Algebra & Number Theory

Volume 13

2019

No. 2

G-valued local deformation rings and global lifts

Rebecca Bellovin and Toby Gee

msp

ALGEBRA AND NUMBER THEORY 13:2 (2019)
dx.doi.org/10.2140/ant.2019.13.333

G-valued local deformation rings and global lifts

Rebecca Bellovin and Toby Gee

We study G-valued Galois deformation rings with prescribed properties, where G is an arbitrary (not necessarily connected) reductive group over an extension of $\mathbb{Z}_l$ for some prime l. In particular, for the Galois groups of p-adic local fields (with p possibly equal to l) we prove that these rings are generically regular, compute their dimensions, and show that functorial operations on Galois representations give rise to well-defined maps between the sets of irreducible components of the corresponding deformation rings. We use these local results to prove lower bounds on the dimension of global deformation rings with prescribed local properties. Applying our results to unitary groups, we improve results in the literature on the existence of lifts of mod l Galois representations, and on the weight part of Serre's conjecture.

1. Introduction

The study of Galois deformation rings was initiated in [Mazur 1989], and was crucial to the proof of Fermat's last theorem in [Wiles 1995], and in particular to the modularity lifting theorems proved in [Wiles 1995; Taylor and Wiles 1995]. Many generalisations of these modularity lifting theorems have been proved over the last 25 years, and it has become increasingly important to consider Galois representations valued in reductive groups other than GL_n. From the point of view of the Langlands program, it is particularly important to be able to use disconnected groups, as the L-groups of nonsplit groups are always disconnected. In particular, it is important to study the structure of local deformation rings for general reductive groups, and to prove lifting results for global deformation rings. We briefly review the history of such results in Section 1.1, but we firstly explain the main theorems of this paper.

Gee was supported in part by a Leverhulme Prize, EPSRC grant EP/L025485/1, Marie Curie Career Integration Grant 303605, by ERC Starting Grant 306326, and by a Royal Society Wolfson Research Merit Award.
MSC2010: primary 11F80; secondary 11F85.
Keywords: Galois deformations.

We begin with a result about local deformation rings. Let $K/\mathbb{Q}_p$ be a finite extension, let $\mathcal{O}$ be the ring of integers in a finite extension E of $\mathbb{Q}_l$ with residue field $\mathbb{F}$, where l is possibly equal to p, and let G be a (not necessarily connected) reductive group over $\mathcal{O}$. Given a representation $\bar{\rho} : \mathrm{Gal}_K \to G(\mathbb{F})$, we consider liftings of $\bar{\rho}$ of some inertial type τ, and in the case $l = p$, some p-adic Hodge type $\boldsymbol{v}$. There is a corresponding universal framed deformation ring $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}$, and we prove the following result (as well as a variant for "fixed determinant ψ" deformations).

Theorem A (Theorem 3.3.2). *Fix an inertial type τ, and if $l = p$ then fix a p-adic Hodge type $\boldsymbol{v}$. Then $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}[1/l]$ is generically regular. In addition, $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}$ is equidimensional of dimension*

$$1 + \dim_E G + \delta_{l=p} \dim_E (\mathrm{Res}_{E\otimes K/E}\, G)/P_{\boldsymbol{v}},$$

and $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v},\psi}$ is equidimensional of dimension

$$1 + \dim_E G^{\mathrm{der}} + \delta_{l=p} \dim_E (\mathrm{Res}_{E\otimes K/E}\, G)/P_{\boldsymbol{v}}.$$

(We are abusing notation here; $P_{\boldsymbol{v}}$ is a $(\mathrm{Res}_{E\otimes K/E}\, G)^{\circ}_{\overline{E}}$-conjugacy class of parabolic subgroups of $\mathrm{Res}_{E\otimes K/E}\, G$, and we choose a representative defined over E to compute the dimension of the quotient.) We are also able to describe the regular locus of $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}[1/l]$ precisely in terms of the corresponding Weil–Deligne representations; see Corollary 3.3.4. In the case that $G = \mathrm{GL}_n$ and $l = p$ this is a theorem of Kisin [2008], and results for general groups (but with more restrictive hypotheses than those of Theorem A) were previously proved by Balaji [2013] and Bellovin [2016].

Combining Theorem A with results of [Balaji 2013], we obtain the following result (see Section 4 for any unfamiliar notation or terminology — in particular, $\mathfrak{g}^0_{\mathbb{F}}$ denotes the $\mathbb{F}$-points of the Lie algebra of the derived subgroup of G); in the case of potentially crystalline representations, this is the main result of [loc. cit.].

Theorem B (Proposition 4.2.6). *Let F be totally real, assume that $l > 2$, let S be a finite set of places of F containing all places dividing $l\infty$, and let $\bar{\rho} : \mathrm{Gal}_{F,S} \to G(\overline{\mathbb{F}}_l)$ be a representation admitting a universal deformation ring. Fix inertial types at all places $v \in S$, and Hodge types at all places $v \mid l$, in such a way that the corresponding local deformation rings are nonzero, and let R^{univ} denote the corresponding fixed determinant universal deformation ring for $\bar{\rho}$.*

Assume that $\bar{\rho}$ is odd, and that $H^0(\mathrm{Gal}_{F,S}, (\mathfrak{g}^0_{\mathbb{F}})^*(1)) = 0$. Suppose also that for each place $v \mid l$ the corresponding Hodge type is regular. Then R^{univ} has Krull dimension at least 1.

We use this result to improve on some results about automorphic forms on unitary groups proved using the methods of [Barnet-Lamb et al. 2014]. Beginning with [Clozel et al. 2008], Galois deformations were considered for representations valued in a certain disconnected group $\mathcal{G}_n$, whose connected component is $\mathrm{GL}_n \times \mathrm{GL}_1$ (this group is related to the L-group of a unitary group, see [Buzzard and Gee 2014, §8]). In the case that $G = \mathcal{G}_n$, Theorem B generalises [Barnet-Lamb et al. 2014, Proposition 1.5.1], removing restrictions on the places in S (which were chosen to split in the splitting field of the corresponding unitary group, in order to reduce the local deformation theory to the GL_n case).

We deduce corresponding improvements to a number of results proved using the methods of [loc. cit.], such as the following general result about Serre weights for rank-2 unitary groups, which removes a "split ramification" hypothesis on the ramification of $\bar{r}$ at places away from l.

Theorem C (Theorem 5.2.2). *Let F be an imaginary CM field with maximal totally real subfield F^+, and suppose that F/F^+ is unramified at all finite places, that each place of F^+ above l splits in F, and that $[F^+ : \mathbb{Q}]$ is even. Suppose that l is odd, that $\bar{r} : G_{F^+} \to \mathcal{G}_2(\overline{\mathbb{F}}_l)$ is irreducible and modular, and that $\bar{r}(G_{F(\zeta_l)})$ is adequate.*

Then the set of Serre weights for which $\bar{r}$ is modular is exactly the set of weights given by the sets $W(\bar{r}|_{G_{F_v}})$, $v \mid l$.

(See Remark 5.2.3 for a discussion of further improvements to this result that could be made by techniques orthogonal to those of this paper.) These results are also crucially applied in [Calegari et al. 2018], where they are used to construct lifts of representations valued in $\mathcal{G}_n$ which have prescribed ramification at certain inert places.

1.1. ***A brief historical overview.*** We now give a very brief overview of some of the developments in the deformation theory of Galois representations, which was introduced for representations valued in GL_n in [Mazur 1989]; we apologise for the many important papers that we do not discuss here for reasons of space. The abstract parts of this deformation theory were generalised to arbitrary reductive groups in [Tilouine 1996]. However, for applications to the Langlands program (and in particular to proving automorphy lifting theorems), one needs to study conditions on Galois deformations coming from p-adic Hodge theory.

This was initially done in a somewhat ad hoc fashion, mostly for the group GL_2 and mostly for conditions coming from p-divisible groups, culminating in [Breuil et al. 2001], which used a detailed study of some particular such deformation rings to complete the proof of the Taniyama–Shimura–Weil conjecture. This situation changed with [Kisin 2008], which proved the existence of local deformation rings for GL_n corresponding to general p-adic Hodge theoretic conditions (namely being potentially crystalline or semistable of a given inertial type), and determined the structure of their generic fibres, in particular showing that they are generically regular, and computing their dimensions.

The results of [Kisin 2008] were generalised in [Balaji 2013] to the case of general reductive groups G under the hypothesis of being potentially crystalline, and in [Bellovin 2016] to the case that G is connected, and the inertial type is totally ramified. In the potentially crystalline case the generic fibres of the deformation rings can easily be shown to be regular, whereas in the potentially semistable case, one has to gain some control of the singularities, which is why there are additional restrictions in the theorems of [loc. cit.]. Our Theorem A is a common generalisation of these results to the case that G is possibly disconnected, and the representation is potentially semistable with no condition on the inertial type. (We also simultaneously handle the case that $p \neq l$.)

Another important application of Galois deformation theory to the Langlands program is to prove results showing that mod l representations of the Galois groups of number fields admit lifts to characteristic 0 with prescribed local properties; for example, such results were an important part of Khare and Wintenberger's proof of Serre's conjecture. The first such results were proved in [Ramakrishna 2002] for GL_2, and this method has now been generalised to a wide class of reductive groups; see in particular [Patrikis 2016; Booher 2019a; 2019b]. However, it has two disadvantages: it loses control of the local properties at a finite set of places, and it only applies in cases where formally smooth deformation rings exist.

A different approach was found in [Khare and Wintenberger 2009], which observed that in conjunction with the theory of potential modularity, such lifting results can be deduced from a lower bound on the Krull dimension of a global deformation ring, which was provided by the results of [Böckle 1999]. Kisin [2007] improved on the results of [Böckle 1999], proving a result about presentations of global deformation rings over local ones for GL_n, and deducing a lower bound on the dimensions of global deformation rings. These results were generalised to general reductive groups by Balaji [2013], and given our Theorem A, results such as Theorem B are essentially immediate from Balaji's.

Finally, [Booher and Patrikis 2017] (independently and contemporaneously) proved similar results to those of this paper in the case $l \neq p$ by a related but different method; rather than constructing a large enough supply of unobstructed points, as in this paper, they instead show that all points can be path connected to unobstructed points. We refer to the introduction to [loc. cit.] for a fuller discussion of the difference between the approaches.

1.2. ***Some details.*** We now explain our local results (and their proofs) in more detail. Theorem A is a generalisation of [Kisin 2008, Theorem 3.3.4], which proves the result in the case $l = p$ and $G = \mathrm{GL}_n$. It was previously adapted to the (much easier) case $G = \mathrm{GL}_n$ and $l \neq p$ in [Gee 2011] by using Weil–Deligne representations in place of the filtered (φ, N)-modules employed in [Kisin 2008]. It was also generalised in [Bellovin 2016] to the case that G is connected, $l = p$, and τ is totally ramified. Our approach is in some sense a synthesis of the approaches of [Gee 2011; Bellovin 2016], in that we treat the cases $l \neq p$ and $l = p$ essentially simultaneously, by using Weil–Deligne representations.

We briefly explain our approach, which in broad outline follows that of [Kisin 2008]. It is relatively straightforward (by passing from Galois representations to Weil–Deligne representations using Fontaine's constructions in the case $l = p$, and Grothendieck's monodromy theorem if $l \neq p$) to reduce Theorem A to analogous statements about moduli spaces of Weil–Deligne representations over l-adic fields. These moduli spaces admit an explicit tangent-obstruction theory given by an analogue of Herr's complex computing Galois cohomology in terms of (φ, Γ)-modules, and the key problem is to prove that the H^2 of this complex generically vanishes. We can think of this H^2 as a coherent sheaf over the moduli space, so by considering its support, we can reduce to the problem of exhibiting sufficiently many points at which the H^2 vanishes (which turn out to be precisely the regular points, which in a standard abuse of terminology we refer to as "smooth points").

Our approach to exhibiting these points is related to that taken in [Bellovin 2016], in that it makes use of the theory of associated cocharacters (see Section 2.3), but it is more streamlined and conceptual (for example, we do not need to consider the case $N = 0$ separately, as was done in [loc. cit.]). Surprisingly (at least to us), it is possible to construct all the smooth points that we need by considering the single Weil–Deligne representation $W_K \to \mathrm{SL}_2(\overline{\mathbb{Q}}_l)$ which is trivial on I_K, takes an arithmetic Frobenius element of W_K to

$$\begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix},$$

where q is the order of the residue field of K, and has

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

It is easy to check that this gives a smooth point of the moduli space of Weil–Deligne representations (while the point with the same representation of W_K but with $N = 0$ is not smooth).

Returning to the case of general G, suppose that the inertial type τ is trivial. If we consider a nilpotent element $N \in \mathrm{Lie}\, G$, the theory of associated cocharacters allows us to construct a particular homomorphism $\mathrm{SL}_2 \to G$ taking $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ to N, and an elementary calculation using the representation theory of $\mathfrak{sl}_2$ shows that the composition of our fixed representation $W_K \to \mathrm{SL}_2(\overline{\mathbb{Q}}_l)$ with this homomorphism defines a smooth point. We obtain further smooth points by multiplication by elements of $G(\overline{\mathbb{Q}}_l)$ of finite order, and this turns out to give us all the smooth points we need (even when G is not connected). (See Remark 2.3.10 for an interpretation of this construction in terms of the SL_2 version of the Weil–Deligne group.)

In the case of general τ we reduce to the same situation by replacing G by the normaliser in G of τ, which is also a reductive group. This use of Weil–Deligne representations is what allows us to remove the assumption made in [Bellovin 2016] that the inertial type is totally ramified, which was used in order to choose coordinates so that the inertial type τ was invariant under Frobenius. (Similarly, it clarifies the calculations made for GL_n in [Kisin 2008], as the semilinear algebra becomes linear algebra.) Under this assumption, when studying the structure of the moduli space of G-valued (φ, N, τ)-modules one could exploit the fact that Φ was in the centraliser $Z_G(\tau)$ and N was in $\mathrm{Lie}\, Z_G(\tau)$. Passing to Weil–Deligne representations r lets us argue similarly for general τ: a generator Φ of the unramified quotient of the Weil group normalises the inertial type and N is centralised by the inertial type. Since $Z_G(r|_{I_{L/K}})$ has finite index in the normaliser $N_G(r|_{I_{L/K}})$, we see that N is again in the Lie algebra of the algebraic group containing Φ.

In view of the functorial nature of our construction of smooth points, we are able to produce points on each irreducible component of the generic fibre of the deformation ring which are furthermore "very smooth" in the sense that they give rise to smooth points after restriction to any finite extension K'/K (these points were called "robustly smooth" in [Barnet-Lamb et al. 2014] when $p \neq l$). In particular, the images of such points on the corresponding deformation rings for $\mathrm{Gal}_{K'}$ lie on only one irreducible component, so that we obtain a well-defined "base change" map between irreducible components. We prove a similar result for the maps between deformation rings induced by morphisms of algebraic groups $G \to G'$ (see Section 3.5 for this, and for the case of base change). In particular, this allows one to talk about taking tensor products of components of deformation rings, which is frequently convenient when applying the Harris tensor product trick; see for example [Calegari et al. 2018].

We end this introduction by explaining the structure of the paper. In Section 2, we prove our main results about the structure of the moduli spaces of Weil–Deligne representations; we explain the tangent-obstruction theory and exhibit smooth points, and study the relationship with Galois representations. In doing so we remove the connectedness hypothesis on G made in [Bellovin 2016], by studying exact tensor-filtrations on fibre functors for disconnected reductive groups. We do this via a functor of points approach, using the dynamic approach to parabolic subgroups discussed in [Conrad et al. 2010, §I.2.1]. In Section 3 we deduce our results on the local structure of Galois deformation rings, which we then combine with the results of [Balaji 2013] to prove our lower bound on the dimension of a global deformation ring in Section 4. Finally, in Section 5 we specialise these results to the case of unitary groups.

1.3. *Notation and conventions.* All representations considered in this paper are assumed to be continuous with respect to the natural topologies, and we will never draw attention to this.

If K is a field then we write $\mathrm{Gal}_K := \mathrm{Gal}(\overline{K}/K)$ for its absolute Galois group, where $\overline{K}$ is a fixed choice of algebraic closure; we will regard all algebraic extensions of K as subfields of $\overline{K}$ without further comment, so that in particular we can take the compositum of any two such extensions. If L/K is a Galois extension then we write $\mathrm{Gal}_{L/K} := \mathrm{Gal}(L/K)$, a quotient of Gal_K. If K is a number field and v is a place of K then we fix an embedding $\overline{K} \hookrightarrow \overline{K}_v$, so that we have a homomorphism $\mathrm{Gal}_{K_v} \to \mathrm{Gal}_K$. If S is a finite set of places of a number field K, then we let $K(S)$ be the maximal extension of K (inside $\overline{K}$) which is unramified outside S, and write $\mathrm{Gal}_{K,S} := \mathrm{Gal}(K(S)/K)$.

If $K/\mathbb{Q}_p$ is a finite extension for some prime p then we write I_K for the inertia subgroup of Gal_K, W_K for the Weil group, and f_K for the inertial degree of $K/\mathbb{Q}_p$. We let φ denote the arithmetic Frobenius on $\overline{\mathbb{F}}_p$, so that we have an exact sequence

$$1 \to I_K \to W_K \to \langle \varphi^{f_K} \rangle \to 1,$$

and we let $v : W_K \to \mathbb{Z}$ be the function such that $v(g) = i$ if the image of g modulo I_K is $\varphi^{i f_K}$. Recall that a Weil–Deligne representation of W_K is a pair (r, N) consisting of a finite-dimensional representation $r : W_K \to \mathrm{End}(V)$ and a (necessarily nilpotent) endomorphism $N \in \mathrm{End}(V)$ satisfying

$$\rho(g)N = p^{v(g) f_K} N \rho(g)$$

for all $g \in W_K$.

1.3.1. *Parabolic subgroups.* If G is a finite-type affine group scheme over A, and $\lambda : \mathbb{G}_m \to G$ is a cocharacter of G, then there is a subgroup $P_G(\lambda)$ of G associated to λ as follows. Following [Conrad et al. 2010, §I.2.1], for any A-algebra A' we define the functors

$$P_G(\lambda)(A') = \{g \in G(A') \mid \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists}\},$$
$$U_G(\lambda)(A') = \{g \in P_G(\lambda)(A') \mid \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} = 1\}.$$

We also let $Z_G(\lambda)$ denote the scheme-theoretic centraliser of λ. All of these functors are representable by subgroup schemes of G, and they are smooth if G is smooth. By construction, the formation of $P_G(\lambda)$, $U_G(\lambda)$, and $Z_G(\lambda)$ commutes with base change on A.

The cocharacter λ induces a grading on the Lie algebra $\mathfrak{g} := \mathrm{Lie}\, G$. Let $\mathfrak{g}_n := \{v \in \mathfrak{g} \mid \mathrm{Ad}(\lambda(t))(v) = t^n v\}$ and let $\mathfrak{g}_{\geq 0} := \bigoplus_{n \geq 0} \mathfrak{g}_n$. Then $\mathrm{Lie}\, P_G(\lambda) = \mathfrak{g}_{\geq 0}$, $\mathrm{Lie}\, U_G(\lambda) = \mathfrak{g}_{\geq 1}$, and $\mathrm{Lie}\, Z_G(\lambda) = \mathfrak{g}_0$.

The multiplication map $Z_G(\lambda) \ltimes U_G(\lambda) \to P_G(\lambda)$ is an isomorphism. Furthermore, the fibres of $U_G(\lambda)$ are unipotent and connected. If the morphism $G \to \operatorname{Spec} A$ has connected reductive fibres, then $P_G(\lambda)$ is a parabolic subgroup scheme with connected fibres, $U_G(\lambda)$ is its unipotent radical, and $Z_G(\lambda)$ is connected and reductive.

1.3.2. *Deformation rings.* Let l be prime, and let $\mathcal{O}$ be the ring of integers in a finite extension $E/\mathbb{Q}_l$ with residue field $\mathbb{F}$. Write $\mathrm{CNL}_{\mathcal{O}}$ for the category of complete local noetherian $\mathcal{O}$-algebras with residue field $\mathbb{F}$.

Let Γ be either the absolute Galois group Gal_K of a finite extension K of $\mathbb{Q}_l$ for some p (possibly equal to l), or a group $\mathrm{Gal}_{K,S}$ where S is a finite set of places of a number field K.

Let G be a smooth affine group scheme over $\mathcal{O}$ whose geometric fibres are reductive (but not necessarily connected), and fix a homomorphism $\bar{\rho} : \Gamma \to G(\mathbb{F})$. A *framed deformation* of $\bar{\rho}$ to a ring $A \in \mathrm{CNL}_{\mathcal{O}}$ is a homomorphism $\rho : \Gamma \to G(A)$ whose reduction modulo $\mathfrak{m}_A$ is equal to $\bar{\rho}$. The functor of framed deformations is represented by the universal framed deformation $\mathcal{O}$-algebra $R^{\square}_{\bar{\rho}}$, an object of $\mathrm{CNL}_{\mathcal{O}}$ [Balaji 2013, Theorem 1.2.2].

Suppose from now on for the rest of the paper that the centre Z_G of G is smooth over $\mathcal{O}$. Write $\mathfrak{g}_{\mathbb{F}}$ and $\mathfrak{z}_{\mathbb{F}}$ for the $\mathbb{F}$-points of the Lie algebras of G and Z_G respectively; Γ acts on $\mathfrak{g}_{\mathbb{F}}$ via the adjoint action composed with $\bar{\rho}$. A *deformation* of $\bar{\rho}$ to A is a $(\ker(G(A) \to G(\mathbb{F})))$-conjugacy class of framed deformations of $\bar{\rho}$ to A. If $H^0(\Gamma, \mathfrak{g}_{\mathbb{F}}) = \mathfrak{z}_{\mathbb{F}}$, then the functor of deformations is represented by the universal framed deformation $\mathcal{O}$-algebra $R_{\bar{\rho}}$, an object of $\mathrm{CNL}_{\mathcal{O}}$; see [Balaji 2013, Theorem 1.2.2] or [Tilouine 1996, Theorem 3.3], together with Comment (2) following [loc. cit., Theorem 3.3].

We will also consider "fixed determinant" versions of these (framed) deformations rings. Let G^{ab} and G^{der} respectively denote the abelianisation and derived subgroup of G, and write $\mathrm{ab} : G \to G^{\mathrm{ab}}$ for the natural map. Write $\mathfrak{g}^0_{\mathbb{F}}$ for the $\mathbb{F}$-points of the Lie algebra of G^{der}. Fix a homomorphism $\psi : \Gamma \to G^{\mathrm{ab}}(\mathcal{O})$ such that $\mathrm{ab} \circ \bar{\rho} = \bar{\psi}$. We let $R^{\square,\psi}_{\bar{\rho}}$ denote the quotient of $R^{\square}_{\bar{\rho}}$ corresponding to deformations ρ with $\mathrm{ab} \circ \rho = \psi$ and $R^{\psi}_{\bar{\rho}}$ denote the quotient of $R_{\bar{\rho}}$ corresponding to framed deformations ρ with $\mathrm{ab} \circ \rho = \psi$.

We write G° for the connected component of G containing the identity. We will always consider representations up to G°-conjugacy, rather than G-conjugacy; note that this is compatible with our definition of deformations, as an element of $(\ker(G(A) \to G(\mathbb{F})))$ is necessarily contained in $G^\circ(A)$.

We for the most part allow any coefficient field E, although for some constructions in p-adic Hodge theory we need to allow it to be sufficiently large; we will comment when we do this. The effect of replacing E with a finite extension E' with ring of integers $\mathcal{O}'$ is simply to replace $R^{\square}_{\bar{\rho}}$ and $R_{\bar{\rho}}$ with $R^{\square}_{\bar{\rho}} \otimes_{\mathcal{O}} \mathcal{O}'$ and $R_{\bar{\rho}} \otimes_{\mathcal{O}} \mathcal{O}'$ respectively.

2. Moduli of Weil–Deligne representations

Let $K/\mathbb{Q}_p$ be a finite extension, and let l be a prime, possibly equal to p. In this section we prove analogues for l-adic Weil–Deligne representations of some results on moduli spaces of weakly admissible modules from [Kisin 2008; Bellovin 2016], and remove some hypotheses imposed in those papers; in particular, we allow our groups to be disconnected, and we work with arbitrary inertial types (rather than totally ramified types). In the case that $l = p$ we relate our moduli spaces to those for weakly admissible modules. In Section 3 we will use these results to study the generic fibres of deformation rings in both the case $l = p$ and the case $l \neq p$.

2.1. ***Moduli of Weil–Deligne representations.*** Let $K/\mathbb{Q}_p$ be a finite extension, and let L/K be a finite Galois extension. As in Section 1.3, we let $E/\mathbb{Q}_l$ be a finite extension for some prime l, with ring of integers $\mathcal{O}$. We also continue to let G be a (not necessarily connected) reductive group over $\mathcal{O}$; in fact, throughout this section we will be working with l inverted, and we will write G for G_E without further comment. We write $\mathfrak{g}_E$ for the Lie algebra of G.

A morphism of G-torsors $f : D \to D'$ over an E-scheme X is a morphism of the underlying X-schemes which is equivariant for the action of G_X. Such a morphism is necessarily an isomorphism. The G-equivariant automorphisms of D, which we denote by $\mathrm{Aut}_G(D)$, form a group, and it makes sense to talk about homomorphisms $r : W_K \to \mathrm{Aut}_G(D)$. We also define a sheaf of automorphism groups $\underline{\mathrm{Aut}}_G(D)$ over X; if X' is an X-scheme, its X'-points are given by $\underline{\mathrm{Aut}}_G(D)(X') := \mathrm{Aut}_G(D_{X'})$. This is a representable functor, since $\underline{\mathrm{Aut}}_G(D)$ is étale-locally isomorphic to G_X, which is affine. We abuse notation by writing $\underline{\mathrm{Aut}}_G(D)$ for the group scheme, as well.

