

# *Algebra & Number Theory*

Volume 13

2019

No. 4

**Cycle integrals of modular functions,  
Markov geodesics and a conjecture of Kaneko**

Paloma Bengoechea and Özlem Imamoglu



# Cycle integrals of modular functions, Markov geodesics and a conjecture of Kaneko

Paloma Bengoechea and Özlem Imamoglu

In this paper we study the values of modular functions at the Markov quadratics which are defined in terms of their cycle integrals along the associated closed geodesics. These numbers are shown to satisfy two properties that were conjectured by Kaneko. More precisely we show that the values of a modular function  $f$ , along any branch  $B$  of the Markov tree, converge to the value of  $f$  at the Markov number which is the predecessor of the tip of  $B$ . We also prove an interlacing property for these values.

## 1. Introduction

A well known theorem of Dirichlet asserts that for any irrational number  $x$ , there are infinitely many rational numbers  $p/q$  satisfying  $|x - p/q| < 1/q^2$ . For irrational numbers that are algebraic, thanks to a theorem of Roth [1955], the exponent 2 is optimal. The constant factor, on the other hand, can be improved and a classical theorem of Hurwitz asserts that for every irrational number  $x$  there exist infinitely many rational numbers  $p/q$  satisfying

$$\left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}.$$

The constant  $1/\sqrt{5}$  is best possible but if we exclude as  $x$  the numbers that are  $\text{PGL}(2, \mathbb{Z})$ -equivalent to the golden ratio  $(1 + \sqrt{5})/2$ , the constant  $1/\sqrt{5}$  improves to  $1/\sqrt{8}$ . If we also exclude the numbers that are  $\text{PGL}(2, \mathbb{Z})$ -equivalent to  $\sqrt{2}$ , then the constant improves to  $5/\sqrt{221}$ . By proceeding in this way, one obtains the Lagrange spectrum defined by

$$L := \{v(x)\}_{x \in \mathbb{R}} \subseteq [0, 1/\sqrt{5}] \quad \text{with} \quad v(x) = \liminf_{q \rightarrow \infty} q \|qx\|,$$

where  $\|x\|$  denotes the distance from a real number  $x$  to a closest integer. The quantity  $v(x)$  provides a measure of approximation of  $x$  by the rationals. For almost all  $x \in \mathbb{R}$  we have  $v(x) = 0$  and when  $v(x) > 0$  we call  $x$  badly approximable. Real quadratic irrationals are badly approximable, the worst ones being the golden ratio and its  $\text{PGL}(2, \mathbb{Z})$ -equivalents, followed by  $\sqrt{2}$  and its  $\text{PGL}(2, \mathbb{Z})$ -equivalents, etc. The Lagrange spectrum is not discrete (see [Hall 1947]) but the part of the spectrum in the subinterval  $(\frac{1}{3}, 1/\sqrt{5}]$  corresponding to classes of worst irrational numbers is, with  $\frac{1}{3}$  as its only accumulation point.

Bengoechea's research is supported by SNF grant 173976.

MSC2010: primary 11F03; secondary 11J06.

Keywords: modular forms, cycle integrals, markov numbers, j-invariant.

$L \cap (\frac{1}{3}, 1/\sqrt{5}]$  is well understood thanks to the work of Markov [1879; 1880] which connects this question of Diophantine approximation to the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz. \tag{1}$$

The set of Markov triples comprising the positive integer solutions  $(x, y, z)$  of (1) can be obtained starting with  $(1, 1, 1), (1, 1, 2), (2, 1, 5)$  and then proceeding recursively going from  $(x, y, z)$  to the new triples obtained by Vieta involutions  $(z, y, 3yz - x)$  and  $(x, z, 3xz - y)$ . The Markov numbers are the greatest coordinates of Markov triples. They form the Markov sequence

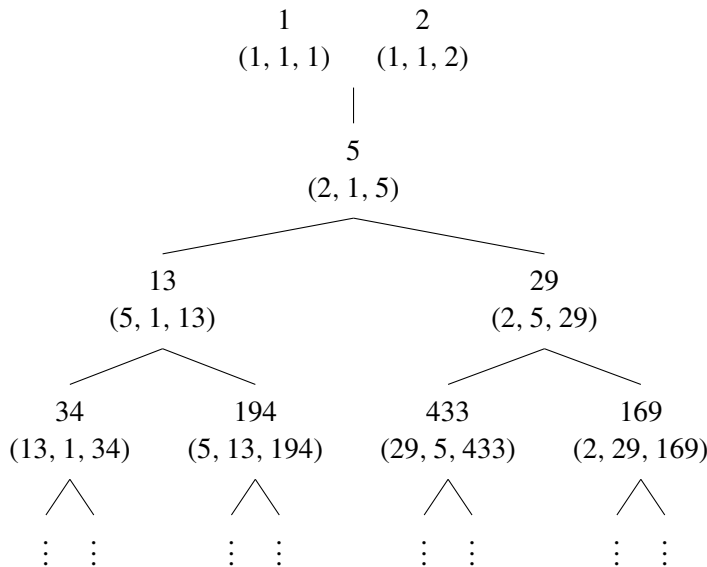
$$\{m_i\}_{i=1}^\infty = \{1, 2, 5, 13, 29, 34, 89, 169, 194, \dots\}.$$

The Markov number  $m_i$  is associated to a quadratic irrationality

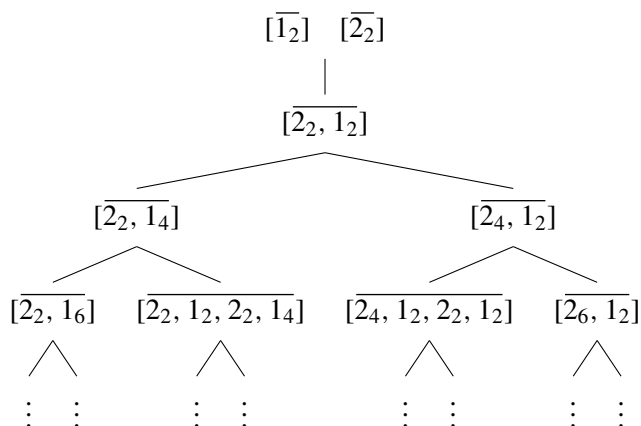
$$\theta_i = \frac{3m_i - 2k_i + \sqrt{9m_i^2 - 4}}{2m_i},$$

where  $k_i$  is an integer that satisfies  $a_i k_i \equiv b_i \pmod{m_i}$  and  $(a_i, b_i, m_i)$  is a solution to (1) with  $m_i$  maximal. Since  $k_i$  is uniquely defined modulo  $m_i$ ,  $\theta_i$  is uniquely defined modulo 1. Markov showed that  $v(\theta_i) = \sqrt{9 - 4/m_i^2}$ , and  $L \cap (\frac{1}{3}, 1/\sqrt{5}] = \{v(\theta_i)\}_{i \geq 1}$ . Moreover, any  $x \in \mathbb{R}$  for which  $v(x) \in L \cap (\frac{1}{3}, 1/\sqrt{5}]$  is  $\text{PGL}(2, \mathbb{Z})$ -equivalent to a Markov quadratic  $\theta_i$ .

Markov numbers come with a tree structure, inherited from Vieta involutions, that arranges them as



Here  $(a, b, c)$  is a solution to (1). The Markov quadratics inherit the same tree structure which can be given in terms of their continued fractions as



where  $b_n$  means that  $b$  is repeated  $n$  times. We note that it is more convenient to write  $[\overline{1_2}]$  instead of  $[\overline{1}]$  in connection with the conjunction operator in (5). The fact that all of the partial quotients of Markov quadratics are 1 or 2 and many of their other properties can be found in [Aigner 2013; Bombieri 2007; Malyshev 1977] (See for example Corollary 1.27 in [Aigner 2013].)

Markov numbers arise in many different contexts: see [Bourgain et al. 2016b; 2016a; Ghosh and Sarnak 2017] for some recent developments regarding the Markov surfaces.

The main goal of this paper is to study the values of modular functions along the tree associated to the Markov quadratics.

Let  $\Gamma = \text{PSL}(2, \mathbb{Z})$ . For a general quadratic irrationality  $w \in \mathbb{Q}(\sqrt{D})$  and a modular function  $f$  for  $\Gamma$ , the “value” of  $f$  at  $w$  is defined in terms of the integral of  $f$  along the geodesic cycle  $C_w \subset \Gamma \backslash \mathcal{H}$  associated to  $w$ . More precisely

$$f(w) := \int_{C_w} f(z) ds,$$

where  $ds$  is the hyperbolic arc length. We can normalize the number  $f(w)$  by the length of the geodesic  $C_w$  and define

$$f^{\text{nor}}(w) := \frac{f(w)}{2 \log \varepsilon_D},$$

where  $\varepsilon_D$  is the fundamental unit (see Section 2A).

The values of modular functions at real quadratic irrationalities were introduced in [Duke et al. 2011] and independently in [Kaneko 2009]. In [Duke et al. 2011] their averages over ideal classes were shown to be coefficients of mock modular forms whereas Kaneko [2009] studied their individual values  $f^{\text{nor}}(w)$  (in the case that the modular function is the Klein’s  $j$  invariant), and based on numerical calculations he made several interesting observations and conjectures.

