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# An explicit bound for the least prime ideal in the Chebotarev density theorem

Jesse Thorner and Asif Zaman

We prove an explicit version of Weiss' bound on the least norm of a prime ideal in the Chebotarev density theorem, which is a significant improvement on the work of Lagarias, Montgomery, and Odlyzko. As an application, we prove the first explicit, nontrivial, and unconditional upper bound for the least prime represented by a positive-definite primitive binary quadratic form. We also consider applications to elliptic curves and congruences for the Fourier coefficients of holomorphic cuspidal modular forms.

## 1. Introduction and statement of results

In 1837, Dirichlet proved that if $a, q \in \mathbb{Z}$ and $\gcd(a,q) = 1$, then there are infinitely many primes $p \equiv a \pmod{q}$. In light of this result, it is natural to ask how big the first such prime, say $P(a,q)$, is. Assuming the generalized Riemann hypothesis (GRH) for Dirichlet $L$-functions, Lamzouri, Li, and Soundararajan [Lamzouri et al. 2015] proved that for all $q \geq 4$,

$$P(a,q) \leq (\varphi(q)\log q)^2, \tag{1-1}$$

where $\varphi$ is Euler's totient function. Nontrivial, unconditional upper bounds are significantly harder to prove. The first such bound on $P(a,q)$ is due to Linnik [1944a; 1944b], who proved that for some absolute constant $c_1 > 0$,

$$P(a,q) \ll q^{c_1} \tag{1-2}$$

with an absolute and computable implied constant. Admissible values of $c_1$ are now known explicitly. Building on the work of Heath-Brown [1992], Xylouris [2011] proved that one may take $c_1 = 5.2$ unconditionally. (Xylouris improved this to $c_1 = 5$ in his Ph.D. thesis.) For a detailed history of the unconditional progress toward (1-1), see [Heath-Brown 1992, Section 1].

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A broad generalization of (1-2) lies in the context of the Chebotarev density theorem. Let $L/F$ be a Galois extension of number fields with Galois group $G$. To each prime ideal $\mathfrak{p}$ of $F$ which is unramified in $L$, there corresponds a certain conjugacy class of automorphisms in $G$ which are attached to the prime ideals of $L$ lying above $\mathfrak{p}$. We denote this conjugacy class using the Artin symbol $\left[\frac{L/F}{\mathfrak{p}}\right]$. For a conjugacy class $C \subset G$, let

$$\pi_C(x, L/F) := \#\left\{\mathfrak{p} : \mathfrak{p} \text{ is unramified in } L, \left[\frac{L/F}{\mathfrak{p}}\right] = C, \mathrm{N}_{F/\mathbb{Q}}\,\mathfrak{p} \le x\right\}.$$

The Chebotarev density theorem asserts that

$$\pi_C(x, L/F) \sim \frac{|C|}{|G|} \int_2^x \frac{dt}{\log t}.$$

In analogy with (1-2), it is natural to bound the quantity

$$P(C, L/F) := \min\left\{\mathrm{N}_{F/\mathbb{Q}}\,\mathfrak{p} : \mathfrak{p} \text{ unramified in } L, \left[\frac{L/F}{\mathfrak{p}}\right] = C, \mathrm{N}_{F/\mathbb{Q}}\,\mathfrak{p} \text{ a rational prime}\right\}. \tag{1-3}$$

Under GRH for Hecke $L$-functions, Lagarias and Odlyzko [1977] proved a bound for $P(C, L/F)$; Bach and Sorenson [1996] made this bound explicit, proving that

$$P(C, L/F) \le (4 \log D_L + 2.5[L : \mathbb{Q}] + 5)^2, \tag{1-4}$$

where $D_L = |\mathrm{disc}(L/\mathbb{Q})|$. (This can be improved assuming Artin's conjecture; see work of V. K. Murty [1994, Equation 2].) We note that if $L = \mathbb{Q}(e^{2\pi i/q})$ for some integer $q \ge 1$ and $F = \mathbb{Q}$, then one recovers a bound of the same analytic quality as (1-1), though the constants are slightly larger.

The first nontrivial, unconditional bound on $P(C, L/F)$ is due to Lagarias, Montgomery, and Odlyzko [Lagarias et al. 1979]; they proved $P(C, L/F) \le 2D_L^{c_2}$ for some absolute constant $c_2 > 0$. Recently, Zaman [2017b] explicitly bounded $c_2$, proving that[1]

$$P(C, L/F) \ll D_L^{40}. \tag{1-5}$$

The bound (1-5), up to quality of the exponent, is commensurate with the best known bounds when $L$ is a quadratic extension of $F = \mathbb{Q}$, which reduces to the problem of bounding the least quadratic nonresidue. We observe, however, that if $q$ is prime, $L = \mathbb{Q}(e^{2\pi i/q})$, and $F = \mathbb{Q}$, then (1-5) states that $P(a, q) \ll q^{40(q-2)}$, which is much worse than (1-2).

Weiss [1983] significantly improved the results in [Lagarias et al. 1979]. Let $A$ be any abelian subgroup of $G$ such that $A \cap C$ is nonempty, let $\widehat{A}$ be the character

[1]Unless mentioned otherwise, all implied constants in all asymptotic inequalities $f \ll g$ or $f = O(g)$ are absolute and computable.

group of $A$, and let $K = L^A$ be the subfield of $L$ fixed by $A$. Let the $K$-integral ideal $\mathfrak{f}_\chi$ be the conductor of a character $\chi \in \widehat{A}$, and let

$$\mathcal{Q}(L/K) = \max\{\mathrm{N}_{K/\mathbb{Q}}\,\mathfrak{f}_\chi : \chi \in \widehat{A}\}. \tag{1-6}$$

Weiss proved that for certain absolute constants $c_3 > 0$ and $c_4 > 0$,

$$P(C, L/F) \le 2[K:\mathbb{Q}]^{c_3[K:\mathbb{Q}]}(D_K\mathcal{Q}(L/K))^{c_4}. \tag{1-7}$$

To see how (1-7) compares to (1-5), we observe that if $A$ is a cyclic subgroup of $G$, then

$$D_L^{1/|A|} \le D_K\mathcal{Q}(L/K) \le D_L^{1/\varphi(|A|)}.$$

(See [Bach and Sorenson 1996, Lemma 4.2] for a proof of the upper bound; the lower bound holds for all $A$ and follows from the conductor-discriminant formula.) Furthermore, if $F = \mathbb{Q}$ and $L = \mathbb{Q}(e^{2\pi i/q})$, then one may take $\widehat{A}$ to be the full group of Dirichlet characters modulo $q$, in which case $K = F = \mathbb{Q}$ and $\mathcal{Q}(L/K) = q$. Thus Weiss proves a bound on $P(C, L/F)$, which provides a "continuous transition" from (1-2) to (1-5). In particular, (1-2) follows from (1-7).

In this paper, we prove the following bound on $P(C, L/F)$, which makes (1-7) explicit.

**Theorem 1.1.** *Let $L/F$ be a Galois extension of number fields with Galois group $G$, let $C \subset G$ be a conjugacy class, and let $P(C, L/F)$ be defined by (1-3). Let $A \subset G$ be an abelian subgroup such that $A \cap C$ is nonempty, $K = L^A$ be the fixed field of $A$, and $\mathcal{Q} = \mathcal{Q}(L/K)$ be defined by (1-6). Then*

$$P(C, L/F) \ll D_K^{694}\mathcal{Q}^{521} + D_K^{232}\mathcal{Q}^{367}[K:\mathbb{Q}]^{290[K:\mathbb{Q}]}.$$

**Remarks.**

- Theorem 1.1 immediately implies that $P(a, q) \ll q^{521}$. For historical context, this is slightly better than Jutila's bound [1970] on $P(a, q)$, which was over 25 years after Linnik's original theorem.
- The bound we obtain on $P(C, L/F)$ follows immediately from the effective lower bound on $\pi_C(x, L/F)$ given by (3-2), which is of independent interest. See [Zaman 2017a, Theorem 1.3.1] for a related lower bound.
- If $[K:\mathbb{Q}] \le 2(\log D_K)/\log\log D_K$, then $P(C, L/F) \ll D_K^{694}\mathcal{Q}^{521}$. Situations where $[K:\mathbb{Q}] > 2(\log D_K)/\log\log D_K$ are rare; the largest class of known examples involve infinite $p$-class tower extensions, which were first studied by Golod and Šafarevič [1964].
- If $L/K$ is unramified, then $\mathcal{Q} = 1$ and $D_K = D_L^{1/|A|}$. Thus

  $$P(C, L/F) \ll D_L^{694/|A|} + D_L^{232/|A|}[K:\mathbb{Q}]^{290[K:\mathbb{Q}]}.$$

  If $[K:\mathbb{Q}] \le 2(\log D_K)/\log\log D_K$, this improves (1-5) when $|A| \ge 18$.

We now consider some specific applications of Theorem 1.1, the first of which is a bound on the least prime represented by a positive-definite primitive binary quadratic form $Q(x, y) \in \mathbb{Z}[x, y]$ of discriminant $D$. It follows from (1-7) that the least such prime $p$ satisfies $p \ll |D|^{c_5}$ for some positive absolute constant $c_5$; see Kowalski and Michel [2002] for a similar observation. Ditchen [2013] proved, on average over $D \not\equiv 0 \pmod 8$, that $p \ll_\epsilon |D|^{20/3+\epsilon}$, and Zaman [2016b] showed $p \ll_\epsilon |D|^{9.5+\epsilon}$ in an exceptional case. However, a nontrivial unconditional explicit bound on the least prime represented by $Q$ for *all* such quadratic forms has not been calculated before now. Such a bound follows immediately from Theorem 1.1.

**Theorem 1.2.** *Let $Q(x, y) \in \mathbb{Z}[x, y]$ be a positive-definite primitive binary quadratic form of discriminant $D$. There exists a prime $p \nmid D$ represented by $Q(x, y)$ such that $p \ll |D|^{694}$. In particular, if $n$ is a fixed positive integer, there exists a prime $p \nmid n$ represented by $x^2 + ny^2$ such that $p \ll n^{694}$.*

We now consider applications to the study of the group of points on an elliptic curve over a finite field. Let $E/\mathbb{Q}$ be an elliptic curve without complex multiplication (CM), and let $N_E$ be the conductor of $E$. The order and group structure of $E(\mathbb{F}_p)$, the group of $\mathbb{F}_p$-rational points on $E$, frequently appears when doing arithmetic over $E$. Thus we are interested in understanding the distribution of values and divisibility properties of $\#E(\mathbb{F}_p)$.

V. K. Murty [1994] and Li [2012] proved unconditional and GRH-conditional bounds on the least prime that does not split completely in a number field. This yields bounds on the least prime $p \nmid \ell N_E$ such that $\ell \nmid \#E(\mathbb{F}_p)$, where $\ell \geq 11$ is prime. As an application of Theorem 1.1, we prove a complementary result on the least $p \nmid \ell N_E$ such that $\ell \mid \#E(\mathbb{F}_p)$. To state the result, we define $\omega(N_E) = \#\{p : p \mid N_E\}$ and $\operatorname{rad}(N_E) = \prod_{p \mid N_E} p$.

**Theorem 1.3.** *Let $E/\mathbb{Q}$ be a non-CM elliptic curve of conductor $N_E$, and let $\ell \geq 11$ be prime. There exists a prime $p \nmid \ell N_E$ such that*

$$p \ll \ell^{(5300+1600\omega(N_E))\ell^2} \operatorname{rad}(N_E)^{1900\ell^2} \quad \textit{and} \quad \ell \mid \#E(\mathbb{F}_p).$$

**Remark.** The proof is easily adapted to allow for elliptic curves over other number fields; we omit further discussion for brevity.

One of the first significant results in the study of the distribution of values of $\#E(\mathbb{F}_p)$ is due to Hasse, who proved that if $p \nmid N_E$, then $|p+1-\#E(\mathbb{F}_p)| < 2\sqrt{p}$. For a prime $\ell$, the distribution of the primes $p$ such that $\#E(\mathbb{F}_p) \equiv p+1 \pmod \ell$ can also be studied using the mod $\ell$ Galois representations associated to $E$.

**Theorem 1.4.** *Let $E/\mathbb{Q}$ be a non-CM elliptic curve of squarefree conductor $N_E$, and let $\ell \geq 11$ be prime. There exists a prime $p \nmid \ell N_E$ such that*

$$\#E(\mathbb{F}_p) \equiv p+1 \pmod \ell \quad \textit{and} \quad p \ll \ell^{(4600+1200\omega(N_E))\ell} N_E^{2100\ell}.$$

Theorem 1.4 will follow from a more general result on congruences for the Fourier coefficients of certain holomorphic cuspidal modular forms. Let

$$f(z)=\sum_{n=1}^{\infty}a_f(n)e^{2\pi inz}$$

be a cusp form of integral weight $k_f\geq 2$, level $N_f\geq 1$, and nebentypus $\chi_f$. Suppose further that $f$ is a normalized eigenform for the Hecke operators. We call such a cusp form $f$ a newform; for each newform $f$, the map $n\mapsto a_f(n)$ is multiplicative. Suppose $a_f(n)\in\mathbb{Z}$ for all $n\geq 1$. In this case, $\chi_f$ is trivial when $f$ does not have CM, and $\chi_f$ is a nontrivial real character when $f$ does have CM. Moreover, when $k_f=2$, $f$ is the newform associated to an isogeny class of elliptic curves $E/\mathbb{Q}$. In this case, $N_f=N_E$, and for any prime $p\nmid N_E$, we have that $a_f(p)=p+1-\#E(\mathbb{F}_p)$.

**Theorem 1.5.** *Let $f(z)=\sum_{n=1}^{\infty}a_f(n)e^{2\pi inz}\in\mathbb{Z}[\![e^{2\pi iz}]\!]$ be a non-CM newform of even integral weight $k_f\geq 2$, level $N_f$, and trivial nebentypus. Let $\ell\geq 3$ be a prime such that (12-1) holds and* $\gcd(k_f-1,\ell-1)=1$. *For any residue class $a$ modulo $\ell$, there exists a prime $p\nmid \ell N_f$ such that*

$$a_f(p)\equiv a \pmod{\ell}\quad\textit{and}\quad p\ll \ell^{(4600+1200\omega(N_f))\ell}\operatorname{rad}(N_f)^{2100\ell}.$$

**Remarks.**

- Equation (12-1) is a fairly mild condition regarding whether the modulo $\ell$ reduction of a certain representation is surjective. This condition is satisfied by all but finitely many choices of $\ell$. See Section 12 for further details.
- The proofs of Theorems 1.3–1.5 are easily adapted to allow composite moduli $\ell$ as well as elliptic curves and modular forms with CM. Moreover, the proofs can be easily modified to study the mod $\ell$ distribution of the trace of Frobenius for elliptic curves over number fields other than $\mathbb{Q}$. We omit further discussion for brevity.
- Using (1-5), the least prime $p$ such that $a_f(p)\equiv a \pmod{\ell}$ satisfies the bound $p\ll \ell^{120\ell^3(1+\omega(N_f))}\operatorname{rad}(N_f)^{40(\ell^3-1)}$ for any choice of $a$. Thus Theorem 1.5 constitutes an improvement over (1-5) for $\ell\geq 11$.
- If $r_{24}(n)$ is the number of representations of $n$ as a sum of 24 squares, then $691r_{24}(p)=16(p^{11}+1)+33152\tau(p)$, where Ramanujan's function $\tau(n)$ is the $n$-th Fourier coefficient of $\Delta(z)$, the unique non-CM newform of weight 12 and level 1. If $\ell\notin\{2,3,5,7,23,691\}$ is such that $\ell\not\equiv 1 \pmod{11}$, then by Theorem 1.5, there exists $p\neq\ell$ such that

$$691r_{24}(p)\equiv 16(p^{11}+1)\pmod{\ell}\quad\text{and}\quad p\ll\ell^{4600\ell}.$$

## 2. Notation and auxiliary estimates

**2A.** ***Notation.*** We use the following notation throughout the paper.

- $K$ is a number field.
- $\mathcal{O}_K$ is the ring of integers of $K$.
- $n_K = [K : \mathbb{Q}]$ is the degree of $K/\mathbb{Q}$.
- $D_K$ is the absolute value of the discriminant of $K$.
- $\mathrm{N} = \mathrm{N}_{K/\mathbb{Q}}$ is the absolute field norm of $K$.
- $\zeta_K(s)$ is the Dedekind zeta function of $K$.
- $\mathfrak{q}$ is an integral ideal of $K$.
- $\mathrm{Cl}(\mathfrak{q}) = I(\mathfrak{q})/P_{\mathfrak{q}}$ is the narrow ray class group of $K$ modulo $\mathfrak{q}$.
- $\chi$, or $\chi \pmod{\mathfrak{q}}$, is a character of $\mathrm{Cl}(\mathfrak{q})$, referred to as a Hecke character or ray class character of $K$.
- $\delta(\chi)$ is the indicator function of the trivial character.
- $\mathfrak{f}_\chi$ is the conductor of $\chi$; that is, it is the maximal integral ideal such that $\chi$ is induced from a primitive character $\chi^* \pmod{\mathfrak{f}_\chi}$.
- $D_\chi = D_K \mathrm{N}\mathfrak{f}_\chi$.
- $L(s, \chi)$ is the Hecke $L$-function associated to $\chi$.
- $H$, or $H \pmod{\mathfrak{q}}$, is a subgroup of $\mathrm{Cl}(\mathfrak{q})$, or equivalently of $I(\mathfrak{q})$, containing $P_{\mathfrak{q}}$. The group $H$ is referred to as a congruence class group of $K$.
- $\chi \pmod{H}$ is a character $\chi \pmod{\mathfrak{q}}$ satisfying $\chi(H) = 1$.
- $Q = Q_H = \max\{\mathrm{N}\mathfrak{f}_\chi : \chi \pmod{H}\}$ is the maximum conductor of $H$.
- $\mathfrak{f}_H = \mathrm{lcm}\{\mathfrak{f}_\chi : \chi \pmod{H}\}$ is the conductor of $H$.
- $H^* \pmod{\mathfrak{f}_H}$ is the primitive congruence class group inducing $H$.
- $h_H = [I(\mathfrak{q}) : H]$.

We also adhere to the convention that all implied constants in all asymptotic inequalities $f \ll g$ or $f = O(g)$ are absolute with respect to $H$ and $K$. If an implied constant depends on a parameter, such as $\epsilon$, then we use $\ll_\epsilon$ and $O_\epsilon$ to denote that the implied constant depends at most on $\epsilon$. All implied constants will be effectively computable. Finally, all sums over integral ideals of $K$ will be over nonzero integral ideals.

**2B.** ***Hecke L-functions.*** For a more detailed reference on Hecke $L$-functions, see [Lagarias et al. 1979]. Strictly speaking, a Hecke character $\chi$ is a function on $\mathrm{Cl}(\mathfrak{q})$ but, by pulling back the domain of $\chi$ and extending it by zero, we regard $\chi$ as a function on integral ideals of $K$. We use this convention throughout the paper.

The *Hecke $L$-function of* $\chi$, denoted $L(s, \chi)$, is defined as

$$L(s, \chi) = \sum_{\mathfrak{n}} \chi(\mathfrak{n}) \mathrm{N}\mathfrak{n}^{-s} = \prod_{\mathfrak{p}} \Big(1 - \frac{\chi(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^s}\Big)^{-1} \tag{2-1}$$

for $\mathrm{Re}\{s\} > 1$, where the sum is over integral ideals $\mathfrak{n}$ of $K$ and the product is over prime ideals $\mathfrak{p}$ of $K$. Recall that the Dedekind zeta function $\zeta_K(s)$ is the primitive Hecke $L$-function associated to the trivial character $\chi_0$; that is,

$$\zeta_K(s) = \sum_{\mathfrak{n}} (\mathrm{N}\mathfrak{n})^{-s} = \prod_{\mathfrak{p}} \Bigl(1 - \frac{1}{\mathrm{N}\mathfrak{p}^s}\Bigr)^{-1} \tag{2-2}$$

for $\mathrm{Re}\{s\} > 1$. Returning to $L(s, \chi)$, assume that $\chi$ is primitive for the remainder of this subsection, unless otherwise specified. Define the *completed Hecke $L$-function* $\xi(s, \chi)$ by

$$\xi(s, \chi) = [s(s-1)]^{\delta(\chi)} D_\chi^{s/2} \gamma_\chi(s) L(s, \chi), \tag{2-3}$$

where $D_\chi = D_K \mathrm{N}\mathfrak{f}_\chi$, $\delta(\chi)$ is the indicator function of the trivial character, and $\gamma_\chi(s)$ is the *gamma factor of* $\chi$ defined by

$$\gamma_\chi(s) = \Bigl[\pi^{-s/2}\Gamma\Bigl(\frac{s}{2}\Bigr)\Bigr]^{a(\chi)} \cdot \Bigl[\pi^{-(s+1)/2}\Gamma\Bigl(\frac{s+1}{2}\Bigr)\Bigr]^{b(\chi)}. \tag{2-4}$$

Here $a(\chi)$ and $b(\chi)$ are certain nonnegative integers satisfying

$$a(\chi) + b(\chi) = n_K. \tag{2-5}$$

It is a classical fact that $\xi(s, \chi)$ is entire of order 1 and satisfies the functional equation

$$\xi(s, \chi) = w(\chi)\xi(1 - s, \bar{\chi}), \tag{2-6}$$

where $w(\chi) \in \mathbb{C}$ is the *root number* of $\chi$ satisfying $|w(\chi)| = 1$. The zeros of $\xi(s, \chi)$ are the *nontrivial zeros* $\rho$ of $L(s, \chi)$ and are known to satisfy $0 < \mathrm{Re}\{\rho\} < 1$. The *trivial zeros* $\omega$ of $L(s, \chi)$ are given by

$$\operatorname*{ord}_{s=\omega} L(s, \chi) = \begin{cases} a(\chi) - \delta(\chi) & \text{if } \omega = 0, \\ b(\chi) & \text{if } \omega = -1, -3, -5, \dots, \\ a(\chi) & \text{if } \omega = -2, -4, -6, \dots, \end{cases} \tag{2-7}$$

and arise as poles of the gamma factor of $L(s, \chi)$. Since $\xi(s, \chi)$ is entire of order 1, it admits a Hadamard product factorization given by

$$\xi(s, \chi) = e^{A(\chi)+B(\chi)s} \prod_{\rho} \Bigl(1 - \frac{s}{\rho}\Bigr) e^{s/\rho}. \tag{2-8}$$

**Lemma 2.1.** *Let $\chi$ be a primitive Hecke character. Then*

$$-\mathrm{Re}\Bigl\{\frac{L'}{L}(s, \chi)\Bigr\} = \frac{1}{2}\log D_\chi + \mathrm{Re}\Bigl\{\frac{\delta(\chi)}{s-1} + \frac{\delta(\chi)}{s}\Bigr\} - \sum_{\rho} \mathrm{Re}\Bigl\{\frac{1}{s-\rho}\Bigr\} + \mathrm{Re}\Bigl\{\frac{\gamma_\chi'}{\gamma_\chi}(s)\Bigr\},$$

*where the sum is over all nontrivial zeros $\rho$ of $L(s, \chi)$.*

*Proof.* See [Lagarias and Odlyzko 1977, Lemma 5.1], for example. ☐

By similar arguments, there exists an explicit formula for higher derivatives of $-\frac{L'}{L}(s,\chi)$.