Definition 2.1.1. Let $G\text{-}\mathrm{WD}_E(L/K)$ be the category cofibred in groupoids over E-Alg whose fibre over an E-algebra A is a G-torsor D over A together with a pair (r, N), where now $r : W_K \to \mathrm{Aut}_G(D)$ is a representation of the Weil group such that $r|_{I_L}$ is trivial, $N \in \mathrm{Lie}\,\underline{\mathrm{Aut}}_G(D)$, and $N = p^{-v(g) f_K} \mathrm{Ad}(r(g))(N)$ for all $g \in W_K$.

Requiring D to be a trivial G-torsor equipped with a trivialising section lets us define a representable functor covering $G\text{-}\mathrm{WD}_E(L/K)$, as follows. The exact sequence

$$0 \to I_K \to W_K \to \langle \varphi^{f_K} \rangle \cong \mathbb{Z} \to 0$$

is noncanonically split, and choosing a splitting is the same as choosing a lift $g_0 \in W_K$ of φ^{f_K}. Thus, to specify a representation $r : W_K \to \mathrm{Aut}_G(D)$, it suffices to specify $r|_{I_K}$ and $r(g_0)$ (which we denote by Φ). Since we are interested in representations which are trivial on I_L, we may replace $r|_{I_K}$ with $r|_{I_{L/K}}$. For an E-algebra A, we let $\mathrm{Rep}_A\, I_{L/K}$ denote the set of A-linear representations of $I_{L/K}$ on $G(A)$.

Definition 2.1.2. Choose $g_0 \in W_K$ lifting φ^{f_K}. We let $Y_{L/K,\varphi,\mathcal{N}}$ be the functor on the category of E-algebras whose A-points are triples

$$(\Phi, N, \tau) \in G(A) \times \mathfrak{g}_E(A) \times \mathrm{Rep}_A\, I_{L/K}$$

which satisfy

- $N = p^{-f_K} \mathrm{Ad}(\Phi)(N)$,
- $\Phi \circ \tau(g) \circ \Phi^{-1} = \tau(g_0 g g_0^{-1})$ for all $g \in I_{L/K}$, and
- $N = \mathrm{Ad}(\tau(g))(N)$ for all $g \in I_{L/K}$.

To go from $Y_{L/K,\varphi,\mathcal{N}}$ to $G\text{-}\mathrm{WD}_E(L/K)$, we need to forget the trivialising section and also forget g_0; the representation associated to (Φ, N, τ) is given by

$$r(g_0^n h) = \Phi^n \tau(h),$$

where $n \in \mathbb{Z}$ and $h \in I_K$.

The functor $Y_{L/K,\varphi,\mathcal{N}}$ is visibly represented by a finite-type affine scheme over E, and there is an action of G on $Y_{L/K,\varphi,\mathcal{N}}$ given by changing the trivialising section; explicitly,

$$a \cdot (\Phi, N, \{\tau(g)\}_{g \in I_{L/K}}) := (a\Phi a^{-1}, \mathrm{Ad}(a)(N), \{a\tau(g)a^{-1}\}_{g \in I_{L/K}}).$$

Recall that if Z is an E-scheme equipped with a left-action of an algebraic group H over E, then for any E-scheme S, the groupoid $[Z/H](S)$ over S is the category

$$[Z/H](S) := \{\text{Left } H\text{-bundle } D \to S \text{ and } H\text{-equivariant morphism } D \to Z\}.$$

A morphism $f : D \to D'$ in this fibre category is a morphism of H-torsors over S.

Lemma 2.1.3. *The quotient stack $[Y_{L/K,\varphi,\mathcal{N}}/G]$ is equivalent to the groupoid $G\text{-}\mathrm{WD}_E(L/K)$.*

Proof. We choose $g_0 \in W_K$ lifting φ^{f_K}. Given an A-valued point of $G\text{-}\mathrm{WD}_E(L/K)$ with underlying G-torsor D, the base change $D \times_A D \to D$ (which is projection on the first factor) is a trivial G-torsor (with G acting on the second factor). The identity morphism $D \xrightarrow{\sim} D$ induces a canonical trivialising section $D \to D \times_A D$, namely the diagonal. Pulling back r and N to $D \times_A D$, writing them in coordinates (with respect to the trivialising section), and writing $\tau := r|_{I_{L/K}}$ and $\Phi := r(g_0)$ gives us a morphism $D \to Y_{L/K,\varphi,\mathcal{N}}$.

We need to check that the morphism $D \to Y_{L/K,\varphi,\mathcal{N}}$ is G-equivariant. If A' is an A-algebra, the morphism $D \to Y_{L/K,\varphi,\mathcal{N}}$ carries $x \in D(A')$ to the fibre of (Φ, N, τ) over x. The fibre of $D \times_A D \to D$ over x is a copy of $D_{A'}$, together with a section (defined by taking the fibre of the diagonal over x). If $g \in G(A')$, the fibre of $D \times_A D \to D$ over $g \cdot x$ is also a copy of $D_{A'}$, but the section has been multiplied by g. Thus, our "change-of-basis" formula for triples (Φ, N, τ) implies that the morphism $D \to Y_{L/K,\varphi,\mathcal{N}}$ is G-equivariant, as required. □

Similarly, we let $Y_{L/K,\mathcal{N}}$ denote the functor on the category of E-algebras parametrising pairs

$$(N, \tau) \in \mathfrak{g}_E(A) \times \mathrm{Rep}_A\, I_{L/K}$$

such that $N = \mathrm{Ad}(\tau(g))(N)$ for all $g \in I_{L/K}$; and we let $Y_{L/K}$ be the functor on the category of E-algebras, whose A-points are $\mathrm{Rep}_A\, I_{L/K}$.

Let K'/K be a finite extension, and write L'/K' for the compositum of K' and L. Then L'/K' is Galois, with Galois group $\mathrm{Gal}_{L'/K'} \subset \mathrm{Gal}_{L/K}$. There are versions of the above functors for L'/K' which we write $Y_{L'/K',\varphi,\mathcal{N}}$, $Y_{L'/K',\mathcal{N}}$, and $Y_{L'/K'}$. Restriction of Weil–Deligne representations from W_K to $W_{K'}$ induces morphisms $Y_{L/K,\varphi,\mathcal{N}} \to Y_{L'/K',\varphi,\mathcal{N}}$, $Y_{L/K,\mathcal{N}} \to Y_{L'/K',\mathcal{N}}$ and $Y_{L/K} \to Y_{L'/K'}$.

2.2. ***A tangent-obstruction theory for G-$\mathrm{WD}_E(L/K)$.*** Choose an object $D_A \in G\text{-}\mathrm{WD}_E(L/K)$ with coefficients in an E-algebra A, and let $\operatorname{ad} D_A$ denote the Weil–Deligne module induced on Lie $\underline{\mathrm{Aut}}_G D_A$. Choose $g_0 \in W_K$ which lifts φ^{f_K} and write $\Phi := r(g_0)$, let $\mathrm{Ad}(\Phi)$ denote the action on $\operatorname{ad} D_A$ given by differentiating the homomorphism $\underline{\mathrm{Aut}}_G D_A \to \underline{\mathrm{Aut}}_G D_A$ given by $g \mapsto \Phi g \Phi^{-1}$, and let ad_N act by $x \mapsto [N, x]$. If $G = \mathrm{GL}_n$ and D_A is the trivial torsor, these actions become $x \mapsto \Phi \circ x \circ \Phi^{-1}$ and $x \mapsto N \circ x - x \circ N$, respectively. Then we have an anticommutative diagram:

$$\begin{array}{ccc} (\operatorname{ad} D_A)^{I_{L/K}} & \xrightarrow{1-\mathrm{Ad}(\Phi)} & (\operatorname{ad} D_A)^{I_{L/K}} \\ \downarrow{\scriptstyle \mathrm{ad}_N} & & \downarrow{\scriptstyle \mathrm{ad}_N} \\ (\operatorname{ad} D_A)^{I_{L/K}} & \xrightarrow{p^{-f_K}\mathrm{Ad}(\Phi)-1} & (\operatorname{ad} D_A)^{I_{L/K}} \end{array}$$

Here $g \in I_{L/K}$ acts on $\operatorname{ad} D_A$ via $\mathrm{Ad}(\tau(g))$; note that the minus sign in p^{-f_K} arises because g_0 is a lift of arithmetic Frobenius. This diagram does not depend on our choice of g_0, because any two lifts of φ^{f_K} differ by an element of $I_{L/K}$, which acts trivially on $(\operatorname{ad} D_A)^{I_{L/K}}$.

The total complex $C^\bullet(D_A)$ of this double complex controls the deformation theory of objects of G-$\mathrm{WD}_E(L/K)$. We write $H^i(\operatorname{ad} D_A)$ for the cohomology groups of $C^\bullet(D_A)$. The following result will be proved in a very similar way to [Kisin 2008, Proposition 3.1.2], which is an analogous result for semilinear representations in the case $G = \mathrm{GL}_n$.

Proposition 2.2.1. *Let A be a local E-algebra with maximal ideal $\mathfrak{m}_A$ and let $I \subset A$ be an ideal with $I\mathfrak{m}_A = (0)$. Let $D_{A/I}$ be an object of G-$\mathrm{WD}_E(L/K)$ with coefficients in A/I, with Weil–Deligne representation $(\bar{r}, \bar{N})$. Then:*

(1) *If $H^2(\operatorname{ad} D_{A/\mathfrak{m}_A}) = 0$, then there exists an object D_A in G-$\mathrm{WD}_E(L/K)$ with coefficients in A, such that $(A/I) \otimes_A D_A \cong D_{A/I}$.*

(2) *The set of isomorphism classes of liftings of $D_{A/I}$ to D_A is either empty or a torsor under $I \otimes_{A/\mathfrak{m}_A} H^1(\operatorname{ad} D_{A/\mathfrak{m}_A})$.*

We begin by proving a preliminary lemma.

Lemma 2.2.2. *Let D_A be a G-torsor over A, and suppose there is a representation $\bar{r} : W_K \to \mathrm{Aut}_G(D_{A/I})$ such that $\bar{r}|_{I_L}$ is trivial. Then there is a representation $r : W_K \to \mathrm{Aut}_G(D_A)$ such that $r|_{I_L}$ is trivial and r lifts $\bar{r}$. Moreover, the set of infinitesimal automorphisms of r (as a lift of $\bar{r}$) is a torsor under*

$$H^0(W_K/I_L, I \otimes_{A/\mathfrak{m}_A} \operatorname{ad} D_{A/\mathfrak{m}_A}^{I_L}) = I \otimes_{A/\mathfrak{m}_A} \operatorname{ad} D_{A/\mathfrak{m}_A}^{W_K},$$

and the set of lifts of $\bar{r}$ is a torsor under

$$H^1(W_K/I_K, I \otimes_{A/\mathfrak{m}_A} \operatorname{ad} D_{A/\mathfrak{m}_A}^{I_{L/K}}).$$

Proof. An isomorphism $\bar{f} : D_{A/I} \to D_{A/I}$ lifts to an isomorphism $f : D_A \to D_A$, and the set of such lifts is a torsor under either a left- or right-action of $H^0(A, I \otimes_{A/\mathfrak{m}_A} D_{A/\mathfrak{m}_A})$ by [Bellovin 2016, Lemma 3.5]. Thus, for each $g \in W_K$, we can lift the map $\bar{r}(g) : D_{A/I} \to D_{A/I}$ to an isomorphism $r(g) : D_A \to D_A$.

The assignment

$$(g_1, g_2) \mapsto r(g_1)r(g_2)r(g_1g_2)^{-1}$$

is a 2-cocycle of W_K/I_L valued in $I \otimes_{A/\mathfrak{m}_A} \operatorname{ad} D_{A/\mathfrak{m}_A}$. Since we are in characteristic 0, and $I_{L/K}$ is a finite group, the Hochschild–Serre spectral sequence implies that for each $i > 0$, we have an isomorphism

$$H^i(W_K/I_K, I \otimes_{A/\mathfrak{m}_A} \operatorname{ad} D_{A/\mathfrak{m}_A}^{I_{L/K}}) \xrightarrow{\sim} H^i(W_K/I_L, I \otimes_{A/\mathfrak{m}_A} \operatorname{ad} D_{A/\mathfrak{m}_A}).$$

In particular,

$$H^2(W_K/I_L, I \otimes_{A/\mathfrak{m}_A} \operatorname{ad} D_{A/\mathfrak{m}_A}) \cong H^2(\widehat{\mathbb{Z}}, I \otimes_{A/\mathfrak{m}_A} \operatorname{ad} D_{A/\mathfrak{m}_A}^{I_{L/K}}) = 0,$$

so $\bar{r}$ lifts to a representation $r : W_K \to \operatorname{Aut}_G(D_A)$ with $r|_{I_L} = 0$, as claimed.

An isomorphism $f : D_A \to D_A$ is an infinitesimal automorphism of r if and only if it is the identity modulo I and $r(g) \circ f = f \circ r(g)$ for all $g \in W_K$. Equivalently, f is an element of $I \otimes_{A/\mathfrak{m}_A} \operatorname{ad} D_{A/\mathfrak{m}_A}$ fixed by W_K, and since I is a vector space over $A/\mathfrak{m}_A$, this is equivalent to $f \in I \otimes_{A/\mathfrak{m}_A} \operatorname{ad} D_{A/\mathfrak{m}_A}^{W_K}$, as desired.

Finally, if $r' : W_K \to \operatorname{Aut}_G(D)$ is another such lift, then $g \mapsto r'(g)r(g)^{-1}$ is a 1-cocycle of W_K/I_L valued in $I \otimes_{A/\mathfrak{m}_A} \operatorname{ad} D_{A/\mathfrak{m}_A}$. But

$$H^1(W_K/I_L, I \otimes_{A/\mathfrak{m}_A} \operatorname{ad} D_{A/\mathfrak{m}_A}) \cong H^1(W_K/I_K, I \otimes_{A/\mathfrak{m}_A} \operatorname{ad} D_{A/\mathfrak{m}_A}^{I_{L/K}}),$$

so we are done. □

Proof of Proposition 2.2.1. By [Bellovin 2016, Lemma 3.4], the underlying G-torsor $D_{A/I}$ lifts to a G-torsor D_A over $\operatorname{Spec} A$, and D_A is unique up to isomorphism, and by Lemma 2.2.2, $\bar{r}$ lifts to a representation $r : W_K \to \operatorname{Aut}_G(D_A)$. Moreover, by [loc. cit., Lemma 3.7], $\bar{N} \in \operatorname{ad} D_{A/I}$ lifts to some $N \in \operatorname{ad} D_A$ such that $\operatorname{Ad}(r(g))(N) = N$ for all $g \in I_{L/K}$, and any two lifts differ by an element of $I \otimes_{A/\mathfrak{m}_A} (\operatorname{ad} D_{A/\mathfrak{m}_A})^{I_{L/K}}$.

Now D_A, together with r and N, is an object of $G\text{-}\mathrm{WD}_E(L/K)$ if and only if $N = p^{-f_K} \operatorname{Ad}(\Phi)(N)$, where $\Phi := r(\varphi^{f_K})$. We define

$$h := N - p^{-f_K} \operatorname{Ad}(\Phi)(N) \in I \otimes_{A/\mathfrak{m}_A} \operatorname{ad} D_{A/\mathfrak{m}_A}^{I_{L/K}}.$$

If $H^2(\operatorname{ad} D_{A/\mathfrak{m}_A}) = 0$, then by definition there exist $f, g \in I \otimes_{A/\mathfrak{m}_A} \operatorname{ad} D_{A/\mathfrak{m}_A}^{I_{L/K}}$ such that $h = \operatorname{ad}_{\bar{N}}(f) + (p^{-f_K} \operatorname{Ad}(\bar{\Phi}) - 1)(g)$. We can view f and g either as elements of $\operatorname{Aut}_G(D_A)$ (congruent to the identity modulo I) or as elements of its tangent space. Thus we claim that if we define $\widetilde{N} := N + g$ and $\widetilde{\Phi} := f^{-1} \circ \Phi$, then $\widetilde{N} = p^{-f_K} \operatorname{Ad}(\widetilde{\Phi})(\widetilde{N})$. Indeed,

$$\begin{aligned}
&\widetilde{N} - p^{-f_K} \operatorname{Ad}(\widetilde{\Phi})(\widetilde{N}) \\
&\qquad = N + g - p^{-f_K}(\operatorname{Ad}(1 - f) \circ \operatorname{Ad}(\Phi))(N + g) \\
&\qquad = N + g - p^{-f_K} \operatorname{Ad}(\Phi)(N) - p^{-f_K} \operatorname{Ad}(\Phi)(g) + p^{-f_K}[f, \operatorname{Ad}(\Phi)(N)] + p^{-f_K}[f, \operatorname{Ad}(\Phi)(g)] \\
&\qquad = \operatorname{ad}_{\bar{N}}(f) + p^{-f_K}[f, \operatorname{Ad}(\Phi)(N)] \\
&\qquad = [h, f] = 0.
\end{aligned}$$

Here we have used that $f, g, h \in I \otimes_{A/\mathfrak{m}_A} \operatorname{ad} D_{A/\mathfrak{m}_A}^{\mathrm{Gal}_{L/K}}$ and $I \cdot I \subset I\mathfrak{m}_A = 0$, so the Lie brackets $[f, \operatorname{Ad}(\Phi)(g)]$ and $[h, f]$ vanish. This proves part (1).

Now suppose that $\widetilde{N} = p^{-f_K} \operatorname{Ad}(\widetilde{\Phi})(\widetilde{N})$, and let $f, g \in I \otimes_{A/\mathfrak{m}_A} \operatorname{ad} D_{A/\mathfrak{m}_A}^{I_{L/K}}$. Define $\widetilde{N}' := N + g$ and $\widetilde{\Phi}' := f^{-1} \circ \widetilde{\Phi}$. Then

$$\begin{aligned}
&\widetilde{N}' - p^{-f_K} \operatorname{Ad}(\widetilde{\Phi}')(\widetilde{N}') \\
&\qquad = \widetilde{N} + g - p^{-f_K} \operatorname{Ad}(\widetilde{\Phi})(\widetilde{N}) - p^{-f_K} \operatorname{Ad}(\widetilde{\Phi})(g) + p^{-f_K}[f, \operatorname{Ad}(\widetilde{\Phi})(\widetilde{N})] + p^{-f_K}[f, \operatorname{Ad}(\widetilde{\Phi})(g)] \\
&\qquad = (1 - p^{-f_K} \operatorname{Ad}(\widetilde{\Phi}))(g) + [f, \widetilde{N}] \\
&\qquad = -(p^{-f_K} \operatorname{Ad}(\Phi) - 1)(g) - \operatorname{ad}_N(f).
\end{aligned}$$

Thus, $\widetilde{\Phi}', \widetilde{N}'$ give another lift if and only if $(f, g) \in \ker(d^1)$.

Moreover, if $(\widetilde{\Phi}', \widetilde{N}')$ is another lift, it is isomorphic to $(\widetilde{\Phi}, \widetilde{N})$ if and only if there is some $j \in I \otimes_{A/\mathfrak{m}_A} D_{A/\mathfrak{m}_A}^{I_{L/K}}$ such that

$$\widetilde{N}' = \operatorname{Ad}(1 + j)(\widetilde{N}) \quad \text{and} \quad (1 + j)\widetilde{\Phi} = \widetilde{\Phi}'(1 + j).$$

This is equivalent to

$$\widetilde{N} - \widetilde{N}' = \operatorname{ad}_N(j) \quad \text{and} \quad \widetilde{\Phi}(\widetilde{\Phi}')^{-1} = 1 - (1 - \operatorname{Ad}(\Phi))(j).$$

In other words, $(\widetilde{\Phi}, \widetilde{N})$ and $(\widetilde{\Phi}', \widetilde{N}')$ differ by an element of $\operatorname{im}(d^0)$, as required. □

2.3. ***Construction of smooth points.*** We wish to show that "most" points of $Y_{L/K,\varphi,\mathcal{N}}$ are smooth, and so are their images in $Y_{L'/K',\varphi,\mathcal{N}}$ for any finite extension K'/K. In this section we will consider a single fixed extension K'/K, and in Section 2.4 below we will deduce a result for all extensions K'/K simultaneously.

We begin by fixing an inertial type $\tau : I_{L/K} \to G(E)$. This amounts to considering the fibre of $Y_{L/K,\varphi,\mathcal{N}} \to Y_{L/K}$ over the point corresponding to τ. Next, we observe that if we can find $r : W_K \to G(E)$ such that $r|_{I_K} = \tau$, then $\Phi := r(g_0)$ is an element of the algebraic group defined over E

$$N_G(\tau) := \{h \in G \mid hr(g)h^{-1} \in r(I_{L/K}) \text{ for all } g \in I_{L/K}\}.$$

Note that Φ is not necessarily an element of the centraliser

$$Z_G(\tau) := \{h \in G \mid hr(g)h^{-1} = r(g) \text{ for all } g \in I_{L/K}\}.$$

However, since $I_{L/K}$ is finite (and in particular has only finitely many automorphisms), $Z_G(\tau) \subset N_G(\tau)$ has finite index; so we have $Z_G(\tau)^\circ = N_G(\tau)^\circ$ and $\operatorname{Lie} Z_G(\tau) = \operatorname{Lie} N_G(\tau)$. In particular, this implies that $N_G(\tau)$ and $Z_G(\tau)$ are reductive:

Theorem 2.3.1. *The normaliser $N_G(\tau) := \{h \in G \mid hr(g)h^{-1} \in r(I_{L/K}) \text{ for all } g \in I_{L/K}\}$ of $\tau(I_{L/K})$ is a reductive group.*

Proof. Since we are working over a field of characteristic 0, it is enough to prove that the connected component of the identity $N_G(\tau)^\circ = Z_G(\tau)^\circ = Z_{G^\circ}(\tau)^\circ$ is reductive. But reductivity for the latter group follows from [Prasad and Yu 2002, Theorem 2.1], which states that when a finite group acts on a connected reductive group, the connected component of the identity of the fixed points is reductive. □

Remark 2.3.2. Prasad and Yu prove their result under the assumption that the characteristic of the ground field does not divide the order of the group. Conrad, Gabber, and Prasad prove a more general result [Conrad et al. 2010, Proposition A.8.12], assuming only that the algebraic group acting is geometrically linearly reductive.

Our hypotheses imply that $N \in \operatorname{Lie} Z_G(\tau)$ and $\Phi \in N_G(\tau)$. However, if (r, N) exists and has the correct inertial type, the set of $\Phi \in G(E)$ compatible with $r|_{I_{L/K}}$ and N is a torsor under $Z_G(\tau) \cap Z_G(N)$.

We now briefly recall the theory of associated cocharacters over a field of characteristic 0; we refer the reader to [Jantzen 2004] (in particular Section 5) for further details and proofs. We will not draw attention to the assumption that our ground field has characteristic 0 below (but we will frequently use it); on the other hand, we do explain why the results that we are recalling hold over arbitrary fields of characteristic 0.

If $N \in \mathfrak{g}$ is nilpotent, a cocharacter $\lambda : \mathbb{G}_m \to G$ is said to be *associated* to N if

- $\operatorname{Ad}(\lambda(t))(N) = t^2 N$, and
- λ takes values in the derived subgroup of a Levi subgroup $L \subset G$ for which $N \in \mathfrak{l} := \operatorname{Lie} L$ is distinguished (that is, every torus contained in $Z_L(N)$ is contained in the centre of L).

By [McNinch 2004, Theorem 26], for any N there exists a cocharacter associated to N which is defined over the same field as N. Any two cocharacters associated to N are conjugate under the action of $Z_G(N)^\circ$.

An $\mathfrak{sl}_2$*-triple* is as usual a nonzero triple (X, H, Y) of elements of $\mathfrak{g}$ such that $[H, X] = 2X$, $[H, Y] = -2Y$, and $[X, Y] = H$. The Jacobson–Morozov theorem [Bourbaki 2005, Chapter VIII, §11, Proposition 2] states that for a nonzero nilpotent element N in a semisimple Lie algebra, an $\mathfrak{sl}_2$-triple (N, H, Y) always exists, and any two such triples (N, H, Y) and (N, H', Y') are conjugate under the action of $Z_G(N)^\circ$ [loc. cit., Chapter VIII, §11, Proposition 1]. Given a pair (N, H) such that $[H, N] = 2N$ and $H \in [N, \mathfrak{g}]$, it is possible to construct an $\mathfrak{sl}_2$-triple (N, H, Y) [loc. cit., Chapter VIII, §11, Lemme 6] (or the zero triple if $N = H = 0$). Since SL_2 is simply connected, this implies that there is a homomorphism $\mathrm{SL}_2 \to G$ which sends the "standard" basis for $\mathfrak{sl}_2$ to (N, H, Y).

If we let $\lambda : \mathbb{G}_m \to \mathrm{SL}_2 \to G$ be the composition of the cocharacter $t \mapsto \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$ with this homomorphism $\mathrm{SL}_2 \to G$, then λ is associated to N. Moreover, the association $\lambda \mapsto d\lambda(1)$ sends cocharacters associated to N to elements H such that $[H, N] = 2N$ and $H \in [N, \mathfrak{g}]$, and this is an injective map [Jantzen 2004, Proposition 5.5] (this reference assumes that the ground field is algebraically closed, but this hypothesis is not used). Thus (in characteristic 0) associated cocharacters are a group-theoretic analogue of the Jacobson–Morozov theorem.

We use the following properties of associated cocharacters; the given reference assumes the ground field is algebraically closed, but these statements can all be checked after extension of the ground field.

Proposition 2.3.3 [Jantzen 2004, 5.9–11]. *Let G be a connected reductive group, let $N \in \mathfrak{g}$ be a nilpotent element, and let $\lambda : \mathbb{G}_m \to G$ be an associated cocharacter for N. Then*:

(1) *The associated parabolic* $P_G(\lambda)$ *depends only on* N, *not on the choice of associated cocharacter.*

(2) *We have* $Z_G(N) \subset P_G(\lambda)$. *In particular,* $Z_G(N) = Z_{P_G(\lambda)}(N)$.

(3) $Z_G(N) = (U_G(\lambda) \cap Z_G(N)) \rtimes (Z_G(\lambda) \cap Z_G(N))$.

