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We provide the first steps towards a new relative trace formula proof of the celebrated formula of Waldspurger relating the square of a toric period integral on PGL_2 to the central value of an L-function.

Introduction

In this paper we present the first steps towards a new and independent proof of Waldspurger's remarkable formula for toric periods of automorphic forms on PGL_2. This formula, which involves the period integral of a cusp form $\varphi \in \sigma$ on the group PGL_2 — or an inner form — over a number field F along a nonsplit torus $\mathrm{T} \subset \mathrm{PGL}_2$, can be loosely written as

$$|\mathscr{P}_\pi(\phi)|^2 = (*)L\big(\tfrac{1}{2}, \pi\big).$$

Here, $\mathbb{A}$ is the ring of adeles over F, $\pi = \chi \otimes \sigma$ is an automorphic cuspidal representation of $\mathrm{T}(\mathbb{A}) \times \mathrm{PGL}_2(\mathbb{A})$, $L\big(\tfrac{1}{2}, \pi\big)$ is the central critical value of the Rankin–Selberg L-function attached to $\pi = \chi \otimes \sigma$, and the period integral $\mathscr{P}_\pi(\varphi)$ is defined as

$$\mathscr{P}_\pi(\phi) := \int_{\mathrm{T}(F)\backslash \mathrm{T}(\mathbb{A})} \chi(t)\varphi(t)\,dt = \int_{\mathrm{T}(F)\backslash \mathrm{T}(\mathbb{A})} \phi(t)\,dt,$$

where we write $\phi = \chi \otimes \varphi \in \pi$. For the purpose of this introduction, we will be vague about the nature of the quantity $(*)$, except to say that it can be made explicit and contains both local and global information about the automorphic cuspidal representation π and the vector ϕ.

This result has a storied past. The formula in question was first shown by Waldspurger [1985] using methods of the theta correspondence. Since then, there have been many other proofs given. Most notably, Jacquet [1986; 1987] provided an ambitious alternate approach to this theorem; see also [Jacquet and Chen 2001]. The method relies on writing down and comparing two relative trace formulas, foregoing the constructive methods of the theta correspondence and instead deducing the result from a spectral identity. The technique is appealingly general. Indeed, by adapting these constructions, Jacquet and Rallis later offered a possible avenue of attack on a high-rank relative of Waldspurger's formula: the Gan–Gross–Prasad (GGP) conjecture for unitary groups. This has been wildly successful. See [Jacquet and Rallis 2011; Yun 2011; Zhang 2014a; 2014b; Chaudouard and Zydor 2016; Xue 2017a; Beuzart-Plessis 2016].

Given the large number of distinct proofs of Waldspurger's formula, it is reasonable to wonder whether there is any benefit to seeing another. We believe there is, for the following reason. The main appeal of our approach, over others, is that it is tuned to the interpretation of Waldspurger's formula as the $n = 2$ case of a more general period relation: the Gross–Prasad conjecture for $\mathrm{SO}_n \times \mathrm{SO}_{n+1}$. Unlike its cousin, the GGP conjecture for unitary groups, orthogonal Gross–Prasad has so far resisted analysis through trace formula techniques. And, although discussion in this paper remains restricted to Waldspurger's (known) case of $\mathrm{T} \times \mathrm{PGL}_2 = \mathrm{SO}_2 \times \mathrm{SO}_3$, we hope that our work — a new relative trace formula proof of Waldspurger's formula — can be expanded to offer an approach to the high-rank problem. We hope to directly address this generalization in future work.

For now, though, let us come back to earth and describe the trace formulas and comparison we have in mind. We change notation slightly, and think of the group $\mathrm{T} \times \mathrm{PGL}_2$ (or an inner form $\mathrm{T} \times \mathrm{PB}^\times$) via exceptional isomorphism instead as $\mathrm{SO}_2 \times \mathrm{SO}_3$. That is, we fix a nondegenerate quadratic space V of dimension 3 together with an orthogonal decomposition $V = W + Fe$, with e is a nonisotropic vector, and consider the group $\mathrm{SO}_W \times \mathrm{SO}_V$. Our first relative trace formula distribution J takes as input a function f on $(\mathrm{SO}_W \times \mathrm{SO}_V)(\mathbb{A})$. It is designed to encode the period integral $\mathscr{P}$ as part of its spectral expansion; it also has a geometric expansion in terms of orbital integrals, and together we write (loosely)

$$\sum_\pi J_\pi(f) = J(f) = \sum_\gamma J_\gamma(f).$$

In the above, $J_\pi(f)$ is the global spherical character, a distribution that encodes the period integral $\mathscr{P}_\pi$ (the left-hand side of Waldspurger's formula), while $J_\gamma(f)$ is the orbital integral, which is defined for (regular semisimple) γ by

$$J_\gamma(f) = \int_{O_W(\mathbb{A})} F(h^{-1}\gamma h)\, dh,$$

where

$$F(x) = \int_{\mathrm{SO}_W(\mathbb{A})} f((h,h)(1,x))\, dh.$$

We also construct a second, more interesting trace formula I. This distribution lives on $\mathrm{GL}_2 \times \mathrm{GL}_2$, and takes as input a test function f' on $(\mathrm{GL}_2 \times \mathrm{GL}_2)(\mathbb{A})$, a Schwartz function $\Phi \in S(\mathbb{A})$, and a "doubly even" Schwartz function $\Psi \in S^{++}(\mathbb{A}_E)$, where $E = F[\sqrt{d_W}]$ is the discriminant algebra of W (equivalently, the splitting field for the torus T). It is designed to admit a spectral expansion into a sum of distributions I_Π, which themselves encode the L-function side of Waldspurger's formula (roughly, I_Π computes $L\left(\frac{1}{2}, \Pi\right)$), but only for those Π on $\mathrm{GL}_2 \times \mathrm{GL}_2$ which are functorial lifts from $\mathrm{SO}_2 \times \mathrm{SO}_3$). I also possesses a geometric expansion, and in total we write

$$\sum_\Pi I_\Pi(f' \otimes \Phi \otimes \Psi) = I(f' \otimes \Phi \otimes \Psi) = \sum_{\gamma'} s(\gamma') I_{\gamma'}(f' \otimes \Phi \otimes \Psi).$$

There is something quite odd happening here. At first glance, the distribution $I(f' \otimes \Phi \otimes \Psi)$ seems to have no chance of admitting a geometric expansion in the usual sense of the relative trace formula, where orbits of a group action appear through a double coset computation. However $I(f' \otimes \Phi \otimes \Psi)$ amazingly still unfolds into a sum of factorizable integrals, with sum indexed over some formal objects γ'. These γ' appear through a condition on Fourier coefficients. To that end, although there is some temptation to call these objects "global orbits" — as a sum over these γ' makes up the "geometric side" of our trace formula — we instead elect to call them *global tones* and their associated factorizable integrals $I_{\gamma'}(f' \otimes \Phi \otimes \Psi)$ *tonal integrals*. We identify a Zariski open set of tones, called the regular semisimple locus. For such a regular semisimple tone $\gamma = (\alpha; \beta, \zeta)$ the tonal integral is defined as

$$I_{\gamma'}(f' \otimes \Phi \otimes \Psi) = \int_{\mathrm{GL}_2(\mathbb{A})} \int_{\mathrm{GL}_2^{(2)}(\mathbb{A})} f'(g^{-1}a(\alpha)h, g^{-1}) \mathscr{W}\left(a\left(\frac{\beta}{\alpha}\right)g\right) \boldsymbol{R}^\psi \overline{\Psi}(\zeta) \, dh \, dg.$$

We postpone a precise discussion of the terms appearing above to the body of this paper.

If we vary the quadratic spaces (W, V) over all pairs $V = W \oplus Fe$ with the same fixed discriminants d_W and d_V, then there is a bijection between regular semisimple orbits γ, which appear in the geometric expansion of J, and regular semisimple tones γ', which appear in the geometric expansion of I. This bijection has a remarkable property.

Theorem 0.1. *Given a Schwartz function $f = (f_{(W,V)}) \in \bigoplus_{(W,V)} S((\mathrm{SO}_W \times \mathrm{SO}_V)(\mathbb{A}))$, there exists a finite sum $\sum_i f_i' \otimes \Phi_i \otimes \Psi_i$ so that, for all $\gamma \leftrightarrow \gamma'$ corresponding global regular semisimple orbits and tones,*

$$J_\gamma(f) = \sum_i I_{\gamma'}'(f_i \otimes \Phi_i \otimes \Psi_i).$$

Conversely, given $\sum_i f_i' \otimes \Phi_i \otimes \Psi_i$, there exists $f = (f_{(W,V)})$ satisfying the above equality for all $\gamma \leftrightarrow \gamma'$.

Although we have stated this result in global language, this is really a theorem about the relationship between the local versions of J_γ and $I_{\gamma'}$. See Theorem 3.4, which resolves the problem of "smooth transfer", as well as Theorems 3.7 and 3.15, which provide the "fundamental lemma" in the cases when E/F is unramified or split respectively.

It is worth noting that these results, which are the main content of the paper, still fall quite short of reproving Waldspurger's theorem. The main obstacle is the regularization of the trace formula I. In the sequel [Krishna $\geq$ 2019] we complete this global analysis and, as a consequence, rederive Waldspurger's result.

Outline of the paper.   The body of this paper runs in reverse order to our discussion above. Namely, we first describe the local results of smooth transfer and the fundamental lemma in Part I, and only then describe the trace formulas motivating these results in Part II. Our discussion in Part II is entirely formal, and should more or less be viewed only as motivation for the work we do in Part I. The analysis of [Krishna $\geq$ 2019] entirely supersedes this part; still, as the regularization of the trace formula is quite foreboding, we believe it worthwhile to include this simple, albeit naive, discussion.

In Section 1 we first parametrize the orbits of $SO_3 /^{\text{conj}} SO_2$ and describe the space of orbital integrals Ω, viewed as functions on this quotient, completely explicitly. Then, in Section 2 we define Ω', the space of tonal integrals, and identify it as a space of functions on the set of tones. As a consequence, we derive our result on smooth transfer, which we state carefully in Section 3 as Theorem 3.4. The remainder of Section 3 in concerned with the fundamental lemma. This is stated and proved separately in these cases of E/F unramified or E/F split as Theorems 3.7 and 3.15.

In Section 4, we construct the relative trace formula J in its naive form. Our presentation is a little different from what one may consider the obvious approach, as we make some effort to incorporate the outer automorphism of O_2 into the trace formula. This is, in some sense, unnecessary, but appears to be the correct philosophical setup in light of the necessary corrections to the Ichino–Ikeda conjecture (the analogue of Waldspurger's formula in the high-rank Gross–Prasad case); see [Xue 2017b]. Finally, in Section 5 we set up the (naive form of the) trace formula I and decompose it into a sum of tonal integrals. As ingredients, we briefly review some of the theory of Rankin–Selberg convolution on $GL_2 \times GL_2$ as well as the theory behind the integral representation of the symmetric square L-function for representations of GL_2.

It may be helpful for the reader to violate convention and read Part II of the paper before tackling the computations in Part I. The two parts are largely independent, and can be read in any order.

Notation.   Throughout this paper, we adopt the following notation:

• F will denote either a number field or a local field of characteristic 0. In the local setting, $\mathcal{O}$ will denote the ring of integers of F. In the global setting, $\mathbb{A}$ will denote its ring of adeles.

• E/F will be a quadratic extension. When F is a number field, we ask that E be a field. When F is local, we allow E to be split. When E/F is local, we write $\mathcal{O}_E$ for the integers of E, while when E/F is global, we write $\mathbb{A}_E$ for the adeles of E.

• Capitalized roman letters (e.g., G, GL_n, $GSpin_V$) denote the appropriate algebraic groups over F. When there is no real risk of confusion, and particularly when we are working in case of F a local field, we will follow the typical abuse of notation and write G for both the algebraic group and its F-points G(F).

- We define, given our quadratic extension E/F, the algebraic group U(1) to be

$$\mathrm{U}(1) = \{g \in \mathrm{Res}_{E/F} \mathrm{GL}_1 : \bar{g}g = 1\},$$

where $\bar{\cdot}$ is the nontrivial element of $\mathrm{Gal}(E/F)$ and Res denotes Weil restriction.

- When F is a local field, we will frequently talk about the space of Schwartz functions on the F points $\mathrm{G}(F)$ of an algebraic group G. We denote this space by $S(\mathrm{G}(F))$. It is defined as follows:

  - If F is non-Archimedean, $S(\mathrm{G}(F)) = C_c^\infty(\mathrm{G}(F))$ is the space of locally constant and compactly supported $\mathbb{C}$-valued functions.

  - If F is $\mathbb{R}$ or $\mathbb{C}$, then we set

  $$S(\mathrm{G}(F)) := \{f \in C^\infty(\mathrm{G}(F)) : |Df| \text{ is bounded for all polynomial differential operators } D\}.$$

- We use the notation

$$\mathbb{Z} = \mathbb{Z}_{\mathrm{GL}_2} = \left\{ \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} : z \in F^\times \right\}, \quad \mathrm{N} = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in F \right\},$$

$$\mathrm{A} = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} : a_i \in F^\times \right\}, \quad \mathrm{B} = \mathrm{N}\,\mathrm{A}, \quad \mathrm{P} = \left\{ \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} : a \in F^\times, x \in F \right\}$$

for subgroups of GL_2 and similarly use the shorthand

$$z(z) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}, \quad a(t) = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}, \quad d(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

for elements of these subgroups. We will also occasionally refer to A as $\mathrm{T}_{\mathrm{GL}_2}$ throughout.

- In the global setting of F a number field, we will use $[\cdot]$ to denote the automorphic quotient. So, for instance, we write

$$[\mathrm{GL}_2] := \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}).$$

In various situations, we include, as is typical, the letter P to indicate that we additionally mod out by the adelic points of the center. Thus, for instance,

$$[\mathrm{PGL}_2] := \mathbb{Z}(\mathbb{A})\,\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}) = \mathrm{PGL}_2(F)\backslash \mathrm{PGL}_2(\mathbb{A}).$$

We hope this does not cause confusion, as it conflicts slightly with our notation P for the mirabolic of GL_2.

- Finally, we use

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

to denote fixed representatives of the nontrivial Weyl-group element for GL_2 and for SL_2 respectively.

Part I. Local theory

In this part, we study the local integrals appearing on the "geometric" sides of the trace formulas of Part II. These results are used crucially to establish the trace formula comparison and the resulting spectral identities that form the backbone of our proof.

1. Orbits and orbital integrals for $SO_2 \times SO_3$

1A. *Preliminaries on quadratic spaces.* Let us review some easy facts on quadratic spaces of dimensions 2 and 3. For this subsection we let F denote an arbitrary field of characteristic not 2. Throughout this subsection, and indeed this paper, we will use W to denote a nondegenerate quadratic space of dimension 2 over F and V to denote one of dimension 3. If there is an embedding $W \hookrightarrow V$ then we say that the pair (W, V) is *relevant*. Given any nondegenerate quadratic space $U = (U, Q_U)$ of dimension n, we call

$$d_U := (-1)^{\frac{n(n-1)}{2}} \det(Q_U),$$

or its class in $F^\times / (F^\times)^2$, the *discriminant* of U. Given such a U, we say a space U' is a *pure inner form* of U if

$$\dim U = \dim U' = n$$

and

$$d_U = d_{U'}$$

in $F^\times / (F^\times)^2$.

Remark 1.1. Note that our language of relevant pairs is consistent with the terminology laid out in Section 2 of [Gan et al. 2012]; in their language, given (W, V) with decomposition $V = W \oplus W^\perp$, a pair (W', V') is a relevant pure inner form of (W, V) if $V' = W \oplus W'^\perp$, $d(W) = d(W')$ and $d(V) = d(V')$, *and* $W^\perp \cong W'^\perp$. This last condition is immediate when W is codimension 1 in V.

We begin with an elementary observation.

Lemma 1.2. *Quadratic spaces of dimensions* 2 *and* 3 *admit the following explicit descriptions*:

(1) *Every nondegenerate quadratic space W of dimension* 2 *is of the form*

$$(W, Q_W) \cong (E, \varepsilon \, \mathrm{N}),$$

where E/F is a quadratic étale algebra over F defined by $E = F[t]/(t^2 - d_W)$ and $\varepsilon \in F^\times / \mathrm{N}\, E^\times$.

(2) *Every nondegenerate quadratic space V of dimension* 3 *is of the form*

$$(V, Q_V) \cong (B^{\mathrm{Tr}=0}, -d \, \mathrm{N}),$$

where B/F is a (possibly split) quaternion algebra over F, $B^{\mathrm{Tr}=0} \subset B$ is the subspace of elements with reduced trace 0, N *denotes the reduced norm* $\mathrm{N} : B \to F$, *and d is an element of $F^\times / (F^\times)^2$ with $d = d_V$.*

Given a pair $(d_W^\circ, d_V^\circ) \in (F^\times/(F^\times)^2)^2$, there is a unique relevant pair of quadratic spaces (W_0, V_0) of dimensions 2 and 3 and with discriminants $d_{W_0} = d_W^\circ$, $d_{V_0} = d_V^\circ$ so that V_0 is *split*, i.e., has an isotropic vector. We call this the *quasisplit relevant pair* associated to (d_W°, d_V°). Explicitly, we may take a model of these spaces to be

$$(W_0, Q_{W_0}) = \left(E, -\frac{d_V^\circ}{d_W^\circ} \mathrm{N}\right)$$

and

$$V_0 = W_0 \oplus Fe,$$

where $E = F[t]/(t^2 - d_W^\circ)$ and $Q_{V_0}(e) = d_V^\circ/d_W^\circ$.[1] It is easy to see that (W_0, V_0) as defined above are quasisplit; if we denote by ι the above identification of W_0 with E and write $s = \iota(1)$, then $u := s + e$ is visibly an isotropic vector of V_0.

Lemma 1.2 can be thought of as a restatement of the exceptional isomorphism $\mathrm{SO}_{V_0} \cong \mathrm{PGL}_2$. For later computation, it is useful to put coordinates on this group. To do so we exploit this isomorphism, first restating it as an observation about Clifford algebras.

1A1. *Clifford algebras and* GSpin. Recall that for any quadratic space $U = (U, Q_U)$, the Clifford algebra is defined by

$$C(U) = TU/I_{Q_U},$$

where TU denotes the tensor algebra of U and I_{Q_U} is the ideal generated by all elements of the form $u \otimes u - Q(u)$. This is a $\mathbb{Z}/2$ graded algebra; i.e.,

$$C(U) = C_0(U) \oplus C_1(U),$$

with even part denoted by $C_0(U)$. The Clifford algebra defines a $\mathbb{G}_m$ extension GSpin_U of SO_U when U is nondegenerate by

$$\mathrm{GSpin}_U = \{g \in C_0(U)^\times : gug^{-1} \in U \text{ for all } u \in U\},$$

which comes equipped with a map $\rho : \mathrm{GSpin}_U \to \mathrm{SO}_U$ defined by $\rho(g).u = gug^{-1}$. We write

$$1 \to \mathbb{G}_m \to \mathrm{GSpin}_U \xrightarrow{\rho} \mathrm{SO}_U \to 1.$$

By Hilbert's Theorem 90, this map is surjective on F-points; i.e., $\rho : \mathrm{GSpin}_U(F) \twoheadrightarrow \mathrm{SO}_U(F)$.

When we restrict this construction to the simple case above, where $U = W, V$ is either 2- or 3-dimensional, then Lemma 1.2 implies that the Clifford algebra construction unravels into a familiar picture.

Lemma 1.3. *Let W and V be nondegenerate quadratic spaces of dimensions 2 and 3 respectively, satisfying $V = W \oplus Fe$ for a nonisotropic vector e. Then:*

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[1]From here on, we fix two discriminants d_W° and d_V° and consider only relevant quadratic spaces $W \hookrightarrow V$ associated to this pair; we also slightly simplify notation, writing d_W for d_W° and d_V for d_V°.

(1) $C_0(W) \cong E$, where $E = F[t]/(t^2 - d_W)$. *Moreover,*

$$\mathrm{GSpin}_W = C_0(W)^\times \cong \mathrm{Res}_{E/F}\, \mathbb{G}_m$$

   is a torus.

(2) $C_0(V) \cong B$, where B/F *is the quaternion algebra of* [Lemma 1.2](). *Moreover,*

$$\mathrm{GSpin}_V = C_0(V)^\times \cong B^\times$$

   is an inner form of GL_2.

We can put coordinates on $\mathrm{GSpin}_V = C_0(V)^\times$; every element $x \in C_0(V)$ can be uniquely written as

$$x = z + we,$$

where $z \in C_0(W)$ and $w \in W$. When $(W, V) = (W_0, V_0)$ is the unique quasisplit inner form corresponding to a choice of two discriminants d_W, d_V, then we can pin down an explicit isomorphism $\mathrm{Mat}_{2\times 2} \xrightarrow{\sim} C_0(V_0)$; for example

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{a+d}{2} + \frac{b+cd_W}{2d_W}i + \left(\frac{d_W}{d_V}\frac{a-d}{2}s + \frac{b-cd_W}{2d_V}s' \right)e, \tag{1-1}$$

where here $s = \iota(1)$, $s' = \iota(\sqrt{d_W})$, and $i = (d_W/d_V)ss'$ lies in $C_0(W_0)$ and satisfies $i^2 = d_W$. In the above, ι denotes the identification of $(E, -(d_V/d_W)\,\mathrm{N}) \xrightarrow{\iota} W_0$. That the map $\mathrm{Mat}_{2\times 2} \to C_0(V_0)$ is an isomorphism is easy to see; we leave a proof of this assertion to the reader, with the reminder that, as defined above, s, s', and e all anticommute with one another.

1B. *Orbits.* With this language available, let us now compute the quotient space(s) that appear in our analysis.

Let $V = W \oplus Fe$ be as above, and denote by $\mathrm{G} = \mathrm{SO}_W \times \mathrm{SO}_V$ the product of special orthogonal groups. Inside of G is the subgroup $\mathrm{H} = \Delta\,\mathrm{SO}_W$, the diagonally embedded torus. The analysis of [Part II]() forces us to determine the double coset space

$$\mathrm{H}(F)\backslash \mathrm{G}(F)/\mathrm{H}(F) = \Delta\,\mathrm{SO}_W(F)\backslash \mathrm{SO}_W(F) \times \mathrm{SO}_V(F)/\Delta\,\mathrm{SO}_W(F).$$

To do so, we simply note that, by the map $(x, y) \mapsto x^{-1}y$, we can identify

$$\Delta\,\mathrm{SO}_W(F)\backslash \mathrm{SO}_W(F) \times \mathrm{SO}_V(F)/\Delta\,\mathrm{SO}_W(F) \xrightarrow{\sim} \mathrm{SO}_V(F)/^{\mathrm{conj}}\,\mathrm{SO}_W(F),$$

where on the right we are considering the quotient of $\mathrm{SO}_V(F)$ up to conjugation by $\mathrm{SO}_W(F)$.

We can explicitly parametrize orbits of $\mathrm{SO}_W(F)$ acting on $\mathrm{SO}_V(F)$ by conjugation. Let Σ denote a fixed choice of representatives for $F^\times/(F^\times)^2$.

Proposition 1.4. *The set of orbits satisfies*

$$\mathrm{SO}_V(F)/^{\mathrm{conj}}\,\mathrm{SO}_W(F) \cong \left\{ (\alpha; z, b) : \alpha \in \Sigma,\ z \in E/\{\pm 1\},\ b \in \frac{d_V}{d_W}Q(W),\ \mathrm{N}z + b = \alpha \right\}.$$

Proof. We first pull back the picture from $\mathrm{SO}_V(F)$ to $\mathrm{GSpin}_V(F)$, since it is on the latter group that we can easily put coordinates. The projection map $\rho : \mathrm{GSpin}_V(F) \to \mathrm{SO}_V(F)$, which consists of scaling by the central $\mathbb{G}_m$, gives

$$\rho : F^\times \backslash \mathrm{GSpin}_V(F)/^{\mathrm{conj}} \mathrm{SO}_W(F) \xrightarrow{\sim} \mathrm{SO}_V(F)/^{\mathrm{conj}} \mathrm{SO}_W(F).$$

We can simplify the quotient on the left-hand side somewhat. Define

$$\mathrm{GSpin}_V^\Sigma(F) := \{x \in \mathrm{GSpin}_V(F) : \mathrm{N}\, x \in \Sigma\},$$

where N is the spinor norm $\mathrm{N} : \mathrm{GSpin}_V \to \mathbb{G}_m$. This is not an algebraic variety or a group, despite notation. However, the restriction of ρ to this set gives us

$$\rho : \{\pm 1\} \backslash \mathrm{GSpin}_V^\Sigma(F) \xrightarrow{\sim} \mathrm{SO}_V(F)$$

and so we write

$$\rho : \{\pm 1\} \backslash \mathrm{GSpin}_V^\Sigma(F)/^{\mathrm{conj}} \mathrm{SO}_W(F) \xrightarrow{\sim} \mathrm{SO}_V(F)/^{\mathrm{conj}} \mathrm{SO}_W(F).$$

Let us compute the orbits of $\mathrm{SO}_W(F)$ acting on $\mathrm{GSpin}_V^\Sigma(F)$ by conjugation. Given an element $x = z + ze$ satisfying $\mathrm{N}\, x = \mathrm{N}\, z + (d_V/d_W)Q(w) = \alpha$ for some $\alpha \in \Sigma$, conjugating by an element $z' \in C_0(W)^\times$ gives

$$s'(z + we)s'^{-1} = z + (\rho(z').w)e$$

and orbits of $\mathrm{SO}_W(F)$ on $\mathrm{GSpin}_V^\Sigma(F)$ are parametrized by the set of triples

$$\left\{ (\alpha; z, b) : \alpha \in \Sigma,\, z \in E,\, b \in \frac{d_V}{d_W} Q(W),\, \mathrm{N}\, z + b = \alpha \right\}.$$

Finally, quotienting out by multiplication by $\{\pm 1\}$ concludes the proof. $\qquad\square$

The discussion above can be summarized. We identify, given $x \in \mathrm{SO}_V(F)$, its associated $(\alpha; z, b)$ as follows: first lift x arbitrarily to an element of $\mathrm{GSpin}_V^\Sigma(F)$, which we also denote by x. This is determined up to ± 1, and so in particular, determines an $\alpha \in \Sigma$. If we write this lift $x = z + we$, then the invariants by conjugation are z, up to sign, and $b := (d_V/d_W)Q(w)$.

We can now identify a "good" locus of orbits. Recall the following definition from invariant theory.

Definition 1.5. An element x of SO_W acting on SO_V by conjugation is *regular semisimple* (*r.s.s.*) if both:

(1) (x is regular) The stabilizer of x is 0-dimensional.

(2) (x is semisimple) The orbit of x is (Zariski) closed.

We similarly call the orbit of x *regular semisimple* if x is.

Let us identify the locus of regular semisimple orbits explicitly.

Lemma 1.6. *An orbit of $x \in \mathrm{SO}_V(F)$ is regular semisimple if and only if its corresponding b is nonzero.*

Proof. If $b = 0$, then $x = z + we$ satisfies $Q(w) = 0$. Thus, either $w = 0$, in which case $x = z$ has stabilizer all of SO_W (x is not regular) or W is a split 2-dimensional quadratic space and w is a nonzero isotropic vector. In this case, the element z lies in the closure of the orbit of $x = z + we$ but is not in the orbit itself, i.e., x is not semisimple.

Conversely, it is apparent that $x = z + we$ is nonregular if and only if $w = 0$. If $x = z + we$ is nonsemisimple, then there exists $x' = z' + w'e$ which lies in the closure of the orbit of x but not in the orbit itself. It is clear that $z' = z$. Now the orbit $SO_W.w$ of w in W is closed in W if W is nonsplit. Thus the only remaining case to consider is when W is split: then, $SO_W.w$ is nonclosed only if $Q(w) = 0$. □

Remark 1.7. When $x \in SO_V$ has $b \neq 0$, the stabilizer of x in SO_W is usually trivial. However, it can be finite. For instance, if $x = we$ in $\mathrm{GSpin}_V^{\Sigma}(F)$ with $Q(w) \neq 0$, then this is a regular semisimple element and has stabilizer $\{\pm 1\} \subset SO_W$ of order 2.

We write

$$(SO_V(F)/^{\mathrm{conj}} SO_W(F))^{\mathrm{r.s.s.}}$$

for the collection of regular semisimple orbits of $SO_W(F)$ on $SO_V(F)$.

The condition that b lie in $(d_V/d_W)Q(W)$ in Proposition 1.4 may appear artificial. This can be remedied by considering all *relevant pure inner forms* simultaneously.

