_{n\to\infty} \log_n(\dim \operatorname{Ext}^n_A(T,T)) + 1.$$

Let GKdim denote the Gelfand–Kirillov dimension of an algebra (see [Krause and Lenagan 1985] and [McConnell and Robson 1987, Chapter 8]). Under some reasonable hypotheses, one can show

$$\operatorname{cx}(A) = \operatorname{GKdim}\Bigl(\bigoplus_{n=0}^{\infty} \operatorname{Ext}^n_A(T,T)\Bigr).$$

It is easy to see that $\operatorname{cx}(A)$ is an derived invariant. We extend the definition of the complexity to any triangulated category (Definition 8.2(4)).

Theorem 0.5. *Let A be a finite dimensional quiver algebra $\Bbbk Q/(R)$ with relations R and let $\mathcal{A}$ be the bounded derived category of finite dimensional left A-modules. Then*

$$\operatorname{fpg}(\mathcal{A}) \le \operatorname{cx}(A) - 1.$$

This theorem is a consequence of Theorems 8.3 and 8.4(1). The equality $\operatorname{fpg}(\mathcal{A}) = \operatorname{cx}(A) - 1$ holds under some hypotheses (Theorem 8.4(2)).

Frobenius–Perron function. If $\mathcal{T}$ is a triangulated category with Serre functor S, we have an fp-function

$$\mathrm{fp} : \mathbb{Z}^2 \to \mathbb{R} \cup \{\pm\infty\}$$

which is defined by

$$\mathrm{fp}(a,b) := \operatorname{fpd}(\Sigma^a \circ S^b) \in \mathbb{R} \cup \{\pm\infty\}.$$

Then $\operatorname{fpd}(\mathcal{T})$ is the value of the fp-function at $(1,0)$.

The fp-function for the projective line $\mathbb{P}^1$ and the quiver A_2 are given in the Examples 5.1 and 5.4 respectively.

The statements in Theorem 0.3, Questions 0.4 and 7.11 indicate that $\operatorname{fp}(1,0)$ predicts the representation type of $\mathcal{T}$ for certain triangulated categories. It is expected that values of the fp-function at other points in $\mathbb{Z}^2$ are sensitive to other properties of $\mathcal{T}$.

Properties. The paper contains some basic properties of fpd. Let us mention one of them, whose proof can be found in Proposition 3.6.

Proposition 0.6 (Serre duality). *Let $\mathcal{C}$ be a* Hom*-finite category with Serre functor S. Let σ be an endofunctor of $\mathcal{C}$*:

(1) *If σ has a right adjoint $\sigma^!$, then*

$$\operatorname{fpd}(\sigma) = \operatorname{fpd}(\sigma^! \circ S).$$

(2) *If σ is an equivalence with quasiinverse σ^{-1}, then*

$$\operatorname{fpd}(\sigma) = \operatorname{fpd}(\sigma^{-1} \circ S).$$

(3) *If $\mathcal{C}$ is n-Calabi–Yau, then we have a duality, for all i,*

$$\operatorname{fpd}(\Sigma^i) = \operatorname{fpd}(\Sigma^{n-i}).$$

Computations. Our third goal is to develop methods for computation. To use fp-invariants, we need to compute as many examples as possible. In general it is extremely difficult to calculate useful invariants for derived categories, as the definitions of these invariants are quite sophisticated. We develop some techniques for computing fp-invariants. In Sections 4–5, we compute the fp-dimension for some nontrivial examples.

Other significant applications. In addition to the results above, various Frobenius–Perron invariants of endofunctors have applications in study of other important objects/structures such as tensor triangulated categories in the sense of [Balmer 2005, Definition 1.1]. Let Q be a finite acyclic quiver and $\Bbbk Q$ be its path algebra. Let $\mathcal{T}_Q$ denote the bounded derived category $D^b(\mathrm{Mod}_{f.d.} - \Bbbk Q)$ of finite dimensional representations of Q. Note that every path algebra $\Bbbk Q$ of a finite quiver Q is naturally equipped with a weak bialgebra structure (where the coalgebra structure is similar to the one given in [Nikshych and Vainerman 2002, Example 2.5]), which implies that $\mathcal{T}_Q$ is a tensor triangulated category. One significant application of $\operatorname{fpv}(\sigma)$ Definition 2.3(4) (for various endofunctors σ) is to prove that two nonisomorphic acyclic finite quivers are not tensor triangulated equivalent. For example, let Q_1 and Q_2 be two nonisomorphic quivers of the same underlying ADE Dynkin graph. It is well-known that $\mathcal{T}_{Q_1}$ and $\mathcal{T}_{Q_2}$ are triangulated equivalent via Bernstein, Gelfand and Ponomarev reflection functors [Bernstein et al. 1973] (also see [Happel 1987]). Now using $\operatorname{fpv}(\sigma)$ it can be shown that $\mathcal{T}_{Q_1}$ and $\mathcal{T}_{Q_2}$ are *not* tensor triangulated equivalent. Details are given in [Zhang and Zhou $\geq$ 2019]. By using other known invariants such as the Balmer spectrum [2005], it is difficult for us to distinguish the tensor triangulated structures of $\mathcal{T}_{Q_1}$ and $\mathcal{T}_{Q_2}$ where these are triangulated equivalent.

Conventions.

(1) Usually Q means a quiver.

(2) $\mathcal{T}$ is a (pre-)triangulated category with suspension functor $\Sigma = [1]$.

(3) If A is an algebra over the base field $\Bbbk$, then $\mathrm{Mod}_{f.d.} -A$ denotes the category of finite dimensional left A-modules.

(4) If A is an algebra, then we use $\mathfrak{A}$ for the abelian category $\mathrm{Mod}_{f.d.} -A$.

(5) When $\mathfrak{A}$ is an abelian category, we use $\mathcal{A}$ for the bounded derived category $D^b(\mathfrak{A})$.

This paper is organized as follows. We provide background material in Section 1. The basic definitions are introduced in Section 2. Some basic properties are given in Section 3. We prove Theorem 0.1 and Proposition 0.6 in Section 3, see Theorem 3.2 and Proposition 3.6 respectively. Corollary 0.2 is an immediate consequence of Theorem 0.1. Section 4 deals with some derived categories of modules over commutative rings. In Section 5, we work out the fp-theories of the projective line and quiver A_2, as well as an example of nonintegral fpd. In Section 6, we develop some techniques to handle the fpd of projective curves and prove the tame-wild dichotomy of projective curves in terms of fpd. Theorem 0.3 is proved in Section 7 where representation types are discussed. Section 8 focuses on the complexity of algebras and categories and Theorem 0.5 is proved there. We continue to develop the fp-theory in our companion paper [Chen et al. 2019]. Some examples can be found in [Wicks 2019; Zhang and Zhou $\geq$ 2019].

1. Preliminaries

Classical definitions. Let A be an $n \times n$-matrix over complex numbers $\mathbb{C}$. The *spectral radius* of A is defined to be

$$\rho(A) := \max\{|r_1|, |r_2|, \dots, |r_n|\} \in \mathbb{R}$$

where $\{r_1, r_2, \dots, r_n\}$ is the complete set of eigenvalues of A. When each entry of A is a positive real number, $\rho(A)$ is also called the *Perron root* or the *Perron–Frobenius eigenvalue* of A. When applying ρ to the adjacency matrix of a graph (or a quiver), the spectral radius of the adjacency matrix is sometimes called the *Frobenius–Perron dimension* of the graph (or the quiver).

Let us mention a classical result concerning the spectral radius of simple graphs. A finite graph G is called *simple* if it has no loops and no multiple edges. Smith [1970] formulated the following result:

Theorem 1.1 [Dokuchaev et al. 2013, Theorem 1.3]. *Let G be a finite, simple, and connected graph with adjacency matrix A:*

(1) *$\rho(A) = 2$ if and only if G is one of the extended Dynkin diagrams of type $\tilde{A}\tilde{D}\tilde{E}$.*

(2) *$\rho(A) < 2$ if and only if G is one of the Dynkin diagrams of type ADE.*

To save space we refer to [Dokuchaev et al. 2013] and [Happel et al. 1980] for the diagrams of the ADE and $\tilde{A}\tilde{D}\tilde{E}$ quivers/graphs.

In order to include some infinite-dimensional cases, we extend the definition of the spectral radius in the following way.

Let $A := (a_{ij})_{n\times n}$ be an $n\times n$-matrix with entries a_{ij} in $\overline{\mathbb{R}} := \mathbb{R}\cup\{\pm\infty\}$. Define $A' = (a'_{ij})_{n\times n}$ where

$$a'_{ij} = \begin{cases} a_{ij} & a_{ij}\neq \pm\infty, \\ x_{ij} & a_{ij}=\infty, \\ -x_{ij} & a_{ij}=-\infty. \end{cases}$$

In other words, we are replacing ∞ in the (i, j)-entry by a finite real number, called x_{ij}, in the (i, j)-entry. And every x_{ij} is considered as a variable or a function mapping $\mathbb{R}\to\mathbb{R}$.

Definition 1.2. Let A be an $n\times n$-matrix with entries in $\overline{\mathbb{R}}$. The *spectral radius* of A is defined to be

$$\rho(A) := \liminf_{\text{all } x_{ij}\to\infty} \rho(A') \in \overline{\mathbb{R}}. \tag{E1.2.1}$$

Remark 1.3. It also makes sense to use lim sup instead of lim inf in (E1.2.1). We choose to take lim inf in this paper.

Here is an easy example.

Example 1.4. Let $A = \left(\begin{smallmatrix} 1 & -\infty \\ 0 & 2\end{smallmatrix}\right)$. Then $A' = \left(\begin{smallmatrix} 1 & -x_{12} \\ 0 & 2\end{smallmatrix}\right)$. It is obvious that

$$\rho(A) = \lim_{x_{12}\to\infty} \rho(A') = \lim_{x_{12}\to\infty} 2 = 2.$$

$\Bbbk$-linear categories. If $\mathcal{C}$ is a $\Bbbk$-linear category, then $\operatorname{Hom}_{\mathcal{C}}(M, N)$ is a $\Bbbk$-module for all objects M, N in $\mathcal{C}$. If $\mathcal{C}$ is also abelian, then $\operatorname{Ext}^i_{\mathcal{C}}(M, N)$ are $\Bbbk$-modules for all $i\geq 0$. Let dim be the $\Bbbk$-vector space dimension.

Remark 1.5. One can generalize the notion of fpd to categories that are not $\Bbbk$-linear. Even when a category $\mathcal{C}$ is not $\Bbbk$-linear, it might still make sense to define a set map on the Hom-sets of the category $\mathcal{C}$, say

$$\partial : \{\operatorname{Hom}_{\mathcal{C}}(M, N) \mid M, N\in\mathcal{C}\} \to \mathbb{Z}_{\geq 0}\cup\{\infty\}.$$

We call such a map a *dimension function*. The definition of Frobenius–Perron dimension given in the next section can be modified using ∂ instead of dim to fit this very weak version of a dimension function.

Frobenius–Perron dimension of a quiver. In this subsection we recall some known elementary definitions and facts.

Definition 1.6. Let Q be a quiver:

(1) If Q has finitely many vertices, then the *Frobenius–Perron dimension* of Q is defined to be

$$\operatorname{fpd} Q := \rho(A(Q))$$

where $A(Q)$ is the adjacency matrix of Q.

(2) Let Q be any quiver. The *Frobenius–Perron dimension* of Q is defined to be

$$\operatorname{fpd} Q := \sup\{\operatorname{fpd} Q'\}$$

where Q' runs over all finite subquivers of Q.

See [Erdmann and Solberg 2011, Propositions 2.1 and 3.2] for connections between fpd of a quiver and its representation types, as well as its complexity. We need the following well-known facts in linear algebra.

Lemma 1.7. (1) *Let B be a square matrix with nonnegative entries and let A be a principal minor of B. Then $\rho(A) \le \rho(B)$.*

(2) *Let $A := (a_{ij})_{n\times n}$ and $B := (b_{ij})_{n\times n}$ be two square matrices such that $0 \le a_{ij} \le b_{ij}$ for all i, j. Then $\rho(A) \le \rho(B)$.*

Let Q be a quiver with vertices $\{v_1, \dots, v_n\}$. An oriented cycle based at a vertex v_i is called *indecomposable* if it is not a product of two oriented cycles based at v_i. For each vertex v_i let θ_i be the number of indecomposable oriented cycles based at v_i. Define the *cycle number* of a quiver Q to be

$$\Theta(Q) := \max\{\theta_i \mid \forall i\}.$$

The following result should be well known.

Theorem 1.8. *Let Q be a quiver and let $\Theta(Q)$ be the cycle number of Q:*

(1) $\operatorname{fpd}(Q) = 0$ *if and only if* $\Theta(Q) = 0$, *namely, Q is acyclic.*

(2) $\operatorname{fpd}(Q) = 1$ *if and only if* $\Theta(Q) = 1$.

(3) $\operatorname{fpd}(Q) > 1$ *if and only if* $\Theta(Q) \ge 2$.

The proof is not hard, and to save space, it is omitted.

2. Definitions

Throughout the rest of the paper, let $\mathcal{C}$ denote a $\Bbbk$-linear category. A functor between two $\Bbbk$-linear categories is assumed to preserve the $\Bbbk$-linear structure. For simplicity, $\dim(A, B)$ stands for $\dim \operatorname{Hom}_{\mathcal{C}}(A, B)$ for any objects A and B in $\mathcal{C}$.

The set of finite subsets of nonzero objects in $\mathcal{C}$ is denoted by Φ and the set of subsets of n nonzero objects in $\mathcal{C}$ is denoted by Φ_n for each $n \ge 1$. It is clear that $\Phi = \bigcup_{n\ge 1} \Phi_n$. We do not consider the empty set as an element of Φ.

Definition 2.1. Let $\phi := \{X_1, X_2, \dots, X_n\}$ be a finite subset of nonzero objects in $\mathcal{C}$, namely, $\phi \in \Phi_n$. Let σ be an endofunctor of $\mathcal{C}$:

(1) The *adjacency matrix* of (ϕ, σ) is defined to be

$$A(\phi, \sigma) := (a_{ij})_{n\times n} \quad \text{where } a_{ij} := \dim(X_i, \sigma(X_j)) \ \forall i, j.$$

(2) An object M in $\mathcal{C}$ is called a *brick* [Assem et al. 2006, Definition 2.4, Chapter VII] if

$$\operatorname{Hom}_{\mathcal{C}}(M, M) = \Bbbk.$$

[Neeman 2001, Definition 1.1.2], an object M in $\mathcal{C}$ is called an *atomic* object if it is a brick and satisfies

$$\operatorname{Hom}_{\mathcal{C}}(M, \Sigma^{-i}(M)) = 0 \quad \forall i > 0. \tag{E2.1.1}$$

(3) $\phi \in \Phi$ is called a *brick set* (respectively, an *atomic set*) if each X_i is a brick (respectively, atomic) and

$$\dim(X_i, X_j) = \delta_{ij}$$

for all $1 \leq i, j \leq n$. The set of brick (respectively, atomic) n-object subsets is denoted by $\Phi_{n,b}$ (respectively, $\Phi_{n,a}$). We write $\Phi_b = \bigcup_{n \geq 1} \Phi_{n,b}$ (respectively, $\Phi_a = \bigcup_{n \geq 1} \Phi_{n,a}$). Define the *b-height* of $\mathcal{C}$ to be

$$h_b(\mathcal{C}) = \sup\{n \mid \Phi_{n,b} \text{ is nonempty}\}$$

and the *a-height* of $\mathcal{C}$ (when $\mathcal{C}$ is pretriangulated) to be

$$h_a(\mathcal{C}) = \sup\{n \mid \Phi_{n,a} \text{ is nonempty}\}.$$

Remarks 2.2. (1) A brick may not be atomic. Let A be the algebra

$$\Bbbk\langle x, y\rangle/(x^2, y^2 - 1, xy + yx).$$

This is a 4-dimensional Frobenius algebra (of injective dimension zero). There are two simple left A-modules

$$S_0 := A/(x, y - 1), \quad \text{and} \quad S_1 := A/(x, y + 1).$$

Let M_i be the injective hull of S_i for $i = 0, 1$. (Since A is Frobenius, M_i is projective.) There are two short exact sequences

$$0 \to S_0 \to M_0 \xrightarrow{f} S_1 \to 0 \quad \text{and} \quad 0 \to S_1 \xrightarrow{g} M_1 \to S_0 \to 0.$$

It is easy to check that $\operatorname{Hom}_A(M_i, M_j) = \Bbbk$ for all $0 \leq i, j \leq 1$. Let $\mathcal{A}$ be the derived category $D^b(\operatorname{Mod}_{f.d.} - A)$ and let X be the complex

$$\cdots \to 0 \to M_0 \xrightarrow{g \circ f} M_1 \to 0 \to \cdots$$

An easy computation shows that $\operatorname{Hom}_{\mathcal{A}}(X, X) = \Bbbk = \operatorname{Hom}_{\mathcal{A}}(X, X[-1])$. So X is a brick, but not atomic.

(2) A brick object is called a *Schur* object by several authors, see [Carroll and Chindris 2015; Chindris et al. 2015]. It is also called *endosimple* by others, see [van Roosmalen 2008; 2016].

(3) The definition of an atomic object in a triangulated category is similar to (and slightly weaker than) the definition of a point-object given by Bondal and Orlov [2001, Definition 2.1]. In particular, an atomic object only satisfies (ii) and (iii) of that definition with $k(P) = \Bbbk$. Note that a point-object is defined on a

triangulated category with Serre functor. In this paper we do not automatically assume the existence of a Serre functor in general.

Definition 2.3. Retain the notation as in Definition 2.1, and we use Φ_b as the testing objects. When $\mathcal{C}$ is a pretriangulated category, Φ_b is automatically replaced by Φ_a unless otherwise stated:

(1) The *n-th Frobenius–Perron dimension* of σ is defined to be

$$\operatorname{fpd}^n(\sigma) := \sup_{\phi \in \Phi_{n,b}} \{\rho(A(\phi, \sigma))\}.$$

If $\Phi_{n,b}$ is empty, then by convention, $\operatorname{fpd}^n(\sigma) = -\infty$.

(2) The *Frobenius–Perron dimension* of σ is defined to be

$$\operatorname{fpd}(\sigma) := \sup_n \{\operatorname{fpd}^n(\sigma)\} = \sup_{\phi \in \Phi_b} \{\rho(A(\phi, \sigma))\}.$$

(3) The *Frobenius–Perron growth* of σ is defined to be

$$\operatorname{fpg}(\sigma) := \sup_{\phi \in \Phi_b} \{\limsup_{n \to \infty} \log_n(\rho(A(\phi, \sigma^n)))\}.$$

By convention, $\log_n 0 = -\infty$.

(4) The *Frobenius–Perron curvature* of σ is defined to be

$$\operatorname{fpv}(\sigma) := \sup_{\phi \in \Phi_b} \{\limsup_{n \to \infty} (\rho(A(\phi, \sigma^n)))^{1/n}\}.$$

This is motivated by the concept of the *curvature* of a module over an algebra due to Avramov [1998].

(5) We say σ has fp-*exponential growth* (respectively, fp-*subexponential growth*) if $\operatorname{fpv}(\sigma) > 1$ (respectively, $\operatorname{fpv}(\sigma) \leq 1$).

In this above definition, we implicitly assume that

the isom-classes of brick objects (respectively, atomic objects) form a set,

otherwise, $\sup_{\phi \in \Phi_b}$ (respectively, $\sup_{\phi \in \Phi_a}$) is not defined. This assumption is automatic if the category $\mathcal{C}$ is essentially small. But, even when $\mathcal{C}$ is not essentially small, one can check the above assumption in many cases.

Sometimes we prefer to have all information from the Frobenius–Perron dimension. We make the following definition.