In this paper we prove two of Kaneko’s conjectures which involve the values of modular functions at the Markov quadratics. Let  $B$  be any branch of the Markov tree where with a branch we mean a path on the tree without any zigzags. Our first theorem shows that if  $w_n^B$  is the  $n$ -th Markov quadratic on a branch  $B$  and  $w_0^B$  is the predecessor of the tip of  $B$  then the normalized values  $f^{\text{nor}}(w_n^B)$ , for any modular

function  $f$ , converge to the value  $f^{\text{nor}}(w_0^B)$ . (For more precise definitions of the tip of a branch and its predecessor see [Section 3A](#).) More precisely:

**Theorem 1.1.** *Let  $f$  be a modular function defined on  $\mathcal{H}$ . For any branch  $B$  of the Markov tree we have*

$$\lim_{n \rightarrow \infty} f^{\text{nor}}(w_n^B) = f^{\text{nor}}(w_0^B).$$

Our second theorem proves an eventual monotonicity result which also partially proves the interlacing property of the values for the Markov quadratics that was conjectured by Kaneko.

**Theorem 1.2.** *Let  $f$  be a modular function on  $\mathcal{H}$ , let  $B$  be any branch of the Markov tree. Then there exists a constant  $N_{f,B}$  such that, for all  $n \geq N_{f,B}$ , the real and imaginary parts of  $f^{\text{nor}}(w_{n+1}^B)$  lie between the real and respectively imaginary parts of  $f^{\text{nor}}(w_0^B)$  and  $f^{\text{nor}}(w_n^B)$ .*

The rest of the paper is organized as follows. In the next section we give the preliminaries about cycle integrals and continued fractions. In [Section 3](#), we give the basic properties of the Markov quadratics and the Markov tree. In [Sections 4](#) and [5](#) we study the values of modular functions on the Markov tree and prove [Theorems 1.1](#) and [1.2](#) respectively.

## 2. Preliminaries

**2A. Cycle integrals.** Let  $w$  be a real quadratic irrationality and  $\tilde{w}$  be its conjugate, so  $w$  and  $\tilde{w}$  are the roots of a quadratic equation

$$ax^2 + bx + c = 0 \quad (a, b, c \in \mathbb{Z}, \quad (a, b, c) = 1)$$

with discriminant  $D = b^2 - 4ac > 0$ . We change  $[a, b, c]$  to  $-[a, b, c]$  if necessary and write

$$w = \frac{-b + \sqrt{D}}{2a}, \quad \tilde{w} = \frac{-b - \sqrt{D}}{2a}.$$

The geodesic  $S_w$  in  $\mathcal{H}$  joining  $w$  and  $\tilde{w}$  is given by the equation

$$a|z|^2 + b \operatorname{Re}(z) + c = 0 \quad (z \in \mathcal{H}).$$

The stabilizer  $\Gamma_w$  of  $w$  in  $\Gamma$  preserves the quadratic form  $Q_w = [a, b, c]$ , and hence  $S_w$ . The group  $\Gamma_w$  is infinite cyclic; it corresponds to the group  $U_D^2$  of units of norm one of  $\mathbb{Q}(\sqrt{D})$  via the isomorphism:

$$\Gamma_w \longrightarrow U_D^2, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a - cw)^2. \tag{2}$$

We denote by  $A_w$  the generator of  $\Gamma_w$ ,

$$A_w = \begin{pmatrix} \frac{1}{2}(t - bu) & -cu \\ au & \frac{1}{2}(t + bu) \end{pmatrix},$$

where  $(t, u)$  is the smallest positive solution to Pell's equation  $t^2 - Du^2 = 4$ , and we denote by  $\varepsilon$  the generator of the infinite cyclic part of  $U_D$  whose square corresponds to  $A_w$  by the isomorphism [\(2\)](#).

For any modular function  $f$ , since the group  $\Gamma_w$  preserves the expression  $f(z)Q_w(z, 1)^{-1} dz$ , one can define the cycle integral of  $f$  along  $C_w = S_w/\Gamma_w$ , also viewed as the “value” of  $f$  at  $w$ , by the complex number

$$f(w) := \int_{C_w} \frac{\sqrt{D} f(z)}{Q_w(z, 1)} dz. \tag{3}$$

The factor  $\sqrt{D}$  is introduced here for convenience but is also natural since with the constant function  $f \equiv 1$ , (3) gives the length of the geodesic  $C_w$ . The integral defining  $f(w)$  is  $\Gamma$ -invariant and can in fact be taken along any path in  $\mathcal{H}$  from  $z_0$  to  $A_w^{-1}z_0$ , where  $z_0$  is any point in  $\mathcal{H}$ . Note that this gives an orientation on  $S_w$  from  $w$  to  $\tilde{w}$ , which is counterclockwise if  $a > 0$  and clockwise if  $a < 0$ . We normalize the number  $f(w)$  by the length of the geodesic  $C_w$  which is given by

$$\int_{C_w} \frac{\sqrt{D}}{Q_w(z, 1)} dz = 2 \log \varepsilon$$

and we define the normalized value as

$$f^{\text{nor}}(w) := \frac{f(w)}{2 \log \varepsilon}.$$

**2B. The “+” and “-” continued fractions.** Let  $(b_0, b_1, b_2, \dots)$  denote the “-” continued fraction

$$(b_0, b_1, b_2, \dots) = b_0 - \frac{1}{b_1 - \frac{1}{b_2 - \frac{1}{\ddots}}}$$

and  $[a_0, a_1, a_2, \dots]$  be the “+” continued fraction

$$[a_0, a_1, a_2, \dots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}$$

Every real number  $w$  has a “-” continued fraction expansion  $w = (b_0, b_1, b_2, \dots)$  with  $b_i \in \mathbb{Z}$  and  $b_i \geq 2$  for  $i \geq 1$  and a unique “+” continued fraction expansion  $w = [a_0, a_1, a_2, \dots]$  with  $a_i \in \mathbb{Z}$  and  $a_i \geq 1$  for  $i \geq 1$ . The “-” continued fraction expansion of  $w$  is obtained by setting  $w_0 = w$  and inductively  $b_i = \lceil w_i \rceil$ ,  $w_{i+1} = 1/(b_i - w_i) = ST^{-b_i}(w_i)$ , where  $S(x) = -1/x$  and  $T(x) = x + 1$ . The “+” continued fraction expansion is obtained by setting  $a_i = \lfloor w_i \rfloor$ ,  $w_{i+1} = 1/(w_i - a_i) = \varepsilon T^{-a_i}(w_i)$ , where  $\varepsilon(x) = 1/x$ . Hence the “-” continued fraction is given by transformations of  $\Gamma$  on the real line, whereas the “+” continued fraction corresponds to transformations of  $GL(2, \mathbb{Z})$ . To go from the “+” to the “-” continued fraction expansions, the general rule is

$$[a_0, a_1, a_2, \dots] \rightarrow (a_0 + 1, \underbrace{2, \dots, 2}_{a_1-1}, a_2 + 2, \underbrace{2, \dots, 2}_{a_3-1}, a_4 + 2, \dots). \tag{4}$$

It is well known that a real number  $w$  is a quadratic irrationality if and only if its “−” continued fraction expansion (or equivalently, its “+” continued fraction) is eventually periodic:

$$w = (b_0, b_1, \dots, b_k, \overline{b_{k+1}, \dots, b_{k+r}}),$$

where the line over  $b_{k+1}, \dots, b_{k+r}$  denotes the period. We say that  $w$  is *purely periodic* when all the partial quotients repeat. It will be useful for the rest of the paper to remember the following statements:

- (I) Two quadratic irrationalities have the same “−” period if and only if they are  $\Gamma$ -equivalent.
- (II)  $w$  has a purely periodic “−” continued fraction expansion if and only if  $0 < \tilde{w} < 1 < w$ , where  $\tilde{w}$  is the conjugate of  $w$ .
- (III) If  $w = (\overline{b_0, \dots, b_r})$ , then  $1/\tilde{w} = (\overline{b_r, \dots, b_0})$ .

These statements and more information about negative continued fractions can be found in [Zagier 1981, p. 126 ff].

The following lemma gives an upper bound for the distance between two real numbers in terms of the number of first partial quotients for which they coincide.