**Lemma 2.2.** *Let $\chi$ be a Hecke character (not necessarily primitive) and $k \geq 1$ be a positive integer. Then*

$$(-1)^{k+1}\frac{d^k}{ds^k}\frac{L'}{L}(s,\chi) = \sum_{\mathfrak{p}}\sum_{m=1}^{\infty}(\log \mathrm{N}\mathfrak{p})\chi(\mathfrak{p})\frac{(\log \mathrm{N}\mathfrak{p}^m)^k}{(\mathrm{N}\mathfrak{p}^m)^s} = \frac{\delta(\chi)k!}{(s-1)^{k+1}} - \sum_{\omega}\frac{k!}{(s-\omega)^{k+1}}$$

*for* $\mathrm{Re}\{s\} > 1$, *where the first sum is over prime ideals $\mathfrak{p}$ of $K$ and the second sum is over all zeros $\omega$ of $L(s,\chi)$, including trivial ones, counted with multiplicity.*

*Proof.* By standard arguments, this follows from the Hadamard product (2-8) of $\xi(s,\chi)$ and the Euler product of $L(s,\chi)$. See [Lagarias et al. 1979, (5.2) and (5.3)], for example. □

**2C. *Explicit L-function estimates.*** In order to obtain explicit results, we must have explicit bounds on a few important quantities. First, we record a bound for $L(s,\chi)$ in the critical strip $0 < \mathrm{Re}\{s\} < 1$ via a Phragmén–Lindelöf type convexity estimate due to Rademacher.

**Lemma 2.3** [Rademacher 1959]. *Let $\chi$ be a primitive Hecke character and take $\eta \in \left(0, \frac{1}{2}\right]$. Then for $s = \sigma + it$,*

$$|L(s,\chi)| \ll \left|\frac{1+s}{1-s}\right|^{\delta(\chi)}\zeta_{\mathbb{Q}}(1+\eta)^{n_K}\left(\frac{D_\chi}{(2\pi)^{n_K}}(3+|t|)^{n_K}\right)^{(1+\eta-\sigma)/2}$$

*uniformly in the strip $-\eta \leq \sigma \leq 1+\eta$.*

Next, we record an explicit bound on the digamma function and $\frac{\gamma_\chi'}{\gamma_\chi}(s)$.

**Lemma 2.4.** *Let $s = \sigma + it$ with $\sigma > 1$ and $t \in \mathbb{R}$. Then* $\mathrm{Re}\left\{\frac{\Gamma'}{\Gamma}(s)\right\} \leq \log|s| + \sigma^{-1}$ *and, for any Hecke character $\chi$,*

$$\mathrm{Re}\left\{\frac{\gamma_\chi'}{\gamma_\chi}(s)\right\} \leq \frac{n_K}{2}(\log(|s|+1) + \sigma^{-1} - \log\pi).$$

*In particular, for* $1 < \sigma \leq 6.2$ *and* $|t| \leq 1$, *we have* $\mathrm{Re}\left\{\frac{\gamma_\chi'}{\gamma_\chi}(s)\right\} \leq 0$.

*Proof.* The first estimate follows from [Ono and Soundararajan 1997, Lemma 4]. The second estimate is a straightforward consequence of the first combined with the definition of $\gamma_\chi(s)$ in (2-4). The third estimate is contained in [Ahn and Kwon 2014, Lemma 3]. □

Next, we establish some bounds on the number of zeros of $L(s,\chi)$ in a circle.

**Lemma 2.5.** *Let $\chi$ be a Hecke character. Let $s = \sigma + it$ with $\sigma > 1$ and $t \in \mathbb{R}$. For $r > 0$, denote*

$$N_\chi(r; s) := \#\{\rho = \beta + i\gamma : 0 < \beta < 1,\ L(\rho, \chi) = 0,\ |s - \rho| \le r\}. \tag{2-9}$$

*If $0 < r \le 1$, then*

$$N_\chi(r; s) \le \{4 \log D_K + 2 \log \mathrm{N}\mathfrak{f}_\chi + 2n_K \log(|t| + 3) + 4 + 4\delta(\chi)\} \cdot r + 4 + 4\delta(\chi).$$

*Proof.* Without loss, we may assume $\chi$ is primitive. Observe that

$$N_\chi(r; s) \le N_\chi(r; 1 + it) \le N_\chi(2r; 1 + r + it),$$

so it suffices to bound the latter quantity. Now, if $s_0 = 1 + r + it$, notice

$$N_\chi(2r; s_0) \le 4r \sum_{|s_0-\rho|\le 2r} \mathrm{Re}\Big\{\frac{1}{s_0 - \rho}\Big\} \le 4r \sum_{\rho} \mathrm{Re}\Big\{\frac{1}{s_0 - \rho}\Big\}.$$

Applying Lemmas 2.1 and 2.4 twice and noting $\mathrm{Re}\{\frac{L'}{L}(s_0, \chi)\} \le -\frac{\zeta_K'}{\zeta_K}(1 + r)$ via their respective Euler products, the above is

$$\begin{aligned}&\le 4r\bigg(\mathrm{Re}\Big\{\frac{L'}{L}(s_0, \chi)\Big\} + \frac{1}{2}\log D_\chi + \mathrm{Re}\Big\{\frac{\gamma_\chi'}{\gamma_\chi}(s_0)\Big\} + \delta(\chi)\mathrm{Re}\Big\{\frac{1}{s_0} + \frac{1}{s_0-1}\Big\}\bigg)\\ &\le \{4 \log D_K + 2 \log \mathrm{N}\mathfrak{f}_\chi + 2n_K \log(|t| + 3) + 4 + 4\delta(\chi)\} \cdot r + 4 + 4\delta(\chi)\end{aligned}$$

as $D_\chi = D_K \mathrm{N}\mathfrak{f}_\chi$. For details on estimating $-\frac{\zeta_K'}{\zeta_K}(1 + r)$, see Lemma 2.10. □

To improve the bound in Lemma 2.5, we exhibit an explicit inequality involving the logarithmic derivative of $L(s, \chi)$ comparable with [Kadiri and Ng 2012, Theorem 2] for the Dedekind zeta function.

**Proposition 2.6.** *Let $0 < \epsilon < \frac{1}{4}$, $T \ge 1$, and $s = \sigma + it$. For a primitive Hecke character $\chi$, define a multiset of nontrivial zeros of $L(s, \chi)$ by*

$$\mathcal{Z}_{r,t} = \{\rho = \beta + i\gamma : L(\rho, \chi) = 0, |1 + it - \rho| \le r\}.$$

*Then, for $0 < r < \epsilon$,*

$$\begin{aligned}-\mathrm{Re}\Big\{\frac{L'}{L}(s, \chi)\Big\} \le \Big(\frac{1}{4} + \frac{\epsilon}{\pi} + 5\epsilon^{10}\Big)\mathcal{L}_\chi + (4\epsilon^2 + 80\epsilon^{10})\mathcal{L}_\chi'&\\ + \delta(\chi)\mathrm{Re}\Big\{\frac{1}{s-1}\Big\} - \sum_{\rho\in\mathcal{Z}_{r,t}} \mathrm{Re}\Big\{\frac{1}{s-\rho}\Big\} + O_\epsilon(n_K)&\end{aligned} \tag{2-10}$$

*and*

$$-\mathrm{Re}\Big\{\frac{L'}{L}(s, \chi)\Big\} \le \Big(\frac{1}{4} + \frac{\epsilon}{\pi} + 5\epsilon^{10}\Big)\mathcal{L}_\chi + \delta(\chi)\mathrm{Re}\Big\{\frac{1}{s-1}\Big\} + O_\epsilon(n_K) \tag{2-11}$$

*uniformly in the region $1 < \sigma \le 1 + \epsilon$ and $|t| \le T$, where $\mathcal{L}_\chi = \log D_\chi + n_K \log(T + 3)$ and $\mathcal{L}_\chi' = \log D_K + \mathcal{L}_\chi$.*

*Proof.* This result is a modified version of [Zaman 2016a, Lemma 4.3] which is motivated by [Heath-Brown 1992, Lemma 3.1]. The main improvements are the valid range of $\sigma$ and $t$. Consequently, we sketch the argument found in [Zaman 2016a] highlighting the necessary modifications. Assume $\chi$ is nontrivial. Apply [Heath-Brown 1992, Lemma 3.2] with $f(z) = L(z,\chi)$, $a = s$, and $R = 1-\eta$, where $\eta = \eta_{s,\chi} \in \left(0, \frac{1}{10}\right)$ is chosen sufficiently small so that $L(w,\chi)$ has no zeros on the circle $|w-s| = R$. Then

$$-\mathrm{Re}\left\{\frac{L'}{L}(s,\chi)\right\} = -\sum_{|s-\rho|<R} \mathrm{Re}\left\{\frac{1}{s-\rho} - \frac{s-\rho}{R^2}\right\} - J, \tag{2-12}$$

where

$$J := \int_0^{2\pi} \frac{\cos\theta}{\pi R} \cdot \log|L(s+Re^{i\theta},\chi)|\,d\theta.$$

To bound $J$ from below, write

$$J = \int_0^{\pi/2} + \int_{\pi/2}^{3\pi/2} + \int_{3\pi/2}^{2\pi} = J_1 + J_2 + J_3,$$

say, so we may consider each contribution separately. For $J_1$, notice by [Zaman 2016a, Lemma 2.5],

$$\log|L(s+Re^{i\theta},\chi)| \le \log\zeta_K(\sigma + R\cos\theta) \ll n_K \log\left(\frac{1}{\sigma-1+R\cos\theta}\right).$$

Write $\left[0, \frac{\pi}{2}\right] = \left[0, \frac{\pi}{2} - (\sigma-1)\right] \cup \left[\frac{\pi}{2} - (\sigma-1), \frac{\pi}{2}\right] = I_1 \cup I_2$, say. Then

$$\begin{aligned} J_1 &= \int_{I_1} + \int_{I_2} \\ &\ll n_K \int_{I_1} \cos\theta \log\left(\frac{1}{\cos\theta}\right) d\theta + n_K \log\left(\frac{1}{\sigma-1}\right) \int_{I_2} \cos\theta\, d\theta \ll_\epsilon n_K. \end{aligned}$$

A similar argument holds for $J_3$ so $J_1 + J_3 \ll_\epsilon n_K$. For $J_2$, consider $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. As $1 < \sigma \le 1+\epsilon$ and $R < 1$, we have $0 < \sigma + R\cos\theta \le 1+\epsilon$. Hence, by Lemma 2.3,

$$\log|L(s+Re^{i\theta},\chi)| \le \tfrac{1}{2}\mathcal{L}_\chi(-R\cos\theta + \epsilon) + O_\epsilon(n_K).$$

Thus,

$$J_2 \ge \frac{\mathcal{L}_\chi}{2\pi R} \int_{\pi/2}^{3\pi/2} (-R\cos^2\theta + \epsilon\cos\theta)\,d\theta + O_\epsilon(n_K),$$

yielding overall

$$J \ge -\left(\frac{1}{4} + \frac{\epsilon}{\pi R}\right)\mathcal{L}_\chi + O_\epsilon(n_K). \tag{2-13}$$

For the sum over zeros in (2-12), observe that the terms are nonnegative, so (2-11) follows immediately from (2-12) and (2-13) after taking $\eta \to 0$, which implies $R \to 1$. To prove (2-10), consider $0 < r < \epsilon$. By the same observation, we may restrict our sum over zeros from $|s-\rho|<R$ to a smaller circle within it: $|1+it-\rho|\le r$. As $r<\epsilon<\frac{1}{4}$ by assumption, we discard the zeros outside this smaller circle. For such zeros $\rho$ satisfying $|1+it-\rho|\le r$, notice $\mathrm{Re}\{s-\rho\}=\sigma-\beta\le\epsilon+r<2\epsilon$. This implies, by Lemma 2.5, that

$$\sum_{|1+it-\rho|\le r} \mathrm{Re}\Big\{\frac{s-\rho}{R^2}\Big\} \le \frac{2\epsilon}{R^2}\cdot\{(2\mathcal{L}_\chi'+8)r+8\} \le \frac{4\epsilon^2}{R^2}\mathcal{L}_\chi' + O(1). \tag{2-14}$$

Thus, (2-10) immediately follows[2] upon combining (2-12), (2-13), and (2-14), and taking $\eta\to 0$, which implies $R\to 1$. This completes the proof for $\chi$ nontrivial.

For $\chi=\chi_0$ trivial, similarly proceed with [Heath-Brown 1992, Lemma 3.2] with $f(z)=((z-1)/(z+1))\zeta_K(z)$ and $a=z$, but the choice of $R$ is different due to the simple pole of the Dedekind zeta function. Observe that the circles $|w-1|=\epsilon^{10}$ and $|w-s|=R$ are disjoint for at least one of the following:

(i) all $R\in(1-\epsilon^{10},1)$, or

(ii) all $R\in(1-5\epsilon^{10},1-4\epsilon^{10})$.

In the case of (i), choose $R=1-\eta$ for $\eta=\eta_{s,\chi}$ sufficiently small so that $L(w,\chi)$ has no zeros on the circle $|w-s|=R$. Similarly for (ii), take $R=1-4\epsilon^{10}-\eta$.

Continuing with the same arguments, the only difference occurs when bounding $J_1$ and similarly $J_3$, in which case one must estimate

$$\int_0^{\pi/2}\frac{\cos\theta}{\pi R}\log\left|\frac{s-1+Re^{i\theta}}{s+1+Re^{i\theta}}\right|d\theta.$$

By our choice of $R$, the quantity in the logarithm is $\asymp_\epsilon 1$, and hence the above is $O_\epsilon(1)$. The remainder of the argument is the same, except at the final step one must take $R\to 1$ in case (i) and $R\to 1-4\epsilon^{10}$ in case (ii). The latter case yields the additional $\epsilon^{10}$ terms appearing in (2-10). □

**Lemma 2.7.** *Let $\chi$ be a Hecke character and $0<r<\epsilon<\frac{1}{4}$. If $s=\sigma+it$ with $1<\sigma<1+\epsilon$ and $N_\chi(r;s)$ by (2-9), then, letting $\phi=1+\frac{4}{\pi}\epsilon+16\epsilon^2+340\epsilon^{10}$,*

$$N_\chi(r;s)\le\phi\big(2\log D_K+\log\mathrm{N}\mathfrak{f}_\chi+n_K\log(|t|+3)+O_\epsilon(n_K)\big)\cdot r+4+4\delta(\chi).$$

*Proof.* This is analogous to Lemma 2.5 except that we bound $N_\chi(r;1+it)$ instead of $N_\chi(2r;1+r+it)$, and further, we apply Proposition 2.6 in place of Lemmas 2.1 and 2.4. □

[2] One actually obtains (2-10) without the extra $\epsilon^{10}$ terms.

**2D.** ***Arithmetic sums.*** We estimate various sums over integral ideals of $K$, which requires some additional notation. It is well-known that the Dedekind zeta function $\zeta_K(s)$, defined by (2-2), has a simple pole at $s = 1$. Thus, we may define

$$\kappa_K := \operatorname*{Res}_{s=1} \zeta_K(s) \quad \text{and} \quad \gamma_K := \kappa_K^{-1} \lim_{s\to 1}\Big(\zeta_K(s) - \frac{\kappa_K}{s-1}\Big) \tag{2-15}$$

so the Laurent expansion of $\zeta_K(s)$ at $s = 1$ is given by

$$\zeta_K(s) = \frac{\kappa_K}{s-1} + \kappa_K\gamma_K + O_K(|s-1|).$$

We refer to $\gamma_K$ as the *Euler–Kronecker constant of* $K$. (See [Ihara 2006] for details on $\gamma_K$.)

**Lemma 2.8.** *For $x > 0$ and $0 < \epsilon < \frac{1}{2}$,*

$$\bigg|\sum_{\mathrm{N}\mathfrak{n}<x} \frac{1}{\mathrm{N}\mathfrak{n}}\Big(1 - \frac{\mathrm{N}\mathfrak{n}}{x}\Big)^{n_K} - \kappa_K\bigg(\log x - \sum_{j=1}^{n_K}\frac{1}{j}\bigg) - \kappa_K\gamma_K\bigg| \ll_\epsilon (n_K^{n_K} D_K)^{1/4+\epsilon} x^{-1/2}.$$

*Proof.* The quantity we wish to bound equals

$$\begin{aligned}\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \zeta_K(s+1)\frac{x^s}{s}\frac{n_K!}{\prod_{j=1}^{n_K}(s+j)}\,ds \\ = \frac{n_K!}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \zeta_K(s+1)\frac{\Gamma(s)}{\Gamma(n_K+1+s)}x^s\,ds.\end{aligned}$$

Applying Lemma 2.3, Stirling's formula, and $\zeta_{\mathbb{Q}}(1+\epsilon)^{n_K} \ll e^{O_\epsilon(n_K)}$, the result follows. □

**Corollary 2.9.** *Let $\epsilon > 0$ be arbitrary. If $x \geq 3(n_K^{n_K} D_K)^{1/2+\epsilon}$, then*

$$\sum_{\mathrm{N}\mathfrak{n}<x}\frac{1}{\mathrm{N}\mathfrak{n}} \geq \Big\{1 - \frac{1}{1+2\epsilon} + O_\epsilon\Big(\frac{1}{\log x}\Big)\Big\}\cdot\kappa_K\log x.$$

*Proof.* It suffices to assume that $\kappa_K \geq 1/\log x$. From Lemma 2.8, it follows that

$$\frac{1}{\kappa_K}\sum_{\mathrm{N}\mathfrak{n}<x}\frac{1}{\mathrm{N}\mathfrak{n}} \geq \log x - \sum_{j=1}^{n_K}\frac{1}{j} + \gamma_K + O_\epsilon(x^{-\epsilon/8}\log x),$$

by our assumption on $x$. By [Ihara 2006, Proposition 3],

$$\gamma_K \geq -\frac{1}{2}\log D_K + \frac{\gamma_{\mathbb{Q}} + \log 2\pi}{2}\cdot n_K - 1,$$

where $\gamma_{\mathbb{Q}} = 0.5772\ldots$ is Euler's constant. Since $\sum_{1\le j\le n_K} j^{-1} \le \log n_K + 1$,

$$\begin{aligned}\frac{1}{\kappa_K}\sum_{\mathrm{N}\mathfrak{n}<x}\frac{1}{\mathrm{N}\mathfrak{n}} &\\ &\ge (\log x)\{1+O_\epsilon(x^{-\epsilon/8})\} - \frac{1}{2}\log D_K + \frac{\gamma_{\mathbb{Q}}+\log 2\pi}{2}\cdot n_K - \log n_K - 2\\ &\ge (\log x)\Big\{1-\frac{1}{1+2\epsilon}+O_\epsilon((\log x)^{-1})\Big\},\end{aligned}$$

by our assumption on $x$. □

Taking the logarithmic derivative of $\zeta_K(s)$ yields in the usual way

$$-\frac{\zeta_K'}{\zeta_K}(s) = \sum_{\mathfrak{n}\subseteq\mathcal{O}_K}\frac{\Lambda_K(\mathfrak{n})}{(\mathrm{N}\mathfrak{n})^s} \tag{2-16}$$

for $\mathrm{Re}\{s\} > 1$, where $\Lambda_K(\cdot)$ is the von Mangoldt $\Lambda$-function of the field $K$ defined by

$$\Lambda_K(\mathfrak{n}) = \begin{cases}\log \mathrm{N}\mathfrak{p} & \text{if } \mathfrak{n} \text{ is a power of a prime ideal } \mathfrak{p},\\ 0 & \text{otherwise.}\end{cases} \tag{2-17}$$

Using this identity, we prove an elementary lemma.

**Lemma 2.10.** *For $y \ge 3$ and $0 < r < 1$,*

(i) $-\dfrac{\zeta_K'}{\zeta_K}(1+r) = \displaystyle\sum_{\mathfrak{n}}\frac{\Lambda_K(\mathfrak{n})}{\mathrm{N}\mathfrak{n}^{1+r}} \le \frac{1}{2}\log D_K + \frac{1}{r} + 1,$ *and*

(ii) $\displaystyle\sum_{\mathrm{N}\mathfrak{n}\le y}\frac{\Lambda_K(\mathfrak{n})}{\mathrm{N}\mathfrak{n}} \le e\log(eD_K^{1/2}y).$

*Proof.* Part (i) follows from Lemmas 2.1 and 2.4, (2-16), and $\mathrm{Re}\{(1+r-\rho)^{-1}\} \ge 0$. Part (ii) follows from (i) by taking $r = (\log y)^{-1}$. □

Finally, we end this section with a bound for $h_H$ in terms of $n_K$, $D_K$, and $Q = Q_H$, and a comparison between $Q$ and $\mathrm{N}\mathfrak{f}_H$.

**Lemma 2.11.** *Let $H$ be a congruence class group of $K$. For $\epsilon > 0$,*

$$h_H \le e^{O_\epsilon(n_K)} D_K^{1/2+\epsilon} Q^{1+\epsilon}.$$

*Proof.* Observe, by the definitions of $Q$ and $\mathfrak{f}_H$ in Section 2A, that for a Hecke character $\chi \pmod{H}$ we have $\mathfrak{f}_\chi \mid \mathfrak{f}_H$ and $\mathrm{N}\mathfrak{f}_\chi \le Q$. Hence,

$$h_H = \sum_{\chi \,(\mathrm{mod}\, H)} 1 \le \sum_{\substack{\mathrm{N}\mathfrak{f}\le Q\\ \mathfrak{f}\mid\mathfrak{f}_H}}\ \sum_{\chi\,(\mathrm{mod}\,\mathfrak{f})} 1 = \sum_{\substack{\mathrm{N}\mathfrak{f}\le Q\\ \mathfrak{f}\mid\mathfrak{f}_H}} \#\mathrm{Cl}(\mathfrak{f}).$$

Recall the classical bound $\#\mathrm{Cl}(\mathfrak{f}) \le 2^{n_K} h_K \mathrm{N}\mathfrak{f}$, where $h_K$ is the (broad) class number of $K$. (See [Milne 2013, Theorem 1.7], for example.) Bounding the class number using Minkowski's bound (see [Weiss 1983, Lemma 1.12], for example), we deduce that

$$h_H \leq \sum_{\substack{\mathrm{N}\mathfrak{f} \leq Q \\ \mathfrak{f} \mid \mathfrak{f}_H}} e^{O_\epsilon(n_K)} D_K^{1/2+\epsilon} \mathrm{N}\mathfrak{f} \leq e^{O_\epsilon(n_K)} D_K^{1/2+\epsilon} Q^{1+\epsilon} \sum_{\mathfrak{f} \mid \mathfrak{f}_H} \frac{1}{(\mathrm{N}\mathfrak{f})^\epsilon}.$$

For the remaining sum, notice $\sum_{\mathfrak{f} \mid \mathfrak{f}_H} (\mathrm{N}\mathfrak{f})^{-\epsilon} \leq \prod_{\mathfrak{p} \mid \mathfrak{f}_H} (1-\mathrm{N}\mathfrak{p}^{-\epsilon})^{-1} \leq e^{O(\omega(\mathfrak{f}_H))}$, where $\omega(\mathfrak{f}_H)$ is the number of prime ideals $\mathfrak{p}$ dividing $\mathfrak{f}_H$. From [Weiss 1983, Lemma 1.13], we have $\omega(\mathfrak{f}_H) \ll O_\epsilon(n_K) + \epsilon \log(D_K Q)$, whence the desired estimate follows after rescaling $\epsilon$. □

**Remark.** Weiss [1983, Lemma 1.16] achieves a comparable bound with $Q^{1+\epsilon}$ replaced by $\mathrm{N}\mathfrak{f}_H$. This seemingly minor difference will in fact improve the range of $T$ in Theorem 3.2.

**Lemma 2.12.** *Let $H$ be a congruence class group of $K$. Then $Q \leq \mathrm{N}\mathfrak{f}_H \leq Q^2$.*

**Remark.** The lower bound is achieved when $H = P_{\mathfrak{f}_H}$. We did not investigate the tightness of the upper bound, as this estimate is sufficient for our purposes.