Definition 2.4. Let $\mathcal{C}$ be a category and σ be an endofunctor of $\mathcal{C}$:

(1) The *Frobenius–Perron theory* (or fp-theory) of σ is defined to be the set

$$\{\operatorname{fpd}^n(\sigma^m)\}_{n \geq 1, m \geq 0}.$$

(2) The *Frobenius–Perron series* (or fp-series) of σ is defined to be

$$\mathrm{FP}(u,t,\sigma) := \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \mathrm{fpd}^n(\sigma^m)t^m u^n.$$

Remark 2.5. To define Frobenius–Perron dimension, one only needs have an assignment $\tau : \Phi_n \to M_{n\times n}(\mathrm{Mod}-\Bbbk)$, for every $n \geq 1$, satisfying the property

if ϕ_1 is a subset of ϕ_2, then $\tau(\phi_1)$ is a principal submatrix of $\tau(\phi_2)$.

Then we define the adjacency matrix of $\phi \in \Phi_n$ to be

$$A(\phi,\tau) = (a_{ij})_{n\times n} \quad \text{where } a_{ij} = \dim(\tau(\phi))_{ij}\ \forall i, j.$$

Then the Frobenius–Perron dimension of τ is defined in the same way as in Definition 2.3. If there is a sequence of τ_m, the Frobenius–Perron series of $\{\tau_m\}$ is defined in the same way as in Definition 2.4 by replacing σ^m by τ_m. See Example 2.6 next.

Example 2.6. (1) Let $\mathfrak{A}$ be a $\Bbbk$-linear abelian category. For each $m \geq 1$ and $\phi = \{X_1, \dots, X_n\}$, define

$$E^m : \phi \to (\mathrm{Ext}^m_{\mathfrak{A}}(X_i, X_j))_{n\times n}.$$

By convention, let $\mathrm{Ext}^0_{\mathfrak{A}}(X_i, X_j)$ denote $\mathrm{Hom}_{\mathfrak{A}}(X_i, X_j)$. Then, for each $m \geq 0$, one can define the Frobenius–Perron dimension of E^m as mentioned in Remark 2.5.

(2) Let $\mathfrak{A}$ be the $\Bbbk$-linear abelian category $\mathrm{Mod}_{f.d.}-A$ where A is a finite dimensional commutative algebra over a base field $\Bbbk$. For each $m \geq 1$ and $\phi = \{X_1, \dots, X_n\}$, define

$$T_m : \phi \to (\mathrm{Tor}^A_m(X_i, X_j))_{n\times n}.$$

By convention, let $\mathrm{Tor}^A_0(X_i, X_j)$ denote $X_i \otimes_A X_j$. Then, for each $m \geq 0$, one can define the Frobenius–Perron dimension of T_m as mentioned in Remark 2.5.

Definition 2.7. (1) Let $\mathfrak{A}$ be an abelian category. The *Frobenius–Perron dimension* of $\mathfrak{A}$ is defined to be

$$\mathrm{fpd}\,\mathfrak{A} := \mathrm{fpd}(E^1)$$

where $E^1 := \mathrm{Ext}^1_{\mathfrak{A}}(-,-)$ is defined as in Example 2.6(1). The *Frobenius–Perron theory* of $\mathfrak{A}$ is the collection

$$\{\mathrm{fpd}^m(E^n)\}_{m\geq 1, n\geq 0}$$

where $E^n := \mathrm{Ext}^n_{\mathcal{A}}(-,-)$ is defined as in Example 2.6(1).

(2) Let $\mathcal{T}$ be a pretriangulated category with suspension Σ. The *Frobenius–Perron dimension* of $\mathcal{T}$ is defined to be

$$\mathrm{fpd}\,\mathcal{T} := \mathrm{fpd}(\Sigma).$$

The *Frobenius–Perron theory* of $\mathcal{T}$ is the collection

$$\{\mathrm{fpd}^m(\Sigma^n)\}_{m\geq 1, n\in\mathbb{Z}}.$$

The fp-*global dimension* of $\mathcal{T}$ is defined to be

$$\mathrm{fpgldim}\,\mathcal{T} := \sup\{n \mid \mathrm{fpd}(\Sigma^n) \neq 0\}.$$

If $\mathcal{T}$ possesses a Serre functor S, the *Frobenius–Perron S-theory* of $\mathcal{T}$ is the collection

$$\{\mathrm{fpd}^m(\Sigma^n \circ S^w)\}_{m\geq 1, n, w\in\mathbb{Z}}.$$

Remarks 2.8. (1) The Frobenius–Perron dimension (respectively, Frobenius–Perron theory, fp-global dimension) can be defined for suspended categories [Keller and Vossieck 1987] and pre-n-angulated categories [Geiss et al. 2013] in the same way as Definition 2.7(2) since there is a suspension functor Σ.

(2) When $\mathfrak{A}$ is an abelian category, another way of defining the Frobenius–Perron dimension $\mathrm{fpd}\,\mathfrak{A}$ is as follows. We first embed $\mathfrak{A}$ into the derived category $D^b(\mathfrak{A})$. The suspension functor Σ of $D^b(\mathfrak{A})$ maps $\mathfrak{A}$ to $\mathfrak{A}[1]$ (so it is not a functor of $\mathfrak{A}$). The adjacency matrix $A(\phi, \Sigma)$ is still defined as in Definition 2.1(1) for brick sets ϕ in $\mathfrak{A}$. Then we can define

$$\mathrm{fpd}(\Sigma|_{\mathfrak{A}}) := \sup_{\phi\in\Phi_b, \phi\subset\mathfrak{A}} \{\rho(A(\phi, \Sigma))\}$$

as in Definition 2.3(2) by considering only the brick sets in $\mathfrak{A}$. Now $\mathrm{fpd}(\mathfrak{A})$ agrees with $\mathrm{fpd}(\Sigma|_{\mathfrak{A}})$.

The following lemma is clear.

Lemma 2.9. *Let $\mathfrak{A}$ be an abelian category and $n \geq 1$. Then $\mathrm{fpd}^n(D^b(\mathfrak{A})) \geq \mathrm{fpd}^n(\mathfrak{A})$. A similar statement holds for* fpd, fpg *and* fpv.

Proof. This follows from the fact that there is a fully faithful embedding $\mathfrak{A} \to D^b(\mathfrak{A})$ and that E^1 on $\mathfrak{A}$ agrees with Σ on $D^b(\mathfrak{A})$. □

For any category $\mathcal{C}$ with an endofunctor σ, we define the σ*-quiver* of $\mathcal{C}$, denoted by $Q^\sigma_{\mathcal{C}}$, as follows:

(1) the vertex set of $Q^\sigma_{\mathcal{C}}$ consists of bricks in $\Phi_{1,b}$ in $\mathcal{C}$ (respectively, atomic objects in $\Phi_{1,a}$ when $\mathcal{C}$ is pretriangulated), and

(2) the arrow set of $Q^\sigma_{\mathcal{C}}$ consists of $n_{X,Y}$-arrows from X to Y, for all $X, Y \in \Phi_{1,b}$ (respectively, in $\Phi_{1,a}$), where $n_{X,Y} = \dim(X, \sigma(Y))$.

If σ is E^1, this quiver is denoted by $Q^{E^1}_{\mathcal{C}}$, which will be used in later sections.

The following lemma follows from the definition.

Lemma 2.10. *Retain the above notation. Then* $\mathrm{fpd}\,\sigma \leq \mathrm{fpd}\, Q^\sigma_{\mathcal{C}}$.

The fp-theory was motivated by the Frobenius–Perron dimension of objects in tensor or fusion categories [Etingof et al. 2015], see the following example.

Example 2.11. First we recall the definition of the Frobenius–Perron dimension given in [Etingof et al. 2015, Definitions 3.3.3 and 6.1.6]. Let $\mathcal{C}$ be a finite semisimple $\Bbbk$-linear tensor category. Suppose that $\{X_1, \dots, X_n\}$ is the complete list of nonisomorphic simple objects in $\mathcal{C}$. Since $\mathcal{C}$ is semisimple, every object X in $\mathcal{C}$ is a direct sum

$$X = \bigoplus_{i=1}^{n} X_i^{\oplus a_i}$$

for some integers $a_i \geq 0$. The tensor product on $\mathcal{C}$ makes its Grothendieck ring $\mathrm{Gr}(\mathcal{C})$ a $\mathbb{Z}_+$-ring [loc. cit., Definition 3.1.1]. For every object V in $\mathcal{C}$ and every j, write

$$V \otimes_{\mathcal{C}} X_j \cong \bigoplus_{i=1}^{n} X_i^{\oplus a_{ij}} \tag{E2.11.1}$$

for some integers $a_{ij} \geq 0$. In the Grothendieck ring $\mathrm{Gr}(\mathcal{C})$, the left multiplication by V sends X_j to $\sum_{i=1}^{n} a_{ij} X_i$. Then, by [loc. cit., Definition 3.3.3], the *Frobenius–Perron dimension* of V is defined to be

$$\mathrm{FPdim}(V) := \rho((a_{ij})_{n\times n}). \tag{E2.11.2}$$

In fact the Frobenius–Perron dimension is defined for any object in a $\mathbb{Z}_+$-ring.

Next we use Definition 2.3(2) to calculate the Frobenius–Perron dimension. Let σ be the tensor functor $V \otimes_{\mathcal{C}} -$ that is a $\Bbbk$-linear endofunctor of $\mathcal{C}$. If ϕ is a brick subset of $\mathcal{C}$, then ϕ is a subset of $\phi_n := \{X_1, \dots, X_n\}$. For simplicity, assume that ϕ is $\{X_1, \dots, X_s\}$ for some $s \leq n$. It follows from (E2.11.1) that

$$\mathrm{Hom}_{\mathcal{C}}(X_i, \sigma(X_j)) = \Bbbk^{\oplus a_{ij}} \quad \forall i, j.$$

Hence the adjacency matrix of (ϕ_n, σ) is

$$A(\phi_n, \sigma) = (a_{ij})_{n\times n}$$

and the adjacency matrix of (ϕ, σ) is a principal minor of $A(\phi_n, \sigma)$. By Lemma 1.7(1), $\rho(A(\phi, \sigma)) \leq \rho(A(\phi_n, \sigma))$. By Definition 2.3(2), the *Frobenius–Perron dimension* of the functor $\sigma = V \otimes_{\mathcal{C}} -$ is

$$\mathrm{fpd}(V \otimes_{\mathcal{C}} -) = \sup_{\phi \in \Phi_b} \{\rho(A(\phi, \sigma))\} = \rho(A(\phi_n, \sigma)) = \rho((a_{ij})_{n\times n}),$$

which agrees with (E2.11.2). This justifies calling $\mathrm{fpd}(V \otimes_{\mathcal{C}} -)$ the Frobenius–Perron dimension of V.

3. Basic properties

For simplicity, “Frobenius–Perron” is abbreviated to “fp”.

Embeddings. It is clear that the fp-series and the fp-dimensions are invariant under equivalences of categories. We record this fact below. Recall that the Frobenius–Perron series $\mathrm{FP}(u, t, \sigma)$ of an endofunctor σ is defined in Definition 2.4(2).

Lemma 3.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of categories. Let $\sigma_{\mathcal{C}}$ and $\sigma_{\mathcal{D}}$ be endofunctors of $\mathcal{C}$ and $\mathcal{D}$ respectively. Suppose that $F \circ \sigma_{\mathcal{C}}$ is naturally isomorphic to $\sigma_{\mathcal{D}} \circ F$. Then* $\mathrm{FP}(u, t, \sigma_{\mathcal{C}}) = \mathrm{FP}(u, t, \sigma_{\mathcal{D}})$.

Let $\overline{\mathbb{R}}_+$ denote the set of nonnegative real numbers union with $\{\pm\infty\}$. Let

$$f(u,t) := \sum_{m,n=0}^{\infty} f_{m,n} t^m u^n \quad \text{and} \quad g(u,t) := \sum_{m,n=0}^{\infty} g_{m,n} t^m u^n$$

be two elements in $\overline{\mathbb{R}}_+[[u,t]]$. We write $f \le g$ if $f_{m,n} \le g_{m,n}$ for all m, n.

Theorem 3.2. *Let $F : \mathcal{C} \to \mathcal{D}$ be a faithful functor that preserves brick subsets*:

(1) *Let $\sigma_{\mathcal{C}}$ and $\sigma_{\mathcal{D}}$ be endofunctors of $\mathcal{C}$ and $\mathcal{D}$ respectively. Suppose that $F \circ \sigma_{\mathcal{C}}$ is naturally isomorphic to $\sigma_{\mathcal{D}} \circ F$. Then* $\mathrm{FP}(u, t, \sigma_{\mathcal{C}}) \le \mathrm{FP}(u, t, \sigma_{\mathcal{D}})$.

(2) *Let $\tau_{\mathcal{C}}$ and $\tau_{\mathcal{D}}$ be assignments of $\mathcal{C}$ and $\mathcal{D}$ respectively satisfying the property in Remark 2.5. Suppose that $\rho(A(\phi, \tau_{\mathcal{C}})) \le \rho(A(F(\phi), \tau_{\mathcal{D}}))$ for all $\phi \in \Phi_{n,b}(\mathcal{C})$ and all n. Then* $\mathrm{FP}(u, t, \tau_{\mathcal{C}}) \le \mathrm{FP}(u, t, \tau_{\mathcal{D}})$.

Proof. (1) For every $\phi = \{X_1, \dots, X_n\} \in \Phi_n(\mathcal{C})$, let $F(\phi)$ be $\{F(X_1), \dots, F(X_n)\}$ in $\Phi_n(\mathcal{D})$. By hypothesis, if $\phi \in \Phi_{n,b}(\mathcal{C})$, then $F(\phi)$ is in $\Phi_{n,b}(\mathcal{D})$. Let $A = (a_{ij})$ (respectively, $B = (b_{ij})$) be the adjacency matrix of $(\phi, \sigma_{\mathcal{C}})$ (respectively, of $(F(\phi), \sigma_{\mathcal{D}})$). Then, by the faithfulness of F,

$$a_{ij} = \dim(X_i, \sigma_{\mathcal{C}}(X_j)) \le \dim(F(X_i), F(\sigma_{\mathcal{C}}(X_j))) = \dim(F(X_i), \sigma_{\mathcal{D}}(F(X_j))) = b_{ij}.$$

By Lemma 1.7(2),

$$\rho(A(\phi, \sigma_{\mathcal{C}})) =: \rho(A) \le \rho(B) := \rho(A(F(\phi), \sigma_{\mathcal{D}})). \tag{E3.2.1}$$

By definition,

$$\mathrm{fpd}^n(\sigma_{\mathcal{C}}) \le \mathrm{fpd}^n(\sigma_{\mathcal{D}}). \tag{E3.2.2}$$

Similarly, for all n, m, $\mathrm{fpd}^n(\sigma_{\mathcal{C}}^m) \le \mathrm{fpd}^n(\sigma_{\mathcal{D}}^m)$. The assertion follows.

(2) The proof of part (2) is similar. □

Theorem 0.1 follows from Theorem 3.2.

(a-)Hereditary algebras and categories. Recall that the global dimension of an abelian category $\mathfrak{A}$ is defined to be

$$\operatorname{gldim} \mathfrak{A} := \sup\{n \mid \mathrm{Ext}^n_{\mathfrak{A}}(X, Y) \neq 0, \text{ for some } X, Y \in \mathfrak{A}\}.$$

The global dimension of an algebra A is defined to be the global dimension of the category of left A-modules. An algebra (or an abelian category) is called *hereditary* if it has global dimension at most one.

There is a nice property concerning the indecomposable objects in the derived category of a hereditary abelian category (see [loc. cit., Section 2.5]).

Lemma 3.3. *Let $\mathfrak{A}$ be a hereditary abelian category. Then every indecomposable object in the derived category $D(\mathfrak{A})$ is isomorphic to a shift of an object in $\mathfrak{A}$.*

Note that every brick (or atomic) object in an additive category is indecomposable. Based on the property in the above lemma, we make a definition.

Definition 3.4. An abelian category $\mathfrak{A}$ is called *a-hereditary* (respectively, *b-hereditary*) if every atomic (respectively, brick) object X in the bounded derived category $D^b(\mathfrak{A})$ is of the form $M[i]$ for some object M in $\mathfrak{A}$ and $i \in \mathbb{Z}$. The object M is automatically a brick object in $\mathfrak{A}$.

By Lemma 3.11(2), if A is a finite dimensional local algebra, then the category $\mathrm{Mod}_{f.d.}-A$ of finite dimensional A-modules is a-hereditary. If A is not $\Bbbk$, then $\mathrm{Mod}_{f.d.}-A$ is not hereditary. Another such example is given in Lemma 4.1.

If α is an autoequivalence of an abelian category $\mathfrak{A}$, then it extends naturally to an autoequivalence, denoted by $\bar{\alpha}$, of the derived category $\mathcal{A} := D^b(\mathfrak{A})$. The main result in this subsection is the following. Recall that the b-height of $\mathfrak{A}$, denoted by $h_b(\mathfrak{A})$, is defined in Definition 2.1(3) and that the Frobenius–Perron global dimension of $\mathcal{A}$, denoted by $\operatorname{fpgldim} \mathcal{A}$, is defined in Definition 2.7(2).

Theorem 3.5. *Let $\mathfrak{A}$ be an a-hereditary abelian category with an auto-equivalence α. For each n, define $n' = \min\{n, h_b(\mathfrak{A})\}$. Let $\mathcal{A}$ be $D^b(\mathfrak{A})$:*

(1) *If $m < 0$ or $m > \operatorname{gldim} \mathfrak{A}$, then*

$$\operatorname{fpd}(\Sigma^m \circ \bar{\alpha}) = 0.$$

As a consequence, $\operatorname{fpgldim} \mathcal{A} \leq \operatorname{gldim} \mathfrak{A}$.

(2) *For each n,*

$$\operatorname{fpd}^n(\alpha) \leq \operatorname{fpd}^n(\bar{\alpha}) \leq \max_{1 \leq i \leq n'} \{\operatorname{fpd}^i(\alpha)\}. \tag{E3.5.1}$$

If $\operatorname{gldim} \mathfrak{A} < \infty$, then

$$\operatorname{fpd}^n(\bar{\alpha}) = \max_{1 \leq i \leq n'} \{\operatorname{fpd}^i(\alpha)\}. \tag{E3.5.2}$$

(3) *Let $g := \operatorname{gldim} \mathfrak{A} < \infty$. Let β be the assignment $(X, Y) \to (\operatorname{Ext}^g_{\mathfrak{A}}(X, \alpha(Y)))$. Then*

$$\operatorname{fpd}^n(\Sigma^g \circ \bar{\alpha}) = \max_{1 \leq i \leq n'} \{\operatorname{fpd}^i(\beta)\}. \tag{E3.5.3}$$

(4) *For every hereditary abelian category $\mathfrak{A}$, we have $\operatorname{fpd}(\mathcal{A}) = \operatorname{fpd}(\mathfrak{A})$.*

Proof. (1) Since $\mathcal{A}$ is a-hereditary, every atomic object in $\mathcal{A}$ is of the form $M[i]$.

Case 1: $m < 0$. Write ϕ as $\{M_1[d_1], \ldots, M_n[d_n]\}$ where d_i is decreasing and M_i is in $\mathfrak{A}$. Then, for $i \leq j$,

$$a_{ij} = \operatorname{Hom}_{\mathcal{A}}(M_i[d_i], (\Sigma^m \circ \bar{\alpha}) M_j[d_j]) = \operatorname{Hom}_{\mathcal{A}}(M_i, \alpha(M_j)[d_j - d_i + m]) = 0$$

since $d_j - d_i + m < 0$. Thus the adjacency matrix $A := (a_{ij})_{n \times n}$ is strictly lower triangular. As a consequence, $\rho(A) = 0$. By definition, $\operatorname{fpd}(\Sigma^m \circ \bar{\alpha}) = 0$.