Proposition 1.8. *Taking the disjoint union over all pairs (W, V) of relevant quadratic spaces with fixed discriminants, we find*

$$\coprod_{(W,V)} (SO_V(F)/^{\mathrm{conj}} SO_W(F))^{\mathrm{r.s.s.}} \cong \{(\alpha; z, b) : \alpha \in \Sigma, z \in E/\{\pm 1\}, b \in F^\times, \mathrm{N}\, z + b = \alpha\}.$$

Proof. One needs only to check the all relevant pure inner forms, i.e., relevant pairs (W, V) having the same fixed discriminants are parametrized by elements of $F^\times/\mathrm{N}\, E^\times$. This follows immediately from Lemma 1.2.   □

We should also say a brief word about a related quotient set, namely

$$SO_V(F)/^{\mathrm{conj}} O_W(F).$$

It should be clear that an element of $SO_V(F)$ is O_W-*regular semisimple* if and only if it is SO_W-*regular semisimple* (what we were calling, and will continue to call, simply *regular semisimple*). It follows quickly that, as above, we have

$$\coprod_{(W,V)} (SO_V(F)/^{\mathrm{conj}} O_W(F))^{\mathrm{r.s.s.}} \cong \{(\alpha; z, b) : \alpha \in \Sigma,\ z \in E/\{\pm 1\} \times \{\mathrm{id}, \bar{\cdot}\},\ b \in F^\times,\ \mathrm{N}\, z + b = \alpha\},$$

where $\bar{\cdot}$ denotes the nontrivial automorphism of E/F. Note that $\mathrm{N}\, z = z\bar{z}$ for $z \in E/\{\pm 1\} \times \{\mathrm{id}, \bar{\cdot}\}$ is well-defined, i.e., is independent of choice of representative of z in E.

1B1. *GIT quotients.* For the sake of conceptual clarity, it is worth packaging the set-theoretic orbit analysis above in the more algebraic language of invariant theory. We will not use this interpretation seriously, but we believe it is worth keeping in mind. We omit all proofs, as they do not differ significantly from those given above.

Recall that given an affine variety X together with a right action of a group G, the GIT quotient, denoted by $X /\!\!/ G$, is simply

$$X /\!\!/ G = \operatorname{Spec}(F[X]^G),$$

where $F[X]$ denotes the coordinate ring of X.

We can thus rewrite our description of the quotient sets above in invariant-theoretic language as follows:

$$\mathrm{SO}_V /\!\!/^{\mathrm{conj}} \mathrm{SO}_W \cong \{\pm 1\} \sqcup\!\!\!\sqcup \operatorname{Spin}_V /\!\!/^{\mathrm{conj}} \mathrm{SO}_W \cong \{\pm 1\} \sqcup\!\!\!\sqcup \operatorname{Res}_{E/F} \mathbb{G}_a \cong \operatorname{Spec}(F[A, B, C]/(C^2 - AB)),$$

where the last isomorphism sends $a + bi = a + b\sqrt{d}_W \mapsto (A, B, C) = (a^2, b^2, ab)$.

Similarly, we have

$$\mathrm{SO}_V /\!\!/^{\mathrm{conj}} \mathrm{O}_W \cong \{\pm 1\} \sqcup\!\!\!\sqcup \operatorname{Spin}_V /\!\!/^{\mathrm{conj}} \mathrm{O}_W \cong \{\pm 1\} \times \{\mathrm{id}, \bar{\cdot}\} \sqcup\!\!\!\sqcup \operatorname{Res}_{E/F} \mathbb{G}_a \cong \operatorname{Spec}(F[X, Y]),$$

where the last isomorphism sends $x + yi = x + y\sqrt{d_W} \mapsto (X, Y) = (x^2, y^2)$.

The regular semisimple loci are also easy to identify: namely we have the open subvarieties

$$(\mathrm{SO}_V /\!\!/^{\mathrm{conj}} \mathrm{SO}_W)^{\mathrm{r.s.s.}} = \{\pm 1\} \sqcup\!\!\!\sqcup (\operatorname{Res}_{E/F} \mathbb{G}_a - \mathrm{U}(1)) = \operatorname{Spec}\left(F\left[A, B, C, \frac{1}{A - d_W B - 1}\right] / (C^2 - AB)\right)$$

and

$$(\mathrm{SO}_V /\!\!/^{\mathrm{conj}} \mathrm{O}_W)^{\mathrm{r.s.s.}} = \operatorname{Spec}\left(F\left[X, Y, \frac{1}{X - d_W Y - 1}\right]\right).$$

Furthermore, we have identifications

$$\coprod_{(W,V)} (\mathrm{SO}_V(F)/^{\mathrm{conj}} \mathrm{SO}_W(F))^{\mathrm{r.s.s.}} \xrightarrow{\sim} (\mathrm{SO}_V /\!\!/^{\mathrm{conj}} \mathrm{SO}_W)^{\mathrm{r.s.s.}}(F),$$

$$(\alpha; x + iy, b) \mapsto \left(\frac{x^2}{\alpha}, \frac{y^2}{\alpha}, \frac{xy}{\alpha}\right),$$

and

$$\coprod_{(W,V)} (\mathrm{SO}_V(F)/^{\mathrm{conj}} \mathrm{O}_W(F))^{\mathrm{r.s.s.}} \xrightarrow{\sim} (\mathrm{SO}_V /\!\!/^{\mathrm{conj}} \mathrm{O}_W)^{\mathrm{r.s.s.}}(F),$$

$$(\alpha; x + iy, b) \mapsto \left(\frac{x^2}{\alpha}, \frac{y^2}{\alpha}\right).$$

1C. *Local orbital integrals and the space Ω.* Let us now define one of our main objects of study: the space of functions living on these orbits. So let F now denote a local field of characteristic 0.

Given a function $f = f_{(W,V)} \in C_c^\infty(\mathrm{G}(F))$, consider the averaged function

$$F(x) = F_{(W,V)}(x) := \int_{\mathrm{H}(F)} f_{(W,V)}(h(1, x)) \, dh,$$

which lives in $C_c^\infty(\mathrm{SO}_V(F))$. Note that the map

$$C_c^\infty(G(F)) \to C_c^\infty(\mathrm{SO}_V(F)),$$
$$f \mapsto F,$$

is clearly surjective.

We can now define the orbital integral as follows.

Definition 1.9. The *orbital integral* of f is the function $J(f, \gamma)$ on $(\mathrm{SO}_V(F)/^{\mathrm{conj}} \mathrm{O}_W(F))^{\mathrm{r.s.s.}}$ given by

$$J(\gamma, f) = \int_{\mathrm{O}_W(F)} F(h^{-1}\gamma h)\, dh,$$

which we will occasionally also write as $J_\gamma(f)$.

We record for completeness the following very easy fact.

Lemma 1.10. *If γ in $\mathrm{SO}_V(F)$ is regular semisimple, then*

$$J(\gamma, f) = \int_{\mathrm{O}_W(F)} F(h^{-1}\gamma h)\, dh$$

converges absolutely.

Proof.
$$\int_{\mathrm{O}_W(F)} F(h^{-1}\gamma h)\, dh = |\operatorname{Stab}_\gamma(F)| \int_{\operatorname{Stab}_\gamma(F)\backslash \mathrm{O}_W(F)} F(h^{-1}\gamma h)\, dh$$

The latter integral runs over a closed subset of $\mathrm{SO}_V(F)$, and F is compactly supported.   □

By abuse of notation, we will also sometimes denote by f a tuple of functions, $f = (f_{(W,V)})$ indexed by relevant quadratic spaces (W, V) with fixed discriminants. Observe that, if E/F is nonsplit, then there are only two such relevant pairs of quadratic spaces — one quasisplit, and one nonquasisplit — while if E/F is a split extension, there is only one such pair (W, V) — the split pair. In this setting, we will also denote by $J(\gamma, f)$ or $J_\gamma(f)$ the function on $\coprod_{(W,V)} (\mathrm{SO}_V(F)/^{\mathrm{conj}} \mathrm{O}_W(F))^{\mathrm{r.s.s.}}$ given by

$$J(\gamma, f) = \sum_{(W,V)} J(\gamma, f_{(W,V)}).$$

We hope that context will be enough to distinguish between these competing notations.

Definition 1.11. The space Ω of orbital integrals is the space of functions on

$$\coprod_{(W,V)} (\mathrm{SO}_V(F)/^{\mathrm{conj}} \mathrm{O}_W(F))^{\mathrm{r.s.s.}} = (\mathrm{SO}_V /\!\!/^{\mathrm{conj}} \mathrm{O}_W)^{\mathrm{r.s.s.}}(F) = F^2 - \{(X, Y) : X - d_V Y = 1\}$$

which are of the form $J(f, \gamma)$ for some $f = (f_{(W,V)})$ smooth and compactly supported.

The main goal of this subsection is to give an explicit description of Ω. This is the content of the following result.

Theorem 1.12. *Suppose that E/F is a field extension (i.e., is nonsplit). Then*

$$\Omega = \left\{ \phi_1(\alpha; z, b) + \phi_2(\alpha; z, b)\omega\left(\frac{b}{\alpha}\right) \right\},$$

where each function $\phi_i(\alpha; \cdot, \cdot)$ is smooth and compactly supported on the smooth variety $\mathrm{N}\, z + b = \alpha$, and satisfies $\phi_i(\alpha; -z, b) = \phi_i(\alpha; -z, b)$ and $\phi_i(\alpha; \bar{z}, b) = \phi_i(\alpha; z, b)$, and where $\omega = \omega_{E/F}$ is the nontrivial quadratic character of $F^\times$ associated to the extension E/F.

  If E/F is split, then

$$\Omega = \left\{ \phi_1(\alpha; z, b) + \phi_2(\alpha; z, b) \log\left|\frac{b}{\alpha}\right| \right\},$$

where ϕ_i are as above.

Remark 1.13. One can also restate Theorem 1.12 in terms of the coordinates on the GIT quotient given by

$$(\mathrm{SO}_V /\!\!/^{\mathrm{conj}} \mathrm{O}_W)^{\mathrm{r.s.s.}}(F) = F^2 - \{(X, Y) : X - d_V Y = 1,$$

which may be psychologically helpful; however, it is the theorem in the form written above which is most useful to us.

Remark 1.14. We have been considering the space Ω, which consists of all functions of the form $J(\cdot, f)$ for $(f_{(W,V)}) \in \bigoplus_{(W,V)} C_c^\infty((\mathrm{SO}_W \times \mathrm{SO}_V)(F))$. We could instead consider Ω_S, which consists of all functions of the form $J(\cdot, f)$, where $f \in \bigoplus_{(W,V)} S((\mathrm{SO}_W \times \mathrm{SO}_V)(F))$ is a Schwartz function. This differs from Ω only when F is Archimedean; in that case, the above theorem is still true if we replace Ω with Ω_S and force all ϕ_i to be Schwartz.

  To show Theorem 1.12, it is helpful to first analyze a "toy model" of our space of orbital integrals. It captures all of the essentials of our situation.

1C1. *The toy model* (*nonsplit case*). Consider first the nonsplit case. Let E/F be a quadratic field extension of characteristic 0 local fields. Fix $\varepsilon \in F^\times$ a representative of the nontrivial class in $F^\times / \mathrm{N}\, E^\times$ and consider the pair of maps

$$\pi_0 : E \to F, \qquad \pi_1 : E \to F,$$
$$z \mapsto \mathrm{N}\, z, \qquad\quad z \mapsto \varepsilon\, \mathrm{N}\, z.$$

We define the *space of toy orbital integrals* Ω^{toy} to be the push-forward, under $\pi = \pi_0 \coprod \pi_1$, of the space of compactly supported smooth functions on $E \coprod E$. More concretely, we define for $f = (f_0, f_1) \in C_c^\infty(E \coprod E)$ the *toy orbital integral* by

$$J^{\mathrm{toy}}(f, \gamma) = J^{\mathrm{toy}}(f_0, \gamma) + J^{\mathrm{toy}}(f_1, \gamma),$$

where

$$J^{\mathrm{toy}}(f_0, \gamma) = \begin{cases} \int_{\mathrm{U}(1)(F)} f_0(uz)\, du & \text{if } \gamma = \mathrm{N}\, z, \\ 0 & \text{if } \gamma \notin \mathrm{N}\, E^\times \end{cases}$$

and

$$J^{\text{toy}}(f_1, \gamma) = \begin{cases} \int_{\text{U}(1)(F)} f_1(uz)\, du & \text{if } \gamma = \varepsilon\, \text{N}\, z, \\ 0 & \text{if } \gamma \notin \varepsilon\, \text{N}\, E^\times \end{cases}$$

and where all integrals are taken with respect to the Haar measure on $\text{U}(1)(F)$, normalized so that $\text{vol}(\text{U}(1)(F)) = 1$. The space Ω^{toy} is the space of all functions on $F^\times$ of the form $J^{\text{toy}}(f, \cdot)$. When $F = \mathbb{R}$ is Archimedean, an analysis of the space Ω^{toy} has been carried out in [Casselman and Tian 2013]. Let us explain the non-Archimedean case, which is easier. The results we state will also apply to the case described in [loc. cit.].

Since $\text{U}(1)(F)$ is compact, the space Ω^{toy} is extremely easy to understand. For instance, it is clear that, given any $f_0 \in C_c^\infty(E)$, $J^{\text{toy}}(f_0, \gamma)$ is a smooth function on $\text{N}\, E^\times$, that

$$\lim_{\substack{\gamma \to 0 \\ \gamma \in \text{N}\, E^\times}} J^{\text{toy}}(f_0, \gamma) = f_0(0),$$

and that $J^{\text{toy}}(f_0, \gamma) = 0$ for all $|\gamma|$ sufficiently large. Moreover, any function J_0 which satisfies these three conditions, i.e.,

(1) has support in $F^\times$ contained in $\text{N}\, E^\times$, and is smooth on $\text{N}\, E^\times$,

(2) has a limit

$$L_0 = \lim_{\substack{\gamma \to 0 \\ \gamma \in \text{N}\, E^\times}} J_0(\gamma),$$

(3) is zero for $|\gamma|$ sufficiently large,

can occur as an orbital integral $J^{\text{toy}}(f_0, \gamma)$: simply take $f_0(z) = J_0(\text{N}\, z)$ if $z \neq 0$, and set $f_0(0) = L_0$.

The analogous statements are also true for $J^{\text{toy}}(f_1, \gamma)$. All together this shows that

$$\Omega^{\text{toy}} = \{A_1(\gamma) + A_2(\gamma)\omega(\gamma) : A_i \in C_c^\infty(F)\},$$

or, more accurately, is the space of functions on $F^\times$ obtained by restricting such functions $A_1 + A_2\omega$ to $F^\times$.

Deducing this description on Ω^{toy} from our analysis of the integrals $J^{\text{toy}}(f_i, \gamma)$ is very straightforward. Given J_0 and J_1 satisfying the three conditions above (for J_1, we must replace the set $\text{N}\, E^\times$ with $\varepsilon\, \text{N}\, E^\times$ everywhere), extend J_i to smooth compactly supported functions $\tilde{J}_i$ on F satisfying

$$\tilde{J}_0|_{\text{N}\, E^\times} = J_0 \quad \text{and} \quad \tilde{J}_1|_{\varepsilon\, \text{N}\, E^\times} = J_1.$$

Then set $A_1 = \frac{1}{2}(\tilde{J}_0 + \tilde{J}_1)$ and $A_2 = \frac{1}{2}(\tilde{J}_0 - \tilde{J}_1)$.

Summarizing, we have shown:

Proposition 1.15. *When E/F is a (nonsplit) quadratic extension of local fields, the space Ω^{toy} is exactly the space of functions on $F^\times$ which occur as restrictions of functions of the form*

$$A_1(\gamma) + A_2(\gamma)\omega(\gamma),$$

where $A_i \in C_c^\infty(F)$ and $\omega = \omega_{E/F}$ is the quadratic character associated to E/F.

1C2. *The toy model* (*split case*). Let us now consider the split case $E = F \times F$. We again have a map

$$\pi : E = F \times F \to F,$$

$$(x, y) \mapsto \mathrm{N}(x, y) = xy,$$

which is surjective. Given $f \in C_c^\infty(F \times F)$ we define the *toy orbital integral* as a function of $\gamma \in F^\times$ by

$$J^{\mathrm{toy}}(f, \gamma) = \int_{F^\times} f(\gamma t, t^{-1}) \, d^\times t,$$

where

$$d^\times t = \frac{1}{1 - q^{-1}} \frac{dt}{|t|}$$

is the Haar measure on $F^\times$, normalized so that $\mathrm{vol}(\mathcal{O}^\times) = 1$ (we have set dt to give $\mathcal{O}$ volume 1). Furthermore, we call Ω^{toy} the space of all functions on $F^\times$ which are of the form $J^{\mathrm{toy}}(f, \cdot)$ for some $f \in C_c^\infty(F \times F)$.

The following result is slightly harder to prove than the corresponding statement in the nonsplit case.

Proposition 1.16. *Let $E = F \times F$. The space Ω^{toy} is exactly the space of functions on $F^\times$ which occur as restrictions of functions of the form*

$$A_1(\gamma) + A_2(\gamma) \log |\gamma|,$$

where $A_i \in C_c^\infty(F)$.

For a complete and detailed proof of this fact in the Archimedean case $F = \mathbb{R}, \mathbb{C}$, see [Casselman and Tian 2013]. This fact is also discussed, albeit briefly, in both the Archimedean and non-Archimedean settings, in [Sakellaridis 2013].

The argument presented in [Casselman and Tian 2013] more or less runs through verbatim in the non-Archimedean case as well; for completeness we record an elementary version of it (in the non-Archimedean setting) here.

The main point is to determine the asymptotics of functions $J(x) \in \Omega^{\mathrm{toy}}$ as $x \to 0$. To find this, it is enough to apply the Mellin transform and examine the location and multiplicities of possible poles.

Keeping this in mind, let us digress for a moment to make some recollections on the non-Archimedean Mellin transform. Let F be a finite extension of $\mathbb{Q}_p$, with uniformizer ϖ and residue field $\mathbb{F}_q$. We take the decomposition $F^\times = \varpi^{\mathbb{Z}} \mathcal{O}^\times$ and say $\eta : F^\times \to \mathbb{C}^\times$ is a *normalized* character if $\eta(\varpi) = 1$. Any character $\chi : F^\times \to \mathbb{C}^\times$ can be uniquely written as $\chi(t) = \eta(t)|t|^s$, with η a normalized character and $s \in \mathbb{C}/(2\pi i / \log q)\mathbb{Z}$; thus we can, and will, think of the space $\mathfrak{X} = \{\chi : F^\times \to \mathrm{U}_1(\mathbb{R})\}$ of characters as a disjoint union of cylinders, indexed by normalized η, and with coordinate on each cylinder given by $s \in \mathbb{C}/(2\pi i / \log q)\mathbb{Z}$.

The Mellin transform of a function J on F is the function on $\mathfrak{X}$ defined by

$$\mathcal{M}J(\eta | \cdot |^s) = \int_{F^\times} \eta(t)|t|^s J(t) \, d^\times t,$$

assuming this integral converges.

For functions J which are smooth and compactly supported on $F^\times$ (in particular, as functions on F they vanish on a neighborhood of 0), the integral above converges absolutely and uniformly for s in a compact set. Moreover, since J has a conductor $1 + \varpi^k \mathbb{O}$, i.e., there exists a minimal k so that J is invariant under multiplication by $1 + \varpi^k \mathbb{O}$, it is easy to see that $\mathcal{M}J(\eta \cdot |^s) = 0$ for all $\eta \notin (\mathbb{O}^\times/(1 + \varpi^k \mathbb{O}))^*$. Summarizing: for $J \in C_c^\infty(F^\times)$, we have

- $\mathcal{M}J(\eta \cdot |^s)$ is an entire function of s,

- $\mathcal{M}J(\eta \cdot |^s) = 0$ for all but finitely many η.

Let $\mathscr{E}$ denote the space of functions $f(\eta \cdot |^s)$ on $\mathfrak{X}$ which satisfy these two conditions. Via Pontryagin duality, there is an inverse Mellin transform which identifies

$$J(x) = \frac{\log q}{2\pi i} \sum_\eta \eta(x)^{-1} \int_0^{\frac{2\pi}{\log q}} \mathcal{M}J(\eta \cdot |^{c+iy})|x|^{-(c+iy)} \, dy,$$

where $c \in \mathbb{R}$ is arbitrary (strictly speaking, Pontryagin duality shows this equality for $c = 0$, but since $\mathcal{M}J$ is entire we are free to shift contours). All together, we find

$$\mathcal{M} : C_c^\infty(F^\times) \xrightarrow{\sim} \mathscr{E}.$$

For our purposes, we must consider Mellin transforms of a larger class of functions than $C_c^\infty(F^\times)$. Consider, for any λ a real number, the functions $C_{(\lambda)}^\infty(F^\times)$ on $F^\times$ satisfying the following:

(1) J is uniformly locally constant on $F^\times$; i.e., there exists an open subgroup $U \subset F^\times$ for which $J(x) = J(ux)$ for all $u \in U$ and $x \in F^\times$.

(2) J vanishes for $|x|$ sufficiently large.

(3) $|J(x)||x^{-\lambda}|$ is bounded as $x \to 0$.

For such functions, $\mathcal{M}J(\eta \cdot |^s)$ is analytic in the half-plane $\mathrm{Re}(s) > \lambda$ and 0 for all but finitely many η. There is again a Mellin inversion formula: for any $J \in C_{(\lambda)}^\infty(F^\times)$, we have

$$J(x) = \frac{\log q}{2\pi i} \sum_\eta \eta(x)^{-1} \int_0^{\frac{2\pi}{\log q}} \mathcal{M}J(\eta \cdot |^{c+iy})|x|^{-(c+iy)} \, dy$$

for any $c > \lambda$. The (finite) sum is over all normalized characters η.

We can apply this, and shift contours far to the left to derive the following proposition.

Proposition 1.17. *Suppose that $J \in C_{(\lambda)}^\infty(F^\times)$ has the property that $\mathcal{M}J(\eta \cdot |^s)$ is meromorphic in s for all s and η, and has only finitely many poles (η_i, s_i). If we write the principal part of $\mathcal{M}J(\eta \cdot |^s)$ at each pole as*

$$\sum_{k=-N_i}^{-1} a_{i,k}(s - s_i)^k$$

then near $x = 0$, $J(x)$ satisfies

$$\left| f(x) - \sum_i \sum_{k=-N_1}^{-1} \eta_i(x)^{-1}(-1)^{k-1} \frac{a_{i,k}(\log|x|)^{1-k}|x|^{-s_i}}{(1-k)!} \right| \ll_M |x|^M$$

for any M.

Proof. This is a direct consequence of the Mellin inversion formula of the previous paragraph. □

Remark 1.18. We will write the conclusion of the proposition simply as

$$f(x) \sim \sum_i \sum_{k=-N_1}^{-1} \eta_i(x)^{-1}(-1)^{k-1} \frac{a_{i,k}(\log|x|)^{1-k}|x|^{-s_i}}{(1-k)!}$$

and say that the right-hand side is the *asymptotic expansion* of f near $x = 0$.

Remark 1.19. In fact, under the assumptions of the above proposition, we can even say more: the Mellin transform of the difference

$$f(x) - \sum_i \sum_{k=-N_1}^{-1} \eta_i(x)^{-1}(-1)^{k-1} \frac{a_{i,k}(\log|x|)^{1-k}|x|^{-s_i}}{(1-k)!} \mathbb{1}_{\mathscr{O}}(x)$$

is clearly in $\mathscr{E}$; hence

$$f(x) = \sum_i \sum_{k=-N_1}^{-1} \eta_i(x)^{-1}(-1)^{k-1} \frac{a_{i,k}(\log|x|)^{1-k}|x|^{-s_i}}{(1-k)!} \mathbb{1}_{\mathscr{O}}(x) + g(x)$$

for some $g(x) \in C_c^\infty(F^\times)$; we can also restate this conclusion as saying that $f(x)$ is of the form

$$f(x) = \sum_i \sum_{k=-N_1}^{-1} A_{i,k}(x)\eta_i(x)^{-1}(-1)^{k-1} \frac{(\log|x|)^{1-k}|x|^{-s_i}}{(1-k)!}$$

for some $A_{i,k}(x) \in C_c^\infty(F)$ satisfying $A_{i,k}(0) = a_{i,k}$.

With these tools in hand, let us now give a proof of Proposition 1.16 when F is a p-adic field.

Proof. Let $f \in C_c^\infty(F \times F)$. Since f is compactly supported on F^2, there exists M such that $f(x, y) = 0$ if $|x| \geq M$ or $|y| \geq M$. Since $f(x, y)$ is locally constant, by compactness of $\{x : |x| \leq M\}$ and $\{y : |y| \leq M\}$ there also exists k so that $f(x, y) = f(x, 0)$ for all $y \in \varpi^k \mathscr{O}$ and $f(x, y) = f(0, y)$ for all $x \in \varpi^k \mathscr{O}$.

If we set $\phi_1(x) = f(x, 0)$ and $\phi_2(y) = f(0, y)$, then we can write

$$f(x, y) = \phi_1(x) \mathbb{1}_{\varpi^k \mathscr{O}}(y) + \phi_2(y) \mathbb{1}_{\varpi^k \mathscr{O}}(x) - f(0, 0) \mathbb{1}_{\varpi^k \mathscr{O} \times \varpi^k \mathscr{O}}(x, y) + g(x, y),$$

where $g(x, y) \in C_c^\infty(F^2)$ is zero on a neighborhood of $\{xy = 0\}$.

The orbital integral of any function $g(x, y)$ is easy to understand: since $g \in C_c^\infty(F^\times \times F^\times)$ and the map

$$\pi|_{F^\times \times F^\times} : F^\times \times F^\times \to F^\times,$$

$$(x, y) \mapsto xy,$$

is simply composition of the isomorphism $(x, y) \mapsto (xy, y)$ and projection onto the first factor, it follows that the orbital integral of g must lie in $C_c^\infty(F^\times)$. Conversely, it is easy to see that given a function $C_c^\infty(F^\times)$, there exists $g(x, y) \in C_c^\infty(F^\times \times F^\times)$ whose orbital integral is the given function.

So let us assume that $f(x, y)$ is of the form

$$f(x, y) = \phi_1(x)\mathbb{1}_{\varpi^k \mathcal{O}}(y) + \phi_2(y)\mathbb{1}_{\varpi^k \mathcal{O}}(x) - f(0, 0)\mathbb{1}_{\varpi^k \mathcal{O} \times \varpi^k \mathcal{O}}(x, y)$$

and, for the sake of notation, let's define

$$f_1(x, y) = \phi_1(x)\mathbb{1}_{\varpi^k \mathcal{O}}(y), \quad f_2(x, y) = \phi_2(y)\mathbb{1}_{\varpi^k \mathcal{O}}(x), \quad f_3(x, y) = f(0, 0)\mathbb{1}_{\varpi^k \mathcal{O} \times \varpi^k \mathcal{O}}(x, y).$$

Let us compute the Mellin transform of the orbital integral J of f. This is

$$\mathcal{M}J(\eta|\cdot|^s) = \int_{F^\times} \eta(\gamma)|\gamma|^s \int_{F^\times} f(\gamma t, t^{-1})\, d^\times t\, d^\times \gamma$$

$$= \int_{F^\times \times F^\times} \eta(xy)|xy|^s f(x, y)\, d^\times x\, d^\times y.$$

Each term f_1, f_2, f_3 contributes, respectively, the following products of Tate integrals:

$$\int_{F^\times} \eta(x)|x|^s \phi_1(x)\, d^\times x \int_{F^\times} \mathbb{1}_{\varpi^k \mathcal{O}}(y)\, d^\times y = Z(s, \eta, \phi_1)Z(s, \eta, \mathbb{1}_{\varpi^k \mathcal{O}}),$$

$$\int_{F^\times} \eta(x)|x|^s \mathbb{1}_{\varpi^k \mathcal{O}}(x)\, d^\times x \int_{F^\times} \phi_2(y)\, d^\times y = Z(s, \eta, \mathbb{1}_{\varpi^k \mathcal{O}})Z(s, \eta, \phi_2),$$

$$f(0, 0)\int_{F^\times} \eta(x)|x|^s \mathbb{1}_{\varpi^k \mathcal{O}}(x)\, d^\times x \int_{F^\times} \mathbb{1}_{\varpi^k \mathcal{O}}(y)\, d^\times y = f(0, 0)Z(s, \eta, \mathbb{1}_{\varpi^k \mathcal{O}})Z(s, \eta, \mathbb{1}_{\varpi^k \mathcal{O}}).$$

Now, recall that if η is nontrivial then these Tate integrals are all entire; hence $\mathcal{M}J(\eta \cdot |^s)$ is an entire function of s when $\eta \neq 1$. It also visibly vanishes for all but finitely many η.