*Proof.* The arguments here are motivated by [Weiss 1983, Lemma 1.13]. Without loss, we may assume $H$ is primitive. Since $Q = Q_H = \max\{\mathrm{N}\mathfrak{f}_\chi : \chi \pmod{H}\}$ and $\mathfrak{f}_H = \mathrm{lcm}\{\mathfrak{f}_\chi : \chi \pmod{H}\}$, the lower bound is immediate. For the upper bound, we apply arguments similar to [Weiss 1983, Lemma 1.13]. Consider any $\mathfrak{m} \mid \mathfrak{f}_H$. Let $H_\mathfrak{m}$ denote the image of $H$ under the map $I(\mathfrak{f}_H)/P_{\mathfrak{f}_H} \to I(\mathfrak{m})/P_\mathfrak{m}$. This induces a map $I(\mathfrak{f}_H)/H \to I(\mathfrak{m})/H_\mathfrak{m}$, which, since $H$ is primitive, must have nontrivial kernel. Hence, characters of $I(\mathfrak{m})/H_\mathfrak{m}$ induce characters of $I(\mathfrak{f}_H)/H$.

Now, for $\mathfrak{p} \mid \mathfrak{f}_H$, choose $e = e_\mathfrak{p} \geq 1$ maximal so that $\mathfrak{p}^e \mid \mathfrak{f}_H$. Define $\mathfrak{m}_\mathfrak{p} := \mathfrak{f}_H \mathfrak{p}^{-1}$ and consider the induced map $I(\mathfrak{f}_H)/H \to I(\mathfrak{m}_\mathfrak{p})/H_{\mathfrak{m}_\mathfrak{p}}$ with kernel $V_\mathfrak{p}$. Since $H$ is primitive, $V_\mathfrak{p}$ must be nontrivial and hence $\#V_\mathfrak{p} \geq 2$. Observe that the characters $\chi$ of $I(\mathfrak{f}_H)/H$ such that $\mathfrak{p}^e \nmid \mathfrak{f}_\chi$ are exactly those which are trivial on $V_\mathfrak{p}$ and hence are $h_H/\#V_\mathfrak{p}$ in number. For a given $\mathfrak{p}$, this yields

$$\frac{h_H}{2} \leq h_H\Big(1 - \frac{1}{\#V_\mathfrak{p}}\Big) = \sum_{\substack{\chi \pmod{H} \\ \mathfrak{p}^{e_\mathfrak{p}} \,\|\, \mathfrak{f}_\chi}} 1.$$

Multiplying both sides by $\log(\mathrm{N}\mathfrak{p}^{e_\mathfrak{p}})$ and summing over $\mathfrak{p} \mid \mathfrak{f}_H$, we have

$$\begin{aligned}\frac{1}{2} h_H \log \mathrm{N}\mathfrak{f}_H = \frac{h_H}{2} \sum_{\mathfrak{p} \mid \mathfrak{f}_H} \log(\mathrm{N}\mathfrak{p}^{e_\mathfrak{p}}) &\leq \sum_{\mathfrak{p} \mid \mathfrak{f}_H} \sum_{\substack{\chi \pmod{H} \\ \mathfrak{p}^{e_\mathfrak{p}} \,\|\, \mathfrak{f}_\chi}} \log \mathrm{N}\mathfrak{p}^{e_\mathfrak{p}} \\ &\leq \sum_{\chi \pmod{H}} \log \mathrm{N}\mathfrak{f}_\chi \leq h_H \log Q.\end{aligned}$$

Comparing both sides, we deduce $\mathrm{N}\mathfrak{f}_H \leq Q^2$ as desired. □

**Lemma 2.13.** *Let $H$ be a congruence subgroup of $K$ and $\epsilon > 0$ be arbitrary. Then*

$$\sum_{\mathfrak{p} \mid \mathfrak{f}_H} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}} \le (2\epsilon)^{-1} n_K + \epsilon \log Q.$$

*Proof.* This follows from [Zaman 2016a, Lemma 2.4] and Lemma 2.12. □

## 3. Proof of Theorem 1.1 and Linnik's three principles

**3A. *Proof of Theorem 1.1.*** The primary goal in this paper is to prove the following result, from which Theorem 1.1 follows.

**Theorem 3.1.** *Let $K$ be a number field, let $H \pmod{\mathfrak{q}}$ be a congruence class group of $K$, and let $\mathfrak{f}_H$ be the conductor of $H$. Let $I(\mathfrak{q})$ be the group of fractional ideals of $K$ which are coprime to $\mathfrak{q}$ and let $\mathcal{C} \in I(\mathfrak{q})/H$ be arbitrary. Let $\chi \pmod{H}$ be a character of $I(\mathfrak{q})/H$ of conductor $\mathfrak{f}_\chi$. Finally, let $h_H = [I(\mathfrak{q}) : H]$, $Q = \max\{\mathrm{N}_{K/\mathbb{Q}}\mathfrak{f}_\chi : \chi \pmod{H}\}$, and $\mathfrak{m}$ be the product of prime ideals dividing $\mathfrak{q}$ but not $\mathfrak{f}_H$. If*

$$x \ge D_K^{694} Q^{521} + D_K^{232} Q^{367} n_K^{290 n_K} + (D_K Q n_K^{n_K})^{1/1000} \mathrm{N}_{K/\mathbb{Q}}\mathfrak{m}, \tag{3-1}$$

*and $D_K Q [K:\mathbb{Q}]^{[K:\mathbb{Q}]}$ is sufficiently large, then*

$$\#\{\mathfrak{p} \in \mathcal{C} : \deg(\mathfrak{p}) = 1,\ \mathrm{N}_{K/\mathbb{Q}}\,\mathfrak{p} \le x\} \gg (D_K Q n_K^{n_K})^{-5} \frac{x}{h_H \log x}.$$

Assuming Theorem 3.1, we now prove Theorem 1.1.

*Proof of Theorem 1.1.* The proof proceeds exactly as in [Weiss 1983, Theorem 6.1]. Let $L/F$ be a finite Galois extension of number fields with Galois group $G$, and let $C \subset G$ be a given conjugacy class. Let $A \subset G$ be an abelian subgroup such that $A \cap C$ is nonempty, and let $K = L^A$ be the fixed field of $A$. Let $\mathfrak{f}_{L/K}$ be the conductor of $L/K$, and let $\mathfrak{m}$ be the product of prime ideals $\mathfrak{P}$ in $K$ which are unramified in $L$ but such that the prime $\mathfrak{p}$ of $F$ lying under $\mathfrak{P}$ is ramified in $L$. If $\left[\frac{L/K}{\mathfrak{P}}\right]$ denotes the Artin symbol, then the Artin map $\mathfrak{P} \mapsto \left[\frac{L/K}{\mathfrak{P}}\right]$ induces a group homomorphism $I(\mathfrak{m}\mathfrak{f}_{L/K}) \to A$ because the conjugacy classes in $A$ are singletons; thus if $H$ is the kernel of the homomorphism, then the canonical map $\omega : I(\mathfrak{m}\mathfrak{f}_{L/K})/H \to A$ is an isomorphism. Moreover, $H$ is a congruence class group modulo the ideal $\mathfrak{m}\mathfrak{f}_{L/K}$ of $K$ with $\mathfrak{f}_H = \mathfrak{f}_{L/K}$.

Choose $\sigma_0 \in C \cap A$. Using $\omega$, $\sigma_0$ determines a coset of $I(\mathfrak{m}\mathfrak{f}_H)/H$; thus by Theorem 3.1, if (3-1) holds and $D_K Q n_K^{n_K}$ is sufficiently large, then

$$\#\left\{\mathrm{N}_{K/\mathbb{Q}}\mathfrak{P} \le x : \deg(\mathfrak{P}) = 1,\ \left[\frac{L/K}{\mathfrak{P}}\right] = \{\sigma_0\}\right\} \gg (D_K Q n_K^{n_K})^{-5} \frac{x}{h_H \log x}.$$

Let $\mathfrak{p}$ be a prime ideal of $F$ lying under $\mathfrak{P}$. By the definition of $\mathfrak{m}$, $\mathfrak{p}$ is unramified in $L$ and $\mathrm{N}_{K/\mathbb{Q}}\mathfrak{P} = \mathrm{N}_{F/\mathbb{Q}}\,\mathfrak{p}$ because $\deg(\mathfrak{P}) = 1$. Furthermore, $[L/F, \mathfrak{p}] = C$. Thus if $x$ satisfies (3-1),

$$\#\Big\{\mathfrak{p} : \deg(\mathfrak{p}) = 1,\ \Big[\frac{L/F}{\mathfrak{p}}\Big] = C,\ \mathrm{N}_{F/\mathbb{Q}}\mathfrak{p} \le x\Big\} \gg (D_K Q n_K^{n_K})^{-5} \frac{x}{h_H \log x}.$$

As in [Weiss 1983, Theorem 6.1], $\mathrm{N}_{K/\mathbb{Q}}\mathfrak{m} \le D_K$ and $h_H = [L : K]$. By the definition of $Q$ and the definition of $H$, we have that $Q = \mathcal{Q}$, so

$$\pi_C(x, L/F) \gg (D_K \mathcal{Q} n_K^{n_K})^{-5} \frac{x}{[L : K] \log x} \tag{3-2}$$

whenever $D_K Q n_K^{n_K}$ is sufficiently large and $x \ge D_K^{694} \mathcal{Q}^{521} + D_K^{232} \mathcal{Q}^{367} n_K^{290 n_K} + D_K \mathcal{Q} n_K^{n_K}$. Since $D_K \mathcal{Q} n_K^{n_K} \le D_L n_L^{n_L}$ and there are only finitely many number fields $L$ with $D_L n_L^{n_L}$ not sufficiently large, we may enlarge the implied constant in Theorem 1.1 to allow for those exceptions and complete the proof. □

**3B.** ***The key ingredients.*** To outline our proof of Theorem 3.1, we recall the modern approach to proving Linnik's bound on the least prime in an arithmetic progression. In order to obtain small explicit values of $c_1$ in (1-2), one typically requires three principles, explicit versions of which are recorded in [Heath-Brown 1992, Section 1]:

- A zero-free region for Dirichlet $L$-functions: if $q$ is sufficiently large, then the product $\prod_{\chi \pmod q} L(s, \chi)$ has at most one zero in the region

$$s = \sigma + it, \qquad \sigma \ge 1 - \frac{0.10367}{\log(q(2+|t|))}. \tag{3-3}$$

  If such a zero exists, it is real and simple and its associated character is also real.

- A "log-free" zero density estimate: if $q$ is sufficiently large, $\epsilon > 0$, and we define $N(\sigma, T, \chi) = \#\{\rho = \beta + i\gamma : L(\rho, \chi) = 0,\ |\gamma| \le T,\ \beta \ge \sigma\}$, then

$$\sum_{\chi \pmod q} N(\sigma, T, \chi) \ll_\epsilon (qT)^{(12/5+\epsilon)(1-\sigma)}, \qquad T \ge 1. \tag{3-4}$$

- The zero repulsion phenomenon: if $q$ is sufficiently large, $\lambda > 0$ is sufficiently small, $\epsilon > 0$, and the exceptional zero in the region (3-3) exists and equals $1 - \lambda/\log q$, then $\prod_{\chi \pmod q} L(s, \chi)$ has no other zeros in the region

$$\sigma \ge 1 - \frac{\big(\frac{2}{3} - \epsilon\big)(\log \lambda^{-1})}{\log(q(2 + |t|))}. \tag{3-5}$$

  If such an exceptional zero exists, then it is real and simple and it corresponds with a nontrivial real character $\chi$.

Number field variants of these principles were proved by Fogels [1962a; 1962b], but his proof did not maintain the necessary field uniformity. To prove (1-7), Weiss developed variants of these principles with effective number field dependence; the effective field dependence is critical for the proof of (1-7). To prove Theorem 3.1, we make Weiss' field-uniform results explicit.

**3C. *The zero density estimate.*** In Sections 4–6, we prove an explicit version of Weiss' variant of (3-4) for Hecke characters using the power sum method. Assume the notation in the previous section, and define

$$N(\sigma, T, \chi) := \#\{\rho = \beta + i\gamma : L(\rho, \chi) = 0,\ \sigma < \beta < 1,\ |\gamma| \le T\},$$

where the nontrivial zeros $\rho$ of $L(s, \chi)$ are counted with multiplicity. Weiss [1983, Corollary 4.4] proved that there exists an absolute constant $c_6 > 0$ such that if $\frac{1}{2} \le \sigma < 1$ and $T \ge n_K^2 h_H^{1/n_K}$, then

$$\sum_{\chi \pmod H} N(\sigma, T, \chi) \ll (e^{O(n_K)} D_K^2 Q T^{n_K})^{c_6}. \tag{3-6}$$

We prove the following.

**Theorem 3.2.** *Let $H$ be a congruence class group of a number field $K$. If $\frac{1}{2} \le \sigma < 1$ and $T \ge \max\{n_K^{5/6}(D_K^{4/3} Q^{4/9})^{-1/n_K}, 1\}$, then*

$$\sum_{\chi \pmod H} N(\sigma, T, \chi) \ll \{e^{O(n_K)} D_K^2 Q T^{n_K+2}\}^{81(1-\sigma)}. \tag{3-7}$$

*If $1 - 10^{-3} \le \sigma < 1$, then one may replace* 81 *with* 73.5.

**Remarks.**

- Theorem 3.2 noticeably improves Weiss' density estimate (3-6) in the range of $T$. If $n_K \le 2(\log D_K)/\log\log D_K$, then Theorem 3.2 holds for $T \ge 1$. Thus we may take $T \ge 1$ for most choices of $K$.
- We see from Minkowski's lower bound for $D_K$ and the valid range of $T$ that the $e^{O(n_K)}$ factor is always negligible, regardless of how $n_K$ compares to $(\log D_K)/\log\log D_K$.

It is instructive to compare the two primary methods for proving log-free zero density estimates. The basic idea behind the proof of (3-4) (the so-called mollifier method) is to construct a Dirichlet polynomial which detects zeros by assuming large values at the zeros of a Dirichlet $L$-function. The optimal Dirichlet polynomial for this task looks like a smoothed version of $\mu(n)$, where

$$\mu(n) = \begin{cases} (-1)^r & \text{if } n \text{ is squarefree with } r \text{ prime factors,} \\ 0 & \text{otherwise,} \end{cases}$$

is the usual Möbius function. In order to efficiently sum the large values contributed by each of the detected zeros, one relies on the fact that the partial sums of $\mu(n)$ exhibit significant cancellation. To see why this is true, observe that the prime number theorem (with the error term of Hadamard and de la Vallée Poussin) is equivalent to the statement that there exists an absolute constant $c_7 > 0$ such that if $x$ is sufficiently large, then

$$\sum_{n \le x} \mu(n) \ll x \exp(-c_7 (\log x)^{1/2}). \tag{3-8}$$

The fact that (3-8) is a part of the proofs of the log-free zero density estimates in [Graham 1977; Heath-Brown 1992; Iwaniec and Kowalski 2004; Jutila 1977] may not be immediately obvious. After summing the mollified Dirichlet polynomials over all characters $\chi \pmod q$ and applying duality, one must ultimately minimize the quadratic form

$$S(x) = \sum_{n \le x} \Bigl( \sum_{d \mid n} \lambda_d \Bigr)^2$$

subject to the constraint

$$\lambda_d = \begin{cases} \mu(d) \min\Bigl\{1, \dfrac{\log(z_2/d)}{\log(z_2/z_1)}\Bigr\} & \text{if } 1 \le d \le z_2, \\ 0 & \text{if } d > z_2, \end{cases}$$

where $1 < z_1 < z_2$ are given real numbers. (For example, see [Iwaniec and Kowalski 2004, pp. 430–431].) Each of [Graham 1977; Heath-Brown 1992; Iwaniec and Kowalski 2004; Jutila 1977] uses the work of Graham [1978] to estimate $S(x)$ with this choice of $\lambda_d$; Graham proved that

$$S(x) \le \frac{x}{\log(z_2/z_1)} \Bigl( 1 + O\Bigl( \frac{1}{\log(z_2/z_1)} \Bigr) \Bigr). \tag{3-9}$$

At several points in the proof, Graham uses the asymptotic prime number theorem in the form (3-8).

For a number field $K$, let $\mu_K(\mathfrak{n})$ be the extension of the Möbius function to the prime ideals of $K$. For the sake of simplicity, suppose that the Dedekind zeta function $\zeta_K(s)$ has no Landau–Siegel zero. The effective form of the prime ideal theorem proven in [Lagarias and Odlyzko 1977] is equivalent to the statement that there exists an absolute constant $c_8 > 0$ such that if $\log x \gg n_K (\log D_K)^2$, then

$$\sum_{\mathrm{N}\mathfrak{n} \le x} \mu_K(\mathfrak{n}) \ll x \exp\Bigl( -c_8 \Bigl( \frac{\log x}{n_K} \Bigr)^{1/2} \Bigr).$$

Therefore, to generalize (3-9) to the Möbius function of $K$, $x$ needs to be larger than any polynomial in $D_K$ before the partial sums of $\mu_K(\mathfrak{n})$ up to $x$ begin to exhibit cancellation. Thus if one extends the preceding arguments to prove an analogue of (3-4) for the Hecke characters of $K$, then the ensuing log-free zero density estimate will not have the $K$-uniformity which is necessary to prove Theorem 3.1.

Turán developed an alternative formulation of log-free zero density estimates. The idea is to take high derivatives of $L'/L(s, \chi)$. This produces a large sum of complex numbers involving zeros of $L(s,\chi)$, which can be bounded below by the Turán power sum method (see Proposition 5.1). The integral of a certain zero-detecting polynomial (which is not defined in terms of the Möbius function) gives an upper bound for these high derivatives. Thus, when a certain zero-detecting polynomial (which is not defined in terms of the Möbius function) encounters a zero of $L(s,\chi)$, its integral will be bounded away from zero because of the lower bound given by the power sum method. The contributions from the detected zeros up to height $T$ are summed efficiently using a particular large sieve inequality (see Section 4).

The advantage of using the power sum method in our proofs lies in the fact that Turán's lower bound for power sums is a purely Diophantine result, independent of the number fields in our proofs; this allows for noticeably better field uniformity than the mollifier method. The disadvantage is that the lower bound in the power sum method is quite small, which, for example, would inflate the constant $\frac{12}{5}$ in (3-4). To our knowledge, the power sum method is the only tool available that produces a $K$-uniform log-free zero density estimate of the form (3-4) which is strong enough to deduce a conclusion as strong as Theorem 1.1. Limitations to the power sum method indicate a genuine obstacle to any substantive improvements in the constants in Theorem 1.1 when using these methods.

To prove the large sieve inequality (4-4) used in the proof of Theorem 3.2, we use bounds in Section 2 for certain sums over integral ideals, which require smoothing with a kernel that is $n_K$ times differentiable. Unfortunately, the smoothing introduces the powers of ${n_K}^{n_K}$ (see the comments immediately preceding [Weiss 1983, Section 1]). As mentioned after Theorem 1.1, the factor of ${n_K}^{n_K}$ is negligible if $n_K$ is small compared to $(\log D_K)/\log\log D_K$, which is expected to be the case in most applications.

**3D. *Zero repulsion.*** In Section 7, we prove an explicit variant of the zero repulsion phenomenon for Hecke $L$-functions, also known as the Deuring–Heilbronn phenomenon.

**Theorem 3.3.** *Let $H$ be a congruence class group of $K$. Let $\psi \pmod{H}$ be a real Hecke character and suppose $L(s,\psi)$ has a real zero $\beta_1$. Let $T \geq 1$ be arbitrary, $\chi \pmod{H}$ an arbitrary Hecke character, and $\rho' = \beta' + i\gamma'$ a zero of $L(s,\chi)$ satisfying $\frac{1}{2} \leq \beta' < 1$ and $|\gamma'| \leq T$. Then, for $\epsilon > 0$ arbitrary,*

$$\beta' \leq 1 - \frac{\log\left(\dfrac{c_\epsilon}{(1-\beta_1)\log(D_K \cdot Q \cdot T^{n_K} e^{O_\epsilon(n_K)})}\right)}{b_1 \log D_K + b_2 \log Q + b_3 n_K \log T + O_\epsilon(n_K)}$$

*for some absolute, effective constant $c_\epsilon > 0$ and*

$$(b_1, b_2, b_3) = \begin{cases} (48+\epsilon, 60+\epsilon, 24+\epsilon) & \text{if } \psi \text{ is quadratic}, \\ (24+\epsilon, 12+\epsilon, 12+\epsilon) & \text{if } \psi \text{ is trivial.} \end{cases}$$

**Remark.** Other versions of the zero repulsion phenomenon by Kadiri and Ng [2012] and Zaman [2016a] apply for an asymptotically smaller range of $\beta'$ and $|\gamma'| \le 1$.

In Section 8, we collect all existing results and our new theorems on the distribution of zeros of Hecke $L$-functions and package them into versions required for the proof of Theorem 3.1. The necessary explicit zero-free regions for Hecke $L$-functions have already been established in previous work of Zaman [2016a; 2017a], which improved on [Ahn and Kwon 2014; Kadiri 2012], and are valid in a certain neighborhood of $s = 1$. In Sections 9–11, we use Theorems 3.2 and 3.3, along with the aforementioned work of Zaman, to prove Theorem 3.1. In Section 12, we prove Theorems 1.2–1.5 using Theorem 1.1.

## 4. Mean values of Dirichlet polynomials

Gallagher [1970] proved the following mean value results for Dirichlet polynomials.

**Theorem.** *Let $\{a_n\}$ be a sequence of complex numbers such that $\sum_{n\ge 1} n|a_n|^2 < \infty$.*

(1) *If $T \ge 1$, then*

$$\sum_{\chi \pmod q} \int_{-T}^{T} \Big|\sum_{n=1}^{\infty} a_n \chi(n) n^{it}\Big|^2 \, dt \ll \sum_{n=1}^{\infty} (qT + n)|a_n|^2. \tag{4-1}$$

(2) *Let $R \ge 2$, and assume $a_n = 0$ if $n$ has any prime factor less than $R$. If $T \ge 1$, then*

$$\sum_{q\le R} \log\frac{R}{q} \sideset{}{^*}\sum_{\chi \pmod q} \int_{-T}^{T} \Big|\sum_{n=1}^{\infty} a_n \chi(n) n^{it}\Big|^2 \, dt \ll \sum_{n=1}^{\infty} (R^2 T + n)|a_n|^2. \tag{4-2}$$

*Here, $\sum^*$ denotes the restriction to primitive characters $\chi \pmod q$.*

In (4-2), the $\log(R/q)$ weighting on the left-hand side (which arises from the support of $a_n$) turns out to be decisive in some applications, such as the proof of (1-2). To prove Theorem 3.2, we need a $K$-uniform analogue of (4-1) when $a_n$ is supported as in (4-2). Weiss used the Selberg sieve to prove such a result in his Ph.D. thesis [1980, Theorem 3′, p. 98].

**Theorem** (Weiss). *Let $b(\cdot)$ be a complex-valued function on the integral ideals $\mathfrak{n}$ of $K$, and suppose that $\sum_{\mathfrak{n}} (\mathrm{N}\mathfrak{n})|b(\mathfrak{n})|^2 < \infty$. Let $T \gg 1$. Suppose that $b(\mathfrak{n}) = 0$ when $\mathfrak{n}$ has a prime ideal factor $\mathfrak{p}$ with $\mathrm{N}\mathfrak{p} \le z$, and define $V(z) = \sum_{\mathrm{N}\mathfrak{n}\le z} \mathrm{N}\mathfrak{n}^{-1}$. If $0 < \epsilon < \frac{1}{2}$, then*

$$\sum_{\chi(H)=1} \int_{-T}^{T} \Big|\sum_{\mathfrak{n}} b(\mathfrak{n})\chi(\mathfrak{n})\mathrm{N}\mathfrak{n}^{-it}\Big|^2 \, dt \ll \sum_{\mathfrak{n}} |b(\mathfrak{n})|^2 \Big(\frac{\kappa_K}{V(z)}\mathrm{N}\mathfrak{n} + c(\epsilon)(n_K^{n_K} D_K Q T^{n_K} z^4)^{1/2+\epsilon} h_H T\Big)$$

*for some constant $c(\epsilon) > 0$ depending only on $\epsilon$.*

**Remark.** Assuming the Lindelöf hypothesis for Hecke $L$-functions, the upper bound improves to

$$\ll \sum_{\mathfrak{n}} |b(\mathfrak{n})|^2 \Big( \frac{\kappa_K}{V(z)} \mathrm{N}\mathfrak{n} + c(\epsilon)(D_K Q)^{\epsilon} h_H T^{1+\epsilon n_K} z^{2+\epsilon} \Big).$$

This appears to be optimal when using the Selberg sieve, considering that when $K = \mathbb{Q}$, the second term is roughly $(qTz^2)^{1+\epsilon}$. For related unconditional results, see [Duke 1989, Section 1].