**Lemma 2.1.** *If the “−” continued fraction expansions of  $u$  and  $v$  coincide in the first  $r + 1$  partial quotients and their “+” continued fraction expansions have only 1’s and 2’s, then*

$$|u - v| \leq 10 \left( \frac{2}{1 + \sqrt{5}} \right)^{2r}.$$

*Proof.* Let  $u$  and  $v$  be as in the statement of the lemma. Then one can see, by applying the rule (4), that also the “+” continued fraction expansions of  $u$  and  $v$  coincide in the first  $r + 1$  partial quotients. Hence, if we set  $a_0, \dots, a_r$  to be those partial quotients, the rational number  $p/q = [a_0, \dots, a_r]$  is a convergent of both  $u$  and  $v$ . Then it is well known that

$$\left| u - \frac{p}{q} \right| \leq \frac{1}{q^2}, \quad \left| v - \frac{p}{q} \right| \leq \frac{1}{q^2}$$

and

$$q \geq \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^r.$$

Therefore,

$$|u - v| \leq \left| u - \frac{p}{q} \right| + \left| v - \frac{p}{q} \right| \leq 10 \left( \frac{2}{1 + \sqrt{5}} \right)^{2r}. \quad \square$$

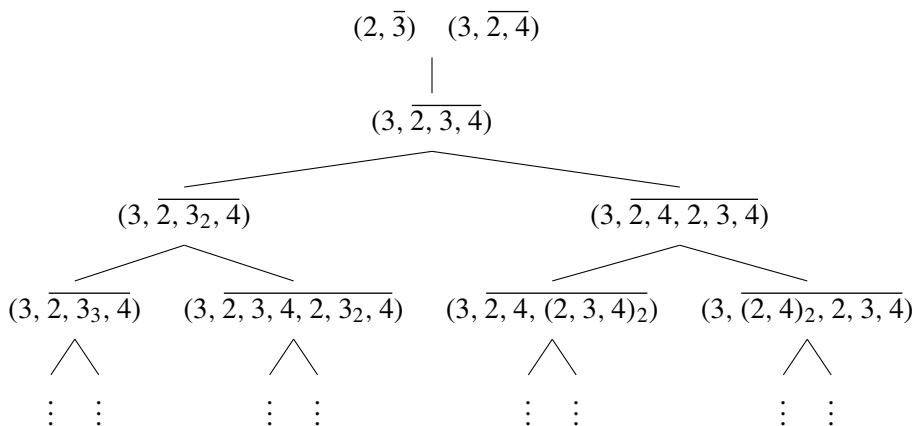
### 3. Markov Tree

**3A. Markov’s quadratics.** Let  $\{m_i\}_{i=1}^\infty = \{1, 2, 5, 13, 29, 34, 89, 169, 194, \dots\}$  be the set of Markov numbers. As in the introduction, for each Markov number  $m_i$ , we let

$$\theta_i = \frac{3m_i - 2k_i + \sqrt{9m_i^2 - 4}}{2m_i}$$

be the Markov quadratic where  $k_i$  is an integer that satisfies  $a_i k_i \equiv b_i \pmod{m_i}$  and  $(a_i, b_i, m_i)$  is a solution to (1) with  $m_i$  maximal. Changing the representative for  $k_i \pmod{m_i}$  does not change the  $\Gamma$  orbit of  $\theta_i$ . In Markov’s theory, only  $\text{PGL}(2, \mathbb{Z})$ -equivalence classes are relevant, which implies that the order of  $(a_i, b_i)$  does not matter. Since we need  $\Gamma$ -equivalence, which distinguishes nonreal  $f(\theta_i)$  and its conjugate, here the order of  $(a_i, b_i)$  becomes relevant. We fix it so that  $\text{Im}(f(w)) > 0$ .

The Markov tree  $\mathcal{T}$  associated to the Markov quadratics given in the introduction is in terms of the “+” continued fractions. Since the cycle integrals are  $\Gamma$  and not  $\text{PGL}(2, \mathbb{Z})$  invariant, we will rather work with the “-” continued fraction. By following the rule (4), the Markov tree  $\mathcal{T}$  becomes in the “-” continued fraction



Note that each branch (a path with no zigzags) in the tree  $\mathcal{T}$  comes with a left or right orientation. We call a branch a left (right) branch if starting from its first vertex on the top and going downwards the branch leans towards left (right). Since no zigzag paths are allowed, each branch has a unique orientation. For example, the branch with the quadratics  $(3, \overline{2, 3, 4}), (3, \overline{2, 3_2, 4}), (3, \overline{2, 3_3, 4})$  is a left branch, whereas the branch with  $(3, \overline{2, 3, 4}), (3, \overline{2, 4, 2, 3, 4}), (3, \overline{(2, 4)_2, 2, 3, 4})$  is a right branch. We call the first vertex at the top of any branch its tip. Except for the two singular cases of  $(2, \overline{3})$  and  $(3, \overline{2, 4})$ , each Markov number lies both on a right and a left branch but it is the tip of only a left or a right branch, except for  $(3, \overline{2, 3, 4})$  which is the tip of both the leftmost and the rightmost branches.

In the case of “+” continued fractions we consider a *conjunction operation* of two periods as

$$[\overline{s_0, \dots, s_n}] \odot [\overline{t_0, \dots, t_m}] = [\overline{s_0, \dots, s_n, t_0, \dots, t_m}]. \tag{5}$$

All Markov quadratics can be constructed by using this operation, starting with  $[\overline{1_2}]$  and  $[\overline{2_2}]$ . Indeed, each Markov quadratic is the result of the conjunction operation of its predecessor on the same branch and the predecessor of the tip of the branch.

For the “-” continued fraction, the rule is also the conjunction of periods except for the leftmost branch, where the  $n$ -th Markov quadratic is  $(3, \overline{2, 3_n, 4})$ . Indeed, let  $x = [\overline{s_0, \dots, s_n}] = (b_0, \overline{b_1, \dots, b_k})$  and  $y = [\overline{t_0, \dots, t_m}] = (c_0, \overline{c_1, \dots, c_\ell})$ . For any branch different from the rightmost branch, by applying (4) together with the observation that  $s_n = t_m = 1$  are in odd positions, so they do not contribute in the



“–” expansion, we obtain

$$x \odot y = (b_0, \overline{b_1, \dots, b_{k-1}, t_0 + 2, c_1, \dots, c_{\ell-1}, s_0 + 2}).$$

But  $t_0$  is equal to 1 on the leftmost branch and 2 on any other branch, and  $s_0 = 2$ . For the rightmost branch, (4) also gives

$$x \odot y = (b_0, \overline{b_1, \dots, b_{k-1}, 4, c_1, \dots, c_{\ell-1}, s_0 + 2})$$

and  $s_0 = 2$ .

Throughout the paper, we denote by  $w_n^B$  ( $n \geq 1$ ) the  $n$ -th Markov quadratic on a branch  $B$  of the tree and  $w_0^B$  the left (right) predecessor of the tip  $w_1^B$  of  $B$  if  $B$  is a left (right) branch. For example, if  $B = L$  is the leftmost branch, then  $w_0^L = (2, \overline{3})$ ,  $w_1^L = (3, \overline{2, 3, 4})$ ,  $w_2^L = (3, \overline{2, 3_2, 4})$ ,  $w_3^L = (3, \overline{2, 3_3, 4})$ , etc. If  $B = R$  is the rightmost branch, then  $w_0^R = (3, \overline{2, 4})$ ,  $w_1^R = (3, \overline{2, 3, 4})$ ,  $w_2^R = (3, \overline{2, 4, 2, 3, 4})$ ,  $w_3^R = (3, \overline{(2, 4)_2, 2, 3, 4})$ , etc.

The  $n$ -th Markov quadratic on a left branch  $B \neq L$  can be written as

$$w_n^B = (3, \overline{a_1, \dots, a_s, (b_1, \dots, b_r)_n}), \tag{6}$$

where  $w_0^B = (3, \overline{b_1, \dots, b_r})$  and  $a_1, \dots, a_s$  depend only on  $B$ . On a right branch  $B$ , we have

$$w_n^B = (3, \overline{(b_1, \dots, b_r)_{n-1}, a_1, \dots, a_s}), \tag{7}$$

and on the leftmost branch  $L$  we have

$$w_n^L = (3, \overline{2, 3_n, 4}). \tag{8}$$

**Remark 3.1.** The leftmost branch in the Markov tree is also called the Fibonacci branch since the associated Markov numbers on this branch are the odd indexed Fibonacci numbers. Similarly the rightmost branch is associated with the odd indexed Pell numbers which are defined by the recurrence  $P_0 = 0$ ,  $P_1 = 1$  and  $P_{n+1} = 2P_n + P_{n-1}$  (see [Aigner 2013, p. 49]).

**3B. The cycle of quadratics of a Markov number.** For any quadratic irrationality  $w$ , it is known that the hyperbolic element  $A_w$  is conjugate to a word in  $T$  and  $V$ , where

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

If in particular  $w = w_n^B$  is a quadratic on  $\mathcal{T}$  ( $n \geq 0$ ), then the associated hyperbolic element  $A_{w_n^B}$  can be written as a word in  $T$  and  $V$ . More specifically,  $A_{w_n^B} = A_0^{-1} \cdots A_{\ell_n}^{-1}$ , where  $A_0 = I$  and  $A_i \in \{T^{-1}, V^{-1}\}$  for  $1 \leq i \leq \ell_n$  are given by the algorithm:

$$w_{n,0}^B = w_n^B, \quad w_{n,i+1}^B = A_{i+1}(w_{n,i}^B) \quad (i \geq 0),$$

where

$$A_{i+1} = \begin{cases} T^{-1} & \text{if } \lfloor w_{n,i}^B \rfloor \geq 1, \\ V^{-1} & \text{otherwise.} \end{cases}$$

Hence

$$w_{n,i}^B = A_i \cdots A_0 w_n^B, \quad i = 0, \dots, \ell_n, \tag{9}$$

and  $\ell_n$  is the length of the word  $A_{w_n^B}$ , or equivalently, the length of the cycle of quadratics  $\{w_{n,i}^B\}_i$  of  $w_n^B$ . As the following example demonstrates, this procedure applied to a Markov quadratic in fact cycles back and hence terminates.