Case 2: $m > \operatorname{gldim}\mathfrak{A}$. Write ϕ as $\{M_1[d_1], \dots, M_n[d_n]\}$ where d_i is increasing and M_i is in $\mathfrak{A}$. Then, for $i \geq j$,

$$a_{ij} = \operatorname{Hom}_{\mathcal{A}}(M_i[d_i], (\Sigma^m \circ \bar{\alpha})M_j[d_j]) = \operatorname{Hom}_{\mathcal{A}}(M_i, \alpha(M_j)[d_j - d_i + m]) = 0$$

since $d_j - d_i + m > \operatorname{gldim}\mathfrak{A}$. Thus the adjacency matrix $A := (a_{ij})_{n\times n}$ is strictly upper triangular. As a consequence, $\rho(A) = 0$. By definition, $\operatorname{fpd}(\Sigma^m \circ \bar{\alpha}) = 0$.

(2) Let F be the canonical fully faithful embedding $\mathfrak{A} \to \mathcal{A}$. By Theorem 3.2 and (E3.2.2),

$$\operatorname{fpd}^n(\alpha) \leq \operatorname{fpd}^n(\bar{\alpha}).$$

For the other assertion, write ϕ as a disjoint union $\phi_{d_1} \cup \cdots \cup \phi_{d_s}$ where d_i is strictly decreasing and the subset ϕ_{d_i} consists of objects of the form $M[d_i]$ for $M \in \mathfrak{A}$. For any objects $X \in \phi_{d_i}$ and $Y \in \phi_{d_j}$ for $i < j$, $\operatorname{Hom}_{\mathcal{A}}(X, Y) = 0$. Thus the adjacency matrix of $(\phi, \bar{\alpha})$ is of the form

$$A(\phi, \bar{\alpha}) = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ * & A_{22} & 0 & \cdots & 0 \\ * & * & A_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ * & * & * & \cdots & A_{ss} \end{pmatrix} \tag{E3.5.4}$$

where each A_{ii} is the adjacency matrix $A(\phi_{d_i}, \bar{\alpha})$. For each ϕ_{d_i}, we have

$$A(\phi_{d_i}, \bar{\alpha}) = A(\phi_{d_i}[-d_i], \bar{\alpha}) = A(\phi_{d_i}[-d_i], \alpha)$$

which implies that

$$\rho(A_{ii}) \leq \operatorname{fpd}^{s_i}(\alpha) \leq \max_{1\leq j\leq n'} \operatorname{fpd}^j(\alpha)$$

where s_i is the size of A_{ii} and $n' = \min\{n, h_b(\mathfrak{A})\}$. By using the matrix (E3.5.4),

$$\rho(A(\phi, \bar{\alpha})) = \max_i\{\rho(A_{ii})\} \leq \max_{1\leq j\leq n'} \operatorname{fpd}^j(\alpha).$$

Then (E3.5.1) follows.

Suppose now that $g := \operatorname{gldim}\mathfrak{A} < \infty$. Let $\phi \in \Phi_{n,a}(\mathcal{A})$. Pick any $M \in \Phi_{1,b}(\mathfrak{A})$. Then, for $g' \gg g$, $\phi' := \phi \cup \{M[g']\} \in \Phi_{n+1,a}(\mathcal{A})$. By Lemma 1.7(1), $\rho(A(\phi', \bar{\alpha})) \geq \rho(A(\phi, \bar{\alpha}))$. Hence $\operatorname{fpd}^n(\bar{\alpha})$ is increasing as n increases. Therefore (E3.5.2) follows from (E3.5.1).

(3) The proof is similar to the proof of part (2). Let F be the canonical fully faithful embedding $\mathfrak{A} \to \mathcal{A}$. By Theorem 3.2(2) and (E3.2.2),

$$\operatorname{fpd}^n(\beta) \leq \operatorname{fpd}^n(\Sigma^g \circ \bar{\alpha}).$$

By the argument at the end of proof of part (2), $\operatorname{fpd}^n(\Sigma^g \circ \bar{\alpha})$ increases when n increases. Then

$$\max_{1\leq j\leq n'} \operatorname{fpd}^j(\beta) \leq \operatorname{fpd}^n(\Sigma^g \circ \bar{\alpha}).$$

For the other direction, write ϕ as a disjoint union $\phi_{d_1} \cup \cdots \cup \phi_{d_s}$ where d_i is strictly increasing and ϕ_{d_i} consists of objects of the form $M[d_i]$ for $M \in \mathfrak{A}$. For objects $X \in \phi_{d_i}$ and $Y \in \phi_{d_j}$ for $i < j$, $\mathrm{Hom}_{\mathcal{A}}(X, \Sigma^g(\alpha(Y))) = 0$. Let $\gamma = \Sigma^g \circ \bar{\alpha}$. Then the adjacency matrix of (ϕ, γ) is of the form (E3.5.4), namely,

$$A(\phi,\gamma) = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ * & A_{22} & 0 & \cdots & 0 \\ * & * & A_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ * & * & * & \cdots & A_{ss} \end{pmatrix}$$

where each A_{ii} is the adjacency matrix $A(\phi_{d_i}, \gamma)$. For each ϕ_{d_i}, we have

$$A(\phi_{d_i}, \gamma) = A(\phi_{d_i}[-d_i], \gamma) = A(\phi_{d_i}[-d_i], \beta)$$

which implies that

$$\rho(A_{ii}) \le \mathrm{fpd}^{s_i}(\beta) \le \max_{1 \le j \le n'} \mathrm{fpd}^j(\beta)$$

where s_i is the size of A_{ii}. By using matrix (E3.5.4),

$$\rho(A(\phi,\gamma)) = \max_i \{\rho(A_{ii})\} \le \max_{1 \le j \le n'} \mathrm{fpd}^j(\beta).$$

The assertion follows.

(4) Take α to be the identity functor of $\mathfrak{A}$ and $g = 1$ (since $\mathfrak{A}$ is hereditary). By (E3.5.3), we have

$$\mathrm{fpd}^n(\Sigma) = \max_{1 \le i \le n'} \{\mathrm{fpd}^i(E^1)\}.$$

By taking $\sup_n$, we obtain that $\mathrm{fpd}(E^1) = \mathrm{fpd}(\Sigma)$. The assertion follows. □

Categories with Serre functor. Recall from [Keller 2008, Section 2.6] that if a Hom-finite category $\mathcal{C}$ has a Serre functor S, then there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(X, Y)^* \cong \mathrm{Hom}_{\mathcal{C}}(Y, S(X))$$

for all $X, Y \in \mathcal{C}$. A (pre-)triangulated Hom-finite category $\mathcal{C}$ with Serre functor S is called *n-Calabi–Yau* if there is a natural isomorphism

$$S \cong \Sigma^n.$$

(In [Keller 2008, Section 2.6] it is called *weakly n-Calabi–Yau*.) We now prove Proposition 0.6.

Proposition 3.6 (Serre duality). *Let $\mathcal{C}$ be a* Hom-*finite category with Serre functor S. Let σ be an endofunctor of $\mathcal{C}$*:

(1) *If σ has a right adjoint $\sigma^!$, then*

$$\mathrm{fpd}(\sigma) = \mathrm{fpd}(\sigma^! \circ S).$$

(2) *If σ is an equivalence with quasiinverse σ^{-1}, then*

$$\operatorname{fpd}(\sigma) = \operatorname{fpd}(\sigma^{-1} \circ S).$$

(3) *If $\mathcal{C}$ is (pre-)triangulated and n-Calabi–Yau, then we have a duality*

$$\operatorname{fpd}(\Sigma^{i}) = \operatorname{fpd}(\Sigma^{n-i})$$

for all i.

Proof. (1) Let $\phi = \{X_1, \dots, X_n\} \in \Phi_{n,b}$ and let $A(\phi, \sigma)$ be the adjacency matrix with (i, j)-entry $a_{ij} = \dim(X_i, \sigma(X_j))$. By Serre duality,

$$a_{ij} = \dim(X_i, \sigma(X_j)) = \dim(\sigma(X_j), S(X_i)) = \dim(X_j, (\sigma^! \circ S)(X_i)),$$

which is the (j, i)-entry of the adjacency matrix $A(\phi, \sigma^! \circ S)$. Then $\rho(A(\phi, \sigma)) = \rho(A(\phi, \sigma^! \circ S))$. It follows from the definition that $\operatorname{fpd}^n(\sigma) = \operatorname{fpd}^n(\sigma^! \circ S)$ for all $n \geq 1$. The assertion follows from the definition.

(2) and (3) These are consequences of part (1). □

Opposite categories.

Lemma 3.7. *Let σ be an endofunctor of $\mathcal{C}$ and suppose that σ has a left adjoint σ^*. Consider σ^* as an endofunctor of the opposite category $\mathcal{C}^{\mathrm{op}}$ of $\mathcal{C}$. Then*

$$\operatorname{fpd}^n(\sigma|_{\mathcal{C}}) = \operatorname{fpd}^n(\sigma^*|_{\mathcal{C}^{\mathrm{op}}})$$

for all n.

Proof. Let $\phi := \{X_1, \dots, X_n\}$ be a brick subset of $\mathcal{C}$ (which is also a brick subset of $\mathcal{C}^{\mathrm{op}}$). Then

$$\dim_{\mathcal{C}}(X_i, \sigma(X_j)) = \dim_{\mathcal{C}}(\sigma^*(X_i), X_j) = \dim_{\mathcal{C}^{\mathrm{op}}}(X_j, \sigma^*(X_i)),$$

which implies that the adjacency matrix of σ^* as an endofunctor of $\mathcal{C}^{\mathrm{op}}$ is the transpose of the adjacency matrix of σ. The assertion follows. □

Definition 3.8. (1) Two pretriangulated categories $(\mathcal{T}_i, \Sigma_i)$, for $i = 1, 2$, are called fp-*equivalent* if

$$\operatorname{fpd}^n(\Sigma_1^m) = \operatorname{fpd}^n(\Sigma_2^m)$$

for all $n \geq 1$, $m \in \mathbb{Z}$:

(2) Two algebras are fp-*equivalent* if their bounded derived categories of finitely generated modules are fp-equivalent.

(3) Two pretriangulated categories with Serre functors $(\mathcal{T}_i, \Sigma_i, S_i)$, for $i = 1, 2$, are called fp-*S-equivalent* if

$$\operatorname{fpd}^n(\Sigma_1^m \circ S_1^k) = \operatorname{fpd}^n(\Sigma_2^m \circ S_2^k)$$

for all $n \geq 1$, $m, k \in \mathbb{Z}$.

Proposition 3.9. *Let $\mathcal{T}$ be a pretriangulated category*:

(1) *$\mathcal{T}$ and $\mathcal{T}^{\mathrm{op}}$ are* fp*-equivalent.*

(2) *Suppose S is a Serre functor of $\mathcal{T}$. Then $(\mathcal{T}, S)$ and $(\mathcal{T}^{\mathrm{op}}, S^{\mathrm{op}})$ are* fp*-S-equivalent.*

Proof. (1) Let Σ be the suspension of $\mathcal{T}$. Then $\mathcal{T}^{\mathrm{op}}$ is also pretriangulated with suspension functor being $\Sigma^{-1} = \Sigma^*$ (restricted to $\mathcal{T}^{\mathrm{op}}$). The assertion follows from Lemma 3.7.

(2) Note that the Serre functor of $\mathcal{T}^{\mathrm{op}}$ is equal to $S^{-1} = S^*$ (restricted to $\mathcal{T}^{\mathrm{op}}$). The assertion follows by Lemma 3.7. □

Corollary 3.10. *Let A be a finite dimensional algebra*:

(1) *A and A^{op} are* fp*-equivalent.*

(2) *Suppose A has finite global dimension. In this case, the bounded derived category of finite dimensional A-modules has a Serre functor. Then A and A^{op} are* fp*-S-equivalent.*

Proof. (1) Since A is finite dimensional, the $\Bbbk$-linear dual induces an equivalence of triangulated categories between $D^b(\mathrm{Mod}_{f.d.}-A)^{\mathrm{op}}$ and $D^b(\mathrm{Mod}_{f.d.}-A^{\mathrm{op}})$. The assertion follows from Proposition 3.9(1).

(2) The proof is similar, using Proposition 3.9(2) instead. □

There are examples where $\mathcal{T}$ and $\mathcal{T}^{\mathrm{op}}$ are not triangulated equivalent, see Example 3.12. In this paper, a $\Bbbk$-algebra A is called *local* if A has a unique maximal ideal $\mathfrak{m}$ and $A/\mathfrak{m} \cong \Bbbk$. The following lemma is easy and well known.

Lemma 3.11. *Let A be a finite dimensional local algebra over $\Bbbk$. Let $\mathfrak{A}$ be the category* $\mathrm{Mod}_{f.d.}-A$ *and $\mathcal{A}$ be $D^b(\mathfrak{A})$*:

(1) *Let X be an object in $\mathcal{A}$ such that $\mathrm{Hom}_{\mathcal{A}}(X, X[-i]) = 0$ for all $i > 0$. Then X is of the form $M[n]$ where M is an object in $\mathfrak{A}$ and $n \in \mathbb{Z}$.*

(2) *Every atomic object in $\mathcal{A}$ is of the form $M[n]$ where M is a brick object in $\mathfrak{A}$ and $n \in \mathbb{Z}$. Namely, $\mathfrak{A}$ is a-hereditary.*

Proof. (2) is an immediate consequence of part (1). We only prove part (1).

On the contrary we suppose that $H^m(X) \neq 0$ and $H^n(X) \neq 0$ for some $m < n$. Since X is a bounded complex, we can take m to be minimum of such integers and n to be the maximum of such integers. Since A is local, there is a nonzero map from $H^n(X) \to H^m(X)$, which induces a nonzero morphism in $\mathrm{Hom}_{\mathcal{A}}(X, X[m-n])$. This contradicts the hypothesis. □

Example 3.12. Let m, n be integers ≥ 2. Define $A_{m,n}$ to be the algebra

$$\Bbbk\langle x_1, x_2\rangle/(x_1^m, x_2^n, x_1 x_2).$$

It is easy to see that $A_{m,n}$ is a finite dimensional local connected graded algebra generated in degree 1 (with $\deg x_1 = \deg x_2 = 1$). If $A_{m,n}$ is isomorphic to $A_{m',n'}$ as algebras, by [Bell and Zhang 2017, Theorem 1], these are isomorphic as graded algebras. Suppose $f : A_{m,n} \to A_{m',n'}$ is an isomorphism of graded algebras and write

$$f(x_1) = ax_1 + bx_2, \quad f(x_2) = cx_1 + dx_2.$$

Then the relation $f(x_1)f(x_2) = 0$ forces $b = c = 0$. As a consequence, $m = m'$ and $n = n'$. So we have proven that

(1) $A_{m,n}$ is isomorphic to $A_{m',n'}$ if and only if $m = m'$ and $n = n'$.

Next we claim that

(2) the derived category $D^b(\mathrm{Mod}_{f.d.} - A_{m,n})$ is not triangulated equivalent to $D^b(\mathrm{Mod}_{f.d.} - A^{\mathrm{op}}_{m,n})$, if $m \neq n$.

Let m, n, m', n' be integers ≥ 2. Suppose that $D^b(\mathrm{Mod}_{f.d.} - A_{m,n})$ is triangulated equivalent to $D^b(\mathrm{Mod}_{f.d.} - A_{m',n'})$. Since $A_{m,n}$ is local, by [Yekutieli 1999, Theorem 2.3], every tilting complex over $A_{m,n}$ is of the form $P[n]$ where P is a progenerator over $A_{m,n}$. As a consequence, $A_{m,n}$ is Morita equivalent to $A_{m',n'}$. Since both $A_{m,n}$ and $A_{m',n'}$ are local, Morita equivalence implies that $A_{m,n}$ is isomorphic to $A_{m',n'}$. By part (1), $m = m'$ and $n = n'$. In other words, if $(m, n) \neq (m', n')$, then $D^b(\mathrm{Mod}_{f.d.} - A_{m,n})$ is not triangulated equivalent to $D^b(\mathrm{Mod}_{f.d.} - A_{m',n'})$. As a consequence, if $m \neq n$, then $D^b(\mathrm{Mod}_{f.d.} - A_{m,n})$ is not triangulated equivalent to $D^b(\mathrm{Mod}_{f.d.} - A_{n,m})$. By definition, $A^{\mathrm{op}}_{m,n} \cong A_{n,m}$. Therefore claim (2) follows.

We can show that $D^b(\mathrm{Mod}_{f.d.} - A)$ is dual to $D^b(\mathrm{Mod}_{f.d.} - A^{\mathrm{op}})$ by using the $\Bbbk$-linear dual. In other words, $D^b(\mathrm{Mod}_{f.d.} - A)^{\mathrm{op}}$ is triangulated equivalent to $D^b(\mathrm{Mod}_{f.d.} - A^{\mathrm{op}})$. Therefore the following is a consequence of part (2).

(3) Suppose $m \neq n$ and let $\mathcal{A}$ be $D^b(\mathrm{Mod}_{f.d.} - A_{m,n})$. Then $\mathcal{A}$ is not triangulated equivalent to $\mathcal{A}^{\mathrm{op}}$. But by Proposition 3.9(1), $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ are fp-equivalent.

4. Derived category over a commutative ring

Throughout this section A is a commutative algebra and $\mathcal{A} = D^b(\mathrm{Mod} - A)$. (In other sections $\mathcal{A}$ usually denotes $D^b(\mathrm{Mod}_{f.d.} - A)$.)

Lemma 4.1. *Let A be a commutative algebra. Let X be an atomic object in $\mathcal{A}$. Then X is of the form $M[i]$ for some simple A-module M and some $i \in \mathbb{Z}$. As a consequence*, $\mathrm{Mod} - A$ *is a-hereditary.*

Proof. Consider X as a bounded above complex of projective A-modules. Since A is commutative, every $f \in A$ induces naturally a morphism of X by multiplication. For each i, $H^i(X)$ is an A-module. We have natural morphisms of A-algebras

$$A \to \mathrm{Hom}_{\mathcal{A}}(X, X) \to \mathrm{End}_A(H^i(X)).$$

By definition, $\operatorname{Hom}_{\mathcal{A}}(X, X) = \Bbbk$. Thus $\operatorname{Hom}_{\mathcal{A}}(X, X) = A/\mathfrak{m}$ for some ideal $\mathfrak{m}$ of A that has codimension 1. Hence the A-action on $H^i(X)$ factors through the map $A \to A/\mathfrak{m}$. This means that $H^i(X)$ is a direct sum of $A/\mathfrak{m}$.

Let $n = \sup X$ and $m = \inf X$. Then $H^m(X) = (A/\mathfrak{m})^{\oplus s}$ and $H^n(X) = (A/\mathfrak{m})^{\oplus t}$ for some $s, t > 0$. If $m < n$, then

$$\operatorname{Hom}_{\mathcal{A}}(X, X[m-n]) \cong \operatorname{Hom}_{\mathcal{A}}(X[n], X[m]) \cong \operatorname{Hom}_{\mathcal{A}}(H^n(X), H^m(X)) \neq 0$$

which contradicts (E2.1.1). Therefore $m = n$ and $X = M[n]$ for $M := H^n(X)$. Since X is atomic, M has only one copy of $A/\mathfrak{m}$. □

Lemma 4.2. *Let A be a noetherian commutative algebra. Let X and Y be two atomic objects in $\mathcal{A}$. Then* $\operatorname{Hom}_{\mathcal{A}}(X, Y) \neq 0$ *if and only if there is an ideal $\mathfrak{m}$ of A of codimension 1 such that $X \cong A/\mathfrak{m}[m]$ and $Y \cong A/\mathfrak{m}[n]$ for some $0 \le n - m \le \operatorname{projdim} A/\mathfrak{m}$.*

Proof. By Lemma 4.1, $X \cong A/\mathfrak{m}[m]$ for some ideal $\mathfrak{m}$ of codimension 1 and some integer m. Similarly, $Y \cong A/\mathfrak{n}[n]$ for ideal $\mathfrak{n}$ of codimension 1 and integer n.