When $\eta = 1$, the only pole of the Tate integral occurs at $s = 0 \in \mathbb{C}/(2\pi i / \log q)\mathbb{Z}$ — even further, we have

$$Z(s, 1, \mathbb{1}_{\varpi^k \mathcal{O}}) = \frac{q^{-ks}}{1 - q^{-s}} = \frac{1}{\log q}\frac{1}{s} + \frac{1}{2} + \text{h.o.t.},$$

where the last line is simply the first two terms of the Taylor expansion around 0 (h.o.t. is shorthand for "higher-order terms"). Similarly, we can observe that

$$Z(s, 1, \phi_i) = \frac{f(0, 0)}{\log q}\frac{1}{s} + \lim_{s \to 0}\left(\frac{d}{ds}s Z(s, 1, \phi_i)\right) + \text{h.o.t.}.$$

It follows that the principal part of $\mathcal{M}J(|\cdot|^s)$ near 0 is exactly

$$\frac{1}{\log q}\left(f(0,0)s^{-2}+\frac{1}{2}\left(\lim_{s\to 0}\left(\frac{d}{ds}sZ(s,1,f(\cdot,0))\right)+\lim_{s\to 0}\left(\frac{d}{ds}sZ(s,1,f(0,\cdot))\right)\right)s^{-1}\right)$$

and hence, applying Proposition 1.17 (really the remark following that proposition) we find that

$$J(\gamma)=A_1(\gamma)+A_2(\gamma)\log|\gamma|,$$

where

$$A_1(0)=\frac{1}{2\log q}\left(\lim_{s\to 0}\left(\frac{d}{ds}sZ(s,1,f(\cdot,0))\right)+\lim_{s\to 0}\left(\frac{d}{ds}sZ(s,1,f(0,\cdot))\right)\right)$$

and

$$A_2(0)=-\frac{1}{\log q}f(0,0).$$

Moreover, by our discussion it is clear that any function of the form $J(\gamma)=A_1(\gamma)+A_2(\gamma)\log|\gamma|$ can occur as the orbital integral of some f; one needs only to choose f so that conditions on $A_1(0)$ and $A_2(0)$ above are met, and then one can freely modify our initial choice of f by a test function $g\in C_c^\infty(F^\times\times F^\times)$ to obtain the desired $A_i(x)$. $\qquad\square$

Remark 1.20. Following the thread of Remark 1.14, we should remark that both Propositions 1.15 and 1.16 have variants where the test functions for orbital integrals are allowed to be Schwartz functions (not just compactly supported and smooth functions). This version of Proposition 1.16 involving Schwartz functions is verbatim the result appearing in [Casselman and Tian 2013]. The statements of the results are the same; merely allow the A_i to be in $S(F)$. The result of Remark 1.14 then follows from these variants and the argument in the next subsection.

1C3. *Reduction to the toy model.* We would now like to deduce Theorem 1.12 from our results on the toy model. This is quite straightforward, because the action of O_W on SO_V by conjugation *is* in some sense the action of $U_1(F)$ (for E/F nonsplit) or $F^\times$ (for E/F split) on W occurring in the toy model.

But let us be careful. Let $x\in GSpin_V$ be a semisimple but nonregular point; i.e., let $x=z\in E^\times\subset GSpin_V$. The tangent space $T_x GSpin_V$ can be identified with the whole even Clifford algebra $C_0(V)$. Let U be an open SO_W stable neighborhood of 0 in $C_0(V)$, together with an SO_W equivariant embedding $\rho:U\hookrightarrow GSpin_V$ satisfying $\rho(0)=x$. By the analytic Luna slice theorem [Aizenbud and Gourevitch 2009, Theorem 2.3.17], such a (U,ρ) always exist, and in fact we can even take U to have saturated image; i.e., $\rho(U)$ is the preimage under $GSpin_V(F)\to(GSpin_V\mathbin{/\!\!/}SO_W)(F)$ of an open subset.

A quick aside: the use of the analytic Luna slice theorem is in some sense gross overkill. However, in the interest of brevity, we simply cite it and move on rather than trying to explicitly write down such a (U,ρ).

In any case, this construction shows that the behavior of orbital integrals of functions $f\in\coprod_{(W,V)}GSpin_V$ by SO_W-conjugation is the same as that of functions on $\coprod_{(W,V)}C_0(V)$ by SO_W-conjugation. But we

have the diagram

$$\coprod_{(W,V)} C_0(V)(F) \xrightarrow{\ \varphi\ } \coprod_{(W,V)} W(F)$$
$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$
$$(C_0(V) /\!\!/ \mathrm{SO}_W)(F) \longrightarrow (W /\!\!/ \mathrm{SO}_W)(F)$$

where $\varphi : z + we \mapsto w$. However, this makes it apparent that the map φ is nothing more than projection onto the second factor, $C_0(V) = E \times W \to W$, and so the left downward arrow in the above diagram is nothing more than the right downward arrow, together with an untouched factor of E.

However, the right downward arrow $\coprod_{(W,V)} W(F) \to (W /\!\!/ \mathrm{SO}_W)(F)$ is exactly the map appearing in our toy model. Thus, orbital integrals for $\mathrm{GSpin}_V /^{\mathrm{conj}} \mathrm{SO}_W$ are exactly those functions of the form

$$\phi_1(z,b) + \phi_2(z,b)\omega(b)$$

in the nonsplit case and

$$\phi_1(z,b) + \phi_2(z,b)\log|b|$$

in the split case (here, $\phi_i(z,b)$ are smooth and compactly supported on $\{\mathrm{N}\,z + b \neq 0\}$). Integrating over the central G_m, and averaging over the outer automorphism given by conjugation by $\sigma \in \mathrm{O}_W / \mathrm{SO}_W$ completes the proof of Theorem 1.12.

2. Tones and tonal integrals for $\mathrm{GL}_2 \times \mathrm{GL}_2$

We now study the local integrals appearing on the "geometric" side of the $\mathrm{GL}_2 \times \mathrm{GL}_2$ trace formula. This will require a few preliminaries.

2A. *Useful facts from the representation theory of* $\mathrm{GL}_2(F)$. The definition of the local tonal integral requires some knowledge of the Weil representation and of Whittaker functions for GL_2. We give a brief tour of these topics. Throughout we consider a fixed local field F of characteristic 0 and a fixed pair (d_W, d_V) of discriminants, and hence also an E/F defined by $E = F[\sqrt{d_W}]$.

2A1. *The Weil representation of* $\mathrm{GL}_2^{(2)}(F)$. We follow the exposition of [Takeda 2014]. In the interest of concision, we omit most details and instead direct the reader to the literature.

Given any ring R, let $\mathrm{GL}_2^{(2)}(R)$ be the subgroup of $\mathrm{GL}_2(R)$ consisting of R-valued 2×2 matrices with square determinant. (Although notation suggests otherwise, $\mathrm{GL}_2^{(2)}$ is in fact not an algebraic subgroup of GL_2. This fact will not cause any real difficulties.) Similarly, for any subgroup of GL_2, let the superscript $^{(2)}$ denote its intersection with $\mathrm{GL}_2^{(2)}$.

Let F be a characteristic-0 local field as above. The theory of the Weil representation [1964] defines a double cover $\widetilde{\mathrm{SL}}_2(F)$ of $\mathrm{SL}_2(F)$ often called the metaplectic group. It is easy to extend "the" cocycle

$$\tau : \mathrm{SL}_2(F) \times \mathrm{SL}_2(F) \to \{\pm 1\}$$

corresponding to this central extension to a cocycle τ for $\mathrm{GL}_2^{(2)}(F)$; this defines a metaplectic double cover $\widetilde{\mathrm{GL}}_2^{(2)}(F)$. Set-theoretically,

$$\widetilde{\mathrm{GL}}_2^{(2)}(F) = \mathrm{GL}_2^{(2)}(F) \times \{\pm 1\}$$

with multiplication given by

$$(h, \zeta)(h', \zeta') = (hh', \tau(h, h')\zeta\zeta').$$

This central extension comes equipped with two important set-theoretic sections, which, following [Takeda 2014], we denote by κ and s

$$\kappa, s : \mathrm{GL}_2^{(2)}(F) \to \widetilde{\mathrm{GL}}_2^{(2)}(F).$$

(The distinction between the two sections is largely unimportant. They appear as the "obvious" set-theoretic sections for two different (but cohomologous) choices of cocycle used to define $\widetilde{\mathrm{GL}}_2^{(2)}(F)$. For our local considerations, we use s.)

Like $\widetilde{\mathrm{SL}}_2(F)$, the group $\widetilde{\mathrm{GL}}_2^{(2)}(F)$ also comes equipped with a Weil representation r^ψ (which depends on a choice of additive character $\psi : F \to \mathrm{U}_1(\mathbb{R})$).

The representation r^ψ can be realized as an action on the space $S^+(F)$ of even Schwartz functions on F. It is given by the formulas

$$r^\psi\left(s\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) f(x) = \gamma(\psi)\hat{f}(x),$$

$$r^\psi\left(s\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\right) f(x) = \psi(bx^2) f(x),$$

$$r^\psi\left(s\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) f(x) = |a|^{\frac{1}{2}} \mu_\psi(a) f(ax),$$

$$r^\psi\left(s\begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix}\right) f(x) = |a|^{-\frac{1}{2}} f(a^{-1}x),$$

$$r^\psi(\xi) f(x) = \xi f(x).$$

Here ξ lies in the central $\{\pm 1\}$ used to define the double cover. In the above formulas, $\hat{f}(x) = \int_F f(y)\psi(2xy)\,dy$ is the Fourier transform, with dy a self-dual measure on F, and $\gamma(\psi)$ is the Weil index of the character of second degree $x \mapsto \psi(x^2)$. We define

$$\mu_\psi(a) = \gamma(\psi_a)/\gamma(\psi).$$

The properties of the Weil index and of μ_ψ are nicely explained in [Ranga Rao 1993]. We will not need them seriously.

We are interested not in r^ψ, but in the "squared" representation $R^\psi = r^\psi \,\hat{\otimes}\, r^{\psi - d_W}$. As the product of two genuine representations of $\widetilde{\mathrm{GL}}_2^{(2)}$, this descends to a representation of $\mathrm{GL}_2^{(2)}(F)$. The action can

be realized on the space of Schwartz functions on $E = F[\sqrt{d_W}]$ which are invariant under $z \mapsto -z$ and $z \mapsto \bar{z}$; we denote this space of functions as $S^{++}(E)$. $\boldsymbol{R}^\psi$ acts by the formulas

$$\boldsymbol{R}^\psi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\Psi(z) = \gamma(\psi, N)\hat{\Psi}(z),$$

$$\boldsymbol{R}^\psi\left(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\right)\Psi(z) = \psi(b\,N\,z)\Psi(z),$$

$$\boldsymbol{R}^\psi\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right)\Psi(z) = |a|\omega(a)\Psi(az),$$

$$\boldsymbol{R}^\psi\left(\begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix}\right)\Psi(z) = |a|^{-1}\Psi(a^{-1}z).$$

Here, $\hat{\Psi}(z) = \int_E \Psi(z')\psi(N(z + z') - N(z) - N(z'))\, dz'$ is the Fourier transform, with dz' normalized so as to be self-dual. N is of course the norm map from $E \to F$.

Remark 2.1. Observe that, by the above formulas, $\overline{\boldsymbol{R}^\psi(h)\Psi(z)} = \boldsymbol{R}^{\psi^{-1}}(h)\bar{\Psi}(z)$.

We will call $\boldsymbol{R}^\psi$ the *Weil representation for* $\mathrm{GL}_2^{(2)}$ and $\boldsymbol{r}^\psi$ the *Weil representation for* $\widetilde{\mathrm{GL}}_2^{(2)}$.

2A2. *Whittaker functions for* GL_2. Let us now record some recollections on Whittaker functions for GL_2.

Let $\eta = (\eta_1, \eta_2) : A(F) \to U_1(\mathbb{R})$ be a character of the diagonal maximal torus of GL_2, with η_i normalized characters. Consider the space of smooth sections

$$I(s, \eta) = \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)}(\eta \otimes \delta^{s-\frac{1}{2}}) \cap L^2$$

$$= \left\{ f_s : \mathrm{GL}_2(F) \to \mathbb{C} : \begin{array}{l} f_s(ang) = \eta(a)\delta(a)^s f_s(g) \text{ for all } a \in A(F), n \in N(F), g \in G(F), \\ f_s \text{ is smooth and } \int_K |f_s(k)|^2\, dk < \infty \end{array} \right\}$$

In the above, Ind denotes normalized induction, and K is the usual choice of maximal compact subgroup of $\mathrm{GL}_2(F)$. A choice of $f_s \in I(s, \eta)$ for varying s is said to be a *standard* or *flat section* if $f_s|_K$ does not depend on s.

We can construct elements in $I(s, \eta)$ explicitly in the following fashion: let $\Phi \in S(F^2)$, and define

$$F_s(g) = F_s(g, \Phi; s, \eta) := \eta_1(\det g)|\det g|^s \int_{F^\times} \Phi((0, t)g)\eta_1\eta_2^{-1}(t)|t|^{2s}\, d^\times t.$$

Here $d^\times t$ is Haar measure on $F^\times$, normalized so that $\mathrm{vol}(\mathcal{O}^\times)) = 1$. F_s for fixed Φ is not a flat section, but it is K-finite, and one can achieve such sections by convolving flat sections with functions $\phi \in C_c^\infty(G(F))$. See [Jacquet 2004]. Note that the integral defining F_s is a Tate integral for $L(2s, \eta_1\eta_2^{-1})$, and hence converges absolutely as long as $\mathrm{Re}(s) > 0$.

By [Jacquet and Zagier 1987], every K-finite vector in $I(s, \eta)$ can be written as a finite combination

$$f_s(g) = \frac{1}{L(2s, \eta_1\eta_2^{-1})} \sum_i P_i(s) F_s(g, \Phi_i; s, \eta), \tag{2-1}$$

where $P_i(s)$ is the reciprocal of a polynomial in s and q^{-s} which has no zeros in $\mathrm{Re}(s) > 0$.

We will work largely with $I(s, \eta)^\infty := S(\mathrm{GL}_2(F))I(s, \eta)^{\mathrm{K}\text{-fin}}$, where $S(\mathrm{GL}_2(F))$ denotes the Schwartz space of $\mathrm{GL}_2(F)$ and this acts on the space of K-finite vectors in $I(s, \eta)$ via the left action of functions on the group on representations. $I(s, \eta)^\infty$ is the smooth (admissible moderate growth) Fréchet globalization of the $(\mathfrak{g}, \mathrm{K})$-module of K-finite vectors $I(s, \eta)^{\mathrm{K}\text{-fin}}$. (Of course, these concerns only manifest when F is Archimedean!)

Recall the following definition.

Definition 2.2. Given a section $f_s \in I(s, \eta)$, the *Whittaker function associated to f_s* is the function given by

$$\mathscr{W}_{f_s}^{\psi^{-1}}(g) := \int_F f_s(wn(x)g)\psi^{-1}(x)\, dx,$$

assuming this integral converges.

The convergence of this integral for elements $f_s \in I(s, \eta)^\infty$ for most values of s can analyzed in a relatively straightforward manner. It reduces to understanding convergence for K-finite vectors, and hence by (2-1) to understanding

$$\mathscr{W}_{F_s}^{\psi^{-1}}(g) = \eta_1(-1)\eta_1(\det g)|\det g|^s \int_F \psi(-x) \int_{F^\times} \Phi((t, tx)g)\eta_1\eta_2^{-1}(t)|t|^{2s}\, d^\times t\, dx$$

$$= \eta_1(-1)\eta_1(\det g)|\det g|^s \int_{F^\times} \widehat{(g.\Phi)}((t, t^{-1}))\eta_1\eta_2^{-1}(t)|t|^{2s-1}\, d^\times t,$$

where $\hat{}$ denotes partial Fourier transform in the second variable. Since the above integral converges for all s, the convergence of the integral defining $\mathscr{W}_{f_s}^{\psi^{-1}}(g)$ is established for all $f_s \in I(\eta, s)^\infty$.

Indeed, essentially the same calculation also allows us to examine the asymptotics of

$$\mathscr{W}_{F_s}^{\psi^{-1}}(a(b)) = \eta_1(-1)\eta_1(\det g)|\det g|^s \int_{F^\times} \hat{\Phi}((bt, t^{-1}))\eta_1\eta_2^{-1}(t)|t|^{2s-1}\, d^\times t$$

as $b \to 0$. For every fixed s, we are looking at the integral

$$\int_{F^\times} \hat{\Phi}((bt, t^{-1}))\eta_1\eta_2^{-1}(t)|t|^{2s-1}\, d^\times t$$

and, as in our analysis of the split orbital integral above, the behavior of this function of b, as $b \to 0$ can be read off of the locations and multiplicities of the poles of its Mellin transform. Continuing this line of thought leads to the following proposition. In the non-Archimedean case, this proposition is no more and no less than the computation of the Kirillov model for principal series representations of GL_2.

We use $K(s, \eta)$ to denote the space of functions on $F^\times$ of the form $\mathscr{W}_{f_s}^{\psi^{-1}}(a(\cdot))$ for $f_s \in I(s, \eta)^\infty$.

Proposition 2.3. *Every element $K \in K(s, \eta)$ has the following form for some choices of $A_i \in S(F)$:*

- *If $\eta_1 = \eta_2 = \eta$ are equal and $s \neq \frac{1}{2}$,*

$$K(b) = A_1(b)\eta(b)|b|^s + A_2(b)\eta(b)|b|^{1-s}.$$

- *If $\eta_1 = \eta_2 = \eta$ are equal and $s = \frac{1}{2}$,*

$$K(b) = |b|^{\frac{1}{2}} (A_1(b)\eta(b) + A_2(b)\eta(b) \log |b|).$$

- *If $\eta_1 \neq \eta_2$,*

$$K(b) = A_1(b)\eta_1(b)|b|^s + A_2(b)\eta_2(b)|b|^{1-s}.$$

Moreover, given choices of $A_i \in S(F)$, there always exists $K \in K(s, \eta)$ of the above form, as appropriate.

Proof. For $F = \mathbb{R}, \mathbb{C}$ this is the analysis of the "toy model" which was analyzed in [Casselman and Tian 2013], although twisted by a character. It is also discussed in Section 3 of [Jacquet 2004]. For F non-Archimedean, this can be viewed as a consequence of the computation of the Kirillov model of a principal series. Alternatively, one can see this via an argument nearly identical to the one described above for Proposition 1.16. $\qquad\square$

Not surprisingly, this proposition will play an analogous role in our analysis of tonal integrals to that played by the "toy model" in our discussion of orbital integrals.

2B. *Tones.* Preliminaries out of the way, we are now ready to begin defining our local integrals. As one will see in Part II, the sum appearing in the "geometric side" of the $GL_2 \times GL_2$ trace formula is indexed not by orbits of a group action, but rather by what we will call *tones*. To that end, we propose the following artificial-looking definition.

Definition 2.4. A *tone* γ' consists of a triple

$$\gamma' = (\alpha; \zeta, \beta),$$

with $\alpha \in \Sigma$, a fixed set of representatives of $F^\times/(F^\times)^2$, $\zeta \in E/(\{\pm 1\} \times \{1, \bar{\ }\})$, and $\beta \in F$, satisfying

$$N\zeta + \beta = \alpha.$$

We say that a tone γ' is *regular semisimple* if $\beta \neq 0$.

Of course, one can choose different representatives of $F^\times/(F^\times)^2$. If we replace α by $t^2\alpha$, then we must replace a tone

$$\gamma = (\alpha; \zeta, \beta) \mapsto t.\gamma := (t^2\alpha; t\zeta, t^2\beta)$$

by its scaled version $t.\gamma$. It will be a simple but important feature of our definition of tonal integrals that they are invariant under this scaling action.

2C. *Local tonal integrals and the space Ω'.* We are finally in the position to define the tonal integral. Throughout, whenever we write a Whittaker function $\mathscr{W}_{F_s}^{\psi^{-1}}$, we take $\eta = (1, \omega) = (1, \omega^{-1})$, where $\omega = \omega_{E/F}$ is the quadratic character associated to E/F.

Definition 2.5. Let $f' \in C_c^\infty((\mathrm{GL}_2 \times \mathrm{GL}_2)(F))$, and let $\Phi \in S(F^2)$ and $\Psi \in S^{++}(E)$. Given a regular semisimple tone γ', the tonal integral of $f' \otimes \Phi \otimes \Psi$ is the expression

$$I(s, \gamma'; f' \otimes \Phi \otimes \Psi) = I_{\gamma'}(s; f' \otimes \Phi \otimes \Psi)$$

$$:= |\alpha|^{\frac{1}{2}} \int f'((g^{-1}, g^{-1})(a(\alpha), 1)(h, 1)) \mathscr{W}_{F_s}^{\psi^{-1}} \left(a\left(\frac{\beta}{\alpha}\right) g \right) \overline{\boldsymbol{R}^{\psi^{-1}}(h) \Psi(\zeta)}$$

$$= |\alpha|^{\frac{1}{2}} \int f'((g^{-1}, g^{-1})(a(\alpha), 1)(h, 1)) \mathscr{W}_{F_s}^{\psi^{-1}} \left(a\left(\frac{\beta}{\alpha}\right) g \right) \boldsymbol{R}^{\psi}(h) \overline{\Psi}(\zeta),$$

where the final equality follows from Remark 2.1. In the above, the integration is over $g \in \mathrm{GL}_2(F)$ and $h \in \mathrm{GL}_2^{(2)}(F)$. In interest of space, we omit writing the symbols dg, dh. We write $\Omega'(s)$ for the space of finite linear combinations of tonal integrals, viewed as a space of function on the locus of regular semisimple tones.

When $f' = f_1' \otimes f_2'$, we often write this integral in an expanded, slightly different form

$$I_{\gamma'}(s; f' \otimes \Phi \otimes \Psi) = |\alpha|^{\frac{1}{2}} \int f_1'(g^{-1}h) f_2'(g^{-1}a(\alpha)^{-1}) \mathscr{W}_{F_s}^{\psi^{-1}}(a(\beta)g) \boldsymbol{R}^{\psi}(h) \overline{\Psi}(\zeta).$$

Remark 2.6. The integral defining the tonal integral clearly converges absolutely — f' is compactly supported and smooth.

Remark 2.7. A simple change of variables quickly shows that

$$I_{t.\gamma'}(s; f' \otimes \Phi \otimes \Psi) = I_{\gamma'}(s; f' \otimes \Phi \otimes \Psi),$$

so indeed, as alluded to above, the tonal integral is invariant under the scaling action $\gamma' \mapsto t.\gamma'$.

We are most concerned with the behavior of $I_{\gamma'}(s; f' \otimes \Phi \otimes \Psi)$ at $s = \frac{1}{2}$; to this end, we simply write

$$I(\gamma', f' \otimes \Phi \otimes \Psi) = I_{\gamma'}(f' \otimes \Phi \otimes \Psi) := I_{\gamma'}\left(\tfrac{1}{2}; f' \otimes \Phi \otimes \Psi\right)$$

and

$$\Omega' := \Omega'\left(\tfrac{1}{2}\right).$$

We can give a complete description of Ω'. With our description of the asymptotics of Whittaker functions at hand, this becomes extremely straightforward.

Theorem 2.8. *Suppose E/F is a field extension. Then*

$$\Omega' = \left|\frac{b}{\alpha}\right|^{\frac{1}{2}} \left(\phi_1(\alpha; \zeta, \beta) + \phi_2(\alpha; \zeta, \beta) \omega\left(\frac{\beta}{\alpha}\right) \right),$$

where $\omega = \omega_{E/F}$ is the quadratic character corresponding to E/F and ϕ_i are Schwartz functions. If $E = F \times F$ is split,

$$\Omega' = \left|\frac{b}{\alpha}\right|^{\frac{1}{2}} \left(\phi_1(\alpha; \zeta, \beta) + \phi_2(\alpha; \zeta, \beta) \log\left|\frac{\beta}{\alpha}\right| \right).$$

Proof. It is enough consider each α. Thus, fix α — we will show that $I((\alpha; \zeta, \beta), f' \otimes \Phi \otimes \Psi)$ has the claimed behavior near $\beta = 0$, and that all such functions are (finite linear combinations of) tonal integrals. We have

$$I_{\gamma'}(f' \otimes \Phi \otimes \Psi) = |\alpha|^{\frac{1}{2}} \int f'((g^{-1}, g^{-1})(a(\alpha), 1)(h, 1)) \mathscr{W}_{F_s}^{\psi^{-1}} \left(a\left(\frac{\beta}{\alpha} \right) g \right) \boldsymbol{R}^{\psi}(h) \overline{\Psi}(\zeta);$$

if we write $f'_\alpha \in C_c^\infty(\mathrm{GL}_2(F) \times \mathrm{GL}_2^{(2)}(F))$ for the function

$$f'_\alpha(x, y) = f'((x^{-1}, x^{-1})(a(\alpha), 1)(y, 1))$$

then the proof falls out. Consider the space $V := K\left(\frac{1}{2}, (1, \omega)\right) \widehat{\otimes} S^{++}(E)$. Here $\widehat{\otimes}$ denotes the completed tensor product.[2] This is a Fréchet space. There is an obvious left action λ of $\mathrm{GL}_2^{(2)}(F) \times \mathrm{GL}_2^{(2)}(F)$ on V via

$$\lambda(x, y)(K \otimes \Psi) = R(x)K \otimes \boldsymbol{R}^{\psi}(y)\Psi,$$

where R denotes the action of $\mathrm{GL}_2(F)$ on $K\left(\frac{1}{2}, (1, \omega)\right)$ given by the right action on the Whittaker model of $I\left(\frac{1}{2}, (1, \omega)\right)$. Now, simply applying the Dixmier–Malliavin theorem [1978] in the context of this group action allows us to conclude. $\square$

Remark 2.9. Given Proposition 2.3, the above theorem is completely elementary when F is a non-Archimedean local field, since all representations are smooth. It is only when F is real or complex that any functional-analytic subtleties manifest.

3. Matching and the fundamental lemma

We can relate the two spaces Ω and Ω' of orthogonal orbital integrals and general linear tonal integrals to one another by a matching of regular semisimple orbits to regular semisimple tones. This matching induces a "transfer of smooth functions" which identifies Ω_S and Ω'. Moreover, this transfer can be completely explicated when test functions in question are taken to be elements of the respective spherical Hecke algebras; this explication, also known as the fundamental lemma, is the main goal of this section.

3A. *Matching of orbits and transfer of smooth functions.* Thus far, our notation has been extremely suggestive of the following definition.

Definition 3.1. Fix the same set of representatives Σ of $F^\times/(F^\times)^2$. Let

$$\gamma \in \coprod_{(W, V)} (\mathrm{SO}_V(F)/^{\mathrm{conj}} \mathrm{O}_W(F))^{\mathrm{r.s.s.}}$$

be a regular semisimple orbit and let

$$\gamma' = (\alpha; \zeta, \beta), \quad \beta \neq 0,$$

---

[2] Since both $K\left(\frac{1}{2}, (1, \omega)\right)$ and $S^{++}(E)$ are nuclear spaces, this notation is unambiguous.

be a regular semisimple tone. We say that γ and γ' *match* if they are identical as tuples, i.e., if when we write γ in terms of coordinates on GSpin_V, the corresponding tuple $(\alpha; z, b)$ is the tone γ'. We write $\gamma \leftrightarrow \gamma'$ for matching regular semisimple orbits and tones.

Furthermore, given this definition of matching, Theorems 1.12 and 2.8 have a suggestive corollary. We first make a simple definition.

Definition 3.2. Given $\gamma' = (\alpha; \zeta, \beta)$ a regular semisimple tone, the *transfer factor* corresponding to γ is the function

$$t(\gamma') = \left| \frac{b}{\alpha} \right|^{\frac{1}{2}}.$$

Given this, we have the following definition

Definition 3.3. Let $f = (f_{(W,V)})$ be a tuple of functions on $\coprod_{(W,V)} \mathrm{G}_{(W,V)}(F)$, where the indexing set is over a class of relevant pure inner forms (W, V) corresponding to a fixed pair of discriminants (d_W, d_V). Let $\sum f_i' \otimes \Phi_i \otimes \Psi_i$ be a finite sum of tensors, where f_i is a function on $\mathrm{G}'(F)$, $\Phi_i \in S(F^2)$, $\Psi_i \in S^{++}(E)$. Then we say that $(f_{(W,V)})$ and $\sum f_i' \otimes \Phi_i \otimes \Psi_i$ are *smooth transfers of one another*, or more simply, that they *match*, if, for all matching regular semisimple $\gamma \leftrightarrow \gamma'$, we have

$$t(\gamma')J(\gamma, f) = \sum_i I(\gamma', f_i' \otimes \Phi_i \otimes \Psi_i).$$

For shorthand, we write $f \leftrightarrow \sum f_i' \otimes \Phi_i \otimes \Psi_i$ if the two match.

Theorems 1.12 and 2.8 ensure that there are many matching functions. The following theorem is an obvious corollary of the aforementioned results.