This result is interesting in its own right, but to make the result more practical for the applications at hand, Weiss chose $b(\mathfrak{n})$ to be supported on the prime ideals $\mathfrak{p}$ such that $y < \mathrm{N}\mathfrak{p} \le y^{c_9}$. Then, Weiss set $z = y^{1/3}$ and chose $\log y \ge c_{10} \log(D_K Q T^{n_K})$ and $\epsilon = \frac{1}{3}$. By Corollary 2.9 and taking $c_9$ and $c_{10}$ to be sufficiently large, Weiss' result reduces to

$$\sum_{\chi(H)=1} \int_{-T}^{T} \Big| \sum_{y < \mathrm{N}\mathfrak{p} \le y^{c_9}} b(\mathfrak{p}) \chi(\mathfrak{n}) \mathrm{N}\mathfrak{n}^{-it} \Big|^2 dt \ll \frac{1}{\log y} \sum_{y < \mathfrak{p} \le y^{c_9}} |b(\mathfrak{p})|^2 \mathrm{N}\mathfrak{p}.$$

Weiss [1983, Corollary 3.8] recast this estimate with more generality.

**Corollary 4.1** (Weiss). *Let $b(\cdot)$ be a complex-valued function on the prime ideals $\mathfrak{p}$ of $K$ such that $\sum_{\mathfrak{p}} (\mathrm{N}\mathfrak{p}) |b(\mathfrak{p})|^2 < \infty$ and $b(\mathfrak{p}) = 0$ whenever $\mathrm{N}\mathfrak{p} \le y$. Let $H$ be a primitive congruence class group of $K$. If $y \ge (h_H n_K^{2n_K} D_K Q T^{2n_K})^8$, then*

$$\sum_{\chi(H)=1} \int_{-T}^{T} \Big| \sum_{\mathfrak{p}} b(\mathfrak{p}) \chi(\mathfrak{n}) \mathrm{N}\mathfrak{n}^{-it} \Big|^2 dt \ll \frac{1}{\log y} \sum_{\mathfrak{p}} |b(\mathfrak{p})|^2 \mathrm{N}\mathfrak{p}.$$

The exponent 8 in the range of $y$ in Corollary 4.1 is large enough to influence the value of $c_6$ in (3-6), which affects $c_3$ and $c_4$ in (1-7). In this section, we improve Corollary 4.1 so that it does not influence the exponents in Theorem 3.2.

**Theorem 4.2.** *Let $\upsilon \ge \epsilon > 0$ be arbitrary. Let $b(\cdot)$ be a complex-valued function on the prime ideals $\mathfrak{p}$ of $K$ such that $\sum_{\mathfrak{p}} (\mathrm{N}\mathfrak{p}) |b(\mathfrak{p})|^2 < \infty$ and $b(\mathfrak{p}) = 0$ whenever $\mathrm{N}\mathfrak{p} \le y$. Let $H$ be a primitive congruence class group of $K$. If $T \ge 1$ and*

$$y \ge C_\epsilon \{ h_H n_K^{(5/4+\upsilon)n_K} D_K^{3/2+\upsilon} Q^{1/2} T^{n_K/2+1} \}^{1+\epsilon} \tag{4-3}$$

*for some sufficiently large $C_\epsilon > 0$, then*

$$\begin{aligned} \sum_{\chi \ (\mathrm{mod}\ H)} \int_{-T}^{T} \Big| \sum_{\mathfrak{p}} b(\mathfrak{p}) \chi(\mathfrak{p}) \mathrm{N}\mathfrak{p}^{-it} \Big|^2 dt \\ \le \bigg( \frac{5\pi \{1 - \frac{1}{1+\upsilon}\}^{-1}}{\frac{1}{1+\epsilon} \log\big(\frac{y}{h_H}\big) - \mathcal{L}'} + O_\epsilon(y^{-\epsilon/2}) \bigg) \sum_{\mathfrak{p}} \mathrm{N}\mathfrak{p} |b(\mathfrak{p})|^2, \end{aligned} \tag{4-4}$$

*where $\mathcal{L}' = \frac{1}{2}\log D_K + \frac{1}{2}\log Q + \frac{1}{4} n_K \log n_K + \big(\frac{1}{2} n_K + 1\big) \log T + O_\epsilon(1)$.*

**Remark.** Taking $\upsilon = \epsilon$ and using Lemma 2.11, we improve the range of $y$ in Corollary 4.1 to

$$y \gg e^{O_\epsilon(n_K)}\{n_K^{5/4n_K} D_K^2 Q^{3/2} T^{n_K/2+1}\}^{1+\epsilon}.$$

**4A.** ***Preparing for the Selberg sieve.*** To apply the Selberg sieve, we require several weighted estimates involving Hecke characters. Before we begin, we highlight the necessary properties of our weight $\Psi$.

**Lemma 4.3.** *For $T \geq 1$, let $A = T\sqrt{2n_K}$. Define*

$$\widehat{\Psi}(s) = \left[\frac{\sinh(s/A)}{s/A}\right]^{2n_K}$$

*and let*

$$\Psi(x) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \widehat{\Psi}(s)x^{-s}\,ds$$

*be the inverse Mellin transform of $\widehat{\Psi}(s)$. Then*:

(i) $0 \leq \Psi(x) \leq A/2$ *and $\Psi(x)$ is a compactly supported function vanishing outside the interval* $e^{-2n_K/A} \leq x \leq e^{2n_K/A}$.

(ii) $\widehat{\Psi}(s)$ *is an entire function.*

(iii) *For all complex $s = \sigma + it$,* $|\widehat{\Psi}(s)| \leq (A/|s|)^{2n_K} e^{|\sigma|/A}$.

(iv) *For* $|s| \leq A$, $|\widehat{\Psi}(s)| \leq (1 + |s|^2/(5A^2))^{2n_K}$.

(v) *Uniformly for* $|\sigma| \leq A/\sqrt{2n_K}$, $|\widehat{\Psi}(s)| \ll 1$.

(vi) *Let $\{b_m\}_{m\geq 1}$ be a sequence of complex numbers with $\sum_m |b_m| < \infty$. Then*

$$\int_{-T}^{T}\Big|\sum_m b_m m^{-it}\Big|^2\,dt \leq \frac{5\pi}{2}\int_0^\infty \Big|\sum_m b_m \Psi\Big(\frac{x}{m}\Big)\Big|^2 \frac{dx}{x}.$$

*Proof.* For (i)–(v), see [Weiss 1983, Lemma 3.2]; in his notation, $\Psi(x) = H_{2n_K}(x)$ with parameter $A = T\sqrt{2n_K}$. Statement (vi) follows easily from the proof of [Weiss 1983, Corollary 3.3]. □

For the remainder of this section, assume:

- $H \pmod{\mathfrak{q}}$ is an arbitrary *primitive* congruence class group of $K$.
- $0 < \epsilon < \frac{1}{2}$ and $T \geq 1$ is arbitrary.
- $\Psi$ is the weight function of Lemma 4.3.

Next, we establish improved analogues of [Weiss 1983, Lemmas 3.4 and 3.6 and Corollary 3.5].

**Lemma 4.4.** *Let $\chi \pmod{H}$ be a Hecke character. For $x > 0$,*

$$\Big|\sum_{\mathfrak{n}} \frac{\chi(\mathfrak{n})}{\mathrm{N}\mathfrak{n}} \cdot \Psi\Big(\frac{x}{\mathrm{N}\mathfrak{n}}\Big) - \delta(\chi)\frac{\varphi(\mathfrak{q})}{\mathrm{N}\mathfrak{q}}\kappa_K\Big| \ll_\epsilon \{n_K^{n_K/4} D_K^{1/2} Q^{1/2} T^{n_K/2+1}\}^{1+\epsilon}.$$

*Proof.* The quantity we wish to bound equals

$$\frac{1}{2\pi i}\int_{-1-i\infty}^{-1+i\infty} L(s+1,\chi)\widehat{\Psi}(s)x^s\, ds. \tag{4-5}$$

If $\chi \pmod{\mathfrak{q}}$ is induced by the primitive character $\chi^* \pmod{\mathfrak{f}_\chi}$, then

$$L(s,\chi) = L(s,\chi^*) \prod_{\mathfrak{p}|\mathfrak{q},\ \mathfrak{p}\nmid\mathfrak{f}_\chi} (1-\chi^*(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-s}).$$

Thus $|L(it,\chi)| \le 2^{\omega(\mathfrak{q})}|L(it,\chi^*)|$, where $\omega(\mathfrak{q})$ is the number of distinct prime ideal divisors of $\mathfrak{q}$. Since $H \pmod{\mathfrak{q}}$ is primitive, $\omega(\mathfrak{q}) \le 6e^{4/\epsilon}n_K + \frac{\epsilon}{2}\log(D_K Q)$ by [Weiss 1983, Lemma 1.13]. So, for $\mathrm{Re}\{s\} = -1$,

$$|L(s+1,\chi)| \ll e^{O_\epsilon(n_K)}(D_K Q)^{\epsilon/2}|L(s+1,\chi^*)|.$$

Thus, by Lemma 2.3, (4-5) is

$$\ll e^{O_\epsilon(n_K)}(D_K Q)^{1/2+\epsilon}x^{-1}\int_0^\infty (1+|t|)^{(1/2+\epsilon)n_K}|\widehat{\Psi}(-1+it)|\, dt$$

as $D_\chi \le D_K Q$. By Lemma 4.3(iii) and (iv), this integral is

$$\ll \int_0^{A/2}(1+|t|)^{(1/2+\epsilon)n_K}|\widehat{\Psi}(-1+it)|\, dt + \int_{A/2}^\infty (1+|t|)^{(1/2+\epsilon)n_K}|\widehat{\Psi}(-1+it)|\, dt,$$

which is $\ll e^{O(n_K)}A^{(1/2+\epsilon)n_K+1}$. Collecting the above estimates, the claimed bound, up to a factor of $\epsilon$, follows upon recalling $A = T\sqrt{2n_K}$ and noting $e^{O(n_K)} \ll_\epsilon (n_K^{n_K})^\epsilon$. □

**Corollary 4.5.** *Let $C$ be a coset of $H$, and let $\mathfrak{d}$ be an integral ideal coprime to $\mathfrak{q}$. For all $x > 0$, we have*

$$\Big|\sum_{\mathfrak{n}\in C,\ \mathfrak{d}|\mathfrak{n}} \frac{1}{\mathrm{N}\mathfrak{n}}\Psi\Big(\frac{x}{\mathrm{N}\mathfrak{n}}\Big) - \frac{\varphi(\mathfrak{q})}{\mathrm{N}\mathfrak{q}}\frac{\kappa_K}{h_H}\cdot\frac{1}{\mathrm{N}\mathfrak{d}}\Big| \ll_\epsilon \{n_K^{n_K/4} D_K^{1/2} Q^{1/2} T^{n_K/2+1}\}^{1+\epsilon}\cdot\frac{1}{x}.$$

*Proof.* The proof is essentially the same as that of [Weiss 1983, Corollary 3.5], except for the fact that we have an improved bound in Lemma 4.4. □

We now apply the Selberg sieve. For $z \ge 1$, define

$$S_z = \{\mathfrak{n} : \mathfrak{p} \mid \mathfrak{n} \Rightarrow \mathrm{N}\mathfrak{p} > z\} \quad \text{and} \quad V(z) = \sum_{\mathrm{N}\mathfrak{n}\le z}\frac{1}{\mathrm{N}\mathfrak{n}}. \tag{4-6}$$

**Lemma 4.6.** *Let $C$ be a coset of $H$. For $x > 0$ and $z \geq 1$,*

$$\sum_{\mathfrak{n}\in C\cap S_z} \frac{1}{\mathrm{N}\mathfrak{n}} \Psi\Big(\frac{x}{\mathrm{N}\mathfrak{n}}\Big) \leq \frac{\kappa_K}{h_H V(z)} + O_\epsilon\bigg(\frac{\{n_K^{n_K/4} D_K^{1/2} Q^{1/2} T^{n_K/2+1}\}^{1+\epsilon} z^{2+2\epsilon}}{x}\bigg).$$

*Proof.* The proof is essentially the same as that of [Weiss 1983, Lemma 3.6], except for the fact that we have an improved bound in Lemma 4.4. □

**4B.** ***Proof of Theorem 4.2.*** Let $z$ be a parameter satisfying $1 \leq z \leq y$, which we will specify later. Extend $b(\mathfrak{n})$ to all integral ideals $\mathfrak{n}$ of $K$ by zero. Applying Lemma 4.3 and writing

$$b_m = \sum_{\mathrm{N}\mathfrak{n}=m} b(\mathfrak{n})\chi(\mathfrak{n}),$$

for each Hecke character $\chi \pmod H$, it follows that

$$\sum_{\chi \pmod H} \int_{-T}^{T} \Big|\sum_{\mathfrak{n}} b(\mathfrak{n})\chi(\mathfrak{n})\mathrm{N}\mathfrak{n}^{-it}\Big|^2 dt \leq \frac{5\pi}{2}\int_0^\infty \sum_{\chi \pmod H} \Big|\sum_{\mathfrak{n}} b(\mathfrak{n})\chi(\mathfrak{n})\Psi\Big(\frac{x}{\mathrm{N}\mathfrak{n}}\Big)\Big|^2 \frac{dx}{x}. \tag{4-7}$$

By the orthogonality of characters and the Cauchy–Schwarz inequality,

$$\sum_{\chi \pmod H} \Big|\sum_{\mathfrak{n}} b(\mathfrak{n})\chi(\mathfrak{n})\Psi\Big(\frac{x}{\mathrm{N}\mathfrak{n}}\Big)\Big|^2 \leq h_H \sum_{C\in I(\mathfrak{q})/H} \Big(\sum_{\mathfrak{n}\in C} \mathrm{N}\mathfrak{n}|b(\mathfrak{n})|^2\Psi\Big(\frac{x}{\mathrm{N}\mathfrak{n}}\Big)\Big) \sum_{\mathfrak{n}\in C\cap S_z} \frac{\Psi(x/\mathrm{N}\mathfrak{n})}{\mathrm{N}\mathfrak{n}}$$

since $z \leq y$ and $b(\mathfrak{n})$ is supported on prime ideals with norm greater than $y$. For $\delta = \delta(\epsilon) > 0$ sufficiently small and $B_\delta > 0$ sufficiently large, denote

$$M'_\delta = M_\delta z^{2+2\delta} \quad \text{and} \quad M_\delta = B_\delta\{n_K^{n_K/4} D_K^{1/2} Q^{1/2} T^{n_K/2+1}\}^{1+\delta}.$$

By Lemma 4.6, the right-hand side of the preceding inequality is therefore at most

$$\sum_{C\in I(\mathfrak{q})/H} \sum_{\mathfrak{n}\in C} \mathrm{N}\mathfrak{n}|b(\mathfrak{n})|^2\Psi\Big(\frac{x}{\mathrm{N}\mathfrak{n}}\Big)\Big(\frac{\kappa_K}{V(z)} + \frac{h_H M'_\delta}{x}\Big) \leq \sum_{\mathfrak{n}} \mathrm{N}\mathfrak{n}|b(\mathfrak{n})|^2\Psi\Big(\frac{x}{\mathrm{N}\mathfrak{n}}\Big)\Big(\frac{\kappa_K}{V(z)} + \frac{h_H M'_\delta}{x}\Big),$$

By Lemma 4.3(v), if we insert the above estimates into (4-7), then we obtain the bound

$$\sum_{\chi \pmod H} \int_{-T}^{T} \Big|\sum_{\mathfrak{p}} b(\mathfrak{p})\chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-it}\Big|^2 dt$$
$$\le \frac{5\pi}{2} \sum_{\mathfrak{n}} \mathrm{N}\mathfrak{n}|b(\mathfrak{n})|^2 \Big( \frac{\kappa_K}{V(z)} \int_0^\infty \Psi\Big(\frac{x}{\mathrm{N}\mathfrak{n}}\Big) \frac{dx}{x} + h_H M'_\delta \int_0^\infty \frac{1}{x} \Psi\Big(\frac{x}{\mathrm{N}\mathfrak{n}}\Big) \frac{dx}{x} \Big)$$
$$\le \frac{5\pi}{2} \sum_{\mathfrak{n}} \mathrm{N}\mathfrak{n}|b(\mathfrak{n})|^2 \Big( \frac{\kappa_K}{V(z)} |\widehat{\Psi}(0)| + \frac{h_H M'_\delta}{\mathrm{N}\mathfrak{n}} |\widehat{\Psi}(1)| \Big).$$

Since $b(\mathfrak{n})$ is supported on prime ideals whose norm is greater than $y$, the last line of the previous display is

$$\le \frac{5\pi}{2} \Big( \frac{\kappa_K}{V(z)} + O(h_H M_\delta z^{2+2\delta} y^{-1}) \Big) \sum_{\mathfrak{p}} \mathrm{N}\mathfrak{p}|b(\mathfrak{p})|^2.$$

Now, select $z$ satisfying

$$z = \Big( \frac{y^{(1+\delta)/(1+\epsilon)}}{h_H M_\delta} \Big)^{1/(2+2\delta)}, \tag{4-8}$$

so $1 \le z \le y$ and hence

$$\sum_{\chi \pmod H} \int_{-T}^{T} \Big|\sum_{\mathfrak{p}} b(\mathfrak{p})\chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-it}\Big|^2 dt \le \frac{5\pi}{2} \Big( \frac{\kappa_K}{V(z)} + O_\epsilon(y^{-\epsilon/2}) \Big) \sum_{\mathfrak{p}} \mathrm{N}\mathfrak{p}|b(\mathfrak{p})|^2 \tag{4-9}$$

for $\delta = \delta(\epsilon) > 0$ sufficiently small. If $C_\epsilon$ in (4-3) is sufficiently large, then (4-3) and (4-8) imply $z \ge 3(n_K^{n_K} D_K)^{1/2+\upsilon/2}$. Applying Corollary 2.9 to (4-9), it follows that

$$\sum_{\chi \pmod H} \int_{-T}^{T} \Big|\sum_{\mathfrak{p}} b(\mathfrak{p})\chi(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-it}\Big|^2 dt \le \Big( \frac{5\pi\upsilon}{2\{1+\upsilon\}\log z + O_\epsilon(1)} + O_\epsilon(y^{-\epsilon/2}) \Big) \sum_{\mathfrak{p}} \mathrm{N}\mathfrak{p}|b(\mathfrak{p})|^2$$

since $\upsilon \ge \epsilon > 0$. Finally, by (4-3) and (4-8),

$$2\log z \ge \frac{1}{1+\epsilon} \log\Big(\frac{y}{h_H}\Big) - \frac{1}{2}\Big\{ \log D_K + \log Q + \frac{1}{2} n_K \log n_K + (n_K+2)\log T + O_\epsilon(1) \Big\}.$$

Putting this estimate into the previous inequality gives the conclusion. □

## 5. Detecting the zeros of Hecke $L$-functions

**5A. *Notation.*** We first specify some additional notation to be used throughout this section.

*Arbitrary quantities.*

- Let $H \pmod{\mathfrak{q}}$ be a primitive congruence class group.
- Let $\epsilon \in \left(0, \frac{1}{8}\right)$ and $\phi = 1 + \frac{4}{\pi}\epsilon + 16\epsilon^2 + 340\epsilon^{10}$.
- Let $T \geq 1$. Define $Q = Q_H$ and

$$\mathcal{L} = \mathcal{L}_{T,\epsilon} := \log D_K + \tfrac{1}{2}\log Q + \left(\tfrac{1}{2}n_K + 1\right)\log(T+3) + \Theta n_K, \tag{5-1}$$

  where $\Theta = \Theta(\epsilon) \geq 1$ is sufficiently large depending on $\epsilon$.
- Let $\lambda_0 > \frac{1}{20}$. Suppose $\tau \in \mathbb{R}$ and $\lambda > 0$ satisfy

$$\lambda_0 \leq \lambda \leq \tfrac{1}{16}\mathcal{L} \quad \text{and} \quad |\tau| \leq T. \tag{5-2}$$

  Furthermore, let $r = \frac{\lambda}{\mathcal{L}}$.

*Fixed quantities.*

- Let $\alpha, \eta, \omega \in (0,1)$ be fixed.
- Define $A \geq 1$, so that $A_1 = \sqrt{A^2+1}$ satisfies

$$A_1 = 2\left(4e\left(1 + \frac{1}{\alpha}\right)\right)^{\alpha}(1+\eta). \tag{5-3}$$

- Let $x = e^{X\mathcal{L}}$ and $y = e^{Y\mathcal{L}}$ with $X, Y > 0$ given by

$$\begin{aligned} Y = Y_\lambda &= \frac{1}{eA_1} \cdot \frac{1}{\alpha}\left\{2\phi A + \frac{8}{\lambda}\right\}, \\ X = X_\lambda &= \frac{2\log\left(\frac{2A_1}{1-\omega}\right)}{(1-\omega)} \cdot \frac{1+\alpha}{\alpha}\left\{2\phi A + \frac{8}{\lambda}\right\}, \end{aligned} \tag{5-4}$$

  and $\alpha, \eta, \omega$ chosen so that $2 < Y < X$. Notice $X = X_\lambda$ and $Y = Y_\lambda$ depend on the arbitrary quantities $\epsilon$ and $\lambda$, but they are uniformly bounded above and below in terms of $\alpha$, $\eta$, and $\omega$, i.e., $X \asymp 1$ and $Y \asymp 1$. For this reason, while $X$ and $Y$ are technically not fixed quantities, they may be treated as such.

**5B. *Statement of results.***

*Detecting zeros.* The first goal of this section is to prove the following proposition.

**Proposition 5.1.** *Let $\chi \pmod{H}$ be a Hecke character. Suppose $L(s,\chi)$ has a nontrivial zero $\rho$ satisfying*

$$|1 + i\tau - \rho| \leq r = \frac{\lambda}{\mathcal{L}}. \tag{5-5}$$

*Further assume*

$$J(\lambda) := \frac{W_1\lambda + W_2}{A_1(1+\eta)^{k_0}} < 1, \tag{5-6}$$

*where*

$$\begin{aligned}
X &= X_\lambda, \qquad Y = Y_\lambda,\\
k_0 &= k_0(\lambda) = \alpha^{-1}(2\phi A\lambda + 8),\\
W_1 &= W_1(\lambda) = 8A_1\big(1+\tfrac{1}{k_0}\big) + 2eA_1\big(Y + \tfrac{1}{2} + \{2X+1\}e^{-\omega\lambda X}\big) + O(\epsilon),\\
W_2 &= W_2(\lambda) = 2e\omega^{-1}A_1 e^{-\omega\lambda X} + 18 + O(\epsilon).
\end{aligned}$$

*If* $\lambda < \frac{\epsilon}{A_1}\mathcal{L}$ *and* $2 < Y < X$, *then*

$$r^4 \log\Big(\frac{x}{y}\Big) \int_y^x \Big| \sum_{y \le \mathrm{N}\mathfrak{p} < u} \frac{\chi(\mathfrak{p}) \log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1+i\tau}} \Big|^2 \frac{du}{u} + \delta(\chi)\mathbf{1}_{\{|\tau|<Ar\}}(\tau) \ge \Big(\frac{\alpha/(1+\alpha)}{8e2^{1/\alpha}}\Big)^{4\phi A\lambda+16} \frac{(1-J(\lambda))^2}{4}.$$

**Remark.** Note that $W_j(\lambda) \ll 1$ for $j = 1, 2$.

The proof of Proposition 5.1 is divided into two main steps, with the final arguments culminating in Section 5E. The method critically hinges on the following power sum estimate due to Kolesnik and Straus.