(4) $Z_G(\lambda) \cap Z_G(N)$ *is reductive.*

In particular, by Proposition 2.3.3(3), the disconnectedness of $Z_G(N)$ is entirely accounted for by the disconnectedness of $Z_G(\lambda) \cap Z_G(N)$. The connectedness assumption on G for that part is removed in [Bellovin 2016, Proposition 4.9], so we may apply it to groups such as $Z_G(\tau)$ (which is reductive but not necessarily connected).

We will use the following lemma in the proof of Theorem 2.3.6 below.

Lemma 2.3.4. *If* λ *is an associated cocharacter of* N, *then the weight-*2 *part of* $\mathfrak{g}$ *for the adjoint action of* λ *is in the image of* ad_N.

Proof. If $N = 0$, then λ is the constant cocharacter and the corresponding weight-2 subspace is trivial. Otherwise, we may find an $\mathfrak{sl}_2$-triple of the form $(N, d\lambda(1), Y)$ and view $\mathfrak{g}$ as a representation of $\mathfrak{sl}_2$. Then the result follows by the representation theory of $\mathfrak{sl}_2$: if $T \in \mathfrak{g}$ is in the weight-2 part, then $\frac{1}{2}[Y, T]$ is in the weight-0 part and

$$\left[N, \tfrac{1}{2}[Y, T]\right] = \tfrac{1}{2}[[N, Y], T] = \tfrac{1}{2}[d\lambda(1), T] = T,$$

so T is in the image of ad_N. □

Let $f : G \to G'$ be a morphism of reductive groups over E, inducing a morphism $\mathfrak{g} \to \mathfrak{g}'$ on Lie algebras, which we also denote by f. We use the following lemma in the proof of Theorem 2.3.8 below.

Lemma 2.3.5. *If* λ *is an associated cocharacter for* $N \in \mathfrak{g}$, *then* $f \circ \lambda$ *is an associated cocharacter for* $f(N)$.

Proof. It is clear that $d\lambda(1)$ is semisimple. Then there exists some $Y \in \mathfrak{g}$ such that $(N, d\lambda(1), Y)$ is an $\mathfrak{sl}_2$-triple, and therefore there is a homomorphism $\mathrm{SL}_2 \to G$ such that the precomposition with the diagonal is λ. The composition $\mathbb{G}_m \to \mathrm{SL}_2 \to G \to G'$ is $f \circ \lambda$. Moreover, if we consider the composition $\mathrm{SL}_2 \to G \to G'$ and differentiate, we get a map $\mathfrak{sl}_2 \to \mathfrak{g}'$ sending the "standard" basis of $\mathfrak{sl}_2$ to $(f(N), f(d\lambda(1)), f(Y))$. This shows that $[f(d\lambda(1)), f(N)] = 2f(N)$ and $f(d\lambda(1))$ is in the image of $\mathrm{ad}_{f(N)}$. Since $f(d\lambda(1)) = d(f \circ \lambda)(1)$, this shows that $f \circ \lambda$ is associated to $f(N)$, by [Jantzen 2004, Proposition 5.5]. □

If K'/K is a finite extension, we write $H^2_{L'/K'}$ for the coherent sheaf on $Y_{L/K,\varphi,\mathcal{N}}$ given by the cokernel of

$$(\mathrm{ad}\,\mathcal{D})^{I_{L'/K'}} \oplus (\mathrm{ad}\,\mathcal{D})^{I_{L'/K'}} \xrightarrow{\mathrm{ad}_{N_{L'}} - (p^{-f_{K'}} \mathrm{Ad}(\Phi^{f_{K'}/f_K}) - 1)} (\mathrm{ad}\,\mathcal{D})^{I_{L'/K'}},$$

where $(\mathcal{D}, \Phi, N, \tau)$ is the universal object over $Y_{L/K,\varphi,\mathcal{N}}$, the operator $\mathrm{ad}_{N_{L'}}$ acts on the first factor and $(p^{-f_{K'}} \mathrm{Ad}(\Phi^{f_{K'}/f_K}) - 1)$ acts on the second factor. Then the fibre of $H^2_{L'/K'}$ at a closed point of $Y_{L/K,\varphi,\mathcal{N}}$ controls the obstruction theory of the restriction to $W_{K'}$ of the corresponding Weil–Deligne representation.

Theorem 2.3.6. *Let K'/K be a finite extension. Then there is a dense open subscheme $U \subset Y_{L/K,\varphi,\mathcal{N}}$ (possibly depending on K') such that $H^2_{L'/K'}|_U = 0$.*

Proof. Since the support of $H^2_{L'/K'}$ is closed, it suffices to show that if we consider the map $Y_{L/K,\varphi,\mathcal{N}} \to Y_{L/K,\mathcal{N}}$, then each component of the fibre over some point $N \in Y_{L/K,\mathcal{N}}$ contains a point (Φ, N) whose corresponding H^2 vanishes (when viewed as a point of $Y_{L'/K',\varphi,\mathcal{N}}$).

To do this, we consider a new moduli problem $\widetilde{Y}_{L/K,\varphi,\mathcal{N}}$, which by definition is the functor on the category of E-algebras whose A-points are triples

$$(\Phi, N, \tau) \in N_G(\tau) \times \operatorname{Lie} Z_G(\tau) \times \operatorname{Rep}_A I_{L/K}$$

which satisfy $N = p^{-f_K} \operatorname{Ad}(\Phi)(N)$.

This is representable by an affine scheme which we also write as $\widetilde{Y}_{L/K,\varphi,\mathcal{N}}$, and there is a natural morphism $\widetilde{Y}_{L/K,\varphi,\mathcal{N}} \to Y_{L/K,\mathcal{N}}$. Indeed, the map $Y_{L/K,\varphi,\mathcal{N}} \to Y_{L/K,\mathcal{N}}$ factors through the natural inclusion $Y_{L/K,\varphi,\mathcal{N}} \hookrightarrow \widetilde{Y}_{L/K,\varphi,\mathcal{N}}$, and the fibres of $Y_{L/K,\varphi,\mathcal{N}} \to Y_{L/K,\mathcal{N}}$ are closed and open in the fibres of $\widetilde{Y}_{L/K,\varphi,\mathcal{N}} \to Y_{L/K,\mathcal{N}}$. Thus, it suffices to study the fibres of the map $\widetilde{Y}_{L/K,\varphi,\mathcal{N}} \to Y_{L/K,\mathcal{N}}$. (Note that the tangent-obstruction complex for objects of $G\text{-}\mathrm{WD}_E(L/K)$ makes sense over $\widetilde{Y}_{L/K,\varphi,\mathcal{N}}$ as well.)

Choose an associated cocharacter $\lambda : \mathbb{G}_m \to Z_G(\tau)^\circ$ for N, so that in particular $\operatorname{Ad}(\lambda(t))(N) = t^2 N$, and let $\Phi := \lambda(p^{f_K/2})$. Then (Φ, N, τ) is a point of $\widetilde{Y}_{L/K,\varphi,\mathcal{N}}$, and we wish to study the restriction $(\Phi^{f_{K'}/f_K}, N_{L'}, \tau|_{I_{L'/K'}})$.

If D denotes the underlying G-torsor for (Φ, N, τ), and $\operatorname{ad} D$ denotes its pushout via the adjoint representation, then $\operatorname{Ad}(\Phi)$ and $\operatorname{Ad}(\Phi^{f_{K'}/f_K})$ are semisimple operators on $(\operatorname{ad} D)^{I_{L/K}}$ and $(\operatorname{ad} D)^{I_{L'/K'}}$, respectively. Therefore, $p^{-f_K} \operatorname{Ad}(\Phi) - 1$ and $p^{f_{K'}} \operatorname{Ad}(\Phi^{f_{K'}/f_K}) - 1$ are semisimple as well (since they are the difference of commuting semisimple operators in characteristic 0).

Thus, to compute the cokernel of $p^{-f_{K'}} \operatorname{Ad}(\Phi^{f_{K'}/f_K}) - 1$, it suffices to compute its kernel. Now $(\operatorname{ad} D)^{I_{L'/K'}}$ is graded by the adjoint action of $\lambda : \mathbb{G}_m \to Z_G(\tau) \subset Z_G(\tau|_{I_{L'/K'}})$, and if $(\operatorname{ad} D)_k^{I_{L'/K'}}$ denotes the weight-k subspace, then $p^{-f_{K'}} \operatorname{Ad}(\Phi^{f_{K'}/f_K}) - 1$ preserves it, so it suffices to compute

$$\ker(p^{-f_{K'}} \operatorname{Ad}(\Phi^{f_{K'}/f_K}) - 1)|_{(\operatorname{ad} D)_k^{I_{L'/K'}}}$$

for each k. But $p^{-f_{K'}} \operatorname{Ad}(\Phi^{f_{K'}/f_K}) - 1$ acts invertibly unless $k = 2$ (in which case it acts by 0), so the cokernel of $p^{-f_{K'}} \operatorname{Ad}(\Phi^{f_{K'}/f_K}) - 1$ is exactly $(\operatorname{ad} D)_2^{I_{L'/K'}}$. By Lemma 2.3.4, the weight-2 part of $\mathfrak{g}^{I_{L'/K'}}$ is in the image of ad_N, so we conclude that $H^2_{L'/K'}$ vanishes at (Φ, N), and at its image in $Y_{L'/K',\varphi,\mathcal{N}}$.

We need to find similar points on every connected component of the fibre of $\widetilde{Y}_{L/K,\varphi,\mathcal{N}} \to Y_{L/K,\mathcal{N}}$ over $N \in Y_{L/K,\mathcal{N}}$. This fibre is a torsor under $N_G(\tau) \cap Z_G(N)$, and the disconnectedness of $N_G(\tau) \cap Z_G(N)$ is entirely accounted for by the disconnectedness of $N_G(\tau) \cap Z_G(\lambda) \cap Z_G(N)$, by [Bellovin 2016, Proposition 4.9] (applied with $G' = N_G(\tau)$). On each component of $N_G(\tau) \cap Z_G(N)$, we may therefore by [loc. cit., Lemma 5.3] choose a finite-order element $c \in N_G(\tau) \cap Z_G(\lambda) \cap Z_G(N)$. (Note that $N_G(\tau) \cap Z_G(\lambda) \cap Z_G(N) = Z_{N_G(\tau)}(N) \cap Z_{N_G(\tau)}(\lambda)$ is reductive by Proposition 2.3.3.)

We now check that $H^2_{L/K}$ and $H^2_{L'/K'}$ vanish at the points of $\widetilde{Y}_{L/K,\varphi,N}$ and $\widetilde{Y}_{L'/K',\varphi,N}$, respectively, corresponding to $(\Phi\cdot c, N)$.

Firstly, we claim that $p^{-f_{K'}}\operatorname{Ad}((\Phi\cdot c)^{f_{K'}/f_K})-1$ is semisimple, or equivalently, that $\operatorname{Ad}((\Phi\cdot c)^{f_{K'}/f_K})$ is semisimple. For this, it suffices to check that some iterate of $\operatorname{Ad}((\Phi\cdot c)^{f_{K'}/f_K})$ is semisimple (since we are in characteristic 0). Let n be the order of c. Since c and $\Phi=\lambda(p^{f_K/2})$ commute,

$$\operatorname{Ad}(\Phi^{f_{K'}/f_K}\cdot c)^n=\operatorname{Ad}(\Phi^{nf_{K'}/f_K}\cdot c^n)=\operatorname{Ad}(\Phi^{nf_{K'}/f_K}).$$

But since $\operatorname{Ad}(\Phi)$ is semisimple by construction, so is $\operatorname{Ad}(\Phi^{nf_{K'}/f_K})$, as claimed.

Thus, to compute the cokernel of $p^{-f_{K'}}\operatorname{Ad}((\Phi\cdot c)^{f_{K'}/f_K})-1$, it suffices to compute its kernel, which is contained in the kernel of $p^{-nf_{K'}}\operatorname{Ad}(\Phi^{nf_{K'}/f_K})-1$. Since $p^{-nf_{K'}}\operatorname{Ad}(\Phi^{nf_{K'}/f_K})-1$ acts invertibly on each weight space $(\operatorname{ad} D)_k^{I_{L/K}}$ unless $k=2$, the cokernel of $p^{-f_{K'}}\operatorname{Ad}(\Phi^{f_{K'}/f_K}\cdot c)-1$ is contained in $(\operatorname{ad} D)_2^{I_{L/K}}$. Since $(\operatorname{ad} D)_2^{I_{L/K}}$ is again in the image of ad_N by Lemma 2.3.4, we are done. □

Corollary 2.3.7. *The stack $G\text{-}\mathrm{WD}_E(L/K)$ is generically smooth, and is equidimensional of dimension* 0*; equivalently, the scheme $Y_{L/K,\varphi,\mathcal{N}}$ is generically smooth, and is equidimensional of dimension* $\dim G$*. The nonsmooth locus is precisely the locus of Weil–Deligne representations D with $H^2(\operatorname{ad} D)\neq 0$. Moreover, $Y_{L/K,\varphi,\mathcal{N}}$ is locally a complete intersection and reduced.*

Proof. It is enough to prove the statement for $Y_{L/K,\varphi,\mathcal{N}}$. Let $U\subset Y_{L/K,\varphi,\mathcal{N}}$ be the dense open subscheme provided by Theorem 2.3.6 (with $K'=K$). Then at each closed point x of U, it follows from Lemma 2.2.2 and Proposition 2.2.1 that $Y_{L/K,\varphi,\mathcal{N}}$ is formally smooth at x. Furthermore, for any closed point x of $Y_{L/K,\varphi,\mathcal{N}}$ with corresponding Weil–Deligne representation D_x, the dimension of the tangent space at x is $\dim G-\dim H^0(D_x)+\dim H^1(D_x)$. Since the Euler characteristic of $C^\bullet(D_x)$ is 0, this is equal to $\dim G+\dim H^2(\operatorname{ad} D_x)=\dim G$, and the claim about $H^2(\operatorname{ad} D)$ follows immediately.

To see that $Y_{L/K,\varphi,\mathcal{N}}$ is reduced and locally a complete intersection, we proceed as in the proof of [Bellovin 2016, Corollary 5.4]. We have morphisms $Y_{L/K,\varphi,\mathcal{N}}\to Y_{L/K,\mathcal{N}}\to Y_{L/K}$, and the fibre above a point $\tau\in Y_{L/K}$ is defined by the relation $N=p^{-f_K}\operatorname{Ad}(\Phi)(N)$, where $\Phi\in Z_G(\tau)$ and $N\in\operatorname{Lie} Z_G(\tau)$. In other words, the fibre $Y_{L/K,\varphi,\mathcal{N}}|_\tau$ is cut out of the smooth $(2\dim Z_G(\tau))$-dimensional space $Z_G(\tau)\times\operatorname{Lie} Z_G(\tau)$ by $\dim Z_G(\tau)$ equations.

The quotient map $G\to G/Z_G(\tau)\cong Y_{L/K}$ admits sections étale locally. Thus, there is an étale neighborhood $U\to Y_{L/K}$ of τ such that the U-pullback $Y_{L/K,\varphi,\mathcal{N}}\times_{Y_{L/K}}U$ is isomorphic to $U\times Y_{L/K,\varphi,\mathcal{N}}|_\tau$. Since $Y_{L/K,\varphi,\mathcal{N}}\times_{Y_{L/K}}U$ is étale over $Y_{L/K,\varphi,\mathcal{N}}$, it is equidimensional of dimension $\dim G$. On the other hand, it is cut out of the smooth $(\dim U+2\dim Z_G(\tau))$-dimensional space $U\times Z_G(\tau)\times\operatorname{Lie} Z_G(\tau)$ by $\dim Z_G(\tau)$ equations.

Since $\dim U=\dim Y_{L/K}=\dim G-\dim Z_G(\tau)$ and being locally a complete intersection can be checked étale locally, it follows that $Y_{L/K,\varphi,\mathcal{N}}$ is locally a complete intersection. Moreover, schemes which are local complete intersections are Cohen–Macaulay, by [Matsumura 1989, Theorem 21.3], and Cohen–Macaulay schemes which are generically reduced are reduced everywhere, by [loc. cit., Theorem 17.3], so we are done. □

If $G \to G'$ is a morphism of reductive groups over E, then for any family of G-torsors D over Spec A, we can push out to a family D' of G'-torsors. Therefore, the moduli space $Y_{L/K,\varphi,\mathcal{N}}$ of (framed) G-valued Weil–Deligne representations carries a family D' of G'-torsors, and ad $D' := \operatorname{Lie}\operatorname{Aut}_{G'}(D')$ is a coherent sheaf on $Y_{L/K,\varphi,\mathcal{N}}$. Since D is a trivial G-torsor, D' is a trivial G'-torsor. Since pushing out G-torsors to G'-torsors is functorial, D' is a family of G'-valued Weil–Deligne representations and we can construct the complex $C^\bullet(D')$. We let $H^2_{G'}$ denote its cohomology in degree 2.

Theorem 2.3.8. *Let $f : G \to G'$ be a morphism of reductive groups over E. Then there is a dense open subset $U \subset Y_{L/K,\varphi,\mathcal{N}}$ (possibly depending on G') such that $H^2_{G'}|_U = 0$.*

Proof. As in the proof of Theorem 2.3.6, it suffices to construct a point on each connected component of each fibre of the map $Y_{L/K,\varphi,\mathcal{N}} \to Y_{L/K,\mathcal{N}}$ where $H^2_{G'}$ vanishes. In fact, the same points work: by Lemma 2.3.5 the composition $f \circ \lambda$ is an associated cocharacter for $f_*(N)$. Therefore, $H^2_{G'}$ vanishes at the point corresponding to $(\lambda(p^{f_K/2}), N)$. Similarly, if $c \in N_G(\tau) \cap Z_G(\lambda) \cap Z_G(N)$ is a finite-order point, then $H^2_{G'}$ vanishes at the point corresponding to $(\lambda(p^{f_K/2}) \cdot c, N)$. □

Remark 2.3.9. The proofs of Theorems 2.3.6 and 2.3.8 justify the claim we made in the Introduction, that all of the smooth points that we explicitly construct arise from pushing out a single "standard" smooth point for SL_2. Indeed, as discussed above, given an associated cocharacter λ for N, the map $\lambda \mapsto d\lambda(1)$ allows us to determine a homomorphism $\mathrm{SL}_2 \to G$, and we see that the choice of Φ, N made in the proof of Theorem 2.3.6 is the image under this homomorphism of the elements Φ, N for SL_2 discussed in the Introduction.

Remark 2.3.10. The Jacobson–Morozov theorem allows one to think of semisimple Weil–Deligne representations as representations of $W_K \times \mathrm{SL}_2$; see [Gross and Reeder 2010, Proposition 2.2] for a precise statement. From this perspective, our construction of smooth points from associated cocharacters can be summarised as follows: given a nilpotent $N \in \operatorname{Lie} G$, we obtain a map $\mathrm{SL}_2 \to G$, and the corresponding Weil–Deligne representation is obtained by composing with the map

$$W_K \times \mathrm{SL}_2 \to \mathrm{SL}_2$$

which on the first factor is unramified and takes an arithmetic Frobenius to the matrix

$$\begin{pmatrix} p^{f_K} & 0 \\ 0 & p^{-f_K} \end{pmatrix},$$

and is the identity on the second factor.

2.4. *Tate local duality for Weil–Deligne representations.*

If D is a G-valued Weil–Deligne representation over a field E, we can also prove an analogue of Tate local duality for the complex $C^\bullet(D)$. In addition to allowing us to compute with either kernels or cokernels, this pairing allows us to give an explicit characterisation of the smooth locus (see Corollary 2.4.2). Since we only need the pairing between H^0 and H^2, we have not worked out the details of the pairing on H^1s, which for reasons of space we leave to the interested reader.

To construct pairings $H^i((\operatorname{ad} D)^*(1)) \times H^{2-i}(\operatorname{ad} D) \to E(1)$, we use the evaluation pairing

$$\mathrm{ev} : (\operatorname{ad} D)^* \times \operatorname{ad} D \to E.$$

Here the "(1)" means that we multiply the action of $\mathrm{Ad}(\Phi)$ by p^{f_K}; since $(\operatorname{ad} D)^*$ and $(\operatorname{ad} D)^*(1)$ have the same underlying vector space (as do E and $E(1)$), we have an induced pairing $\mathrm{ev}(1) : (\operatorname{ad} D)^*(1) \times \operatorname{ad} D \to E(1)$. Note that if $X \in (\operatorname{ad} D)^*$, $Y \in \operatorname{ad} D$, then

$$\mathrm{ev}(\mathrm{Ad}(\Phi)(X), \mathrm{Ad}(\Phi)(Y)) = \mathrm{ev}(X, Y),$$

and if $X \in (\operatorname{ad} D)^*(1)$, $Y \in \operatorname{ad} D$, then

$$\mathrm{ev}(1)(\mathrm{Ad}(\Phi)(X), \mathrm{Ad}(\Phi)(Y)) = \mathrm{ev}(p^{f_K}\,\mathrm{Ad}(\Phi)(X), \mathrm{Ad}(\Phi)(Y)) = p^{f_K}\,\mathrm{ev}(X, Y) = \mathrm{Ad}(\Phi)(\mathrm{ev}(1)(X, Y)).$$

Proposition 2.4.1. *Let D be as above. Then the evaluation pairing induces a perfect pairing*

$$H^0((\operatorname{ad} D)^*(1)) \times H^2(\operatorname{ad} D) \to E(1).$$

Proof. We first check that the pairing $\mathrm{ev}(1) : (\operatorname{ad} D)^*(1) \times \operatorname{ad} D \to E(1)$ descends to a well-defined pairing $H^0((\operatorname{ad} D)^*(1)) \times H^2(\operatorname{ad} D) \to E(1)$. If $X \in (\operatorname{ad} D)^*(1)^{I_{L/K}}$ is in the kernel of ad_N and the kernel of $1 - \mathrm{Ad}(\Phi)$, and $Y \in (\operatorname{ad} D)^{I_{L/K}}$, then

$$\begin{aligned}\mathrm{ev}(1)(X, Y + \mathrm{ad}_N(Z)) &= \mathrm{ev}(1)(X, Y) + \mathrm{ev}(1)(X, \mathrm{ad}_N(Z)) \\ &= \mathrm{ev}(1)(X, Y) - \mathrm{ev}(1)(\mathrm{ad}_N(X), Z) \\ &= \mathrm{ev}(1)(X, Y),\end{aligned}$$

and

$$\begin{aligned}\mathrm{ev}(1)(X, Y + (p^{-f_K}\,\mathrm{Ad}(\Phi) - 1)(Z)) &= \mathrm{ev}(1)(X, Y) + \mathrm{ev}(1)(X, p^{-f_K}\,\mathrm{Ad}(\Phi)(Z)) - \mathrm{ev}(1)(X, Z) \\ &= \mathrm{ev}(1)(X, Y) + p^{-f_K}\,\mathrm{ev}(1)(\mathrm{Ad}(\Phi)(X), \mathrm{Ad}(\Phi)(Z)) - \mathrm{ev}(1)(X, Z) \\ &= \mathrm{ev}(1)(X, Y) + \mathrm{ev}(1)(X, Z) - \mathrm{ev}(1)(X, Z) \\ &= \mathrm{ev}(1)(X, Y),\end{aligned}$$

so the pairing is indeed well-defined.

Next, we need to check that this pairing is perfect. Suppose $X \in H^0((\operatorname{ad} D)^*(1))$ and $\mathrm{ev}(1)(X, Y) = 0$ for all $Y \in H^2(\operatorname{ad} D)$. Then $\mathrm{ev}(1)(X, Y) = 0$ for all $Y \in (\operatorname{ad} D)^{I_{L/K}}$, so $X = 0$. This implies that the natural map $H^0((\operatorname{ad} D)^*(1)) \to (H^2(\operatorname{ad} D)^*)(1)$ is injective.

On the other hand, let $f : H^2(\operatorname{ad} D) \to E(1)$ be an element of $(H^2(\operatorname{ad} D)^*)(1)$. By composition, we have a linear functional

$$f : (\operatorname{ad} D)^{I_{L/K}} \to H^2(\operatorname{ad} D) \to E(1).$$

This is an element of $((\operatorname{ad} D)^{I_{L/K}})^*(1)$; we need to show that $\mathrm{ad}_N(f) = (1 - \mathrm{Ad}(\Phi))(f) = 0$. But for any $Y \in (\operatorname{ad} D)^{I_{L/K}}$,

$$\mathrm{ev}(1)(\mathrm{ad}_N(f), Y) = \mathrm{ev}(f, -\mathrm{ad}_N(Y)) = 0$$

since f factors through $H^2(\operatorname{ad} D)$. Similarly, for any $Y \in (\operatorname{ad} D)^{I_{L/K}}$,

$$\begin{aligned}\operatorname{ev}(1)((1-\operatorname{Ad}(\Phi))(f), Y) &= \operatorname{ev}(1)(f, Y) - \operatorname{ev}(1)(\operatorname{Ad}(\Phi)(f), Y) \\ &= \operatorname{ev}(1)(f, Y) - \operatorname{ev}(1)(f, p^{-f_K} \operatorname{Ad}(\Phi)^{-1}(Y)) \\ &= \operatorname{ev}(1)(f, (1-p^{-f_K} \operatorname{Ad}(\Phi)^{-1})(Y)) \\ &= \operatorname{ev}(1)\big(f, (p^{f_K} \operatorname{Ad}(\Phi)-1)(p^{-f_K} \operatorname{Ad}(\Phi^{-1})(Y))\big) = 0\end{aligned}$$

Since $\operatorname{Ad}(\Phi) : (\operatorname{ad} D)^{I_{L/K}} \to (\operatorname{ad} D)^{I_{L/K}}$ is an isomorphism, this suffices. □

Corollary 2.4.2. *The nonsmooth locus of the stack G-$\mathrm{WD}_E(L/K)$ is precisely the locus of Weil–Deligne representations D with $H^0((\operatorname{ad} D)^*(1)) \neq 0$.*

Proof. This is immediate from Corollary 2.3.7 and Proposition 2.4.1. □

We now use Corollary 2.4.2 to deduce that there is a dense set of points of $Y_{L/K,\varphi,\mathcal{N}}$ which give smooth points for every finite extension K'/K.

Definition 2.4.3. A point $x \in Y_{L/K,\varphi,\mathcal{N}}$ is *very smooth* if its image in $Y_{L'/K',\varphi,\mathcal{N}}$ is smooth for every finite extension K'/K.