**Example 3.2.** For example, the cycle of  $w_1^L = (3, \overline{2, 3, 4})$  on the leftmost branch is:

$$\begin{aligned} w_{1,0}^L &= (3, \overline{2, 3, 4}), \\ w_{1,1}^L &= T^{-1}(w_{1,0}^L) = (2, \overline{2, 3, 4}), \\ w_{1,2}^L &= T^{-1}(w_{1,1}^L) = (1, \overline{2, 3, 4}), \\ w_{1,3}^L &= V^{-1}(w_{1,2}^L) = (1, \overline{3, 4, 2}), \\ w_{1,4}^L &= V^{-1}(w_{1,3}^L) = (2, \overline{4, 2, 3}), \\ w_{1,5}^L &= T^{-1}(w_{1,4}^L) = (1, \overline{4, 2, 3}), \\ w_{1,6}^L &= V^{-1}(w_{1,5}^L) = (3, \overline{2, 3, 4}) = w_{1,0}^L. \end{aligned}$$

The length is  $\ell_1 = 6$  and  $A_{w_1^L} = ITTVVTV$ .

From now on we restrict to a left branch but not the leftmost branch. All the following arguments apply in the same way if  $B$  is a right branch or  $B = L$ , the leftmost branch. The small difference in the arguments arise due to the different conjunction operations necessary, which are given in (7) for the right and in (8) for the leftmost branches.

We now consider  $w_n^B$ , in a left branch  $B \neq L$ , written as in (6). Then

$$\ell_n = n\ell_0 + \sum_{i=1}^s (a_i - 1), \tag{10}$$

where

$$\ell_0 = \sum_{i=1}^r (b_i - 1)$$

is the length of the cycle of  $w_0^B$ . The number of partial quotients in the period of  $w_1^B$  is  $s + r$  and the conjunction operation ensures that this is  $\leq 2r$ . Hence  $s \leq r$  and since  $a_i \leq 4$ , we have

$$\ell_n \leq 3r(n + 1). \tag{11}$$

It is convenient to set

$$a = \sum_{i=1}^s (a_i - 1)$$

and

$$\mathbf{p} = (b_1, \dots, b_r), \quad \mathbf{p}_k = (b_1, \dots, b_r)_k, \quad \mathbf{q}_k = (b_r, \dots, b_1)_k,$$

where the subindex  $k$  means that the continued fraction is repeated  $k$  times. With these notations, the cycle of  $w_n^B$  is of the form:

$$\begin{aligned} w_{n,0}^B &= (3, \overline{a_1, \dots, a_s, \mathbf{p}_n}), \\ w_{n,1}^B &= (2, \overline{a_1, \dots, a_s, \mathbf{p}_n}), \\ w_{n,2}^B &= (1, \overline{a_1, \dots, a_s, \mathbf{p}_n}), \\ w_{n,3}^B &= (a_1 - 1, \overline{a_2, \dots, a_s, \mathbf{p}_n, a_1}), \\ &\vdots \\ w_{n,a}^B &= (3, \overline{\mathbf{p}_n, a_1, \dots, a_s}), \\ &\vdots \\ w_{n,a+\ell_0}^B &= (3, \overline{\mathbf{p}_{n-1}, a_1, \dots, a_s, \mathbf{p}}), \\ &\vdots \\ w_{n,a+n\ell_0}^B &= (3, \overline{a_1, \dots, a_s, \mathbf{p}_n}) = w_{n,0}^B. \end{aligned}$$

**Remark 3.3.** One can easily write the continued fraction expansion for the Galois conjugate  $-\tilde{w}_{n,i}^B$  of  $-w_{n,i}^B$  in terms of that of  $w_{n,i}^B$ . Indeed, let  $(d_0, \overline{d_1, \dots, d_m})$  be the continued fraction expansion of  $w_{n,i}^B$ . The quadratic  $ST^{-d_0}(w_{n,i}^B)$  is purely periodic with continued fraction  $(\overline{d_1, \dots, d_m})$  so, by the property (III), its Galois conjugate is  $1/(\overline{d_m, \dots, d_1})$ . Therefore,

$$\tilde{w}_{n,i}^B = T^{d_0} S(1/(\overline{d_m, \dots, d_1})) = -(d_m - d_0, \overline{d_{m-1}, \dots, d_1, d_m}).$$

### 4. Convergence property

In this section we study the values of a modular function on the Markov tree. Let  $B$  be any branch of the tree and  $w_n^B$  be the  $n$ -th Markov quadratic on  $B$ . Let  $A_{w_n^B} = A_0^{-1} \cdots A_{\ell_n}^{-1}$ , where  $A_0 = I$  and  $A_i \in \{T^{-1}, V^{-1}\}$  for  $1 \leq i \leq \ell_n$ . Let  $\rho = e^{\pi i/3}$  and  $z_i = A_0^{-1} \cdots A_i^{-1} \rho^2$ . Then using the modularity of  $f$  we have

$$\begin{aligned} f(w_n^B) &= -\sqrt{D} \sum_{i=0}^{\ell_n-1} \int_{z_i}^{z_{i+1}} \frac{f(z)}{Q_{w_n^B}(z, 1)} dz \\ &= -\sqrt{D} \sum_{i=0}^{\ell_n-1} \int_{\rho^2}^{A_{i+1}^{-1} \rho^2} \frac{f(z)}{(Q_{w_n^B} | A_0^{-1} \cdots A_i^{-1})(z, 1)} dz \\ &= -\sum_{i=0}^{\ell_n-1} \int_{\rho^2}^{A_{i+1}^{-1} \rho^2} f(z) \left( \frac{1}{z - w_{n,i}^B} - \frac{1}{z - \tilde{w}_{n,i}^B} \right) dz. \end{aligned}$$

Since  $V(\rho^2) = T(\rho^2) = \rho$ , we obtain:

**Lemma 4.1.** For  $n \geq 0$  we have

$$f(w_n^B) = \int_{\rho}^{\rho^2} \sum_{i=0}^{\ell_n-1} f(z) \left( \frac{1}{z - w_{n,i}^B} - \frac{1}{z - \tilde{w}_{n,i}^B} \right) dz. \tag{12}$$

**Lemma 4.1** is the main tool we use to estimate the values of modular functions at real quadratic irrationalities.

Throughout the paper, we denote by  $\mathcal{C}$  the arc of circle joining  $\rho^2$  and  $\rho$ . We denote by  $\varepsilon_n^B$  the image of  $A_{w_n}^B$  under the isomorphism (2), so the length of  $C_{w_n^B}$  equals  $2 \log \varepsilon_n^B$ .

Our first goal is to show that the normalized values  $f^{\text{nor}}(w_n^B)$  for any modular function  $f$  along any branch  $B$  converge to the value  $f^{\text{nor}}(w_0^B)$ . We call this property ‘‘convergence property’’ and prove it in this section. The main idea of the proof is to divide the sum in **Lemma 4.1** into several ranges and bound each piece making repeated use of **Lemma 2.1**. For simplicity of the notation, as mentioned before, we restrict to a left but not the leftmost branch. However, the argument in the proof of **Theorem 4.2** applies in the same way if  $B$  is a right branch or  $B = L$ . Only the bound  $\delta_1(r, N)$  will be slightly modified but will still be of the form  $O(rN\lambda^r)$  where  $\lambda = (2/(1 + \sqrt{5}))^2$ . Hence **Corollary 4.4** also remains true for any branch.

**Theorem 4.2.** *Let  $f$  be a modular function,  $B$  be any left branch  $\neq L$  of the Markov tree  $\mathcal{T}$  and  $N \geq 1$ . There exists a complex number  $K = K_{f,B,N}$  such that for all  $n \geq N$ ,*

$$|f(w_n^B) - nf(w_0^B) - K| \leq \delta_1(r, N) \max_{z \in \mathcal{C}} |f(z)|, \tag{13}$$

where

$$\delta_1(r, N) = \frac{80\pi}{3} (2 + r(N + 1)) \left( \frac{2}{1 + \sqrt{5}} \right)^{2(rN-1)} \tag{14}$$

and  $r + 1$  is the number of partial quotients in the period of  $w_0^B$ .

*Proof.* By applying **Lemma 4.1** for  $f(w_n^B)$  and  $f(w_0^B)$  we have:

$$f(w_n^B) - nf(w_0^B) = \int_{\rho}^{\rho^2} f(z) (S_1(n, N, z) + S_2(n, N, z) + S_3(n, N, z)) dz, \tag{15}$$

where

$$\begin{aligned} S_1(n, N, z) &= \sum_{i=0}^{a-1} \frac{1}{z - w_{n,i}^B} + \sum_{i=a+(n-N)\ell_0}^{\ell_n-1} \frac{1}{z - w_{n,i}^B} - \sum_{i=0}^{a+N\ell_0-1} \frac{1}{z - \tilde{w}_{n,i}^B} - N \sum_{i=0}^{\ell_0-1} \left( \frac{1}{z - w_{0,i}^B} - \frac{1}{z - \tilde{w}_{0,i}^B} \right), \\ S_2(n, N, z) &= \sum_{i=a}^{a+(n-N)\ell_0-1} \frac{1}{z - w_{n,i}^B} - (n - N) \sum_{i=0}^{\ell_0-1} \frac{1}{z - w_{0,i}^B}, \\ S_3(n, N, z) &= - \sum_{i=a+N\ell_0}^{a+n\ell_0-1} \frac{1}{z - \tilde{w}_{n,i}^B} + (n - N) \sum_{i=0}^{\ell_0-1} \frac{1}{z - \tilde{w}_{0,i}^B}. \end{aligned}$$

Moreover, we can also write

$$S_1(n, N, z) = S_1(N, N, z) + (S_1(n, N, z) - S_1(N, N, z)). \tag{16}$$

Define

$$K := \int_{\rho}^{\rho^2} f(z)S_1(N, N, z) dz$$

and

$$c(n, z) := |S_1(n, N, z) - S_1(N, N, z)| + |S_2(n, N, z)| + |S_3(n, N, z)|.$$

Then

$$|f(w_n^B) - nf(w_0^B) - K| \leq \int_{\rho}^{\rho^2} c(n, z)|f(z)||dz|. \tag{17}$$

These divisions are guided by the continued fraction expansions of all the terms in the cycle of  $w_n^B$  and  $w_0^B$  and their conjugates. As we will see shortly, the repeated use of [Lemma 2.1](#) will allow us to bound all the other sums after we separate the main term  $K$ .