**Theorem 5.2** [Kolesnik and Straus 1983]. *For any integer* $M \ge 0$ *and complex numbers* $z_1, \dots, z_N$, *there is an integer* $k$ *with* $M+1 \le k \le M+N$ *such that*

$$|z_1^k + \cdots + z_N^k| \ge 1.007\Big(\frac{N}{4e(M+N)}\Big)^N |z_1|^k.$$

Makai [1964] showed that the constant $4e$ is essentially optimal.

*Explicit zero density estimate.* Using Theorem 4.2 and Proposition 5.1, the second and primary goal of this section is to establish an explicit log-free zero density estimate. Recall, for a Hecke character $\chi$,

$$N(\sigma, T, \chi) = \#\{\rho : L(\rho,\chi) = 0,\ \sigma < \mathrm{Re}\{\rho\} < 1,\ |\mathrm{Im}(\rho)| \le T\}, \tag{5-7}$$

where $\sigma \in (0,1)$ and $T \ge 1$.

**Theorem 5.3.** *Let* $\xi \in (1,\infty)$ *and* $\upsilon \in \big(0, \frac{1}{10}\big]$ *be fixed and set* $\sigma = 1 - \frac{\lambda}{\mathcal{L}}$. *Suppose*

$$\lambda_0 \le \lambda < \frac{\epsilon}{\xi A_1}\mathcal{L}, \qquad X > Y > 4.6, \qquad \text{and} \quad T \ge \max\{n_K^{5/6}\big(D_K^{4/3}Q^{4/9}\big)^{-1/n_K}, 1\}, \tag{5-8}$$

*where* $X = X_{\xi\lambda}$ *and* $Y = Y_{\xi\lambda}$. *Then*

$$\sum_{\chi \ (\mathrm{mod}\ H)} N(\sigma, T, \chi) \le \frac{4\xi}{\sqrt{\xi^2-1}} \cdot (C_4\lambda^4 + C_3\lambda^3 + C_1\lambda + C_0)e^{B_1\lambda + B_2} \cdot \{1 - J(\xi\lambda)\}^{-2},$$

*where $J(\cdot)$ is defined by (5-6) satisfying $J(\xi\lambda) < 1$, and*

$$\begin{aligned} B_1 &= 4\phi A\xi \log(4e\alpha^{-1}(1+\alpha)2^{(1+\alpha)/\alpha}), \\ B_2 &= 16\log(4e\alpha^{-1}(1+\alpha)2^{(1+\alpha)/\alpha}), \\ C_4 &= \frac{5\pi e\phi X(X-Y)^2(X+Y+1+\epsilon)\xi^4}{(1-\frac{1}{1+\upsilon})\big(\frac{1}{1+\epsilon}Y-4\big)}, \\ C_3 &= \frac{4}{\phi\xi}C_4, \quad C_1 = 4\phi A\xi, \quad C_0 = 16A+\epsilon. \end{aligned} \tag{5-9}$$

**Remark.**

- In Sections 6 and 8E, we will employ Theorem 5.3 with various choices of parameters $\alpha$, $\eta$, $\upsilon$, $\epsilon$, $\omega$, and $\xi$ depending on the range of $\sigma$. Consequently, this result is written without any explicit choice of the fixed or arbitrary quantities found in Section 5A.
- The quantities $C_4$ and $C_3$ are technically not constants with respect to $\lambda$ or $\epsilon$, but one can see that both are bounded absolutely according to the definitions in Section 5A.

Sections 5C and 5D are dedicated to preparing for the proof of Proposition 5.1 which is contained in Section 5E. The proof of Theorem 5.3 is finalized in Section 5F.

**5C.** ***A large derivative.*** Suppose $\chi \pmod H$ is induced from the primitive character $\chi^*$. Define $F(s) := \frac{L'}{L}(s,\chi^*)$ and $z := 1+r+i\tau$. Using Theorem 5.2, the goal of this subsection is to show $F(s)$ has a large high-order derivative, which we establish in the following lemma.

**Lemma 5.4.** *Keep the above notation and suppose $L(s,\chi)$ has a zero $\rho$ satisfying (5-5). If $\lambda < \frac{\epsilon}{A_1}\mathcal{L}$ and $\mathbf{1}_S$ is the indicator function of a set $S$, then*

$$\delta(\chi)\mathbf{1}_{\{|\tau|<Ar\}}(\tau) + \left|\frac{r^{k+1}}{k!}F^{(k)}(z)\right| \\ \geq \frac{\big(\frac{\alpha}{4e(1+\alpha)}\big)^{2\phi A\lambda+8}}{2^{k+1}}\left\{1 - \frac{\{8\big(1+\frac{1}{k}\big)A_1 + O(\epsilon)\}\lambda + 18}{A_1(1+\eta)^k}\right\}$$

*for some integer $k$ in the range $\frac{1}{\alpha}\cdot(2\phi A\lambda+8) \le k \le \frac{1+\alpha}{\alpha}\cdot(2\phi A\lambda+8)$.*

*Proof.* By [Weiss 1983, Lemma 1.10],

$$F(s) + \frac{\delta(\chi)}{s-1} = \sum_{|1+i\tau-\rho|<1/2} \frac{1}{s-\rho} + G(s)$$

uniformly in the region $|1+i\tau-s| < \frac{1}{2}$, where $G(s)$ is analytic and $|G(s)| \ll \mathcal{L}$ in this region. Differentiating the above formula $k$ times and evaluating at $z = 1+r+i\tau$, we deduce

$$\frac{(-1)^k}{k!}\cdot F^{(k)}(z)+\frac{\delta(\chi)}{(z-1)^{k+1}}=\sum_{|1+i\tau-\rho|<1/2}\frac{1}{(z-\rho)^{k+1}}+O(4^k\mathcal{L})$$

since $r=\frac{\lambda}{\mathcal{L}}<\frac{1}{16}$ by assumption (5-2). The error term arises from bounding $G^{(k)}(z)$ using Cauchy's integral formula with a circle of radius $\frac{1}{4}$. For zeros $\rho$ that satisfy $Ar<|1+i\tau-\rho|<\frac{1}{2}$, notice

$$(A^2+1)r^2<r^2+|1+i\tau-\rho|^2\le|z-\rho|^2\le(r+|1+i\tau-\rho|)^2\le\left(r+\tfrac{1}{2}\right)^2<1.$$

Recalling $A_1=\sqrt{A^2+1}$, it follows by partial summation that

$$\begin{aligned}\sum_{Ar<|1+i\tau-\rho|<1/2}\frac{1}{|z-\rho|^{k+1}}&\le\int_{A_1r}^{1}u^{-k-1}\,dN_\chi(u;z)\\&=(k+1)\int_{A_1r}^{1}\frac{N_\chi(u;z)}{u^{k+2}}\,du+O(\mathcal{L}),\end{aligned}$$

where we bounded $N_\chi(1;z)\ll\mathcal{L}$ using [Lagarias et al. 1979, Lemma 2.2]. By Lemma 2.5, the above is therefore

$$\le(k+1)\int_{A_1r}^{\infty}\frac{4u\mathcal{L}+8}{u^{k+2}}\,du+O(\mathcal{L})\le\frac{4\{1+\frac{1}{k}\}A_1r\mathcal{L}+8}{(A_1r)^{k+1}}+O(\mathcal{L}).$$

By considering cases, one may bound the $\delta(\chi)$-term as follows:

$$r^{k+1}\cdot\left|\frac{\delta(\chi)}{(z-1)^{k+1}}\right|\le\delta(\chi)\cdot\mathbf{1}_{\{|\tau|<Ar\}}(\tau)+\frac{1}{A_1^{k+1}}.\tag{5-10}$$

The above results now yield

$$\begin{aligned}&\delta(\chi)\mathbf{1}_{\{|\tau|<Ar\}}(\tau)+\left|\frac{r^{k+1}F^{(k)}(z)}{k!}\right|\\&\quad\ge\left|\sum_{|1+i\tau-\rho|\le Ar}\frac{r^{k+1}}{(z-\rho)^{k+1}}\right|-\left[\frac{4\{1+\frac{1}{k}\}A_1r\mathcal{L}+9}{A_1^{k+1}}+O((4r)^{k+1}\mathcal{L})\right].\end{aligned}\tag{5-11}$$

To bound the remaining sum over zeros from below, we wish to apply Theorem 5.2. Let

$$N=N_\chi(Ar;1+i\tau)=\#\{\rho:L(\rho,\chi)=0,\ |1+i\tau-\rho|\le Ar\}.$$

Since $\lambda<\frac{\epsilon}{A_1}\mathcal{L}<\frac{\epsilon}{A}\mathcal{L}$ and $\epsilon<\frac{1}{8}$, Lemma 2.7 and (5-1) imply that $N\le 2\phi A\lambda+8$. Define $M:=\lfloor(2\phi A\lambda+8)/\alpha\rfloor$. Thus, from Theorem 5.2 and assumption (5-5),

$$\left|\sum_{|1+i\tau-\rho|\le Ar}\frac{1}{(z-\rho)^{k+1}}\right|\ge\left(\frac{\alpha}{4e(1+\alpha)}\right)^{2\phi A\lambda+8}\frac{1}{(2r)^{k+1}}\tag{5-12}$$

for some $M+1 \le k \le M+N$. To simplify the right-hand side of (5-11), observe that

$$(4r)^{k+1}\mathcal{L} \le 4\lambda(4r)^k \ll \lambda(4\epsilon)^k A_1^{-k} \ll \epsilon\lambda A_1^{-k}, \tag{5-13}$$

since

$$r = \frac{\lambda}{\mathcal{L}} < \frac{\epsilon}{A_1} < \frac{1}{4A_1}$$

by assumption. Moreover, our choice of $A_1$ in (5-3) implies

$$\begin{aligned} A_1^{-(k+1)} &= \Big(\frac{\alpha}{4e(1+\alpha)}\Big)^{\alpha k} \frac{1}{2^k} \cdot \frac{1}{A_1(1+\eta)^k} \\ &\le \Big(\frac{\alpha}{4e(1+\alpha)}\Big)^{2\phi A\lambda+8} \frac{1}{2^{k+1}} \cdot \frac{2}{A_1(1+\eta)^k}, \end{aligned} \tag{5-14}$$

since $\alpha k \ge \alpha(M+1) \ge 2\phi A\lambda + 8$. Incorporating (5-12)–(5-14) into (5-11) yields the desired result. The range of $k$ in Lemma 5.4 is determined by the above choice of $M$ and $N$. $\square$

**5D.** ***Short sum over prime ideals.*** Continuing with the discussion and notation of Section 5C, from the Euler product for $L(s,\chi^*)$, we have

$$F(s) = \frac{L'}{L}(s,\chi^*) = -\sum_{\mathfrak{n}} \chi^*(\mathfrak{n})\Lambda_K(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-s}$$

for $\mathrm{Re}\{s\} > 1$ and where $\Lambda_K(\,\cdot\,)$ is given by (2-17). Differentiating the above formula $k$ times, we deduce

$$\frac{(-1)^{k+1} r^{k+1}}{k!} \cdot F^{(k)}(z) = \sum_{\mathfrak{n}} \frac{\Lambda_K(\mathfrak{n})\chi^*(\mathfrak{n})}{\mathrm{N}\mathfrak{n}^{1+r+i\tau}} \cdot rE_k(r\log \mathrm{N}\mathfrak{n}) \tag{5-15}$$

for any integer $k \ge 1$, where $z = 1+r+i\tau$ and $E_k(u) = u^k/k!$. From Stirling's bound (see [DLM 2010]) in the form

$$k^k e^{-k}\sqrt{2\pi k} \le k! \le k^k e^{-k}\sqrt{2\pi k}\, e^{1/12k},$$

one can verify

$$E_k(u) \le \begin{cases} A_1^{-k} e^{u} & \text{if } u \le \dfrac{k}{eA_1}, \\ A_1^{-k} e^{(1-\omega)u} & \text{if } u \ge \dfrac{2}{1-\omega}\log\Big(\dfrac{2A_1}{1-\omega}\Big)k, \end{cases} \tag{5-16}$$

for any $k \ge 1$ and $A_1 > 1, \omega \in (0,1)$ defined in Section 5A. The goal of this subsection is to bound the infinite sum in (5-15) by an integral average of short sums over prime ideals.

**Lemma 5.5.** *Suppose the integer $k$ is in the range given in Lemma 5.4. If $\lambda < \frac{\epsilon}{A_1}\mathcal{L}$ then*

$$\Big|\sum_{\mathfrak{n}} \frac{\chi^*(\mathfrak{n})\Lambda_K(\mathfrak{n})}{\mathrm{N}\mathfrak{n}^{1+r+i\tau}} \cdot rE_k(r\log \mathrm{N}\mathfrak{n})\Big| \\ \leq r^2\int_y^x \Big|\sum_{y\leq \mathrm{N}\mathfrak{p}<u} \frac{\chi^*(\mathfrak{p})\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1+i\tau}}\Big|\frac{du}{u} \\ + \Big(e\Big[Y+\frac{1}{2}+\{2X+1\}e^{-\omega\lambda X}+O(\epsilon)\Big]\lambda+\frac{e^{1-\omega\lambda X}}{\omega}\Big)A_1^{-k},$$

*where $x = e^{X\mathcal{L}}$ and $y = e^{Y\mathcal{L}}$ with $X = X_\lambda$ and $Y = Y_\lambda$ defined by (5-4).*

*Proof.* First, divide the sum on the left-hand side into four sums:

$$\sum_{\mathfrak{n}} = \sum_{\mathrm{N}\mathfrak{p}<y} + \sum_{y\leq \mathrm{N}\mathfrak{p}<x} + \sum_{\mathrm{N}\mathfrak{p}\geq x} + \sum_{\mathfrak{n}\text{ not prime}} = S_1+S_2+S_3+S_4.$$

Observe that (5-4) and (5-16), along with the range of $k$ in Lemma 5.4, imply that

$$E_k(r\log \mathrm{N}\mathfrak{n}) \leq \begin{cases} A_1^{-k}(\mathrm{N}\mathfrak{n})^r & \text{if } \mathrm{N}\mathfrak{n}\leq y, \\ A_1^{-k}(\mathrm{N}\mathfrak{n})^{(1-\omega)r} & \text{if } \mathrm{N}\mathfrak{n}\geq x. \end{cases} \tag{5-17}$$

Hence, for $S_1$, it follows by Lemma 2.10 that

$$|S_1| \leq rA_1^{-k}\sum_{\mathrm{N}\mathfrak{p}<y}\frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}} \\ \leq rA_1^{-k}\cdot e\log(eD_K^{1/2}y) \leq e\Big(\lambda Y+\frac{\lambda}{2}+\epsilon\Big)A_1^{-k},$$

since $r = \frac{\lambda}{\mathcal{L}} < \epsilon$, $\log D_K \leq \mathcal{L}$, and $y = e^{Y\mathcal{L}}$. Similarly, for $S_3$, apply partial summation using Lemma 2.10 to deduce

$$|S_3| \leq rA_1^{-k}\sum_{\mathrm{N}\mathfrak{p}\geq x}\frac{\log \mathrm{N}\mathfrak{p}}{(\mathrm{N}\mathfrak{p})^{1+\omega r}} \\ \leq rA_1^{-k}\int_x^\infty \frac{\omega r e\log(eD_K^{1/2}t)}{t^{1+\omega r}}\,dt \leq \big(\{X+\tfrac{1}{2}\}\lambda+\omega^{-1}+\epsilon\big)\frac{e^{1-\omega\lambda X}}{A_1^k}.$$

For $S_4$, since $\frac{u^k}{k!} \leq e^u$ for $u > 0$, observe

$$E_k(r\log \mathrm{N}\mathfrak{n}) = \tfrac{1}{k!}(2r)^k\big(\tfrac{1}{2}\log \mathrm{N}\mathfrak{n}\big)^k \leq (2r)^k(\mathrm{N}\mathfrak{n})^{1/2}.$$

Thus, by Lemma 2.10,

$$
\begin{aligned}
|S_4| &\le r\sum_{\mathfrak{p}}\sum_{m\ge 2}\frac{\log \mathrm{N}\mathfrak{p}}{(\mathrm{N}\mathfrak{p}^m)^{1+r}}E_k(r\log \mathrm{N}\mathfrak{p}^m)\\
&\le (2r)^k r\sum_{\mathfrak{p}}\sum_{m\ge 2}\frac{\log \mathrm{N}\mathfrak{p}}{(\mathrm{N}\mathfrak{p}^m)^{1/2+r}}\\
&\ll (2r)^k r\sum_{\mathfrak{p}}\frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1+2r}} \ll \lambda\epsilon A_1^{-k},
\end{aligned}
$$

since $\log D_K \le \mathcal{L}$ and $\mathcal{L}^{-1} \ll r = \frac{\lambda}{\mathcal{L}} < \frac{\epsilon}{A_1}$. Also note that $\epsilon \in \left(0, \frac{1}{8}\right)$ implies $(2\epsilon)^k \ll \epsilon$. Finally, for the main term $S_2$, define

$$
W(u) = W_\chi(u;\tau) := \sum_{y\le \mathrm{N}\mathfrak{p}<u}\frac{\chi(\mathfrak{p})\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1+i\tau}},
$$

so by partial summation,

$$
S_2 = rW(x)x^{-r}E_k(r\log x) - r^2\int_y^x W(u)\,\frac{d}{dt}[e^{-t}E_k(t)]\Big|_{t=r\log u}\frac{du}{u} \tag{5-18}
$$

as $W(y)=0$. Similar to $S_1$, $S_3$, and $S_4$, from (5-17) and Lemma 2.10 it follows

$$
\begin{aligned}
|rW(x)x^{-r}E_k(r\log x)| &\le rA_1^{-k}x^{-\omega r}\sum_{y\le \mathrm{N}\mathfrak{p}<x}\frac{\Lambda_K(\mathfrak{n})}{\mathrm{N}\mathfrak{n}}\\
&\le e\left(\left\{X+\tfrac{1}{2}\right\}\lambda+\epsilon\right)e^{-\omega\lambda X}A_1^{-k}.
\end{aligned}
$$

Observe

$$
\left|\frac{d}{dt}(e^{-t}E_k(t))\right| = |e^{-t}E_{k-1}(t)-e^{-t}E_k(t)| \le e^{-t}[E_{k-1}(t)+E_k(t)] \le 1
$$

from the definition of $E_k(t)$ and since $\sum_{k=0}^{\infty}E_k(t)=e^t$. Hence,

$$
|S_2| \le r^2\int_y^x |W(u)|\,\frac{du}{u} + e\left(\left\{X+\frac{1}{2}\right\}\lambda+\epsilon\right)e^{-\omega\lambda X}A_1^{-k}.
$$

Collecting all of our estimates, we conclude the desired result as $\lambda \ge \lambda_0 \gg 1$. □

**5E.** ***Proof of Proposition 5.1.*** If $\delta(\chi)\mathbf{1}_{\{|\tau|<Ar\}}(\tau) = 1$, then the inequality in Proposition 5.1 holds trivially, as the right-hand side is certainly less than 1. Thus, we may assume otherwise.

Combining Lemmas 5.4 and 5.5 via (5-15), it follows that

$$
r^2\int_y^x\Big|\sum_{y\le \mathrm{N}\mathfrak{p}<u}\frac{\chi^*(\mathfrak{p})\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1+i\tau}}\Big|\,\frac{du}{u} \ge \left(\frac{\alpha}{4e(1+\alpha)}\right)^{2\phi A\lambda+8}\cdot\frac{1}{2^{k+1}}\{1-J(\lambda)\}, \tag{5-19}
$$

after bounding $A_1^{-k}$ as in (5-14) and noting $k \geq k_0$ in the range of Lemma 5.4. By assumption, $J(\lambda) < 1$ and hence the right-hand side of (5-19) is positive. Therefore, squaring both sides and applying Cauchy–Schwarz to the left-hand side gives

$$r^4 \log\Big(\frac{x}{y}\Big) \int_y^x \Big| \sum_{y \leq \mathrm{N}\mathfrak{p} < u} \frac{\chi^*(\mathfrak{p}) \log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1+i\tau}} \Big|^2 \frac{du}{u} \\ \geq \Big(\frac{\alpha}{4e(1+\alpha)}\Big)^{4\phi A\lambda + 16} \cdot \frac{1}{2^{2k+2}} \{1 - J(\lambda)\}^2.$$

By assumption, $y = e^{Y\mathcal{L}} > e^{2\mathcal{L}} \geq \mathrm{N}\mathfrak{f}_\chi$, so it follows that $\chi^*(\mathfrak{p}) = \chi(\mathfrak{p})$ for $y \leq \mathrm{N}\mathfrak{p} < x$. So we may replace $\chi^*$ with $\chi$ in the above sum over prime ideals. Finally, we note $k \leq \frac{1+\alpha}{\alpha}(2\phi A\lambda + 8)$ since $k$ is in the range of Lemma 5.4, yielding the desired result. □

**5F.** ***Proof of Theorem 5.3.*** For $\chi \pmod H$, consider zeros $\rho = \beta + i\gamma$ of $L(s,\chi)$ such that

$$1 - \frac{\lambda}{\mathcal{L}} \leq \beta < 1, \qquad |\gamma| \leq T. \tag{5-20}$$

Let $\lambda^\star = \xi\lambda$ and $r^\star = \frac{\lambda^\star}{\mathcal{L}} = \xi(1-\sigma)$, so by (5-8) we have $r^\star < \frac{\epsilon}{A_1}$. For any zero $\rho = \beta + i\gamma$ of $L(s,\chi)$, define $\Phi_{\rho,\chi}(\tau) := \mathbf{1}_{\{|1+i\tau-\rho| \leq r^\star\}}(\tau)$. If $\rho$ satisfies (5-20) then one can verify by elementary arguments that

$$\frac{1}{r^\star} \int_{-T}^{T} \Phi_{\rho,\chi}(\tau)\, d\tau \geq \frac{\sqrt{\xi^2 - 1}}{\xi}.$$

Applying Proposition 5.1 to such zeros $\rho$, it follows that

$$\int_{-T}^{T} \frac{1}{r^\star} \Phi_{\rho,\chi}(\tau) \\ \times \Big[ (r^\star)^4 \log\Big(\frac{x}{y}\Big) \int_y^x \Big| \sum_{y \leq \mathrm{N}\mathfrak{p} < u} \frac{\chi(\mathfrak{p}) \log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1+i\tau}} \Big|^2 \frac{du}{u} + \delta(\chi) \mathbf{1}_{\{|\tau| < Ar^\star\}}(\tau) \Big] d\tau \\ \geq \frac{\sqrt{\xi^2-1}}{4\xi} \Big(\frac{\alpha}{4e(1+\alpha)2^{(1+\alpha)/\alpha}}\Big)^{2\phi A\xi\lambda + 16} \{1 - J(\xi\lambda)\} =: w(\lambda).$$