Theorem 3.4 (existence of smooth transfers). *Given any $f \in \bigoplus_{(W,V)} S(\mathrm{G}_{(W,V)}(F))$, there exists a matching $\sum f_i' \otimes \Phi_i \otimes \Psi_i$, with $f_i' \in C_c^\infty(\mathrm{G}'(F))$, $\Phi_i \in S(F^2)$, $\Psi_i \in S^{++}(E)$. Conversely, given such $\sum f_i' \otimes \Phi_i \otimes \Psi_i$, there exists a matching $f \in \bigoplus_{(W,V)} S(\mathrm{G}_{(W,V)}(F))$.*

Remark 3.5. Contrary to what notation may suggest, given f, there is not a unique matching $\sum f_i' \otimes \Phi_i \otimes \Psi_i$ (or vice versa).

Remark 3.6. The appearance of the transfer factor $t(\gamma')$ may seem strange; however, it is worth noting that if γ and γ' are global orbits or tones, then $\prod_v t_v(\gamma') = 1$. Thus, the relation between global orbital integrals and global tonal integrals does not see the transfer factor.

With these preliminaries behind us, let us now explicate this transfer for smooth functions for Hecke elements.

3B. *The fundamental lemma: E/F unramified.* In this subsection, we assume that F and d_W are such that E/F is an unramified extension of non-Archimedean local fields of residue characteristic not 2. We also make the simplifying assumption that $d_W, d_V \in \mathcal{O}^\times$; this is not essential and merely simplifies notation. In this setting, there are two relevant pure inner forms corresponding to the pair of discriminants

(d_W, d_V): the first, (W_0, V_0) has V_0 split; the second, (W_1, V_1) has V_1 nonsplit. Concretely, we have (noting that ϖ is a representative of the nontrivial class in $F^\times / \mathrm{N}\, E^\times$)

$$(W_0, Q_0|_{W_0}) \cong \left(E, -\frac{d_V}{d_W}\mathrm{N}\right) \quad \text{and} \quad (W_1, Q_1|_{W_1}) \cong \left(E, -\varpi \frac{d_V}{d_W}\mathrm{N}\right),$$

while we always set

$$V_i \cong W_i \oplus Fe,$$

with $Q_i(e) = d_V/d_W$.

As before, let $\mathrm{G}_i = \mathrm{G}_{(W_i, V_i)}$ denote the group $\mathrm{SO}_{W_i} \times \mathrm{SO}_{V_i}$ and let $\mathrm{G}' = \mathrm{GL}_2 \times \mathrm{GL}_2$. It may help to orient the reader to recall

$$\mathrm{G}_i \cong \begin{cases} (\mathbb{G}_m \backslash \mathrm{Res}_{E/F}\, \mathbb{G}_m) \times \mathrm{PGL}_2 & \text{if } i = 0, \\ (\mathbb{G}_m \backslash \mathrm{Res}_{E/F}\, \mathbb{G}_m) \times \mathrm{PB}^\times & \text{if } i = 1, \end{cases}$$

where B/F is the unique quaternion division algebra over F. In either case, as E/F is unramified, the groups G_i are unramified; hence it makes sense to talk about their maximal compact subgroups $\mathrm{K}_i = \mathrm{G}_i(\mathcal{O})$. Also let $\mathrm{K}' = \mathrm{G}'(\mathcal{O})$ and let

$$\mathscr{H}_i = \mathscr{H}_{\mathrm{G}_i} = \mathscr{H}(\mathrm{G}_i, \mathrm{K}_i),$$
$$\mathscr{H}' = \mathscr{H}_{\mathrm{G}'} = \mathscr{H}(\mathrm{G}', \mathrm{K}')$$

denote the spherical Hecke algebras for G and G' respectively. $\mathscr{H}_1 \cong \mathbb{C}$ as $\mathrm{G}_1(F) = \mathrm{K}_1$ is already compact. Hence, we concentrate attention on the more interesting $\mathscr{H}_0$.

The standard functorial liftings of forms from SO_{W_0} and SO_{V_0} to GL_2 are embodied in a homomorphism of Hecke algebras

$$\mathrm{St} : \mathscr{H}' \to \mathscr{H}_0.$$

We will abuse notation and also write St for the individual maps $\mathscr{H}_{\mathrm{GL}_2} \to \mathscr{H}_{\mathrm{SO}_{W_0}}$ and $\mathscr{H}_{\mathrm{GL}_2} \to \mathscr{H}_{\mathrm{SO}_{V_0}}$ when this does not cause any confusion.

We can now state the fundamental lemma in the case that E/F is unramified.

Theorem 3.7 (the fundamental lemma, E/F unramified). *Let $f' \in \mathscr{H}'$ lie in the spherical Hecke algebra for G', and set $\Phi^\circ = \mathbb{1}_{\mathcal{O}^{\oplus 2}}$ and $\Psi^\circ = \mathbb{1}_{\mathcal{O}_E}$. Then the functions $f' \otimes \Phi^\circ \otimes \Psi^\circ$ and $(\mathrm{St}(f'), 0)$ match each other, in the sense that*

$$I_{\gamma'}\left(\tfrac{1}{2}; f' \otimes \Phi^\circ \otimes \Psi^\circ\right) = \begin{cases} J_\gamma(\mathrm{St}(f')) & \text{if } \gamma' \leftrightarrow \gamma \text{ for } \gamma \in [\mathrm{SO}_{V_0}(F)/\mathrm{SO}_{W_0}(F)]^{\mathrm{r.s.s.}}, \\ 0 & \text{otherwise.} \end{cases}$$

Before we prove this result, let us first describe the map $\mathrm{St} : \mathscr{H}' \to \mathscr{H}$ in some detail. Let us write

$$\mathrm{K}_{\mathrm{GL}_2} = \mathrm{GL}_2(\mathcal{O}),$$
$$\mathrm{K}_{\mathrm{SO}_{V_0}} = \mathrm{SO}_{V_0}(\mathcal{O}).$$

Let us also take, for $m \geq 0$ and $n \in \mathbb{Z}$,

$$T'_{m,n} = \mathbb{1}_{\mathrm{K}_{\mathrm{GL}_2} \varpi^{(m+n,n)} \mathrm{K}_{\mathrm{GL}_2}}.$$

It is apparent by the Cartan decomposition that the collection of $T'_{m,n}$ linearly spans $\mathscr{H}_{\mathrm{GL}_2}$.

If we similarly define $T_m \in \mathcal{H}_{\mathrm{SO}_{V_0}}$ by

$$T_m = \mathbb{1}_{\mathrm{K}_{\mathrm{SO}_{V_0}} \, \varpi^{m\lambda_0} \, \mathrm{K}_{\mathrm{SO}_{V_0}}},$$

where

$$\varpi^{m\lambda_0} = \rho\left(\frac{\varpi^m + 1}{2} + \frac{\varpi^m - 1}{2} \frac{d_W}{d_V} se \right),$$

then, upon noticing that the cocharacter

$$\lambda_0 : t \mapsto \rho\left(\frac{t+1}{2} + \frac{t-1}{2} \frac{d_W}{d_V} se \right)$$

generates $X_*(\mathrm{T}_{\mathrm{SO}_{V_0}})$, it is again clear that the collection of T_m linearly span $\mathcal{H}_{\mathrm{SO}_{V_0}}$.

Now that we have established some notation, we can describe the map St on Hecke algebras.

Lemma 3.8. *The morphism* St *breaks up according to the two factors in* $\mathcal{H}' = \mathcal{H}_{\mathrm{GL}_2} \otimes \mathcal{H}_{\mathrm{GL}_2}$ *as follows*:

(1) *On the first factor, the map*

$$\mathcal{H}_{\mathrm{GL}_2} \to \mathcal{H}_{\mathrm{SO}_{W_0}} = \mathbb{C}$$

is given by

$$T'_{m,n} \mapsto \begin{cases} (-1)^n & \text{if } m = 0, \\ (-1)^n (q^{\frac{m}{2}} + q^{\frac{m-2}{2}}) & \text{if } m \neq 0 \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

(2) *On the second factor, the map*

$$\mathcal{H}_{\mathrm{GL}_2} \to \mathcal{H}_{\mathrm{SO}_{V_0}}$$

is given by

$$T'_{m,n} \mapsto T_m.$$

Proof. The first claim (1) is more or less straightforward. Since $\mathrm{SO}_{W_0}(F)$ is compact, $\mathcal{H}_{\mathrm{SO}_{W_0}} = \mathbb{C}$. The map $\mathrm{St} : \mathcal{H}_{\mathrm{GL}_2} \to \mathcal{H}_{\mathrm{SO}_{W_0}}$ is defined by

$$\mathcal{H}_{\mathrm{GL}_2} \xrightarrow{\mathrm{Sat}_{\mathrm{GL}_2}} \mathcal{H}_{\mathrm{T}_{\mathrm{GL}_2}} \to \mathcal{H}_{\mathrm{SO}_{W_0}} = \mathbb{C}.$$

The first map is the Satake isomorphism for GL_2. The second map is defined as follows: write

$$\mathcal{H}_{\mathrm{T}_{\mathrm{GL}_2}} = \mathbb{C}[X_*(\mathrm{T}_{\mathrm{GL}_2})]^W = \mathbb{C}[X_1, X_2, X_1^{-1}, X_2^{-1}]^W,$$

where $X_1 = \mathbb{1}_{\varpi^{(1,0)} \mathrm{T}_{\mathrm{GL}_2}(\mathbb{O})}$, $X_2 = \mathbb{1}_{\varpi^{(0,1)} \mathrm{T}_{\mathrm{GL}_2}(\mathbb{O})}$ and W acts by switching X_1 and X_2. If we then define

$$\mathcal{H}_{\mathrm{T}_{\mathrm{GL}_2}} = \mathbb{C}[X_1, X_2, X_1^{-1}, X_2^{-1}]^W \to \mathbb{C}[T, T^{-1}] = \mathcal{H}_{\mathrm{Res}_{E/F} \mathbb{G}_m}$$

by sending

$$X_i \mapsto \omega(\varpi)^{i-1} T^{\frac{1}{2}} = (-1)^{i-1} T^{\frac{1}{2}}$$

and restricting to W invariants, then this is exactly the transfer map $\mathcal{H}_{\mathrm{T}_{\mathrm{GL}_2}} \to \mathcal{H}_{\mathrm{Res}_{E/F} \mathbb{G}_m}$ used to define the automorphic induction of forms from $\mathrm{Res}_{E/F} \mathbb{G}_m$ to GL_2. See, as one of many possible references,

[Lafforgue 2010]. To obtain the transfer map for automorphic forms on $SO_{W_0} = \mathbb{G}_m \backslash \operatorname{Res}_{E/F} \mathbb{G}_m$ to GL_2, we simply compose this map with the map $T \mapsto 1$. Applying the Macdonald formula for GL_2 (see Lemma 3.13) gives

$$\operatorname{Sat}_{GL_2}(T'_{m,n}) = \begin{cases} (X_1 X_2)^n & \text{if } m = 0, \\ (X_1 X_2)^n q^{\frac{1}{2}}(X_1 + X_2) & \text{if } m = 1, \\ (X_1 X_2)^n \left(q^{\frac{m}{2}} \sum_{i=0}^{m} X_1^{m-i} X_2^i - q^{\frac{m-2}{2}} X_1 X_2 \sum_{i=0}^{m-2} X_1^{m-2-i} X_2^i\right) & \text{if } m \geq 2 \end{cases}$$

and claim (1) follows immediately.

The same strategy yields the second claim (2). Again, by definition, the morphism $\operatorname{St} : \mathcal{H}_{GL_2} \to \mathcal{H}_{SO_{V_0}}$ factors as follows:

$$\mathcal{H}_{GL_2} \xrightarrow{\operatorname{Sat}_{GL_2}} \mathcal{H}_{T_{GL_2}} \to \mathcal{H}_{T_{SO_{V_0}}} \xrightarrow{\operatorname{Sat}_{SO_{V_0}}^{-1}} \mathcal{H}_{SO_{V_0}},$$

where the first arrow is the Satake isomorphism on GL_2, the second arrow is induced by the obvious $X_*(T_{GL_2}) = X^*(T_{GL_2}) \to X^*(T_{SL_2}) = X_*(T_{SO_{V_0}})$, and the third arrow is the inverse Satake transform on SO_{V_0}. To see that $T'_{m,n}$ is sent to T_m, it remains to use the Macdonald formula on GL_2 and $SO_{V_0} \cong PGL_2$, and to recall that $\mathbb{1}_{\varpi^{(k,l)} T_{GL_2}(\mathcal{O})} \mapsto 1_{\varpi^{(k-l)\lambda_0} T_{SO_{V_0}}(\mathcal{O})}$. We omit the tedious details. $\square$

It will be helpful for the proof of Theorem 3.7 to be more explicit about the double cosets $K_{GL_2} \varpi^{(k,l)} K_{GL_2}$ and $K_{SO_{V_0}} \varpi^{m\lambda_0} K_{SO_{V_0}}$. The characterization for GL_2 is well known: it is the theory of elementary divisors.

Lemma 3.9. *Let $k \geq l$ be integers. We have*

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_{GL_2} \begin{pmatrix} \varpi^k & 0 \\ 0 & \varpi^l \end{pmatrix} K_{GL_2}$$

if and only if both

(1) $\operatorname{val} \det g = k + l$,

(2) $\min(\operatorname{val} a, \operatorname{val} b, \operatorname{val} c, \operatorname{val} d) = l$

hold.

On the other side we must describe $K_{SO_{V_0}}$ double cosets in SO_{V_0} in terms of the coordinates $x = z + we$.
To do this, it helps to describe these double cosets in terms of $\mathcal{O}$-lattices in V_0. Indeed, when we wrote above that

$$K_{SO_{V_0}} = SO_{V_0}(\mathcal{O}),$$

we were really abusing notation and talking about the $\mathcal{O}$-points of an integral model of our group SO_{V_0}, defined in terms of a quadratic form on a lattice. Consider the $\mathcal{O}$-lattice $\mathcal{L}_0 \subset V_0$ given by

$$\mathcal{L}_0 = \mathcal{O}s + \mathcal{O}s' + \mathcal{O}e.$$

It is easy to see that this is self-dual (under our assumption that $d_W, d_V \in \mathcal{O}^\times$), and that $\mathcal{L}_0$ gives rise to a good integral model for SO_{V_0}, i.e., $K_{SO_{V_0}} = SO(\mathcal{L}_0)$.

Just as in Lemma 3.9, we can similarly describe the double cosets $\mathrm{K}_{\mathrm{SO}_{V_0}} \varpi^{m\lambda_0} \mathrm{K}_{\mathrm{SO}_{V_0}}$.

Definition 3.10. Let $\mathscr{L}$ and $\mathscr{L}'$ be two self-dual lattices in $V_0 = (V_0, Q)$. We say that $\mathscr{L}$ and $\mathscr{L}'$ have *relative position* $\varpi^{m\lambda_0}$ (or more simply just m) if $m \geq 0$ is the smallest integer so that

$$\mathscr{L} \subseteq \varpi^{-m} \mathscr{L}'.$$

Since we can identify

$$\mathrm{SO}_{V_0}(F) / \mathrm{K}_{\mathrm{SO}_{V_0}} \xrightarrow{\sim} \{\text{self-dual lattices } \mathscr{L} \subset V_0\},$$
$$g \mapsto g\mathscr{L}_0,$$

it is immediate that

$$\mathrm{K}_{\mathrm{SO}_{V_0}} \varpi^{m\lambda_0} \mathrm{K}_{\mathrm{SO}_{V_0}} = \{g : g\mathscr{L}_0 \text{ and } \mathscr{L}_0 \text{ have relative position } m\}.$$

We now use this observation to deduce a clean description of these double cosets in terms of our coordinates $x = z + we$.

Lemma 3.11. *The inverse image* $\rho^{-1}(\mathrm{K}_{\mathrm{SO}_{V_0}} \varpi^{m\lambda_0} \mathrm{K}_{\mathrm{SO}_{V_0}})$ *in* GSpin_{V_0} *of a double coset is given by*

$$\rho^{-1}(\mathrm{K}_{\mathrm{SO}_{V_0}} \varpi^{m\lambda_0} \mathrm{K}_{\mathrm{SO}_{V_0}}) = \begin{cases} \left\{z + we : \mathrm{val}\left(\dfrac{\mathrm{N}\,z - Q(e)Q(w)}{\mathrm{N}(x)}\right) \geq 0\right\} & \text{if } m = 0, \\[4mm] \left\{z + we : \mathrm{val}\left(\dfrac{\mathrm{N}\,z - Q(e)Q(w)}{\mathrm{N}(x)}\right) = -m\right\} & \text{if } m > 0. \end{cases}$$

Proof. For notational convenience, let's define

$$K_m := \rho^{-1}(\mathrm{K}_{\mathrm{SO}_{V_0}} \varpi^{m\lambda_0} \mathrm{K}_{\mathrm{SO}_{V_0}}) = \{x = z + we : \rho(x)\mathscr{L}_0 \text{ has relative position } m \text{ with respect to } \mathscr{L}_0\}$$

and

$$K'_m := \begin{cases} \left\{z + we : \mathrm{val}\left(\dfrac{\mathrm{N}\,z - Q(e)Q(w)}{\mathrm{N}(x)}\right) \geq 0\right\} & \text{if } m = 0, \\[4mm] \left\{z + we : \mathrm{val}\left(\dfrac{\mathrm{N}\,z - Q(e)Q(w)}{\mathrm{N}(x)}\right) = -m\right\} & \text{if } m > 0. \end{cases}$$

We wish to show that $K_m = K'_m$. We expand out

$$\rho(x)(e) = xex^{-1} = \left(\frac{1}{\mathrm{N}\,x} z e \bar{z} - \frac{Q(e)Q(w)}{\mathrm{N}\,x} e\right) + 2\frac{Q(e)}{\mathrm{N}\,x} zw$$
$$= \frac{\mathrm{N}\,z - Q(e)Q(w)}{\mathrm{N}\,x} e + 2\frac{Q(e)}{\mathrm{N}\,x} zw$$

and for $w' \in W$

$$\rho(x)(w') = xw'x^{-1} = \frac{1}{\mathrm{N}\,x}((zw'\bar{z} - Q(e)ww'w) - (zw'w + ww'\bar{z})e)$$
$$= \left(\frac{\mathrm{N}\,z}{\mathrm{N}\,x}\rho(z)(w') + \frac{Q(e)Q(w)}{\mathrm{N}\,x}w' - 2\frac{(w', w)}{\mathrm{N}\,x}w\right) - 2\frac{(w', \bar{z}w)}{\mathrm{N}\,x}e.$$

Observe too that since

$$\mathrm{val}\left(\frac{\mathrm{N}\,z + Q(e)Q(w)}{\mathrm{N}\,x}\right) = \mathrm{val}(1) = 0,$$

we have:

(1) If $m = 0$, then

$$\frac{\mathrm{N}\,z - Q(e)Q(w)}{\mathrm{N}\,x} \in \mathscr{O}$$

if and only if both $\mathrm{N}\,z/(\mathrm{N}\,x)$ and $Q(e)Q(w)/(\mathrm{N}\,x)$ lie in $\mathscr{O}$.

(2) If $m > 0$, then

$$\mathrm{val}\left(\frac{\mathrm{N}\,z - Q(e)Q(w)}{\mathrm{N}\,x}\right) = -m$$

if and only if

$$\mathrm{val}\left(\frac{\mathrm{N}\,z}{\mathrm{N}\,x}\right) = \mathrm{val}\left(\frac{Q(e)Q(w)}{\mathrm{N}\,x}\right) = -m.$$

The claim now follows quickly. Looking at the e-component of $\rho(x)e$ shows that if $x \in K'_m$, then $\rho(x)\mathscr{L}$ has relative position at least m with respect to $\mathscr{L}_0$. Now we examine the sizes of each of the terms

$$2\frac{Q(e)}{\mathrm{N}\,x}zw, \quad \frac{\mathrm{N}\,z}{\mathrm{N}(x)}\rho(z)(w'), \quad \frac{Q(e)Q(w)}{\mathrm{N}(x)}w', \quad 2\frac{(w',w)}{\mathrm{N}\,x}w, \quad 2\frac{(zw',w)}{\mathrm{N}\,x}e$$

appearing in the calculations of $\rho(x)e$ and $\rho(x)w'$ above; for example

$$\left|Q\left(2\frac{Q(e)}{\mathrm{N}\,x}zw\right)\right| = \left|\frac{Q(e)^2 Q(w)\,\mathrm{N}\,z}{(\mathrm{N}\,x)^2}\right| = \left|\frac{Q(e)Q(w)}{\mathrm{N}\,x}\frac{\mathrm{N}\,z}{\mathrm{N}\,x}\right|$$

(we have assumed $|Q(e)| = 1$) and use our above observation to conclude that, when $w' \in \mathscr{L}_0 \cap W$ each term lives in $\varpi^{-m}\mathscr{L}_0$. Thus $K'_m \subseteq K_m$. In fact, the above remarks also show the reverse inclusion $K_m \subseteq K'_m$. We are done.   □

We will require one last lemma before giving the proof of [Theorem 3.7]. It will help with bookkeeping. Before stating it, we make the following easy definition.

Definition 3.12. Let $f \in \mathscr{H}_{\mathrm{GL}_2}$ be a spherical Hecke function. We define $\mathscr{S}f(a,x)$, the *Fourier–Satake transform* of f, by

$$\mathscr{S}f(a,x) := \int_F f(an(y))\psi(-xy)dy.$$

In the above, $a \in \mathrm{T}_{\mathrm{GL}_2}$ and $x \in F$.

It is worth noting that $\mathrm{Sat}_{\mathrm{GL}_2} f(a) = |\delta(a)|^{\frac{1}{2}}\mathscr{S}f(a,0)$, which should offer some justification for our choice of terminology.

One should also note that for each fixed x, $\mathscr{S}f(\cdot,x) \in C_c^{\infty}(\mathrm{T}_{\mathrm{GL}_2}(\mathscr{O})\backslash \mathrm{T}_{\mathrm{GL}_2}(F))$, but it may not be invariant under the Weyl group unless $x = 0$. For each fixed a, $\mathscr{S}f(a,a) \in C_c^{\infty}(F)$.

The following lemma is, for $x = 0$, nothing more or less than a rederivation of the MacDonald formula for GL_2. For general x, we did not know a reference for the result, although we feel it (and relatives for higher-rank groups) must be well-known.

Lemma 3.13 (the MacDonald formula for GL_2). *Suppose $f = T'_{m,0}$ with $m \geq 0$. Then*

$$\mathscr{S}f(a,x) = \begin{cases} q^k \mathbb{1}_{\varpi^k \mathcal{O}}(x) & \text{if } a \in \left(\begin{smallmatrix} \varpi^k & 0 \\ 0 & \varpi^{m-k} \end{smallmatrix}\right) T_{GL_2}(\mathcal{O}), \, k = 0, m, \\ q^k \mathbb{1}_{\varpi^k \mathcal{O}}(x) - q^{k-1} \mathbb{1}_{\varpi^{k-1} \mathcal{O}}(x) & \text{if } a \in \left(\begin{smallmatrix} \varpi^k & 0 \\ 0 & \varpi^{m-k} \end{smallmatrix}\right) T_{GL_2}(\mathcal{O}), \, 0 < k < m, \\ 0 & \text{else.} \end{cases}$$

Proof. As a function of a, $\mathscr{S}f$ clearly depends only on the $T_{GL_2}(\mathcal{O})$ coset of a. Thus, if

$$a \in \begin{pmatrix} \varpi^k & 0 \\ 0 & \varpi^l \end{pmatrix} T_{GL_2}(\mathcal{O}),$$

then

$$\mathscr{S}f(a,x) = \int_F T'_{m,0}\left(\begin{pmatrix} \varpi^k & \varpi^k y \\ 0 & \varpi^l \end{pmatrix}\right) \psi(-xy)\, dy.$$

Lemma 3.9 implies that $l = m - k$ and that this integral runs over y such that $\min(k, k + \mathrm{val}(y), m - k) = 0$. That is, if $k = 0$ or m, then the integration takes place over $y \in \varpi^{-k}\mathcal{O}$, while if $0 < k < m$, then the integration is over $y \in \varpi^{-k}\mathcal{O} - \varpi^{1-k}\mathcal{O}$. Thus,

$$\mathscr{S}f(a,x) = \begin{cases} \int_F \mathbb{1}_{\varpi^{-k}\mathcal{O}}(y)\psi(-xy)\, dy & \text{if } k = 0, m, \\ \int_F \mathbb{1}_{\varpi^{-k}\mathcal{O}}(y) - \mathbb{1}_{\varpi^{1-k}\mathcal{O}}(y)\psi(-xy)\, dy & \text{if } 0 < k < m, \end{cases}$$

$$= \begin{cases} q^k \mathbb{1}_{\varpi^k \mathcal{O}}(x) & \text{if } k = 0, m, \\ q^k \mathbb{1}_{\varpi^k \mathcal{O}}(x) - q^{k-1} \mathbb{1}_{\varpi^{k-1}\mathcal{O}}(x) & \text{if } 0 < k < m, \end{cases}$$

as desired. $\qquad\square$

We can now show Theorem 3.7.

Proof. It suffices to show the result for $f' = f'_1 \otimes f'_2 \in \mathcal{H}'$ with $f'_i = T'_{m_i,0}$ and $\Phi^\circ = \mathbb{1}_{\mathcal{O}^{\oplus 2}}$ and $\Psi^\circ = \mathbb{1}_{\mathcal{O}_E}$. Thus, we compute the local tonal integral explicitly. For clarity we abbreviate our notation slightly, writing the spherical Whittaker function as

$$\mathscr{W}^\circ := \mathscr{W}_{F_{1/2}^\circ}^{\psi^{-1}}$$

and writing the Weil representation as

$$R := R^\psi.$$

To help with readability, we break the computation in several steps.

<u>Step 1</u>: relate the tonal integral to the Fourier–Satake transforms of f'_1 and f'_2.

Consider the local tonal integral

$$I_{\gamma'}\left(\tfrac{1}{2}; f' \otimes \Phi \otimes \Psi\right) := |\alpha|^{\frac{1}{2}} \int f'_1(g^{-1}h) f'_2(g^{-1}a(\alpha)^{-1}) \mathscr{W}^\circ(a(\beta)g) R(h)\overline{\Psi}(\zeta)$$

or simply just $I_{\gamma'}$. We can analyze this using the Iwasawa decomposition. Write

$$g = n(x_1)t_1k_1,$$
$$h = n(x_2)t_2k_2,$$

where $t_1 \in T_{\mathrm{GL}_2}$, $t_2 \in T_{\mathrm{GL}_2^{(2)}}$, $x_i \in F$, and $k_1 \in \mathrm{GL}_2(\mathscr{O})$, $k_2 \in \mathrm{GL}_2^{(2)}(\mathscr{O})$, with notation as laid out in the [Introduction](). Then the tonal integral breaks into

$$I_{\gamma'} = |\alpha|^{\frac{1}{2}} \int f_1'(t_1^{-1}n(x_2-x_1)t_2) f_2'(t_1^{-1}n(-x_1)a(\alpha)^{-1}) \mathscr{W}^\circ(a(\beta)n(x_1)t_1)\boldsymbol{R}(n(x_2)t_2)\Psi^\circ(\zeta)|\delta(t_1t_2)|^{-1}.$$

We simplify this by first changing variables $x_2 \mapsto x_2 + x_1$, followed by $x_1 \mapsto -x_1/\alpha$ and $x_2 \mapsto \delta(t_2)x_2$. This gives

$$I_{\gamma'} = |\alpha|^{-\frac{1}{2}} \int f_1'(t_1^{-1}t_2n(x_2))\psi(\mathrm{N}\zeta\delta(t_2)x_2) f_2'(t_1^{-1}a(\alpha)^{-1}n(x_1))\psi\left(-\frac{\beta+\mathrm{N}\zeta}{\alpha}x_1\right)$$
$$\times \mathscr{W}^\circ(a(\beta)t_1)\boldsymbol{R}(t_2)\Psi^\circ(\zeta)|\delta(t_1)|^{-1}$$

and we can recognize the integrals over x_1 and x_2 as Fourier–Satake transforms. Thus

$$I_{\gamma'} = |\alpha|^{-\frac{1}{2}} \int \mathscr{S}f_1'(t_1^{-1}t_2, -\mathrm{N}\zeta\delta(t_2))\mathscr{S}f_2'(t_1^{-1}a(\alpha)^{-1}, 1)\mathscr{W}^\circ(a(\beta)t_1)\boldsymbol{R}(t_2)\Psi^\circ(\zeta)|\delta(t_1)|^{-1}.$$

<u>Step 2</u>: rewrite the tonal integral as a finite sum.