Suppose $\operatorname{Hom}_{\mathcal{A}}(X, Y) \neq 0$. If $\mathfrak{m} \neq \mathfrak{n}$, then clearly $\operatorname{Hom}_{\mathcal{A}}(X, Y) = 0$. Hence $\mathfrak{m} = \mathfrak{n}$. Further, $\operatorname{Ext}^{n-m}_A(A/\mathfrak{m}, A/\mathfrak{m}) \cong \operatorname{Hom}_{\mathcal{A}}(X, Y) \neq 0$ implies that $0 \le n - m \le \operatorname{projdim} A/\mathfrak{m}$. The converse can be proved in a similar way by passing to a localization. □

If A is an affine commutative ring over $\Bbbk$, then every simple A-module is 1-dimensional. Hence $(A/\mathfrak{m})[i]$ is a brick (and atomic) object in $\mathcal{A}$ for every $i \in \mathbb{Z}$ and every maximal ideal $\mathfrak{m}$ of A. The fp-global dimension $\operatorname{fpgldim}(\mathcal{A})$ is defined in Definition 2.7(2).

Proposition 4.3. *Let A be an affine commutative domain of global dimension $g < \infty$:*

(1) $\operatorname{fpd}(\mathcal{A}) = g$.

(2) $\operatorname{fpd}(\Sigma^i) = \binom{g}{i}$ *for all i.*

(3) $\operatorname{fpgldim}(\mathcal{A}) = g$.

Proof. (1) By Lemma 4.1, every atomic object is of the form $M[i]$ for some $M \cong A/\mathfrak{m}$ where $\mathfrak{m}$ is an ideal of codimension 1, and $i \in \mathbb{Z}$. It is well-known that

$$\dim \operatorname{Ext}^i_A(A/\mathfrak{m}, A/\mathfrak{m}) = \binom{g}{i} \quad \forall i. \tag{E4.3.1}$$

If $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are two different maximal ideals, then

$$\operatorname{Ext}^i_A(A/\mathfrak{m}_1, A/\mathfrak{m}_2) = 0 \tag{E4.3.2}$$

for all i. Let ϕ be an atomic n-object subset. We can decompose ϕ into a disjoint union $\phi_{A/\mathfrak{m}_1} \cup \cdots \cup \phi_{A/\mathfrak{m}_s}$ where $\phi_{A/\mathfrak{m}}$ consists of objects of the form $A/\mathfrak{m}[i]$ for $i \in \mathbb{Z}$. It follows from (E4.3.2) that the adjacency matrix is a block-diagonal matrix. Hence, we only need to consider the case when $\phi = \phi_{A/\mathfrak{m}}$ after we use the reduction similar to the one used in the proof of Theorem 3.5. Let $\phi = \phi_{A/\mathfrak{m}} = \{A/\mathfrak{m}[d_1], \dots, A/\mathfrak{m}[d_m]\}$

where d_i is increasing. By Lemma 4.2, we have $d_{i+1} - d_i > g$, or $d_i + g < d_{i+1}$, for all $i = 1, \dots, m-1$. Under these conditions, the adjacency matrix is lower triangular with each diagonal being g. Thus $\operatorname{fpd}(\Sigma) = g$.

The proof of (2) is similar and (3) is a consequence of (2). □

Suggested by Theorem 3.5, we could introduce some secondary invariants as follows. The *stabilization index* of a triangulated category $\mathcal{T}$ is defined to be

$$\operatorname{SI}(\mathcal{T}) = \min\{n \mid \operatorname{fpd}^{n'} \mathcal{T} = \operatorname{fpd} \mathcal{T}\ \forall n' \geq n\}.$$

The *global stabilization index* of $\mathcal{T}$ is defined to be

$$\operatorname{GSI}(\mathcal{T}) = \max\{\operatorname{SI}(\mathcal{T}') \mid \text{for all thick triangulated full subcategories } \mathcal{T}' \subseteq \mathcal{T}\}.$$

It is clear that both stabilization index and global stabilization index can be defined for an abelian category.

Similar to Proposition 4.3, one can show the following. Suppose that A is affine. For every i, let

$$d_i := \sup\{\dim \operatorname{Ext}^i_A(A/\mathfrak{m}, A/\mathfrak{m}) \mid \text{maximal ideals } \mathfrak{m} \subseteq A\}.$$

Proposition 4.4. *Let A be an affine commutative algebra. Then, for each i, $\operatorname{fpd}(\Sigma^i) = d_i < \infty$ and $\rho(A(\phi, \Sigma^i)) \leq d_i$ for all $\phi \in \Phi_{n,a}$. As a consequence, for each integer i, the following hold:*

(1) $\operatorname{fpd}(\Sigma^i) = \operatorname{fpd}^1(\Sigma^i)$. *Hence the stabilization index of $\mathcal{A}$ is* 1.

(2) $\operatorname{fpd}(\Sigma^i)$ *is a finite integer.*

Theorem 4.5. *Let A be an affine commutative algebra and $\mathcal{A}$ be $D^b(\operatorname{Mod} A)$. Let $\mathcal{T}$ be a triangulated full subcategory of $\mathcal{A}$ with suspension $\Sigma_{\mathcal{T}}$. Let i be an integer:*

(1) $\operatorname{fpd}(\Sigma^i_{\mathcal{T}}) = \operatorname{fpd}^1(\Sigma^i_{\mathcal{T}})$. *As a consequence, the global stabilization index of $\mathcal{A}$ is* 1.

(2) $\operatorname{fpd}(\Sigma^i_{\mathcal{T}})$ *is a finite integer.*

(3) *If $\mathcal{T}$ is isomorphic to $D^b(\operatorname{Mod}_{f.d.} - B)$ for some finite dimensional algebra B, then B is Morita equivalent to a commutative algebra.*

Proof. (1) and (2) are similar to Proposition 4.4.

(3) Since B is finite dimensional, it is Morita equivalent to a basic algebra. So we can assume B is basic and show that B is commutative. Write B as a $\Bbbk Q/(R)$ where Q is a finite quiver with admissible ideal $R \subseteq (\Bbbk Q)_{\geq 2}$. We will show that B is commutative.

First we claim that each connected component of Q consists of only one vertex. Suppose not. Then Q contains distinct vertices v_1 and v_2 with an arrow $\alpha : v_1 \to v_2$. Let S_1 and S_2 be the simple modules corresponding to v_1 and v_2 respectively. Then $\{S_1, S_2\}$ is an atomic set in $\mathcal{T}$. The arrow represents a nonzero element in $\operatorname{Ext}^1_B(S_1, S_2)$. Hence

$$\operatorname{Hom}_{\mathcal{T}}(S_1, S_2[1]) \cong \operatorname{Ext}^1_B(S_1, S_2) \neq 0.$$

By Lemma 4.2, S_1 is isomorphic to a complex shift of S_2. But this is impossible. Therefore, the claim holds.

It follows from the claim in the last paragraph that $B = B_1 \oplus \cdots \oplus B_n$ where each B_i is a finite dimensional local ring corresponding to a vertex, say v_i. Next we claim that each B_i is commutative. Without loss of generality, we can assume $B_i = B$.

Now let ι be the fully faithful embedding from

$$\iota : \mathcal{T} := D^b(\mathrm{Mod}_{f.d.} - B) \to \mathcal{A} := D^b(\mathrm{Mod} - A).$$

Let S be the unique simple left B-module. Then, by Lemma 4.1, there is a maximal ideal $\mathfrak{m}$ of A such that $\iota(S) = A/\mathfrak{m}[w]$ for some $w \in \mathbb{Z}$. After a shift, we might assume that $\iota(S) = A/\mathfrak{m}$. The left B-module B has a composition series such that each simple subquotient is isomorphic to S, which implies that, as a left A-module, $\iota(B)$ is generated by $A/\mathfrak{m}$ in $\mathcal{A}$. By induction on the length of B, one sees that, for every $n \in \mathbb{Z}$, $H^n(\iota(B))$ is a left $A/\mathfrak{m}^d$-module for some $d \gg 0$ (we can take $d = \mathrm{length}({}_B B)$). Since $\mathrm{Hom}_{\mathcal{A}}(\iota(B), \iota(B)[-i]) = \mathrm{Hom}_{\mathcal{T}}(B, B[-i]) = 0$ for all $i > 0$, the proof of Lemma 3.11(2) shows that $\iota(B) \cong M[m]$ for some left $A/\mathfrak{m}^d$-module M and $m \in \mathbb{Z}$. Since there are nonzero maps from S to B and from B to S, we have nonzero maps from $A/\mathfrak{m}$ to $\iota(B)$ and from $\iota(B)$ to $A/\mathfrak{m}$. This implies that $m = 0$. Since B is local (and then $B/\mathfrak{m}_B$ is 1-dimensional for the maximal ideal $\mathfrak{m}_B$), this forces that $M = A/I$ where I is an ideal of A containing $\mathfrak{m}^d$. Finally,

$$B^{\mathrm{op}} = \mathrm{End}_B(B) \cong \mathrm{End}_{\mathcal{A}}(A/I, A/I) = \mathrm{End}_A(A/I, A/I) \cong A/I$$

which is commutative. Hence B is commutative. $\square$

5. Examples

In this section we give three examples.

Frobenius–Perron theory of projective line $\mathbb{P}^1 := \mathbf{Proj}\, \Bbbk[t_0, t_1]$.

Example 5.1. Let $\mathrm{coh}(\mathbb{P}^1) =: \mathfrak{A}$ denote the category of coherent sheaves on $\mathbb{P}^1$. We will calculate the fp dimension of various functors.

Proposition 5.1.1. *Every brick object X in $\mathfrak{A}$ (namely, satisfying $\mathrm{Hom}_{\mathbb{P}^1}(X, X) = \Bbbk$) is either $\mathcal{O}(m)$ for some $m \in \mathbb{Z}$ or $\mathcal{O}_p$ for some $p \in \mathbb{P}^1$.*

The above fact is well known and follows easily from Grothendieck theorem (see also [Brüning and Burban 2007, Example 3.18]).

Let ϕ be in $\Phi_{n,b}(\mathrm{coh}(\mathbb{P}^1))$. If $n = 1$ or ϕ is a singleton, then there are two cases: either $\phi = \{\mathcal{O}(m)\}$ or $\phi = \{\mathcal{O}_p\}$. Let E^1 be the functor $\mathrm{Ext}^1_{\mathbb{P}^1}(-,-)$. In the first case, $\rho(A(\phi, E^1)) = 0$ because $\mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{O}(m), \mathcal{O}(m)) = 0$, and in the second case, $\rho(A(\phi, E^1)) = 1$ because $\mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{O}_p, \mathcal{O}_p) = 1$.

If $|\phi| > 1$, then $\mathcal{O}(m)$ can not appear in ϕ as $\mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}(m), \mathcal{O}(m')) \neq 0$ and $\mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}(m), \mathcal{O}_p) \neq 0$ for all $m \leq m'$ and $p \in \mathbb{P}^1$. Hence, ϕ is a collection of $\mathcal{O}_p$ for finitely many distinct points p's. In this case, the adjacency matrix is the identity $n \times n$-matrix and $\rho(A(\phi, E^1)) = 1$. Therefore

$$\mathrm{fpd}^n(\mathrm{coh}(\mathbb{P}^1)) = \mathrm{fpd}(\mathrm{coh}(\mathbb{P}^1)) = 1 \tag{E5.1.1}$$

for all $n \geq 1$. Since $\operatorname{coh}(\mathbb{P}^1)$ is hereditary, by Theorem 3.5(3,4), we obtain that

$$\operatorname{fpd}^n(D^b(\operatorname{coh}(\mathbb{P}^1))) = \operatorname{fpd}(D^b(\operatorname{coh}(\mathbb{P}^1))) = 1 \tag{E5.1.2}$$

for all $n \geq 1$.

Let K_2 be the Kronecker quiver

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd} \tag{E5.1.3}$$

By a result of Beilinson [1978], the derived category $D^b(\operatorname{Mod}_{f.d.} -\Bbbk K_2)$ is triangulated equivalent to $D^b(\operatorname{coh}(\mathbb{P}^1))$. As a consequence,

$$\operatorname{fpd}(D^b(\operatorname{Mod}_{f.d.} -\Bbbk K_2)) = \operatorname{fpd}(D^b(\operatorname{coh}(\mathbb{P}^1))) = 1. \tag{E5.1.4}$$

It is easy to see, or by Theorem 1.8(1),

$$\operatorname{fpd} K_2 = 0$$

where fpd of a quiver is defined in Definition 1.6.

This implies that

$$\operatorname{fpd}(D^b(\operatorname{Mod}_{f.d.} -\Bbbk K_2)) > \operatorname{fpd} K_2. \tag{E5.1.5}$$

Next we consider some general auto-equivalences of $D^b(\operatorname{coh}(\mathbb{P}^1))$. Let

$$(m) : \operatorname{coh}(\mathbb{P}^1) \to \operatorname{coh}(\mathbb{P}^1)$$

be the auto-equivalence induced by the shift of degree m of the graded modules over $\Bbbk[t_0, t_1]$ and let Σ be the suspension functor of $D^b(\operatorname{coh}(\mathbb{P}^1))$. Then the Serre functor S of $D^b(\operatorname{coh}(\mathbb{P}^1))$ is $\Sigma \circ (-2)$. Let σ be the functor $\Sigma^a \circ (b)$ for some $a, b \in \mathbb{Z}$. By Theorem 3.5(1),

$$\operatorname{fpd}^n(\Sigma^a \circ (b)) = 0 \quad \forall a \neq 0, 1.$$

For the rest we consider $a = 0$ or 1. By Theorem 3.5(2,3), we only need to consider fpd on $\operatorname{coh}(\mathbb{P}^1)$.

If ϕ is a singleton $\{\mathcal{O}(n)\}$, then the adjacency matrix is

$$A(\phi, \sigma) = \dim(\mathcal{O}, \Sigma^a \mathcal{O}(b)) = \begin{cases} 0 & a = 0, b < 0, \\ b+1 & a = 0, b \geq 0, \\ 0 & a = 1, b \geq -1, \\ -b-1 & a = 1, b < -1. \end{cases}$$

This follows from the well-known computation of $H^i_{\mathbb{P}^1}(\mathcal{O}(m))$ for $i = 0, 1$ and $m \in \mathbb{Z}$. (It also follows from a more general computation [Artin and Zhang 1994, Theorem 8.1].) If $\phi = \{\mathcal{O}_p\}$ for some $p \in \mathbb{P}^1$, then the adjacency matrix is

$$A(\phi, \sigma) = \dim(\mathcal{O}_p, \Sigma^a(\mathcal{O}_p)) = 1 \quad \text{for } a = 0, 1.$$

It is easy to see from the above computation that

$$\operatorname{fpd}^1(\Sigma^a \circ (b)) = \begin{cases} 1 & a=0, b<0, \\ b+1 & a=0, b\geq 0, \\ 1 & a=1, b\geq -1, \\ -b-1 & a=1, b<-1. \end{cases} \tag{E5.1.6}$$

Now we consider the case when $n>1$. If $\phi \in \Phi_{n,b}(\operatorname{coh}(\mathbb{P}^1))$, ϕ is a collection of $\mathcal{O}_p$ for finitely many distinct p's. In this case, the adjacency matrix $A(\phi, \Sigma^a \circ (b))$ is the identity $n\times n$-matrix for $a=0,1$, and $\rho(A(\phi,\sigma))=1$. Therefore

$$\operatorname{fpd}^n(\Sigma^a \circ (b)) = 1 \tag{E5.1.7}$$

for all $n>1$, when restricted to the category $\operatorname{coh}(\mathbb{P}^1)$.

It follows from Theorem 3.5(2,3) that:

Claim 5.1.2. *Consider $\Sigma^a \circ (b)$ as an endofunctor of $D^b(\operatorname{coh}(\mathbb{P}^1))$. For $a,b\in\mathbb{Z}$ and $n\geq 1$, we have*

$$\operatorname{fpd}^n(\Sigma^a \circ (b)) = \begin{cases} 0 & a\neq 0,1, \\ 1 & a=0, b<0, \\ b+1 & a=0, b\geq 0, \\ 1 & a=1, b\geq -1, \\ -b-1 & a=1, b<-1. \end{cases} \tag{E5.1.8}$$

Since $S=\Sigma\circ(-2)$, we have the following

$$\operatorname{fpd}^n(\Sigma^a \circ S^b) = \operatorname{fpd}^n(\Sigma^{a+b}\circ(-2b)) = \begin{cases} 0 & a+b\neq 0,1, \\ 1 & a+b=0, b>0, \\ -(2b-1) & a+b=0, b\leq 0, \\ 1 & a+b=1, b\leq 0, \\ 2b-1 & a+b=1, b>0. \end{cases} \tag{E5.1.9}$$

Claim 5.1.3. *Since $D^b(\operatorname{coh}(\mathbb{P}^1))$ and $D^b(\operatorname{Mod}_{f.d.}-\Bbbk K_2)$ are equivalent,* (E5.1.9) *agrees with the* fp-*theory of $D^b(\operatorname{Mod}_{f.d.}-\Bbbk K_2)$.*

Frobenius–Perron theory of the quiver A_2. We start with the following example.

Example 5.2. Let A be the $\mathbb{Z}$-graded algebra $\Bbbk[x]/(x^2)$ with $\deg x=1$. Let $\mathcal{C}:=\operatorname{gr}-A$ be the category of finitely generated graded left A-modules. Let $\sigma:=(-)$ be the degree shift functor of $\mathcal{C}$. It is clear that σ is an autoequivalence of $\mathcal{C}$. Let $\mathfrak{A}$ be the additive subcategory of $\mathcal{C}$ generated by $\sigma^n(A)=A(n)$ for all $n\in\mathbb{Z}$. Note that $\mathfrak{A}$ is not abelian and that every object in $\mathfrak{A}$ is of the form $\bigoplus_{n\in\mathbb{Z}} A(n)^{\oplus p_n}$ for some integers $p_n\geq 0$. Since the Hom-set in the graded module category consists of homomorphisms of degree zero, we have

$$\operatorname{Hom}_{\mathfrak{A}}(A, A(n)) = \begin{cases} \Bbbk & n=0,1, \\ 0 & \text{otherwise.} \end{cases}$$

In the following diagram each arrow represents a 1-dimensional Hom for all possible Hom-set for different objects $A(n)$

$$\cdots \to A(-2) \to A(-1) \to A(0) \to A(1) \to A(2) \to \cdots \tag{E5.2.1}$$

where the number of arrows from $A(m)$ to $A(n)$ agrees with $\dim \operatorname{Hom}(A(m), A(n))$. It is easy to see that the set of indecomposable objects is $\{A(n)\}_{n\in\mathbb{Z}}$, which is also the set of bricks in $\mathfrak{A}$.

Lemma 5.3. *Retain the notation as in Example 5.2. When restricting σ onto the category $\mathfrak{A}$, we have, for every $m \geq 1$,*

$$\operatorname{fpd}^m(\sigma^n) = \begin{cases} 1 & n = 0, 1, \\ 0 & \textit{otherwise.} \end{cases} \tag{E5.3.1}$$

Proof. When $n = 0$, (E5.3.1) is trivial. Let $n = 1$. For each set $\phi \in \Phi_{m,b}$, we can assume that $\phi = \{A(d_1), A(d_2,), \ldots, A(d_m)\}$ for a strictly increasing sequence $\{d_i \mid i = 1, 2, \ldots, m\}$. For any $i < j$, the (i, j)-entry of the adjacency matrix is

$$a_{ij} = \dim(A(d_i), A(d_j + 1)) = 0.$$

Thus $A(\phi, \sigma)$ is a lower triangular matrix with

$$a_{ii} = \dim(A(d_i), A(d_i + 1)) = 1.$$

Hence $\rho(A(\phi, \sigma)) = 1$. So $\operatorname{fpd}^m(\sigma) = 1$.