Note $x = e^{X\mathcal{L}}$ and $y = e^{Y\mathcal{L}}$, where $X = X_{\lambda^\star}$ and $Y = Y_{\lambda^\star}$. Summing over all zeros $\rho$ of $L(s,\chi)$ satisfying (5-20), we have that

$$w(\lambda) N(\sigma, T, \chi) \\ \leq (X - Y)(2\phi r^\star \mathcal{L} + 8)(r^\star)^3 \mathcal{L} \int_y^x \Big( \int_{-T}^{T} \Big| \sum_{y \leq \mathrm{N}\mathfrak{p} < u} \frac{\chi(\mathfrak{p}) \log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{1+i\tau}} \Big|^2 d\tau \Big) \frac{du}{u} \\ + \delta(\chi)(4\phi A r^\star \mathcal{L} + 16A) \tag{5-21}$$

since, for $|\tau| \leq T$ and $r^\star < \epsilon$,

$$\sum_{\rho : L(\rho,\chi)=0} \Phi_{\rho,\chi}(\tau) = N_\chi(r^\star; 1+i\tau) \leq 2\phi r^\star \mathcal{L} + 8$$

by Lemma 2.7. From the conditions on $Y$ and $T$ in (5-8) and the definition of $\mathcal{L}$ in (5-1), observe that, for $\nu = \nu(\epsilon) > 0$ sufficiently small, Lemma 2.11 implies

$$y = e^{Y\mathcal{L}} \geq C_\nu \{h_H n_K^{(5/4+2\upsilon)n_K} D_K^{3/2+2\upsilon} Q^{1/2} T^{n_K/2+1}\}^{1+\nu}$$

since $\upsilon \leq \frac{1}{10}$ and $\Theta = \Theta(\epsilon) \geq 1$ is sufficiently large. Therefore, we may sum (5-21) over $\chi \pmod H$ and apply Theorem 4.2 with $b(\mathfrak{p}) = (\log \mathrm{N}\mathfrak{p})/\mathrm{N}\mathfrak{p}$ for $y \leq \mathrm{N}\mathfrak{p} < u$ to deduce

$$\begin{aligned} w(\lambda) \sum_{\chi \pmod H} N(\sigma, T, \chi) &\\ \leq \Bigl(C'(2\phi r^\star \mathcal{L} + 8)(r^\star)^3 + O_\epsilon\Bigl(\frac{(r^\star)^4 \mathcal{L}^2}{e^{\epsilon Y \mathcal{L}/2}}\Bigr)\Bigr) \int_y^x \sum_{y \leq \mathrm{N}\mathfrak{p} < u} \frac{(\log \mathrm{N}\mathfrak{p})^2}{\mathrm{N}\mathfrak{p}} \frac{du}{u} &\\ + 4A\phi r^\star \mathcal{L} + 16A, & \qquad (5\text{-}22) \end{aligned}$$

where

$$C' = 5\pi(X-Y)\Bigl(1 - \frac{1}{1+\upsilon}\Bigr)^{-1}\Bigl(\frac{1}{1+\epsilon}Y - 4\Bigr)^{-1}.$$

To calculate $C'$, we replaced $\mathcal{L}'$ (as found in Theorem 4.2) by observing from Lemma 2.11 that $\mathcal{L}' + \frac{1}{1+\epsilon}\log h_H \leq 4\mathcal{L}$ (since $T \geq \max\{n_K^{5/6}(D_K^{4/3} Q^{4/9})^{-1/n_K}, 1\}$ and $\Theta = \Theta(\epsilon)$ is sufficiently large). For the remaining integral in (5-22), notice by Lemma 2.10 that

$$\begin{aligned} \int_y^x \sum_{y \leq \mathrm{N}\mathfrak{p} < u} \frac{(\log \mathrm{N}\mathfrak{p})^2}{\mathrm{N}\mathfrak{p}} \frac{du}{u} &\leq \log x \int_y^x e \log(e D_K^{1/2} u) \frac{du}{u} \\ &\leq \frac{e}{2} X(X-Y)\Bigl(X + Y + 1 + \frac{2}{\mathcal{L}}\Bigr)\mathcal{L}^3. \end{aligned}$$

Substituting this estimate in (5-22) and recalling $r^\star = \frac{\lambda^\star}{\mathcal{L}} = \frac{\xi\lambda}{\mathcal{L}}$, we have shown

$$\begin{aligned} w(\lambda) \sum_{\chi \pmod H} N(\sigma, T, \chi) &\\ \leq 2\phi C''\xi^4 \cdot \lambda^4 + 8C''\xi^3 \cdot \lambda^3 + 4\phi A \xi \cdot \lambda + 16A + O_\epsilon(\lambda^3 \mathcal{L} e^{-\epsilon \mathcal{L}}), & \end{aligned}$$

where

$$C'' = \frac{e}{2} X(X-Y)\Bigl(X + Y + 1 + \frac{2}{\mathcal{L}}\Bigr) C'.$$

Since $\mathcal{L} \geq \Theta$ and $\Theta$ is sufficiently large depending on $\epsilon$, the big-O error term above and the quantity $\frac{2}{\mathcal{L}}$ in $C''$ may both be bounded by $\epsilon$. This completes the proof of Theorem 5.3. $\square$

## 6. Log-free zero density estimate

Having established Theorem 5.3, in this section we prove Theorem 3.2.

***Proof of Theorem 3.2:*** Without loss, we may assume $H \pmod{\mathfrak{q}}$ is primitive because $Q = Q_H = Q_{H'}, h_H = h_{H'}$ and

$$\sum_{\chi \pmod{H}} N(\sigma, T, \chi) = \sum_{\chi \pmod{H'}} N(\sigma, T, \chi)$$

if $H'$ induces $H$. Suppose $\frac{1}{2} \leq \sigma \leq 1 - \frac{0.05}{4}$. By a naive application of [Lagarias et al. 1979, Lemma 2.1], one can verify that for $T \geq 1$,

$$\begin{aligned}\sum_{\chi \pmod{H}} N(\sigma, T, \chi) &\ll h_H T \log(D_K Q T^{n_K}) \\ &\ll (e^{O(n_K)} D_K^2 Q T^{n_K+2})^{81(1-\sigma)} \end{aligned} \tag{6-1}$$

after bounding $h_H$ with Lemma 2.11.

Now, let $\epsilon \in \left(0, \frac{1}{8}\right)$ be fixed and define $\mathcal{L}$ as in (5-1). Suppose $1 - \frac{\epsilon}{4} < \sigma < 1$. Let $R \geq 1$ be fixed and sufficiently large. By applying the bound in Lemma 2.11 to [Weiss 1983, Theorem 4.3], we deduce that for $T \geq 1$,

$$\sum_{\chi \pmod{H}} N\left(1 - \frac{R}{\mathcal{L}}, T, \chi\right) \ll 1, \tag{6-2}$$

so it suffices to bound $\sum_{\chi(H)=1} N(\sigma, T, \chi)$ in the range

$$1 - \frac{\epsilon}{4} < \sigma < 1 - \frac{R}{\mathcal{L}}, \tag{6-3}$$

or equivalently, if $\sigma = 1 - \frac{\lambda}{\mathcal{L}}$, in the range

$$R < \lambda < \frac{\epsilon}{4}\mathcal{L}.$$

According to Theorem 5.3 and the notation defined in Section 5A, select

$$\xi = 1 + 10^{-5}, \quad \upsilon = 10^{-5}, \quad \eta = 10^{-5}, \quad \omega = 10^{-5}, \quad \text{and} \quad \alpha = 0.15.$$

It follows that the constants $B_2, C_0, C_1, C_3, C_4$ in Theorem 5.3 are bounded absolutely,

$$X > Y > 4.6, \quad B_1 \leq 146.15\phi, \quad \text{and} \quad \xi A_1 < 4,$$

where $\phi = 1 + \frac{4}{\pi}\epsilon + 16\epsilon^2 + 340\epsilon^{10}$. Moreover, since $\lambda > R$,

$$J(\xi\lambda) \ll \frac{\lambda}{(1+10^{-5})^{\lambda}} \ll \frac{R}{(1+10^{-5})^{R}},$$

and therefore $J(\xi\lambda) < \frac{1}{2}$ for $R$ sufficiently large. Thus, by Theorem 5.3,

$$\sum_{\chi \pmod{H}} N(\sigma, T, \chi) \ll \lambda^4 e^{146.15\phi\lambda} \ll e^{146.2\phi\lambda} = e^{146.2\phi(1-\sigma)\mathcal{L}} \tag{6-4}$$

for $\sigma$ satisfying (6-3) and $T \geq \max\{n_K^{5/6} D_K^{-4/3n_K} Q^{-4/9n_K}, 1\}$. To complete the proof of Theorem 3.2, it remains to choose $\epsilon$ in (6-4). If $\epsilon = 0.05$ then $146.2\phi < 162 = 2 \cdot 81$, yielding the desired result when combined with (6-1). If $\epsilon = 10^{-3}$ then $146.2\phi < 147 = 2 \cdot 73.5$ as claimed. $\square$

## 7. Zero repulsion: the Deuring–Heilbronn phenomenon

In this section, we prove Theorem 3.3 and establish the Deuring–Heilbronn phenomenon for $L$-functions of Hecke characters $\chi \pmod{H}$ where $H \pmod{\mathfrak{q}}$ is a (not necessarily primitive) congruence class group. We will critically use the following power sum inequality.

**Theorem 7.1** (Lagarias–Montgomery–Odlyzko). *Let $\epsilon > 0$ and a sequence of complex numbers $\{z_n\}_n$ be given. Suppose that $|z_n| \leq |z_1|$ for all $n \geq 1$. Define $M := \frac{1}{|z_1|} \sum_n |z_n|$. Then there exists $m_0$ with $1 \leq m_0 \leq (12+\epsilon)M$ such that*

$$\mathrm{Re}\left\{\sum_{n=1}^{\infty} z_n^{m_0}\right\} \geq \frac{\epsilon}{48+5\epsilon} |z_1|^{m_0}.$$

*Proof.* This is a modified version of [Lagarias et al. 1979, Theorem 4.2]; see [Zaman 2017b, Theorem 2.3] for details. $\square$

We prepare for the application of this result by establishing a few preliminary estimates and then end this section with the proof of Theorem 3.3.

### 7A. *Preliminaries.*

**Lemma 7.2.** *Let $\chi \pmod{\mathfrak{q}}$ be a Hecke character. For $\sigma \geq 2$ and $t \in \mathbb{R}$,*

$$-\mathrm{Re}\left\{\frac{L'}{L}(\sigma+it, \chi)\right\} \leq -\mathrm{Re}\left\{\frac{L'}{L}(\sigma+it, \chi^*)\right\} + \frac{1}{2^\sigma - 1}(n_K + \log \mathrm{N}\mathfrak{q}),$$

*where $\chi^*$ is the primitive character inducing $\chi$.*

*Proof.* By definition,

$$L(s, \chi) = P(s, \chi) L(s, \chi^*), \quad \text{where } P(s, \chi) = \prod_{\mathfrak{p} \mid \mathfrak{q},\, \mathfrak{p} \nmid \mathfrak{f}_\chi} \left(1 - \frac{\chi^*(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^s}\right),$$

so it suffices to show $\left|\frac{P'}{P}(s,\chi)\right| \le \frac{1}{2^\sigma-1}(n_K+\log \mathrm{N}\mathfrak{q})$. Observe, by elementary arguments,

$$\left|\frac{P'}{P}(s,\chi)\right| = \left|\sum_{\mathfrak{p}\mid\mathfrak{q},\,\mathfrak{p}\nmid\mathfrak{f}_\chi}\sum_{k=1}^{\infty}\frac{\chi^*(\mathfrak{p}^k)\log \mathrm{N}\mathfrak{p}^k}{k(\mathrm{N}\mathfrak{p}^k)^s}\right| \le \sum_{\mathfrak{p}\mid\mathfrak{q}}\frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^\sigma-1} \le \frac{1}{1-2^{-\sigma}}\cdot\frac{1}{2^{\sigma-1}}\sum_{\mathfrak{p}\mid\mathfrak{q}}\frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}}.$$

From [Zaman 2016a, Lemma 2.4],

$$\sum_{\mathfrak{p}\mid\mathfrak{q}}\frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}} \le \sqrt{n_K\log \mathrm{N}\mathfrak{q}} \le \frac{n_K}{2}+\frac{\log \mathrm{N}\mathfrak{q}}{2}.$$

Combining this fact with the previous inequality gives the desired estimate. □

**Lemma 7.3.** *Let $\chi \pmod{\mathfrak{q}}$ be a Hecke character. For $\sigma > 1$ and $t\in\mathbb{R}$,*

$$\sum_{\omega \text{ trivial}}\frac{1}{|\sigma+it-\omega|^2} \le \begin{cases} \left(\frac{1}{2\sigma}+\frac{1}{\sigma^2}\right)\cdot n_K & \text{if } \chi \text{ is primitive,}\\ \left(\frac{1}{2\sigma}+\frac{1}{\sigma^2}\right)\cdot n_K + \left(\frac{1}{2\sigma}+\frac{2}{\sigma^2\log 2}\right)\cdot\log \mathrm{N}\mathfrak{q} & \text{unconditionally,}\end{cases}$$

*where the sum is over all trivial zeros $\omega$ of $L(s,\chi)$ counted with multiplicity.*

*Proof.* Suppose $\chi \pmod{\mathfrak{q}}$ is induced by the primitive character $\chi^* \pmod{\mathfrak{f}_\chi}$. Then

$$L(s,\chi) = P(s,\chi)L(s,\chi^*),\quad \text{where } P(s,\chi) = \prod_{\mathfrak{p}\mid\mathfrak{q},\,\mathfrak{p}\nmid\mathfrak{f}_\chi}\left(1-\frac{\chi^*(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^s}\right),$$

for all $s\in\mathbb{C}$. Thus, the trivial zeros of $L(s,\chi)$ are either zeros of the finite Euler product $P(s,\chi)$ or trivial zeros of $L(s,\chi^*)$. We consider each separately. From (2-7) and (2-5), observe

$$\sum_{\substack{\omega \text{ trivial}\\ L(\omega,\chi^*)=0}}\frac{1}{|\sigma+it-\omega|^2} \le a(\chi)\sum_{k=0}^{\infty}\frac{1}{(\sigma+2k)^2+t^2}+b(\chi)\sum_{k=0}^{\infty}\frac{1}{(\sigma+2k+1)^2+t^2} \le n_K\sum_{k=0}^{\infty}\frac{1}{(\sigma+2k)^2} \le \left(\frac{1}{2\sigma}+\frac{1}{\sigma^2}\right)n_K.$$

Now, if $\chi$ is primitive then $P(s,\chi)\equiv 1$ and hence never vanishes. Otherwise, notice the zeros of each $\mathfrak{p}$-factor in the Euler product of $P(s,\chi)$ are totally imaginary and are given by $a_\chi(\mathfrak{p})i+2\pi i\mathbb{Z}/\log \mathrm{N}\mathfrak{p}$ for some $0\le a_\chi(\mathfrak{p})<2\pi/\log \mathrm{N}\mathfrak{p}$. Translating these zeros $\omega \mapsto \omega + it$ amounts to choosing another representative $0 \le b_\chi(\mathfrak{p};t) < 2\pi/\log \mathrm{N}\mathfrak{p}$. Therefore,

$$\sum_{\substack{\omega \text{ trivial} \\ P(\omega,\chi)=0}} \frac{1}{|\sigma+it-\omega|^2} \le 2 \sum_{\mathfrak{p}|\mathfrak{q},\, \mathfrak{p}\nmid\mathfrak{f}_\chi} \sum_{k=0}^{\infty} \frac{1}{\sigma^2+(2\pi k/\log \mathrm{N}\mathfrak{p})^2} \le \Big(\frac{1}{2\sigma}+\frac{2}{\sigma^2\log 2}\Big)\log \mathrm{N}\mathfrak{q},$$

as required. □

**Lemma 7.4.** *Let $H \pmod{\mathfrak{q}}$ be a congruence class group of $K$. Suppose $\psi \pmod{H}$ is real and $\chi \pmod{H}$ is arbitrary. For $\sigma = \alpha + 1$ with $\alpha \ge 1$ and $t \in \mathbb{R}$,*

$$\sum_{\substack{\rho \\ \zeta_K(\rho)=0}} \frac{1}{|\sigma-\rho|^2} + \sum_{\substack{\rho \\ L(\rho,\psi)=0}} \frac{1}{|\sigma-\rho|^2} + \sum_{\substack{\rho \\ L(\rho,\chi)=0}} \frac{1}{|\sigma+it-\rho|^2} + \sum_{\substack{\rho \\ L(\rho,\psi\chi)=0}} \frac{1}{|\sigma+it-\rho|^2}$$
$$\le \frac{1}{\alpha}\cdot\Big[\frac{1}{2}\log(D_K^3 Q^2 D_\psi) + \Big(\log(\alpha+2) + \frac{2}{\alpha+1} + \frac{1}{2^{\alpha+1}-1} - 2\log\pi\Big)n_K + n_K\log(\alpha+2+|t|) + \frac{2}{2^{\alpha+1}-1}\log Q + \frac{4}{\alpha} + \frac{4}{\alpha+1}\Big],$$

*where the sums are over all nontrivial zeros of the corresponding $L$-functions.*

**Remark.** If $\psi$ is trivial, notice that the left-hand side equals

$$2\Big(\sum_{\substack{\rho \\ \zeta_K(\rho)=0}} \frac{1}{|\sigma-\rho|^2} + \sum_{\substack{\rho \\ L(\rho,\chi)=0}} \frac{1}{|\sigma+it-\rho|^2}\Big).$$

This additional factor of 2 will be useful to us later.

*Proof.* Suppose $\psi$ and $\chi$ are induced from the primitive characters $\psi^*$ and $\chi^*$, respectively. From the identity $0 \le (1+\psi^*(\mathfrak{n}))(1+\mathrm{Re}\{\chi^*(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-it}\})$, it follows that

$$0 \le -\mathrm{Re}\Big\{\frac{\zeta_K'}{\zeta_K}(\sigma) + \frac{L'}{L}(\sigma,\psi^*) + \frac{L'}{L}(\sigma+it,\chi^*) + \frac{L'}{L}(\sigma+it,\psi^*\chi^*)\Big\}. \tag{7-1}$$

The first three $L$-functions are primitive, but $\xi := \psi^*\chi^*$ is a character modulo $[\mathfrak{f}_\chi,\mathfrak{f}_\psi]$, the least common multiple of $\mathfrak{f}_\psi$ and $\mathfrak{f}_\chi$, and hence is not necessarily primitive. Thus, by Lemma 7.2, we deduce

$$0 \le -\mathrm{Re}\Big\{\frac{\zeta_K'}{\zeta_K}(\sigma) + \frac{L'}{L}(\sigma,\psi^*) + \frac{L'}{L}(\sigma+it,\chi^*) + \frac{L'}{L}(\sigma+it,\xi^*)\Big\} + \frac{n_K+\log \mathrm{N}[\mathfrak{f}_\chi,\mathfrak{f}_\psi]}{2^\sigma-1}.$$

Note $\mathrm{N}[\mathfrak{f}_\chi, \mathfrak{f}_\psi] \le Q^2$ since $\psi$ and $\chi$ are both characters trivial on the congruence subgroup $H$, and therefore the norms of their respective conductors are bounded by $Q$. Using this bound, we apply Lemmas 2.1 and 2.4 to each of the primitive $L$-function terms, yielding

$$0 \le \frac{1}{2}\log(D_K D_\psi D_\chi D_\xi) + \frac{2}{2^\sigma - 1}\log Q + n_K \log(\sigma + 1 + |t|) + A_\sigma n_K$$
$$-\mathrm{Re}\Bigg\{ \sum_{\substack{\rho \\ \zeta_K(\rho)=0}} \frac{1}{\sigma-\rho} + \sum_{\substack{\rho \\ L(\rho,\psi)=0}} \frac{1}{\sigma-\rho} + \sum_{\substack{\rho \\ L(\rho,\chi)=0}} \frac{1}{\sigma+it-\rho} + \sum_{\substack{\rho \\ L(\rho,\psi\chi)=0}} \frac{1}{\sigma+it-\rho} \Bigg\}$$
$$+ \frac{1+\delta(\psi)}{\alpha} + \frac{1+\delta(\psi)}{\alpha+1} + \mathrm{Re}\Bigg\{ \frac{\delta(\chi)+\delta(\chi\psi)}{\alpha+it} + \frac{\delta(\chi)+\delta(\chi\psi)}{\alpha+1+it} \Bigg\}, \tag{7-2}$$

where $A_\sigma = \log(\sigma+1) + \frac{2}{\sigma} + \frac{1}{2^\sigma - 1} - 2\log\pi$. Since $0 < \beta < 1$, we notice

$$\mathrm{Re}\Bigg\{\frac{1}{\sigma+it-\rho}\Bigg\} \ge \frac{\alpha}{|\sigma+it-\rho|^2} \quad \text{and} \quad \mathrm{Re}\Bigg\{\frac{1}{\alpha+it} + \frac{1}{\alpha+1+it}\Bigg\} \le \frac{1}{\alpha} + \frac{1}{\alpha+1}.$$

Further, $D_\chi$ and $D_\xi$ are both $\le D_K Q$, since $\xi = \psi^*\chi^*$ induces the character $\psi\chi \pmod{\mathfrak{q}}$, which is trivial on $H$. Rearranging (7-2) and employing all of the subsequent observations gives the desired conclusion. □

**7B.** ***Proof of Theorem 3.3.*** If $\widetilde{H} \pmod{\mathfrak{m}}$ induces $H \pmod{\mathfrak{q}}$, then a character $\chi \pmod{H}$ is induced by a character $\tilde{\chi} \pmod{\widetilde{H}}$. It follows that

$$L(s,\chi) = L(s,\tilde{\chi}) \prod_{\mathfrak{p}|\mathfrak{q},\, \mathfrak{p}\nmid\mathfrak{m}} \left(1 - \frac{\tilde{\chi}(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^s}\right)$$

for all $s \in \mathbb{C}$. This implies that the nontrivial zeros of $L(s,\chi)$ are the same nontrivial zeros of $L(s,\tilde{\chi})$. Therefore, without loss of generality, we may assume $H \pmod{\mathfrak{q}}$ is primitive.

We divide the proof according to whether $\psi$ is quadratic or trivial. The arguments in each case are similar but require some minor differences.

*Case 1: $\psi$ is quadratic.* Let $m$ be a positive integer, $\alpha \ge 1$, and $\sigma = \alpha + 1$. From the inequality $0 \le (1 + \psi^*(\mathfrak{n}))(1 + \mathrm{Re}\{\chi^*(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-i\gamma'}\})$ and Lemma 2.2 with $s = \sigma + i\gamma'$, it follows that

$$\mathrm{Re}\Bigg\{\sum_{n=1}^{\infty} z_n^m\Bigg\} \le \frac{1}{\alpha^m} - \frac{1}{(\alpha+1-\beta_1)^{2m}}$$
$$+ \mathrm{Re}\Bigg\{ \frac{\delta(\chi)+\delta(\psi\chi)}{(\alpha+i\gamma')^{2m}} - \frac{\delta(\chi)+\delta(\psi\chi)}{(\alpha+1+i\gamma'-\beta_1)^{2m}} \Bigg\}, \tag{7-3}$$

where $z_n = z_n(\gamma')$ satisfies $|z_1| \geq |z_2| \geq \cdots$ and runs over the multisets

$$\begin{gathered}\{(\sigma-\omega)^{-2} : \omega \text{ is any zero of } \zeta_K(s)\},\\ \{(\sigma-\omega)^{-2} : \omega \neq \beta_1 \text{ is any zero of } L(s,\psi^*)\},\\ \{(\sigma+i\gamma'-\omega)^{-2} : \omega \neq \beta_1 \text{ is any zero of } L(s,\chi^*)\},\\ \{(\sigma+i\gamma'-\omega)^{-2} : \omega \neq \beta_1 \text{ is any zero of } L(s,\psi^*\chi^*)\}.\end{gathered} \tag{7-4}$$

Note that the multisets include trivial zeros of the corresponding $L$-functions, and $\psi^*\chi^*$ is a Hecke character (not necessarily primitive) modulo the least common multiple of $\mathfrak{f}_\chi$ and $\mathfrak{f}_\psi$. With this choice, it follows that

$$\big(\alpha+\tfrac{1}{2}\big)^{-2} \leq (\alpha+1-\beta')^{-2} \leq |z_1| \leq \alpha^{-2}. \tag{7-5}$$

The right-hand side of (7-3) may be bounded via the observation

$$\begin{aligned}\left|\frac{1}{(\alpha+it)^{2m}} - \frac{1}{(\alpha+it+1-\beta_1)^{2m}}\right| &\leq \alpha^{-2m}\left|1-\frac{1}{\big(1+\frac{1-\beta_1}{\alpha+it}\big)^{2m}}\right|\\ &\ll \alpha^{-2m-1} m(1-\beta_1),\end{aligned}$$

whence

$$\operatorname{Re}\bigg\{\sum_{n=1}^{\infty} z_n^m\bigg\} \ll \alpha^{-2m-1} m(1-\beta_1). \tag{7-6}$$

On the other hand, by Theorem 7.1, for $\epsilon > 0$, there exists some $m_0 = m_0(\epsilon)$ with $1 \leq m_0 \leq (12+\epsilon)M$ such that

$$\operatorname{Re}\bigg\{\sum_{n=1}^{\infty} z_n^{m_0}\bigg\} \geq \frac{\epsilon}{50}|z_1|^{m_0} \geq \frac{\epsilon}{50}(\alpha+1-\beta')^{-2m_0} \geq \frac{\epsilon}{50}\alpha^{-2m_0}\exp\Big(-\frac{2m_0}{\alpha}(1-\beta')\Big),$$

where $M = |z_1|^{-1}\sum_{n=1}^{\infty}|z_n|$. Comparing with (7-6) for $m = m_0$, it follows that

$$\exp\Big(-(24+2\epsilon)\frac{M}{\alpha}(1-\beta')\Big) \ll_\epsilon \frac{M}{\alpha}(1-\beta_1). \tag{7-7}$$

Therefore, it suffices to bound $\frac{M}{\alpha}$ and optimize over $\alpha \geq 1$.