If we now write $t_1 = z(z_1)^{-1}a(a_1)^{-1}$ and $t_2 = z(z_2)d(a_2)$, then after a change of variables $z_2 \mapsto z_1^{-1}z_2$ we get

$$I_{\gamma'} = |\alpha|^{-\frac{1}{2}} \int \mathscr{S}f_1'\big(z(z_2)a(a_1)d(a_2), -\mathrm{N}(a_2\zeta)\big)\mathscr{S}f_2'(z(z_1)a(a_1)a(\alpha)^{-1}, 1)$$
$$\times \omega(z_2a_2)|a_1a_2|\mathscr{W}^\circ(a(\beta a_1^{-1}))\Psi^\circ(a_2\zeta). \quad (3\text{-}1)$$

The computation of $I_{\gamma'}$ now reduces to evaluating a finite sum. Write $a_i = \varpi^{k_i}u_i$ and $z_i = \varpi^{l_i}v_i$, where $u_i, v_i \in \mathscr{O}^\times$ are units. Then

$$I_{\gamma'} = |\alpha|^{-\frac{1}{2}} \sum \mathscr{S}f_1'\left(\begin{pmatrix} \varpi^{l_2+k_1+k_2} & 0 \\ 0 & \varpi^{l_2-k_2} \end{pmatrix}, \mathrm{N}(\varpi^{k_2}\zeta)\right)\mathscr{S}f_2'\left(\begin{pmatrix} \varpi^{l_1+k_1-\mathrm{val}\,\alpha} & 0 \\ 0 & \varpi^{l_1} \end{pmatrix}, 1\right)$$
$$\times (-1)^{l_2+k_2}q^{-k_1-k_2}\mathscr{W}^\circ(a(\beta\varpi^{-k_1}))\Psi^\circ(\varpi^{k_2}\zeta),$$

where there are a number of restrictions in the sum, namely

- the determinant conditions

$$m_1 = 2l_2 + k_1, \quad m_2 = 2l_1 + k_1 - \mathrm{val}\,\alpha,$$

  and

- the support conditions: if we set $r_1 = l_2 + k_1 + k_2$ (hence $m_1 - r_1 = l_2 - k_2$) and $r_2 = l_1 + k_1 - \mathrm{val}\,\alpha$ (hence $m_2 - r_1 = l_1$), then

$$0 \le r_1 \le m_1, \quad 0 \le r_2 \le m_2.$$

<u>Step 3</u>: change variables in the sum.

We can write the k_i and l_i in terms of the r_i and the fixed m_i; this yields

$$l_1 = m_2 - r_2,$$
$$l_2 = \tfrac{1}{2}(m_2 - m_1 - \operatorname{val}\alpha) + m_1 - r_2,$$
$$k_1 = 2r_2 - m_2 + \operatorname{val}\alpha,$$
$$k_2 = \tfrac{1}{2}(m_2 - m_1 - \operatorname{val}\alpha) + r_1 - r_2$$

(note that $T = T(m_1, m_2, \alpha) := \tfrac{1}{2}(m_2 - m_1 - \operatorname{val}\alpha)$ must be an integer, otherwise $I_{\gamma'}$ is zero for parity reasons) and so

$$I_{\gamma'} = \sum_{r_1=0}^{m_1} \sum_{r_2=0}^{m_2} \mathscr{S} f_1' \left(\begin{pmatrix} \varpi^{r_1} & 0 \\ 0 & \varpi^{m_1-r_1} \end{pmatrix}, \mathrm{N}(\varpi^{T+r_1-r_2}\zeta) \right) \mathscr{S} f_2' \left(\begin{pmatrix} \varpi^{r_2} & 0 \\ 0 & \varpi^{m_2-r_2} \end{pmatrix}, 1 \right)$$

$$\times (-1)^{m_1-r_1} q^{\frac{m_1}{2}-r_1+\frac{m_2}{2}-r_2} \mathscr{W}^\circ(a(\beta\varpi^{m_2-2r_2-\operatorname{val}\alpha})) \Psi^\circ(\varpi^{T+r_1-r_2}\zeta)$$

or, better,

$$I_{\gamma'} = \sum_{r_2=0}^{m_2} q^{\frac{m_2}{2}-r_2} \mathscr{S} f_2' \left(\begin{pmatrix} \varpi^{r_2} & 0 \\ 0 & \varpi^{m_2-r_2} \end{pmatrix}, 1 \right) \mathscr{W}^\circ(a(\beta\varpi^{m_2-2r_2-\operatorname{val}\alpha}))$$

$$\times \sum_{r_1=0}^{m_1} (-1)^{m_1-r_1} q^{\frac{m_1}{2}-r_1} \mathscr{S} f_1' \left(\begin{pmatrix} \varpi^{r_1} & 0 \\ 0 & \varpi^{m_1-r_1} \end{pmatrix}, \mathrm{N}(\varpi^{T+r_1-r_2}\zeta) \right) \Psi^\circ(\varpi^{T+r_1-r_2}\zeta).$$

<u>Step 4</u>: exploit cancellation.

Let us first examine the inner sum over r_1. This is

$$(-1)^{m_1} q^{\frac{m_1}{2}} \sum_{r_1=0}^{m_1} (-1)^{r_1} q^{-r_1} \mathscr{S} f_1' \left(\begin{pmatrix} \varpi^{r_1} & 0 \\ 0 & \varpi^{m_1-r_1} \end{pmatrix}, \mathrm{N}(\varpi^{T+r_1-r_2}\zeta) \right) \Psi^\circ(\varpi^{T+r_1-r_2}\zeta).$$

If we apply Lemma 3.13, this yields

$$(-1)^{m_1} q^{\frac{m_1}{2}} \Bigg(\mathbb{1}_{\mathscr{O}}(\mathrm{N}(\varpi^{T-r_2}\zeta)) \mathbb{1}_{\mathscr{O}_E}(\varpi^{T-r_2}\zeta)$$

$$+ \sum_{r_1=1}^{m_1-1} (-1)^{r_1} \big(\mathbb{1}_{\varpi^{r_1}\mathscr{O}}(\mathrm{N}(\varpi^{T+r_1-r_2}\zeta)) - q^{-1} \mathbb{1}_{\varpi^{r_1-1}\mathscr{O}}(\mathrm{N}(\varpi^{T+r_1-r_2}\zeta)) \big) \mathbb{1}_{\mathscr{O}_E}(\varpi^{T+r_1-r_2}\zeta)$$

$$+ (-1)^{m_1} \mathbb{1}_{\varpi^{m_1}\mathscr{O}}(\mathrm{N}(\varpi^{T+m_1-r_2}\zeta)) \mathbb{1}_{\mathscr{O}_E}(\varpi^{T+m_1-r_2}\zeta) \Bigg).$$

Observe that

$$\mathbb{1}_{\varpi^{r_1}\mathscr{O}}(\mathrm{N}(\varpi^{T+r_1-r_2}\zeta)) = \mathbb{1}_{\varpi^{r_2-T-\lfloor r_1/2 \rfloor}\mathscr{O}_E}(\zeta),$$
$$\mathbb{1}_{\varpi^{r_1-1}\mathscr{O}}(\mathrm{N}(\varpi^{T+r_1-r_2}\zeta)) = \mathbb{1}_{\varpi^{r_2-T-\lceil r_1/2 \rceil}\mathscr{O}_E}(\zeta),$$
$$\mathbb{1}_{\mathscr{O}_E}(\varpi^{T+r_1-r_2}\zeta) = \mathbb{1}_{\varpi^{r_2-r_1-T}\mathscr{O}_E}(\zeta).$$

Thus, this inner sum becomes

$$(-1)^{m_1} q^{\frac{m_1}{2}} \left(\mathbb{1}_{\varpi^{r_2-T} \mathcal{O}_E}(\zeta) + \sum_{r_1=1}^{m_1-1} (-1)^{r_1} \left(\mathbb{1}_{\varpi^{r_2-T-\lfloor r_1/2 \rfloor} \mathcal{O}_E}(\zeta) - q^{-1} \mathbb{1}_{\varpi^{r_2-T-\lceil r_1/2 \rceil} \mathcal{O}_E}(\zeta) \right) \right.$$
$$\left. + (-1)^{m_1} \mathbb{1}_{\varpi^{r_2-T-\lfloor m_1/2 \rfloor} \mathcal{O}_E}(\zeta) \right),$$

which, if m_1 is odd, telescopes to 0. If m_1 is even, then it simplifies dramatically to

$$(q^{\frac{m_1}{2}} + q^{\frac{m_1-2}{2}}) \mathbb{1}_{\varpi^{r_2-T-m_1/2} \mathcal{O}_E}(\zeta) = (q^{\frac{m_1}{2}} + q^{\frac{m_1-2}{2}}) \mathbb{1}_{\varpi^{(\mathrm{val}\,\alpha - m_2)/2 + r_2} \mathcal{O}_E}(\zeta).$$

<u>Step 5</u>: apply the Casselman–Shalika formula and simplify.

Returning to $I_{\gamma'}$, we have shown that, if m_1 is even, then

$$I_{\gamma'} = (q^{\frac{m_1}{2}} + q^{\frac{m_1-2}{2}}) \sum_{r_2=0}^{m_2} q^{\frac{m_2}{2}-r_2} \mathscr{S} f_2' \left(\begin{pmatrix} \varpi^{r_2} & 0 \\ 0 & \varpi^{m_2-r_2} \end{pmatrix}, 1 \right)$$
$$\times \mathscr{W}^\circ(a(\beta \varpi^{m_2-2r_2-\mathrm{val}\,\alpha})) \mathbb{1}_{\varpi^{(\mathrm{val}\,\alpha - m_2)/2 + r_2} \mathcal{O}_E}(\zeta).$$

Now, if we denote by $\mathcal{O}^{\mathrm{even}}$ the subset of elements of $\mathcal{O}$ with even valuation, then the Casselman–Shalika formula gives

$$\mathscr{W}^\circ(a(t)) = |t|^{\frac{1}{2}} \mathbb{1}_{\mathcal{O}^{\mathrm{even}}}(t).$$

If we apply this together with [Lemma 3.13](), in total we find

$$I_{\gamma'} = \left| \frac{\beta}{\alpha} \right|^{\frac{1}{2}} (q^{\frac{m_1}{2}} + q^{\frac{m_1-2}{2}})$$
$$\times \left(\mathbb{1}_{\mathcal{O}}(1) \mathbb{1}_{\varpi^{\mathrm{val}\,\alpha - m_2} \mathcal{O}^{\mathrm{even}}}(\beta) \mathbb{1}_{\varpi^{(\mathrm{val}\,\alpha - m_2)/2} \mathcal{O}_E}(\zeta) \right.$$
$$+ \sum_{r_2=1}^{m_2-1} (q^{r_2} \mathbb{1}_{\varpi^{r_2} \mathcal{O}}(1) - q^{r_2-1} \mathbb{1}_{\varpi^{r_2-1} \mathcal{O}}(1)) \mathbb{1}_{\varpi^{\mathrm{val}\,\alpha - m_2 + 2r_2} \mathcal{O}^{\mathrm{even}}}(\beta) \mathbb{1}_{\varpi^{(\mathrm{val}\,\alpha - m_2)/2 + r_2} \mathcal{O}_E}(\zeta)$$
$$\left. + q^{m_2} \mathbb{1}_{\varpi^{m_2} \mathcal{O}}(1) \mathbb{1}_{\varpi^{\mathrm{val}\,\alpha + m_2} \mathcal{O}^{\mathrm{even}}}(\beta) \mathbb{1}_{\varpi^{(\mathrm{val}\,\alpha + m_2)/2} \mathcal{O}_E}(\zeta) \right),$$

which clearly simplifies to

$$I_{\gamma'} = \left| \frac{\beta}{\alpha} \right|^{\frac{1}{2}} (q^{\frac{m_1}{2}} + q^{\frac{m_1-2}{2}}) \left(\mathbb{1}_{\varpi^{\mathrm{val}\,\alpha - m_2} \mathcal{O}^{\mathrm{even}}}(\beta) \mathbb{1}_{\varpi^{(\mathrm{val}\,\alpha - m_2)/2} \mathcal{O}_E}(\zeta) \right.$$
$$\left. - \mathbb{1}_{\varpi^{\mathrm{val}\,\alpha - m_2 + 2} \mathcal{O}^{\mathrm{even}}}(\beta) \mathbb{1}_{\varpi^{(\mathrm{val}\,\alpha - m_2)/2 + 1} \mathcal{O}_E}(\zeta) \right).$$

Applying [Lemma 3.11]() and noting that $\mathrm{SO}_{W_0}(F) \subset \mathrm{K}_{\mathrm{SO}_{V_0}}$ allows us to conclude. $\qquad\square$

Remark 3.14. If one is only interested in the fundamental lemma for the unit element of the Hecke algebra, then the computation above greatly simplifies. We encourage any reader unhappy with the ugly manipulations above to work out this example, which becomes nearly trivial.

3C. *The fundamental lemma: E/F split.* In this subsection, we again assume F is a non-Archimedean local field, but now ask that $E \cong F \times F$ is split, i.e., that $d_W \in (F^\times)^2$. There is now only one relevant

pure inner form corresponding to the pair of discriminants (d_W, d_V); we denote it simply by (W, V) and the corresponding product of split special orthogonal groups by $G = SO_W \times SO_V$. We have

$$G \cong \mathbb{G}_m \times PGL_2$$

since

$$(W, Q) \cong \left(F \times F, -\frac{d_V}{d_W} N \right),$$

where $N : (x, y) \mapsto xy$ is visibly split.

Because SO_W is now a split torus of SO_V, in this subsection we now use a different identification of $C_0(V)$ with $Mat_{2\times 2}$ than previously. Rather than (1-1), we choose the isomorphism

$$z + we = (a, d) + \iota(b, c)e \mapsto \begin{pmatrix} a & -\frac{d_V}{d_W} b \\ \frac{d_V}{d_W} c & d \end{pmatrix}. \tag{3-2}$$

In the above, $(a, d), (b, c) \in E = F \times F$. The choice of identification (3-2) will simplify expressions quite significantly.

Write $G' = GL_2 \times GL_2$ and denote the corresponding spherical Hecke algebras for G and G' by

$$\mathscr{H} = \mathscr{H}(G, K),$$
$$\mathscr{H}' = \mathscr{H}(G', K').$$

As before, there is a standard map of Hecke algebras

$$St : \mathscr{H}' \to \mathscr{H};$$

in this subsection, we will show that in this split case, this again respects formation of orbital and tonal integrals.

Theorem 3.15 (the fundamental lemma, E/F split). *Let $f' \in \mathscr{H}'$ lie in the spherical Hecke algebra for G', and set $\Phi^\circ = \mathbb{1}_{\mathscr{O}\oplus 2}$ and $\Psi^\circ = \mathbb{1}_{\mathscr{O}_E}$. Then the functions $f' \otimes \Phi^\circ \otimes \Psi^\circ$ and $St(f')$ match each other, in the sense that*

$$I_{\gamma'}\left(\tfrac{1}{2}; f' \otimes \Phi^\circ \otimes \Psi^\circ\right) = J_\gamma(St(f'))$$

for all $\gamma' \leftrightarrow \gamma$ matching regular semisimple tones and orbits.

Once again, it is helpful to explicate the transfer map St on Hecke algebras. We denote by

$$T'_{m,n} := \mathbb{1}_{K_{GL_2} \, \varpi^{(m+n,m)} \, K_{GL_2}},$$
$$S_k := \mathbb{1}_{\varpi^{k\mu_0} K_{SO_W}},$$
$$T_m := \mathbb{1}_{K_{SO_V} \, \varpi^{m\mu_0} K_{SO_V}}$$

the obvious generators of $\mathscr{H}_{GL_2}$, $\mathscr{H}_{SO_W}$, and $\mathscr{H}_{SO_V}$ respectively. In the above, if we denote by $i = (l, -l)$ a fixed element in $E \cong F \times F$ with l in F satisfying $l^2 = d_W$, then

$$\mu_0 : \mathbb{G}_m \to \mathrm{SO}_W,$$

$$t \mapsto \rho\left(\frac{t+1}{2}l + \frac{t-1}{2}i\right) = \rho\left(\frac{t+1}{2} + \frac{t-1}{2}(1,-1)\right),$$

is a generator of both $X_*(\mathrm{SO}_W)$ and $X_*(\mathrm{T}_{\mathrm{SO}_V})$.

Lemma 3.16. *The morphism* St *breaks up according to the two factors in* $\mathcal{H}' = \mathcal{H}_{\mathrm{GL}_2} \otimes \mathcal{H}_{\mathrm{GL}_2}$ *as follows*:

(1) *On the first factor*, *the map*

$$\mathcal{H}_{\mathrm{GL}_2} \to \mathcal{H}_{\mathrm{SO}_W}$$

is given by

$$T'_{m,n} \mapsto \begin{cases} S_0 & \text{if } m = 0, \\ q^{\frac{1}{2}}(S_1 + S_{-1}) & \text{if } m = 1, \\ q^{\frac{m}{2}} \sum_{i=0}^{m} S_{m-2i} - q^{\frac{m-2}{2}} \sum_{i=0}^{m-2} S_{m-2-2i} & \text{if } m \geq 1. \end{cases}$$

(2) *On the second factor*, *the map*

$$\mathcal{H}_{\mathrm{GL}_2} \to \mathcal{H}_{\mathrm{SO}_V}$$

is given by

$$T'_{m,n} \mapsto T_m.$$

Proof. Part (1) follows directly from the Macdonald formula for GL_2. St is given as the composition

$$\mathcal{H}_{\mathrm{GL}_2} \xrightarrow{\mathrm{Sat}_{\mathrm{GL}_2}} \mathcal{H}_{\mathrm{T}_{\mathrm{GL}_2}} \to \mathcal{H}_{\mathrm{SO}_W}.$$

If we write

$$\mathcal{H}_{\mathrm{T}_{\mathrm{GL}_2}} = \mathbb{C}[X_*(\mathrm{T}_{\mathrm{GL}_2})]^W = \mathbb{C}[X_1, X_2, X_1^{-1}, X_2^{-1}]^W,$$

with $X_1 = \mathbb{1}_{\varpi^{(1,0)} \mathrm{T}_{\mathrm{GL}_2}(\mathbb{O})}$, $X_2 = \mathbb{1}_{\varpi^{(0,1)} \mathrm{T}_{\mathrm{GL}_2}(\mathbb{O})}$, then this second map above is simply

$$\mathcal{H}_{\mathrm{T}_{\mathrm{GL}_2}} = \mathbb{C}[X_1, X_2, X_1^{-1}, X_2^{-1}]^W \to \mathbb{C}[S, S^{-1}] = \mathcal{H}_{\mathrm{SO}_W},$$

which sends X_1 to S and X_2 to S^{-1}. Applying the Macdonald formula for GL_2 gives

$$\mathrm{St}(T'_{m,n}) = \begin{cases} 1 & \text{if } m = 0, \\ q^{\frac{1}{2}}(S + S^{-1}) & \text{if } m = 1, \\ q^{\frac{m}{2}} \sum_{i=0}^{m} S^{m-2i} - q^{\frac{m-2}{2}} \sum_{i=0}^{m-2} S^{m-2-2i} & \text{if } m \geq 2. \end{cases}$$

For the second claim (2), see Lemma 3.8. □

This information in hand, we can now show Theorem 3.15. The calculation of tonal integrals is similar to the one given in the proof of Theorem 3.7, so we will be a bit terse. However, unlike in the proof of that theorem, we will not include a step-by-step breakdown of the argument. This is since the basic structure of the computation is so similar (although this is a bit more ferocious). We adopt the same notation as in Theorem 3.7.

Proof. As before, let $f' = f_1' \otimes f_2' \in \mathcal{H}'$ satisfy $f_i' = T_{m_i,0}'$, $\Phi^\circ = \mathbb{1}_{\mathcal{O}\oplus 2}$ and $\Psi^\circ = \mathbb{1}_{\mathcal{O}_E}$.

We again reduce the computation of $I_{\gamma'}$ to evaluating the expression in (3-1), which we recall states

$$I_{\gamma'} = |\alpha|^{-\frac{1}{2}} \int \mathscr{S}f_1'(z(z_2)a(a_1)d(a_2), -\mathrm{N}(a_2\zeta))\mathscr{S}f_2'(z(z_1)a(a_1)a(\alpha)^{-1}, 1)$$
$$\times \omega(z_2 a_2)|a_1 a_2|\mathscr{W}^\circ(a(\beta a_1^{-1}))\Psi^\circ(a_2\zeta).$$

Once again, this is a finite sum. Write $a_i = \varpi^{k_i}u_i$ and $z_i = \varpi^{l_i}v_i$, where $u_i, v_i \in \mathcal{O}^\times$ are units. Then, as $\omega = 1$,

$$I_{\gamma'} = |\alpha|^{-\frac{1}{2}} \sum \mathscr{S}f_1'\left(\begin{pmatrix} \varpi^{l_2+k_1+k_2} & 0 \\ 0 & \varpi^{l_2-k_2} \end{pmatrix}, \mathrm{N}(\varpi^{k_2}\zeta)\right)\mathscr{S}f_2'\left(\begin{pmatrix} \varpi^{l_1+k_1-\mathrm{val}\,\alpha} & 0 \\ 0 & \varpi^{l_1} \end{pmatrix}, 1\right)$$
$$\times \mathscr{W}^\circ(a(\beta\varpi^{-k_1}))\Psi^\circ(\varpi^{k_2}\zeta),$$

where this sum is over all k_i, l_i. Changing variables in the sum as in the proof of Theorem 3.7, we end up with, if we set $T = T(m_1, m_2, \alpha) := \frac{1}{2}(m_2 - m_1 - \mathrm{val}\,\alpha)$,

$$I_{\gamma'} = \sum_{r_1=0}^{m_1} \sum_{r_2=0}^{m_2} \mathscr{S}f_1'\left(\begin{pmatrix} \varpi^{r_1} & 0 \\ 0 & \varpi^{m_1-r_1} \end{pmatrix}, \mathrm{N}(\varpi^{T+r_1-r_2}\zeta)\right)\mathscr{S}f_2'\left(\begin{pmatrix} \varpi^{r_2} & 0 \\ 0 & \varpi^{m_2-r_2} \end{pmatrix}, 1\right)$$
$$\times q^{\frac{m_1}{2}-r_1+\frac{m_2}{2}-r_2}\mathscr{W}^\circ(a(\beta\varpi^{m_2-2r_2-\mathrm{val}\,\alpha}))\Psi^\circ(\varpi^{T+r_1-r_2}\zeta)$$

or, better written,

$$I_{\gamma'} = \sum_{r_2=0}^{m_2} q^{\frac{m_2}{2}-r_2}\mathscr{S}f_2'\left(\begin{pmatrix} \varpi^{r_2} & 0 \\ 0 & \varpi^{m_2-r_2} \end{pmatrix}, 1\right)\mathscr{W}^\circ(a(\beta\varpi^{m_2-2r_2-\mathrm{val}\,\alpha}))$$
$$\times \sum_{r_1=0}^{m_1} q^{\frac{m_1}{2}-r_1}\mathscr{S}f_1'\left(\begin{pmatrix} \varpi^{r_1} & 0 \\ 0 & \varpi^{m_1-r_1} \end{pmatrix}, \mathrm{N}(\varpi^{T+r_1-r_2}\zeta)\right)\Psi^\circ(\varpi^{T+r_1-r_2}\zeta).$$

Now we must proceed slightly differently from before. The "inner sum" in this calculation does not admit the same sort of cancellation as in Theorem 3.7, so further bookkeeping is essential.

Let us write the inner sum in the above expression as

$$A(r_2, \zeta) := q^{\frac{m_1}{2}} \sum_{r_1=0}^{m_1} q^{-r_1}\mathscr{S}f_1'\left(\begin{pmatrix} \varpi^{r_1} & 0 \\ 0 & \varpi^{m_1-r_1} \end{pmatrix}, \mathrm{N}(\varpi^{T+r_1-r_2}\zeta)\right)\Psi^\circ(\varpi^{T+r_1-r_2}\zeta).$$

The tonal integral is thus

$$I_{\gamma'} = \sum_{r_2=0}^{m_2} q^{\frac{m_2}{2}-r_2}\mathscr{S}f_2'\left(\begin{pmatrix} \varpi^{r_2} & 0 \\ 0 & \varpi^{m_2-r_2} \end{pmatrix}, 1\right)\mathscr{W}^\circ(a(\beta\varpi^{m_2-2r_2-\mathrm{val}\,\alpha}))A(r_2, \zeta)$$

and applying Lemma 3.13, this gives

$$I_{\gamma'} = \begin{cases} q^{\frac{m_2}{2}}\mathscr{W}^\circ(a(\beta\varpi^{m_2-\mathrm{val}\,\alpha}))A(0, \zeta) & \text{if } m_2 = 0, 1, \\ q^{\frac{m_2}{2}}\mathscr{W}^\circ(a(\beta\varpi^{m_2-\mathrm{val}\,\alpha}))A(0, \zeta) - q^{\frac{m_2-2}{2}}\mathscr{W}^\circ(a(\beta\varpi^{m_2-2-\mathrm{val}\,\alpha}))A(1, \zeta) & \text{if } m_2 > 1. \end{cases}$$

So let us now compute the inner sums $A(0, \zeta)$ and $A(1, \zeta)$. If we once again apply [Lemma 3.13](#), we find $A(r_2, \zeta)$ is given by

$$q^{\frac{m_1}{2}}\Bigg(\mathbb{1}_{\mathscr{O}}(\mathrm{N}(\varpi^{T-r_2}\zeta))\mathbb{1}_{\mathscr{O}_E}(\varpi^{T-r_2}\zeta)$$
$$+ \sum_{r_1=1}^{m_1-1}\big(\mathbb{1}_{\varpi^{r_1}\mathscr{O}}(\mathrm{N}(\varpi^{T+r_1-r_2}\zeta)) - q^{-1}\mathbb{1}_{\varpi^{r_1-1}\mathscr{O}}(\mathrm{N}(\varpi^{T+r_1-r_2}\zeta))\big)\mathbb{1}_{\mathscr{O}_E}(\varpi^{T+r_1-r_2}\zeta)$$
$$+ \mathbb{1}_{\varpi^{m_1}\mathscr{O}}(\mathrm{N}(\varpi^{T+m_1-r_2}\zeta))\mathbb{1}_{\mathscr{O}_E}(\varpi^{T+m_1-r_2}\zeta)\Bigg).$$

Observe that, as we are in the split case $E = F \times F$, when we write $\zeta = (\xi_1, \xi_2)$

$$\mathbb{1}_{\varpi^{r_1}\mathscr{O}}(\mathrm{N}(\varpi^{T+r_1-r_2}\zeta)) = \mathbb{1}_{\{(\xi_1,\xi_2):\mathrm{val}(\xi_1)+\mathrm{val}(\xi_2)\geq 2r_2-r_1-2T\}}(\zeta),$$
$$\mathbb{1}_{\varpi^{r_1-1}\mathscr{O}}(\mathrm{N}(\varpi^{T+r_1-r_2}\zeta)) = \mathbb{1}_{\{(\xi_1,\xi_2):\mathrm{val}(\xi_1)+\mathrm{val}(\xi_2)\geq 2r_2-r_1-2T-1\}}(\zeta),$$
$$\mathbb{1}_{\mathscr{O}_E}(\varpi^{T+r_1-r_2}\zeta) = \mathbb{1}_{\{(\xi_1,\xi_2):\mathrm{val}(\xi_i)\geq r_2-r_1-T\}}(\zeta).$$

Thus, $A(r_2, \zeta)$, $0 \leq r_2 \leq 1$, becomes

$$q^{\frac{m_1}{2}}\Bigg(\mathbb{1}_{\{\mathrm{val}(\xi_1)+\mathrm{val}(\xi_2)\geq 2r_2-2T\}}(\zeta)\mathbb{1}_{\{\mathrm{val}(\xi_i)\geq r_2-T\}}(\zeta)$$
$$+ \sum_{r_1=1}^{m_1-1}\big(\mathbb{1}_{\{\mathrm{val}(\xi_1)+\mathrm{val}(\xi_2)\geq 2r_2-r_1-2T\}}(\zeta) - q^{-1}\mathbb{1}_{\{\mathrm{val}(\xi_1)+\mathrm{val}(\xi_2)\geq 2r_2-r_1-2T-1\}}(\zeta)\big)\mathbb{1}_{\{\mathrm{val}(\xi_i)\geq r_2-r_1-T\}}(\zeta)$$
$$+ \mathbb{1}_{\{\mathrm{val}(\xi_1)+\mathrm{val}(\xi_2)\geq 2r_2-m_1-2T\}}(\zeta)\mathbb{1}_{\{(\xi_1,\xi_2):\mathrm{val}(\xi_i)\geq r_2-m_1-T\}}(\zeta)\Bigg).$$

We can write this compactly. Define, for $0 \leq i \leq m_1$,

$$B_i(r_2, \zeta) := \mathbb{1}_{\{\mathrm{val}(\xi_1)+\mathrm{val}(\xi_2)\geq 2r_2-i-2T\}}(\zeta)\mathbb{1}_{\{\mathrm{val}(\xi_i)\geq r_2-i-T\}}(\zeta).$$

We can rewrite our expression as

$$A(r_2, \zeta) = q^{\frac{m_1}{2}}\Bigg(B_0(r_2, \zeta) + \sum_{r_1=1}^{m_1-1}\big(B_{r_1}(r_2, \zeta) - q^{-1}B_{r_1-1}(r_2, \varpi\zeta)\big) + B_{m_1}(r_2, \zeta)\Bigg)$$
$$= q^{\frac{m_1}{2}}\sum_{r_1=0}^{m_1}B_{r_1}(r_2, \zeta) - q^{\frac{m_1-2}{2}}\sum_{r_1=1}^{m_1-1}B_{r_1-1}(r_2, \varpi\zeta),$$

and thus, after noting that

$$B_i(1, \zeta) = B_i(0, \varpi^{-1}\zeta),$$

we find

$$I_{\gamma'} = \begin{cases} q^{\frac{m_2}{2}}\mathscr{W}^{\circ}(a(\beta\varpi^{m_2-\mathrm{val}\,\alpha}))A(0, \zeta) & \text{if } m_2 = 0, 1, \\ q^{\frac{m_2}{2}}\mathscr{W}^{\circ}(a(\beta\varpi^{m_2-\mathrm{val}\,\alpha}))A(0, \zeta) - q^{\frac{m_2-2}{2}}\mathscr{W}^{\circ}(a(\beta\varpi^{m_2-2-\mathrm{val}\,\alpha}))A(0, \varpi^{-1}\zeta) & \text{if } m_2 > 1. \end{cases}$$

It will be also be helpful to introduce the notation

$$C_k(\zeta) := \sum_{r_1=0}^{k} B_{r_1}(0, \zeta).$$

When we combine this with the Casselman–Shalika formula, which in this split setting gives

$$\mathscr{W}^\circ(a(t)) = |t|^{\frac{1}{2}} (1 + \mathrm{val}(t)) \mathbb{1}_{\mathscr{O}}(t),$$

we find that

$$I_{\gamma'} = \left| \frac{\beta}{\alpha} \right|^{\frac{1}{2}} \left(1 + m_2 + \mathrm{val}\left(\frac{\beta}{\alpha}\right) \right) \mathbb{1}_{\varpi^{-m_2}\mathscr{O}}\left(\frac{\beta}{\alpha}\right) \left(q^{\frac{m_1}{2}} C_{m_1}(\zeta) - q^{\frac{m_1-2}{2}} C_{m_1-2}(\varpi\zeta) \right) \qquad (3\text{-}3)$$

if $m_2 = 0, 1$, while

$$I_{\gamma'} = \left| \frac{\beta}{\alpha} \right|^{\frac{1}{2}} \left(q^{\frac{m_1}{2}} \left(\left(1 + m_2 + \mathrm{val}\left(\frac{\beta}{\alpha}\right) \right) \mathbb{1}_{\varpi^{-m_2}\mathscr{O}}\left(\frac{\beta}{\alpha}\right) C_{m_1}(\zeta) \right. \right.$$

$$\left. - \left(-1 + m_2 + \mathrm{val}\left(\frac{\beta}{\alpha}\right) \right) \mathbb{1}_{\varpi^{-(m_2-2)}\mathscr{O}}\left(\frac{\beta}{\alpha}\right) C_{m_1}(\varpi^{-1}\zeta) \right)$$

$$- q^{\frac{m_1-2}{2}} \left(\left(1 + m_2 + \mathrm{val}\left(\frac{\beta}{\alpha}\right) \right) \mathbb{1}_{\varpi^{-m_2}\mathscr{O}}\left(\frac{\beta}{\alpha}\right) C_{m_1-2}(\varpi\zeta) \right.$$

$$\left. \left. - \left(-1 + m_2 + \mathrm{val}\left(\frac{\beta}{\alpha}\right) \right) \mathbb{1}_{\varpi^{-(m_2-2)}\mathscr{O}}\left(\frac{\beta}{\alpha}\right) C_{m_1-2}(\zeta) \right) \right) \qquad (3\text{-}4)$$

if $m_2 > 1$.