Similarly, $\operatorname{fpd}^m(\sigma^n) = 0$ when $n > 1$ as $\dim(A(d_i), A(d_i + 2)) = 0$ for all i.

Let $n < 0$. Let $\phi = \{A(d_1), A(d_2,), \ldots, A(d_m)\} \in \Phi_{m,b}$ where d_i are strictly decreasing. Then $a_{ij} = \dim(A(d_i), A(d_j + n)) = 0$ for all $i \leq j$. Thus $\rho(A(\phi, \sigma^n)) = 0$ and (E5.3.1) follows in this case. □

Example 5.4. Consider the quiver A_2

$$\bullet_2 \to \bullet_1. \tag{E5.4.1}$$

Let P_i (respectively, I_i) be the projective (respectively, injective) left $\Bbbk A_2$-modules corresponding to vertices i, for $i = 1, 2$, It is well-known that there are only three indecomposable left modules over A_2, with Auslander–Reiten quiver (or AR-quiver, for short)

$$P_2 \to P_1(= I_2) \to I_1 \tag{E5.4.2}$$

where each arrow represents a nonzero homomorphism (up to a scalar) [Schiffler 2014, Example 1.13, pages 24–25]. The AR-translation (or translation, for short) τ is determined by $\tau(I_1) = P_2$. Let $\mathcal{T}$ be $D^b(\mathrm{Mod}_{f.d.} - \Bbbk A_2)$. The Auslander–Reiten theory can be extended from the module category to the derived category. It is direct that, in $\mathcal{T}$, we have the AR-quiver of all indecomposable objects

$$\begin{tikzcd}[column sep=small] \cdots & P_2[-1] \arrow[dr] & & I_1[-1] \arrow[dr] & & P_1 = I_2 \arrow[dr] & & P_2[1] \arrow[dr] & & I_1[1] & \cdots \\ & & P_1[-1] \arrow[ur] & & P_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & \end{tikzcd} \tag{E5.4.3}$$

The above represents all possible nonzero morphisms (up to a scalar) between nonisomorphic indecomposable objects in $\mathcal{T}$. Note that $\mathcal{T}$ has a Serre functor S and that the AR-translation τ can be extended to a functor of $\mathcal{T}$ such that $S = \Sigma \circ \tau$ [Reiten and Van den Bergh 2002, Proposition I.2.3] or [Crawley-Boevey 1992, Remarks(2), page 23]. After identifying

$$P_2[i] \leftrightarrow A(3i), \quad P_1[i] \leftrightarrow A(3i+1), \quad I_1[i] \leftrightarrow A(3i+2),$$

(E5.4.3) agrees with (E5.2.1). Using the above identification, at least when restricted to objects, we have

$$\Sigma(A(i)) \cong A(i+3), \tag{E5.4.4}$$

$$\tau(A(i)) \cong A(i-2), \tag{E5.4.5}$$

$$S(A(i)) \cong A(i+1). \tag{E5.4.6}$$

It follows from the definition of the AR-quiver [Auslander et al. 1995, VII] that the degree of τ is -2, see also [Assem et al. 2006, Picture on page 131]. Equation (E5.4.5) just means that the degree of τ is -2.

By (E5.4.6), the Serre functor S satisfies the property of σ defined in Example 5.2. By Lemma 5.3 or (E5.3.1), we have

$$\operatorname{fpd}^n(\Sigma^a \circ S^b) = \operatorname{fpd}^n(\sigma^{3a+b}) = \begin{cases} 1 & 3a+b = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the fp-S-theory of $\mathcal{T}$ is given as above.

In particular, we have proven

$$\operatorname{fpgldim}(D^b(\mathrm{Mod}_{f.d.} - \Bbbk A_2)) = \operatorname{fpd}(D^b(\mathrm{Mod}_{f.d.} - \Bbbk A_2)) = \operatorname{fpd}(\Sigma) = 0,$$

which is less than $\operatorname{gldim} \Bbbk A_2 = 1$.

An example of nonintegral Frobenius–Perron dimension. In the next example, we "glue" K_2 in (E5.1.3) and A_2 in (E5.4.1) together.

Example 5.5. Let G_2 be the quiver

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, bend left=40, "\beta"] \arrow[r, bend left=15, "\gamma"'] & \underset{2}{\bullet} \arrow[l, bend left=40, "\alpha"] \end{tikzcd} \tag{E5.5.1}$$

consisting of two vertices 1 and 2, with arrow $\alpha : 2 \to 1$ and $\beta, \gamma : 1 \to 2$ satisfying relations

$$R: \quad \beta\alpha = \gamma\alpha = 0, \quad \alpha\beta = \alpha\gamma = 0. \tag{E5.5.2}$$

Note that (G_2, R) is a quiver with relations. The corresponding quiver algebra with relations is a 5-dimensional algebra

$$A = \Bbbk e_1 + \Bbbk e_2 + \Bbbk\alpha + \Bbbk\beta + \Bbbk\gamma.$$

We can use the following matrix form to represent the algebra A

$$A = \begin{pmatrix} \Bbbk e_1 & \Bbbk\alpha \\ \Bbbk\beta + \Bbbk\gamma & \Bbbk e_2 \end{pmatrix}.$$

For each $i = 1, 2$, let S_i be the left simple A-module corresponding to the vertex i and P_i be the projective cover of S_i. Then $P_1 \cong Ae_1$ is isomorphic to the first column of A, namely $\left(\begin{smallmatrix} \Bbbk e_1 \\ \Bbbk\beta+\Bbbk\gamma \end{smallmatrix}\right)$, and $P_2 \cong Ae_2$ is isomorphic to the second column of A, namely $\left(\begin{smallmatrix} \Bbbk\alpha \\ \Bbbk e_2 \end{smallmatrix}\right)$.

We will show that the Frobenius–Perron dimension of the category of finite dimensional representations of (G_2, R) is $\sqrt{2}$, by using several lemmas below that contain some detailed computations.

Lemma 5.6. *Let $V = (V_1, V_2)$ be a representation of (G_2, R). Let $\overline{W} = \operatorname{im}\alpha$ and $K = \ker\alpha$. Take a $\Bbbk$-space decomposition $V_2 = W \oplus K$ where $W \cong \overline{W}$. Then there is a decomposition of (G_2, R)-representations $V \cong (\overline{W} \oplus T, W \oplus K) \cong (\overline{W}, W) \oplus (T, K)$ where α is the identity when restricted to W (and identifying W with $\overline{W}$) and is zero when restricted to K, where β and γ are zero when restricted to $\overline{W}$.*

Proof. Since $\overline{W} = \operatorname{im}\alpha$, $V_2 \cong W \oplus K$ where $K = \ker\alpha$ and $W \cong \overline{W}$. Write $V_1 = \overline{W} \oplus T$ for some $\Bbbk$-subspace $T \subseteq V_1$. The assertion follows by using the relations in (E5.5.2). □

Recall that A_2 is the quiver given in (E5.4.1) and K_2 is the Kronecker quiver given in (E5.1.3). By the above lemma, the subrepresentation (W, W) (where we identify $\overline{W}$ with W) is in fact a representation of $\left(\begin{smallmatrix} \Bbbk e_1 & \Bbbk\alpha \\ 0 & \Bbbk e_2 \end{smallmatrix}\right) (\cong \Bbbk A_2)$ and the subrepresentation (T, K) is a representation of $\left(\begin{smallmatrix} \Bbbk e_1 & 0 \\ \Bbbk\beta+\Bbbk\gamma & \Bbbk e_2 \end{smallmatrix}\right) (\cong \Bbbk K_2)$.

Let I_n be the $n \times n$-identity matrix. Let $Bl(\lambda)$ denote the block matrix

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}.$$

Lemma 5.7. *Suppose $\Bbbk$ is of characteristic zero. The following is a complete list of indecomposable representations of (G_2, R).*

(1) $P_2 \cong (\Bbbk, \Bbbk)$, *where* $\alpha = I_1$ *and* $\beta = \gamma = 0$.

(2) $X_n(\lambda) = (K, K)$ *with* $\dim K = n$, *where* $\alpha = 0$, $\beta = I_n$ *and* $\gamma = Bl(\lambda)$ *for some* $\lambda \in \Bbbk$.

(3) $Y_n = (K, K)$ *with* $\dim K = n$, *where* $\alpha = 0$, $\beta = Bl(0)$ *and* $\gamma = I_n$.

(4) $S_{2,n} = (T, K)$ *with* $\dim T = n$ *and* $\dim K = n + 1$, *where* $\alpha = 0$, $\beta = (I_n, 0)$ *and* $\gamma = (0, I_n)$.

(5) $S_{1,n} = (T, K)$ *with* $\dim T = n + 1$ *and* $\dim K = n$, *where* $\alpha = 0$, $\beta = (I_n, 0)^\tau$ *and* $\gamma = (0, I_n)^\tau$.

As a consequence, $\Bbbk G_2/(R)$ *is of tame representation type* (*Definition 7.1*).

Proof. (1) By Lemma 5.6, this is the only case that could happen when $\alpha \neq 0$. Now we assume $\alpha = 0$.

(2), (3), (4) and (5) If $\alpha = 0$, then we are working with representations of Kronecker quiver K_2 (E5.1.3). The classification follows from a classical result of Kronecker [Benson 1991, Theorem 4.3.2].

By (1)–(5), for each integer n, there are only finitely many 1-parameter families of indecomposable representations of dimension n. Therefore A is of tame representation type. □

The following is a consequence of Lemma 5.7 and a direct computation.

Lemma 5.8. *Retain the hypotheses of Lemma 5.7. The following is a complete list of brick representations of (G_2, R):*

(1) $P_2 \cong (\Bbbk, \Bbbk)$, *where* $\alpha = I_1$ *and* $\beta = \gamma = 0$.

(2) $X_1(\lambda) = (\Bbbk, \Bbbk)$, *where* $\alpha = 0$, $\beta = I_1$ *and* $\gamma = \lambda I_1$ *for some* $\lambda \in \Bbbk$.

(3) $Y_1 = (\Bbbk, \Bbbk)$, *where* $\alpha = 0$, $\beta = 0$ *and* $\gamma = I_1$.

(4) $S_{2,n}$ *for* $n \geq 0$.

(5) $S_{1,n}$ *for* $n \geq 0$.

The set $\Phi_{1,b}$ *consists of the above objects.*

Let $X_1(\infty)$ denote Y_1. We have the following short exact sequences of (G_2, R)-representations

$$0 \longrightarrow S_1 \longrightarrow P_2 \longrightarrow S_2 \longrightarrow 0,$$

$$0 \longrightarrow S_2 \longrightarrow X_1(\lambda) \longrightarrow S_1 \longrightarrow 0,$$

$$0 \longrightarrow S_2^{n+1} \longrightarrow S_{2,n} \longrightarrow S_1^n \longrightarrow 0,$$

$$0 \longrightarrow S_2^n \longrightarrow S_{1,n} \longrightarrow S_1^{n+1} \longrightarrow 0,$$

$$0 \longrightarrow S_2^2 \longrightarrow S_{2,n} \longrightarrow S_{1,n-1} \longrightarrow 0,$$

where $n \geq 1$ for the last exact sequence, and have the following nonzero Homs, where $A = \Bbbk G_2/(R)$:

$$\begin{aligned}
\operatorname{Hom}_A(X_1(\lambda), S_{1,n}) &\neq 0 \quad && \forall n \geq 1, \\
\operatorname{Hom}_A(S_{2,n}, X_1(\lambda)) &\neq 0 \quad && \forall n \geq 1, \\
\operatorname{Hom}_A(S_{2,m}, S_{2,n}) &\neq 0 \quad && \forall m \leq n, \\
\operatorname{Hom}_A(S_{1,n}, S_{1,m}) &\neq 0 \quad && \forall m \leq n, \\
\operatorname{Hom}_A(S_{2,n}, S_{1,m}) &\neq 0 \quad && \forall m + n \geq 1.
\end{aligned}$$

Lemma 5.9. *Retain the hypotheses of Lemma 5.7. The following is the complete list of zero hom-sets between brick representations of G_2 in both directions*:

(1) $\mathrm{Hom}_A(X_1(\lambda), X_1(\lambda')) = \mathrm{Hom}_A(X_1(\lambda'), X_1(\lambda)) = 0$ *if* $\lambda \neq \lambda'$ *in* $\Bbbk \cup \{\infty\}$.

(2) $\mathrm{Hom}_A(S_1, S_2) = \mathrm{Hom}_A(S_2, S_1) = 0$.

As a consequence, if $\phi \in \Phi_{n,b}$ *for some* $n \geq 2$, *then* $\phi = \{S_1, S_2\}$ *or* $\phi = \{X_1(\lambda_i)\}_{i=1}^n$ *for different parameters* $\{\lambda_1, \dots, \lambda_n\}$.

We also need to compute the Ext^1_A-groups.

Lemma 5.10. *Retain the hypotheses of Lemma 5.7. Let* $\lambda \neq \lambda'$ *be in* $\Bbbk \cup \{\infty\}$:

(1) $\mathrm{Ext}^1_A(X_1(\lambda), X_1(\lambda)) = \mathrm{Hom}_A(X_1(\lambda), X_1(\lambda)) = \Bbbk$.

(2) $\mathrm{Ext}^1_A(X_1(\lambda), X_1(\lambda')) = \mathrm{Hom}_A(X_1(\lambda), X_1(\lambda')) = 0$.

(3) $\begin{pmatrix} \mathrm{Ext}^1_A(S_1, S_1) & \mathrm{Ext}^1_A(S_1, S_2) \\ \mathrm{Ext}^1_A(S_2, S_1) & \mathrm{Ext}^1_A(S_2, S_2) \end{pmatrix} = \begin{pmatrix} 0 & \Bbbk^{\oplus 2} \\ \Bbbk & 0 \end{pmatrix}$.

(4) $\mathrm{Ext}^1_A(P_2, P_2) = 0$.

(5) $\dim \mathrm{Ext}^1_A(S_{2,n}, S_{2,n}) \leq 1$ *for all* n.

(6) $\dim \mathrm{Ext}^1_A(S_{1,n}, S_{1,n}) \leq 1$ *for all* n.

Remarks 5.11. In fact, one can show the following stronger version of Lemma 5.10(5) and (6):

(5′) $\mathrm{Ext}^1_A(S_{2,n}, S_{2,n}) = 0$ for all n.

(6′) $\mathrm{Ext}^1_A(S_{1,n}, S_{1,n}) = 0$ for all n.

Proof of Lemma 5.10. (1) and (2) Consider a minimal projective resolution of $X_1(\lambda)$

$$P_1 \to P_2 \xrightarrow{f_\lambda} P_1 \to X_1(\lambda) \to 0$$

where f_λ sends $e_2 \in P_2$ to $\gamma - \lambda\beta \in P_1$. More precisely, we have

$$\begin{pmatrix} \Bbbk e_1 \\ \Bbbk\beta + \Bbbk\gamma \end{pmatrix} \xrightarrow{e_1 \to \alpha} \begin{pmatrix} \Bbbk\alpha \\ \Bbbk e_2 \end{pmatrix} \xrightarrow{e_2 \to \gamma - \lambda\beta} \begin{pmatrix} \Bbbk e_1 \\ \Bbbk\beta + \Bbbk\gamma \end{pmatrix} \to P_1/(\Bbbk(\gamma - \lambda\beta)) \to 0.$$

Applying $\mathrm{Hom}_A(-, X_1(\lambda'))$ to the truncated projective resolution of the above, we obtain the following complex

$$\Bbbk \xleftarrow{0} \Bbbk \xleftarrow{g} \Bbbk \to 0.$$

If g is zero, this is case (1). If $g \neq 0$, this is case (2).

(3) The proof is similar to the above by considering minimal projective resolutions of S_1 and S_2.

(4) This is clear since P_2 is a projective module.

(5) and (6) Let S be either $S_{2,n}$ or $S_{1,n}$. By Example 5.1, $\mathrm{fpd}(\mathrm{Mod}_{f.d} - \Bbbk K_2) = 1$. This implies that

$$\dim \mathrm{Ext}^1_{\Bbbk K_2}(S, S) \leq 1$$

where S is considered as an indecomposable K_2-module.

Let us make a comment before we continue the proof. Following a more careful analysis, one can actually show that

$$\mathrm{Ext}^1_{\Bbbk K_2}(S, S) = 0.$$

Using this fact, the rest of the proof would show the assertions (5',6') in Remarks 5.11.

Now we continue the proof. There is a projective cover $P_1^b \xrightarrow{f} S$ so that $\ker f$ is a direct sum of finitely many copies of S_2. Since P_2 is the projective cover of S_2, we have a minimal projective resolution

$$\to P_2^a \to P_1^b \to S \to 0$$

for some a, b. In the category $\mathrm{Mod}_{f.d} - \Bbbk K_2$, we have a minimal projective resolution of S

$$0 \to S_2^a \to P_1^b \to S \to 0$$

where S_2 is a projective $\Bbbk K_2$-module. Hence we have a morphism of complexes

$$\begin{tikzcd}
 \arrow[r] & P_2^a \arrow[r] \arrow[d] & P_1^b \arrow[r] \arrow[d, "="] & S \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & S_2^a \arrow[r] & P_1^b \arrow[r] & S \arrow[r] & 0
\end{tikzcd}$$

Applying $\mathrm{Hom}_A(-, S)$ to above, we obtain that

$$\begin{tikzcd}
\cdots & \mathrm{Hom}_A(P_2^a, S) \arrow[l] & \mathrm{Hom}_A(P_1^b, S) \arrow[l, "h"'] & \mathrm{Hom}_A(S, S) \arrow[l] & 0 \arrow[l] \\
\cdots & \mathrm{Hom}_A(S_2^a, S) \arrow[l] \arrow[u, "g"] & \mathrm{Hom}_A(P_1^b, S) \arrow[l, "f"'] \arrow[u, "="'] & \mathrm{Hom}_A(S, S) \arrow[l] \arrow[u, "="'] & 0 \arrow[l]
\end{tikzcd}$$

Note that g is an isomorphism. Since $\dim \mathrm{Ext}^1_{\Bbbk K_2}(S, S) \le 1$, the cokernel of f has dimension at most 1. Since g is an isomorphism, the cokernel of h has dimension at most 1. This implies that $\mathrm{Ext}^1_A(S, S)$ has dimension at most 1. □

Proposition 5.12. *Let* $\mathfrak{A}$ *be the category* $\mathrm{Mod}_{f.d.} - A$ *where* A *is as in Example 5.5:*

(1)
$$\mathrm{fpd}^n\, \mathfrak{A} = \begin{cases} \sqrt{2} & n = 2, \\ 1 & n \neq 2. \end{cases}$$

As a consequence, $\mathrm{fpd}\, \mathfrak{A} = \sqrt{2}$.

(2) $\mathrm{SI}(\mathfrak{A}) = 2$.

(3) $\mathrm{fpd}\, \mathcal{A} \ge \sqrt{2}$.

Proof. (1) This is a consequence of Lemmas 5.9 and 5.10. Parts (2) and (3) follow from part (1). □

Remarks 5.13. Let A be the algebra given in Example 5.5. We list some facts, comments and questions:
(1) The algebra A is nonconnected $\mathbb{N}$-graded Koszul.