By (7-4), $M$ is a sum involving nontrivial and trivial zeros of certain $L$-functions. For the nontrivial zeros, we employ Lemma 7.4 with $D_\psi = D_K \mathrm{N}\mathfrak{f}_\psi \leq D_K Q$ since $\psi$ is quadratic. For the trivial zeros, apply Lemma 7.3 in the "primitive" case for $\zeta_K(s)$, $L(s,\psi^*)$, $L(s,\chi^*)$ and in the "unconditional" case for $L(s,\psi^*\chi^*)$. In the latter case, we additionally observe that, as $H \pmod{\mathfrak{q}}$ is primitive, $\log \mathrm{N}\mathfrak{q} \leq 2\log Q$

by Lemma 2.12. Combining these steps along with (7-5), it follows that

$$\frac{M}{\alpha} \leq \frac{(\alpha+1/2)^2}{\alpha^2}\cdot\Big[2\log D_K+\Big(\frac{3}{2}+\frac{2\alpha}{2\alpha+2}+\frac{4\alpha}{(\alpha+1)^2\log 2}+\frac{2}{2^{\alpha+1}-1}\Big)\log Q \\ +\Big(\log(\alpha+2)+\log(\alpha+3)+2-2\log\pi+\frac{4\alpha}{(\alpha+1)^2}+\frac{1}{2^{\alpha+1}-1}\Big)n_K \\ +n_K\log T+\frac{4}{\alpha}+\frac{4}{\alpha+1}\Big], \tag{7-8}$$

for $\alpha \geq 1$. Note that, in applying Lemma 7.4, we used that $\log(\alpha+2+T) \leq \log(\alpha+3)+\log T$ for $T \geq 1$. Finally, select $\alpha$ sufficiently large, depending on $\epsilon > 0$, so the right-hand side of (7-8) is

$$\leq \Big(2+\frac{\epsilon}{100}\Big)\log D_K+\Big(2.5+\frac{\epsilon}{100}\Big)\log Q+\Big(1+\frac{\epsilon}{100}\Big)n_K\log T+O_\epsilon(n_K).$$

Incorporating the resulting bounds into (7-7) completes the proof of Theorem 3.3 for $\psi$ quadratic.

*Case 2: $\psi$ is trivial.* Begin with the inequality $0 \leq 1+\mathrm{Re}\{\chi^*(\mathfrak{n})(\mathrm{N}\mathfrak{n})^{-i\gamma'}\}$. This similarly implies

$$\mathrm{Re}\Big\{\sum_{n=1}^{\infty} z_n^m\Big\} \leq \frac{1}{\alpha^m}-\frac{1}{(\alpha+1-\beta_1)^{2m}} \\ +\mathrm{Re}\Big\{\frac{\delta(\chi)}{(\alpha+i\gamma')^{2m}}-\frac{\delta(\chi)}{(\alpha+1+i\gamma'-\beta_1)^{2m}}\Big\} \tag{7-9}$$

for a new choice $z_n = z_n(\gamma')$ satisfying $|z_1| \geq |z_2| \geq \cdots$ and which runs over the multisets

$$\begin{gathered}\{(\sigma-\omega)^{-2} : \omega \neq \beta_1 \text{ is any zero of } \zeta_K(s)\},\\ \{(\sigma+i\gamma'-\omega)^{-2} : \omega \neq \beta_1 \text{ is any zero of } L(s,\chi^*)\}.\end{gathered} \tag{7-10}$$

Following the same arguments as before, we may arrive at (7-7) for the new quantity $M = |z_1|^{-1}\sum_{n=1}^{\infty}|z_n|$. To bound the nontrivial zeros arising in $M$, apply Lemma 7.4 with $D_\psi = D_K$ since $\psi$ is trivial. For the trivial zeros, apply Lemma 7.3 in the "primitive" case for both $\zeta_K(s)$ and $L(s,\chi^*)$. It follows from (7-5) that, for $\alpha \geq 1$,

$$\frac{M}{\alpha} \leq \frac{(\alpha+1/2)^2}{\alpha^2}\cdot\Big[\log D_K+\Big(\frac{1}{2}+\frac{1}{2^{\alpha+1}-1}\Big)\log Q \\ +\frac{1}{2}n_K\log T+\frac{2}{\alpha}+\frac{2}{\alpha+1}+\Big(\frac{1}{2}\log(\alpha+2)+\frac{1}{2}\log(\alpha+3)+1 \\ -\log\pi+\frac{2\alpha}{(\alpha+1)^2}+\frac{1/2}{2^{\alpha+1}-1}\Big)n_K\Big]. \tag{7-11}$$

Again, we select $\alpha$ sufficiently large, depending on $\epsilon > 0$, so the right-hand side of (7-11) is

$$\leq \Big(1+\frac{\epsilon}{50}\Big)\log D_K + \Big(0.5+\frac{\epsilon}{50}\Big)\log Q + \Big(0.5+\frac{\epsilon}{50}\Big)n_K\log T + O_\epsilon(n_K).$$

Incorporating the resulting bound into (7-7) completes the proof of Theorem 3.3. □

**Remark.** To obtain a more explicit version of Theorem 3.3, the only difference in the proof is selecting an explicit value of $\alpha$ in the final step of each case. The possible choice of $\alpha$ is somewhat arbitrary because the coefficients of $\log D_K$, $\log Q$, and $n_K$ in (7-8) and (7-11) cannot be simultaneously minimized. Hence, in the interest of having relatively small coefficients of comparable size for all quantities, one could choose the value $\alpha = 18$.

## 8. Zeros in low-lying rectangles

Analogous to [Heath-Brown 1992] for the classical case, most of the key numerical estimates we use to prove Theorem 3.1 pertain to zeros in a "low-lying" rectangle. In this section, we record the relevant existing results and establish some new ones. These encompass the required three principles in Section 3 and will be applied in the final arguments for the proof of Theorem 3.1. We begin with some notation.

**8A.** ***Logarithmic quantity.*** Let $\delta_0 > 0$ be fixed and sufficiently small. For the remainder of the paper, define

$$\mathscr{L} := \begin{cases} \big(\tfrac{1}{3}+\delta_0\big)\log D_K + \big(\tfrac{19}{36}+\delta_0\big)\log Q \\ \qquad\qquad + \big(\tfrac{5}{12}+\delta_0\big)n_K\log n_K & \text{if } n_K^{5n_K/6} \geq D_K^{4/3}Q^{4/9}, \\ \big(1+\delta_0\big)\log D_K + \big(\tfrac{3}{4}+\delta_0\big)\log Q \\ \qquad\qquad + \delta_0 n_K\log n_K & \text{otherwise.} \end{cases} \tag{8-1}$$

Notice that

$$\begin{aligned} \mathscr{L} &\geq (1+\delta_0)\log D_K + \big(\tfrac{3}{4}+\delta_0\big)\log Q + \delta_0 n_K\log n_K, \\ \mathscr{L} &\geq \big(\tfrac{5}{12}+\delta_0\big)n_K\log n_K, \end{aligned} \tag{8-2}$$

unconditionally. For $T_\star \geq 1$ fixed,[3] set $T_0 := \max\{n_K^{5/6}(D_K^{4/3}Q^{4/9})^{-1/n_K}, T_\star\}$. We compare $\mathcal{L} = \mathcal{L}_{T_0,\delta_0}$ given by (5-1) with $\mathscr{L}$ and deduce $\mathcal{L} \leq \mathscr{L}$ for $\mathscr{L}$ sufficiently large. This observation implies that

$$N\Big(1-\frac{\lambda}{\mathscr{L}}, T, \chi\Big) \leq N\Big(1-\frac{\lambda}{\mathcal{L}}, T, \chi\Big) \tag{8-3}$$

for $\lambda > 0$ and $N(\sigma, T, \chi)$ defined in (5-7). We will utilize this fact in Section 8E.

[3]For the purposes of this paper, setting $T_\star = 1$ would suffice, but we avoid this choice to make the results of Section 8 more widely applicable.

**8B.** ***Low-lying zeros.*** Next we specify some important zeros of $\prod_{\chi \pmod H} L(s,\chi)$ which will be used for the remainder of the paper. Consider the multiset of zeros given by

$$\mathcal{Z} := \Big\{ \rho \in \mathbb{C} : \prod_{\chi \pmod H} L(\rho,\chi) = 0,\ 0 < \mathrm{Re}\{\rho\} < 1,\ |\mathrm{Im}(\rho)| \le T_\star \Big\}. \tag{8-4}$$

We select three important zeros in $\mathcal{Z}$ as follows:

- Choose $\rho_1 \in \mathcal{Z}$ such that $\mathrm{Re}\{\rho_1\}$ is maximal. Let $\chi_1$ be its associated Hecke character, so $L(\rho_1,\chi_1) = 0$. Let
$$\rho_1 = \beta_1 + i\gamma_1 = \Big(1 - \frac{\lambda_1}{\mathscr{L}}\Big) + i\frac{\mu_1}{\mathscr{L}},$$
where $\beta_1 = \mathrm{Re}\{\rho_1\}$, $\gamma_1 = \mathrm{Im}\{\rho_1\}$, $\lambda_1 > 0$, and $\mu_1 \in \mathbb{R}$.
- Choose $\rho' \in \mathcal{Z} \setminus \{\rho_1, \overline{\rho_1}\}$ satisfying $L(\rho',\chi_1) = 0$ such that $\mathrm{Re}\{\rho'\}$ is maximal.[4] Similarly, let
$$\rho' = \beta' + i\gamma' = \Big(1 - \frac{\lambda'}{\mathscr{L}}\Big) + i\frac{\mu'}{\mathscr{L}}.$$
- Let $\mathcal{Z}_1$ be the multiset of zeros of $L(s,\chi_1)$ contained in $\mathcal{Z}$. Choose $\rho_2 \in \mathcal{Z} \setminus \mathcal{Z}_1$ such that $\mathrm{Re}\{\rho_2\}$ is maximal. Let $\chi_2$ be its associated Hecke character, so $L(\rho_2,\chi_2) = 0$. Similarly, let
$$\rho_2 = \beta_2 + i\gamma_2 = \Big(1 - \frac{\lambda_2}{\mathscr{L}}\Big) + i\frac{\mu_2}{\mathscr{L}}.$$

**8C.** ***Zero-free regions.*** With the above notation, we may introduce the first of three principles. We record the current best-known existing explicit result regarding zero-free regions of Hecke $L$-functions.

**Theorem 8.1** (Zaman). *For $\mathscr{L}$ sufficiently large, we have* $\min\{\lambda', \lambda_2\} > 0.2866$. *If* $\lambda_1 < 0.0875$ *then* $\rho_1$ *is a simple real zero of* $\prod_{\chi \pmod H} L(s,\chi)$ *and is associated with a real character* $\chi_1$.

*Proof.* When $T_\star = 1$ and $H = P_{\mathfrak{q}}$, in which case $Q = \mathrm{N}\mathfrak{q}$, this is implied by [Zaman 2016a, Theorems 1.1 and 1.3] since $\mathscr{L}$ satisfies (8-2). For general congruence subgroups $H$ and any fixed $T_\star \ge 1$, the argument, which occurs in [Zaman 2017a], is achieved by modifying [Zaman 2016a] as follows:

- Assume $H \pmod{\mathfrak{q}}$ is primitive, i.e., $\mathfrak{f}_H = \mathfrak{q}$.
- Restrict to characters $\chi \pmod{\mathfrak{q}}$ satisfying $\chi(H) = 1$ throughout.
- Redefine $\mathcal{L}$ and $\mathcal{L}^*$ in [Zaman 2016a, (3.2)] to replace $\log \mathrm{N}\mathfrak{q}$ with $\log Q$.

[4] If $\rho_1$ is real then $\rho' \in \mathcal{Z} \setminus \{\rho_1\}$ instead with the other conditions remaining the same.

- Substitute applications of [Zaman 2016a, Lemma 2.4] with Lemma 2.13 since $\mathfrak{q} = \mathfrak{f}_H$. When estimating certain sums, this allows one to transfer from imprimitive characters $\chi \pmod{H}$ to primitive ones.
- Modify [Zaman 2016a, Lemma 3.2] so that the special value $T_0(\mathfrak{q})$, in that lemma's notation, instead satisfies $T_\star \le T_0(\mathfrak{q}) \le \frac{1}{10} T_\star \mathcal{T}$; one can achieve this by analogously supposing, for a contradiction, that each region $\alpha \le \sigma \le 1$ and $T_\star 10^j \le |t| \le T_\star 10^{j+1}$ for $0 \le j < J$ with $J = [\log \mathcal{T} / \log 10]$ contains at least one zero of $\prod_{\chi \pmod{H}} L(s, \chi)$. After applying [Zaman 2016a, (3.4)] with $T = T_\star \mathcal{T}$, the rest of the argument follows similarly. ☐

**8D.** ***Zero repulsion.*** Here we record two explicit estimates for zero repulsion when an exceptional zero exists.

**Theorem 8.2** (Zaman). *If $\lambda_1 < 0.0875$ then unconditionally, for $\mathscr{L}$ sufficiently large,* $\min\{\lambda', \lambda_2\} > 0.44$. *If $\eta \le \lambda_1 < 0.0875$ then, for $\mathscr{L}$ sufficiently large depending on $\eta > 0$,* $\min\{\lambda', \lambda_2\} > 0.2103 \log(1/\lambda_1)$.

*Proof.* When $T_\star = 1$ and $H = P_{\mathfrak{q}}$, this is contained in [Zaman 2016a, Theorem 1.4] since $\mathscr{L}$ satisfies (8-2). Similarly to the proof of Theorem 8.1, one may modify [Zaman 2016a] to deduce the same theorem for general congruence subgroups $H$ and any fixed $T_\star \ge 1$. ☐

Theorem 8.2 is not equipped to deal with exceptional zeros $\rho_1$ extremely close to 1 due to the requirement $\lambda_1 \ge \eta$. Thus, we require a more widely applicable version of zero repulsion; this is precisely the purpose of Theorem 3.3, which we restate here in the current notation.

**Theorem 8.3.** *Let $T \ge 1$ be arbitrary. Suppose $\chi_1$ is a real character and $\rho_1$ is a real zero. For $\chi \pmod{H}$, let $\rho \ne \rho_1$ be any nontrivial zero of $L(s, \chi)$ satisfying $\frac{1}{2} \le \operatorname{Re}\{\rho\} = 1 - \frac{\lambda}{\mathscr{L}} < 1$ and $|\operatorname{Im}\{\rho\}| \le T$. For $\mathscr{L}$ sufficiently large depending on $\epsilon > 0$ and $T$, we have $\lambda > \log(c_\epsilon / \lambda_1)/(80 + \epsilon)$, where $c_\epsilon > 0$ is an effective constant depending only on $\epsilon$.*

*Proof.* This follows immediately from Theorem 3.3 since

$$(48 + \epsilon) \log D_K + (60 + \epsilon) \log Q + (24 + \epsilon) n_K \log T + O_\epsilon(n_K) \le (80 + 2\epsilon)\mathscr{L}$$

for $\mathscr{L}$ sufficiently large depending on $\epsilon$ and $T$. ☐

The repulsion constant $\frac{1}{80+\epsilon} \approx 0.0125$ in Theorem 8.3 is much smaller than 0.2103 in Theorem 8.2. This deficiency follows from using power sum arguments; see the remarks following Theorem 3.3. We now quantify how close an exceptional zero $\rho_1$ can be to 1.

**Theorem 8.4** [Stark 1974]. *Unconditionally,* $\lambda_1 \gg e^{-24\mathscr{L}/5}$.

*Proof.* This follows from (8-1), (8-2), and the proof of [Stark 1974, Theorem 1′, p. 148]. □

**8E. *Log-free zero density estimates.*** First, we restate a slightly weaker form of Theorem 3.2 in the current notation.

**Theorem 8.5.** *Let $T \geq 1$ be arbitrary. If $0 < \lambda < \mathscr{L}$ then*

$$\sum_{\chi \pmod{H}} N\Big(1 - \frac{\lambda}{\mathscr{L}}, T, \chi\Big) \ll e^{162\lambda}$$

*provided $\mathscr{L}$ is sufficiently large depending on $T$.*

*Proof.* This follows from (8-1) and Theorem 3.2. □

In addition to Theorem 8.5, we require a completely explicit zero density estimate for "low-lying" zeros. Define[5]

$$\begin{aligned}\mathcal{N}(\lambda) = \mathcal{N}_H(\lambda) &:= \sum_{\chi \pmod{H}} N\Big(1 - \frac{\lambda}{\mathscr{L}}, T_\star, \chi\Big)\\ &= \sum_{\chi \pmod{H}} \#\Big\{\rho : L(\rho,\chi) = 0,\ 1 - \frac{\lambda}{\mathscr{L}} < \operatorname{Re}\{\rho\} < 1,\ |\operatorname{Im}\{\rho\}| \leq T_\star\Big\}. \qquad (8\text{-}5)\end{aligned}$$

By Theorem 8.1, observe that $\mathcal{N}(0.0875) \leq 1$ and $\mathcal{N}(0.2866) \leq 2$. In light of these bounds, we exhibit explicit numerical estimates for $\mathcal{N}(\lambda)$ in the range with $0.287 \leq \lambda \leq 1$. For each fixed value of $\lambda$, we apply Theorem 5.3 with $\upsilon = 0.1$ and $\epsilon \in (0, 10^{-5})$ assumed to be fixed and sufficiently small, and obtain a bound for $\mathcal{N}(\lambda\mathscr{L}/\mathcal{L})$. By (8-3), the same bound holds for $\mathcal{N}(\lambda)$. By performing numerical experimentation over the remaining parameters $(\alpha, \eta, \omega, \xi)$ using MATLAB, we roughly optimize the bound in Theorem 5.3 and generate Table 1. Note that we have verified $J(\xi\lambda) < 1$ and $X_{\xi\lambda} > Y_{\xi\lambda} > 4.6$ in each case.

Based on Table 1, we may also establish an explicit estimate for $\mathcal{N}(\lambda)$ by specifying parameters in Theorem 5.3.

**Theorem 8.6.** *Let $\epsilon_0 > 0$ be fixed and sufficiently small. If $0 < \lambda < \epsilon_0\mathscr{L}$ then $\mathcal{N}(\lambda) \leq e^{162\lambda+188}$ for $\mathscr{L}$ sufficiently large. If $0 < \lambda \leq 1$ then $\mathcal{N}(\lambda)$ is also bounded as in Table 1.*

*Proof.* For $\lambda \leq 0.2866$, the result is immediate as $\mathcal{N}(0.2866) \leq 2$ by Theorem 8.1. For $0.2866 \leq \lambda \leq 1$, one can directly verify the desired bound by using Table 1. Now, consider $\lambda \geq 1$. Apply Theorem 5.3 with

$$\begin{array}{llll} T = T_0, & \lambda_0 = 1, & \alpha = 0.1549, & \eta = 0.05722, \\ \epsilon = 10^{-5}, & \upsilon = 0.1, & \xi = 1.0030, & \omega = 0.02074. \end{array}$$

[5] Note $\mathcal{N}(\lambda)$ defined here is *not* the same as $N(\lambda)$ as defined in [Zaman 2016a]. Instead, one has $N(\lambda) \leq \mathcal{N}(\lambda)$.

| $\lambda$ | $\log N(\lambda) \leq$ | $\alpha$ | $\eta$ | $\omega$ | $\xi$ | $J(\xi\lambda)$ | $Y_{\xi\lambda}$ | $X_{\xi\lambda}$ |
|---|---|---|---|---|---|---|---|---|
| 0.287 | 198.1 | 0.3448 | 0.09955 | 0.03466 | 1.0082 | 0.46 | 5.8 | 993 |
| 0.288 | 198.3 | 0.3444 | 0.09943 | 0.03462 | 1.0082 | 0.46 | 5.8 | 991 |
| 0.289 | 198.5 | 0.3441 | 0.09931 | 0.03458 | 1.0082 | 0.46 | 5.8 | 988 |
| 0.290 | 198.7 | 0.3437 | 0.09918 | 0.03454 | 1.0082 | 0.46 | 5.8 | 986 |
| 0.291 | 198.9 | 0.3433 | 0.09906 | 0.03450 | 1.0082 | 0.46 | 5.8 | 984 |
| 0.292 | 199.1 | 0.3429 | 0.09894 | 0.03446 | 1.0081 | 0.46 | 5.8 | 982 |
| 0.293 | 199.3 | 0.3426 | 0.09882 | 0.03442 | 1.0081 | 0.46 | 5.8 | 979 |
| 0.294 | 199.5 | 0.3422 | 0.09870 | 0.03439 | 1.0081 | 0.46 | 5.8 | 977 |
| 0.295 | 199.8 | 0.3418 | 0.09859 | 0.03435 | 1.0081 | 0.46 | 5.8 | 975 |
| 0.296 | 200.0 | 0.3415 | 0.09847 | 0.03431 | 1.0081 | 0.46 | 5.8 | 973 |
| 0.297 | 200.2 | 0.3411 | 0.09835 | 0.03427 | 1.0080 | 0.46 | 5.8 | 970 |
| 0.298 | 200.4 | 0.3408 | 0.09823 | 0.03423 | 1.0080 | 0.46 | 5.8 | 968 |
| 0.299 | 200.6 | 0.3404 | 0.09811 | 0.03420 | 1.0080 | 0.46 | 5.8 | 966 |
| 0.300 | 200.8 | 0.3400 | 0.09800 | 0.03416 | 1.0080 | 0.46 | 5.8 | 964 |
| 0.325 | 205.9 | 0.3316 | 0.09518 | 0.03326 | 1.0075 | 0.47 | 5.8 | 914 |
| 0.350 | 211.0 | 0.3240 | 0.09257 | 0.03242 | 1.0071 | 0.47 | 5.7 | 871 |
| 0.375 | 216.0 | 0.3171 | 0.09014 | 0.03163 | 1.0067 | 0.47 | 5.7 | 833 |
| 0.400 | 220.9 | 0.3108 | 0.08787 | 0.03090 | 1.0064 | 0.48 | 5.7 | 800 |
| 0.425 | 225.7 | 0.3054 | 0.08678 | 0.02878 | 1.0061 | 0.46 | 5.6 | 769 |
| 0.450 | 230.4 | 0.2998 | 0.08373 | 0.02956 | 1.0059 | 0.48 | 5.6 | 744 |
| 0.475 | 235.1 | 0.2948 | 0.08184 | 0.02895 | 1.0056 | 0.48 | 5.6 | 720 |
| 0.500 | 239.8 | 0.2903 | 0.08006 | 0.02837 | 1.0054 | 0.49 | 5.6 | 699 |
| 0.550 | 249.0 | 0.2821 | 0.07677 | 0.02729 | 1.0050 | 0.49 | 5.5 | 661 |
| 0.600 | 258.0 | 0.2748 | 0.07379 | 0.02631 | 1.0046 | 0.50 | 5.5 | 629 |
| 0.650 | 266.9 | 0.2684 | 0.07109 | 0.02542 | 1.0043 | 0.50 | 5.4 | 602 |
| 0.700 | 275.6 | 0.2627 | 0.06862 | 0.02460 | 1.0041 | 0.50 | 5.4 | 579 |
| 0.750 | 284.3 | 0.2576 | 0.06634 | 0.02383 | 1.0039 | 0.51 | 5.4 | 559 |
| 0.800 | 292.9 | 0.2529 | 0.06424 | 0.02313 | 1.0037 | 0.51 | 5.4 | 541 |
| 0.850 | 301.4 | 0.2486 | 0.06230 | 0.02247 | 1.0035 | 0.51 | 5.3 | 525 |
| 0.900 | 309.8 | 0.2447 | 0.06049 | 0.02186 | 1.0033 | 0.51 | 5.3 | 510 |
| 0.950 | 318.2 | 0.2412 | 0.05880 | 0.02128 | 1.0032 | 0.52 | 5.3 | 497 |
| 1.000 | 326.5 | 0.2378 | 0.05722 | 0.02074 | 1.0030 | 0.52 | 5.3 | 486 |

**Table 1.** Bounds for $\mathcal{N}(\lambda)$ using Theorem 5.3 with $\upsilon = 0.1$ and $\epsilon \in (0, 10^{-5}]$.

This choice of values is motivated by the last row of Table 1, but with a more suitable choice for $\alpha$. With this selection, one can check that for any $\lambda \geq 1$,

$$4.61 \leq Y_{\xi\lambda} \leq 9.2, \qquad 264 \leq X_{\xi\lambda} \leq 526, \qquad J(\xi\lambda) \leq 0.272.$$

These inequalities can be verified by elementary arguments involving the definitions in Section 5A and (5-6). In particular, for any $\lambda \geq 1$, the assumptions of Theorem 5.3 are satisfied for all $1 \leq \lambda < \epsilon_0 \mathscr{L}$.