There is one last manipulation required before we can identify this as the orbital integral of $\mathrm{St}(T_{m_1} \otimes T_{m_2})$. This requires recognizing that

$$C_{m_1}(\zeta) := \sum_{r_1=0}^{m_1} B_{r_1}(0, \zeta)$$

$$= \sum_{i=0}^{m_1} \mathbb{1}_{\{\mathrm{val}(\xi_1) \geq -T - m_1 + i, \mathrm{val}(\xi_2) \geq -T - i\}}(\zeta)$$

$$= \sum_{i=0}^{m_1} \mathbb{1}_{\{\mathrm{val}(\xi_1) \geq \frac{1}{2}(\mathrm{val}\,\alpha - (m_1 - 2i) - m_2), \mathrm{val}(\xi_2) \geq \frac{1}{2}(\mathrm{val}\,\alpha + (m_1 - 2i) - m_2)\}}(\zeta).$$

Similarly

$$C_{m_1-2}(\varpi\zeta) = \sum_{j=0}^{m_1-2} \mathbb{1}_{\{\mathrm{val}(\xi_1) \geq -T + 1 - m_1 + j, \mathrm{val}(\xi_2) \geq -T - 1 - i\}}(\zeta)$$

$$= \sum_{j=0}^{m_1-2} \mathbb{1}_{\{\mathrm{val}(\xi_1) \geq \frac{1}{2}(\mathrm{val}\,\alpha - (m_1 - 2 - 2j) - m_2), \mathrm{val}(\xi_2) \geq \frac{1}{2}(\mathrm{val}\,\alpha + (m_1 - 2 - 2j) - m_2)\}}(\zeta).$$

If we apply these expansions to (3-3) and (3-4), then we write, for example when $m_2 > 1$,

$$
\begin{aligned}
I_{\gamma'} = \left|\frac{\beta}{\alpha}\right|^{\frac{1}{2}} \Bigg(q^{\frac{m_1}{2}} \sum_{i=0}^{m_1} \Bigg(& \left(1+m_2+\mathrm{val}\left(\frac{\beta}{\alpha}\right)\right) \mathbb{1}_{\varpi^{-m_2}\mathcal{O}} \\
& \times \left(\frac{\beta}{\alpha}\right) \mathbb{1}_{\{\mathrm{val}(\xi_1)\geq\frac{1}{2}(\mathrm{val}\,\alpha-(m_1-2i)-m_2),\,\mathrm{val}(\xi_2)\geq\frac{1}{2}(\mathrm{val}\,\alpha+(m_1-2i)-m_2)\}}(\zeta) \\
& -\left(-1+m_2+\mathrm{val}\left(\frac{\beta}{\alpha}\right)\right) \mathbb{1}_{\varpi^{-(m_2-2)}\mathcal{O}} \\
& \times \left(\frac{\beta}{\alpha}\right) \mathbb{1}_{\{\mathrm{val}(\xi_1)\geq\frac{1}{2}(\mathrm{val}\,\alpha-(m_1-2i)-m_2)+1,\,\mathrm{val}(\xi_2)\geq\frac{1}{2}(\mathrm{val}\,\alpha+(m_1-2i)-m_2)+1\}}(\zeta) \Bigg) \\
-q^{\frac{m_1-2}{2}} \sum_{j=0}^{m_1-2} \Bigg(& \left(1+m_2+\mathrm{val}\left(\frac{\beta}{\alpha}\right)\right) \mathbb{1}_{\varpi^{-m_2}\mathcal{O}} \\
& \times \left(\frac{\beta}{\alpha}\right) \mathbb{1}_{\{\mathrm{val}(\xi_1)\geq\frac{1}{2}(\mathrm{val}\,\alpha-(m_1-2-2j)-m_2),\,\mathrm{val}(\xi_2)\geq\frac{1}{2}(\mathrm{val}\,\alpha+(m_1-2-2j)-m_2)\}}(\zeta) \\
& -\left(-1+m_2+\mathrm{val}\left(\frac{\beta}{\alpha}\right)\right) \mathbb{1}_{\varpi^{-(m_2-2)}\mathcal{O}} \\
& \times \left(\frac{\beta}{\alpha}\right) \mathbb{1}_{\{\mathrm{val}(\xi_1)\geq\frac{1}{2}(\mathrm{val}\,\alpha-(m_1-2-2j)-m_2)+1,\,\mathrm{val}(\xi_2)\geq\frac{1}{2}(\mathrm{val}\,\alpha+(m_1-2-2j)-m_2)+1\}}(\zeta) \Bigg) \Bigg).
\end{aligned}
$$

By the following lemma, these expressions are exactly those appearing in the computation of orbital integrals. $\square$

Lemma 3.17. *Let*

$$
f = f_W \otimes f_V = S_l \otimes T_m \in \mathcal{H}.
$$

Then $J_\gamma(f)$ *is given by, for* $\gamma = \rho((a,d)+\iota(b,c)e)$,

$$
\begin{aligned}
& J_\gamma(f) \\
& = \left(1+m+\mathrm{val}\left(\frac{Q(\iota(b,c))}{N\gamma}\right)\right) \mathbb{1}_{\varpi^{-m}\mathcal{O}}\left(\frac{Q(\iota(b,c))}{N\gamma}\right) \mathbb{1}_{\{\mathrm{val}(a)\geq\frac{1}{2}(\mathrm{val}(N\gamma)-l-m),\,\mathrm{val}(d)\geq\frac{1}{2}(\mathrm{val}(N\gamma)+l-m)\}}((a,d))
\end{aligned}
$$

if $m = 0, 1$, *while*

$$
\begin{aligned}
J_\gamma(f) = & \left(1+m+\mathrm{val}\left(\frac{Q(\iota(b,c))}{N\gamma}\right)\right) \mathbb{1}_{\varpi^{-m}\mathcal{O}} \\
& \times \left(\frac{Q(\iota(b,c))}{N\gamma}\right) \mathbb{1}_{\{\mathrm{val}(a)\geq\frac{1}{2}(\mathrm{val}(N\gamma)-l-m),\,\mathrm{val}(d)\geq\frac{1}{2}(\mathrm{val}(N\gamma)+l-m)\}}((a,d)) \\
& -\left(-1+m+\mathrm{val}\left(\frac{Q(\iota(b,c))}{N\gamma}\right)\right) \mathbb{1}_{\varpi^{-(m-2)}\mathcal{O}} \\
& \times \left(\frac{Q(\iota(b,c))}{N\gamma}\right) \mathbb{1}_{\{\mathrm{val}(a)\geq\frac{1}{2}(\mathrm{val}(N\gamma)-l-m)+1,\,\mathrm{val}(d)\geq\frac{1}{2}(\mathrm{val}(N\gamma)+l-m)+1\}}((a,d))
\end{aligned}
$$

if $m > 1$.

Proof. Recall that $J_\gamma(f)$ is defined by first setting

$$F(x) = \int_{SO_W(F)} f_W(h) f_V(hx) \, dh$$

and then taking

$$J_\gamma(f) = \int_{SO_W(F)} F(h^{-1}\gamma h) \, dh.$$

So note first that for $f = S_l \otimes T_m$ we have (since we have taken $\text{vol}(SO_W(\mathcal{O})) = 1$)

$$F(x) = \mathbb{1}_{\varpi^{-l\mu_0} \, K_{SO_V} \, \varpi^{m\mu_0} \, K_{SO_V}}(x),$$

which does not lie in $\mathcal{H}_{SO_V}$. Nevertheless, we may compute its regular semisimple orbital integrals. Suppose $\gamma = \rho((a,d) + \iota(b,c)e)$; then by the identification (3-2) and Lemma 3.9, we know that $\gamma \in \varpi^{-l\mu_0} \, K_{SO_V} \, \varpi^{m\mu_0} \, K_{SO_V}$ if and only if

$$l + \text{val}\left(ad - \frac{d_V^2}{d_W^2}bc\right) - 2\min\left(l + \text{val}(a), l + \text{val}\left(\frac{d_V}{d_W}b\right), \text{val}\left(\frac{d_V}{d_W}c\right), \text{val}(d)\right) = m,$$

which is better written as

$$\min\left(l + \text{val}(a), l + \text{val}\left(\frac{d_V}{d_W}b\right), \text{val}\left(\frac{d_V}{d_W}c\right), \text{val}(d)\right) = \frac{\text{val}(N\gamma) + l - m}{2}. \qquad (\star)$$

Observe that $\frac{1}{2}(\text{val}(N\gamma) + l - m)$ must be an integer in order for γ to lie in $\varpi^{-l\mu_0} \, K_{SO_V} \, \varpi^{m\mu_0} \, K_{SO_V}$.

Note that the right-hand side of the above expression $(\star)$ does not change when we conjugate γ by an element $h = (t, 1) \in SO_W(F)$, while such conjugation replaces the left-hand side with

$$\min\left(k + \text{val}(a), k + \text{val}\left(\frac{d_V}{d_W}t^{-1}b\right), \text{val}\left(\frac{d_V}{d_W}tc\right), \text{val}(d)\right).$$

Keeping this in mind, let us determine which γ are conjugate to an element of $\varpi^{-l\mu_0} \, K_{SO_V} \, \varpi^{m\mu_0} \, K_{SO_V}$.

Let us assume now that $m > 1$; the analysis for $m = 0, 1$ is similar but even easier. There are a number of possible cases to consider.

(1) $\dfrac{Q(\iota(b,c))Q(e)}{N\gamma} = -\dfrac{d_V^2}{d_W^2}\dfrac{bc}{N\gamma} \notin \varpi^{-m}\mathcal{O}.$

This implies

$$l + \text{val}\left(\frac{d_V}{d_w}b\right) + \text{val}\left(\frac{d_V}{d_W}c\right) < \text{val}(N\gamma) + l - m;$$

hence at least one of $l + \text{val}((d_V/d_W)b)$, $\text{val}((d_V/d_W)c)$ is less than $\frac{1}{2}(\text{val}(N\gamma) + l - m)$. Thus γ can never be conjugate to an element of $\varpi^{-l\mu_0} \, K_{SO_V} \, \varpi^{m\mu_0} \, K_{SO_V}$.

(2) $\dfrac{Q(\iota(b,c))Q(e)}{N\gamma} \in \varpi^{-m}\mathcal{O} - \varpi^{-m+2}\mathcal{O}.$

In this case, since we have

$$\mathrm{N}\,\gamma = ad - \frac{d_V^2}{d_W^2} bc,$$

this forces $\mathrm{val}(ad/(\mathrm{N}\,\gamma)) = -m, -m+1$ as well. Thus, in order to have γ conjugate to an element satisfying $(\star)$, it must be the case that at least one of $l + \mathrm{val}(a)$ or $\mathrm{val}(d)$ must be exactly $\frac{1}{2}(\mathrm{val}(\mathrm{N}\,\gamma) + l - m)$.

(3) $\dfrac{Q(\iota(b,c))Q(e)}{\mathrm{N}\,\gamma} \in \varpi^{-m+2}\mathscr{O}$, at least one of $l + \mathrm{val}(a), \mathrm{val}(d) < \frac{1}{2}(\mathrm{val}(\mathrm{N}\,\gamma) + l - m)$.

Here, it is clear that γ can never be conjugate to an element satisfying $(\star)$.

(4) $\dfrac{Q(\iota(b,c))Q(e)}{\mathrm{N}\,\gamma} \in \varpi^{-m+2}\mathscr{O}$, at least one of $l + \mathrm{val}(a), \mathrm{val}(d) = \frac{1}{2}(\mathrm{val}(\mathrm{N}\,\gamma) + l - m)$.

We implicitly assume both $l + \mathrm{val}(a), \mathrm{val}(d) \geq \frac{1}{2}(\mathrm{val}(\mathrm{N}\,\gamma) + l - m)$ so that we are not in case (3). Then γ is always conjugate to an element satisfying $(\star)$.

(5) $\dfrac{Q(\iota(b,c))Q(e)}{\mathrm{N}\,\gamma} \in \varpi^{-m+2}\mathscr{O}$, with $l + \mathrm{val}(a), \mathrm{val}(d) > \frac{1}{2}(\mathrm{val}(\mathrm{N}\,\gamma) + l - m)$.

Here again, γ is always conjugate to an element satisfying $(\star)$.

In each case, we can easily determine the value of $J_\gamma(f)$. In (1) and (3), it is clear that $J_\gamma(f) = 0$. In cases (2) and (4), we get $1 + \mathrm{val}(Q(\iota(b,c))Q(e)/(\mathrm{N}\,\gamma)) + m$. Finally, in case (5), we find $J_\gamma(f) = 2$. This is exactly the claimed formula. $\qquad\square$

Part II. Trace formulas, done naively

In this part, we present the global motivations underlying the calculations in Part I. Throughout this part, unless otherwise specified, F will be a number field. We will describe the construction of two trace formulas, the "geometric sides" of which will involve the orbital and tonal integrals studied in Part I.

However, a caveat: *all manipulations will be purely formal.* We give little to no thought towards convergence of various sums and integrals, and we will freely interchange these as needed. This failing will be remedied in the sequel to this paper, where we will give a complete discussion of the "correct" form of these trace formulas, and where we confront these analytic issues and provide suitable regularized versions of these trace formula identities.

Because of this, some words should be said in defense of including this discussion at all — after all, with these naive manipulations there are no real results in this part, only motivation! We mention only that the regularized forms of the trace formulas we allude to above appear quite obscure, and discussion of them lacks the clarity that we feel underlies our approach, and which is apparent in this naive discussion.

That said, let us begin with a description of our first relative trace formula, which is designed to encode "the periods side" of Waldspurger's formula.

4. The relative trace formula on $SO_2 \times SO_3$

The setup for our trace formula here mostly follows the basic pattern for all relative trace formulas. At first approximation, we simply consider the group $G = SO_W \times SO_V$ together with its diagonally embedded subgroup $H = \Delta SO_W$, and write down the relative trace formula corresponding to the double coset space $H \backslash G / H$. This gives a trace formula whose spectral side appears well-adapted to encode "the periods side" of Waldspurger's formula and whose geometric side carries a natural interpretation.

However, there is a slight hiccup with this clean approach. This only really rears its head not in the case of $SO_2 \times SO_3$, but in the more general study of SO_n periods for forms on SO_{n+1} when n is large. For instance, the naive analogue of Waldspurger's formula in high rank — the Ichino–Ikeda conjecture [2010] — is not quite correct as stated, and must be slightly modified. See [Xue 2017b] Conjecture 6.2.1 for a corrected version of the Ichino–Ikeda formula; see also Conjecture 6.3.1 of that paper for another variant, which reinterprets the corrected Ichino–Ikeda formula as a formula for periods on $O_n \times O_{n+1}$.

So what is this hiccup? The root cause of the matter is the existence of the outer automorphism of SO_n, when $n = 2r$ is even, which we may view as given by conjugation by a representative of the nontrivial element in O_n / SO_n, and which brings about, when $n = 2r$ is large, the failure of multiplicity 1 in the cuspidal spectrum of SO_n.

As one might imagine, this does not cause any serious problems in the Waldspurger case of $SO_2 \times SO_3$ (SO_2 certainly has multiplicity 1!) and could more or less be ignored in this setting; i.e., we could easily deal with the trace formula in the naive manner described above. We believe that this would be philosophically incorrect. To that end, we will incorporate the outer automorphism of SO_2 into the setup of our trace formula, which will force us to deviate slightly from the clear outline presented in the first paragraph of this section.

The basic idea behind this correction is simple. Since the most natural setting for the (corrected) $SO_n \times SO_{n+1}$ Ichino–Ikeda conjecture is not on $SO_n \times SO_{n+1}$ at all, but rather on $O_n \times O_{n+1}$ (again, see Conjecture 6.3.1 of [Xue 2017b]), we would like to "push forward" the relative trace formula on $O_W \times O_V$ corresponding to the double coset space $\Delta O_W \backslash O_W \times O_V / \Delta O_W$ from $O_W \times O_V$ to $G = SO_W \times SO_V$. This will give a distribution on G with a spectral side designed to encode periods on $O_W \times O_V$, as desired.

Let us begin by giving a few simple definitions.

4A. *The period integral and the global spherical character.* We assume here that F is a number field. Let our notation be as above: $G = SO_W \times SO_V$ and $H = \Delta SO_W = \{(h, \iota(h)) : h \in SO_W\}$ is the diagonally embedded SO_W. In the future, we will omit writing the embedding $\iota : SO_W \hookrightarrow SO_V$. Let $\pi = \chi \otimes \sigma$ be an irreducible automorphic cuspidal representation of G; i.e., let χ be an automorphic character $\chi : SO_W(F) \backslash SO_W(\mathbb{A}) \to \mathbb{C}^\times$ and σ an irreducible automorphic cuspidal representation of SO_V.

We must consider the period integral of Waldspurger–Gross–Prasad applied to forms in π. This is defined by

$$\mathscr{P} : \pi = \chi \otimes \sigma \to \mathbb{C},$$

$$\chi \otimes \phi \mapsto \int_{[H]} \chi(h)\phi(h) \, dh.$$

Here $\phi \in \sigma$ and dh is the measure on $H(\mathbb{A}) = SO_W(\mathbb{A})$ normalized to give $\text{vol}([H]) = 1$. The notation $\phi \in \sigma$ is unambiguous, since by the multiplicity-one theorem for $SO_V \cong PB^\times$ there is only one realization of σ in the space of automorphic forms.

The period integral $\mathcal{P}(\phi)$ is absolutely convergent when π is cuspidal.

4A1. *The global spherical character.* Let π be an automorphic cuspidal representation of $G = SO_W \times SO_V$. We define the *global spherical character* by taking for $f \in C_c^\infty(G(\mathbb{A}))$

$$J_\pi(f) = \sum_{\varphi \in \text{ON}(\pi)} \mathcal{P}(\pi(f)\varphi)\overline{\mathcal{P}(\varphi)}.$$

Here the sum runs over a fixed orthonormal basis of $\text{ON}(\pi) \subset \pi \subset L^2([G])$, and the operator $\pi(f)$ is defined by

$$\pi(f)\varphi = \int_{G(\mathbb{A})} f(g)\pi(g)\varphi \, dg.$$

J_π is visibly a distribution of positive type, in the following sense: if $f = a * a^\vee$ for a function a, where $a^\vee(g) = \overline{a(g^{-1})}$, then

$$J_\pi(f) \geq 0.$$

Moreover, the following lemma is easily apparent:

Lemma 4.1. *Let $\mathcal{P}$ denote the period integral, viewed as an element in $\text{Hom}_{H(\mathbb{A})}(\pi, \mathbb{C})$. Then $\mathcal{P} = 0$ if and only if $J_\pi = 0$.*

This should be interpreted as the statement that the distribution J_π encodes the period integral, viewed as a functional on π.

4A2. *A variant for full orthogonal groups.* We also define, for future use, the corresponding objects for full orthogonal groups $O_W \times O_V$. Let $\tilde{\pi} = \tilde{\chi} \otimes \tilde{\sigma}$ be a cuspidal representation of $O_W \times O_V$ (note that $\tilde{\cdot}$ does not mean contragredient here!), and consider the functional

$$\tilde{\mathcal{P}} : \tilde{\pi} \to \mathbb{C},$$
$$\tilde{\varphi} \mapsto \int_{[\Delta O_W]} \tilde{\varphi}(\tilde{h}) \, d\tilde{h}.$$

As above, we can define a global spherical character

$$\tilde{J}_{\tilde{\pi}}(\tilde{f}) = \sum_{\tilde{\varphi} \in \text{ON}(\tilde{\pi})} \tilde{\mathcal{P}}(\tilde{\pi}(\tilde{f})\tilde{\varphi})\overline{\tilde{\mathcal{P}}\tilde{\varphi}}.$$

4B. *The $SO_W \times SO_V$ trace formula.* Consider the homomorphism

$$D : O_W \times O_W \to \mu_2,$$
$$(h_1, h_2) \mapsto \det h_1 \det h_2,$$

and define the disconnected group $\boldsymbol{H} = (H^2)^\pm$ by setting

$$(H^2)^\pm := \ker(D).$$

We can also write this group as the semidirect product $(H^2)^{\pm} = (H \times H) \rtimes \{1, \sigma\}$, where σ is the element of O_W induced by the nontrivial automorphism $\bar{\cdot} \in \mathrm{Gal}(E/F)$ acting through the identification $W \cong (E, \varepsilon \, N)$. The semidirect product is given by the diagonal action

$$\sigma(h_1, h_2)\sigma^{-1} = ({}^{\sigma}h_1, {}^{\sigma}h_2).$$

We use $\boldsymbol{h} = (h_1 s, h_2 s)$, $s \in \{1, \sigma\}$, to denote an element of $\boldsymbol{H} = (H^2)^{\pm}$.

We will identify the group $\{1, \sigma\}$ with μ_2 as follows. Denote by $s : \mu_2 \to O_W$ the map

$$t \mapsto s_t,$$

where

$$s_t = \begin{cases} 1 & \text{if } t = 1, \\ \sigma & \text{if } t = -1. \end{cases}$$

We also define another map $z : \mu_2 \to O_V$, written

$$t \mapsto z_t.$$

Here $z_t \in O_V$ is the transformation which is the identity on W and multiplication by t on $W^{\perp} = Fe$. It is worth noting by construction that any element z_t commutes with any element of O_W.

We take Haar measures dh_v on $\mathrm{SO}_W(F_v)$ and dt_v on $\mu_2(F_v)$ so that for almost all places

$$\mathrm{vol}(K_{\mathrm{SO}_W, v}) = 1,$$

while for all v,

$$\mathrm{vol}(\mu_2(F_v)) = 1.$$

We also take Haar measures $dh = \prod dh_v$ and $dt = \prod dt_v$ on $\mathrm{SO}_W(\mathbb{A})$ and $\mu_2(\mathbb{A})$ — by scaling dh_v at finitely many places, we can ensure that

$$\mathrm{vol}([\mathrm{SO}_W]) = 1,$$

while, without scaling dt_v, we still have

$$\mathrm{vol}([\mu_2]) = \tfrac{1}{2}.$$

Together, these induce a Haar measure $d\boldsymbol{h}$ on $\boldsymbol{H}(\mathbb{A})$ and a corresponding measure on the automorphic quotient $[\boldsymbol{H}]$ via

$$\int_{[\boldsymbol{H}]} \phi(\boldsymbol{h}) \, d\boldsymbol{h} = \int_{[\mu_2]} \int_{[\mathrm{SO}_W]} \int_{[\mathrm{SO}_W]} \phi(h_1 s_t, h_2 s_t) \, dh_1 \, dh_2 \, dt$$

for ϕ a smooth function on $[\boldsymbol{H}]$.

We can now define our trace formula. Let $f \in C_c^{\infty}(\mathrm{G}(\mathbb{A}))$ be a test function, and let $x = (x_1, x_2)$, $y = (y_1, y_2)$ be elements of $\mathrm{G}(\mathbb{A}) = \mathrm{SO}_W(\mathbb{A}) \times \mathrm{SO}_V(\mathbb{A})$. Consider the usual kernel function

$$K_f(x, y) = \sum_{\delta \in \mathrm{G}(F)} f(x^{-1}\delta y).$$

It is easy to see that this is formally represented by the sum

$$K_f(x, y) = \sum_{\varphi \in \text{ON}(L^2([G]))} (R(f)\varphi)(x)\overline{\varphi(y)}.$$

Here $\text{ON}(L^2([G]))$ denotes a fixed orthonormal basis of automorphic forms of the L^2 space of $G(F) \backslash G(\mathbb{A})$ and $R(f)\varphi$ is the right regular action of Hecke operators, namely

$$R(f)\varphi(x) = \int_{G(\mathbb{A})} f(y)\varphi(xy) \, dy.$$

We must consider the following:

Definition 4.2. The $\text{SO}_W \times \text{SO}_V$ *trace formula distribution* $J(f)$ is defined formally as

$$J(f) = \int_{[\mu_2]} \int_{[\text{SO}_W]} \int_{[\text{SO}_W]} K_f((h_1 s_t, h_1 s_t), (h_2 s_t, h_2 s_t)) \, dh_1 \, dh_2 \, dt. \tag{4-1}$$

There is small inconsistency in our notation: we defined K_f as a function on $(\text{SO}_W(\mathbb{A}) \times \text{SO}_V(\mathbb{A}))^2$, while above we seem to be evaluating this at elements of $O_W(\mathbb{A})^2$. However, the expression (4-1) can, and should, be made sense of in the following way:

$$\int_{[\mu_2]} \int_{[\text{SO}_W]} \int_{[\text{SO}_W]} K_f((h_1 s_t, h_1 s_t), (h_2 s_t, h_2 s_t)) \, dh_1 \, dh_2 \, dt$$

$$= \int_{[\mu_2]} \int_{[\text{SO}_W]} \int_{[\text{SO}_W]} \sum_{(\delta_1, \delta_2) \in G(F)} f(s_t^{-1} h_1^{-1} \delta_1 h_2 s_t, s_t^{-1} h_1^{-1} \delta_2 h_2 s_t) \, dh_1 \, dh_2 \, dt$$

$$= \int_{[\mu_2]} \int_{[\text{SO}_W]} \int_{[\text{SO}_W]} \sum_{(\delta_1, \delta_2) \in G(F)} f(^{s_t}(h_1^{-1} \delta_1 h_2), {}^{s_t}(h_1^{-1} \delta_2 h_2)) \, dh_1 \, dh_2 \, dt.$$

If we omitted the integration over $[\mu_2]$ in our definition of $J(f)$, then it would be extremely easy to interpret the corresponding distribution spectrally. Simply write (formally) $L^2([G]) = \bigoplus \pi$, where the "sum" is over all automorphic representations of G. This gives

$$\int_{[\text{SO}_W]} \int_{[\text{SO}_W]} K_f((h_1, h_1), (h_2, h_2)) \, dh_1 \, dh_2 = \int_{[\text{SO}_W]} \int_{[\text{SO}_W]} \sum_{\varphi \in \text{ON}(L^2([G]))} (R(f)\varphi)(h_1)\overline{\varphi(h_2)} \, dh_1 \, dh_2$$

$$= \sum_{\pi} \sum_{\varphi \in \text{ON}(\pi)} \mathscr{P}(\pi(f)\varphi)\overline{\mathscr{P}(\varphi)}$$

$$= \sum_{\pi} J_{\pi}(f)$$

and so spectrally, this distribution is nothing more than the sum of all the global spherical characters J_{π}.