(2) The minimal projective resolutions of S_1 and S_2 are

$$\cdots \to P_1^{\oplus 4} \to P_2^{\oplus 4} \to P_1^{\oplus 2} \to P_2^{\oplus 2} \to P_1 \to S_1 \to 0,$$

and

$$\cdots \to P_2^{\oplus 4} \to P_1^{\oplus 2} \to P_2^{\oplus 2} \to P_1 \to P_2 \to S_2 \to 0.$$

(3) For $i \geq 0$, we have:

$$\mathrm{Ext}^i_A(S_1, S_1) = \begin{cases} \Bbbk^{\oplus 2^{i/2}} & i \text{ is even,} \\ 0 & i \text{ is odd.} \end{cases} \quad \mathrm{Ext}^i_A(S_1, S_2) = \begin{cases} 0 & i \text{ is even,} \\ \Bbbk^{\oplus 2^{(i+1)/2}} & i \text{ is odd.} \end{cases}$$

$$\mathrm{Ext}^i_A(S_2, S_2) = \begin{cases} \Bbbk^{\oplus 2^{i/2}} & i \text{ is even,} \\ 0 & i \text{ is odd.} \end{cases} \quad \mathrm{Ext}^i_A(S_2, S_1) = \begin{cases} 0 & i \text{ is even,} \\ \Bbbk^{\oplus 2^{(i-1)/2}} & i \text{ is odd.} \end{cases}$$

(4) One can check that every algebra of dimension 4 or less has either infinite or integral fpd. Hence, A is an algebra of smallest $\Bbbk$-dimension that has finite nonintegral (or irrational) fpd. It is unknown if there is a finite dimensional algebra A such that $\mathrm{fpd}(\mathrm{Mod}_{f.d.} -A)$ is transcendental.

(5) Several authors have studied the connection between tame-wildness and complexity [Bergh and Solberg 2010; Erdmann and Solberg 2011; Farnsteiner 2007; Feldvoss and Witherspoon 2011; Külshammer 2013; Rickard 1990]. The algebra A is probably the first explicit example of a tame algebra with infinite complexity.

(6) It follows from part (3) that the fp-curvature of $\mathcal{A} := D^b(\mathrm{Mod}_{f.d.} -A)$ is $\sqrt{2}$ (some details are omitted). As a consequence, $\mathrm{fpg}(\mathcal{A}) = \infty$. By Theorem 8.3, the complexity of A is ∞. We don't know what $\mathrm{fpd}\,\mathcal{A}$ is.

6. σ-decompositions

We fix a category $\mathcal{C}$ and an endofunctor σ. For a set of bricks B in $\mathcal{C}$ (or a set of atomic objects when $\mathcal{C}$ is triangulated), we define

$$\mathrm{fpd}^n\mid_B(\sigma) = \sup\{\rho(A(\phi, \sigma)) \mid \phi := \{X_1, \ldots, X_n\} \in \Phi_{n,b} \text{ and } X_i \in B\ \forall i\}.$$

Let $\Lambda := \{\lambda\}$ be a totally ordered set. We say a set of bricks B in $\mathcal{C}$ has a σ*-decomposition* $\{B^\lambda\}_{\lambda \in \Lambda}$ (based on Λ) if the following hold:

(1) B is a disjoint union $\bigcup_{\lambda \in \Lambda} B^\lambda$.

(2) If $X \in B^\lambda$ and $Y \in B^\delta$ with $\lambda < \delta$, $\mathrm{Hom}_{\mathcal{C}}(X, \sigma(Y)) = 0$.

The following lemma is easy.

Lemma 6.1. *Let n be a positive integer. Suppose that B has a σ-*decomposition* $\{B^\lambda\}_{\lambda \in \Lambda}$. Then*

$$\mathrm{fpd}^n\mid_B(\sigma) \leq \sup_{\lambda \in \Lambda, m \leq n} \{\mathrm{fpd}^m\mid_{B^\lambda}(\sigma)\}.$$

Proof. Let ϕ be a brick set that is used in the computation of $\operatorname{fpd}^n|_B(\sigma)$. Write

$$\phi = \phi_{\lambda_1} \cup \cdots \cup \phi_{\lambda_s} \tag{E6.1.1}$$

where λ_i is strictly increasing and $\phi_{\lambda_i} = \phi \cap B^{\lambda_i}$. For any objects $X \in \phi^{\lambda_i}$ and $Y \in \phi^{\lambda_j}$, where $\lambda_i < \lambda_j$, by definition, $\operatorname{Hom}_{\mathcal{C}}(X, \sigma(Y)) = 0$. Listing the objects in ϕ in the order that suggested by (E6.1.1), then the adjacency matrix of (ϕ, σ) is of the form

$$A(\phi,\sigma) = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ * & A_{22} & 0 & \cdots & 0 \\ * & * & A_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ * & * & * & \cdots & A_{ss} \end{pmatrix}$$

where each A_{ii} is the adjacency matrix $A(\phi_{\lambda_i}, \sigma)$. By definition,

$$\rho(A_{ii}) \leq \operatorname{fpd}^{s_i}|_{B^{\lambda_i}}(\sigma)$$

where s_i is the size of A_{ii}, which is no more than n. Therefore

$$\rho(A(\phi,\sigma)) = \max_i\{\rho(A_{ii})\} \leq \sup_{\lambda\in\Lambda, m\leq n}\{\operatorname{fpd}^m|_{B^\lambda}(\sigma)\}.$$

The assertion follows. $\square$

We give some examples of σ-decompositions.

Example 6.2. Let $\mathfrak{A}$ be an abelian category and $\mathcal{A}$ be the derived category $D^b(\mathfrak{A})$. Let $[n]$ be the n-fold suspension Σ^n:

(1) Suppose that α is an endofunctor of $\mathfrak{A}$ and $\bar{\alpha}$ is the induced endofunctor of $\mathcal{A}$. For each $n \in \mathbb{Z}$, let $B^n := \{M[-n] \mid M \text{ is a brick in } \mathfrak{A}\}$ and $B := \bigcup_{n\in\mathbb{Z}} B^n$. If $M_i[-n_i] \in B^{n_i}$, for $i = 1, 2$, such that $n_1 < n_2$, then

$$\operatorname{Hom}_{\mathcal{A}}(M_1[-n_1], \bar{\alpha}(M_2[-n_2])) = \operatorname{Ext}^{n_1-n_2}_{\mathfrak{A}}(M_1, \alpha(M_2)) = 0.$$

Then B has a $\bar{\alpha}$-decomposition $\{B^n\}_{n\in\mathbb{Z}}$ based on $\mathbb{Z}$.

(2) Suppose $g := \operatorname{gldim}\mathfrak{A} < \infty$. Let σ be the functor $\Sigma^g \circ \bar{\alpha}$. For each $n \in \mathbb{Z}$, let $B^n := \{M[n] \mid M \text{ is a brick in } \mathfrak{A}\}$ and $B := \bigcup_{n\in\mathbb{Z}} B^n$. If $M_i[n_i] \in B^{n_i}$, for $i = 1, 2$, such that $n_1 < n_2$, then

$$\operatorname{Hom}_{\mathcal{A}}(M_1[n_1], \sigma(M_2[n_2])) = \operatorname{Ext}^{n_2-n_1+g}_{\mathfrak{A}}(M_1, \alpha(M_2)) = 0.$$

Then B has a σ-decomposition $\{B^n\}_{n\in\mathbb{Z}}$ based on $\mathbb{Z}$.

Example 6.3. Let C be a smooth projective curve and let $\mathfrak{A}$ be the category of coherent sheaves over C. Every coherent sheaf over C is a direct sum of a torsion subsheaf and a locally free subsheaf. Define

$$B^0 = \{T \text{ is a torsion brick object in } \mathfrak{A}\}, \quad B^{-1} = \{F \text{ is a locally free brick object in } \mathfrak{A}\}, \quad B = B^{-1} \cup B^0.$$

Let σ be the functor $E^1 := \mathrm{Ext}^1_{\mathfrak{A}}(-,-)$. If $F \in B^{-1}$ and $T \in B^0$, then

$$\mathrm{Ext}^1_{\mathfrak{A}}(F, T) = 0.$$

Hence, B has an E^1-decomposition based on the totally ordered set $\Lambda := \{-1, 0\}$.

The next example is given in [Brüning and Burban 2007].

Example 6.4. Let C be an elliptic curve. Let $\mathfrak{A}$ be the category of coherent sheaves over C and $\mathcal{A}$ be the derived category $D^b(\mathfrak{A})$.

First we consider coherent sheaves. Let Λ be the totally ordered set $\mathbb{Q} \cup \{+\infty\}$. The slope of a coherent sheaf $X \neq 0$ [loc. cit., Definition 4.6] is defined to be

$$\mu(X) := \frac{\chi(X)}{\mathrm{rk}(X)} \in \Lambda$$

where $\chi(X)$ is the Euler characteristic of X and $\mathrm{rk}(X)$ is the rank of X. If X and Y are bricks such that $\mu(X) < \mu(Y)$, by [loc. cit., Corollary 4.11], X and Y are semistable, and thus by [loc. cit., Proposition 4.9(1)], $\mathrm{Hom}_{\mathfrak{A}}(Y, X) = 0$. By Serre duality (namely, Calabi–Yau property),

$$\mathrm{Hom}_{\mathcal{A}}(X, Y[1]) = \mathrm{Ext}^1_{\mathfrak{A}}(X, Y) = \mathrm{Hom}_{\mathfrak{A}}(Y, X)^* = 0. \tag{E6.4.1}$$

Write $B = \Phi_{1,b}(\mathfrak{A})$ and B^λ be the set of (semistable) bricks with slope λ. Then $B = \bigcup_{\lambda \in \Lambda} B^\lambda$. By (E6.4.1), $\mathrm{Ext}^1_{\mathfrak{A}}(X, Y) = 0$ when $X \in B^\lambda$ and $Y \in B^\nu$ with $\lambda < \nu$. Hence B has an E^1-decomposition. By Lemma 6.1, for every $n \geq 1$,

$$\mathrm{fpd}^n(E^1) = \mathrm{fpd}^n\,|_B(E^1) \leq \sup_{\lambda \in \Lambda, m \leq n} \{\mathrm{fpd}^m\,|_{B^\lambda}(E^1)\}.$$

Next we compute $\mathrm{fpd}^n\,|_{B^\lambda}(E^1)$. Let SS^λ be the full subcategory of $\mathfrak{A}$ consisting of semistable coherent sheaves of slope λ. By [loc. cit., Summary], SS^λ is an abelian category that is equivalent to SS^∞. Therefore one only needs to compute $\mathrm{fpd}^n\,|_{B^\infty}(E^1)$ in the category SS^∞. Note that SS^∞ is the abelian category of torsion sheaves and every brick object in SS^∞ is of the form $\mathcal{O}_p$ for some $p \in C$. In this case, $A(\phi, E^1)$ is the identity matrix. Consequently, $\rho(A(\phi, E^1)) = 1$. This shows that $\mathrm{fpd}^n\,|_{B^\lambda}(E^1) = \mathrm{fpd}^n\,|_{B^\infty}(E^1) = 1$ for all $n \geq 1$. It is clear that $\mathrm{fpd}^n(E^1) \geq \mathrm{fpd}^n\,|_{B^\infty}(E^1) = 1$. Combining with Lemma 6.1, we obtain that $\mathrm{fpd}^n(E^1) = 1$ for all n. (The above approach works for functors other than E^1.)

Finally we consider the fp-dimension for the derived category $\mathcal{A}$. It follows from Theorem 3.5(3) that

$$\mathrm{fpd}^n(\Sigma) = \mathrm{fpd}^n(E^1) = 1$$

for all $n \geq 1$. By definition,

$$\mathrm{fpd}(\mathcal{A}) = \mathrm{fpd}(\mathfrak{A}) = 1.$$

As we explained before fpd is an indicator of the representation types of categories.

Drozd and Greuel [2001] studied a tame-wild dichotomy for vector bundles on projective curves and introduced the notion of VB-finite, VB-tame and VB-wild similar to the corresponding notion in the representation theory of finite dimensional algebras.

Let C be a connected smooth projective curve, Drozd and Greuel [2001] showed the following:

(a) C is VB-finite if and only if C is $\mathbb{P}^1$.

(b) C is VB-tame if and only if C is elliptic (that is, of genus 1).

(c) C is VB-wild if and only if C has genus $g \geq 2$.

We now prove an fp-version of [Drozd and Greuel 2001, Theorem 1.6]. We thank Max Lieblich for providing ideas in the proof of Proposition 6.5(3).

Proposition 6.5. *Suppose $\Bbbk = \mathbb{C}$. Let $\mathbb{X}$ be a connected smooth projective curve and let g be the genus of $\mathbb{X}$:*

(1) *If $g = 0$ or $\mathbb{X} = \mathbb{P}^1$, then* $\operatorname{fpd} D^b(\operatorname{coh}(\mathbb{X})) = 1$.

(2) *If $g = 1$ or $\mathbb{X}$ is an elliptic curve, then* $\operatorname{fpd} D^b(\operatorname{coh}(\mathbb{X})) = 1$.

(3) *If $g \geq 2$, then* $\operatorname{fpd} D^b(\operatorname{coh}(\mathbb{X})) = \infty$.

Proof. (1) The assertion follows from (E5.1.4).

(2) The assertion follows from Example 6.4.

(3) By Theorem 3.5(4), $\operatorname{fpd}(D^b(\operatorname{coh}(\mathbb{X}))) = \operatorname{fpd}(\operatorname{coh}(\mathbb{X}))$. Hence it suffices to show that $\operatorname{fpd}(\operatorname{coh}(\mathbb{X})) = \infty$.

For each n, let $\{x_i\}_{i=1}^n$ be a set of n distinct points on $\mathbb{X}$. By [Drozd and Greuel 2001, Lemma 1.7], we might further assume that $2x_i \not\sim x_j + x_k$ for all $i \neq j$, as divisors on $\mathbb{X}$. Write $\mathcal{E}_i := \mathcal{O}(x_i)$ for all i. By [loc. cit., page 11], $\operatorname{Hom}_{\mathcal{O}_{\mathbb{X}}}(\mathcal{E}_i, \mathcal{E}_j) = 0$ for all $i \neq j$, which is also a consequence of a more general result [Huybrechts and Lehn 1997, Proposition 1.2.7]. It is clear that $\operatorname{Hom}_{\mathcal{O}_{\mathbb{X}}}(\mathcal{E}_i, \mathcal{E}_i) = \Bbbk$ for all i. Let ϕ_n be the set $\{\mathcal{E}_1, \dots, \mathcal{E}_n\}$. Then it is a brick set of nonisomorphic vector bundles on $\mathbb{X}$ (which are stable with $\operatorname{rank}(\mathcal{E}_i) = \deg(\mathcal{E}_i) = 1$ for all i).

Define the sheaf $\mathcal{H}_{ij} = \mathcal{H}om(\mathcal{E}_i, \mathcal{E}_j)$ for all i, j. Then $\deg(\mathcal{H}_{ij}) = 0$. By the Riemann–Roch theorem, we have

$$\begin{aligned} 0 &= \deg(\mathcal{H}_{ij}) \\ &= \chi(\mathcal{H}_{ij}) - \operatorname{rank}(\mathcal{H}_{ij})\chi(\mathcal{O}_{\mathbb{X}}) \\ &= \dim \operatorname{Hom}_{\mathcal{O}_{\mathbb{X}}}(\mathcal{E}_i, \mathcal{E}_j) - \dim \operatorname{Ext}^1_{\mathcal{O}_{\mathbb{X}}}(\mathcal{E}_i, \mathcal{E}_j) - (1-g) \\ &= \delta_{ij} - \dim \operatorname{Ext}^1_{\mathcal{O}_{\mathbb{X}}}(\mathcal{E}_i, \mathcal{E}_j) + (g-1), \end{aligned}$$

which implies that $\dim \operatorname{Ext}^1_{\mathcal{O}_{\mathbb{X}}}(\mathcal{E}_i, \mathcal{E}_j) = g - 1 + \delta_{ij}$. This formula was also given in [Drozd and Greuel 2001, page 11 before Lemma 1.7] when $i \neq j$.

Define the matrix A_n with entries $a_{ij} := \dim \operatorname{Ext}^1_{\mathcal{O}_{\mathbb{X}}}(\mathcal{E}_i, \mathcal{E}_j) = g - 1 + \delta_{ij}$, which is the adjacency matrix of (ϕ_n, E^1). This matrix has entries g along the diagonal and entries $g-1$ everywhere else. Therefore the vector $(1, \dots, 1)$ is an eigenvector for this matrix with eigenvalue $n(g-1)+1$. So $\rho(A_n) \geq n(g-1)+1 \geq n+1$. Since we can define ϕ_n for arbitrarily large n, we must have $\operatorname{fpd}(\operatorname{coh}(\mathbb{X})) = \infty$. □

Question 6.6. Let $\mathbb{X}$ be a smooth irreducible projective curve of genus $g \geq 2$. Is $\operatorname{fpd}^n(\mathbb{X})$ finite for each n? If yes, do these invariants recover g?

Proposition 6.7. *Suppose $\Bbbk = \mathbb{C}$. Let $\mathbb{Y}$ be a smooth projective scheme of dimension at least 2. Then*

$$\operatorname{fpd}^1(\operatorname{coh}(\mathbb{Y})) = \operatorname{fpd}(\operatorname{coh}(\mathbb{Y})) = \operatorname{fpd}^1(D^b(\operatorname{coh}(\mathbb{Y}))) = \operatorname{fpd}(D^b(\operatorname{coh}(\mathbb{Y}))) = \infty.$$

Proof. It is clear that $\operatorname{fpd}^1(\operatorname{coh}(\mathbb{Y}))$ is smallest among these four invariants. It suffices to show that $\operatorname{fpd}^1(\operatorname{coh}(\mathbb{Y})) = \infty$.

It is well-known that $\mathbb{Y}$ contains an irreducible projective curve $\mathbb{X}$ of arbitrarily large (either geometric or arithmetic) genus, see, for example, [Ciliberto et al. 2016, Theorem 0.1] or [Chen 1997, Theorems 1 and 2]. Let $\mathcal{O}_{\mathbb{X}}$ be the coherent sheaf corresponding to the curve $\mathbb{X}$ and let g be the arithmetic genus of $\mathbb{X}$. In the abelian category $\operatorname{coh}(\mathbb{X})$, we have

$$\dim \operatorname{Ext}^1_{\mathcal{O}_{\mathbb{X}}}(\mathcal{O}_{\mathbb{X}}, \mathcal{O}_{\mathbb{X}}) = \dim H^1(\mathbb{X}, \mathcal{O}_{\mathbb{X}}) = g.$$

Since $\operatorname{coh}(\mathbb{X})$ is a full subcategory of $\operatorname{coh}(\mathbb{Y})$, we have

$$\dim \operatorname{Ext}^1_{\mathcal{O}_{\mathbb{Y}}}(\mathcal{O}_{\mathbb{X}}, \mathcal{O}_{\mathbb{X}}) \geq \dim \operatorname{Ext}^1_{\mathcal{O}_{\mathbb{X}}}(\mathcal{O}_{\mathbb{X}}, \mathcal{O}_{\mathbb{X}}) = g.$$

By taking $\phi = \{\mathcal{O}_{\mathbb{X}}\}$, one sees that $\operatorname{fpd}^1(\operatorname{coh}(\mathbb{Y})) \geq \operatorname{fpd}^1(\operatorname{coh}(\mathbb{X})) \geq g$ for all such $\mathbb{X}$. Since g can be arbitrarily large, the assertion follows. □

7. Representation types

Representation types. We first recall some known definitions and results.

Definition 7.1. Let A be a finite dimensional algebra:

(1) We say A is of *finite representation type* if there are only finitely many isomorphism classes of finite dimensional indecomposable left A-modules.

(2) We say A is *tame* or *of tame representation type* if it is not of finite representation type, and for every $n \in \mathbb{N}$, all but finitely many isomorphism classes of n-dimensional indecomposables occur in a finite number of one-parameter families.