Lemma 2.4.4. *Fix a finite extension E'/E. There is a finite extension K'/K (which depends only on E') such that $H^2_{L'/K'}$ vanishes at $x \in Y_{L/K,\varphi,\mathcal{N}}(E')$ if and only if x is very smooth.*

Proof. Suppose (D, Φ, N, τ) corresponds to a point of $Y_{L/K,\varphi,\mathcal{N}}$ such that $H^2_{L''/K''}$ does not vanish at its image in $Y_{L''/K'',\varphi,\mathcal{N}}$. By Corollary 2.4.2, this holds if and only if $H^0((\operatorname{ad} D)^*(1))$ does not vanish.

Thus, it suffices to consider the injectivity of

$$1 - p^{f_{K''}} \operatorname{Ad}(\Phi^{f_{K''}/f_K})^* : (\operatorname{ad} D)^{I_{L''/K''}} \to (\operatorname{ad} D)^{I_{L''/K''}}$$

on $\ker(\operatorname{ad}_N)$, where $\operatorname{Ad}(\Phi^{f_{K''}/f_K})^*$denotes the dual of $\operatorname{Ad}(\Phi^{f_{K''}/f_K})$. If this map is not injective, this implies that $p^{f_K} \operatorname{Ad}(\Phi)^*$ has a generalised eigenvalue λ satisfying $\lambda^{f_{K''}/f_K} = 1$. But the characteristic polynomial of $\operatorname{Ad}(\Phi)$ acting on $\operatorname{ad} D$ has degree $\dim \operatorname{ad} D = \dim G$ and there are only finitely many roots of unity with minimal polynomial of bounded degree over E'. It follows that there are only a finite number of possibilities for λ.

In other words, to check whether $1 - p^{f_{K''}} \operatorname{Ad}(\Phi^{f_{K''}/f_K})^*$ has a nontrivial kernel for any finite extension K''/K, it suffices to consider some fixed K' such that $f_{K'}/f_K$ is divisible by all n such that $\phi(n) \leq \dim G$ and such that $\tau|_{I_{L'/K'}}$ is trivial (where $\phi(n)$ denotes Euler's totient function), as required. □

Corollary 2.4.5. *The set of closed points of G-$\mathrm{WD}_E(L/K)$ which are very smooth is Zariski dense.*

Proof. Let E'/E be a finite extension such that $Y_{L/K,\varphi,\mathcal{N}}(E')$ is Zariski dense in $Y_{L/K,\varphi,\mathcal{N}}$. By Lemma 2.4.4, there is a finite extension K'/K such that $x \in Y_{L/K,\varphi,\mathcal{N}}(E')$ is very smooth if $H^2_{L'/K'}$ vanishes at x. By Theorem 2.3.6, there is a Zariski dense open subscheme $U \subset Y_{L/K,\varphi,\mathcal{N}}$ such that $H^2_{L'/K'}|_U = 0$. But then the intersection $U \cap Y_{L/K,\varphi,\mathcal{N}}(E')$ is a Zariski dense subset of $Y_{L/K,\varphi,\mathcal{N}}$ consisting of very smooth points, so we are done. □

2.5. ***l-adic Hodge theory.*** We suppose in this subsection that $l \neq p$. We briefly recall some results from [Fontaine 1994], which will allow us to relate l-adic representations of Gal_K to Weil–Deligne representations.

Recall that by a theorem of Grothendieck, a continuous representation $\rho : \mathrm{Gal}_K \to \mathrm{GL}_d(E)$ is automatically potentially semistable, in the sense that there is a finite extension L/K such that $\rho|_{I_L}$ is unipotent. After making a choice of a compatible system of l-power roots of unity in $\overline{K}$, we see from [loc. cit., Propositions 1.3.3, 2.3.4] that there is an equivalence of Tannakian categories between the category of E-linear representations of Gal_K which become semistable over L, and the full subcategory of Weil–Deligne representations (r, N) of W_K over E with the properties that $r|_{I_L}$ is trivial and the roots of the characteristic polynomial of any arithmetic Frobenius element of W_L are l-adic units (such an equivalence is given by the functor $\widehat{\underline{WD}}_{pst}$ of [loc. cit., §2.3.7]).

2.6. ***The case $l = p$: (φ, N)-modules.*** In this section we let $l = p$, and we explain the relationship between Weil–Deligne representations and (φ, N)-modules. Let K_0, L_0 be the maximal unramified subfields of K, L respectively, of respective degrees f_K, f_L over $\mathbb{Q}_p$. Let $E/\mathbb{Q}_p$ be a finite extension, which is large enough that it contains the image of all embeddings $L_0 \hookrightarrow E$, so that we may identify $E \otimes_{\mathbb{Q}_p} L_0$ with $\bigoplus_{L_0 \hookrightarrow E} E$. Let φ denote the arithmetic Frobenius.

If D is a $\mathrm{Res}_{E\otimes_{\mathbb{Q}_p} L_0/E}\, G$-torsor over $\operatorname{Spec} A$, we may also view D as a G-torsor over $A \otimes_{\mathbb{Q}_p} L_0$. Then any automorphism $g : L_0 \to L_0$ extends to an automorphism of $A \otimes_{\mathbb{Q}_p} L_0$, and we may pull D back to a G-torsor g^*D over $A \otimes_{\mathbb{Q}_p} L_0$. Then we may view g^*D as a $\mathrm{Res}_{E\otimes_{\mathbb{Q}_p} L_0/E}\, G$-torsor over $\operatorname{Spec} A$, which we also denote by g^*D. In particular, we may pull D back by Frobenius and obtain another $\mathrm{Res}_{E\otimes_{\mathbb{Q}_p} L_0/E}\, G$-torsor φ^*D over $\operatorname{Spec} A$.

This motivates us to define the following groupoid on E-algebras.

Definition 2.6.1. The category of G-valued $(\varphi, N, \mathrm{Gal}_{L/K})$-modules, which we denote by $G\text{-}\mathrm{Mod}_{L/K,\varphi,N}$, is the groupoid whose fibre over an E-algebra A consists of a $\mathrm{Res}_{E\otimes L_0/E}\, G$-torsor D over A, equipped with

- an isomorphism $\Phi : \varphi^*D \xrightarrow{\sim} D$,
- a nilpotent element $N \in \mathrm{Lie}\,\mathrm{Aut}_G\, D$, and
- for each $g \in \mathrm{Gal}_{L/K}$, an isomorphism $\tau(g) : g^*D \xrightarrow{\sim} D$.

These are required to satisfy the following compatibilities:

(1) $\underline{\mathrm{Ad}}\Phi(N) = \frac{1}{p}N$.

(2) $\underline{\mathrm{Ad}}\tau(g)(N) = N$ for all $g \in \mathrm{Gal}_{L/K}$.

(3) $\tau(g_1g_2) = \tau(g_1) \circ g_1^*\tau(g_2)$ for all $g_1, g_2 \in \mathrm{Gal}_{L/K}$.

(4) $\tau(g) \circ g^*\Phi = \Phi \circ \varphi^*\tau(g)$ for all $g \in \mathrm{Gal}_{L/K}$.

Here $\underline{\mathrm{Ad}}\Phi$ and $\underline{\mathrm{Ad}}\tau(g)$ are "twisted adjoint" actions on $\mathrm{Lie}\,\mathrm{Aut}_G\, D$; after pushing out Y by a representation $\sigma \in \mathrm{Rep}_E(G)$, they are given by $M \mapsto \Phi_\sigma \circ M \circ \Phi_\sigma^{-1}$ and $M \mapsto \tau(g)_\sigma \circ M \circ \tau(g)_\sigma^{-1}$, respectively.

Note that the action of $\mathrm{Gal}_{L/K}$ on scalars factors through the abelian quotient $\langle \varphi^{f_K} \rangle$, which also commutes with φ, so $(g_1 g_2)^* = g_1^* \circ g_2^*$ and $g^* \varphi^* = \varphi^* g^*$.

Requiring D to be a trivial $\mathrm{Res}_{E \otimes L_0/E}$-torsor equipped with a trivialising section lets us define a representable functor which covers $G\text{-}\mathrm{Mod}_{L/K,\varphi,N,\tau}$, as follows.

Definition 2.6.2. Let $X_{L/K,\varphi,\mathcal{N}}$ denote the functor on the category of E-algebras whose A-points are triples

$$(\Phi, N, \tau) \in (\mathrm{Res}_{E\otimes L_0/E}\, G)(A) \times (\mathrm{Res}_{E\otimes L_0/E}\, \mathfrak{g}_E)(A) \times \mathrm{Rep}_{A \otimes L_0} \mathrm{Gal}_{L/K}$$

which satisfy

- $N = p\underline{\mathrm{Ad}}(\Phi)(N)$,
- $\tau(g) \circ \Phi = \Phi \circ \tau(g)$, and
- $\underline{\mathrm{Ad}}(\tau(g))(N) = N$ for all $g \in \mathrm{Gal}_{L/K}$.

This functor is visibly representable by a finite-type affine scheme over E, which we also denote by $X_{L/K,\varphi,\mathcal{N}}$. Moreover, there is a left action of $\mathrm{Res}_{E\otimes L_0/E}\, G$ on $X_{L/K,\varphi,\mathcal{N}}$ coming from changing the choice of trivialising section. Explicitly,

$$a \cdot (\Phi, N, \{\tau(g)\}_{g \in \mathrm{Gal}_{L/K}}) = (a\Phi\varphi(a)^{-1}, \mathrm{Ad}(a)(N), \{a\tau(g)(g\cdot a)^{-1}\}_{g\in \mathrm{Gal}_{L/K}}).$$

As in Lemma 2.1.3, we have the following:

Lemma 2.6.3. *The stack quotient $[X_{L/K,\varphi,\mathcal{N}} / \mathrm{Res}_{E\otimes L_0/E}\, G]$ is isomorphic to $G\text{-}\mathrm{Mod}_{L/K,\varphi,N}$.*

Proof. The proof follows as in Lemma 2.1.3. □

Given a $(\varphi, N, \mathrm{Gal}_{L/K})$-module, there is a standard recipe due to Fontaine for constructing a Weil–Deligne representation, and there is an analogous construction for $\mathrm{Res}_{E\otimes L_0/E}\, G$-torsors. Indeed, let A be an E-algebra. Given a $\mathrm{Res}_{E\otimes L_0/E}\, G$-torsor D over A, and an embedding $\sigma : L_0 \hookrightarrow E$, the σ-isotypic part is a G-torsor over A which we denote by D_σ. Moreover, if N_σ denotes the σ-isotypic component of N, then $N_\sigma \in \mathrm{Lie}\,\mathrm{Aut}_G(D_\sigma)$ is nilpotent.

Given an isomorphism $\Phi : \varphi^* D \xrightarrow{\sim} D$, the composition $\Phi^{f_L} := \Phi \circ \varphi^*(\Phi) \circ \cdots \circ (\varphi^{f_L - 1})^*(\Phi)$ restricts to an isomorphism $D_\sigma \to D_\sigma$ for each σ.

Lemma 2.6.4. *For any σ and any E-algebra A, the association $(D, \Phi) \rightsquigarrow (D_\sigma, \Phi^{f_L})$ defines an equivalence of categories between $\mathrm{Res}_{E\otimes L_0/E}\, G$-torsors D over A equipped with an isomorphism $\Phi : \varphi^* D \xrightarrow{\sim} D$, and G-torsors D_σ over A equipped with an isomorphism $\Phi'_\sigma : D_\sigma \xrightarrow{\sim} D_\sigma$.*

Proof. Write the embeddings $\sigma_i : L_0 \hookrightarrow E$, $i \in \mathbb{Z}/f_L\mathbb{Z}$, with the numbering chosen so that $\sigma_1 = \sigma$, and Φ induces isomorphisms $\sigma_i : D_{i+1} \xrightarrow{\sim} D_i$ for each i (where we write D_i for D_{σ_i}).

Let $A \to A'$ be an fpqc cover trivialising D, so that $D_{A'}$ is a trivial torsor and we may choose a section. Then we can write $\Phi = (\Phi_1, \ldots, \Phi_{f_L})$.

We define

$$\underline{a} := (1, (\Phi_2 \cdots \Phi_{f_L})^{-1}, (\Phi_3 \cdots \Phi_{f_L})^{-1}, \ldots, \Phi_{f_L}^{-1}).$$

Then if we multiply our choice of trivialising section by $\underline{a}$, we replace Φ by

$$\underline{a}\Phi\varphi(\underline{a})^{-1} = (\Phi_1 \cdots \Phi_{f_L}, 1, \ldots, 1).$$

Thus, we can recover $(D_{A'}, \Phi)$ from $((D_\sigma)_{A'}, \Phi^{f_L})$.

Furthermore, $D_{A'}$ is equipped with a descent datum, since it is the base change of D. Therefore, $(D_i)_{A'}$ has a descent datum, and since $(D_i)_{A'} \to \operatorname{Spec} A'$ is affine, it is effective.

Now suppose that $f = (f_1, \ldots, f_{f_L}) : D \xrightarrow{\sim} D'$ is an isomorphism of $\operatorname{Res}_{E\otimes L_0/E} G$-torsors equipped with isomorphisms $\Phi : \varphi^* D \xrightarrow{\sim} D$ and $\Phi' : \varphi^* D' \xrightarrow{\sim} D'$. We obtain a corresponding isomorphism $f_{A'} : D_{A'} \xrightarrow{\sim} D'_{A'}$, together with a covering datum. Then each $f_i : D_i \xrightarrow{\sim} D'_i$ is an isomorphism of G-torsors, and we have

$$f_i \circ \Phi_i = \Phi'_i \circ f_{i+1} : D_{i+1} \to D'_i.$$

Multiplying the trivialising section of $D_{A'}$ by $\underline{a}$ and multiplying the trivialising section of $D_{A'}$ by $\underline{a}'$ has the effect of replacing $\underline{f}$ with $\underline{a}' \circ \underline{f} \circ \underline{a}^{-1}$. Then if we let $\underline{a}$ and $\underline{a}'$ be as above, $\underline{f}$ becomes $(f_1, \ldots, f_1)$. Thus, we can also recover morphisms of pairs $(D, \Phi) \to (D', \Phi')$ from the associated morphisms of pairs $(D_i, \Phi^{f_L}) \to (D'_i, (\Phi')^{f_L})$, as required. □

Now suppose that D is a $\operatorname{Res}_{E\otimes L_0/E} G$-torsor equipped with an isomorphism $\Phi : \varphi^* D \xrightarrow{\sim} D$, and suppose in addition that D is equipped with a semilinear action τ of $\operatorname{Gal}_{L/K}$, compatible with Φ in the sense that $\Phi \circ \varphi^*\tau(g) = \tau(g) \circ g^*(\Phi)$ for all $g \in \operatorname{Gal}_{L/K}$. For each σ, we will construct a Weil–Deligne representation on D_σ which is trivial on I_L.

There is a surjective map $W_K \twoheadrightarrow \operatorname{Gal}_{L/K}$ which restricts to a surjection $I_K \twoheadrightarrow I_{L/K}$. If $g \in W_K$, we write $\bar{g}$ for its image in $\operatorname{Gal}_{L/K}$. For $g \in W_K$, we have an isomorphism

$$\tau(\bar{g}) : g^* D \xrightarrow{\sim} D$$

and we have an isomorphism

$$\Phi^{-v(g) f_K} := D \xrightarrow{\Phi^{-1}} \varphi^* D \xrightarrow{\varphi^*\Phi^{-1}} \cdots \xrightarrow{(g\varphi^{-1})^*\Phi^{-1}} g^* D.$$

Accordingly, we define $r(g) : D_\sigma \xrightarrow{\sim} D_\sigma$ to be the restriction of

$$r(g) := \tau(\bar{g}) \circ \Phi^{-v(g) f_K} : D \xrightarrow{\sim} D.$$

Note that $r|_{I_L}$ is trivial.

Lemma 2.6.5. *Let D be a G-torsor and let $r : W_K \to \operatorname{Aut}_G(D)$ be a homomorphism such that $r|_{I_L}$ is trivial. Then $r(W_L)$ centralises $r(W_K)$.*

Proof. Let $g \in W_K$ and let $h \in W_L$. Then $v(ghg^{-1}h^{-1}) = 0$, so $ghg^{-1}h^{-1} \in I_K$. Moreover, $W_L \subset W_K$ is a normal subgroup, so that $ghg^{-1}h^{-1} \in W_L$. But $I_K \cap W_L = I_L$, so $r(ghg^{-1}h^{-1}) = 1$, as required. □

We now prove the equivalence between Weil–Deligne representations and (φ, N)-modules. In the case that $G = \mathrm{GL}_n$, the following lemma is [Breuil and Schneider 2007, Proposition 4.1].

Lemma 2.6.6. *The map $r : W_K \to \mathrm{Aut}_G(D_\sigma)$ is a homomorphism, and $(D, \Phi, N, \tau) \rightsquigarrow (D_\sigma, r, N_\sigma)$ is an equivalence of categories between $G\text{-}\mathrm{Mod}_{L/K,\varphi,N}$ and $G\text{-}\mathrm{WD}_E(L/K)$.*

Proof. Since $\tau(\bar{g}) \circ g^*(\Phi) = \Phi \circ \varphi^*(\tau(\bar{g}))$, we have $\Phi^{-1} \circ \tau(\bar{g}) = \varphi^*(\tau(\bar{g})) \circ g^*(\Phi^{-1})$ as isomorphisms $g^*D \xrightarrow{\sim} \varphi^*D$. It follows that

$$\begin{aligned} r(g_1)r(g_2) &= (\tau(\bar{g}_1) \circ \Phi^{-v(g_1)f_K}) \circ (\tau(\bar{g}_2) \circ \Phi^{-v(g_2)f_K}) \\ &= \tau(\bar{g}_1) \circ (\varphi^{v(g_1)f_K})^*(\tau(\bar{g}_2) \circ \Phi^{-v(g_1g_2)f_K}) \\ &= \tau(\overline{g_1g_2}) \circ \Phi^{-v(g_1g_2)f_K} = r(g_1g_2) \end{aligned}$$

and r is a homomorphism. Another short computation shows that

$$N_\sigma = p^{-v(g)f_K}\,\mathrm{Ad}(r(g))(N_\sigma),$$

so that (E_σ, r, N_σ) is a G-valued Weil–Deligne representation.

The association $(D, \Phi, N, \tau) \rightsquigarrow (D_\sigma, r, N_\sigma)$ is clearly functorial. Moreover, if $f : D \to D'$ is a morphism of G-valued $(\varphi, N, \mathrm{Gal}_{L/K})$-modules, then $\Phi' \circ \varphi^*(f) = f \circ \Phi$. This implies that f is determined by its restriction $f|_{D_\sigma}$ to the σ-isotypic piece, and therefore, the functor is fully faithful.

We need to check that this functor is essentially surjective. In other words, we need to check that we can construct (D, Φ, N, τ) from (D_σ, r, N_σ). To do so, we number the embeddings as σ_i, as in the proof of Lemma 2.6.4. For each element $h \in I_{L/K}$, we fix a lift to an element $\tilde{h} \in I_K$; note that since $r|_{I_L}$ is trivial, $r(\tilde{h})$ is independent of the choice of $\tilde{h}$.

To construct $\Phi^{f_L}|_{D_i}$ from r, we observe that if $g_0 \in W_K$ lifts φ^{f_K} and (D_i, r, N_i) is in the essential image of our functor, then

$$r(g_0^{f_L/f_K}) = \tau(\bar{g}_0^{f_L/f_K})\Phi^{-f_L}.$$

But $\bar{g}_0^{f_L/f_K} \in I_{L/K}$, so we can define $\Phi^{f_L}|_{D_i} := r(g_0^{f_L/f_K})^{-1} r(\widetilde{\bar{g}_0^{f_L/f_K}})$.

We need to check that $\Phi^{f_L}|_{D_i}$ does not depend on our choice of g_0. Indeed, if $h \in I_K$, then

$$(g_0h)^{f_L/f_K} = h_1 \cdots h_{f_L/f_K-1} g_0^{f_L/f_K},$$

where $h_i := g_0^i h g_0^{-i} \in I_K$, so we may write $(g_0h)^{f_L/f_K} = h' g_0^{f_L/f_K}$ for some $h' \in I_K$. Then $r(\tilde{\bar{h}}') = r(h')$, so

$$r((g_0h)^{f_L/f_K})^{-1} r(\widetilde{\overline{g_0h}^{f_L/f_K}}) = r(g_0^{f_L/f_K})^{-1} r(h')^{-1} r(\tilde{\bar{h}}') r(\widetilde{\bar{g}_0^{f_L/f_K}}) = r(g_0^{f_L/f_K})^{-1} r(\widetilde{\bar{g}_0^{f_L/f_K}}),$$

as required.

Lemma 2.6.4 now implies that we can construct (D, Φ) from $(D_i, \Phi^{f_L}|_{D_i})$. Since $W_K \to \mathrm{Gal}_{L/K}$ is surjective, we define for $g \in \mathrm{Gal}_{L/K}$

$$\tau(g) := r(\tilde{g}) \circ \Phi^{v(\tilde{g})f_K} = r(\tilde{g}) \circ (\Phi \circ \cdots \circ (\varphi^{-1})^* g^* \Phi)$$

as a map $D_{i+v(g)f_K} \to D_i$. We need to check that this is well-defined. Note that the kernel of $W_K \to \mathrm{Gal}_{L/K}$ is W_L, and if $h \in W_L$, then $v(h) = (f_L/f_K) \cdot i$ for some $i \in \mathbb{Z}$. Thus, for any $h \in W_L$,

$$r(\tilde{g}h) \circ \Phi^{v(\tilde{g}h)f_K} = r(\tilde{g})r(h) \circ \Phi^{i \cdot f_L} \circ \Phi^{v(\tilde{g})f_K},$$

so it suffices to show that $r(h)\circ\Phi^{i\cdot f_L}=1$. Since $r|_{I_L}$ is trivial, it suffices to consider the case $i=1$, i.e., h generates the unramified quotient of W_L. But then

$$r(h)\circ\Phi^{f_L}=r(h)r(g_0^{f_L/f_K})^{-1}r(\widetilde{\bar{g}_0^{f_L/f_K}});$$

on the one hand $hg_0^{-f_L/f_K}\in I_K$ and $\widetilde{\bar{g}_0^{f_L/f_K}}\in I_K$, and on the other hand $g_0^{-f_L/f_K}\widetilde{\bar{g}_0^{f_L/f_K}}\in W_L$. It follows that

$$hg_0^{-f_L/f_K}\widetilde{\bar{g}_0^{f_L/f_K}}\in I_K\cap W_L=I_L$$

and the result follows.

We can also construct $\tau(g):D_{j+v(\tilde{g})f_K}\xrightarrow{\sim}D_j$ for the remaining σ_j-isotypic factors. Indeed, the desired compatibility between Φ and τ forces us to set

$$\varphi^*\tau(g):=\Phi^{-1}\circ\tau(g)\circ g^*\Phi:D_{i+v(\tilde{g})f_K+1}\xrightarrow{\sim}D_{i+1} \tag{2-6-1}$$

(and we proceed inductively).

We need to check that this is well-defined. More precisely, we need to check that $(\varphi^{f_L})^*\tau(g)=\tau(g)$ for all $g\in\mathrm{Gal}_{L/K}$. In other words, we need to check that

$$\tau(g)\circ(g^*\Phi\circ\varphi^*g^*\Phi\circ\cdots\circ(\varphi^{f_L-1})^*g^*\Phi)=(\Phi\circ\varphi^*\Phi\circ\cdots\circ(\varphi^{f_L-1})^*\Phi)\circ\tau(g)$$

as isomorphisms $D_{i+v(\tilde{g})f_K}\xrightarrow{\sim}D_i$, or equivalently that

$$\tau(g)\circ g^*\Phi^{f_L}=\Phi^{f_L}\circ\tau(g).$$

But

$$\begin{aligned}\tau(g)\circ g^*\Phi^{f_L}&=(r(\tilde{g})\circ\Phi^{v(\tilde{g})f_K})\circ g^*(\Phi^{f_L})\\&=r(\tilde{g})\circ\Phi^{f_L}\circ\Phi^{v(\tilde{g})f_K}\\&=r(\tilde{g})\cdot r(g_0^{-f_L/f_K}\widetilde{\bar{g}_0^{f_L/f_K}})\circ\Phi^{v(\tilde{g})f_K}\\&=r(g_0^{-f_L/f_K}\widetilde{\bar{g}_0^{f_L/f_K}})\cdot r(\tilde{g})\circ\Phi^{v(\tilde{g})f_K}\\&=\Phi^{f_L}\circ\tau(g).\end{aligned}$$

Here we used Lemma 2.6.5 and the fact that $g_0^{-f_L/f_K}\widetilde{\bar{g}_0^{f_L/f_K}}\in W_L$.

It remains to show that τ is a semilinear representation, or more precisely, that $\tau(g_1g_2)=\tau(g_1)\circ g_1^*\tau(g_2)$ for all $g_1,g_2\in\mathrm{Gal}_{L/K}$. Now by (2-6-1) we see that

$$\begin{aligned}\tau(g_1)\circ g_1^*\tau(g_2)&=\tau(g_1)\circ\big(((g_1\varphi^{-1})^*\Phi^{-1}\circ\cdots\circ\Phi^{-1})\circ\tau(g_2)\circ(g_2^*\Phi\circ\cdots\circ(g_1\varphi^{-1})^*g_2^*\Phi)\big)\\&=\tau(g_1)\circ((g_1\varphi^{-1})^*\Phi^{-1}\circ\cdots\circ\Phi^{-1})\circ\tau(g_2)\circ g_2^*(\Phi\circ\cdots\circ(g_1\varphi^{-1})^*\Phi)\\&=r(\tilde{g}_1)\circ r(\tilde{g}_2)\circ\Phi^{v(\tilde{g}_2)f_K}\circ g_2^*\Phi^{v(\tilde{g}_1)f_K}\\&=r(\tilde{g}_1)r(\tilde{g}_2)\circ\Phi^{v(\tilde{g}_1\tilde{g}_2)f_K}\\&=\tau(g_1g_2),\end{aligned}$$

as required.