The same argument, but applied to $O_W \times O_V$, gives the spectral decomposition of $J(f)$, as we will now explain.

Consider the distribution $\tilde{J}$, defined by, for a function $\tilde{f} \in C_c^\infty((O_W \times O_V)(\mathbb{A}))$,

$$\tilde{J}(\tilde{f}) := \int_{[O_W]} \int_{[O_W]} K_{\tilde{f}}(\tilde{h}_1, \tilde{h}_2) \, d\tilde{h}_1 \, d\tilde{h}_2,$$

where

$$K_{\tilde{f}}(x, y) = \sum_{\delta \in (O_W \times O_V)(F)} \tilde{f}(x^{-1}\delta y)$$

is the Selberg kernel function for $O_W \times O_V$. Here $d\tilde{h}_i = d\tilde{h}$ denote the same measure on $[O_W]$, defined using our fixed measures dt on $[\mu_2]$ and dh on $[SO_W]$ as above via

$$\int_{[O_W]} \phi(\tilde{h}) \, d\tilde{h} = \int_{[\mu_2]} \int_{[SO_W]} \phi(hs_t) \, dh \, dt.$$

By the same argument as above, $\tilde{J}(\tilde{f})$ formally decomposes as a sum of global spherical characters $\tilde{J}_{\tilde{\pi}}(\tilde{f})$ for $O_W \times O_V$. Some simple manipulations will also allow us to relate $\tilde{J}(\tilde{f})$ to the distribution $J(f)$.

Expand out $\tilde{J}(\tilde{f})$ to get

$$\tilde{J}(\tilde{f}) = \int_{[O_W]} \int_{[O_W]} K_{\tilde{f}}(\tilde{h}_1, \tilde{h}_2) \, d\tilde{h}_1 \, d\tilde{h}_2$$

$$= \int_{[\mu_2]^2} \int_{[SO_W]^2} \sum_{\delta \in G(F)} \sum_{\substack{\tau_1 \in \mu_2(F) \\ \tau_2 \in \mu_2(F)}} \tilde{f}(s_{t_1}^{-1} h_1^{-1} \delta_1 s_{\tau_1} h_2 s_{t_2}, s_{t_1}^{-1} h_1^{-1} \delta_2 s_{\tau_2} h_2 s_{t_2}) \, dh_1 \, dh_2 \, dt_1 \, dt_2.$$

Changing variables in the sum $\delta_1 \mapsto s_{\tau_1} \delta_1 s_{\tau_1}^{-1}$, $\delta_2 \mapsto s_{\tau_1} \delta_2 s_{\tau_1}^{-1}$, followed by $s_{\tau_2} \mapsto s_{\tau_1} s_{\tau_2}$, and finally followed by $h_1 \mapsto s_{\tau_1} h_1 s_{\tau_1}^{-1}$ gives

$$\tilde{J}(\tilde{f})$$
$$= \int_{[\mu_2]^2} \int_{[SO_W]} \int_{[SO_W]} \sum_{\delta \in G(F)} \sum_{\substack{\tau_1 \in \mu_2(F) \\ \tau_2 \in \mu_2(F)}} \tilde{f}((s_{\tau_1} s_{t_1})^{-1} h_1^{-1} \delta_1 h_2 s_{t_2}, (s_{\tau_1} s_{t_1})^{-1} h_1^{-1} \delta_2 s_{\tau_2} h_2 s_{t_2}) \, dh_1 \, dh_2 \, dt_1 \, dt_2.$$

If we unfold the integration over one copy of $[\mu_2]$ and change variables $s_{t_1} \mapsto s_{t_2} s_{t_1}^{-1}$, we find that

$$\tilde{J}(\tilde{f})$$
$$= \int_{[\mu_2]} \int_{[SO_W]^2} \sum_{\delta \in G(F)} \left(\int_{\mu_2(\mathbb{A})} \sum_{\tau_2 \in \mu_2(F)} \tilde{f}(s_{t_1}^{-1} h_1^{-1} \delta_1 h_2 s_{t_2}, s_{t_1}^{-1} h_1^{-1} \delta_2 s_{\tau_2} h_2 s_{t_2}) \, dt_1 \right) dh_1 \, dh_2 \, dt_2$$

$$= \int_{[\mu_2]} \int_{[SO_W]^2} \sum_{\delta \in G(F)} \left(\int_{\mu_2(\mathbb{A})} \sum_{\tau_2 \in \mu_2(F)} \tilde{f}(s_{t_1} s_{t_2}^{-1} h_1^{-1} \delta_1 h_2 s_{t_2}, s_{t_1} s_{t_2}^{-1} h_1^{-1} \delta_2 s_{\tau_2} h_2 s_{t_2}) \, dt_1 \right) dh_1 \, dh_2 \, dt_2.$$

Finally, we change $\delta_2 \mapsto \delta_2 s_{\tau_2}^{-1} z_{\tau_2}$, noting that z_{τ_2} commutes with elements in SO_W, and write

$$\tilde{J}(\tilde{f})$$

$$= \int_{[\mu_2]} \int_{[\mathrm{SO}_W]^2} \sum_{\delta \in G(F)} \left(\int_{\mu_2(\mathbb{A})} \sum_{\tau_2 \in \mu_2(F)} \tilde{f}(s_{t_1} s_{t_2}^{-1} h_1^{-1} \delta_1 h_2 s_{t_2}, s_{t_1} s_{t_2}^{-1} h_1^{-1} \delta_2 h_2 s_{t_2} z_{\tau_2}) \, dt_1 \right) dh_1 \, dh_2 \, dt_2$$

$$= \int_{[\mu_2]} \int_{[\mathrm{SO}_W]^2} \sum_{\delta \in G(F)} f(s_{t_2}^{-1} h_1^{-1} \delta_1 h_2 s_{t_2}, s_{t_2}^{-1} h_1^{-1} \delta_2 h_2 s_{t_2}) \, dh_1 \, dh_2 \, dt_2$$

$$= J(f),$$

where, given $\tilde{f}$, we take

$$f(a, b) = \int_{\mu_2(\mathbb{A})} \sum_{\tau \in \mu_2(F)} \tilde{f}(s_t a, s_t b z_\tau) \, dt.$$

Since the map $\tilde{f} \mapsto f$ is clearly surjective, we find that $J(f)$ does indeed have a spectral interpretation: it is essentially the corresponding relative trace formula on $O_W \times O_V$, and so exactly encodes the sum of global spherical characters on $O_W \times O_V$. For more on why these global spherical characters on $O_W \times O_V$ are the correct objects of study (rather than those on $\mathrm{SO}_W \times \mathrm{SO}_V$), we once again refer the reader to [Xue 2017b, Conjecture 6.3.1].

4B1. *The geometric side.* We turn to the geometric expansion of $J(f)$ as a sum of orbital integrals. These are defined purely locally in Part I: see Definition 1.9.

We can also easily define, for F a number field, $\gamma \in \mathrm{SO}_V(F)$, and $f^\sharp = \bigotimes f_v^\sharp \in C_c^\infty(\mathrm{SO}_V(\mathbb{A}))$ a factorizable function, the *global orbital integral* by setting

$$J(\gamma, f^\sharp) = J_\gamma(f^\sharp) := \int_{O_W(\mathbb{A})} f^\sharp(h^{-1} \gamma h) \, dh = \prod_v J_v(\gamma, f_v^\sharp).$$

As in Part I, if $f \in C_c^\infty(G(\mathbb{A}))$, then we associate a corresponding $f^\sharp$ via

$$f^\sharp(x) = \int_{\mathrm{SO}_W(\mathbb{A})} f(h(1, x)) \, dh.$$

Again, it is easy to see that the map

$$C_c^\infty(G(\mathbb{A})) \to C_c^\infty(\mathrm{SO}_V(\mathbb{A})),$$
$$f \mapsto \left(f^\sharp : x \mapsto \int_{\mathrm{SO}_W(\mathbb{A})} f(h(1, x)) \, dh \right),$$

is surjective. We often abuse notation and interchangeably write $J(\gamma, f)$ for $J(\gamma, f^\sharp)$.

Let us return to discussing the geometric expansion of $J(f)$. We can decompose $J(f)$ as a sum of global orbital integrals — in practice, this is little more than the double coset computation

$$\Delta \, \mathrm{SO}_W \backslash (\mathrm{SO}_W \times \mathrm{SO}_V) / \Delta \, \mathrm{SO}_W \xrightarrow{\sim} \mathrm{SO}_V /^{\mathrm{conj}} \mathrm{SO}_W,$$

$$(a, b) \mapsto a^{-1} b,$$

but where we also keep track of the outer automorphism of SO_W. Better said: consider the right action of $\boldsymbol{H}$ on G given, if $\boldsymbol{h} = (h_1 s, h_2 s)$, by

$$(a, b).\boldsymbol{h} = (s^{-1} h_1^{-1} a h_2 s, s^{-1} h_1^{-1} b h_2 s).$$

Then, we have

$$(\mathrm{SO}_W \times \mathrm{SO}_V)/\boldsymbol{H} \xrightarrow{\sim} \mathrm{SO}_V /^{\mathrm{conj}} \mathrm{O}_W,$$

$$(a, b) \mapsto a^{-1} b.$$

Recall that we have defined

$$J(f) = \int_{[\boldsymbol{H}]} \sum_{\delta \in \mathrm{G}(F)} f(\delta.\boldsymbol{h}) \, d\boldsymbol{h}.$$

Denote by $\boldsymbol{H}_\delta$ the stabilizer in $\boldsymbol{H}$ of $\delta \in \mathrm{G}$. We can take as orbit representatives of $\mathrm{G}(F)/\boldsymbol{H}(F)$ elements of the form $\delta = (1, \gamma)$, with γ a representative of $\mathrm{SO}_V(F)/^{\mathrm{conj}} \mathrm{O}_W(F)$, and for such δ

$$\boldsymbol{H}_\delta = \Delta(\mathrm{O}_W)_\gamma,$$

where by $(\mathrm{O}_W)_\gamma$, we mean the stabilizer in O_W of $\gamma \in \mathrm{SO}_V$ under the conjugation action. Note that this group can be identified either as O_W, SO_W, μ_2, or 1. When γ is regular semisimple, it is either 1 or μ_2.

In any case, we may unfold the integral defining $J(f)$ to find

$$J(f) = \sum_{\delta \in \mathrm{G}(F)/\boldsymbol{H}(F)} \mathrm{vol}([\boldsymbol{H}_\delta]) \int_{\boldsymbol{H}_\delta(\mathbb{A}) \backslash \boldsymbol{H}(\mathbb{A})} f(\delta.\boldsymbol{h}) \, d\boldsymbol{h}$$

$$= \sum_{\gamma \in \mathrm{SO}_V(F)/^{\mathrm{conj}} \mathrm{O}_W(F)} \mathrm{vol}([(\mathrm{O}_W)_\gamma]) \int_{(\mathrm{O}_W)_\gamma(\mathbb{A}) \backslash \mathrm{O}_W(\mathbb{A})} F(h^{-1} \gamma h) \, dh$$

or, even more simply,

$$J(f) = \sum_{\gamma \in (\mathrm{SO}_V(F)/^{\mathrm{conj}} \mathrm{SO}_W(F))^{\mathrm{r.s.s.}}} J_\gamma(f) + \text{non-r.s.s orbital integrals}.$$

Thus, the orbital integrals studied in Part I are exactly the terms appearing in the geometric side of our first trace formula.

5. The relative trace formula on $\mathrm{GL}_2 \times \mathrm{GL}_2$

Having now produced a trace formula distribution which encodes "the periods side" of Waldspurger's formula, it remains to construct one which controls "the L-function side". This is the goal of this section. Our distribution in question will live on the group $\mathrm{G}' := \mathrm{GL}_2 \times \mathrm{GL}_2$. To define it, we will require some ingredients.

5A. *Preliminaries.* Let $\Pi = \Pi_1 \otimes \Pi_2$ be an irreducible automorphic cuspidal representation of $\mathrm{G}'(\mathbb{A})$; i.e., let Π_i be irreducible cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$, and denote by ω_1 and ω_2 their (unitary) central characters. The main ingredients for our trace formula will be certain period integrals $\mathscr{P}_{\mathrm{RS}}$ and

$\mathscr{P}_{\text{dist}}$ of forms $\phi' \in \Pi$. These are integrals taken along two subgroups $H'_1 = \Delta \, \text{GL}_2$ and $H'_2 = \text{GL}_2^{(2)} \times N$, but weighted by various functions (Eisenstein series and theta functions respectively).

Thus, before we can proceed, we must review some basics on how these periods are defined, what they compute, and why we include them in our construction.

5A1. *Rankin–Selberg convolution on* GL_2. We begin with a flurry of definitions regarding Eisenstein series, and briefly survey their role in the theory of Rankin–Selberg convolution for $\text{GL}_2 \times \text{GL}_2$.

We first confront the global analogue of Section 2A2. Let $\Phi \in S(\mathbb{A}^2)$ be a Schwartz function on $\mathbb{A}^2$, and let ω be a unitary central character of GL_2:

$$\omega : F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times.$$

Recall that we denote by

$$P = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \in \text{GL}_2 \right\}$$

the usual "mirabolic" subgroup of GL_2, and by

$$B = \mathbb{Z} \, P = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \text{GL}_2 \right\}$$

the full Borel subgroup containing P. As in Section 2A2, we may define (via the identification $\mathbb{A}^2 - \{0\} \cong P \backslash \text{GL}_2$) a section F_s by the Tate integral

$$F_s(g) = |\det g|^s \int_{\mathbb{A}^\times} \Phi((0, t)g) \omega(t) |t|^{2s} \, d^\times t.$$

Again, $F_s \in \text{Ind}_{B(\mathbb{A})}^{\text{GL}_2(\mathbb{A})} (\delta^{s - \frac{1}{2}} (1, \omega^{-1}))$, where

$$\delta \left(\begin{pmatrix} a & * \\ & b \end{pmatrix} \right) = \left| \frac{a}{b} \right|$$

is the modular quasicharacter of B,

$$(1, \omega^{-1}) \left(\begin{pmatrix} a & * \\ & b \end{pmatrix} \right) := \omega^{-1}(b),$$

and Ind denotes normalized induction.

Definition 5.1. The Eisenstein series associated to Φ and ω is given, when $\text{Re}(s) \gg 0$, by the expression

$$E(g) = E(s, g) = E(s, (1, \omega^{-1}); g, \Phi) := \sum_{\gamma \in B(F) \backslash \text{GL}_2(F)} F_s(\gamma g).$$

To us, the importance of this Eisenstein series stems entirely from the following application.

Proposition 5.2 (Jacquet, Piatetski-Shapiro, Shalika, Rankin, Selberg). *Let* Π_1 *and* Π_2 *be irreducible automorphic cuspidal representations of* $\text{GL}_2(\mathbb{A})$, *with central characters* ω_1 *and* ω_2. *Let* $\varphi_i \in \Pi_i$ *be*

forms, and let $\Phi \in S(\mathbb{A}^2)$ be a Schwartz function. Then the integral

$$Z(s; \varphi_1, \varphi_2, \Phi) := \int_{\mathbb{Z}(\mathbb{A}) \, \mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbb{A})} \varphi_1(g)\varphi_2(g)E(s, (1, \omega_1^{-1}\omega_2^{-1}); g, \Phi) \, dg$$

$$= \int_{[\mathrm{PGL}_2]} \varphi_1(g)\varphi_2(g)E(s, g) \, dg$$

unfolds to an Euler product

$$Z(s; \varphi_1, \varphi_2, \Phi) = \int_{\mathrm{N}(\mathbb{A}) \backslash \mathrm{GL}_2(\mathbb{A})} \mathscr{W}^{\psi^{-1}}\varphi_1(g)\mathscr{W}^{\psi}\varphi_2(g)\Phi((0, 1)g)|\det g|^s \, dg,$$

where

$$\mathscr{W}^{\psi^{-1}}\varphi_1(g) = \int_{[\mathrm{N}]} \varphi_1(ng)\psi^{-1}(n) \, dn,$$

$$\mathscr{W}^{\psi}\varphi_2(g) = \int_{[\mathrm{N}]} \varphi_2(ng)\psi(n) \, dn$$

are the ψ-th and ψ^{-1}-th Fourier coefficients of φ_1 and φ_2 respectively. Once one fixes factorizations of the global Whittaker functionals of taking ψ-th or ψ^{-1}-th Fourier coefficient into products of local Whittaker functionals, this gives a factorization

$$Z(s; \varphi_1, \varphi_2, \Phi) = \prod_v Z_v(s; \varphi_{1,v}, \varphi_{2,v}, \Phi_v),$$

where each Z_v is a local integral

$$Z_v(s; \varphi_{1,v}, \varphi_{2,v}, \Phi_v) = \int_{\mathrm{N}(F_v) \backslash \mathrm{GL}_2(F_v)} \mathscr{W}_v^{\psi^{-1}}\varphi_{1,v}(g)\mathscr{W}_v^{\psi}\varphi_{2,v}(g)\Phi_v(e_n g)|\det g|^s \, dg$$

which exactly computes the local factors $L_v(s, \Pi_{1,v} \times \Pi_{2,v})$ for unramified places and unramified data.

Remark 5.3. The ramified places can be controlled by nonvanishing results for the local integrals, and thus this integral gives a robust understanding of the analytic properties of $L(s, \Pi_1 \times \Pi_2)$. In fact, even more can be said: the L-function occurs as a "g.c.d." of zeta integrals $Z(s, \varphi_1, \varphi_2, \Phi)$ as $\varphi_1, \varphi_2, \Phi$ vary. The functional equation follows from the functional equation of the Eisenstein series.

Proof. For a very readable discussion and proof in the general case of $\mathrm{GL}_n \times \mathrm{GL}_n$, see [Cogdell and Piatetski-Shapiro 2004]. $\square$

We will interpret the map $\varphi_1 \otimes \varphi_2 \otimes \Phi \mapsto Z(s, \varphi_1, \varphi_2, \Phi)$ as a linear functional, which we denote by

$$\mathscr{P}_{\mathrm{RS}}(s) : \Pi_1 \otimes \Pi_2 \otimes S(\mathbb{A}^2) \to \mathbb{C},$$

$$\varphi_1 \otimes \varphi_2 \otimes \Phi \mapsto Z(s, \varphi_1, \varphi_2, \Phi).$$

5A2. *Fourier coefficients of $E(g, s)$.* The Fourier expansion of the GL_2-Eisenstein series will be used later. The relevant and well-known computation follows.

Lemma 5.4. *Let* $\Phi = \otimes \Phi_v$ *be factorizable Schwartz function, and let* $\mathrm{Re}(s) \gg 0$. *The Eisenstein series*

$$E(g) = E(s, (1, \omega^{-1}); g, \Phi) = \sum_{\gamma \in B(F) \backslash \mathrm{GL}_2(F)} F(\gamma g, \Phi; s, \omega)$$

has α*-th Fourier coefficient given by*

$$E^\alpha(g) = \int_{F \backslash \mathbb{A}} E(n(x)g) \psi(-\alpha x) \, dx$$

$$= \begin{cases} \int_{\mathbb{A}} F_s(wn(x)g) \psi^{-1}(x) \, dx & \text{if } \alpha \neq 0, \\ F_s(g) + \int_{\mathbb{A}} F_s(wn(x)g) \, dx & \text{if } \alpha = 0 \end{cases}$$

$$= \begin{cases} \mathscr{W}_{F_s}^{\psi^{-1}}(g) & \text{if } \alpha \neq 0, \\ F_s(g) + M(s)F_s(g) & \text{if } \alpha = 0. \end{cases}$$

Here $M(s)$ *is the usual intertwining integral between principal series of* GL_2, *given by*

$$M(s) : f_s(g) \mapsto \int_{\mathbb{A}} f_s(wn(x)g) \, dx.$$

Proof. Recall the Eisenstein series is given by

$$E(g) = \sum_{\gamma \in B(F) \backslash \mathrm{GL}_2(F)} F_s(\gamma g).$$

Taking a set of representatives of $B(F) \backslash \mathrm{GL}_2(F)$ to be the identity matrix 1 (the small Bruhat cell) and matrices of the form $wn(t)$ (the large cell), we find

$$E^\alpha(g) = \int_{F \backslash \mathbb{A}} \left(F_s(n(x)g) + \sum_{t \in F} F_s(wn(t)n(x)g) \right) \psi(-\alpha x) \, dx.$$

Note that

$$\int_{F \backslash \mathbb{A}} F_s(n(x)g) \psi(-\alpha x) \, dx = F_s(g) \int_{F \backslash \mathbb{A}} \psi(-\alpha x) \, dx = 0$$

unless $\alpha = 0$.

So suppose first that $\alpha \neq 0$. Then we unfold to find

$$E^\alpha(g) = \int_{F \backslash \mathbb{A}} \sum_{t \in F} F_s(wn(x+t)g) \psi(-\alpha x) \, dx$$

$$= \int_{\mathbb{A}} F_s(wn(x)g) \psi(-\alpha x) \, dx$$

$$= \mathscr{W}_{F_s}^{\psi^{-1}}(a(\alpha)g)$$

so this coefficient is factorizable.

If $\alpha = 0$, the computation is similar. Proceeding as above, we get

$$E^0(g) = F_s(g) + \int_{\mathbb{A}} F_s(wn(x)g) \, dx,$$

as desired.   □

5A3. *The symmetric square L-function for* GL_2. Let us quickly review the construction of the symmetric square L-function for GL_2 due to Shimura and Gelbart–Jacquet. This crucially involves the Weil representation, treated adelically, and so, just as Section 2A1 did for the corresponding local picture, we now give a brief tour of this global theory.

Once again, for these matters we follow [Takeda 2014]. Most details will be omitted. Recall that we write, for any ring R,

$$GL_2^{(2)}(R) = \{g \in GL_2(R) : \det(g) \in (R^\times)^2\}.$$

These are not the R-points of an algebraic group, but we will still use this notation freely.

As in the local story, one can define a double cover $\widetilde{SL}_2(\mathbb{A})$ of $SL_2(\mathbb{A})$ as well as a central extension

$$1 \to \{\pm 1\} \to \widetilde{GL}_2^{(2)}(\mathbb{A}) \to GL_2^{(2)}(\mathbb{A}) \to 1.$$

This central extension comes equipped with two important (partially defined) set-theoretic sections, κ and s

$$\kappa, s : GL_2^{(2)}(\mathbb{A}) \to \widetilde{GL}_2^{(2)}(\mathbb{A}).$$

More precisely, κ is an honest set-theoretic section, defined for all h, while $s(h)$ only makes sense for $h \in B^{(2)}(\mathbb{A})$ or $h \in GL_2^{(2)}(F)$. (For more discussion on κ, s, and the precise cocycle used to write down the global metaplectic group, see [Takeda 2014].)

$\widetilde{GL}_2^{(2)}(\mathbb{A})$ also comes equipped with a Weil representation r^ψ. It can be realized as an action on the space $S^+(\mathbb{A})$ of even Schwartz functions on $\mathbb{A}$, where it is given by the formulas

$$r^\psi\left(s\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) f(x) = \gamma(\psi, \cdot^2)\hat{f}(x),$$

$$r^\psi\left(s\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\right) f(x) = \psi(bx^2)f(x),$$

$$r^\psi\left(s\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) f(x) = |a|^{\frac{1}{2}}\mu_\psi(a)f(ax),$$

$$r^\psi\left(s\begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix}\right) f(x) = |a|^{-\frac{1}{2}}f(a^{-1}x),$$

$$r^\psi(\xi)f(x) = \xi f(x),$$

as in the local setting. The Schwartz space model of r^ψ gives rise to a well-known automorphic realization: to any $f \in S^+(\mathbb{A})$ we attach a theta function

$$\theta^\psi(\tilde{h}, f) = \sum_{\xi \in F} r^\psi(\tilde{h})f(\xi)$$

which is an automorphic form on $\widetilde{GL}_2^{(2)}(\mathbb{A})$.

The representation r^ψ can also be constructed via the residues of Eisenstein series on $\widetilde{GL}_2^{(2)}$. This proceeds as follows. Let $\widetilde{T}^{(2)}(\mathbb{A})$ denote the preimage of the diagonal maximal torus $T^{(2)}(\mathbb{A})$ of $GL_2^{(2)}(\mathbb{A})$

in the double cover. We define a character ω^ψ by

$$\omega^\psi : \widetilde{\mathrm{T}}^{(2)}(\mathbb{A}) \to \mathbb{C}^\times,$$

$$(1, \xi)\boldsymbol{s}(t) \mapsto \xi\mu_\psi(t_1),$$

where $t = \mathrm{diag}(t_1, t_2)$ (again, the definition and properties of μ_ψ can be found in [Ranga Rao 1993, Appendix]). Then, given a section

$$\tilde{f}_s^\psi \in \mathrm{Ind}_{\widetilde{\mathrm{T}}^{(2)}(\mathbb{A})\boldsymbol{s}(\mathrm{N}(\mathbb{A}))}^{\widetilde{\mathrm{GL}}_2^{(2)}(\mathbb{A})} \omega^\psi \otimes \delta^{s-\frac{1}{2}},$$

one can construct an Eisenstein series

$$\widetilde{E}(s; \tilde{h}, \tilde{f}_s^\psi) := \sum_{\gamma \in \mathrm{B}^{(2)}(F) \backslash \mathrm{GL}^{(2)}(F)} \tilde{f}_s^\psi(\boldsymbol{s}(\gamma)\tilde{h})$$

on $\widetilde{\mathrm{GL}}_2^{(2)}(\mathbb{A})$. As $\widetilde{E}(s; \tilde{h}, f_s^\psi)$ lives on $\widetilde{\mathrm{GL}}_2^{(2)}(\mathbb{A})$, but does not descend on $\mathrm{GL}_2^{(2)}(A)$, we refer to it as a *genuine* or *half-integral weight* Eisenstein series. $\widetilde{E}(s; \tilde{h}, f_s^\psi)$ is known to have a pole at $s = \frac{3}{4}$, and in fact $\boldsymbol{r}^\psi$ can be realized as the space of residues of these Eisenstein series at $s = \frac{3}{4}$. For a simple proof of this fact, see [Gelbart and Piatetski-Shapiro 1980].

The importance of the functions $\theta^\psi(\tilde{h}, f)$ and $\widetilde{E}(s; \tilde{h}, f_s)$ rests in the following integral representation of the symmetric square L-function for GL_2.