(3) We say A is *wild* or *of wild representation type* if, for every finite dimensional $\Bbbk$-algebra B, the representation theory of B can be embedded into that of A.

The following is the famous trichotomy result due to Drozd [1980].

Theorem 7.2 (Drozd's trichotomy theorem). *Every finite dimensional algebra is either of finite, tame, or wild representation type.*

Remarks 7.3. (1) An equivalent and more precise definition of a wild algebra is the following. An algebra A is called *wild* if there is a faithful exact embedding of abelian categories

$$\mathrm{Emb}: \mathrm{Mod}_{f.d.}-\Bbbk\langle x, y\rangle \to \mathrm{Mod}_{f.d.}-A \tag{E7.3.1}$$

that preserves indecomposables and respects isomorphism classes (namely, $\mathrm{Emb}(X) \cong \mathrm{Emb}(Y)$ implies that $X \cong Y$).

(2) A stronger notion of wildness is the following. An algebra A is called *strictly wild*, also called *fully wild*, if Emb in part (1) is a fully faithful embedding.

(3) It is clear that strictly wild is wild. The converse is not true.

We collect some celebrated results in terms of representation types of path algebras.

Theorem 7.4. *Let Q be a finite connected quiver*:

(1) [Gabriel 1972] *The path algebra $\Bbbk Q$ is of finite representation type if and only if the underlying graph of Q is a Dynkin diagram of type ADE.*

(2) [Nazarova 1973; Donovan and Freislich 1973] *The path algebra $\Bbbk Q$ is of tame representation type if and only if the underlying graph of Q is an extended Dynkin diagram of type $\tilde{A}\tilde{D}\tilde{E}$.*

Our main goal in this section is to prove Theorem 0.3. We thank Klaus Bongartz for suggesting the following lemma (personal communication).

Lemma 7.5. *Let A be a finite dimensional algebra that is strictly wild. Then, for each integer $a > 0$, there is a finite dimensional brick left A-module N such that $\dim \mathrm{Ext}^1_A(N, N) \geq a$.*

Proof. Let V be the vector space $\bigoplus_{i=1}^{a} \Bbbk x_i$ and let B be the finite dimensional algebra $\Bbbk\langle V\rangle/(V^{\otimes 2})$. By [Bongartz 2016, Theorem 2(i)], there is a fully faithful exact embedding

$$\mathrm{Mod}_{f.d.}-B \to \mathrm{Mod}_{f.d.}-\Bbbk\langle x, y\rangle.$$

Since A is strictly wild, there is a fully faithful exact embedding

$$\mathrm{Mod}_{f.d.}-\Bbbk\langle x, y\rangle \to \mathrm{Mod}_{f.d.}-A.$$

Hence we have a fully faithful exact embedding

$$F: \mathrm{Mod}_{f.d.}-B \to \mathrm{Mod}_{f.d.}-A. \tag{E7.5.1}$$

Let S be the trivial B-module $B/B_{\geq 1}$. It follows from an easy calculation that $\dim \mathrm{Ext}^1_B(S, S) = \dim(V)^* = a$. Since F is fully faithful exact, F induces an injection

$$F: \mathrm{Ext}^1_B(S, S) \to \mathrm{Ext}^1_A(F(S), F(S)).$$

Thus $\dim \mathrm{Ext}^1_A(F(S), F(S)) \geq a$. Since S is simple, it is a brick. Hence, $F(S)$ is a brick. The assertion follows by taking $N = F(S)$. $\square$

Proposition 7.6. (1) *Let A be a finite dimensional algebra that is strictly wild, then*

$$\operatorname{fpd}^1(E^1) = \operatorname{fpd}(\mathfrak{A}) = \operatorname{fpd}(\mathcal{A}) = \infty.$$

(2) *If $A := \Bbbk Q$ is wild, then*

$$\operatorname{fpd}^1(E^1) = \operatorname{fpd}(\mathfrak{A}) = \operatorname{fpd}(\mathcal{A}) = \infty.$$

Proof. (1) For each integer a, by Lemma 7.5, there is a brick N in $\mathfrak{A}$ such that $\operatorname{Ext}^1_{\mathfrak{A}}(N, N) \geq a$. Hence $\operatorname{fpd}^1(E^1) \geq a$. Since a is arbitrary, $\operatorname{fpd}^1(E^1) = \infty$. Consequently, $\operatorname{fpd}(\mathfrak{A}) = \infty$. By Lemma 2.9, $\operatorname{fpd}(\mathcal{A}) = \infty$.

(2) It is well-known that a wild path algebra is strictly wild, see a comment of Gabriel [1975, page 149] or [Ariki 2005, Proposition 7]. The assertion follows from part (1). □

The following lemma is based on a well-understood AR-quiver theory for acyclic quivers of finite representation type and the hammock theory introduced by Brenner [1986]. We refer to [Ringel and Vossieck 1987] if the reader is interested in a more abstract version of the hammock theory.

For a class of quivers including all ADE quivers, there is a convenient (though not essential) way of positioning the vertices as in [Assem et al. 2006, Example IV.2.6]. A quiver Q is called *well-positioned* if the vertices of Q are located so that all arrows are strictly from the right to the left of the same horizontal distance. For example, the following quiver D_n is well positioned:

$$\begin{tikzcd}
 & & & & n-1 \arrow[dl] \\
1 & 2 \arrow[l] & \cdots \arrow[l] & n-2 \arrow[l] & \\
 & & & & n \arrow[ul]
\end{tikzcd}$$

Lemma 7.7. *Let Q be a quiver such that*

(i) *the underlying graph of Q is a Dynkin diagram of type A, or D, or E, and that*

(ii) *Q is well-positioned.*

Let $A = \Bbbk Q$ and let M, N be two indecomposable left A-modules in the AR-quiver of A. Then the following hold:

(1) *There is a standard way of defining the order or degree for indecomposable left A-modules M, denoted by* $\deg M$, *such that all arrows in the AR-quiver have degree* 1, *or equivalently, all arrows are from the left to the right of the same horizontal distance. As in (E5.4.2), when $Q = A_2$,* $\deg P_2 = 0$, $\deg P_1 = 1$ *and* $\deg I_1 = 2$.

(2) *If* $\operatorname{Hom}_A(M, N) \neq 0$, *then* $\deg M \leq \deg N$.

(3) *The degree of the AR-translation τ is -2.*

(4) *If* $\mathrm{Ext}^1_A(M, N) \neq 0$, *then* $\deg M \geq \deg N + 2$.

(5) *There is no oriented cycle in the* E^1*-quiver of* $\mathfrak{A} := \mathrm{Mod}_{f.d.} - \Bbbk Q$, *denoted by* $Q^{E^1}_{\mathfrak{A}}$, *defined before Lemma 2.10.*

(6) $\mathrm{fpd}(\mathfrak{A}) = 0$.

Proof. (1) This is a well-known fact in AR-quiver theory. For each given quiver Q as described in (i) and (ii), one can build the AR-quiver by using the Auslander–Reiten translation τ and *the knitting algorithm*, see [Schiffler 2014, Chapter 3]. Some explicit examples are given in [Gabriel 1980, Chapter 6] and [Schiffler 2014, Chapter 3].

(2) This follows from (1). Note that the precise dimension of $\mathrm{Hom}_A(M, N)$ can be computed by using hammock theory [Brenner 1986; Ringel and Vossieck 1987]. Some examples are given in [Schiffler 2014, Chapter 3].

(3) This follows from the definition of the translation τ in the AR-quiver theory [Auslander et al. 1995, VII]. See also, [Crawley-Boevey 1992, Remarks (2), page 23].

(4) By Serre duality, $\mathrm{Ext}^1_R(M, N) = \mathrm{Hom}_A(N, \tau M)^*$ [Reiten and Van den Bergh 2002, Proposition I.2.3] or [Crawley-Boevey 1992, Lemma 1, page 22]. If $\mathrm{Ext}^1_R(M, N) \neq 0$, then, by Serre duality and part (2), $\deg N \leq \deg \tau M = \deg M - 2$. Hence $\deg M \geq \deg N + 2$.

(5) In this case, every indecomposable module is a brick. Hence the E^1-quiver $Q^{E^1}_{\mathfrak{A}}$ has the same vertices as the AR-quiver. By part (4), if there is an arrow from M to N in the quiver $Q^{E^1}_{\mathfrak{A}}$, then $\deg M \geq \deg N + 2$. This means that all arrows in $Q^{E^1}_{\mathfrak{A}}$ are from the right to the left. Therefore there is no oriented cycle in $Q^{E^1}_{\mathfrak{A}}$.

(6) This follows from part (5), Theorem 1.8(1) and Lemma 2.10. □

Theorem 7.8. *Let* Q *be a finite quiver whose underlying graph is a Dynkin diagram of type* ADE *and let* $A = \Bbbk Q$. *Then* $\mathrm{fpd}(\mathfrak{A}) = \mathrm{fpd}(\mathcal{A}) = \mathrm{fpgldim}(\mathcal{A}) = 0$.

Proof. Since the path algebra A is hereditary, $\mathfrak{A}$ is a-hereditary of global dimension 1. By Theorem 3.5(3), $\mathrm{fpd}(\mathfrak{A}) = \mathrm{fpd}(\mathcal{A})$. If Q_1 and Q_2 are two quivers whose underlying graphs are the same, then, by Bernstein–Gelfand–Ponomarev (BGP) reflection functors [Bernstein et al. 1973], $D^b(\mathrm{Mod}_{f.d.} - \Bbbk Q_1)$ and $D^b(\mathrm{Mod}_{f.d.} - \Bbbk Q_2)$ are triangulated equivalent. Hence we only need prove the statement for one representative. Now we can assume that Q satisfies the hypotheses (i) and (ii) of Lemma 7.7. By Lemma 7.7(6), $\mathrm{fpd}(\mathfrak{A}) = 0$. Therefore $\mathrm{fpd}(\mathcal{A}) = 0$, or equivalently, $\mathrm{fpd}(\Sigma) = 0$. By Theorem 3.5(1), $\mathrm{fpd}(\Sigma^i) = 0$ for all $i \neq 0, 1$. Therefore $\mathrm{fpgldim}(\mathcal{A}) = 0$. □

Weighted projective lines. To prove Theorem 0.3, it remains to show part (2) of the theorem. Our proof uses a result of [Chen et al. 2019] about weighted projective lines, which we now review. Details can be found in [Geigle and Lenzing 1987, Section 1].

For $t \geq 1$, let $\boldsymbol{p} := (p_0, p_1, \ldots, p_t)$ be a $(t+1)$-tuple of positive integers, called the *weight sequence*. Let $\boldsymbol{D} := (\lambda_0, \lambda_1, \ldots, \lambda_t)$ be a sequence of distinct points of the projective line $\mathbb{P}^1$ over $\Bbbk$. We normalize

$\boldsymbol{D}$ so that $\lambda_0 = \infty$, $\lambda_1 = 0$ and $\lambda_2 = 1$ (if $t \geq 2$). Let

$$S := \Bbbk[X_0, X_1, \dots, X_t]/(X_i^{p_i} - X_1^{p_1} + \lambda_i X_0^{p_0}, i = 2, \dots, t).$$

The image of X_i in S is denoted by x_i for all i. Let $\mathbb{L}$ be the abelian group of rank 1 generated by $\overrightarrow{x_i}$ for $i = 0, 1, \dots, t$ and subject to the relations

$$p_0 \overrightarrow{x_0} = \cdots = p_i \overrightarrow{x_i} = \cdots = p_t \overrightarrow{x_t} =: \overrightarrow{c}.$$

The algebra S is $\mathbb{L}$-graded by setting $\deg x_i = \overrightarrow{x_i}$. The corresponding *weighted projective line*, denoted by $\mathbb{X}(\boldsymbol{p}, \boldsymbol{D})$ or simply $\mathbb{X}$, is a noncommutative space whose category of coherent sheaves is given by the quotient category

$$\operatorname{coh}(\mathbb{X}) := \frac{\operatorname{gr}^{\mathbb{L}} - S}{\operatorname{gr}^{\mathbb{L}}_{f.d.} - S}$$

where $\operatorname{gr}^{\mathbb{L}} - S$ is the category of noetherian $\mathbb{L}$-graded left S-modules and $\operatorname{gr}^{\mathbb{L}}_{f.d.} - S$ is the full subcategory of $\operatorname{gr}^{\mathbb{L}} - S$ consisting of finite dimensional modules.

The weighted projective lines are classified into the following three classes:

$$\mathbb{X} \text{ is } \begin{cases} \textit{domestic} & \text{if } \boldsymbol{p} \text{ is } (p, q), (2, 2, n), (2, 3, 3), (2, 3, 4), (2, 3, 5); \\ \textit{tubular} & \text{if } \boldsymbol{p} \text{ is } (2, 3, 6), (3, 3, 3), (2, 4, 4), (2, 2, 2, 2); \\ \textit{wild} & \text{otherwise.} \end{cases}$$

Let $\mathbb{X}$ be a weighted projective curve. Let $\operatorname{Vect}(\mathbb{X})$ be the full subcategory of $\operatorname{coh}(\mathbb{X})$ consisting of all vector bundles. Similar to the elliptic curve case, Example 6.4, one can define the concepts of *degree*, *rank* and *slope* of a vector bundle on a weighted projective curve $\mathbb{X}$, see [Lenzing and Meltzer 1993, Section 2] for details. For each $\mu \in \mathbb{Q} \cup \{\infty\}$, let $\operatorname{Vect}_\mu(\mathbb{X})$ be the full subcategory of $\operatorname{Vect}(\mathbb{X})$ consisting of all vector bundles of slope μ.

Lemma 7.9. *Let $\mathbb{X} = \mathbb{X}(\boldsymbol{p}, \boldsymbol{D})$ be a weighted projective line*:

(1) $\operatorname{coh}(\mathbb{X})$ *is noetherian and hereditary.*

(2)
$$D^b(\operatorname{coh}(\mathbb{X})) \cong \begin{cases} D^b(\operatorname{Mod}_{f.d.} - \Bbbk\tilde{A}_{p,q}) & \textit{if } \boldsymbol{p} = (p, q), \\ D^b(\operatorname{Mod}_{f.d.} - \Bbbk\tilde{D}_n) & \textit{if } \boldsymbol{p} = (2, 2, n), \\ D^b(\operatorname{Mod}_{f.d.} - \Bbbk\tilde{E}_6) & \textit{if } \boldsymbol{p} = (2, 3, 3), \\ D^b(\operatorname{Mod}_{f.d.} - \Bbbk\tilde{E}_7) & \textit{if } \boldsymbol{p} = (2, 3, 4), \\ D^b(\operatorname{Mod}_{f.d.} - \Bbbk\tilde{E}_8) & \textit{if } \boldsymbol{p} = (2, 3, 5). \end{cases}$$

(3) *Let M be a generic simple object in* $\operatorname{coh}(\mathbb{X})$. *Then* $\operatorname{Ext}^1_{\mathbb{X}}(M, M) = 1$.

(4) $\operatorname{fpd}^1(\operatorname{coh}(\mathbb{X})) \geq 1$.

(5) *If $\mathbb{X}$ is tubular or domestic, then* $\operatorname{Ext}^1_{\mathbb{X}}(X, Y) = 0$ *for all* $X \in \operatorname{Vect}_{\mu'}(\mathbb{X})$ *and* $Y \in \operatorname{Vect}_\mu(\mathbb{X})$ *with* $\mu' < \mu$.

(6) *If $\mathbb{X}$ is domestic, then* $\operatorname{Ext}^1_{\mathbb{X}}(X, Y) = 0$ *for all* $X \in \operatorname{Vect}_{\mu'}(\mathbb{X})$ *and* $Y \in \operatorname{Vect}_\mu(\mathbb{X})$ *with* $\mu' \leq \mu$. *As a consequence*, $\operatorname{fpd}(\Sigma|_{\operatorname{Vect}_{\mu'}(\mathbb{X})}) = 0$ *for all* $\mu < \infty$.

(7) *Suppose $\mathbb{X}$ is tubular. Then every indecomposable vector bundle on $\mathbb{X}$ is semistable.*

(8) *Suppose $\mathbb{X}$ is tubular and let $\mu \in \mathbb{Q}$. Then each* $\mathrm{Vect}_{\mu}(\mathbb{X})$ *is a uniserial category. Accordingly indecomposables in* $\mathrm{Vect}_{\mu}(\mathbb{X})$ *decomposes into Auslander–Reiten components, which all are tubes of finite rank.*

Proof. (1) This is well known.

(2) [Geigle and Lenzing 1987, 5.4.1].

(3) Let M be a generic simple object. Then M is a brick and $\mathrm{Ext}^1(M, M) = 1$.

(4) Follows from (3) by taking $\phi := \{M\}$.

(5) This is [Schiffmann 2012, Corollary 4.34(i)] since tubular is also called elliptic in that work.

(6) This is [Schiffmann 2012, Comments after Corollary 4.34] since domestic is also called parabolic in that work. The consequence is clear.

(7) [Geigle and Lenzing 1987, Theorem 5.6(i)].

(8) [Geigle and Lenzing 1987, Theorem 5.6(iii)]. □

We will use the following result which is proved in [Chen et al. 2019].

Theorem 7.10. *Let $\mathbb{X}$ be a weighted projective line*:

(1) *If $\mathbb{X}$ is domestic, then* $\mathrm{fpd}\, D^b(\mathrm{coh}(\mathbb{X})) = 1$.

(2) *If $\mathbb{X}$ is tubular, then* $\mathrm{fpd}\, D^b(\mathrm{coh}(\mathbb{X})) = 1$.

(3) *If $\mathbb{X}$ is wild, then* $\mathrm{fpd}\, D^b(\mathrm{coh}(\mathbb{X})) \geq \dim \mathrm{Hom}_{\mathbb{X}}(\mathcal{O}_{\mathbb{X}}, \mathcal{O}_{\mathbb{X}}(\vec{\omega}))$ *where $\vec{\omega}$ is the dualizing element* [Geigle and Lenzing 1987, Section 1.2].

There is a similar statement for smooth complex projective curves (Proposition 6.5). The authors are interested in answering the following question.

Question 7.11. Let $\mathbb{X}$ be a wild weighted projective line. What is the exact value of $\mathrm{fpd}^n\, D^b(\mathrm{coh}(\mathbb{X}))$?

Tubes. The following example is studied in [Chen et al. 2019], which is dependent on direct linear algebra calculations.

Example 7.12. Let ξ be a primitive n-th root of unity. Let T_n be the algebra

$$T_n := \frac{\Bbbk\langle g, x\rangle}{(g^n - 1, xg - \xi g x)}.$$

This algebra can be expressed by using a group action. Let G be the group

$$\{g \mid g^n = 1\} \cong \mathbb{Z}/(n)$$

acting on the polynomial ring $\Bbbk[x]$ by $g \cdot x = \xi x$. Then T_n is naturally isomorphic to the skew group ring $\Bbbk[x] * G$. Let $\overrightarrow{A_{n-1}}$ denote the cycle quiver with n vertices, namely, the quiver with one oriented cycle connecting n vertices. It is also known that T_n is isomorphic to the path algebra of the quiver $\overrightarrow{A_{n-1}}$. Then $\operatorname{fpd}(\operatorname{Mod}_{f.d.}-T_n) = 1$ by [Chen et al. 2019].

Proof of Theorem 0.3. Part (1) follows from Theorems 7.4(1) and 7.8 and part (3) follows from Proposition 7.6(2). It remains to deal with part (2).