Finally, we construct N. We have N_i, and we use the desired relation $N = p\underline{\mathrm{Ad}}(\Phi)(N)$ to construct the Frobenius-conjugates of N_i. It then follows that for any $g \in \mathrm{Gal}_{L/K}$

$$\begin{aligned}\underline{\mathrm{Ad}}(\tau(g))(N) &= \underline{\mathrm{Ad}}(r(\tilde{g}) \circ \Phi^{v(g)f_K})(N)\\ &= \mathrm{Ad}(r(\tilde{g}) \circ \Phi^{v(g)f_K})(p^{-v(g)f_K}\,\mathrm{Ad}(\Phi^{-v(g)f_K})(N))\\ &= \mathrm{Ad}(r(\tilde{g}))(N) = N\end{aligned}$$

so we are done.

The assignment $(D_i, r, N_i) \rightsquigarrow (D, \Phi, N, \tau)$ is clearly functorial and quasi-inverse to $(D, \Phi, N, \tau) \rightsquigarrow (D_i, r, N_i)$. □

2.7. *Exact ⊗-filtrations for disconnected groups.* In this section we prove some results on tensor filtrations that we will apply to the Hodge filtration in p-adic Hodge theory.

Let G be an affine group scheme over a field k of characteristic 0, let A be a k-algebra, and let η be a fibre functor from $\mathrm{Rep}_k(G)$ to Proj_A. More precisely, $\mathrm{Rep}_k(G)$ is the category of k-linear finite-dimensional representations of G, Proj_A is the category of finite projective A-modules (which we will also think of as being vector bundles on Spec A), and by a "fibre functor" we mean that:

(1) η is k-linear, exact, and faithful.

(2) η is a tensor functor; that is, $\eta(V_1 \otimes_k V_2) = \eta(V_1) \otimes_A \eta(V_2)$.

(3) If **1** denotes the trivial representation of G, then $\eta(\mathbf{1})$ is the trivial A-module of rank 1.

Given a fibre functor $\eta : \mathrm{Rep}_k(G) \to \mathrm{Proj}_A$ and an A-algebra A', there is a natural fibre functor $\eta' : \mathrm{Rep}_k(G) \to \mathrm{Proj}_{A'}$ given by composing η with the natural base extension functor $\iota_{A'} : \mathrm{Proj}_A \to \mathrm{Proj}_{A'}$ sending M to $M \otimes_A A'$.

Definition 2.7.1. Let $\omega, \eta : \mathrm{Rep}_k(G) \rightrightarrows \mathrm{Proj}_A$ be fibre functors. Then $\underline{\mathrm{Hom}}^{\otimes}(\omega, \eta)$ is the functor on A-algebras given by

$$\underline{\mathrm{Hom}}^{\otimes}(\omega, \eta)(A') := \mathrm{Hom}^{\otimes}(\iota_{A'} \circ \omega, \iota_{A'} \circ \eta).$$

Here $\underline{\mathrm{Hom}}^{\otimes}$ refers to natural transformations of functors which preserve tensor products.

Theorem 2.7.2 [Deligne and Milne 1982, Theorem 3.2]. *Let $\omega : \mathrm{Rep}_k(G) \to \mathrm{Vec}_k$ be the natural forgetful functor*:

(1) *For any fibre functor $\eta : \mathrm{Rep}_k(G) \to \mathrm{Proj}_A$, the functor $\underline{\mathrm{Hom}}^{\otimes}(\iota_A \circ \omega, \eta)$ is representable by an affine scheme faithfully flat over* Spec A; *it is therefore a G-torsor.*

(2) *The functor $\eta \rightsquigarrow \underline{\mathrm{Hom}}^{\otimes}(\iota_A \circ \omega, \eta)$ is an equivalence between the category of fibre functors $\eta : \mathrm{Rep}_k(G) \to \mathrm{Proj}_A$ and the category of G-torsors over* Spec A. *The quasi-inverse assigns to any G-torsor X over A the functor η sending any $\rho : G \to \mathrm{GL}(V)$ to the $M \in \mathrm{Proj}_A$ associated to the push-out of X over A.*

Corollary 2.7.3. *Let $\eta : \mathrm{Rep}_k(G) \to \mathrm{Proj}_A$ be a fibre functor, corresponding to a G-torsor $X \to$* Spec A. *Then the functor $\underline{\mathrm{Aut}}^{\otimes}(\eta)$ is representable by the A-group scheme $\mathrm{Aut}_G(X)$. This is a form of G_A.*

We now assume that η is equipped with an exact $\otimes$-filtration; i.e., for each $V \in \mathrm{Rep}_k(G)$, we have a decreasing filtration $\mathcal{F}^\bullet(\eta(V))$ of vector sub-bundles on each $\eta(V)$ such that:

(1) The specified filtrations are functorial in V.

(2) The specified filtrations are tensor-compatible, in the sense that

$$\mathcal{F}^n\eta(V \otimes_k V') = \sum_{p+q=n} \mathcal{F}^p\eta(V) \otimes_A \mathcal{F}^q\eta(V') \subset \eta(V) \otimes_A \eta(V').$$

(3) $\mathcal{F}^n(\eta(\mathbf{1})) = \eta(\mathbf{1})$ if $n \le 0$ and $\mathcal{F}^n(\eta(\mathbf{1})) = 0$ if $n \ge 1$.

(4) The associated functor from $\mathrm{Rep}_k(G)$ to the category of graded projective A-modules is exact.

Equivalently, an exact $\otimes$-filtration of η is the same as a factorisation of η through the category of filtered vector bundles over $\operatorname{Spec} A$.

We define two auxiliary subfunctors of $\underline{\mathrm{Aut}}^\otimes(\eta)$:

- $P_\mathcal{F} = \underline{\mathrm{Aut}}^\otimes_\mathcal{F}(\eta)$ is the functor on A-algebras such that

$$P_\mathcal{F}(A') = \{\lambda \in \underline{\mathrm{Aut}}^\otimes(\eta)(A') \mid \lambda(\mathcal{F}^n\eta(V)) \subset \mathcal{F}^n\eta(V) \text{ for all } V \in \mathrm{Rep}_k(G) \text{ and } n \in \mathbb{Z}\}.$$

- $U_\mathcal{F} = \underline{\mathrm{Aut}}^{\otimes !}_\mathcal{F}(\eta)$ is the functor on A-algebras such that

$$U_\mathcal{F}(A') = \{\lambda \in \underline{\mathrm{Aut}}^\otimes(\eta)(A') \mid (\lambda - \mathrm{id})(\mathcal{F}^n\eta(V)) \subset \mathcal{F}^{n+1}\eta(V) \text{ for all } V \in \mathrm{Rep}_k(G) \text{ and } n \in \mathbb{Z}\}.$$

By [Saavedra Rivano 1972, Chapter IV, 2.1.4.1], these functors are both representable by closed subgroup schemes of $\mathrm{Aut}_G(X)$, and they are smooth if G is. This holds for any affine group G over k (since it is automatically flat); there is no need for reductivity or connectedness hypotheses. Furthermore, $\mathrm{Lie}\, P_\mathcal{F} = \mathcal{F}^0(\mathrm{Lie}\, \underline{\mathrm{Aut}}^\otimes(\eta))$ and $\mathrm{Lie}\, U_\mathcal{F} = \mathcal{F}^1(\mathrm{Lie}\, \underline{\mathrm{Aut}}^\otimes(\eta))$, by the same result.

We also have a notion of a $\otimes$-grading on η: a $\otimes$-*grading* of η is the specification of a grading $\eta(V) = \bigoplus_{n\in\mathbb{Z}} \eta(V)_n$ of vector bundles on each $\eta(V)$ such that:

(1) The specified gradings are functorial in V.

(2) The specified grading are tensor-compatible, in the sense that

$$\eta(V \otimes_k V')_n = \bigoplus_{p+q=n} (\eta(V)_p \otimes_A \eta(V')_q).$$

(3) $\eta(\mathbf{1})_0 = \eta(\mathbf{1})$.

Equivalently, a $\otimes$-grading of η is a factorisation of η through the category of graded vector bundles on $\operatorname{Spec} A$. A $\otimes$-grading induces a homomorphism of A-group schemes $\mathbb{G}_m \to \underline{\mathrm{Aut}}^\otimes(\eta)$.

Given a $\otimes$-grading of η, we may construct a $\otimes$-filtration of η, by setting

$$\mathcal{F}^n\eta(V) = \bigoplus_{n' \ge n} \eta(V)_{n'}.$$

We say that a $\otimes$-filtration $\mathcal{F}^\bullet$ is *splittable* if it arises in this way, and we say that $\mathcal{F}^\bullet$ is *locally splittable* if fpqc-locally on $\operatorname{Spec} A$ it arises in this way. A *splitting* of $\mathcal{F}^\bullet$ is a $\otimes$-grading on η giving rise to $\mathcal{F}^\bullet$.

Given an exact $\otimes$-filtration $\mathcal{F}^\bullet$ on η, we may define a fibre functor $\mathrm{gr}(\eta)$ equipped with a $\otimes$-grading by setting

$$\mathrm{gr}(\eta)(V)_n := \mathcal{F}^n(V)/\mathcal{F}^{n+1}(V).$$

Thus, a splitting of $\mathcal{F}^\bullet$ is equivalent to an isomorphism of filtered fibre functors $\mathrm{gr}(\eta) \cong \eta$.

In fact, by a theorem of Deligne (proved in [Saavedra Rivano 1972, Chapter IV, 2.4]), every $\otimes$-filtration is locally splittable (in fact, splittable Zariski-locally on Spec A), because G is smooth and A has characteristic 0 (this result also holds under various other sets of hypotheses on G and A). Again, this does not require G to be reductive or connected. If $\lambda : \mathbb{G}_m \to \underline{\mathrm{Aut}}^{\otimes}(\eta)$ is a cocharacter splitting the filtration, then $P_{\mathcal{F}} = U_{\mathcal{F}} \rtimes Z_G(\lambda)$, by [loc. cit., Chapter IV, 2.1.5.1]. In particular, λ factors through $P_{\mathcal{F}}$.

If $\mathcal{F}^\bullet$ is a splittable filtration on η, we may consider the functor $\underline{\mathrm{Scin}}(\eta, \mathcal{F}^\bullet)$ of splittings. Then $\underline{\mathrm{Scin}}(\eta, \mathcal{F}^\bullet)$ is the same as the functor $\underline{\mathrm{Isom}}^{\otimes !}_{\mathcal{F}}(\mathrm{gr}_{\mathcal{F}}(\eta), \eta)$, which is the subset of $\underline{\mathrm{Isom}}^{\otimes}_{\mathcal{F}}(\mathrm{gr}_{\mathcal{F}}(\eta), \eta)$ inducing the identity $\mathrm{gr}_{\mathcal{F}}(\eta) \to \mathrm{gr}_{\mathcal{F}}(\eta)$. Thus, $\underline{\mathrm{Scin}}(\eta, \mathcal{F}^\bullet)$ is a left torsor under $U_{\mathcal{F}}$. It follows that the composition $\lambda : \mathbb{G}_m \to P_{\mathcal{F}} \to P_{\mathcal{F}}/U_{\mathcal{F}}$ is independent of the choice of splitting.

In other words, $P_{\mathcal{F}}$ and $U_{\mathcal{F}}$ depend only on the filtration, and if it is locally splittable, there is a homomorphism $\bar{\lambda} : \mathbb{G}_m \to P_{\mathcal{F}}/U_{\mathcal{F}}$ which also only depends on the filtration. If the filtration is actually splittable, a choice of splitting lets us lift $\bar{\lambda}$ to a cocharacter $\lambda : \mathbb{G}_m \to P_{\mathcal{F}}$. In that case, since both $\underline{\mathrm{Scin}}(\eta, \mathcal{F})$ and the set of lifts of cocharacters from $P_{\mathcal{F}}/U_{\mathcal{F}}$ to $P_{\mathcal{F}}$ are torsors under $U_{\mathcal{F}}$ (in the latter case, $U_{\mathcal{F}}$ acts by conjugation), they are isomorphic. In particular, any two cocharacters $\lambda, \lambda' : \mathbb{G}_m \rightrightarrows P_{\mathcal{F}}$ splitting the $\otimes$-filtration $\mathcal{F}$ are conjugate by $U_{\mathcal{F}}$.

Let $\mathcal{G} := \underline{\mathrm{Aut}}^{\otimes}(\eta)$, so that the geometric fibres of $\mathcal{G}$ are isomorphic to $G_{\bar{k}}$. Then for any geometric point $x \in \mathrm{Spec}\, A$, the $G^\circ(\kappa(x))$-conjugacy class of $\mathcal{F}^\bullet_x$ induces a unique $G^\circ(\kappa(x))$-conjugacy class of cocharacters, and this conjugacy class is Zariski-locally constant on Spec A.

Recall that when $\lambda : \mathbb{G}_m \to \mathcal{G}$ is a cocharacter, we defined subgroups $U_{\mathcal{G}}(\lambda) \subset P_{\mathcal{G}}(\lambda) \subset \mathcal{G}$ in Section 1.3.

Proposition 2.7.4. *Suppose that G is a (possibly disconnected) algebraic group. Let $\eta : \mathrm{Rep}_k(G) \to \mathrm{Proj}_A$ be a fibre functor equipped with a splittable exact $\otimes$-filtration $\mathcal{F}^\bullet$, and let $\lambda : \mathbb{G}_m \to \underline{\mathrm{Aut}}^{\otimes}(\eta)$ be a splitting. Let $\mathcal{G}$ denote the group scheme representing $\underline{\mathrm{Aut}}^{\otimes}(\eta)$. Then $P_{\mathcal{F}} = P_{\mathcal{G}}(\lambda)$, $U_{\mathcal{F}} = U_{\mathcal{G}}(\lambda)$, and the fibres of $U_{\mathcal{F}}$ are connected.*

Proof. We consider the map $\mu : \mathbb{G}_m \times P_{\mathcal{F}} \to \underline{\mathrm{Aut}}^{\otimes}(\eta)$ defined by $\mu(t, g) := \lambda(t) g \lambda(t^{-1})$, and for $g \in P_{\mathcal{F}}(A')$, we let $\mu_g : (\mathbb{G}_m)_{A'} \to (\underline{\mathrm{Aut}}^{\otimes}(\eta))_{A'}$ be the restriction $\mu|_{\mathbb{G}_m \times \{g\}}$. Let $\sigma : G \to \mathrm{GL}(V)$ be a representation of G. Then the pushout $\eta(V)$ is a filtered vector bundle, and if $g \in P_{\mathcal{F}}(A')$, the action of g preserves the filtration on $\eta(V)$. The choice of a splitting in particular specifies an isomorphism $\mathrm{gr}^\bullet(\eta(V)) \xrightarrow{\sim} \eta(V)$, and $t \in \mathbb{G}_m(A')$ acts via t^n on $(\eta(V))_n$.

Let $\sigma_*(\lambda)$ denote the corresponding cocharacter $\sigma_*(\lambda) : \mathbb{G}_m \to \mathrm{Aut}_{\mathrm{GL}(V)}(\eta(V))$. Since this cocharacter induces the filtration on $\eta(V)$, we see that the morphism

$$\sigma_*(\mu_g) := \sigma_*(\lambda)(t) g \sigma_*(\lambda)(t^{-1}) : \mathbb{G}_m \to P_{\mathrm{Aut}_{\mathrm{GL}(V)}(\eta(V))}(\sigma_*(\lambda))$$

extends uniquely to a morphism

$$\widetilde{\sigma_*(\mu_g)} : \mathbb{A}^1 \to P_{\mathrm{Aut}_{\mathrm{GL}(V)}(\eta(V))}(\sigma_*(\lambda)).$$

We claim that the collection $\{\widetilde{\sigma_*(\mu_g)}\}_\sigma$ is functorial in σ and tensor-compatible. Indeed, since the collection $\{\widetilde{\sigma_*(\mu_g)}|_{\mathbb{G}_m}\}_\sigma$ is functorial in σ and tensor-compatible, and the extensions to $\mathbb{A}^1$ are unique, it follows that $\{\widetilde{\sigma_*(\mu_g)}\}_\sigma$ is functorial in σ and tensor-compatible. Thus, there is a morphism $\tilde{\mu}_g : \mathbb{A}^1 \to \underline{\mathrm{Aut}}^{\otimes}_{\mathcal{F}}(\eta)$ whose restriction to $\mathbb{G}_m$ is μ_g. It follows that $g \in P_{\mathcal{G}}(\lambda)(A')$.

Suppose in addition that $g \in U_{\mathcal{F}}(A')$. Then for every representation $\sigma : G \to \mathrm{GL}(V)$, g induces the identity map from $\mathrm{gr}^\bullet(\sigma(\mathcal{F}^\bullet))$ to itself. It follows that $\widetilde{\sigma_*(\mu_g)}(0) = \mathbf{1}$ for all σ, and therefore $\tilde{\mu}_g(0) = \mathbf{1}$.

On the other hand, if $g \in P_{\mathcal{G}}(\lambda)(A')$, then the morphism $\mu_g : (\mathbb{G}_m)_{A'} \to \underline{\mathrm{Aut}}^{\otimes}(\eta)_{A'}$ defined by $t \mapsto \lambda(t)g\lambda(t^{-1})$ extends to a morphism $\tilde{\mu}_g : (\mathbb{A}^1)_{A'} \to \underline{\mathrm{Aut}}^{\otimes}(\eta)_{\mathbb{A}'}$. It therefore induces a family of morphisms

$$\sigma_*(\tilde{\mu}_g) : (\mathbb{A}^1)_{A'} \to \mathrm{GL}(V)_{A'}$$

and so $\sigma_*(g) \in P_{\mathrm{Aut}_{\mathrm{GL}(V)}(\eta(V))}(\sigma_*(\lambda))$. But then $\sigma_*(g)$ preserves the filtration on $\eta(V)$ induced by $\sigma_*(\lambda)$; since this holds for all $V \in \mathrm{Rep}_k(G)$, we have $g \in P_{\mathcal{F}}(A')$. A similar argument shows that if $g \in U_{\mathcal{G}}(\lambda)(A')$, then $g \in U_{\mathcal{F}}(A')$.

Finally, since $\tilde{\mu}_g : \mathbb{A}^1 \to \underline{\mathrm{Aut}}^{\otimes}(\eta)$ is a morphism from a connected scheme such that $\tilde{\mu}_g(0) = \mathbf{1}$ and $\tilde{\mu}_g(1) = g$, we see that g is in the connected component of the identity for all $g \in U_{\mathcal{F}}(A')$. □

Lemma 2.7.5. *Let $\mathcal{F}^\bullet$ be a locally splittable exact $\otimes$-filtration on η. Then the geometric fibres of $P_{\mathcal{F}}$ are parabolic subgroups of $G_{\bar{k}}$.*

Proof. We may work locally on Spec A and assume that we have a cocharacter $\lambda : \mathbb{G}_m \to \mathcal{G}_A$ splitting the exact $\otimes$-filtration. Then $P_{\mathcal{F}} \cong P_{\mathcal{G}}(\lambda)$. Since the formation of $P_{\mathcal{G}}(\lambda)$ commutes with base change on A, we may assume that $A = k = \bar{k}$ and $\mathcal{G} = G = G_{\bar{k}}$. Then $P_{G^\circ}(\lambda) \subset G^\circ$ is a parabolic subgroup, so $G^\circ/P_{G^\circ}(\lambda)$ is proper. There is a sequence of maps

$$G^\circ/P_{G^\circ}(\lambda) \to G/P_{G^\circ}(\lambda) \twoheadrightarrow G/P_G(\lambda).$$

Since $G^\circ \subset G$ has finite index, the properness of $G^\circ/P_{G^\circ}(\lambda)$ implies the properness of $G/P_{G^\circ}(\lambda)$. This implies that $G/P_G(\lambda)$ is proper, so $P_G(\lambda) \subset G$ is a parabolic subgroup. □

We will also need the following result:

Theorem 2.7.6 [SGA 3_{II} 1970, Exposé IX, Théorème 3.6]. *Let S be an affine scheme, S_0 a subscheme defined by a nilpotent ideal J, H a group of multiplicative type over S, G a smooth group scheme over S, and $\mu_0 : H \times_S S_0 \to G \times_S S_0$ a homomorphism of S_0-groups.*

Then there exists a homomorphism $\mu : H \to G$ of S-groups which lift μ_0, and any two such lifts are conjugate by an element of $G(S)$ which reduces to the identity modulo J.

Corollary 2.7.7. *Let A be an artin local k-algebra with maximal ideal $\mathfrak{m}_A$, and let $I \subset A$ be an ideal such that $I\mathfrak{m}_A = (0)$. Then if D_A is a G-torsor over A such that the reduction $D_{A/I} := D_A \otimes_A A/I$ is* *equipped with an exact $\otimes$-filtration $\mathcal{F}^\bullet_{A/I}$, then the set of lifts of $\mathcal{F}^\bullet_{A/I}$ to an exact $\otimes$-filtration on D_A is nonempty, and is a torsor under $I \otimes_{A/\mathfrak{m}_A} (\mathrm{ad}\, D_{A/\mathfrak{m}_A}/\mathcal{F}^0_{A/\mathfrak{m}_A}(\mathrm{ad}\, D_{A/\mathfrak{m}_A}))$.*

Proof. Suppose that $D_{A/I}$ is a G-torsor over $\operatorname{Spec} A/I$, equipped with an exact $\otimes$-filtration $\mathcal{F}^\bullet_{A/I}$. Since A/I is local, $\mathcal{F}^\bullet_{A/I}$ is split, so it is induced by a cocharacter $\lambda_{A/I} : \mathbb{G}_m \to \mathrm{Aut}_G(D_{A/I})$. By Theorem 2.7.6, $\lambda_{A/I}$ lifts to a cocharacter $\lambda_A : \mathbb{G}_m \to \mathrm{Aut}_G(D_A)$. Then λ_A induces an exact $\otimes$-filtration $\mathcal{F}^\bullet_A$ on D_A which lifts that on $D_{A/I}$.

Suppose there are two exact $\otimes$-filtrations, $\mathcal{F}^\bullet_A$ and $\mathcal{F}'^\bullet_A$ on D_A lifting $\mathcal{F}^\bullet_{A/I}$, induced by cocharacters λ_A and λ'_A, respectively, which lift $\lambda_{A/I}$. Then λ_A and λ'_A are conjugate by an element of $\mathrm{Aut}_G(D_A)$ which is the identity modulo I. In other words, there is some $j \in \mathrm{ad}\, D_{A/\mathfrak{m}_A} \otimes_{A/\mathfrak{m}_A} I$ such that $\lambda'_A = (1+j)\lambda_A(1-j)$. This implies that $\mathcal{F}^\bullet_A$ and $\mathcal{F}'^\bullet_A$ are conjugate.

On the other hand, conjugation by $1 + j$ preserves $\mathcal{F}^\bullet_A$ if and only if $1 + j \in P_{\mathcal{F}_A}(\mathrm{Aut}_G(D_A))$. This holds if and only if $j \in \mathcal{F}^0_{A/\mathfrak{m}_A} \mathrm{Lie}\,\mathrm{Aut}_G(D_{A/\mathfrak{m}_A}) \otimes_{A/\mathfrak{m}_A} I = \mathcal{F}^0_{A/\mathfrak{m}_A} \mathrm{ad}\, D_{A/\mathfrak{m}_A} \otimes_{A/\mathfrak{m}_A} I$. □

2.8. *p-adic Hodge theory.* Our goal is to study deformations of potentially semistable Galois representations. That is, we wish to consider deformations of representations $\rho : \mathrm{Gal}_K \to G(E)$ such that $\rho|_{\mathrm{Gal}_L}$ is semistable. Such representations can be described by linear algebra. Briefly, for every representation $\sigma : G \to \mathrm{GL}_d$, $\sigma \circ \rho$ is a potentially semistable representation, and $D^L_{\mathrm{st}}(\sigma \circ \rho)$ is a weakly admissible filtered $(\varphi, N, \mathrm{Gal}_{L/K})$-module. The formation of $D^L_{\mathrm{st}}(\sigma \circ \rho)$ is exact and tensor-compatible in σ, and if $\mathbf{1}$ denotes the trivial representation of G, then $D^L_{\mathrm{st}}(\mathbf{1} \circ \rho)$ is the trivial filtered $(\varphi, N, \mathrm{Gal}_{L/K})$-module with coefficients in E.

Therefore, as in [Bellovin 2016, §A.2.8–9], $\sigma \mapsto D^L_{\mathrm{st}}(\sigma \circ \rho)$ is a fibre functor $\eta : \mathrm{Rep}_E(G) \to \mathrm{Proj}_{E \otimes_{\mathbb{Q}_p}} L_0$, and we obtain from ρ a G-torsor $D = D^L_{\mathrm{st}}(\rho)$ over $E \otimes L_0$ equipped with

- an isomorphism $\Phi : \varphi^* D \xrightarrow{\sim} D$,
- a nilpotent element $N \in \mathrm{Lie}\,\mathrm{Aut}_G D$,
- for each $g \in \mathrm{Gal}_{L/K}$, an isomorphism $\tau(g) : g^* D \xrightarrow{\sim} D$,
- a $\mathrm{Gal}_{L/K}$-stable exact $\otimes$-filtration on D_L, or equivalently (by Galois descent), an exact $\otimes$-filtration on the $\mathrm{Res}_{E \otimes K/E}\, G$-torsor $D_L^{\mathrm{Gal}_{L/K}}$ over K.

These satisfy the requisite compatibilities such that forgetting the filtration on $D^L_{\mathrm{st}}(\rho)$ gives us an object of $G\text{-}\mathrm{Mod}_{L/K,\varphi,N}$.

Definition 2.8.1. The category of G-valued filtered $(\varphi, N, \mathrm{Gal}_{L/K})$-modules, which we denote by $G\text{-}\mathrm{Mod}_{L/K,\varphi,N,\mathrm{Fil}}$, is the category cofibred in groupoids over E-Alg whose fibre over an E-algebra A consists of a $\mathrm{Res}_{E \otimes L_0/E}\, G$-torsor D over A, equipped with

- an isomorphism $\Phi : \varphi^* D \xrightarrow{\sim} D$,
- a nilpotent element $N \in \mathrm{Lie}\,\mathrm{Aut}_G D$,
- for each $g \in \mathrm{Gal}_{L/K}$, an isomorphism $\tau(g) : g^* D \xrightarrow{\sim} D$,

- a $\mathrm{Gal}_{L/K}$-stable exact $\otimes$-filtration on D_L, or equivalently, an exact $\otimes$-filtration on the $\mathrm{Res}_{E\otimes K/E}\, G$-torsor $D_L^{\mathrm{Gal}_{L/K}}$ over A.