Proposition 5.5 (Jacquet, Gelbart, Bump, Ginzburg, Takeda). *Let Π be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$, with quadratic central character $\omega = \omega_{E/F}$, with $E = F(\sqrt{d})$. Let $\varphi \in \Pi$, $f \in S^+(\mathbb{A})$, and*

$$\tilde{f}_s^{\psi-d} \in \mathrm{Ind}_{\widetilde{\mathrm{T}}^{(2)}(\mathbb{A})\boldsymbol{s}(\mathrm{N}(\mathbb{A}))}^{\widetilde{\mathrm{GL}}_2^{(2)}(\mathbb{A})} \omega^{\psi-d} \otimes \delta^{s-\frac{1}{2}}.$$

Then the integral

$$Z(s; \varphi, \Psi, f_s, \mathrm{Sym}^2) := \int_{[\mathrm{PGL}_2^{(2)}]} \varphi(h)\theta^\psi(\kappa(h), f)\widetilde{E}(s; \kappa(h), \tilde{f}_s^{\psi-d}) \, dh$$

unfolds to an Euler product

$$Z(s; \varphi, \Psi, f_s, \mathrm{Sym}^2) = \prod_v Z_v(s; \varphi_v, \Psi_v, f_{s,v}, \mathrm{Sym}^2),$$

where the local zeta integrals

$$Z_v(s; \varphi_v, \Psi_v, f_{s,v}, \mathrm{Sym}^2)$$

have the property that, for unramified places v and unramified test data $\varphi_v, \Psi_v, f_{s,v}$,

$$Z_v(s; \varphi_v, \Psi_v, f_{s,v}, \mathrm{Sym}^2) = L_v\left(2s - \frac{1}{2}, \Pi, \mathrm{Sym}^2\right)\zeta_{F,v}(4s-1)^{-1}.$$

Proof. This, and a twisted form of it, are the main theorem of [Takeda 2014] in the more general setting of GL_n. □

Remark 5.6. The restriction above that the central character ω be quadratic is entirely artificial — with some easy modification, the integral above can be defined for any irreducible cuspidal Π, not just those with quadratic central character. It then follows from this representation of $L(s, \Pi, \mathrm{Sym}^2)$ that the symmetric square L-function cannot have a pole at $s = 1$ if $\omega^2 \neq 1$. See [Takeda 2014].

One can take the residue at $s = \frac{3}{4}$ of the above integral representation to find an explicit period integral which detects the existence of a pole of $L(s, \Pi, \mathrm{Sym}^2)$ at $s = 1$. Before we state this result though, let us repackage some of the ingredients slightly.

Consider the 2-dimensional quadratic space (E, N). We can, in a similar manner to what we did above, define the representation $\boldsymbol{R}^\psi$ of $\mathrm{GL}^{(2)}(2, \mathbb{A})$ acting on $S^{++}(\mathbb{A}_E)$, the space of Schwartz functions on $\mathbb{A}_E$ invariant under $z \mapsto \bar{z}$ and $z \mapsto -z$, by the formulas

$$\boldsymbol{R}^\psi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\Psi(z) = \gamma(\psi, \mathrm{N})\hat{\Psi}(z),$$

$$\boldsymbol{R}^\psi\left(\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\right)\Psi(z) = \psi(b\,\mathrm{N}\,z)\Psi(z),$$

$$\boldsymbol{R}^\psi\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right)\Psi(z) = |a|\omega(a)\Psi(az),$$

$$\boldsymbol{R}^\psi\left(\begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix}\right)\Psi(z) = |a|^{-1}\Psi(a^{-1}z).$$

Here, $\hat{\Psi}(z) = \int_{\mathbb{A}_E} \Psi(z')\psi(\mathrm{N}(z + z') - \mathrm{N}(z) - \mathrm{N}(z'))\,dz'$ is the Fourier transform, with dz' normalized so as to be self-dual.

$\boldsymbol{R}^\psi$ has an automorphic realization by considering, given $\Psi \in S^{++}(\mathbb{A}_E)$, the associated theta function

$$\Theta^\psi(h, \Psi) = \sum_{\zeta \in E} (\boldsymbol{R}^\psi(h)\Psi)(\zeta).$$

We are interested in this representation $\boldsymbol{R}^\psi$ entirely because

$$\boldsymbol{R}^\psi \cong \boldsymbol{r}^\psi \hat{\otimes} \boldsymbol{r}^{\psi - d}$$

and because of its appearance in the following proposition.

Proposition 5.7 (Jacquet, Gelbart, Bump, Ginzburg, Takeda). *Let Π be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$. $L(s, \Pi, \mathrm{Sym}^2)$ has a pole at $s = 1$ if and only if for one (equivalently any) ψ we have both*

(1) *the central character ω of Π is of the form $\omega = \omega_{E/F}$ for some $E = F[\sqrt{d}]$ (i.e., $\omega^2 = 1$),*

(2) *the linear functional*

$$\mathscr{P}_{\mathrm{Sym}^2} : \Pi \otimes \boldsymbol{R}^\psi \to \mathbb{C},$$

$$\varphi \otimes \Psi \mapsto \int_{[\mathrm{PGL}_2^{(2)}]} \varphi(h)\Theta^\psi(h, \Psi)\,dh,$$

is not identically zero.

Proof. Indeed, this fact is also discussed in [Takeda 2014]. In addition to this reference, let us make some quick remarks. The result we are after follows directly from the integral representation of Proposition 5.5 and the observation that $\boldsymbol{R}^\psi$ is exactly the representation whose automorphic realization consists of forms like

$$\mathrm{res}_{s=\frac{3}{4}} \, \theta^\psi(\,\cdot\,, \Psi)\, \widetilde{E}(s;\cdot\,, \tilde{f}_s^{\psi-d}).$$

This latter observation is a consequence of the dual identity of $\boldsymbol{r}^\psi$ either as the Weil representation, as described by the action of $\widetilde{\mathrm{GL}}_2^{(2)}(\mathbb{A})$ on $S^+(\mathbb{A})$ and its automorphic realization as theta functions, or as the space of residues of metaplectic Eisenstein series. For more on this, see [Gelbart and Piatetski-Shapiro 1980], particularly Proposition 3.3.3 and their Section 6.   □

Why are we concerned with Proposition 5.7? In Section 5B we will need to characterize those cusp forms on $\mathrm{GL}_2(\mathbb{A})$ which appear as functorial lifts (via the "standard" transfer map) from $\mathrm{SO}_W(\mathbb{A})$. Assuming Langlands functoriality, here is the conjectural answer. According to the Langlands philosophy, an irreducible cuspidal automorphic representation Π of $\mathrm{GL}_2(\mathbb{A})$ is a functorial lift from $\mathrm{SO}_W(\mathbb{A})$ if and only if its corresponding L-parameter factors through the L-group of $\mathrm{SO}_W(\mathbb{A})$. This should occur if and only if we have both

(1) the L-parameter preserves a symmetric bilinear form on $\mathbb{C}^2$,

(2) the determinant of the L-parameter corresponds under class field theory to the character $\omega_{E/F}$.

This should be equivalent to

(1) the symmetric square L-function $L(s, \Pi, \mathrm{Sym}^2)$ has a pole at $s = 1$,

(2) the central character ω of Π is the quadratic character $\omega_{E/F}$.

Summarizing: Π should be a functorial lift from $\mathrm{SO}_W(\mathbb{A})$ if and only if its central character ω happens to be $\omega_{E/F}$ for $E = F[\sqrt{d_W}]$ and $\mathscr{P}_{\mathrm{Sym}^2}$ is not identically zero on $\Pi \otimes \boldsymbol{R}^\psi$. This is what $\mathscr{P}_{\mathrm{Sym}^2}$ is really designed to detect.

Remark 5.8. The soft discussion above, where we relied on some unproven black boxes (such as Langlands functoriality!) to arrive at a conjectural characterization of functorial lifts from $\mathrm{SO}_W(\mathbb{A})$ can be reformulated in a more concrete manner. This is largely because we know the construction of the functorial transfer from $\mathrm{SO}_W(\mathbb{A})$ to $\mathrm{GL}_2(\mathbb{A})$. It is the simplest case of the theta correspondence. Better said, the theta correspondence (with similitude factors) explains our functorial transfer. Note that our representation $\boldsymbol{R}^\psi$ is the restriction to $\mathrm{GL}_2^{(2)}(\mathbb{A})$ of a Weil representation on $\mathbb{A}_E \times \mathrm{GL}_2^{(2)}(\mathbb{A})$ (which itself is the restriction of a representation from the metaplectic double cover of GSp_4). It is then not too hard to see that $\boldsymbol{R}^\psi$ is the sum of all possible dihedral forms with central character $\omega_{E/F}$; hence integration next to $\boldsymbol{R}^\psi$ determines whether Π is a functorial lift from SO_W.

We have avoided this perspective because it so crucially relies on the exceptional isomorphism $\mathrm{SL}_2 = \mathrm{Sp}_2$ at its heart, and thus cannot offer any insight into the same problem for higher-rank even

orthogonal groups SO_n and their functorial transfers to GL_n. On the other hand, the theory of integral representations is bound by no such restriction.

Remark 5.9. It is also worth pointing out that the black boxes referenced in the remark above are largely proven in the case we need, namely that of functorial transfer from a classical group to GL_N. See [Arthur 2013].

5A4. *Fourier coefficients of* $\Theta(h, \Psi)$. For later use, we include the following simple lemma.

Lemma 5.10. *Let* $\Theta(h) = \Theta^{\psi}(h, \Psi)$ *be as above. Then the* α*-th Fourier coefficient is given by*

$$\Theta^{\alpha}(h) = \sum_{\substack{\zeta \in E \\ N\zeta = \alpha}} (\boldsymbol{R}^{\psi}(h)\Psi)(\zeta).$$

Proof. We compute

$$\Theta^{\alpha}(h) = \int_{F \backslash \mathbb{A}} \Theta(n(x)h)\psi(-\alpha x)\, dx$$

$$= \int_{F \backslash \mathbb{A}} \sum_{\zeta \in E} (\boldsymbol{R}^{\psi}(n(x)h)\Psi)(\zeta)\psi(-\alpha x)\, dx$$

$$= \sum_{\zeta \in E} (\boldsymbol{R}^{\psi}(h)\Psi)(\zeta) \int_{F \backslash \mathbb{A}} \psi(N(\zeta))\psi(-\alpha x)\, dx.$$

This last integral is 0 unless $N(\zeta) = \alpha$. By our normalization of measures, it is 1 when $N(\zeta) = \alpha$.   □

5A5. *The exterior square L-function for* GL_2. This is even easier. We record the following very simple fact.

Proposition 5.11. *Let* Π *be an irreducible automorphic cuspidal representation of* GL_2. *Then the exterior square L-function* $L(s, \Pi, \wedge^2)$ *has a pole at* $s = 1$ *if and only if, for* ψ *nontrivial, the linear functional*

$$\mathscr{P}_{\wedge^2} = \mathscr{P}_{\wedge^2}^{\psi} : \Pi \to \mathbb{C},$$
$$\varphi \mapsto \int_{[\mathbb{Z} N]} \varphi(zn)\psi(n)\, dn\, dz,$$

is not identically zero.

Proof. This follows immediately upon noting two things: first, that every irreducible cuspidal representation of $GL_2(\mathbb{A})$ is generic; and second, that for GL_2 the exterior square L-function $L(s, \Pi, \wedge^2)$ is nothing more than the Hecke L-function $L(s, \omega_{\Pi})$, where ω_{Π} is the central character of Π.   □

Just as in the discussion following Proposition 5.7, we can view Proposition 5.11 as identifying which cusp forms on $GL_2(\mathbb{A})$ are functorial transfers from $SO_V(\mathbb{A})$. Once again, this is a consequence of Langlands functoriality. Here, the pole of $L(s, \Pi, \wedge^2)$ at $s = 1$ keeps track of whether or not the L-parameter of Π factors through $Sp_2(\mathbb{C})$, which is the Langlands dual of SO_V.

5B. *The global spherical character.* We are now ready to define the $\mathrm{GL}_2 \times \mathrm{GL}_2$ global spherical character. This is designed to encode the L-function side of Waldspurger's formula. That is, given $\Pi = \Pi_1 \otimes \Pi_2$ a cuspidal representation of $G' = \mathrm{GL}_2 \times \mathrm{GL}_2$, the spherical character I_Π is a distribution that should both

- detect whether Π is a functorial transfer from $G = \mathrm{SO}_W \times \mathrm{SO}_V$ (more accurately, from $O_W \times O_V$),

- compute the Rankin–Selberg L-function $L(s, \Pi) = L(s, \Pi_1 \times \Pi_2)$.

Let us address the first point. If Π is a lift from G, then the central characters of Π_1 and Π_2 are simple enough to determine: namely, the central character of Π_1 must be $\omega = \omega_{E/F}$, i.e., the quadratic automorphic quadratic character of $\mathbb{A}^\times$ corresponding to $E = F[\sqrt{d_W}]$, while the central character of Π_2 must be trivial. Furthermore, our discussion immediately following Propositions 5.7 and 5.11 shows that in addition, $\Pi = \Pi_1 \otimes \Pi_2$ should be a lift from G if and only if the linear functional

$$\mathscr{P}_{\mathrm{dist}} := \mathscr{P}_{\mathrm{Sym}^2}^{\psi^{-1}} \otimes \mathscr{P}_{\wedge^2} : (\Pi_1 \otimes \boldsymbol{R}^{\psi^{-1}}) \otimes \Pi_2 \to \mathbb{C},$$

$$\varphi_1 \otimes \Psi \otimes \varphi_2 \mapsto \int_{[\mathrm{PGL}_2^{(2)}]} \varphi_1(h) \Theta^{\psi^{-1}}(h, \Psi) \, dh \int_{[\mathbb{Z} \mathrm{N}]} \varphi_2(zn) \psi(n) \, dn \, dz,$$

is not identically zero.

Remark 5.12. Note that in the above, we have used ψ^{-1} rather than ψ in the definition of $\mathscr{P}_{\mathrm{Sym}^2}$. This is so that the integrals appearing on the geometric side of our $\mathrm{GL}_2 \times \mathrm{GL}_2$ trace formula are exactly the tonal integrals defined in Part I.

As for the second point, by the discussion of the Rankin–Selberg integral representation, $\mathscr{P}_{\mathrm{RS}}$ exactly computes the L-function we are looking for. This motivates the following definition.

Definition 5.13. Let $f' \in C_c^\infty(G'(\mathbb{A}))$, $\Phi \in \mathscr{S}(\mathbb{A}^{\oplus 2})$, and $\Psi \in \mathscr{S}^{++}(\mathbb{A}_E)$. We define the $\mathrm{GL}_2 \times \mathrm{GL}_2$ *global spherical character* by

$$I_\Pi(s; f' \otimes \Phi \otimes \Psi) = \sum_{\varphi \in \mathrm{ON}(\Pi)} \mathscr{P}_{\mathrm{RS}}(\Pi(f')\varphi \otimes \Phi) \overline{\mathscr{P}_{\mathrm{dist}}(\varphi \otimes \Psi)}.$$

The sum above runs over a fixed orthonormal basis of $\mathrm{ON}(\Pi) \subset \Pi \subset L^2([G'])$, and the operator $\Pi(f)$ is defined by

$$\Pi(f)\varphi = \int_{G'(\mathbb{A})} f(g) \Pi(g)\varphi \, dg.$$

5C. *The* $\mathrm{GL}_2 \times \mathrm{GL}_2$ *trace formula.* Let $f' \in C_c^\infty(G'(\mathbb{A}))$. Consider the kernel function

$$K_{f'}(x, y) = \sum_{\gamma \in G'(F)} f'(x^{-1} \gamma y).$$

This kernel function is formally represented by the sum

$$K_{f'}(x, y) = \sum_{\phi \in \mathrm{ON}(L^2([G']))} (R(f')\phi)(x) \overline{\phi(y)}.$$

Here $\text{ON}(L^2([G']))$ denotes a fixed orthonormal basis of automorphic forms of the L^2 space of $G'(F)\backslash G'(\mathbb{A})$ and $R(f')\phi$ is the right regular representation, namely

$$R(f')\phi(x) = \int_{G'(\mathbb{A})} f'(y)\phi(xy)\,dy.$$

Rather than sum over all ϕ in an orthonormal basis of the full L^2 space, it is helpful to sum only over forms with given central character. That is, let

$$\eta = \eta_1 \otimes \eta_2 : [\mathbb{Z}_{\text{GL}_2} \times \mathbb{Z}_{\text{GL}_2}] = F^\times \backslash \mathbb{A}^\times \times F^\times \backslash \mathbb{A}^\times \to U_1(\mathbb{R})$$

be an automorphic unitary character of the center of G'. Then we define

$$K_{f',\eta}(x, y) = \int_{[Z\times Z]} K_{f'}(x, zy)\eta(z)\,dz.$$

If we write $L^2([G'], \eta)$ for the L^2 completion of the space of square integrable automorphic forms on G' with central character η, then we formally have

$$K_{f',\eta}(x, y) = \sum_{\phi\in\text{ON}(L^2[G'],\eta)} (R(f')\phi)(x)\overline{\phi(y)}.$$

Now let $\eta = \omega \otimes 1$, where $\omega = \omega_{E/F}$.

Definition 5.14. The $\text{GL}_2 \times \text{GL}_2$-*distribution* $I^\psi(s; f'\otimes\Phi\otimes\Psi) = I(s; f'\otimes\Phi\otimes\Psi)$ is defined formally as

$$I(s; f' \otimes \Phi \otimes \Psi)$$
$$= \int_{[N]}\int_{[\text{PGL}_2^{(2)}]}\int_{[\text{PGL}_2]} K_{f',\omega\otimes 1}((g, g), (h, n)) E(s, (1, \omega^{-1}); g, \Phi)\overline{\Theta^{\psi^{-1}}(h, \Psi)}\psi(n^{-1})\,dg\,dh\,dn.$$

Remark 5.15. Note that we are "ignoring" one integration over $[Z]$ that should be appearing in the exterior square period; this is since this integration is hidden in the definition of $K_{f,\omega\otimes 1}$.

The distribution $I^\psi(s; f' \otimes \Phi \otimes \Psi)$ is engineered to decompose formally as a sum of global spherical characters $I_\Pi(s; f' \otimes \Phi \otimes \Psi)$. This is done in the usual way: write $L^2([G'], \omega \otimes 1) = \bigoplus \Pi$, where the "sum" is over all automorphic representations Π of $G' = \text{GL}_2 \times \text{GL}_2$ with central character $\omega \otimes 1$. Then

$$K_{f',\omega\otimes 1}(x, y) = \sum_{\varphi\in\text{ON}(L^2([G'],\omega\otimes 1))} (R(f')\varphi)(x)\overline{\varphi(y)} = \sum_\Pi \sum_{\varphi\in\text{ON}(\Pi)} (\Pi(f')\varphi)(x)\overline{\varphi(y)}$$

and thus

$$I(s; f' \otimes \Phi \otimes \Psi) = \int_{[N]}\int_{[\text{PGL}_2^{(2)}]}\int_{[\text{PGL}_2]} K_{f',\omega\otimes 1}((g, g), (h, n)) E(s, g)\overline{\Theta^{\psi^{-1}}(h)}\psi(n^{-1})\,dg\,dh\,dn$$

$$= \sum_\Pi \sum_{\varphi\in\text{ON}(\Pi)} \mathscr{P}_{\text{RS}}(\Pi(f')\varphi)\overline{\mathscr{P}_{\text{dist}}(\varphi)}$$

$$= \sum_\Pi I_\Pi(s; f' \otimes \Phi \otimes \Psi).$$

Thus I formally decomposes as a sum over all global spherical characters.

5C1. *Decomposition into tones.* Let us now expand $I(s; f' \otimes \Phi \otimes \Psi)$ into a "geometric side".

Theorem 5.16. *The distribution* $I(s) = I(s; f' \otimes \Phi \otimes \Psi)$ *decomposes formally as*

$$I(s) = s(\gamma') \sum_{\gamma' \text{ r.s.s.}} I_{\gamma'}(f' \otimes \Phi \otimes \Psi) + \text{non-r.s.s. tonal integrals}$$

Here the sum is over regular semisimple global tones $\gamma' = (\alpha; \zeta, \beta)$ *and the expression* $I_{\gamma'}$ *(the global tonal integral) is defined by*

$$I_{\gamma'}(f' \otimes \Phi \otimes \Psi) := \int f'\big((g^{-1}, g^{-1})(a(\alpha), 1)(h, 1)\big) \mathscr{W}_{F_s}^{\psi^{-1}}\left(a\left(\frac{\beta}{\alpha}\right)g\right) \mathbf{R}^{\psi} \overline{\Psi}(\zeta),$$

where the integration is over $g \in \mathrm{GL}_2(\mathbb{A})$ *and* $h \in \mathrm{GL}_2^{(2)}(\mathbb{A})$. *For* $\gamma' = (\alpha; \zeta, \beta)$ *a tone, the constant* $s(\gamma')$ *is defined as the size of the fiber over* ζ *of the map*

$$E \to E/\{\pm 1\} \times \{\mathrm{id}, \bar{\cdot}\}.$$

It is either 1, 2, *or* 4.

Finally, *the term "non-r.s.s tonal integrals" refers to the sum*

$$\sum_{\substack{\alpha \in \Sigma \\ \zeta \in E/\{\pm 1\} \times \{\mathrm{id}, \bar{\cdot}\}, \mathrm{N}\zeta = \alpha}} s((\alpha; \zeta, 0)) \int f'\big((g^{-1}, g^{-1})(a(\alpha), 1)(h, 1)\big)(F_s(g) + M(s)F_s(g)) \mathbf{R}^{\psi}(h) \Psi(\zeta).$$

Remark 5.17. Note that if f', Φ, Ψ are all factorizable, then the global tonal integral factors

$$I_{\gamma'}(f' \otimes \Phi \otimes \Psi) = \prod_v I_{\gamma', v}(f'_v \otimes \Phi_v \otimes \Psi_v)$$

as a product of local tonal integrals. In the above equality, we have used the fact that for $\alpha \in \Sigma$, our fixed set of representatives of $F^{\times}/(F^{\times})^2$, we have $\prod_v |\alpha|_v^{\frac{1}{2}} = 1$.

Remark 5.18. For an obvious Zariski open subset of tones, $s(\gamma') = 4$.

Proof of Theorem 5.16. We unfold

$$I(s) = \int \sum_{\gamma = (\gamma_1, \gamma_2) \in G'(F)} f'(g^{-1}\gamma_1 z_1 h, g^{-1}\gamma_2 z_2 n) \omega(z_1) E(g, s) \overline{\Theta(h)} \psi(n^{-1}),$$

where, in the above, integration is over $z_i \in [\mathbb{Z}_{\mathrm{GL}_2}]$, $h \in [\mathrm{PGL}_2^{(2)}]$, $g \in [\mathrm{PGL}_2]$, and $n \in [\mathrm{N}]$. First, change variables $z_1 \mapsto z_2 z_1$. This gives

$$I(s) = \int \sum_{(\gamma_1, \gamma_2) \in G'(F)} f'(g^{-1}z_2\gamma_1 z_1 h, g^{-1}z_2\gamma_2 n) \omega(z_2) E(g, s) \omega(z_1) \overline{\Theta(h)} \psi(n^{-1})$$

$$= \int \sum_{(\gamma_1, \gamma_2) \in G'(F)} f'(g^{-1}z_2\gamma_1 z_1 h, g^{-1}z_2\gamma_2 n) E(z_2^{-1}g, s) \overline{\Theta(z_1 h)} \psi(n^{-1}).$$

Collapsing the integrations over z_1 and z_2 into the integrations over g and h, we can rewrite

$$I(s) = \int \sum_{(\gamma_1, \gamma_2) \in G'} f'(g^{-1}\gamma_1 h, g^{-1}\gamma_2 n) E(g, s) \overline{\Theta(h)} \psi(n^{-1}),$$

where now $g \in [\mathrm{GL}_2]$, $h \in [\mathrm{GL}_2^{(2)}]$, and $n \in [N]$.

Now, we change variables $\gamma_1 \mapsto \gamma_2 \gamma_1$ and unfold the summation in γ_2 to find

$$I(s) = \int \sum_{\gamma_1 \in \mathrm{GL}_2(F)} f'(g^{-1}\gamma_1 h, g^{-1}n) E(g, s) \overline{\Theta(h)} \psi(n^{-1}),$$

where $g \in \mathrm{GL}_2(\mathbb{A})$, $h \in [\mathrm{GL}_2^{(2)}]$, and $n \in [N]$.

We can write every element $\gamma_1 \in \mathrm{GL}_2(F)$ uniquely as $\gamma_1 = a(\alpha)\delta$, where δ runs over $\mathrm{GL}_2^{(2)}(F)$ and where α runs over Σ, a fixed set of representatives of $F^\times/(F^\times)^2$. We unfold in δ and write

$$I(s) = \int \sum_{\alpha \in \Sigma} f'(g^{-1}a(\alpha)h, g^{-1}n) E(g, s) \overline{\Theta(h)} \psi(n^{-1}),$$

where now $g \in \mathrm{GL}_2(\mathbb{A})$, $h \in \mathrm{GL}_2^{(2)}(\mathbb{A})$, and $n \in [N]$.

Change variables $g \mapsto ng$ to find

$$I(s) = \sum_{\alpha \in \Sigma} \int f'(g^{-1}n^{-1}a(\alpha)h, g^{-1}) E(ng, s) \overline{\Theta(h)} \psi(n^{-1}).$$

We write

$$n = n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

with $x \in F\backslash\mathbb{A}$ and conjugate n^{-1} by $a(\alpha)$ to find

$$I(s) = \sum_{\alpha \in \Sigma} \int f'\left(g^{-1}a(\alpha)n(-\alpha^{-1}x)h, g^{-1}\right) E(ng, s) \overline{\Theta(h)} \psi(n^{-1})$$

$$= \sum_{\alpha \in \Sigma} \int f'(g^{-1}a(\alpha)h, g^{-1}) E(ng, s) \overline{\Theta(n(\alpha^{-1}x)h)} \psi(n^{-1}).$$

Now we change variables $x \mapsto \alpha x$, noting that as $\alpha \in F^\times$ the measure on $[N]$ remains unchanged, to arrive at

$$\sum_{\alpha \in \Sigma} \int f'(g^{-1}a(\alpha)h, g^{-1}) E(n(\alpha x)g, s) \overline{\Theta(n(x)h)} \psi(-\alpha x).$$

The integration is running over $g \in \mathrm{GL}_2(\mathbb{A})$, $h \in \mathrm{GL}_2^{(2)}(\mathbb{A})$, and $x \in F\backslash\mathbb{A}$. Note that the integration in x can now pass through f'.

We are led to consider, for each α in our fixed set of representatives of $F^\times/(F^\times)^2$, the inner integral

$$\mathrm{FC}(\alpha) = \mathrm{FC}(\alpha, g, h; \Phi, \Psi, s)$$

$$:= \int_{F\backslash\mathbb{A}} E(n(\alpha x)g, \Phi) \overline{\Theta^{\psi^{-1}}(n(x)h, \Psi)} \psi(-\alpha x)\, dx.$$

We use the Fourier expansions

$$E(n(x)g) = \sum_{\beta' \in F} E^{\beta'}(g)\psi(\beta' x),$$

$$\Theta^{\psi^{-1}}(n(x)h) = \sum_{\beta'' \in F} \Theta^{\psi^{-1},\beta''}(h)\psi(\beta'' x)$$

to determine that

$$\mathrm{FC}(\alpha) = \int_{F \backslash \mathbb{A}} \left(\sum_{\beta' \in F} E^{\beta'}(g)\psi(\alpha\beta' x) \right) \left(\sum_{\beta'' \in F} \overline{\Theta^{\psi^{-1},\beta''}(h)}\psi(-\beta'' x) \right) \psi(-\alpha x) \, dx$$

$$= \sum_{\substack{\beta',\beta'' \in F \\ -\beta'' + \alpha\beta' = \alpha}} E^{\beta'}(g)\overline{\Theta^{\psi^{-1},\beta''}(h)}.$$

That is, this inner integral is a sum of products of Fourier coefficients of the Eisenstein series and theta functions. Using Lemma 5.10 (while keeping in mind that we are using the theta function associated to $\boldsymbol{R}^{\psi^{-1}}$ instead of $\boldsymbol{R}^{\psi}$) we find that

$$\mathrm{FC}(\alpha) = \sum_{\substack{\beta \in F, \zeta \in E \\ \mathrm{N}\zeta + \beta = \alpha}} E^{\frac{\beta}{\alpha}}(g)\overline{(\boldsymbol{R}^{\psi^{-1}}(h)\Psi)(\zeta)},$$

where in the above we have substituted $\beta = \alpha\beta'$.

Substituting this expression back into the decomposition of the distribution I gives

$$I(s; f' \otimes \Phi \otimes \Psi) = \sum_{\alpha \in \Sigma} \sum_{\substack{\beta \in F, \zeta \in E \\ \mathrm{N}\zeta + \beta = \alpha}} \int f'(g^{-1}a(\alpha)h, g^{-1})E^{\frac{\beta}{\alpha}}(g)\overline{(\boldsymbol{R}^{\psi^{-1}}(h)\Psi)(\zeta)},$$

where we integrate over $g \in \mathrm{GL}_2(\mathbb{A})$ and $h \in \mathrm{GL}_2^{(2)}(\mathbb{A})$. Applying Lemma 5.4 and noting that

$$(\boldsymbol{R}^{\psi^{-1}}\Psi)(\zeta') = (\boldsymbol{R}^{\psi^{-1}}\Psi)(-\zeta'),$$

$$\overline{(\boldsymbol{R}^{\psi^{-1}}\Psi)(\zeta')} = (\boldsymbol{R}^{\psi^{-1}}\overline{\Psi})(\bar{\zeta})$$

concludes the calculation. $\square$

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