By Theorem 7.4(2), Q must be of type either $\overrightarrow{A_{n-1}}$, or $\tilde{A}_{p,q}$, or $\tilde{D}_n$, or $\tilde{E}_{6,7,8}$. If Q is of type $\overrightarrow{A_{n-1}}$, the assertion follows from Example 7.12. If Q is of type $\tilde{A}_{p,q}$, $\tilde{D}_n$, or $\tilde{E}_{6,7,8}$, the assertion follows from Lemma 7.9(2) and Theorem 7.10(1). $\square$

8. Complexity

The concept of complexity was first introduced by Alperin and Evens [1981] in the study of group cohomology. Since then the study of complexity has been extended to finite dimensional algebras, Frobenius algebras, Hopf algebras and commutative algebras. First we recall the classical definition of the complexity for finite dimensional algebras and then give a definition of the complexity for triangulated categories. We give the following modified (but equivalent) version, which can be generalized.

Definition 8.1. Let A be a finite dimensional algebra and $T = A/J(A)$ where $J(A)$ is the Jacobson radical of A. Let M be a finite dimensional left A-module:

(1) The *complexity* of M is defined to be

$$\operatorname{cx}(M) := \limsup_{n\to\infty} \log_n(\dim \operatorname{Ext}^n_A(M, T)) + 1.$$

(2) The *complexity* of the algebra A is defined to be

$$\operatorname{cx}(A) := \operatorname{cx}(T).$$

In the original definition of *complexity* by Alperin and Evens [1981] and in most other papers, the dimension of n-syzygies is used instead of the dimension of the Ext^n-groups, but it is easy to see that the asymptotic behavior of these two series are the same, therefore these give rise to the same complexity. It is well-known that $\operatorname{cx}(M) \le \operatorname{cx}(A)$ for all finite dimensional left A-modules M. Next we introduce the notion of a complexity for a triangulated category which is partially motivated by the work in [Bao et al. 2019, Section 4].

Definition 8.2. Let $\mathcal{T}$ be a pretriangulated category. Let d be a real number:

(1) The left subcategory of complexity less than d is defined to be

$${}_d\mathcal{T} := \Big\{X \in \mathcal{T} \mid \lim_{n\to\infty} \tfrac{1}{n^{d-1}} \dim \operatorname{Hom}_{\mathcal{T}}(X, \Sigma^n(Y)) = 0, \forall Y \in \mathcal{T}\Big\}.$$

(2) The right subcategory of complexity less than d is defined to be

$$\mathcal{T}_d := \Big\{X \in \mathcal{T} \mid \lim_{n\to\infty} \tfrac{1}{n^{d-1}} \dim \operatorname{Hom}_{\mathcal{T}}(Y, \Sigma^n(X)) = 0, \forall Y \in \mathcal{T}\Big\}.$$

(3) The *complexity* of $\mathcal{T}$ is defined to be

$$\operatorname{cx}(\mathcal{T}) := \inf\{d \mid {}_d\mathcal{T} = \mathcal{T}\}.$$

(4) The *Frobenius–Perron complexity* of $\mathcal{T}$ is defined to be

$$\operatorname{fpcx}(\mathcal{T}) := \operatorname{fpg}(\Sigma) + 1.$$

Note that it is not hard to show that $\operatorname{cx}(\mathcal{T}) = \inf\{d \mid \mathcal{T}_d = \mathcal{T}\}$.

Theorem 8.3. *Let $\mathcal{T}$ be a pretriangulated category. Then* $\operatorname{fpcx}(\mathcal{T}) \le \operatorname{cx}(\mathcal{T})$.

Proof. Let d be any number strictly larger than $\operatorname{cx}(\mathcal{T})$. We need to show that $\operatorname{fpcx}(\mathcal{T}) \le d$.

Let $\phi \in \Phi_{m,a}$ be an atomic set and let $X := \bigoplus_{X_i \in \phi} X_i$. Then, by definition,

$$\lim_{n\to\infty} \frac{\dim \operatorname{Hom}_{\mathcal{T}}(X, \Sigma^n(X))}{n^{d-1}} = 0.$$

Then there is a constant C such that $\dim \operatorname{Hom}_{\mathcal{T}}(X, \Sigma^n(X)) < Cn^{d-1}$ for all $n > 0$. Since each X_i is a direct summand of X, we have

$$a_{ij}(n) := \dim \operatorname{Hom}_{\mathcal{T}}(X_i, \Sigma^n(X_j)) < Cn^{d-1}$$

for all i, j. This means that each entry $a_{ij}(n)$ in the adjacency matrix of $A(\phi, \Sigma^n)$ is less than Cn^{d-1}. Therefore $\rho(A(\phi, \Sigma^n)) < mCn^{d-1}$. By Definition 2.3(3), $\operatorname{fpg}(\Sigma) \le d-1$. Thus $\operatorname{fpcx}(\mathcal{T}) \le d$ as desired. $\square$

We will prove that the equality $\operatorname{fpcx}(\mathcal{T}) = \operatorname{cx}(\mathcal{T})$ holds under some extra hypotheses. Let A be a finite dimensional algebra with a complete list of simple left A-modules $\{S_1, \dots, S_w\}$. We use n for any integer and i, j for integers between 1 and w. Define, for $i \le j$,

$$p_{ij}(n) := \min\{\dim \operatorname{Ext}^n_A(S_i, S_j), \dim \operatorname{Ext}^n_A(S_j, S_i)\}$$

and

$$P_n := \max\{p_{ij}(n) \mid i \le j\}.$$

We say A satisfies *averaging growth condition* if there are positive integers C and d, independent of the choices of n and (i, j), such that

$$\dim \operatorname{Ext}^n_A(S_i, S_j) \le C \max\{P_{n-d}, P_{n-d+1}, \dots, P_{n+d}\} \tag{E8.3.1}$$

for all n and all $1 \le i, j \le w$.

Theorem 8.4. *Let A be a finite dimensional algebra and $\mathcal{A} = D^b(\operatorname{Mod}_{f.d.} - A)$:*

(1) $\operatorname{cx}(A) = \operatorname{cx}(\mathcal{A})$. *As a consequence*, $\operatorname{cx}(A)$ *is a derived invariant.*

(2) *If A satisfies the averaging growth condition, then* $\operatorname{fpcx}(\mathcal{A}) = \operatorname{cx}(\mathcal{A}) = \operatorname{cx}(A)$. *As a consequence, if A is local or commutative, then* $\operatorname{fpcx}(\mathcal{A}) = \operatorname{cx}(\mathcal{A}) = \operatorname{cx}(A)$.

We will prove Theorem 8.4 after the next lemma.

Let $\mathcal{T}$ be a pretriangulated category with suspension Σ. We use X, Y, Z for objects in $\mathcal{T}$. Fix a family ϕ of objects in $\mathcal{T}$ and a positive number d. Define:

$$ {}_{d}(\phi) = \left\{ X \in \mathcal{T} \mid \lim_{n\to\infty} \tfrac{1}{n^{d-1}} \dim \operatorname{Hom}_{\mathcal{T}}(X, \Sigma^{n}(Y)) = 0, \forall Y \in \phi \right\}. \tag{E8.4.1} $$

$$ (\phi)_{d} = \left\{ X \in \mathcal{T} \mid \lim_{n\to\infty} \tfrac{1}{n^{d-1}} \dim \operatorname{Hom}_{\mathcal{T}}(Y, \Sigma^{n}(X)) = 0, \forall Y \in \phi \right\}. \tag{E8.4.2} $$

$$ {}^{d}(\phi) = \left\{ X \in \mathcal{T} \mid \lim_{n\to\infty} \tfrac{1}{n^{d}} \sum_{i\le n} \dim \operatorname{Hom}_{\mathcal{T}}(X, \Sigma^{i}(Y)) = 0, \forall Y \in \phi \right\}. \tag{E8.4.3} $$

$$ (\phi)^{d} = \left\{ X \in \mathcal{T} \mid \lim_{n\to\infty} \tfrac{1}{n^{d}} \sum_{i\le n} \dim \operatorname{Hom}_{\mathcal{T}}(Y, \Sigma^{i}(X)) = 0, \forall Y \in \phi \right\}. \tag{E8.4.4} $$

Lemma 8.5. *The following are full thick pretriangulated subcategories of $\mathcal{T}$ closed under direct summands*:

(1) ${}_{d}(\phi)$.

(2) $(\phi)_{d}$.

(3) ${}^{d}(\phi)$.

(4) $(\phi)^{d}$.

Proof. We only prove (1). The proofs of other parts are similar. Suppose $X \in {}_{d}(\phi)$. Using the fact $\lim_{n\to\infty} n^{d-1}/(n+1)^{d-1} = 1$, we see that $X[1] = \Sigma(X)$ is in ${}_{d}(\phi)$. Similarly, $X[-1]$ is in ${}_{d}(\phi)$. If $f : X_1 \to X_2$ be a morphism of objects in ${}_{d}(\phi)$, and let X_3 be the mapping cone of f, then, for each $Y \in \phi$, we have an exact sequence

$$ \to \operatorname{Hom}_{\mathcal{T}}(X_1, \Sigma^{n-1}(Y)) \to \operatorname{Hom}_{\mathcal{T}}(X_3, \Sigma^{n}(Y)) \to \operatorname{Hom}_{\mathcal{T}}(X_2, \Sigma^{n}(Y)) \to $$

which implies that $X_3 \in {}_{d}(\phi)$. Therefore ${}_{d}(\phi)$ is a thick pretriangulated subcategory of $\mathcal{T}$. If $X \in {}_{d}(\phi)$ and $X = Y \oplus Z$, it is clear that $Y, Z \in {}_{d}(\phi)$. Therefore ${}_{d}(\phi)$ is closed under taking direct summands. □

Proof of Theorem 8.4. (1) Let $c = \operatorname{cx}(A)$. For every $d < c$, we have that

$$ \limsup_{n\to\infty} \frac{\dim \operatorname{Ext}^{n}_{A}(T, T)}{n^{d-1}} = \infty $$

which implies that $T \notin {}_{d}\mathcal{A}$. Therefore $d \le \operatorname{cx}(\mathcal{A})$.

Conversely, let $d > c$. It follows from the definition that

$$ \limsup_{n\to\infty} \frac{\dim \operatorname{Ext}^{n}_{A}(T, T)}{n^{d-1}} = 0. $$

This means that $T \in (\{T\})_{d}$. Since T generates $\mathcal{A}$, we have $\mathcal{A} = (\{T\})_{d}$. Again, since T generates $\mathcal{A}$, we have $\mathcal{A} = \mathcal{A}_{d} = {}_{d}\mathcal{A}$. By definition, $d \ge \operatorname{cx}(\mathcal{A})$ as desired.

(2) Assume that A satisfies the averaging growth condition. Let

$$\begin{aligned} c_1 &= \operatorname{fpcx}(\mathcal{A}), \\ c_2 &= \limsup_{n\to\infty} \log_n (C \max\{P_{n-d}, P_{n-d+1}, \ldots, P_{n+d}\}) + 1, \\ c_3 &= \limsup_{n\to\infty} \log_n (P_n) + 1, \\ c_4 &= \operatorname{cx}(A) = \operatorname{cx}(\mathcal{A}). \end{aligned}$$

By calculus, we have $c_2 = c_3$. Let ϕ be the atomic set of simple objects $\{S_i\}_{i=1}^w$. Then $\rho(\phi, \Sigma^n) \geq p_{ij}(n)$, for all i, j, by Lemma 1.7(2). So $\rho(\phi, \Sigma^n) \geq P_n$. As a consequence, $c_1 \geq c_3$. Let $T = A/J = \bigoplus_{i=1}^w S_i^{d_i}$ for some finite numbers $\{d_i\}_{i=1}^w$. Let D be $\max_i\{d_i\}$. By the averaging growth condition, namely, (E8.3.1),

$$\begin{aligned} \dim \operatorname{Ext}_A^n(T, T) &= \sum_{i,j} d_i d_j \dim \operatorname{Ext}_A^n(S_i, S_j) \\ &\leq w^2 DC \max\{P_{n-d}, P_{n-d+1}, \ldots, P_{n+d}\} \end{aligned}$$

which implies that $c_4 = \operatorname{cx}(A) = \operatorname{cx}(T) \leq c_2$. Combining with Theorem 8.3, we have $c_1 = c_2 = c_3 = c_4$ as desired.

If A is local, then there is only one simple module S_1. Then (E8.3.1) is automatic. If A is commutative, then $\operatorname{Ext}_A^i(S_i, S_j) = 0$ for all n and all $i \neq j$. Again, in this case, (E8.3.1) is obvious. The consequence follows from the main assertion. □

For all well-studied finite dimensional algebras A, (E8.3.1) holds. For example, the algebra A in Example 5.5 satisfies the averaging growth condition. This can be shown by using the computation given in Remarks 5.13(3). It is natural to ask if every finite dimensional algebra satisfies the averaging growth condition.

Theorem 0.5 follows easily from Theorems 8.3 and 8.4.

Proof of Theorem 0.5. By Definition 8.2(4), Theorems 8.3 and 8.4(1), we have

$$\operatorname{fpg}(\mathcal{A}) = \operatorname{fpcx}(\mathcal{A}) - 1 \leq \operatorname{cx}(\mathcal{A}) - 1 = \operatorname{cx}(A) - 1.$$

The assertion follows. □

Lemma 8.6. (1) *Let $\mathfrak{A}$ be an abelian category and $\mathcal{A} = D^b(\mathfrak{A})$. If* $\operatorname{gldim} \mathfrak{A} < \infty$, *then* $\operatorname{fpcx}(\mathcal{A}) = 0$.

(2) *Let $\mathcal{T}$ be a pretriangulated category. If* $\operatorname{fpgldim} \mathcal{T} < \infty$, *then* $\operatorname{fpcx}(\mathcal{T}) = 0$.

Proof. Both are easy and proofs are omitted. □

We conclude with examples of nonintegral fpg of a triangulated category.

Example 8.7. (1) Let α be any real number in $\{0\} \cup \{1\} \cup [2, \infty)$. By [Krause and Lenagan 1985, Theorem 1.8, or page 14], there is a finitely generated algebra R with $\operatorname{GKdim} R = \alpha$. More precisely, [Krause and Lenagan 1985, Theorem 1.8] implies that there is a 2-dimensional vector space $V \subset R$ that generates R such that, there are positive integers $a < b$, for every $n > 0$,

$$an^{\alpha} < \dim(\Bbbk 1 + V)^n < bn^{\alpha}.$$

Define a filtration $\mathcal{F}$ on R by

$$F_i R = (\Bbbk 1 + V)^i \quad \forall i.$$

Let A be the associated graded algebra $\operatorname{gr} R$ with respect to this grading. Then A is connected graded and generated by two elements in degree 1 and satisfying, for every $n > 0$,

$$an^{\alpha} < \sum_{i=0}^{n} \dim A_i < bn^{\alpha}. \tag{E8.7.1}$$

To match up with the definition of complexity, we further assume that there are $c < d$ such that, for every $n > 0$,

$$cn^{\alpha-1} < \dim A_n < dn^{\alpha-1}. \tag{E8.7.2}$$

This can be achieved, for example, by replacing A by its polynomial extension $A[t]$ (with $\deg t = 1$) and replacing α by $\alpha + 1$.

Next we make A a differential graded (dg) algebra by setting elements in A_i to have cohomological degree i and $d_A = 0$. For this dg algebra, we denote the derived category of left dg A-modules by $\mathcal{A}$. Let $\mathcal{O}$ be the object ${}_A A$ in $\mathcal{A}$. By the definition of the cohomological degree of A, we have

$$\operatorname{Hom}_{\mathcal{A}}(\mathcal{O}, \Sigma^i \mathcal{O}) = A_i \quad \forall i. \tag{E8.7.3}$$

Let $\mathcal{T}$ be the full triangulated subcategory of $\mathcal{A}$ generated by $\mathcal{O}$. (E8.7.3) implies that $\mathcal{O}$ is an atomic object. Now using (E8.7.3) together with (E8.7.2), we obtain that

$$\operatorname{fpcx}(\mathcal{T}) \geq \alpha. \tag{E8.7.4}$$

By (E8.7.2)-(E8.7.3), we have that, for every $d > \alpha$, $\mathcal{O} \in {}_d(\{\mathcal{O}\})$. Since $\mathcal{O}$ generates $\mathcal{T}$, we have ${}_d(\{\mathcal{O}\}) = \mathcal{T}$. The last equation means that $\mathcal{O} \in (\mathcal{T})_d$. Since $\mathcal{O}$ generates $\mathcal{T}$, we have $(\mathcal{T})_d = \mathcal{T}$. By definition, $d > \operatorname{cx}(\mathcal{T})$. Combining these with Theorem 8.3 and (E8.7.4), we have, for every $d > \alpha$,

$$\alpha \leq \operatorname{fpcx}(\mathcal{T}) \leq \operatorname{cx}(\mathcal{T}) < d$$

which implies that $\operatorname{fpcx}(\mathcal{T}) = \operatorname{cx}(\mathcal{T}) = \alpha$. This construction implies that

$$\operatorname{GKdim}\left(\bigoplus_{i=0}^{\infty} \operatorname{Hom}_{\mathcal{T}}(\mathcal{O}, \Sigma^i(\mathcal{O}))\right) = \operatorname{GKdim} A = \alpha. \tag{E8.7.5}$$

(2) We now consider an extreme case. Let $a := \{a_i\}_{i=0}^{\infty}$ be any sequence of nonnegative integers with $a_0 = 1$. Define B to be the dg algebra $\bigoplus B_i$ such that:

(i) $\dim B_i = a_i$ for all i. In particular, $B_0 = \Bbbk$. Elements in B_i have cohomological degree i.

(ii) $\left(\bigoplus_{i>0} B_i\right)^2 = 0$.

(iii) Differential $d_B = 0$.

In this case, $\operatorname{GKdim} B = 0$. Similar to part (1), the derived category of left dg B-modules is denoted by $\mathcal{B}$. Let $\mathcal{O}$ be the object ${}_B B$ in $\mathcal{B}$. Then

$$\operatorname{Hom}_{\mathcal{B}}(\mathcal{O}, \Sigma^i \mathcal{O}) = B_i \quad \forall i,$$

and $\mathcal{O}$ is an atomic object. Let $\mathcal{T}$ be the full triangulated subcategory of $\mathcal{B}$ generated by $\mathcal{O}$. The argument in part (1) shows that

$$\operatorname{fpcx}(\mathcal{T}) = \limsup_{n\to\infty} \log_n(a_n) + 1.$$

Now let r be any real number ≥ 1 and let

$$a_i = \begin{cases} 1 & i = 0, \\ \lfloor i^{r-1} \rfloor & i \geq 1. \end{cases}$$

Then we have $\operatorname{fpcx}(\mathcal{T}) = r$. Let r be any real number ≥ 1 and $a_i = \lfloor r^i \rfloor$ for all $i \geq 0$. Then

$$\operatorname{fpcx}(\mathcal{T}) = \begin{cases} 1 & r = 1, \\ \infty & r > 1. \end{cases}$$

Using a similar method (with details omitted), $\operatorname{fpv}(\mathcal{T}) = r$.

Acknowledgments

The authors would like to thank Klaus Bongartz, Christof Geiss, Ken Goodearl, Claus Michael Ringel, and Birge Huisgen-Zimmermann for many useful conversations on the subject, thank Max Lieblich for the proof of Proposition 6.5(3), and thank the referees for their very careful reading and useful comments. J. Chen was partially supported by the National Natural Science Foundation of China (Grant No. 11571286). Z. Gao was partially supported by the National Natural Science Foundation of China (Grant No. 61401381). E. Wicks and J.J. Zhang were partially supported by the US National Science Foundation (Grant Nos. DMS-1402863 and DMS-1700825). X.-H. Zhang was partially supported by the National Natural Science Foundation of China (Grant No. 11401328). H. Zhu was partially supported by a grant from Jiangsu overseas Research and Training Program for university prominent young and middle-aged Teachers and Presidents, China.

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Communicated by J. Toby Stafford
Received 2018-04-04 Revised 2019-04-04 Accepted 2019-06-25

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Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

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Algebra & Number Theory

Volume 13 No. 9 2019