The $\mathrm{Res}_{E\otimes L_0/E}\, G$-torsor D, together with Φ, N, and $\{\tau(g)\}_{g\in\mathrm{Gal}_{L/K}}$, is required to be an object of $G\text{-}\mathrm{Mod}_{L/K,\varphi,N}$.

Definition 2.8.2. Suppose that $\rho : \mathrm{Gal}_K \to G(E)$ is a potentially semistable Galois representation which becomes semistable when restricted to Gal_L. The *p-adic Hodge type* $\boldsymbol{v}$ of ρ is the $(\mathrm{Res}_{E\otimes K/E}\, G)^\circ(\bar{E})$-conjugacy class of cocharacters $\lambda : \mathbb{G}_m \to (\mathrm{Res}_{E\otimes K/E}\, G)_{\bar{E}}$ which split the $\otimes$-filtration on $D^L_{\mathrm{st}}(\rho)_L^{\mathrm{Gal}_{L/K}}$. We let $P_{\boldsymbol{v}}$ denote the $(\mathrm{Res}_{E\otimes K/E}\, G)^\circ(\bar{E})$-conjugacy class of $P_{\mathrm{Res}_{E\otimes K/E}\, G}(\lambda)$ for $\lambda \in \boldsymbol{v}$.

While we do not need it, for completeness we record the following definition and result, which control the deformation theory of filtered $(\varphi, N, \mathrm{Gal}_{L/K})$-modules. Given an object $D_A \in G\text{-}\mathrm{Mod}_{L/K,\varphi,N,\mathrm{Fil}}$, we consider the diagram

$$\begin{array}{ccccc} (\mathrm{ad}\, D_A)^{\mathrm{Gal}_{L/K}} & \longrightarrow & (\mathrm{ad}\, D_A)^{\mathrm{Gal}_{L/K}} \oplus (\mathrm{ad}\, D_A)^{\mathrm{Gal}_{L/K}} & \longrightarrow & (\mathrm{ad}\, D_A)^{\mathrm{Gal}_{L/K}} \\ \downarrow & & & & \\ (\mathrm{ad}\, D_{A,L}/\mathrm{Fil}^0\mathrm{ad}\, D_{A,L})^{\mathrm{Gal}_{L/K}} & & & & \end{array}$$

where the top line is the total complex of

$$\begin{array}{ccc} (\mathrm{ad}\, D_A)^{\mathrm{Gal}_{L/K}} & \xrightarrow{1-\underline{\mathrm{Ad}}(\Phi)} & (\mathrm{ad}\, D_A)^{\mathrm{Gal}_{L/K}} \\ \downarrow \mathrm{ad}_N & & \downarrow \mathrm{ad}_N \\ (\mathrm{ad}\, D_A)^{\mathrm{Gal}_{L/K}} & \xrightarrow{p\underline{\mathrm{Ad}}(\Phi)-1} & (\mathrm{ad}\, D_A)^{\mathrm{Gal}_{L/K}} \end{array}$$

and the vertical map is the natural quotient map. We let $C^\bullet_{\mathrm{Fil}}$ denote its total complex. Then $C^\bullet_{\mathrm{Fil}}$ controls the deformation theory of D_A:

Proposition 2.8.3. *Let A be an artin local E algebra with maximal ideal $\mathfrak{m}_A$ and let $I \subset A$ be an ideal such that $I\mathfrak{m}_A = (0)$. Let $D_{A/I}$ be an object of $G\text{-}\mathrm{Mod}_{L/K,\varphi,N,\mathrm{Fil}}(A/I)$ and set $D_{A/\mathfrak{m}_A} := D_{A/I} \otimes_{A/I} A/\mathfrak{m}_A$:*

(1) *If $H^2_{\mathrm{Fil}}(D_{A/I}) = 0$, then there exists an object $D_A \in G\text{-}\mathrm{Mod}_{L/K,\varphi,N,\mathrm{Fil}}(A)$ lifting $D_{A/I}$.*

(2) *The set of isomorphism classes of lifts of $D_{A/I}$ to $D_A \in G\text{-}\mathrm{Mod}_{L/K,\varphi,N,\mathrm{Fil}}(A)$ is either empty or a torsor under $H^1_{\mathrm{Fil}}(D_{A/\mathfrak{m}_A}) \otimes_{A/\mathfrak{m}_A} I$.*

Proof. This follows by combining [Bellovin 2016, Proposition 3.2] and Corollary 2.7.7. □

3. Local deformation rings

As in Section 1.3.2, we let $K/\mathbb{Q}_p$ be a finite extension for some prime p, possibly equal to l, and let $\bar{\rho} : \mathrm{Gal}_K \to G(\mathbb{F})$ be a continuous representation. We have a universal framed deformation $\mathcal{O}$-algebra $R^\square_{\bar{\rho}}$, and if we fix a homomorphism $\psi : \Gamma \to G^{\mathrm{ab}}(\mathcal{O})$ such that $\mathrm{ab} \circ \bar{\rho} = \bar{\psi}$, we also have the quotient $R^{\square,\psi}_{\bar{\rho}}$ corresponding to framed deformations ρ with $\mathrm{ab} \circ \rho = \psi$. When we define quotients of $R^\square_{\bar{\rho}}$, there are corresponding quotients of $R_{\bar{\rho}}^{\square,\psi}$, which we will not explicitly define, but will denote by a superscript ψ. An inertial type is by definition a $G^\circ(\bar{E})$-conjugacy class of representations $\tau : I_K \to G(\bar{E})$ with open kernel which admit extensions to Gal_K; any such τ is defined over some finite extension of E. We choose a finite Galois extension L/K for which $\tau|_{I_L}$ is trivial. If E'/E is a finite extension, and $\rho : \mathrm{Gal}_K \to G(E')$ is a representation, which we assume to be potentially semistable if $l = p$, then we say that ρ has type τ if the restriction to I_K (forgetting N) of the corresponding Weil–Deligne representation $\mathrm{WD}(\rho)$ is equivalent to τ.

3.1. *The case $l \neq p$.* Suppose firstly that $l \neq p$. The proof of [Balaji 2013, Proposition 3.0.12] shows that for each τ we may define a $\mathbb{Z}_l$-flat quotient $R_{\bar{\rho}}^{\square,\tau}$ of $R_{\bar{\rho}}^{\square}$ whose characteristic-0 points correspond to representations of type τ. The usual construction of the Weil–Deligne representation associated to a Galois representation makes sense over $R_{\bar{\rho}}^{\square}[1/l]$, so we have a natural morphism

$$\mathrm{Spec}\, R_{\bar{\rho}}^{\square,\tau}[1/l] \to G\text{-}\mathrm{WD}_E(L/K).$$

3.2. *The case $l = p$.* Now suppose that $l = p$. If we fix a p-adic Hodge type $\boldsymbol{v}$ in the sense of Definition 2.8.2 (that is, a $(\mathrm{Res}_{E\otimes K/E}\, G)^\circ(\bar{E})$-conjugacy class of cocharacters $\lambda : \mathbb{G}_m \to (\mathrm{Res}_{E\otimes K/E}\, G)_{\bar{E}}$), and an inertial type τ, then by [Balaji 2013, Proposition 3.0.12] there is a unique $\mathbb{Z}_l$-flat quotient $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}$ of $R_{\bar{\rho}}^{\square}$ with the property that if B is a finite local E-algebra, then a morphism $R_{\bar{\rho}}^{\square} \to B$ factors through $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}$ if and only if the corresponding representation $\rho : \mathrm{Gal}_K \to G(B)$ is potentially semistable with Hodge type $\boldsymbol{v}$ and inertial type τ. For each finite-dimensional representation V of G, we may compose with the representation $\mathrm{Gal}_K \to G(R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}[1/p])$ to obtain a representation $\mathrm{Gal}_K \to \mathrm{GL}(V)(R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}[1/p])$. Then exactly as in [Kisin 2008, Theorem 2.5.5] we obtain a corresponding ($\mathrm{GL}(V)$-valued) filtered $(\varphi, N, \mathrm{Gal}_{L/K})$-module over $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}[1/p]$ (note that we have been working with covariant functors in this paper, while Kisin uses contravariant functors; it is necessary to dualise the construction in [loc. cit., §2.4]). As these filtered $(\varphi, N, \mathrm{Gal}_{L/K})$-modules are exact and tensor-compatible, we obtain a G-valued filtered $(\varphi, N, \mathrm{Gal}_{L/K})$-module over $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}[1/p]$. By Lemma 2.6.6, we again have a natural morphism

$$\mathrm{Spec}\, R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}[1/l] \to G\text{-}\mathrm{WD}_E(L/K).$$

3.3. *Denseness of very smooth points.* We continue to fix an inertial type τ and (if $p = l$) a p-adic Hodge type $\boldsymbol{v}$. For convenience, if $l \neq p$ then for the rest of this section we write $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}$ for $R_{\bar{\rho}}^{\square,\tau}$; this notational convention allows us to treat the cases $l \neq p$ and $l = p$ simultaneously. We study the generic fibre $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}[1/l]$ via the morphism

$$\mathrm{Spec}\, R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}[1/l] \to G\text{-}\mathrm{WD}_E(L/K). \tag{3-3-1}$$

In a standard abuse of terminology, we say that a closed point $x \in \mathrm{Spec}\, R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}[1/l]$ is *smooth* if the (completed) local ring at x is regular. We will see in the proof of Theorem 3.3.2 that these are the points whose images in $G\text{-}\mathrm{WD}_E(L/K)$ are smooth points, which perhaps justifies this terminology. Similarly, we say that x is *very smooth* if for any finite extension K'/K, the image of x in (with obvious notation) $\operatorname{Spec} R^{\square,\tau|_{I_{K'}},\boldsymbol{v}_{K'}}_{\bar\rho|_{G_{K'}}}[1/l]$ is smooth.

As in [Kisin 2009, Proposition 2.3.5], if $x \in \operatorname{Spec} R^{\square,\tau,\boldsymbol{v}}_{\bar\rho}[1/l]$ is a closed point corresponding to a representation ρ_x, then the completed local ring A_x at x pro-represents framed deformations of ρ_x which are potentially semistable of p-adic Hodge type $\boldsymbol{v}$ (if $l = p$), and have inertial type τ.

Proposition 3.3.1. (1) *If x is a closed point of the Jacobson scheme* $\operatorname{Spec} R^{\square,\tau,\boldsymbol{v}}_{\bar\rho}[1/l]$, *then the completion at x of the morphism* (3-3-1) *is formally smooth.*

(2) *The morphism* (3-3-1) *is flat.*

Proof. The formal smoothness follows from the proofs of [Kisin 2008, Lemma 3.2.1, Proposition 3.3.1], which carries over verbatim to our setting (since the morphism of groupoids from framed deformations to unframed deformations is formally smooth). Part (2) then follows from the fact that formally smooth morphisms between locally noetherian schemes are flat, which in turn follows from [EGA IV$_1$ 1964, §0 Théorème 19.7.1]. □

Theorem 3.3.2. *Assume that $R^{\square,\tau,\boldsymbol{v}}_{\bar\rho} \neq 0$. There is a dense open subscheme $U \subset$* $\operatorname{Spec} R^{\square,\tau,\boldsymbol{v}}_{\bar\rho}[1/l]$ *which is regular, and there is a Zariski dense subset of* $\operatorname{Spec} R^{\square,\tau,\boldsymbol{v}}_{\bar\rho}[1/l]$ *consisting of very smooth points. Furthermore,* $\operatorname{Spec} R^{\square,\tau,\boldsymbol{v}}_{\bar\rho}[1/l]$ *is equidimensional of dimension* $\dim G + \delta_{l=p} \dim \operatorname{Res}_{E\otimes K/E} G/P_{\boldsymbol{v}}$, *locally a complete intersection, and reduced.*

Similarly, $\operatorname{Spec} R^{\square,\tau,\boldsymbol{v},\psi}_{\bar\rho}[1/l]$ *contains a regular dense open subscheme and a Zariski dense subset of very smooth points, and is equidimensional of dimension* $\dim G^{\mathrm{der}} + \delta_{l=p} \dim(\operatorname{Res}_{E\otimes K/E} G)/P_{\boldsymbol{v}}$.

Remark 3.3.3. In contrast to previous work (in particular [Kisin 2008; Gee 2011; Bellovin 2016]), we only claim that U is regular, not formally smooth over $\mathbb{Q}_p$. We are grateful to Jeremy Booher and Stefan Patrikis [2017] for drawing our attention to this.

Proof. Since the formation of scheme-theoretic images is compatible with flat base change, the existence of a dense open subscheme U consisting of smooth points follows from Corollary 2.3.7 and Proposition 3.3.1. The existence of a Zariski dense subset of very smooth points follows from Corollary 2.4.5. We claim that if $x \in \operatorname{Spec} R^{\square,\tau,\boldsymbol{v}}_{\bar\rho}[1/l]$ is a closed point in U, then the completion A_x of $R^{\square,\tau,\boldsymbol{v}}_{\bar\rho}[1/l]$ at x is a formally smooth $\mathbb{Q}_p$-algebra, and is in particular regular. Indeed, if $\mathfrak{m}_x$ is the maximal ideal of A_x, then $\operatorname{Spec} A_x/\mathfrak{m}_x^n \subset U$ for all $n \geq 1$ (since U is open). Let B be a local $\mathbb{Q}_p$-algebra with maximal ideal $\mathfrak{m}_B$ and let $I \subset B$ be an ideal such that $I\mathfrak{m}_B = (0)$. If there is a local homomorphism $A_x \to B/I$, let $D_{B/I}$ be the induced object of $G\text{-}\mathrm{WD}_E(L/K)(B/I)$. Then $H^2(\operatorname{ad} D_{B/I}) = 0$, since the homomorphism $A_x \to B/I$ factors through $A/\mathfrak{m}_x^n$ for some n. It follows that $D_{B/I}$ lifts to $D_B \in G\text{-}\mathrm{WD}_E(L/K)(B)$. Since $\operatorname{Spf} A_x \to G\text{-}\mathrm{WD}_E(L/K)$ is formally smooth, D_B is induced from a map $A_x \to B$ lifting $A \to B/I$. Since $R^{\square,\tau,\boldsymbol{v}}_{\bar\rho}[1/l]$ is Noetherian, it follows that the localisation of $R^{\square,\tau,\boldsymbol{v}}_{\bar\rho}[1/l]$ at x is regular [Stacks 2005–, Tag 07NY], so U is regular by [loc. cit., Tag 02IT], as claimed.

Thus, to compute the dimension of $\operatorname{Spec} R^{\square,\tau,\boldsymbol{v}}_{\bar\rho}[1/l]$, it is enough to compute the dimension of the tangent spaces at closed points in U. Let x be such a closed point, let E' be its residue field, and write

A_x for the completion of $R_{\bar\rho}^{\square,\tau,\boldsymbol{v}}[1/l]$ at x. Since the morphism $\operatorname{Spf} A_x \to G\text{-}\mathrm{WD}_E(L/K)$ is formally smooth by Proposition 3.3.1, it is versal at x. More precisely, in the case that $l \neq p$ we see (by the equivalence between Galois representations and Weil–Deligne representations recalled in Section 2.5) that the induced map $\operatorname{Spf} A_x \to G\text{-}\mathrm{WD}_E(L/K)^\wedge_x$ (with the right-hand side denoting the completion of the target at x) is a $\widehat{G}$-torsor, where $\widehat{G}$ is the completion of G_E along the closed subgroup given by the centraliser of the representation corresponding to x, in the sense that there is an evident isomorphism

$$\operatorname{Spf} A_x \times \widehat{G} \xrightarrow{\sim} \operatorname{Spf} A_x \times_{G\text{-}\mathrm{WD}_E(L/K)^\wedge_x} \operatorname{Spf} A_x.$$

In particular, we have $\dim A_x \times_{G\text{-}\mathrm{WD}_E(L/K)^\wedge_x} A_x = \dim A_x + \dim \widehat{G}$, and the claim about the dimension then follows from [Emerton and Gee 2017, Lemma 2.40] and Corollary 2.4.5.

If $l = p$, let $D_x := D^L_{\mathrm{st}}(\rho_x)$; it is equipped with a filtration $\mathcal{F}^\bullet_x$. We consider the set $(\operatorname{Spf} A_x)(E'[\varepsilon])$. Forgetting the framing on liftings is a formally smooth morphism of groupoids and makes the tangent space at x into a Lie G-torsor over the groupoid of unframed deformations. But since $E'[\varepsilon]$ is an artin local E-algebra, by [Bellovin 2016, Proposition 2.4] the category of (unframed) potentially semistable representations of Gal_K over $E'[\varepsilon]$ deforming ρ_x is equivalent to the subcategory of $G\text{-}\mathrm{Mod}_{L/K,\varphi,N,\mathrm{Fil}}(E'[\varepsilon])$ deforming $D^L_{\mathrm{st}}(\rho_x)$.

There is a natural morphism of groupoids

$$G\text{-}\mathrm{Mod}_{L/K,\varphi,N,\mathrm{Fil}} \to G\text{-}\mathrm{Mod}_{L/K,\varphi,N}$$

and therefore a commutative diagram:

$$\begin{array}{ccc} G\text{-}\mathrm{Mod}_{L/K,\varphi,N,\mathrm{Fil}}(E'[\varepsilon]) & \longrightarrow & G\text{-}\mathrm{Mod}_{L/K,\varphi,N}(E'[\varepsilon]) \\ \downarrow & & \downarrow \\ G\text{-}\mathrm{Mod}_{L/K,\varphi,N,\mathrm{Fil}}(E') & \longrightarrow & G\text{-}\mathrm{Mod}_{L/K,\varphi,N}(E') \end{array}$$

By Corollary 2.7.7, the fibres of

$$G\text{-}\mathrm{Mod}_{L/K,\varphi,N,\mathrm{Fil}}(E'[\varepsilon]) \to G\text{-}\mathrm{Mod}_{L/K,\varphi,N}(E'[\varepsilon])$$

over the filtered G-torsor D_x are torsors under $(\operatorname{ad} D_x/\mathcal{F}^0(\operatorname{ad} D_x))^{\mathrm{Gal}_{L/K}}$. Since $G\text{-}\mathrm{Mod}_{L/K,\varphi,N} \cong G\text{-}\mathrm{WD}_E(L/K)$ is equidimensional of dimension 0 and $x \in \operatorname{Spec} R_{\bar\rho}^{\square,\tau,\boldsymbol{v}}[1/l]$ is a smooth point, we conclude that

$$\begin{aligned} \dim A_x &= \dim \operatorname{Lie} G + \dim(\operatorname{ad} D_x/\mathcal{F}^0(\operatorname{ad} D_x))^{\mathrm{Gal}_{L/K}} \\ &= \dim G + \dim \operatorname{Res}_{E\otimes K/E} G/P_{\boldsymbol{v}} \end{aligned}$$

as desired.

To prove that $R_{\bar\rho}^{\square,\tau,\boldsymbol{v}}[1/l]$ is reduced and locally a complete intersection, we consider the fibre product $\operatorname{Spec} R_{\bar\rho}^{\square,\tau,\boldsymbol{v}}[1/l] \times_{G\text{-}\mathrm{WD}_E(L/K)} Y_{L/K,\varphi,\mathcal{N}}$. This is a G-torsor, hence smooth, over $\operatorname{Spec} R_{\bar\rho}^{\square,\tau,\boldsymbol{v}}[1/l]$, so it suffices to prove that this fibre product is reduced and locally a complete intersection. But by Proposition 3.3.1, the natural morphism $\operatorname{Spec} R_{\bar\rho}^{\square,\tau,\boldsymbol{v}}[1/l] \times_{G\text{-}\mathrm{WD}_E(L/K)} Y_{L/K,\varphi,\mathcal{N}} \to Y_{L/K,\varphi,\mathcal{N}}$ is formally smooth, so completed local rings at points of Spec $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}[1/l] \times_{G\text{-}\mathrm{WD}_E(L/K)} Y_{L/K,\varphi,\mathcal{N}}$ are power series rings over completed local rings of $Y_{L/K,\varphi,\mathcal{N}}$. Since the latter are reduced and complete intersections (by Corollary 2.4.5), the same holds for the former.

The corresponding statements for $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v},\psi}$ can be proved in the same way; we leave the details to the reader. □

The following is a generalisation of [Allen 2016a, Theorem D] (which treats the case that $l = p$ and $G = \mathrm{GL}_n$). We let x be a closed point of $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}[1/l]$ with residue field E_x (a finite extension of E), and write $\rho_x : \mathrm{Gal}_K \to G(E_x)$ for the corresponding representation.

Corollary 3.3.4. *The point x is a formally smooth point of $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}[1/l]$ if and only if*

$$H^0((\operatorname{ad}\mathrm{WD}(\rho_x))^*(1)) = 0.$$

Proof. Corollary 2.4.2 implies that the formally smooth points of $G\text{-}\mathrm{WD}_E(L/K)$ are precisely those points x for which $H^0((\operatorname{ad} D_x)^*(1))$. Thus, we need to show that $x \in \operatorname{Spec} R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}[1/l]$ is formally smooth if and only if its image in $G\text{-}\mathrm{WD}_E(L/K)$ is formally smooth.

We have a morphism

$$\operatorname{Spec} R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}[1/l]_x^\wedge \to G\text{-}\mathrm{WD}_E(L/K)_x^\wedge,$$

which is formally smooth by Proposition 3.3.1. But this implies that for any $\mathbb{Q}_p$-finite artin local ring B, the map

$$\operatorname{Spec} R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}[1/l]_x^\wedge(B) \to G\text{-}\mathrm{WD}_E(L/K)_x^\wedge(B)$$

is surjective. Hence, Spec $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}[1/l]_x^\wedge$ is formally smooth if and only if $G\text{-}\mathrm{WD}_E(L/K)_x^\wedge$ is formally smooth. □

Remark 3.3.5. If G is the L-group of a quasisplit reductive group over K, then it seems plausible that the condition of Corollary 3.3.4 could be equivalent to the condition that the (conjectural) L-packet of representations associated to the Frobenius semisimplification of $\mathrm{WD}(\rho_x)$ contains a generic element. In the case that $G = \mathrm{GL}_n$ (where the L-packets are singletons) and $\mathrm{WD}(\rho_x)$ is Frobenius semisimple, this is proved in [Allen 2016a, §1], and in the general case it is closely related to [Gross and Prasad 1992, Conjecture 2.6] (which relates genericity to poles at $s = 1$ of the adjoint L-function).

Remark 3.3.6. In the case that $l \neq p$, the equivalence between Galois representations and Weil–Deligne representations means that we can rewrite the condition in Corollary 3.3.4 as $H^0(\mathrm{Gal}_K, \operatorname{ad}\rho_x^*(1)) = 0$.

We can also consider the quotient $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v},N=0}$, corresponding to the union of the irreducible components of $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}[1/l]$ for which the monodromy operator N vanishes identically (if $l = p$, this is the locus of potentially crystalline representations, and if $l \neq p$, it is the locus of potentially unramified representations).

Theorem 3.3.7. *Fix an inertial type τ, and if $l = p$ then fix a p-adic Hodge type $\boldsymbol{v}$. Assume that $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v},N=0} \neq 0$. Then $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v},N=0}[1/l]$ is regular, and is equidimensional of dimension*

$$\dim_E G + \delta_{l=p} \dim_E(\mathrm{Res}_{E\otimes K/E}\, G)/P_{\boldsymbol{v}}.$$

Similarly $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v},N=0,\psi}[1/l]$ *is regular and equidimensional of dimension*

$$\dim_E G^{\mathrm{der}} + \delta_{l=p} \dim_E(\mathrm{Res}_{E\otimes K/E}\, G)/P_{\boldsymbol{v}}.$$

Proof. This can be proved in exactly the same way as Theorem 3.3.2, replacing the use of the three term complex $\mathcal{C}^\bullet(D)$ considered in Proposition 2.2.1 with the two term complex

$$(\mathrm{ad}\, D_A)^{I_{L/K}} \xrightarrow{1-\mathrm{Ad}(\Phi)} (\mathrm{ad}\, D_A)^{I_{L/K}}$$

concentrated in degrees 0 and 1; see [Kisin 2008, Theorem 3.3.8] for more details in the case that $l = p$ and $G = \mathrm{GL}_n$. □

3.4. *Components of deformation rings.* We now prove the following reassuring lemma, which shows that the components of universal deformation rings are invariant under $G(\mathcal{O})$-conjugacy. It is a generalisation of [Barnet-Lamb et al. 2014, Lemma 1.2.2], which treats the case $G = \mathrm{GL}_n$; the proof there is by an explicit homotopy, while we use the theory of reductive group schemes over $\mathcal{O}$ to construct less explicit homotopies.

Lemma 3.4.1. *Let $h \in G(\mathcal{O}')$ be an element which reduces to the identity modulo the maximal ideal, where $\mathcal{O}'$ is the ring of integers in a finite extension of E. Then conjugation by h induces a map* $\mathrm{Spec}(R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}} \otimes_{\mathcal{O}} \mathcal{O}')[1/l] \to \mathrm{Spec}(R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}} \otimes_{\mathcal{O}} \mathcal{O}')[1/l]$, *and it fixes each irreducible component.*

Before we prove it, we record a preliminary lemma on irreducible components of the generic fibre of $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}$:

Lemma 3.4.2. *Let $A := \mathcal{O}[[X_1, \dots, X_n]]/I$ be the quotient of a power series ring. If $x, x' \in (\mathrm{Spf}\, A)^{\mathrm{rig}}$ lie on the same irreducible component, then they lie on the same irreducible component of* $\mathrm{Spec}\, A[1/l]$.