Let  $\lambda = (2/(1 + \sqrt{5}))^2$ . If we can show that

$$c(n, z) \leq 80(2 + r(N + 1))\lambda^{rN-1} \tag{18}$$

for  $z \in \mathcal{C}$ , then the theorem is proved. Next we show (18).

**Bound for  $|S_2(n, N, z)|$ .** We have that

$$\begin{aligned} |S_2(n, N, z)| \leq & \sum_{k=0}^{n-N-2} \sum_{i=1}^{\ell_0} \frac{|w_{n,2+a+k\ell_0+i}^B - w_{0,2+i}^B|}{|z - w_{n,2+a+k\ell_0+i}^B||z - w_{0,2+i}^B|} + \sum_{i=0}^2 \frac{|w_{n,a+i}^B - w_{0,i}^B|}{|z - w_{n,a+i}^B||z - w_{0,i}^B|} \\ & + \sum_{i=1}^{\ell_0-3} \frac{|w_{n,2+a+(n-N-1)\ell_0+i}^B - w_{0,2+i}^B|}{|z - w_{n,2+a+(n-N-1)\ell_0+i}^B||z - w_{0,2+i}^B|}. \end{aligned}$$

Clearly for any  $z \in \mathcal{C}$  and  $x \in \mathbb{R}$ , we have that  $|z - x| \geq \text{Im}(e^{2\pi i/3}) = \sqrt{3}/2$ . Hence the denominators are bounded below by  $\frac{3}{4}$  when  $z \in \mathcal{C}$  since the points  $w$  are real. The numerators can be bounded by using [Lemma 2.1](#). For  $i = 0, 1, 2$ ,

$$w_{n,a+i}^B = (3 - i, \overline{\mathbf{p}_n, a_1, \dots, a_s}) \quad \text{and} \quad w_{0,i}^B = (3 - i, \overline{\mathbf{p}})$$

coincide at least in the first  $rn + 1$  partial quotients. For each  $0 \leq k \leq n - N - 2$ , we have: For  $1 \leq i \leq b_1 - 1$ ,

$$w_{n,2+a+k\ell_0+i}^B = (b_1 - i, \overline{b_2, \dots, b_r, \mathbf{p}_{n-1-k}, a_1, \dots, a_s, \mathbf{p}_k, b_1}). \tag{19}$$

For the next  $b_2 - 1$  values of  $i$  ( $b_1 \leq i \leq b_1 + b_2 - 2$ ),

$$w_{n,2+a+k\ell_0+i}^B = (b_2 - j, \overline{b_3, \dots, b_r, \mathbf{p}_{n-1-k}, a_1, \dots, a_s, \mathbf{p}_k, b_1, b_2}) \tag{20}$$

with  $1 \leq j \leq b_2 - 1$ . This process goes on until the last  $b_r - 1$  values of  $i$ , where

$$w_{n,2+a+k\ell_0+i}^B = (b_r - j, \overline{\mathbf{p}_{n-1-k}, a_1, \dots, a_s, \mathbf{p}_{k+1}})$$

with  $1 \leq j \leq b_r - 1$ . For  $k = n - N - 1$ , we have the same pattern as before except for the last block of values of  $i$ , where we only have  $b_r - 3$  of them.

Now, for each  $0 \leq k \leq n - N - 1$ , for  $1 \leq i \leq b_1 - 1$ , (19) and

$$w_{0,2+i}^B = (b_1 - i, \overline{b_2, \dots, b_r, b_1})$$

coincide in the first  $rn - rk$  partial quotients. For the next  $b_2 - 1$  values of  $i$ , (20) and

$$w_{0,2+i}^B = (b_2 - j, \overline{b_3, \dots, b_r, b_1, b_2}) \quad (1 \leq j \leq b_2 - 1)$$

coincide in the first  $rn - rk - 1$  partial quotients, similarly for the next  $b_3 - 1$  values of  $i$ ,  $w_{n,2+a+k\ell_0+i}^B$  and  $w_{0,2+i}^B$  coincide in the first  $rn - rk - 2$  partial quotients, etc. Therefore, using Lemma 2.1, for  $z \in \mathcal{C}$ , we have

$$\begin{aligned} |S_2(n, N, z)| &\leq \frac{40}{3} \left( 3\lambda^{rn} + \sum_{i=1}^r (b_i - 1) \sum_{k=0}^{n-N-1} \lambda^{r(n-k)-i} \right) \\ &\leq \frac{40}{3} \left( 3\lambda^{rn} + 3 \left( \sum_{i=1}^r \lambda^{-i} \right) \left( \sum_{k=N+1}^n \lambda^{rk} \right) \right) \\ &\leq \frac{40}{3} \left( 3\lambda^{rn} + 3 \left( \sum_{i=1}^r \lambda^{r-i} \right) \left( \sum_{k=N}^{n-1} \lambda^{rk} \right) \right) \\ &\leq \frac{40}{3} \left( 3\lambda^{rn} + 3 \left( \sum_{i=0}^{r-1} \lambda^i \right) \left( \sum_{k=N}^{n-1} \lambda^{rk} \right) \right) \\ &\leq 40\lambda^{rN} \left( 1 + \frac{1}{(1-\lambda)(1-\lambda^r)} \right) \\ &\leq 120\lambda^{rN}. \end{aligned} \tag{21}$$

In the second inequality we used that  $b_i \leq 4$ , whereas the last inequality follows from the numerical value  $1/1 - \lambda = 1.618\dots$

**Bound for  $|S_3(n, N, z)|$ .** In a similar way we bound  $|S_3(n, N, z)|$ . We have that

$$\begin{aligned} |S_3(n, N, z)| &\leq \sum_{k=N}^{n-2} \sum_{i=1}^{\ell_0} \frac{|\tilde{w}_{n,2+a+k\ell_0+i}^B - \tilde{w}_{0,2+i}^B|}{|z - \tilde{w}_{n,2+a+k\ell_0+i}^B| |z - \tilde{w}_{0,2+i}^B|} + \sum_{i=0}^2 \frac{|\tilde{w}_{n,a+N\ell_0+i}^B - \tilde{w}_{0,i}^B|}{|z - \tilde{w}_{n,a+N\ell_0+i}^B| |z - \tilde{w}_{0,i}^B|} \\ &\quad + \sum_{i=1}^{\ell_0-3} \frac{|\tilde{w}_{n,2+a+(n-1)\ell_0+i}^B - \tilde{w}_{0,2+i}^B|}{|z - \tilde{w}_{n,2+a+(n-1)\ell_0+i}^B| |z - \tilde{w}_{0,2+i}^B|}. \end{aligned}$$

For  $i = 0, 1, 2$ , using Remark 3.3, we have that

$$-\tilde{w}_{n,a+N\ell_0+i}^B = (1 + i, \overline{b_{r-1}, \dots, b_1, q_{N-1}, a_s, \dots, a_1, q_{n-N}, b_r})$$

and

$$-\tilde{w}_{0,i}^B = (1 + i, \overline{b_{r-1}, \dots, b_1, b_r})$$

coincide in the first  $rN$  partial quotients. For each  $N \leq k \leq n - 2$ , we have: For  $1 \leq i \leq b_1 - 1$ ,

$$-\tilde{w}_{n,2+a+k\ell_0+i}^B = (i, \overline{\mathbf{q}_k, a_s \dots, a_1, \mathbf{q}_{n-k}}). \tag{22}$$

For the next  $b_2 - 1$  values of  $i$  ( $b_1 \leq i \leq b_1 + b_2 - 2$ ),

$$-\tilde{w}_{n,2+a+k\ell_0+i}^B = (j, \overline{b_1, \mathbf{q}_k, a_s \dots, a_1, \mathbf{q}_{n-1-k}, b_r, \dots, b_3, b_2}) \tag{23}$$

with  $1 \leq j \leq b_2 - 1$ . This process goes on until the last  $b_r - 1$  values of  $i$ , where

$$-\tilde{w}_{n,2+a+k\ell_0+i}^B = (j, \overline{b_{r-1}, \dots, b_1, \mathbf{q}_k, a_s, \dots, a_1, \mathbf{q}_{n-1-k}, b_r})$$

with  $1 \leq j \leq b_r - 1$ .

For  $k = n - 1$ , we have the same pattern as before except for the last block of values of  $i$ , where we only have  $b_r - 3$  of them. Now, for each  $N \leq k \leq n - 1$ , for the first  $b_1 - 1$  values of  $i$ , (22) coincide with

$$-\tilde{w}_{0,2+i}^B = (i, \overline{\mathbf{q}})$$

in the first  $rk + 1$  partial quotients. For the next  $b_2 - 1$  values of  $i$ , (23) coincide with

$$-\tilde{w}_{0,2+i}^B = (j, \overline{b_1, b_r, \dots, b_2}) \quad (1 \leq j \leq b_2 - 1)$$

in the first  $rk + 2$  partial quotients, for the next  $b_3 - 1$   $i$ -values,  $-\tilde{w}_{n,2+a+k\ell_0+i}^B$  and  $-\tilde{w}_{0,2+i}^B$  coincide in the first  $rk + 3$  partial quotients, etc. Once again using Lemma 2.1, and the fact that  $b_i \leq 4$  together with the numerical value of  $\lambda$ , we have, for  $z \in \mathcal{C}$ ,

$$\begin{aligned} |S_3(n, N, z)| &\leq \frac{40}{3} \left( 3\lambda^{rN-1} + \sum_{i=1}^r (b_i - 1) \sum_{k=N}^{n-1} \lambda^{rk+i-1} \right) \\ &\leq \frac{40}{3} \left( 3\lambda^{rN-1} + 3 \left( \sum_{i=1}^r \lambda^{i-1} \right) \left( \sum_{k=N}^{n-1} \lambda^{rk} \right) \right) \\ &\leq 40\lambda^{rN-1} \left( 1 + \frac{\lambda}{(1-\lambda)(1-\lambda^r)} \right) \\ &\leq 80\lambda^{rN-1}. \end{aligned} \tag{24}$$

**Bound for  $|S_1(n, N, z) - S_1(N, N, z)|$ .** We have

$$\begin{aligned} &|S_1(n, N, z) - S_1(N, N, z)| \\ &\leq \sum_{i=0}^{a-1} \frac{|w_{n,i}^B - w_{N,i}^B|}{|z - w_{n,i}^B| |z - w_{N,i}^B|} + \sum_{i=a}^{\ell_N-1} \frac{|w_{n,i+(n-N)\ell_0}^B - w_{N,i}^B|}{|z - w_{n,i+(n-N)\ell_0}^B| |z - w_{N,i}^B|} + \sum_{i=0}^{a+N\ell_0-1} \frac{|\tilde{w}_{n,i}^B - \tilde{w}_{N,i}^B|}{|z - \tilde{w}_{n,i}^B| |z - \tilde{w}_{N,i}^B|}. \end{aligned}$$

Again the denominators are bounded below by  $\frac{3}{4}$  for  $z \in \mathcal{C}$  and we use Lemma 2.1 to bound the numerators.

For the first term in the first sum, using

$$w_{n,0}^B = (3, \overline{a_1, \dots, a_s, \mathbf{p}_n}) \tag{25}$$

and

$$w_{N,0}^B = (3, \overline{a_1, \dots, a_s, \mathbf{p}_N}), \tag{26}$$

one can see that the successive terms  $w_{n,i}^B$  and  $w_{N,i}^B$  (up to  $i = a - 1$ ) coincide at least in the first  $rN$  partial quotients. This is also true for the second sum, where we have

$$w_{n,a+(n-N)\ell_0}^B = (3, \overline{\mathbf{p}_{n-N}, a_1, \dots, a_s, \mathbf{p}_N}) \quad \text{and} \quad w_{N,a}^B = (3, \overline{\mathbf{p}_N, a_1, \dots, a_s}),$$

as well as for the third and fourth sums, where we can use [Remark 3.3](#) and the continued fractions of [\(25\)](#) and [\(26\)](#), and

$$w_{n,a+n\ell_0}^B = (3, \overline{a_1, \dots, a_s, \mathbf{p}_n}), \quad \text{and} \quad w_{n,a+N\ell_0}^B = (3, \overline{a_1, \dots, a_s, \mathbf{p}_N}),$$

respectively. Hence, using [\(11\)](#), we have

$$|S_1(n, N, z) - S_1(N, N, z)| \leq \frac{80}{3} \ell_N \lambda^{rN-1} \stackrel{(11)}{\leq} 80r(N+1)\lambda^{rN-1}. \tag{27}$$

Finally, since  $\lambda < \frac{2}{3}$ , the bounds [\(21\)](#), [\(24\)](#) and [\(27\)](#) give

$$c(n, z) \leq 80(2+r(N+1))\lambda^{rN-1}. \quad \square$$

In particular, [Theorem 4.2](#) applied to the function  $f = 1$  gives:

**Corollary 4.3.** *Let  $B$  be any left branch  $\neq L$  of  $\mathcal{T}$  and  $N \geq 1$ . For all  $n \geq N$ , there exists  $K = K_{B,N} \in \mathbb{R}$  such that*

$$|\log \varepsilon_n^B - n \log \varepsilon_0^B - K| \leq \delta_1(r, N) \tag{28}$$

with  $\delta_1(r, N)$  and  $r$  as in [\(14\)](#).

The next corollary proves [Theorem 1.1](#) from the introduction.

**Corollary 4.4.** *Let  $f$  be a modular function. For any left branch  $B \neq L$  of  $\mathcal{T}$ ,*

$$\lim_{n \rightarrow \infty} f^{\text{nor}}(w_n^B) = f^{\text{nor}}(w_0^B).$$

*Proof.* It follows from [Theorem 4.2](#) and [Corollary 4.3](#) that  $|f(w_n^B) - nf(w_0^B)|$  and  $|\log \varepsilon_n^B - n \log \varepsilon_0^B|$  are bounded above and below by absolute constants (not depending on  $n$ ). Then

$$0 = \lim_{n \rightarrow \infty} \frac{|f(w_n^B) - nf(w_0^B)|}{\log \varepsilon_n^B} = \lim_{n \rightarrow \infty} \left| \frac{f(w_n^B)}{\log \varepsilon_n^B} - \frac{f(w_0^B)}{\log \varepsilon_0^B} \right|. \quad \square$$

### 5. Interlacing property

In this section we prove [Theorem 1.2](#). As in the proof of the convergence property we restrict again to a left but not the leftmost branch in what follows. The argument applies in the same way to any branch, with the bound  $\delta_2(n, r)$  slightly modified. It will still be of the form  $O(rn\lambda^{rn})$ . Hence [Theorem 1.2](#) applies in fact to any branch of the Markov tree and it is a consequence of the next theorem whose proof is similar to the proof of [Theorem 4.2](#).



**Theorem 5.1.** *Let  $f$  be a modular function. For every left branch  $B \neq L$  of the Markov tree  $\mathcal{T}$  and for all  $n \geq 1$ ,*

$$|f(w_{n+1}^B) - f(w_n^B) - f(w_0^B)| \leq \delta_2(n, r) \max_{z \in \mathcal{C}} |f(z)| \tag{29}$$

where

$$\delta_2(n, r) = \frac{80\pi}{3} (n + 2)r \left( \frac{2}{1 + \sqrt{5}} \right)^{2(rn-1)} \tag{30}$$

and  $r + 1$  is the number of partial quotients in the period of  $w_0^B$ .

*Proof.* Once again, applying Lemma 4.1 and (10) gives

$$f(w_{n+1}^B) - f(w_n^B) = f(w_0^B) + \int_{\rho}^{\rho^2} f(z)R_1(n, z) dz + \int_{\rho}^{\rho^2} f(z)R_2(n, z) dz, \tag{31}$$

where

$$R_1(n, z) = \sum_{i=0}^{a-1} \left( \frac{1}{z - w_{n+1,i}^B} - \frac{1}{z - w_{n,i}^B} \right) + \sum_{i=a}^{\ell_n-1} \left( \frac{1}{z - w_{n+1,\ell_0+i}^B} - \frac{1}{z - w_{n,i}^B} \right) - \sum_{i=0}^{\ell_n-1} \left( \frac{1}{z - \tilde{w}_{n+1,i}^B} - \frac{1}{z - \tilde{w}_{n,i}^B} \right),$$

$$R_2(n, z) = \sum_{i=0}^{\ell_0-1} \left( \frac{1}{z - w_{n+1,a+i}^B} - \frac{1}{z - w_{0,i}^B} - \frac{1}{z - \tilde{w}_{n+1,\ell_n+i}^B} + \frac{1}{z - \tilde{w}_{0,i}^B} \right).$$

Next we give upper bounds for the norms of the two sums above when  $z \in \mathcal{C}$ . We set again  $\lambda = (2/(1 + \sqrt{5}))^2$ .