Proof. If $x = x'$ as points of $(\mathrm{Spf}\, A)^{\mathrm{rig}}$, then by [de Jong 1995, Lemma 7.1.9], $x = x'$ as points of $\mathrm{Spec}\, A[1/l]$. Thus, we may assume that $x \neq x'$. Let $A \to \tilde{A}$ denote the normalisation of A. Then by [Conrad 1999, Theorem 2.1.3], $(\mathrm{Spf}\, \tilde{A})^{\mathrm{rig}} \to (\mathrm{Spf}\, A)^{\mathrm{rig}}$ is a normalisation of the rigid space $(\mathrm{Spf}\, A)^{\mathrm{rig}}$, and x, x' lift to points $\tilde{x}, \tilde{x}' \in (\mathrm{Spf}\, \tilde{A})^{\mathrm{rig}}$ on the same connected component. By [de Jong 1995, Lemma 7.1.9], $\tilde{x}$ and $\tilde{x}'$ correspond to distinct closed points of $\mathrm{Spec}\, \tilde{A}[1/l]$.

If $\tilde{x}$ and $\tilde{x}'$ lie on distinct connected components of $\mathrm{Spec}\, \tilde{A}[1/l]$, there are idempotents $e_x, e_{x'} \in \tilde{A}[1/l]$ such that e_x is 1 at $\tilde{x}$ and 0 at $\tilde{x}'$ and $e_{x'}$ is 1 at $\tilde{x}'$ and 0 at $\tilde{x}$. Again by [loc. cit., Lemma 7.1.9], the natural map $(\mathrm{Spf}\, \tilde{A})^{\mathrm{rig}} \to \mathrm{Spec}\, \tilde{A}[1/l]$ induces isomorphisms on residue fields of closed points. It follows that the pullbacks of e_x and $e_{x'}$ to $(\mathrm{Spf}\, \tilde{A})^{\mathrm{rig}}$ are again idempotents (in the global sections of the structure sheaf of $(\mathrm{Spf}\, \tilde{A})^{\mathrm{rig}}$) such that e_x is 1 at $\tilde{x}$ and 0 at $\tilde{x}'$ and $e_{x'}$ is 1 at $\tilde{x}'$ and 0 at $\tilde{x}$. But this would contradict the fact that $\tilde{x}$ and $\tilde{x}'$ lie on the same connected component of $(\mathrm{Spf}\, \tilde{A})^{\mathrm{rig}}$, so they must actually lie on the same connected component of $\mathrm{Spec}\, \tilde{A}[1/l]$. This in turn implies that they lie on the same irreducible component of $\mathrm{Spec}\, A[1/l]$. □

Proof of Lemma 3.4.1. Let $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}} \otimes_{\mathcal{O}} \mathcal{O}'' \to \mathcal{O}''$ be a homomorphism corresponding to a lift $\rho : \mathrm{Gal}_K \to G(\mathcal{O}'')$, where $\mathcal{O}''$ is the ring of integers in a finite extension of E and contains $\mathcal{O}'$. We continue to write

h for the image of h in $G(\mathcal{O}'')$. There is a finite surjective morphism

$$\operatorname{Spec}(R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}} \otimes_{\mathcal{O}} \mathcal{O}'')[1/l] \to \operatorname{Spec}(R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}} \otimes_{\mathcal{O}} \mathcal{O}')[1/l],$$

so to show that conjugation by h preserves irreducible components of $\operatorname{Spec}(R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}} \otimes_{\mathcal{O}} \mathcal{O}')[1/l]$, it suffices to show that conjugation by h preserves irreducible components of $\operatorname{Spec}(R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}} \otimes_{\mathcal{O}} \mathcal{O}'')[1/l]$. Moreover, by Lemma 3.4.2, it suffices to work with the rigid analytic generic fibre $\operatorname{Spf}(R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}} \otimes_{\mathcal{O}} \mathcal{O}'')^{\mathrm{rig}}$ of $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}} \otimes_{\mathcal{O}} \mathcal{O}''$.

After possibly extending $\mathcal{O}''$, we may assume that G splits over $\mathcal{O}''$. Since h is residually the identity element of G, it is a point of G°. After possibly further increasing $\mathcal{O}''$, there is some Borel subgroup $B_{\mathcal{O}''[1/l]} \subset G^\circ_{\mathcal{O}''[1/l]}$ containing the image of h; it extends to a Borel subgroup $B \subset G^\circ_{\mathcal{O}''}$ which contains h. Since $\mathcal{O}''$ is local, by [Conrad 2014, Proposition 5.2.3] there is a cocharacter $\lambda : (\mathbb{G}_m)_{\mathcal{O}''} \to G^\circ_{\mathcal{O}''}$ such that $B = P_{G^\circ}(\lambda) = U_{G^\circ}(\lambda) \rtimes Z_{G^\circ}(\lambda)$. Write h_z for the projection of h to $Z_{G^\circ}(\lambda)$ and h_u for the projection to $U_{G^\circ}(\lambda)$. Since this decomposition is unique, both h_z and h_u reduce to the identity modulo ϖ (where ϖ is a uniformiser of $\mathcal{O}''$).

Since $Z_{G^\circ}(\lambda)$ is a split torus, there is a map $z_t : (\mathbb{G}_m)_{\mathcal{O}''} \to G^\circ_{\mathcal{O}''}$ which specialises to both h_z and the identity. After analytifying this map, h_z and the identity lie in the same residue disk. Choosing coordinates on this residue disk, and rescaling them if necessary, we obtain a Galois representation $\tilde{\rho} : \mathrm{Gal}_K \to G(\mathcal{O}''[\![T]\!])$ by considering the conjugation map $z_t \rho z_t^{-1} : \mathrm{Gal}_K \to G(\mathcal{O}''[T])$. This induces a homomorphism $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}} \otimes_{\mathcal{O}} \mathcal{O}'' \to \mathcal{O}''[\![T]\!]$, which in turn induces a morphism of rigid spaces $\operatorname{Spf}(\mathcal{O}''[\![T]\!])^{\mathrm{rig}} \to \operatorname{Spf}(R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}} \otimes_{\mathcal{O}} \mathcal{O}'')^{\mathrm{rig}}$. Since the source is irreducible and its image contains points corresponding to both ρ and $h_z \rho h_z^{-1}$, they lie on the same irreducible component of $\operatorname{Spf}(R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}} \otimes_{\mathcal{O}} \mathcal{O}'')^{\mathrm{rig}}$.

Thus, we may assume that $h \in U_{G^\circ}(\lambda)$. By definition, if A is an $\mathcal{O}'$-algebra,

$$U_{G^\circ}(\lambda)(A) = \{g \in G^\circ(A) \mid \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} = 1\},$$

so conjugating h by λ induces a map $u_t : \mathbb{A}^1_{\mathcal{O}''} \to G_{\mathcal{O}''}$ with $u_1 = h$ and $u_0 = 1$. We therefore obtain a Galois representation $\tilde{\rho}' : \mathrm{Gal}_K \to G(\mathcal{O}''\langle T\rangle)$ by l-adically completing the map $u_t \rho u_t^{-1} : \mathrm{Gal}_K \to G(\mathcal{O}''[T])$. Since u_t is the identity modulo ϖ, $\tilde{\rho}'$ in fact lands in $G(\mathcal{O}''\langle \varpi T\rangle)$, and therefore in $G(\mathcal{O}''[\![\varpi T]\!])$. This induces a map $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}} \otimes_{\mathcal{O}} \mathcal{O}'' \to \mathcal{O}''[\![\varpi T]\!]$, and therefore a morphism of rigid spaces $\operatorname{Spf}(\mathcal{O}''[\![\varpi T]\!])^{\mathrm{rig}} \to \operatorname{Spf}(R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}} \otimes_{\mathcal{O}} \mathcal{O}'')^{\mathrm{rig}}$. Since the source is irreducible and its image contains points corresponding to both ρ and $h_u \rho h_u^{-1}$, they lie on the same irreducible component of $\operatorname{Spf}(R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}} \otimes_{\mathcal{O}} \mathcal{O}'')^{\mathrm{rig}}$, as required. □

3.5. ***Tensor products of components, and base change.*** By a "component for $\bar{\rho}$" we mean a choice of τ and $\boldsymbol{v}$ (in the case $l = p$) such that $R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}[1/l] \neq 0$, and a choice of an irreducible component of $\operatorname{Spec} R_{\bar{\rho}}^{\square,\tau,\boldsymbol{v}}[1/l]$.

Let $\bar{r} : \mathrm{Gal}_K \to \mathrm{GL}_n(\mathbb{F})$ and $\bar{s} : \mathrm{Gal}_K \to \mathrm{GL}_m(\mathbb{F})$ be representations, let C be a component for $\bar{r}$ and let D be a component for $\bar{s}$. Let K'/K be a finite extension. The following lemma will be useful in Section 5.

Lemma 3.5.1. *There is a unique component $C \otimes D$ for $\bar{r} \otimes \bar{s}$ with the property that, if $r : \mathrm{Gal}_K \to \mathrm{GL}_n(\overline{\mathbb{Q}}_l)$ and $s : \mathrm{Gal}_K \to \mathrm{GL}_m(\overline{\mathbb{Q}}_l)$ correspond to closed points of C and D respectively, then $r \otimes s$ corresponds* *to a closed point of $C \otimes D$. Similarly, there is a unique component $C|_{K'}$ for $\bar{r}|_{\mathrm{Gal}_{K'}}$ such that for all r, $r|_{\mathrm{Gal}_{K'}}$ corresponds to a closed point of $C|_{K'}$.*

Proof. If a point of $\operatorname{Spec} R_{\bar{r}}^{\square,\tau,\boldsymbol{v}}[1/l]$ or a point of $\operatorname{Spec} R_{\bar{r}\otimes\bar{s}}^{\square,\tau,\boldsymbol{v}}[1/l]$ is smooth, then it lies on a unique irreducible component. Then the first part follows as in the proof of Theorem 3.3.2, replacing the appeal to Corollary 2.4.5 with one to Theorem 2.3.8, applied to the tensor product map

$$\mathrm{GL}_n \times \mathrm{GL}_m \to \mathrm{GL}_{nm}.$$

The second part follows from Theorem 3.3.2 (more precisely, from the existence of very smooth points on each irreducible component). □

In the setting of the previous lemma, we will sometimes say that the component $C \otimes D$ is the tensor product of the components C and D, and that $C|_{K'}$ is the base change to K' of the component C.

4. Global deformation rings

4.1. ***A result of Balaji.*** In this section we recall one of the main results of [Balaji 2013], which we will then combine with the results of Section 3 to prove Proposition 4.2.6, which gives a lower bound for the dimension of certain global deformation rings. In [loc. cit., §4.2] the group G is assumed to be connected, but this is unnecessary. Indeed, the assumption is only made in order to use the results of [Tilouine 1996, §5], where it is also assumed that G is connected; however, this assumption is never used in any of the arguments of [loc. cit., §5], which apply unchanged to general G. Accordingly, we will freely use the results of [Balaji 2013, §4.2] without assuming that G is connected. We assume in this section that E is taken large enough that G_E is quasisplit.

Let F be a number field, and let S be a finite set of places of F containing all of the places dividing $l\infty$. We work in the fixed determinant setting, and accordingly we fix homomorphisms $\bar{\rho} : \mathrm{Gal}_{F,S} \to G(\mathbb{F})$ and $\psi : \mathrm{Gal}_{F,S} \to G^{\mathrm{ab}}(\mathcal{O})$ such that $\mathrm{ab} \circ \bar{\rho} = \bar{\psi}$.

Write $R_{F,S}^{\square,\psi} \in \mathrm{CNL}_{\mathcal{O}}$ for the universal fixed determinant framed deformation $\mathcal{O}$-algebra of $\bar{\rho}$. Let $\Sigma \subset S$ be a subset containing all of the places lying over l. For each $v \in \Sigma$, we let $R_v^{\square,\psi}$ denote the universal fixed determinant framed deformation $\mathcal{O}$-algebra of $\bar{\rho}|_{\mathrm{Gal}_{F_v}}$, and we set

$$R_{\Sigma}^{\square,\psi} := \widehat{\bigotimes}_{v\in\Sigma,\mathcal{O}} R_v^{\square,\psi}.$$

The following result is a special case of [Balaji 2013, Proposition 4.2.5].

Proposition 4.1.1. *Suppose that $H^0(\mathrm{Gal}_{F,S}, (\mathfrak{g}_{\mathbb{F}}^0)^*(1)) = 0$, and let*

$$s := (|\Sigma| - 1)\dim_{\mathbb{F}} \mathfrak{g}_{\mathbb{F}}^0 + \sum_{v\,|\,\infty, v\notin\Sigma} \dim_{\mathbb{F}} H^0(\mathrm{Gal}_{F_v}, \mathfrak{g}_{\mathbb{F}}^0).$$

Then for some $r \geq 0$ there is a presentation

$$R_{F,S}^{\square,\psi} \xrightarrow{\sim} R_{\Sigma}^{\square,\psi}[\![x_1, \ldots, x_r]\!]/(f_1, \ldots, f_{r+s}).$$

4.2. *Global deformation rings of fixed type.* We now combine our local results with Proposition 4.1.1 to prove a lower bound for the Krull dimension of a global deformation ring, following Balaji. This lower bound will only be nontrivial in the following setting.

Definition 4.2.1. If $l > 2$ then we say that $\bar{\rho}$ is *discrete series and odd* if F is totally real, and if for all places $v \mid \infty$ of F we have $\dim_{\mathbb{F}} H^0(\mathrm{Gal}_{F_v}, \mathfrak{g}_{\mathbb{F}}^0) = \dim_E G - \dim_E B$, where B is a Borel subgroup of G.

Remark 4.2.2. Recall that we chose E to be large enough that G_E is quasisplit, so this definition makes sense. The condition that $\bar{\rho}$ is discrete series and odd is needed to make the usual Taylor–Wiles method work; see the introduction to [Clozel et al. 2008]. If G is the L-group of a simply connected group then one can check that this condition is equivalent to F being totally real and $\bar{\rho}$ being odd in the sense of [Gross 2007] (cf. [Balaji 2013, Lemma 4.3.1]). We use the term "discrete series" because the (conjectural) Galois representations associated to tempered automorphic representations which are discrete series at infinite places are expected to satisfy this property; see Section 5 for an example of this, and [Gross 2007] for a more general discussion.

Definition 4.2.3. We say that a p-adic Hodge type $\boldsymbol{v}$ is *regular* if the conjugacy class $P_{\boldsymbol{v}}$ consists of parabolic subgroups of $\mathrm{Res}_{E\otimes K/E}\, G$ whose connected components are Borel subgroups of $(\mathrm{Res}_{E\otimes K/E}\, G)^\circ$.

Remark 4.2.4. If $G = \mathrm{GL}_n$ then Definition 4.2.3 is equivalent to the usual definition, that for each embedding $K \hookrightarrow E$ the Hodge–Tate weights are pairwise distinct.

Remark 4.2.5. If E'/E is a field extension, then

$$(\mathrm{Res}_{E\otimes K/E}\, G)_{E'} \cong \mathrm{Res}_{E'\otimes K/E'}\, G.$$

Furthermore, the formation of $P_{\mathrm{Res}_{E\otimes K/E} G}(\lambda)$ is compatible with extension of scalars from E to E'. Thus, if $\boldsymbol{v}$ is regular after extending scalars, it was regular over E (and $\mathrm{Res}_{E\otimes K/E}\, G$ is automatically quasisplit).

Write S^∞ for the set of finite places in S. For each place $v \in S^\infty$, we fix an inertial type τ_v, and if $v \mid l$ then we fix a Hodge type $\boldsymbol{v}_v$. If $v \nmid l$ (resp. if $v \mid l$), we let $\bar{R}_v$ be a quotient of the corresponding fixed determinant framed deformation ring $R^{\square,\tau_v,\psi}_{\bar{\rho}|_{\mathrm{Gal}_{F_v}}}$ (resp. $R^{\square,\tau_v,\boldsymbol{v}_v,\psi}_{\bar{\rho}|_{\mathrm{Gal}_{F_v}}}$) corresponding to a nonempty union of irreducible components of the generic fibre. Set

$$R^{\square,\mathrm{univ}} := R^{\square,\psi}_{F,S} \otimes_{R^{\square,\psi}_{\Sigma},\mathcal{O}} \widehat{\bigotimes}_{v\in S^\infty} \bar{R}_v;$$

this is nonzero, because we are assuming that each $\bar{R}_v$ is nonzero.

Assume that $H^0(\mathrm{Gal}_{F,S}, \mathfrak{g}_{\mathbb{F}}) = \mathfrak{z}_{\mathbb{F}}$, so that $\bar{\rho}$ admits a universal fixed determinant deformation $\mathcal{O}$-algebra $R^{\psi}_{F,S} \in \mathrm{CNL}_{\mathcal{O}}$, and write R^{univ} for the quotient of $R_{F,S}$ corresponding to $R^{\square,\mathrm{univ}}$ (as in the discussion preceding [Barnet-Lamb et al. 2014, Lemma 1.3.3], this quotient exists by Lemma 3.4.1). In the case that we fix potentially crystalline types at the places $v \mid l$, and do not fix types at places away from l, the following result is [Balaji 2013, Theorem 4.3.2]; the general case follows from the same arguments as those of Balaji, given the input of our local results.

Proposition 4.2.6. *Assume that $l > 2$, that $\bar{\rho}$ is a discrete series and odd (so that in particular F is totally real), and that $H^0(\mathrm{Gal}_{F,S}, (\mathfrak{g}^0_{\mathbb{F}})^*(1)) = 0$. Maintain our assumption that the local deformation rings $\bar{R}_v$ are nonzero.*

Suppose that for each place $v \mid l$ the Hodge type $\boldsymbol{v}_v$ is regular. Then R^{univ} has Krull dimension at least 1.

Proof. By Proposition 4.1.1 (taking $\Sigma = S^\infty$) we see that for some $r \geq \dim_{\mathbb{F}} \mathfrak{g}^0_{\mathbb{F}}$ we have a presentation

$$R^{\square,\mathrm{univ}} \xrightarrow{\sim} \Big(\widehat{\bigotimes}_{v \in S^\infty} \bar{R}_v\Big)[\![x_1, \dots, x_r]\!]/(f_1, \dots, f_{r+s}),$$

where

$$s = (|S^\infty| - 1) \dim_{\mathbb{F}} \mathfrak{g}^0_{\mathbb{F}} + \sum_{v \mid \infty} \dim_{\mathbb{F}} H^0(\mathrm{Gal}_{F_v}, \mathfrak{g}^0_{\mathbb{F}}).$$

Since $R^{\square,\mathrm{univ}}$ is formally smooth over R^{univ} of relative dimension $\dim_{\mathbb{F}} \mathfrak{g}^0_{\mathbb{F}}$, it follows that the Krull dimension of R^{univ} is at least

$$\dim \widehat{\bigotimes}_{v \in S^\infty, \mathcal{O}} \bar{R}_v - |S^\infty| \dim_{\mathbb{F}} \mathfrak{g}^0_{\mathbb{F}} - \sum_{v \mid \infty} \dim_{\mathbb{F}} H^0(\mathrm{Gal}_{F_v}, \mathfrak{g}^0_{\mathbb{F}}),$$

which by Theorem 3.3.2, and our assumption that each Hodge type $\boldsymbol{v}_v$ is regular, is equal to

$$1 + \sum_{v \mid p} [F_v : \mathbb{Q}_p] \dim_E G/B - \sum_{v \mid \infty} \dim_{\mathbb{F}} H^0(\mathrm{Gal}_{F_v}, \mathfrak{g}^0_{\mathbb{F}}),$$

which in turn (by the assumption that $\bar{\rho}$ is discrete series and odd) equals 1, as required. ☐

5. Unitary groups

5.1. *The group $\mathcal{G}_n$.* Let F be a CM field with maximal totally real subfield F^+. In this section we generalise some results of [Barnet-Lamb et al. 2014] on the deformation theory of Galois representations associated to polarised representations of Gal_F, by allowing ramification at primes of F^+ which are inert or ramified in F. This allows us to make cleaner statements, and is also useful in applications; for example, in Theorem 5.2.2 we remove a "split ramification" condition in the proof of the weight part of Serre's conjecture for rank-2 unitary groups. Our results are also needed in [Calegari et al. 2018], where they are used to construct lifts with specified ramification at certain places of F^+ which are inert in F.

Recall from [Clozel et al. 2008] the reductive group $\mathcal{G}_n$ over $\mathbb{Z}$ given by the semidirect product of $\mathcal{G}^0_n = \mathrm{GL}_n \times \mathrm{GL}_1$ by the group $\{1, \jmath\}$, where

$$\jmath(g, a)\jmath^{-1} = (a(g^t)^{-1}, a).$$

We let $\nu : \mathcal{G}_n \to \mathrm{GL}_1$ be the character which sends (g, a) to a and sends $\jmath$ to -1. Our results in this section are for the most part a straightforward application of the results of the earlier sections to the particular case $G = \mathcal{G}_n$, but we need to begin by comparing our definitions to those of [loc. cit.]; we will follow the notation of that paper where possible.

Fix a place $v \mid \infty$. By [Clozel et al. 2008, Lemma 2.1.1], for any ring R there is a natural bijection between the set of homomorphisms $\rho : \mathrm{Gal}_{F^+} \to \mathcal{G}_n(R)$ inducing an isomorphism $\mathrm{Gal}_{F^+} / \mathrm{Gal}_F \xrightarrow{\sim} \mathcal{G}_n/\mathcal{G}_n^0$, and the set of triples $(r, \mu, \langle \cdot, \cdot \rangle)$ where $r : \mathrm{Gal}_F \to \mathrm{GL}_n(R)$, $\mu : \mathrm{Gal}_{F^+} \to R^\times$, and $\langle \cdot, \cdot \rangle : R^n \times R^n \to R$ is a perfect R-linear pairing such that $\langle x, y \rangle = -\mu(c_v)\langle y, x \rangle$, and $\langle r(\delta)x, r^{c_v}(\delta)y \rangle = \mu(\delta)\langle x, y \rangle$ for all $\delta \in \mathrm{Gal}_F$. We refer to such a triple as a μ-polarised representation of Gal_F, and we will sometimes denote it as a pair (r, μ), the pairing being implicit.

This bijection is given by setting $r := \rho|_{\mathrm{Gal}_F}$ (more precisely, the projection of $\rho|_{\mathrm{Gal}_F}$ to $\mathrm{GL}_n(R)$), $\mu := \nu \circ \rho$, and $\langle x, y \rangle = x^t A^{-1} y$, where $\rho(c_v) = (A, -\mu(c_v))\jmath$. If v is a finite place of F^+ which is inert or ramified in F, then we have an induced bijection between representations $\mathrm{Gal}_{F_v^+} \to \mathcal{G}_n(R)$ and μ-polarised representations $\mathrm{Gal}_{F_v} \to \mathrm{GL}_n(R)$.

There is an isomorphism $\mathrm{GL}_1 \to Z_{\mathcal{G}_n}$ given by $g \mapsto (g, g^2) \in \mathrm{GL}_1 \to \mathrm{GL}_1 \subset \mathrm{GL}_n \times \mathrm{GL}_1$, and we have $\mathcal{G}_n^{\mathrm{der}} = \mathrm{GL}_n \times 1$, and $\mathcal{G}_n^{\mathrm{ab}} = \mathrm{GL}_1 \times \{1, \jmath\}$. (It is easy to check by direct calculation that $\mathcal{G}_n^{\mathrm{der}} \subset \mathcal{G}_n^\circ$, and indeed $\mathcal{G}_n^{\mathrm{der}} \subset \mathrm{GL}_n \times 1$. Since $\mathrm{GL}_n^{\mathrm{der}} = \mathrm{SL}_n$, we have $\mathrm{SL}_n \times 1 \subset \mathcal{G}_n^{\mathrm{der}}$, and since $\jmath(1, a)\jmath^{-1}(1, a^{-1}) = (a, 1)$, we also have $\mathrm{GL}_1 \times 1 \subset \mathcal{G}_n^{\mathrm{der}}$, whence $\mathrm{GL}_n \times 1 \subset \mathcal{G}_n^{\mathrm{der}}$. Similarly, one checks easily that $Z_{\mathcal{G}_n} \subset \mathcal{G}_n^\circ$, so that $Z_{\mathcal{G}_n} \subset \mathrm{GL}_1 \times \mathrm{GL}_1$. If $(g, a) \in \mathrm{GL}_1 \times \mathrm{GL}_1$ then $\jmath(g, a)\jmath^{-1} = (ag^{-1}, a)$, so we see that $(g, a) \in Z_{\mathcal{G}_n}$ if and only if $a = g^2$, as required.)

We fix a prime $l > 2$ and a representation $\bar{\rho} : \mathrm{Gal}_{F^+} \to \mathcal{G}_n(\mathbb{F})$ with $\bar{\rho}^{-1}(\mathcal{G}_n^0(\mathbb{F})) = \mathrm{Gal}_F$. We fix a character $\mu : \mathrm{Gal}_{F^+} \to \mathcal{O}^\times$ with $\nu \circ \bar{\rho} = \bar{\mu}$. Write $\psi : \mathrm{Gal}_{F^+} \to \mathcal{G}_n^{\mathrm{ab}}(\mathcal{O})$ for the character taking $g \in \mathrm{Gal}_F$ to $(\mu(g), 1)$ and $g \in \mathrm{Gal}_{F^+} \setminus \mathrm{Gal}_F$ to $(-\mu(g), \jmath)$.

Note that if $R \in \mathrm{CNL}_{\mathcal{O}}$ then a deformation $\rho : \mathrm{Gal}_{F^+} \to \mathcal{G}_n(R)$ of $\bar{\rho}$ has $\mathrm{ab} \circ \rho = \psi$ if and only if $\nu \circ \rho = \mu$, in which case we say that it is μ-polarised. By [Allen 2016b, Proposition 2.2.3], restriction to Gal_F gives an equivalence between the μ-polarised (framed) deformations of $\bar{\rho}$ and the μ-polarised (framed) deformations r of $\bar{r} := \bar{\rho}|_{\mathrm{Gal}_F} : \mathrm{Gal}_F \to \mathrm{GL}_n(\mathbb{F})$, the latter by definition being those r which satisfy $r^c \cong r^\vee \mu$ (where we are writing c for c_v, as r^c is independent of the choice of $v \mid \infty$).

The same equivalence pertains to deformations of $\bar{\rho}|_{\mathrm{Gal}_{F_v^+}}$, where v is inert or ramified in F. On the other hand, if v splits as $\tilde{v}\tilde{v}^c$ in F, then restriction to $\mathrm{Gal}_{F_{\tilde{v}}}$ gives an equivalence between μ-polarised (framed) deformations of $\bar{\rho}|_{\mathrm{Gal}_{F_v^+}}$ and (framed) deformations of $\bar{r}|_{\mathrm{Gal}_{F_{\tilde{v}}}}$; thus at such places the deformation theory of representations valued in $\mathcal{G}_n$ is reduced to the case of GL_n. It is for this reason that [Clozel et al. 2008] and its sequels only permit ramification at places which split in F.

By [loc. cit., Lemma 2.1.3], $\bar{\rho}$ is discrete series and odd in the sense of Definition 4.2.1 if and only if for each place $v \mid \infty$ of F^+ with corresponding complex conjugation $c_v \in \mathrm{Gal}_{F^+}$ we have $\bar{\mu}(c_v) = -1$. This is by definition equivalent to the corresponding polarised representation $(\bar{\rho}|_{\mathrm{Gal}_F}, \bar{\mu})$ being totally odd in the sense of [Barnet-Lamb et al. 2014, §2.1].

Let S be a finite set of places of F^+, including all the places where $\bar{r}$ or μ are ramified, all the infinite places, and all the places dividing l. The following is a generalisation of [loc. cit., Proposition 1.5.1] (which is the case that every finite place in S splits in F, and is actually proved in [Clozel et al. 2008]); note that the assumption that $\bar{\rho}|_{\mathrm{Gal}_{F(\zeta_l)}}$ is absolutely irreducible is missing from the statement of [Barnet-Lamb et al. 2014, Proposition 1.5.1], but should have been included there. Note also that this assumption implies that $\bar{\rho}$

admits a universal deformation ring; indeed, we have $H^0(\mathrm{Gal}_{F^+}, \mathfrak{g}_{\mathbb{F}}) = H^0(\mathrm{Gal}_{F^+}, \mathfrak{gl}_{n,\mathbb{F}} \times \mathfrak{gl}_{1,\mathbb{F}}) = \mathfrak{gl}_{1,\mathbb{F}}$ by Schur's lemma (note that $\mathrm{Gal}(F/F^+)$ acts by -1 on the scalar matrices in $\mathfrak{gl}_{n,\mathbb{F}}$).

Corollary 5.1.1. *Let $l > 2$ be prime, and let $\bar{\rho} : \mathrm{Gal}_{F^+} \to \mathcal{G}_n(\mathbb{F})$ be such that $\bar{\rho}|_{\mathrm{Gal}_{F(\zeta_l)}}$ is absolutely irreducible. Assume that $\bar{\rho}$ is discrete series and odd.*

Let μ be a de Rham lift of $\bar{\mu}$, and let S be a finite set of places of F^+ including all the places at which either $\bar{r}$ or μ is ramified, and all the places dividing $l\infty$. For each finite place $v \in S$, fix an inertial type τ_v, and if $v \mid l$, fix a regular Hodge type $\boldsymbol{v}_v$. Fix quotients of the corresponding local μ-polarised framed deformation rings which correspond to a (nonempty) union of irreducible components of the generic fibre.

Let R^{univ} be the universal deformation ring for μ-polarised deformations of $\bar{\rho}$ which are unramified outside S, and lie on the given union of irreducible components for each finite place $v \in S$. Then R^{univ} has Krull dimension at least 1.

Proof. By Proposition 4.2.6, we need only check that $H^0(\mathrm{Gal}_{F^+,S}, (\mathfrak{gl}_{n,\mathbb{F}})^*(1))$ vanishes, where $\mathfrak{gl}_{n,\mathbb{F}}$ is the Lie algebra of $\mathcal{G}_n^{\mathrm{der}}$. By inflation-restriction this group injects into

$$H^0(\mathrm{Gal}_{F(\zeta_l)}, (\mathfrak{gl}_{n,\mathbb{F}})^*(1))^{\mathrm{Gal}(F(\zeta_l)/F^+)} = H^0(\mathrm{Gal}_{F(\zeta_l)}, (\mathfrak{gl}_{n,\mathbb{F}}))^{\mathrm{Gal}(F(\zeta_l)/F^+)}.$$

Since $\bar{\rho}|_{\mathrm{Gal}_{F(\zeta_l)}}$ is absolutely irreducible by assumption, this group vanishes by Schur's lemma (noting again that $\mathrm{Gal}(F/F^+)$ acts by -1 on the scalar matrices in $\mathfrak{gl}_{n,\mathbb{F}}$). □

5.2. *Existence of lifts and the weight part of Serre's conjecture.* We now prove a strengthening of [Barnet-Lamb et al. 2013, Theorem A.4.1], removing the condition that the places at which our Galois representations are ramified are split in F. We refer the reader to [Barnet-Lamb et al. 2014] for any unfamiliar terminology; in particular, potential diagonalisability is defined in [loc. cit., §1.4], while adequacy and the notion of a polarised Galois representation being potentially diagonalisably automorphic are defined in [loc. cit., §2.1].

Theorem 5.2.1. *Let l be an odd prime not dividing n, and suppose that $\zeta_l \notin F$. Let $\bar{\rho} : \mathrm{Gal}_{F^+} \to \mathcal{G}_n(\mathbb{F})$ be such that $\bar{\rho}|_{\mathrm{Gal}_{F(\zeta_l)}}$ is absolutely irreducible. Assume that $\bar{\rho}$ is discrete series and odd. Let S be a finite set of places of F^+, including all places dividing $l\infty$.*

Let μ be a de Rham lift of $\bar{\mu}$, and let S be a finite set of places of F^+ including all the places at which either $\bar{r}$ or μ is ramified, and all the places dividing $l\infty$. For each finite place $v \in S$, fix an inertial type τ_v, and if $v \mid l$, fix a regular Hodge type $\boldsymbol{v}_v$. Fix quotients of the corresponding local μ-polarised framed deformation rings which correspond to an irreducible component of the generic fibre; if $v \mid l$, assume also that this component is potentially diagonalisable

Assume further that there is a finite extension of CM fields F'/F such that F' does not contain ζ_l, all finite places of $(F')^+$ above S split in F, and $\bar{\rho}(\mathrm{Gal}_{F'(\zeta_l)})$ is adequate, and assume that there exists a lift $\rho' : \mathrm{Gal}_{F^+,S} \to \mathcal{G}_n(\mathcal{O})$ of $\bar{\rho}|_{\mathrm{Gal}_{(F')^+,S}}$ with $\nu \circ \rho' = \mu|_{\mathrm{Gal}_{F^+,S}}$, with the further property that ρ' is potentially diagonalisably automorphic.

Then there is a lift

$$\rho : \mathrm{Gal}_{F^+,S} \to \mathcal{G}_n(\mathcal{O})$$

of $\bar{\rho}$ *such that*:

(1) $\nu \circ \rho = \mu$.

(2) *If* $v \in S$ *is a finite place, then* $\rho|_{G_{F_v^+}}$ *corresponds to a point on our chosen component of the local deformation ring.*

(3) $\rho|_{\mathrm{Gal}_{(F')^+,S}}$ *is potentially diagonalisably automorphic.*

Proof. Let R^{univ} be the universal deformation ring for μ-polarised deformations of $\bar{\rho}$ which are unramified outside S, and lie on the given irreducible component for each finite place $v \in S$. Then R^{univ} has Krull dimension at least 1 by Corollary 5.1.1. We claim that R^{univ} is a finite $\mathcal{O}$-algebra. Admitting this claim, we can choose a homomorphism $R^{\mathrm{univ}} \to E$, and let ρ be the corresponding representation. This satisfies properties (1) and (2) by construction.

Let $R^{\mathrm{univ}}_{F'}$ be the universal deformation ring for $\mu|_{G_{(F')^+,S}}$-polarised deformations of $\bar{r}|_{G_{F',S}}$ which lie on the base changes of our chosen components. By [Barnet-Lamb et al. 2014, Lemma 1.2.3(1)], R^{univ} is a finite $R^{\mathrm{univ}}_{F'}$-algebra, so in order to prove the claim it is enough to show that $R^{\mathrm{univ}}_{F'}$ is a finite $\mathcal{O}$-algebra.

By [Barnet-Lamb et al. 2013, Theorem A.4.1] (with F there taken to equal F'), there is a representation $\rho'' : G_{(F')^+,S} \to \mathcal{G}_n(\mathcal{O})$ corresponding to an $\mathcal{O}$-point of $R^{\mathrm{univ}}_{F'}$, which is furthermore potentially diagonalisably automorphic. Then $R^{\mathrm{univ}}_{F'}$ is a finite $\mathcal{O}$-algebra by [Barnet-Lamb et al. 2014, Theorem 2.3.2]. as required. Finally, property (3) holds by [loc. cit., Theorem 2.3.2] (applied to ρ'' and $\rho|_{G_{(F')^+,S}}$). □

We now apply this result to the weight part of Serre's conjecture for unitary groups. We restrict ourselves to the case $n = 2$, where the existing results in the literature are strongest; our results should also allow the removal of the hypothesis of "split ramification" from results in the literature for higher-rank unitary groups, such as the results of [Barnet-Lamb et al. 2018]. We recall that if $K/\mathbb{Q}_l$ is a finite extension, there is associated to any representation $\bar{\rho} : \mathrm{Gal}_K \to \mathrm{GL}_2(\mathbb{F})$ a set $W(\bar{\rho})$ of Serre weights. A definition of $W(\bar{\rho})$ was first given in [Buzzard et al. 2010] in the case that $K/\mathbb{Q}_l$ is unramified, and various generalisations and alternative definitions have subsequently been proposed. As a result of the main theorems of [Gee et al. 2015; Calegari et al. 2017], all of these definitions are equivalent; we refer the reader to the introductions to those papers for a discussion of the various definitions.

Suppose that F is an imaginary CM field with maximal totally real subfield F^+ such that F/F^+ is unramified at all finite places, that each place of F^+ above l splits in F, and that $[F^+ : \mathbb{Q}]$ is even. Then as in [Barnet-Lamb et al. 2013] we have a unitary group G/F^+ which is quasisplit at all finite places and compact at all infinite places. If $\bar{r} : \mathrm{Gal}_{F^+} \to \mathcal{G}_2(\bar{\mathbb{F}}_l)$ is irreducible, the notion of $\bar{r}$ being modular of a Serre weight is defined in [loc. cit., Definition 2.1.9]. This definition (implicitly) insists that $\bar{r}$ is only ramified at places which split in F, and we relax it as follows: we change the definition of a good compact open subgroup $U \subset G(\mathbb{A}^{\infty}_{F^+})$ in [loc. cit., Definition 2.1.5] to require only that at all places $v \mid l$ we have $U_v = G(\mathcal{O}_{F_v^+})$, and at all places $v \nmid l$ we have $U_v \subset G(\mathcal{O}_{F_v^+})$. (Consequently, we are now considering automorphic forms of arbitrary level away from l, whereas in [loc. cit.] the level is hyperspecial at all places which do not split in F.)

Having made this change, everything in [loc. cit., §2] goes through unchanged, except that all mentions of "split ramification" can be deleted. The following theorem strengthens [Gee et al. 2014, Theorem A], removing a hypothesis on the ramification away from l (and also a hypothesis on the ramification at l, although that could already have been removed thanks to the results of [Gee et al. 2015]).

Theorem 5.2.2. *Let F be an imaginary CM field with maximal totally real subfield F^+, and suppose that F/F^+ is unramified at all finite places, that each place of F^+ above l splits in F, and that $[F^+ : \mathbb{Q}]$ is even. Suppose that l is odd, that $\bar{r} : G_{F^+} \to \mathcal{G}_2(\overline{\mathbb{F}}_l)$ is irreducible and modular, and that $\bar{r}(G_{F(\zeta_l)})$ is adequate.*

Then the set of Serre weights for which $\bar{r}$ is modular is exactly the set of weights given by the sets $W(\bar{r}|_{G_{F_v}})$, $v \mid l$.

Proof. We begin by observing that the proof of [Barnet-Lamb et al. 2013, Theorem 5.1.3] goes through in our more general context (that is, without assuming "split ramification"). Indeed, we have already observed that the results of [loc. cit., §2] are valid in our context, and chasing back through the references, we see that the only change that needs to be made is to relax the hypotheses in [loc. cit., Theorem 3.1.3] by no longer requiring that the places $v \in S$, $v \nmid l$, split in F. This follows by replacing the citation of [loc. cit., Theorem A.4.1] in the proof of [loc. cit., Theorem 3.1.3] with a reference to Theorem 5.2.1 above (after making a further extension of F' to arrange that all of the places of $(F')^+$ lying over S split in F').

This shows that $\bar{r}$ is modular of every weight given by the $W(\bar{r}|_{G_{F_v}})$, $v \mid l$. For the converse, observe that [loc. cit., Corollary 4.1.8] also holds in our context (again, since the results of [loc. cit., §2] go through); the result then follows immediately from [Gee et al. 2015, Theorem 6.1.8]. □

Remark 5.2.3. It is presumably possible to prove in the same way a further strengthening of Theorem 5.2.2 where we allow our unitary group to be ramified at some finite places (and thus allow $[F^+ : \mathbb{Q}]$ to be odd, and F/F^+ to be ramified at some finite places), but to do so would involve a lengthier discussion of automorphic representations on unitary groups, which would take us too far afield.

Remark 5.2.4. We have assumed that the places of F^+ above l split in F, because the weight part of Serre's conjecture has not been considered in the literature for unitary groups which do not split above l (although if l is unramified in F, and we are in the generic semisimple case, such a conjecture is a special case of the conjectures of [Gee et al. 2018]). However, it seems likely that it is possible to formulate and prove a generalisation of Theorem 5.2.2 which removes this assumption, following the ideas of [Gee and Kisin 2014; Gee and Geraghty 2015] (that is, using the Breuil–Mézard conjecture for potentially Barsotti–Tate representations). Again, this would take us too far afield from the main concerns of this paper, so we do not pursue this; and in any case we understand that this will be carried out in forthcoming work of Koziol and Morra.

Acknowledgements

We would like to thank Matthew Emerton for emphasising the importance of Weil–Deligne representations to us, and for his comments on an earlier draft of this paper. We would also like to thank Jeremy Booher, George Boxer, Stefan Patrikis, and Jacques Tilouine for helpful conversations, and Brian Conrad, Mark Kisin and Daniel Le for their comments on an earlier draft. We would like to thank the referees for their careful reading of the paper and their many helpful comments.

References

[Allen 2016a] P. B. Allen, "Deformations of polarized automorphic Galois representations and adjoint Selmer groups", *Duke Math. J.* **165**:13 (2016), 2407–2460. MR Zbl

[Allen 2016b] P. B. Allen, "On automorphic points in polarized deformation rings", 2016. To appear in *Amer. J. Math.* arXiv

[Balaji 2013] S. Balaji, *G-valued potentially semi-stable deformation rings*, Ph.D. thesis, University of Chicago, 2013, available at https://search.proquest.com/docview/1346024943.

[Barnet-Lamb et al. 2013] T. Barnet-Lamb, T. Gee, and D. Geraghty, "Serre weights for rank two unitary groups", *Math. Ann.* **356**:4 (2013), 1551–1598. MR Zbl

[Barnet-Lamb et al. 2014] T. Barnet-Lamb, T. Gee, D. Geraghty, and R. Taylor, "Potential automorphy and change of weight", *Ann. of Math.* (2) **179**:2 (2014), 501–609. MR Zbl

[Barnet-Lamb et al. 2018] T. Barnet-Lamb, T. Gee, and D. Geraghty, "Serre weights for $U(n)$", *J. Reine Angew. Math.* **735** (2018), 199–224. MR Zbl

[Bellovin 2016] R. Bellovin, "Generic smoothness for G-valued potentially semi-stable deformation rings", *Ann. Inst. Fourier (Grenoble)* **66**:6 (2016), 2565–2620. MR Zbl

[Böckle 1999] G. Böckle, "A local-to-global principle for deformations of Galois representations", *J. Reine Angew. Math.* **509** (1999), 199–236. MR Zbl

[Booher 2019a] J. Booher, "Minimally ramified deformations when $l \neq p$", *Compos. Math.* **155**:1 (2019), 1–37. MR Zbl

[Booher 2019b] J. Booher, "Producing geometric deformations of orthogonal and symplectic Galois representations", *J. Number Theory* **195** (2019), 115–158. Zbl

[Booher and Patrikis 2017] J. Booher and S. Patrikis, "G-valued Galois deformation rings when $l \neq p$", 2017. To appear in *Math. Res. Lett.* arXiv

[Bourbaki 2005] N. Bourbaki, *Lie groups and Lie algebras, chapters 7–9*, Springer, 2005. MR Zbl

[Breuil and Schneider 2007] C. Breuil and P. Schneider, "First steps towards p-adic Langlands functoriality", *J. Reine Angew. Math.* **610** (2007), 149–180. MR Zbl

[Breuil et al. 2001] C. Breuil, B. Conrad, F. Diamond, and R. Taylor, "On the modularity of elliptic curves over $\mathbb{Q}$: wild 3-adic exercises", *J. Amer. Math. Soc.* **14**:4 (2001), 843–939. MR Zbl

[Buzzard and Gee 2014] K. Buzzard and T. Gee, "The conjectural connections between automorphic representations and Galois representations", pp. 135–187 in *Automorphic forms and Galois representations, I* (Durham, 2011), edited by F. Diamond et al., London Math. Soc. Lecture Note Ser. **414**, Cambridge Univ. Press, 2014. MR Zbl

[Buzzard et al. 2010] K. Buzzard, F. Diamond, and F. Jarvis, "On Serre's conjecture for mod l Galois representations over totally real fields", *Duke Math. J.* **155**:1 (2010), 105–161. MR Zbl

[Calegari et al. 2017] F. Calegari, M. Emerton, T. Gee, and L. Mavrides, "Explicit Serre weights for two-dimensional Galois representations", *Compos. Math.* **153**:9 (2017), 1893–1907. MR Zbl

[Calegari et al. 2018] F. Calegari, M. Emerton, and T. Gee, "Globally realizable components of local deformation rings", preprint, 2018. arXiv

[Clozel et al. 2008] L. Clozel, M. Harris, and R. Taylor, "Automorphy for some l-adic lifts of automorphic mod l Galois representations", *Publ. Math. Inst. Hautes Études Sci.* **108** (2008), 1–181. MR Zbl

[Conrad 1999] B. Conrad, "Irreducible components of rigid spaces", *Ann. Inst. Fourier* (*Grenoble*) **49**:2 (1999), 473–541. MR Zbl

[Conrad 2014] B. Conrad, "Reductive group schemes", pp. 93–444 in *Autour des schémas en groupes, I*, Panor. Synthèses **42/43**, Soc. Math. France, Paris, 2014. MR Zbl

[Conrad et al. 2010] B. Conrad, O. Gabber, and G. Prasad, *Pseudo-reductive groups*, New Math. Monographs **17**, Cambridge Univ. Press, 2010. MR Zbl

[Deligne and Milne 1982] P. Deligne and J. S. Milne, "Tannakian categories", pp. ii+414 in *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Math. **900**, Springer, 1982. MR Zbl

[EGA IV_1 1964] A. Grothendieck and J. A. Dieudonné, "Eléments de géométrie algébrique, IV: Étude locale des schémas et des morphismes de schémas, I", *Inst. Hautes Études Sci. Publ. Math.* **20** (1964), 5–259. MR Zbl

[Emerton and Gee 2017] M. Emerton and T. Gee, "Dimension theory and components of algebraic stacks", preprint, 2017. arXiv

[Fontaine 1994] J.-M. Fontaine, "Représentations l-adiques potentiellement semi-stables", pp. 321–347 in *Périodes p-adiques* (Bures-sur-Yvette, French, 1988), Astérisque **223**, Soc. Math. France, Paris, 1994. MR Zbl

[Gee 2011] T. Gee, "Automorphic lifts of prescribed types", *Math. Ann.* **350**:1 (2011), 107–144. MR Zbl

[Gee and Geraghty 2015] T. Gee and D. Geraghty, "The Breuil–Mézard conjecture for quaternion algebras", *Ann. Inst. Fourier* (*Grenoble*) **65**:4 (2015), 1557–1575. MR Zbl

[Gee and Kisin 2014] T. Gee and M. Kisin, "The Breuil–Mézard conjecture for potentially Barsotti–Tate representations", *Forum Math. Pi* **2** (2014), art. id. e1. MR Zbl

[Gee et al. 2014] T. Gee, T. Liu, and D. Savitt, "The Buzzard–Diamond–Jarvis conjecture for unitary groups", *J. Amer. Math. Soc.* **27**:2 (2014), 389–435. MR Zbl

[Gee et al. 2015] T. Gee, T. Liu, and D. Savitt, "The weight part of Serre's conjecture for GL(2)", *Forum Math. Pi* **3** (2015), art. id. e2. MR Zbl

[Gee et al. 2018] T. Gee, F. Herzig, and D. Savitt, "General Serre weight conjectures", *J. Eur. Math. Soc.* **20**:12 (2018), 2859–2949. MR Zbl

[Gross 2007] B. H. Gross, "Odd Galois representations", preprint, 2007, available at https://tinyurl.com/grossoddg.

[Gross and Prasad 1992] B. H. Gross and D. Prasad, "On the decomposition of a representation of SO_n when restricted to SO_{n-1}", *Canad. J. Math.* **44**:5 (1992), 974–1002. MR Zbl

[Gross and Reeder 2010] B. H. Gross and M. Reeder, "Arithmetic invariants of discrete Langlands parameters", *Duke Math. J.* **154**:3 (2010), 431–508. MR Zbl

[Jantzen 2004] J. C. Jantzen, "Nilpotent orbits in representation theory", pp. 1–211 in *Lie theory*, edited by J.-P. Anker and B. Orsted, Progr. Math. **228**, Birkhäuser, Boston, 2004. MR Zbl

[de Jong 1995] A. J. de Jong, "Crystalline Dieudonné module theory via formal and rigid geometry", *Inst. Hautes Études Sci. Publ. Math.* **82** (1995), 5–96. MR Zbl

[Khare and Wintenberger 2009] C. Khare and J.-P. Wintenberger, "On Serre's conjecture for 2-dimensional mod p representations of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$", *Ann. of Math.* (2) **169**:1 (2009), 229–253. MR Zbl

[Kisin 2007] M. Kisin, "Modularity of 2-dimensional Galois representations", pp. 191–230 in *Current developments in mathematics*, edited by D. Jerison et al., Int. Press, Somerville, MA, 2007. MR Zbl

[Kisin 2008] M. Kisin, "Potentially semi-stable deformation rings", *J. Amer. Math. Soc.* **21**:2 (2008), 513–546. MR Zbl

[Kisin 2009] M. Kisin, "Moduli of finite flat group schemes, and modularity", *Ann. of Math.* (2) **170**:3 (2009), 1085–1180. MR Zbl

[Matsumura 1989] H. Matsumura, *Commutative ring theory*, 2nd ed., Cambridge Studies in Adv. Math. **8**, Cambridge Univ. Press, 1989. MR Zbl

[Mazur 1989] B. Mazur, "Deforming Galois representations", pp. 385–437 in *Galois groups over* $\mathbb{Q}$ (Berkeley, 1987), edited by Y. Ihara et al., Math. Sci. Res. Inst. Publ. **16**, Springer, 1989. MR Zbl

[McNinch 2004] G. J. McNinch, "Nilpotent orbits over ground fields of good characteristic", *Math. Ann.* **329**:1 (2004), 49–85. MR Zbl

[Patrikis 2016] S. Patrikis, "Deformations of Galois representations and exceptional monodromy", *Invent. Math.* **205**:2 (2016), 269–336. MR Zbl

[Prasad and Yu 2002] G. Prasad and J.-K. Yu, "On finite group actions on reductive groups and buildings", *Invent. Math.* **147**:3 (2002), 545–560. MR Zbl

[Ramakrishna 2002] R. Ramakrishna, "Deforming Galois representations and the conjectures of Serre and Fontaine–Mazur", *Ann. of Math.* (2) **156**:1 (2002), 115–154. MR Zbl

[Saavedra Rivano 1972] N. Saavedra Rivano, *Catégories Tannakiennes*, Lecture Notes in Math. **265**, Springer, 1972. MR Zbl

[SGA $3_{\rm II}$ 1970] M. Demazure and A. Grothendieck, *Schémas en groupes, Tome II: Groupes de type multiplicatif, et structure des schémas en groupes généraux, Exposés VIII–XVIII* (Séminaire de Géométrie Algébrique du Bois Marie 1962–1964), Lecture Notes in Math. **152**, Springer, 1970. MR Zbl

[Stacks 2005–] Stacks project authors, "The Stacks project", electronic reference, 2005–, available at https://tinyurl.com/on7qtt4.

[Taylor and Wiles 1995] R. Taylor and A. Wiles, "Ring-theoretic properties of certain Hecke algebras", *Ann. of Math.* (2) **141**:3 (1995), 553–572. MR Zbl

[Tilouine 1996] J. Tilouine, *Deformations of Galois representations and Hecke algebras*, Narosa, New Delhi, 1996. MR Zbl

[Wiles 1995] A. Wiles, "Modular elliptic curves and Fermat's last theorem", *Ann. of Math.* (2) **141**:3 (1995), 443–551. MR Zbl

Communicated by Samit Dasgupta
Received 2017-10-06 Revised 2018-11-08 Accepted 2018-12-24

r.bellovin@imperial.ac.uk *Department of Mathematics, Imperial College London, London, United Kingdom*

toby.gee@imperial.ac.uk *Department of Mathematics, Imperial College London, London, United Kingdom*

Algebra & Number Theory

msp.org/ant

See inside back cover or msp.org/ant for submission instructions.

The subscription price for 2019 is US $385/year for the electronic version, and $590/year (+$60, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to MSP.

Algebra & Number Theory (ISSN 1944-7833 electronic, 1937-0652 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

ANT peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Algebra & Number Theory

Volume 13 No. 2 2019