**Bound for  $|R_1(n, z)|$ .** For  $z \in \mathcal{C}$ , we have

$$|R_1(n, z)| \leq \sum_{i=0}^{a-1} \frac{|w_{n+1,i}^B - w_{n,i}^B|}{|z - w_{n+1,i}^B||z - w_{n,i}^B|} + \sum_{i=a}^{\ell_n-1} \frac{|w_{n+1,i+\ell_0}^B - w_{n,i}^B|}{|z - w_{n+1,i+\ell_0}^B||z - w_{n,i}^B|} + \sum_{i=0}^{a+n\ell_0-1} \frac{|\tilde{w}_{n+1,i}^B - \tilde{w}_{n,i}^B|}{|z - \tilde{w}_{n+1,i}^B||z - \tilde{w}_{n,i}^B|} + \sum_{i=a+n\ell_0}^{\ell_n-1} \frac{|\tilde{w}_{n+1,i}^B - \tilde{w}_{n,i}^B|}{|z - \tilde{w}_{n+1,i}^B||z - \tilde{w}_{n,i}^B|}.$$

As before we use the bound of  $\frac{3}{4}$  for the denominators and Lemma 2.1 for the numerators. In the first sum using

$$w_{n+1,0}^B = (3, \overline{a_1, \dots, a_s, \mathbf{p}_{n+1}}) \tag{32}$$

and

$$w_{n,0}^B = (3, \overline{a_1, \dots, a_s, \mathbf{p}_n}), \tag{33}$$

one can see that the successive terms  $w_{n+1,i}^B$  and  $w_{n,i}^B$  (up to  $i = a - 1$ ) coincide at least in the first  $rn$  partial quotients. The same is true for the second sum, where

$$w_{n+1,a+\ell_0}^B = (3, \overline{\mathbf{p}_n, a_1, \dots, a_s, \mathbf{p}}) \quad \text{and} \quad w_{n,a}^B = (3, \overline{\mathbf{p}_n, a_1, \dots, a_s}).$$

For the third and fourth sums, we use once again Remark 3.3 together with (32) and (33), and

$$w_{n+1, a+n\ell_0}^B = (3, \overline{a_1, \dots, a_s, \mathbf{p}_{n+1}}) \quad \text{and} \quad w_{n, a+n\ell_0}^B = (3, \overline{a_1, \dots, a_s, \mathbf{p}_n})$$

respectively.

Hence

$$|R_1(n, z)| \leq \frac{80}{3} \ell_n \lambda^{rn-1} \stackrel{(11)}{\leq} 80r(n+1)\lambda^{rn-1}.$$

**Bound for  $|R_2(n, z)|$ .** In a similar way we bound this second sum when  $z \in \mathcal{C}$ :

$$|R_2(n, z)| \leq \sum_{i=0}^{\ell_0-1} \frac{|w_{n+1, a+i}^B - w_{0, i}^B|}{|z - w_{n+1, a+i}^B| |z - w_{0, i}^B|} + \frac{|\tilde{w}_{n+1, \ell_n+i}^B - \tilde{w}_{0, i}^B|}{|z - \tilde{w}_{n+1, \ell_n+i}^B| |z - \tilde{w}_{0, i}^B|}.$$

Again using

$$w_{n+1, a}^B = (3, \overline{\mathbf{p}_{n+1}})$$

and

$$w_{0, 0}^B = (3, \overline{\mathbf{p}}), \tag{34}$$

one can see that all the successive terms  $w_{n+1, a+i}^B$  and  $w_{0, i}^B$  in the sum coincide at least in the first  $rn$  partial quotients. For the conjugate terms, one can see from (34) and

$$w_{n+1, \ell_n}^B = (3, \overline{\mathbf{p}, a_1, \dots, a_s, \mathbf{p}_n})$$

that  $-\tilde{w}_{n+1, \ell_n+i}^B$  and  $-\tilde{w}_{0, i}^B$  coincide as well in the first  $rn$  partial quotients. Hence

$$|R_2(n, z)| \leq \frac{80}{3} \ell_0 \lambda^{rn-1} \stackrel{(11)}{\leq} 80r\lambda^{rn-1}.$$

Therefore,

$$|f(w_{n+1}) - f(w_n) - f(w_0)| \leq \int_{\rho}^{\rho^2} |f(z)| (|R_1(n, z)| + |R_2(n, z)|) |dz| \leq \delta_2(n, r) \max_{z \in \mathcal{C}} |f(z)|$$

with

$$\delta_2(n, r) = \frac{80\pi}{3} r(n+2)\lambda^{rn-1}. \quad \square$$

Theorem 5.1 applied to the function  $f = 1$  gives:

**Corollary 5.2.** For every left branch  $B \neq L$  of  $\mathcal{T}$  and for all  $n \geq 1$ ,

$$|\log \varepsilon_{n+1}^B - \log \varepsilon_n^B - \log \varepsilon_0^B| \leq \delta_2(n, r)$$

with  $\delta_2(n, r)$  and  $r$  as in (30).

We finish this section by giving the proof of Theorem 1.2 in the case that the branch  $B$  is any left branch  $\neq L$ . The proof of the general case goes along the same lines.

**Theorem 5.3.** Let  $f$  be a modular function,  $B$  be any left branch  $\neq L$  of the Markov tree  $\mathcal{T}$ . There exists a constant  $N_{f, B}$  such that, for all  $n \geq N_{f, B}$ , the real and imaginary parts of  $f^{\text{nor}}(w_{n+1}^B)$  lie between the real and imaginary parts respectively of  $f^{\text{nor}}(w_0^B)$  and  $f^{\text{nor}}(w_n^B)$ .

*Proof.* By definition, the inequality

$$\operatorname{Re}(f^{\operatorname{nor}}(w_n^B)) < \operatorname{Re}(f^{\operatorname{nor}}(w_0^B))$$

holds if and only if

$$\operatorname{Re}(f(w_n^B)) \log \varepsilon_0^B < \operatorname{Re}(f(w_0^B)) \log \varepsilon_n^B. \tag{35}$$

Let  $N, M$  be positive constants. For all  $n \geq \max(N, M)$ , we can write

$$\operatorname{Re}(f(w_n^B)) = n \operatorname{Re}(f(w_0^B)) + K_{f,B,N} + \varepsilon_1(n, N), \tag{36}$$

$$\log \varepsilon_n^B = n \log \varepsilon_0^B + K_{1,B,M} + \varepsilon_2(n, M), \tag{37}$$

where  $K_{f,B,N}, K_{1,B,M}$  are the real parts of the constants in [Theorem 4.2](#) and [Corollary 4.3](#) respectively,  $|\varepsilon_1(n, N)| \leq \delta_1(N) \max_{z \in \mathcal{C}} |f(z)|$  and  $|\varepsilon_2(n, M)| \leq \delta_1(M)$ . Therefore (35) is equivalent to

$$\operatorname{Re}(f^{\operatorname{nor}}(w_0^B)) > \frac{K_{f,B,N}}{K_{1,B,M}} + \frac{\varepsilon_1(n, N) - \varepsilon_2(n, M) \operatorname{Re}(f^{\operatorname{nor}}(w_0^B))}{K_{1,B,M}}. \tag{38}$$

There exists a constant  $C_1(f, B)$  depending on  $f$  and  $B$  such that, for  $\max(N, M) \geq C_1(f, B)$ , (38) is equivalent to either

$$\operatorname{Re}(f^{\operatorname{nor}}(w_0^B)) > \frac{K_{f,B,N}}{K_{1,B,M}} \tag{39}$$

or (39) with the strict inequality replaced by  $\geq$ , according to whether the error term in (38) is positive or negative. If we can choose  $N, M \geq C_1(f, B)$  satisfying  $\operatorname{Re}(f^{\operatorname{nor}}(w_0^B)) \neq K_{f,B,N}/K_{1,B,M}$ , then (38) is equivalent to (39) for those  $N, M$ . If we cannot choose such  $N, M$ , then  $K_{f,B,N}, K_{1,B,M}$  would be constants that do not depend on  $N, M$ , and in particular  $\varepsilon_1(n, N) = \varepsilon_2(n, M) = 0$ . Hence, also in this case (38) is equivalent to (39) for all  $N, M \geq C_1(f, B)$ .

In a similar way, the inequality

$$\operatorname{Re}(f^{\operatorname{nor}}(w_n^B)) > \operatorname{Re}(f^{\operatorname{nor}}(w_0^B))$$

is equivalent to

$$\operatorname{Re}(f^{\operatorname{nor}}(w_0^B)) < \frac{K_{f,B,N}}{K_{1,B,M}} \tag{40}$$

for  $N, M$  chosen as before. Since (39) and (40) do not depend on  $n$ , we have either

$$\operatorname{Re}(f^{\operatorname{nor}}(w_n^B)) < \operatorname{Re}(f^{\operatorname{nor}}(w_0^B))$$

simultaneously for all  $n \geq \max(N, M)$  with  $N, M$  chosen as before, or

$$\operatorname{Re}(f^{\operatorname{nor}}(w_n^B)) > \operatorname{Re}(f^{\operatorname{nor}}(w_0^B)).$$

Similarly, the inequality

$$\operatorname{Re}(f^{\operatorname{nor}}(w_{n+1}^B)) < \operatorname{Re}(f^{\operatorname{nor}}(w_n^B))$$

holds if and only if

$$\operatorname{Re}(f(w_{n+1}^B)) \log \varepsilon_n^B < \operatorname{Re}(f(w_n^B)) \log \varepsilon_{n+1}^B. \tag{41}$$

[Theorem 5.1](#) and [Corollary 5.2](#) respectively imply that

$$\operatorname{Re}(f(w_{n+1}^B)) = \operatorname{Re}(f(w_n^B)) + \operatorname{Re}(f(w_0^B)) + \mu(n)$$

with  $|\mu(n)| \leq \delta_2(n) \max_{z \in \mathcal{C}} |f(z)|$  and

$$\log \varepsilon_{n+1}^B = \log \varepsilon_n^B + \log \varepsilon_0^B + \nu(n)$$

with  $|\nu(n)| \leq \delta_2(n)$ . Hence (41) is equivalent to

$$(\operatorname{Re}(f(w_0^B)) + \mu(n)) \log \varepsilon_n^B < \operatorname{Re}(f(w_n^B))(\log \varepsilon_0^B + \nu(n)). \tag{42}$$

Now, there exists a constant  $C_2(f, B) \geq C_1(f, B)$  such that, for  $n \geq C_2(f, B)$ , we have that

$$\operatorname{Re}(f^{\operatorname{nor}}(w_n^B)) \neq \operatorname{Re}(f^{\operatorname{nor}}(w_0^B))$$

and that (42) is equivalent to

$$\operatorname{Re}(f(w_0^B)) \log \varepsilon_n^B < \operatorname{Re}(f(w_n^B)) \log \varepsilon_0^B. \tag{43}$$

Using (36) and (37) again, we obtain that (43) is equivalent to

$$\operatorname{Re}(f^{\operatorname{nor}}(w_0^B)) < \frac{K_{f,B,N}}{K_{1,B,M}}, \tag{44}$$

where  $N, M$  are chosen as before.

Therefore, we finally have that either

$$\operatorname{Re}(f^{\operatorname{nor}}(w_0^B)) < \operatorname{Re}(f^{\operatorname{nor}}(w_{n+1}^B)) < \operatorname{Re}(f^{\operatorname{nor}}(w_n^B))$$

for all  $n \geq \max(C_2(f, B), N, M)$  or

$$\operatorname{Re}(f^{\operatorname{nor}}(w_n^B)) < \operatorname{Re}(f^{\operatorname{nor}}(w_{n+1}^B)) < \operatorname{Re}(f^{\operatorname{nor}}(w_0^B)).$$

The same argument applies to the imaginary parts of  $f^{\operatorname{nor}}(w_{n+1}^B)$ ,  $f^{\operatorname{nor}}(w_n^B)$  and  $f^{\operatorname{nor}}(w_0^B)$ . □

### Acknowledgements

We thank M. Kaneko and the referee for numerous and very helpful comments that improved our exposition.

### References

- [Aigner 2013] M. Aigner, *Markov's theorem and 100 years of the uniqueness conjecture*, Springer, 2013. [MR](#) [Zbl](#)
- [Bombieri 2007] E. Bombieri, "Continued fractions and the Markoff tree", *Expo. Math.* **25**:3 (2007), 187–213. [MR](#) [Zbl](#)
- [Bourgain et al. 2016a] J. Bourgain, A. Gamburd, and P. Sarnak, "Markoff surfaces and strong approximation, I", preprint, 2016. [arXiv](#)
- [Bourgain et al. 2016b] J. Bourgain, A. Gamburd, and P. Sarnak, "Markoff triples and strong approximation", *C. R. Math. Acad. Sci. Paris* **354**:2 (2016), 131–135. [MR](#)

- [Duke et al. 2011] W. Duke, O. Imamoglu, and A. Tóth, “Cycle integrals of the  $j$ -function and mock modular forms”, *Ann. of Math. (2)* **173**:2 (2011), 947–981. [MR](#) [Zbl](#)
- [Ghosh and Sarnak 2017] A. Ghosh and P. Sarnak, “Integral points on Markoff type cubic surfaces”, preprint, 2017. [arXiv](#)
- [Hall 1947] M. Hall, Jr., “On the sum and product of continued fractions”, *Ann. of Math. (2)* **48** (1947), 966–993. [MR](#) [Zbl](#)
- [Kaneko 2009] M. Kaneko, “Observations on the “values” of the elliptic modular function  $j(\tau)$  at real quadratics”, *Kyushu J. Math.* **63**:2 (2009), 353–364. [MR](#) [Zbl](#)
- [Malyshev 1977] A. V. Malyshev, “Markov and Lagrange spectra (a survey of the literature)”, pp. 5–38 in *Studies in number theory, part 4*, vol. 67, “Nauka” Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 1977. In Russian; translated in *J. Soviet Math.* **16**:1 (1981), 767–788. [MR](#)
- [Markov 1879] A. Markoff, “Sur les formes quadratiques binaires indéfinies”, *Math. Ann.* **15** (1879), 381–406. [Zbl](#)
- [Markov 1880] A. Markoff, “Sur les formes quadratiques binaires indéfinies”, *Math. Ann.* **17**:3 (1880), 379–399. [MR](#) [Zbl](#)
- [Roth 1955] K. F. Roth, “Rational approximations to algebraic numbers”, *Mathematika* **2** (1955), 1–20; corrigendum, 168. [MR](#) [Zbl](#)
- [Zagier 1981] D. B. Zagier, *Zetafunktionen und quadratische Körper: eine Einführung in die höhere Zahlentheorie*, Springer, 1981. [MR](#) [Zbl](#)

Communicated by Philippe Michel

Received 2018-03-27    Revised 2018-12-18    Accepted 2019-02-08

[paloma.bengoechea@math.ethz.ch](mailto:paloma.bengoechea@math.ethz.ch)

*Department of Mathematics, ETH Zurich, Switzerland*

[ozlem.imamoglu@math.ethz.ch](mailto:ozlem.imamoglu@math.ethz.ch)

*Department of Mathematics, ETH Zurich, Switzerland*

# Algebra & Number Theory

[msp.org/ant](http://msp.org/ant)

## EDITORS

### MANAGING EDITOR

Bjorn Poonen  
Massachusetts Institute of Technology  
Cambridge, USA

### EDITORIAL BOARD CHAIR

David Eisenbud  
University of California  
Berkeley, USA

### BOARD OF EDITORS

Richard E. Borcherds	University of California, Berkeley, USA	Raman Parimala	Emory University, USA
Antoine Chambert-Loir	Université Paris-Diderot, France	Jonathan Pila	University of Oxford, UK
J-L. Colliot-Thélène	CNRS, Université Paris-Sud, France	Anand Pillay	University of Notre Dame, USA
Brian D. Conrad	Stanford University, USA	Michael Rapoport	Universität Bonn, Germany
Samit Dasgupta	University of California, Santa Cruz, USA	Victor Reiner	University of Minnesota, USA
Hélène Esnault	Freie Universität Berlin, Germany	Peter Sarnak	Princeton University, USA
Gavril Farkas	Humboldt Universität zu Berlin, Germany	Joseph H. Silverman	Brown University, USA
Hubert Flenner	Ruhr-Universität, Germany	Michael Singer	North Carolina State University, USA
Sergey Fomin	University of Michigan, USA	Christopher Skinner	Princeton University, USA
Edward Frenkel	University of California, Berkeley, USA	Vasudevan Srinivas	Tata Inst. of Fund. Research, India
Andrew Granville	Université de Montréal, Canada	J. Toby Stafford	University of Michigan, USA
Joseph Gubeladze	San Francisco State University, USA	Pham Huu Tiep	University of Arizona, USA
Roger Heath-Brown	Oxford University, UK	Ravi Vakil	Stanford University, USA
Craig Huneke	University of Virginia, USA	Michel van den Bergh	Hasselt University, Belgium
Kiran S. Kedlaya	Univ. of California, San Diego, USA	Akshay Venkatesh	Institute for Advanced Study, USA
János Kollár	Princeton University, USA	Marie-France Vignéras	Université Paris VII, France
Philippe Michel	École Polytechnique Fédérale de Lausanne	Kei-Ichi Watanabe	Nihon University, Japan
Susan Montgomery	University of Southern California, USA	Melanie Matchett Wood	University of Wisconsin, Madison, USA
Shigefumi Mori	RIMS, Kyoto University, Japan	Shou-Wu Zhang	Princeton University, USA
Martin Olsson	University of California, Berkeley, USA		

## PRODUCTION

[production@msp.org](mailto:production@msp.org)

Silvio Levy, Scientific Editor

---

See inside back cover or [msp.org/ant](http://msp.org/ant) for submission instructions.

---

The subscription price for 2019 is US \$385/year for the electronic version, and \$590/year (+\$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.


---

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

---

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2019 Mathematical Sciences Publishers

# Algebra & Number Theory

Volume 13    No. 4    2019

---

<a href="#">Artin's criteria for algebraicity revisited</a>	749
JACK HALL and DAVID RYDH	
<a href="#">Differential characters of Drinfeld modules and de Rham cohomology</a>	797
JAMES BORGER and ARNAB SAHA	
<a href="#">Quadratic twists of abelian varieties and disparity in Selmer ranks</a>	839
ADAM MORGAN	
<a href="#">Iwasawa theory for Rankin-Selberg products of <math>p</math>-nonordinary eigenforms</a>	901
KÂZİM BÜYÜKBODUK, ANTONIO LEI, DAVID LOEFFLER and GUHAN VENKAT	
<a href="#">Cycle integrals of modular functions, Markov geodesics and a conjecture of Kaneko</a>	943
PALOMA BENGOCHEA and ÖZLEM IMAMOĞLU	
<a href="#">A finiteness theorem for specializations of dynatomic polynomials</a>	963
DAVID KRUMM	