Now with these estimates, we may deduce upper bounds for $C_4$, $C_3$, $C_1$, $C_0$, $B_2$, and $B_1$ in Theorem 5.3 as follows:

$$\begin{aligned} C_4 &= C_4(\lambda) \le 6.0\times 10^{13}, \qquad & C_1 &\le 17, \qquad & B_2 &\le 154,\\ C_3 &= C_3(\lambda) \le 2.4\times 10^{14}, \qquad & C_0 &\le 65, \qquad & B_1 &\le 156, \end{aligned}$$

for $\lambda \ge 1$. Thus, by Theorem 5.3, for $1 \le \lambda \le \epsilon_0 \mathscr{L}$,

$$N(\lambda) \le 52(6.0\times 10^{13}\cdot\lambda^4 + 2.4\times 10^{14}\cdot\lambda^3 + 17\cdot\lambda + 65)e^{156\lambda+154}.$$

To simplify the expression on the right-hand side, we crudely observe that the above is

$$\begin{aligned} &\le 52\cdot 65\Big(6.0\times 10^{13}\cdot\frac{24}{6^4\cdot 65}\cdot\frac{(6\lambda)^4}{4!}\\ &\qquad\qquad + 2.4\times 10^{14}\cdot\frac{6}{6^3\cdot 65}\cdot\frac{(6\lambda)^3}{3!} + 6\lambda + 1\Big)e^{156\lambda+154}\\ &\le 52\cdot 6.7\times 10^{12}\cdot\Big(\frac{(6\lambda)^4}{4!} + \frac{(6\lambda)^3}{3!} + 6\lambda + 1\Big)e^{156\lambda+154} \le e^{162\lambda+188}, \end{aligned}$$

as desired. □

## 9. Proof of Theorem 3.1: preliminaries

We may finally begin the proof of Theorem 3.1. The arguments below are motivated by [Heath-Brown 1992, Section 10] and mostly follow the structure of [Zaman 2017b, Section 4]. Recall that we retain the notation introduced in Section 8 for the remainder of the paper.

**9A. *Choice of weight.*** We define a weight function (see [Zaman 2017b, Lemmas 2.6 and 2.7]) and describe its properties.

**Lemma 9.1.** *For real numbers $A, B > 0$ and a positive integer $\ell \ge 1$ satisfying $B > 2\ell A$, define*

$$F(z) = F_\ell(z; B, A) = e^{-(B-2\ell A)z}\Big(\frac{1-e^{-Az}}{Az}\Big)^{2\ell}, \tag{9-1}$$

*and let $f(t)$ be the inverse Laplace transform of $F(z)$. Then*:

(i) $0 \le f(t) \le A^{-1}$ *for all* $t \in \mathbb{R}$.

(ii) *The support of $f$ is contained in* $[B - 2\ell A, B]$.

(iii) *For $x > 0$ and $y \in \mathbb{R}$,*

$$|F(x+iy)| \le e^{-(B-2\ell A)x}\Big(\frac{1-e^{-Ax}}{Ax}\Big)^{2\ell} \le e^{-(B-2\ell A)x}.$$

For the entirety of this section, select real numbers $A, B > 0$ and an integer $\ell \geq 1$ satisfying $B > 2\ell A$, and let $F(\cdot) = F_\ell(\cdot\,; B, A)$. The inverse Laplace transform of $F(z)$ is written as $f(t)$, so that $F(z) = \int_0^\infty f(t)e^{-zt}\,dt$. To motivate our choice of $f$, we note that the parameter $\ell$ is chosen to be of size $O(n_K)$, so that $f(t)$ is $O(n_K)$-times differentiable and hence $F(x+iy)$ decays like $|y|^{-O(n_K)}$ for fixed $x > 0$ and as $|y| \to \infty$. This decay rate is necessary when applying log-free zero density estimates such as Theorem 3.2 to bound the contribution of zeros which are high in the critical strip.

**9B.** ***A weighted sum of prime ideals.*** For the congruence class group $H \pmod{\mathfrak{q}}$, let $\mathcal{C}$ be an element of the class group of $H$; that is, $\mathcal{C} \in I(\mathfrak{q})/H$. Using the compactly supported weight $f$, define

$$S := \sum_{\substack{\mathfrak{p} \nmid \mathfrak{q}\mathfrak{D}_K \\ \mathrm{N}\mathfrak{p} \text{ is a rational prime}}} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}} f\Big(\frac{\log \mathrm{N}\mathfrak{p}}{\mathscr{L}}\Big) \cdot \mathbf{1}_{\mathcal{C}}(\mathfrak{p}), \tag{9-2}$$

where $\mathbf{1}_{\mathcal{C}}(\cdot)$ is an indicator function for the coset $\mathcal{C}$, $\mathfrak{D}_K$ is the different of $K$, and the sum is over degree 1 prime ideals $\mathfrak{p}$ of $K$ not dividing $\mathfrak{q}\mathfrak{D}_K$. We reduce the proof of Theorem 3.1 to verifying the following lemma.

**Lemma 9.2.** *Let $\eta > 0$ be sufficiently small and let $\mathfrak{m}$ be the product of prime ideals dividing $\mathfrak{q}$ but not $\mathfrak{f}_H$. If $h_H \mathscr{L}^{-1} S \gg_\eta \min\{1, \lambda_1\}$ for*

$$B \geq \max\Big\{693.5, \frac{\log \mathrm{N}\mathfrak{m}}{\mathscr{L}} + 8\eta\Big\}, \qquad A = \frac{4}{\mathscr{L}}, \qquad \ell = \lfloor \eta\mathscr{L} \rfloor \tag{9-3}$$

*and $\mathscr{L}$ is sufficiently large then Theorem 3.1 holds.*

*Proof.* Select $B = (\log x)/\mathscr{L}$ with $A = 4/\mathscr{L}$ and $\ell = \lfloor \eta\mathscr{L} \rfloor$. From the definition (8-1) of $\mathscr{L}$ and the condition on $x$ in (3-1), one can verify that $B$ satisfies (9-3). Now, since $f$ is supported in $[B - 2\ell A, B]$ and $|f| \leq A^{-1} \leq \mathscr{L}$ by Lemma 9.1,

$$S \leq \mathscr{L} e^{8\eta\mathscr{L}} x^{-1} \log x \,\#\{\mathfrak{p} : \mathrm{N}\mathfrak{p} \leq x,\ \deg(\mathfrak{p}) = 1,\ \mathfrak{p} \in \mathcal{C}\}.$$

Multiplying both sides by $h_H \mathscr{L}^{-1}$ and noting $B$ satisfies (9-3), we conclude

$$\#\{\mathfrak{p} : \mathrm{N}\mathfrak{p} \leq x,\ \deg(\mathfrak{p}) = 1,\ \mathfrak{p} \in \mathcal{C}\} \geq \frac{4S}{\mathscr{L}} \cdot \frac{x e^{-8\eta\mathscr{L}}}{\log x} \gg_\eta e^{-5\mathscr{L}} \cdot \frac{x}{h_H \log x}.$$

by Theorems 8.1 and 8.4. Fixing $\eta$ and noting $\mathscr{L} \leq \log(D_K Q n_K^{n_K})$ yields the conclusion of Theorem 3.1. □

Now, by orthogonality of characters,

$$S = \frac{1}{h_H} \sum_{\chi \pmod{H}} \bar{\chi}(\mathcal{C}) S_\chi, \qquad \text{where } S_\chi := \sum_{\substack{\mathfrak{p} \nmid \mathfrak{q}\mathfrak{D}_K \\ \mathrm{N}\mathfrak{p} \text{ is a rational prime}}} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}} \chi(\mathfrak{p}) f\Big(\frac{\log \mathrm{N}\mathfrak{p}}{\mathscr{L}}\Big). \tag{9-4}$$

We wish to write $S_\chi$ as a contour integral involving a logarithmic derivative of a *primitive* Hecke $L$-function. Before doing so, we define

$$\mathfrak{m} = \prod_{\mathfrak{p} \mid \mathfrak{q},\ \mathfrak{p} \nmid \mathfrak{f}_H} \mathfrak{p}. \tag{9-5}$$

**Lemma 9.3.** *If* $B - 2\ell A > \max\{1, (\log \mathrm{N}\mathfrak{m})/\mathscr{L}\}$ *then*

$$\mathscr{L}^{-1} S_\chi = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} -\frac{L'}{L}(s, \chi^*) F((1-s)\mathscr{L})\, ds + O(A^{-1} e^{-(B-2\ell A)\mathscr{L}/2}),$$

*where* $\chi^*$ *is the primitive Hecke character inducing* $\chi \pmod{H}$.

*Proof.* Observe

$$\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} -\frac{L'}{L}(s, \chi^*) F((1-s)\mathscr{L})\, ds = \mathscr{L}^{-1} \sum_{\mathfrak{n}} \frac{\Lambda(\mathfrak{n})}{\mathrm{N}\mathfrak{n}} \chi^*(\mathfrak{n}) f\Big(\frac{\log \mathrm{N}\mathfrak{n}}{\mathscr{L}}\Big) = \mathscr{L}^{-1} \widetilde{S}_\chi,$$

say. Thus, we must show $\widetilde{S}_\chi$ equals $S_\chi$ up to a negligible contribution from prime ideal powers, prime ideals whose norms are not rational primes, and prime ideals dividing $\mathfrak{q}\mathfrak{D}_K$. For simplicity, denote $X = e^{(B-2\ell A)\mathscr{L}}$.

*Prime ideal powers.* By Lemma 9.1, the contribution of such ideals in $\widetilde{S}_\chi$ is bounded by

$$\sum_{\mathfrak{p}} \sum_{m \geq 2} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^m} f\Big(\frac{\log \mathrm{N}\mathfrak{p}^m}{\mathscr{L}}\Big) \leq A^{-1} \sum_{\mathfrak{p}} \sum_{\substack{m \geq 2 \\ \mathrm{N}\mathfrak{p}^m \geq X}} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^m}.$$

Since a rational prime $p$ splits into at most $n_K$ prime ideals in $K$, the right-hand side is

$$\leq A^{-1} \sum_{p \text{ rational}} \sum_{(p) \subseteq \mathfrak{p}} \sum_{\substack{m \geq 2 \\ \mathrm{N}\mathfrak{p}^m \geq X}} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^m} \leq A^{-1} \sum_{\substack{p \text{ rational} \\ p \geq X^{1/2}}} \frac{1}{p^2} \sum_{(p) \subseteq \mathfrak{p}} \log \mathrm{N}\mathfrak{p} \ll A^{-1} \mathscr{L} X^{-1/2}$$

by partial summation and noting $n_K \ll \mathscr{L}$ from Minkowski's bound.

*Prime ideals with norm not equal to a rational prime.* By Lemma 9.1,

$$\sum_{\substack{\mathfrak{p} \\ \mathrm{N}\mathfrak{p} \text{ not a rational prime}}} \sum_{m=1}^{\infty} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^m} f\Big(\frac{\log \mathrm{N}\mathfrak{p}^m}{\mathscr{L}}\Big) \ll A^{-1} \sum_{\substack{\mathrm{N}\mathfrak{p} \geq X \\ \mathrm{N}\mathfrak{p} \text{ not a rational prime}}} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}}.$$

For $\mathfrak{p}$ appearing in the right-hand sum and lying above the rational prime $p$, notice $\mathrm{N}\mathfrak{p} \geq p^2$. Thus, arguing as in the previous case, we deduce

$$\ll A^{-1} \sum_{\substack{p \geq X^{1/2} \\ p \text{ rational prime}}} \frac{1}{p^2} \sum_{(p) \subseteq \mathfrak{p}} \log \mathrm{N}\mathfrak{p} \ll A^{-1} \mathscr{L} X^{-1/2}.$$

*Prime ideals dividing* $\mathfrak{q}\mathfrak{D}_K$. As $B - 2\ell A > \max\{1, (\log \mathrm{N}\mathfrak{m})/\mathscr{L}\}$, $\mathrm{N}\mathfrak{D}_K \leq D_K \leq e^{\mathscr{L}}$ by (8-2), and $f$ is supported in $[B - 2\ell A, B]$, we have $f((\log \mathrm{N}\mathfrak{p})/\mathscr{L}) = 0$ for $\mathfrak{p} \mid \mathfrak{m}\mathfrak{D}_K$. As $\chi(\mathfrak{p}) = \chi^*(\mathfrak{p})$ for all $\mathfrak{p} \nmid \mathfrak{m}$, this implies that

$$\chi(\mathfrak{p}) f\Big(\frac{\log \mathrm{N}\mathfrak{p}}{\mathscr{L}}\Big) = \chi^*(\mathfrak{p}) f\Big(\frac{\log \mathrm{N}\mathfrak{p}}{\mathscr{L}}\Big)$$

for all prime ideals $\mathfrak{p}$. Combining all of these contributions to compare $S_\chi$ with $\widetilde{S}_\chi$ yields the desired result. □

Applying Lemma 9.3 to (9-4), we deduce

$$\mathscr{L}^{-1} S = \frac{1}{h_H} \sum_{\chi \ (\mathrm{mod}\ H)} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} -\frac{L'}{L}(s, \chi^*) F((1-s)\mathscr{L})\, ds + O(A^{-1} e^{-(B-2\ell A)\mathscr{L}/2}), \tag{9-6}$$

provided $B - 2\ell A > \max\{1, (\log \mathrm{N}\mathfrak{m})/\mathscr{L}\}$.

**9C.** ***A sum over low-lying zeros.*** The next step is to shift the contour in (9-6) and pick up the arising poles. Our objective in this subsection is to reduce the analysis to the "low-lying" zeros of Hecke $L$-functions.

**Lemma 9.4.** *Let $T_\star \geq 1$ be fixed, and let $\rho_1$ and $\chi_1$ be as in Section 8B. If the inequalities $B - 2\ell A > \max\{162, (\log \mathrm{N}\mathfrak{m})/\mathscr{L}\}$, $\ell > (81 n_K + 162)/4$, and $A > 1/\mathscr{L}$ hold and $\mathscr{L}$ is sufficiently large, then*

$$\big|h_H \mathscr{L}^{-1} S - F(0) + \overline{\chi_1}(\mathcal{C}) F((1-\rho_1)\mathscr{L})\big| \leq \sum_{\chi \ (\mathrm{mod}\ H)} \sideset{}{'}\sum_{\rho} |F((1-\rho)\mathscr{L})| + O\Big(\Big(\frac{2}{A T_\star \mathscr{L}}\Big)^{2\ell} T_\star^{40.5 n_K} + e^{-78\mathscr{L}}\Big),$$

*where the sum $\sum'$ indicates a restriction to nontrivial zeros $\rho \neq \rho_1$ of $L(s, \chi)$, counted with multiplicity, satisfying* $0 < \mathrm{Re}\{\rho\} < 1$ *and* $|\mathrm{Im}\{\rho\}| \leq T_\star$.

*Proof.* Shift the contour in (9-6) to the line $\operatorname{Re}\{s\} = -\frac{1}{2}$. For each primitive character $\chi^*$, this picks up the nontrivial zeros of $L(s,\chi)$, the simple pole at $s=1$ when $\chi$ is trivial, and the trivial zero at $s=0$ of $L(s,\chi)$ of order $r(\chi)$. To bound the remaining contour, by [Lagarias et al. 1979, Lemma 2.2] and Lemma 9.1(iii) with [Zaman 2017b, Lemma 2.7], for $\operatorname{Re}\{s\} = -\frac{1}{2}$ we have

$$-\frac{L'}{L}(s,\chi^*) \ll \mathscr{L} + n_K \log(|s|+2) \quad \text{and} \quad |F((1-s)\mathscr{L})| \ll e^{-\frac{3}{2}(B-2\ell A)\mathscr{L}} \cdot |s|^{-2}$$

since $A > 1/\mathscr{L}$. It follows that

$$\frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} -\frac{L'}{L}(s,\chi^*) F((1-s)\mathscr{L})\, ds \ll \mathscr{L} e^{-\frac{3}{2}(B-2\ell A)\mathscr{L}}.$$

Overall, (9-6) becomes

$$\frac{h_H S}{\mathscr{L}} - F(0) + \sum_{\chi \ (\mathrm{mod}\ H)} \bar{\chi}(C) \sum_{\rho} F((1-\rho)\mathscr{L}) \ll \sum_{\chi \ (\mathrm{mod}\ H)} r(\chi) F(\mathscr{L}) + \frac{\mathscr{L}}{e^{(B-2\ell A)\mathscr{L}/2}}, \tag{9-7}$$

where the inner sum over $\rho$ is over all nontrivial zeros of $L(s,\chi)$. From (2-5) and (2-7), notice $r(\chi) \le n_K$. Thus, by Lemma 9.1 and Minkowski's bound $n_K \ll \mathscr{L}$,

$$\frac{1}{h_H} \sum_{\chi \ (\mathrm{mod}\ H)} r(\chi) F(\mathscr{L}) \ll \mathscr{L} e^{-(B-2\ell A)\mathscr{L}}.$$

Since $h_H \ll e^{2\mathscr{L}}$ by Lemma 2.11 and (8-2), it follows from (9-7) that

$$h_H \mathscr{L}^{-1} S = F(0) - \sum_{\chi \ (\mathrm{mod}\ H)} \bar{\chi}(C) \sum_{\rho} F((1-\rho)\mathscr{L}) + O(\mathscr{L} e^{-(B-2\ell A-4)\mathscr{L}/2}).$$

The error term is bounded by $O(e^{-78\mathscr{L}})$ as $B - 2\ell A > 162$. Therefore, it suffices to show

$$Z := \sum_{\chi \ (\mathrm{mod}\ H)} \sum_{k=0}^{\infty} \sum_{\substack{\rho \\ 2^k T_\star \le \operatorname{Im}\{\rho\} < 2^{k+1} T_\star}} |F((1-\rho)\mathscr{L})| \ll \left(\frac{2}{A T_\star \mathscr{L}}\right)^{2\ell} T_\star^{40.5 n_K}.$$

From Lemma 9.1, writing $\rho = \beta + i\gamma$ with $\beta \ge \frac{1}{2}$, observe

$$|F(\rho\mathscr{L})| + |F((1-\rho)\mathscr{L})| \le 2e^{-(B-2\ell A)(1-\beta)\mathscr{L}} \left(\frac{2}{A|\gamma|\mathscr{L}}\right)^{2\ell},$$

and moreover, from Theorem 3.2,

$$\widetilde{N}(\sigma) := \sum_{\chi \ (\mathrm{mod}\ H)} N(\sigma, 2T, \chi) \ll (e^{162\mathscr{L}} T^{81 n_K + 162})^{(1-\sigma)}$$

for $\frac{1}{2} \le \sigma \le 1$, $T \ge 1$, and $\mathscr{L}$ sufficiently large. Thus, by partial summation,

$$\sum_{\chi \pmod H} \sum_{\substack{\rho \\ T \le |\mathrm{Im}\{\rho\}| \le 2T}} |F((1-\rho)\mathscr{L})| \ll \left(\frac{2}{AT\mathscr{L}}\right)^{2\ell} \int_1^{1/2} e^{-(B-2\ell A)(1-\sigma)\mathscr{L}}\, d\tilde{N}(\sigma)$$
$$\ll \left(\frac{2}{A\mathscr{L}}\right)^{2\ell} T^{40.5n_K+81-2\ell}$$

since $B > 2\ell A + 162$. Note we have used that the zeros of $\prod_{\chi \pmod H} L(s,\chi)$ are symmetric across the critical line $\mathrm{Re}\{s\} = \frac{1}{2}$. Overall, we deduce

$$Z \ll \left(\frac{2}{A\mathscr{L}}\right)^{2\ell} T_\star^{40.5n_K+81-2\ell} \sum_{k=0}^{\infty} (2^k)^{40.5n_K+81-2\ell} \ll \left(\frac{2}{AT_\star\mathscr{L}}\right)^{2\ell} T_\star^{40.5n_K},$$

since $\ell > \frac{1}{4}(81n_K + 162)$ and $T_\star$ is fixed, as desired. □

We further restrict the sum over zeros in Lemma 9.4 to zeros $\rho$ close to the line $\mathrm{Re}\{s\} = 1$. To simplify the statement, we also select parameters $\ell$ and $A$ for the weight function.

**Lemma 9.5.** *Let $T_\star \ge 1$ and $\eta \in (0,1)$ be fixed and $1 \le R \le \mathscr{L}$ be arbitrary. Suppose*

$$B - 2\ell A > \max\left\{162, \frac{\log \mathrm{N}\mathfrak{m}}{\mathscr{L}}\right\}, \qquad A = \frac{4}{\mathscr{L}}, \qquad \ell = \lfloor \eta\mathscr{L} \rfloor. \tag{9-8}$$

*If $\mathscr{L}$ is sufficiently large then*

$$\left| h_H \mathscr{L}^{-1} S - F(0) + \overline{\chi_1}(C) F((1-\rho_1)\mathscr{L}) \right|$$
$$\le \sum_{\chi \pmod H} \sideset{}{^\star}\sum_{\rho} |F((1-\rho)\mathscr{L})| + O(e^{-(B-2\ell A-162)R} + (2T_\star)^{-2\eta\mathscr{L}} e^{\eta\mathscr{L}} + e^{-78\mathscr{L}})$$

*where the marked sum $\sum^\star$ runs over zeros $\rho \ne \rho_1$ of $L(s,\chi)$, counting with multiplicity, satisfying $1 - R/\mathscr{L} < \mathrm{Re}\{\rho\} < 1$ and $|\mathrm{Im}\{\rho\}| \le T_\star$.*

*Proof.* For $\mathscr{L}$ sufficiently large depending on $\epsilon$ and $\eta$, the quantities $B$, $A$, and $\ell$ satisfy the assumptions of Lemma 9.4. Denote $B' = B - 2\ell A$. We claim it suffices to show

$$\sum_{\chi \pmod H} \sideset{}{'}\sum_{\mathrm{Re}\{\rho\} \le 1-R/\mathscr{L}} |F((1-\rho)\mathscr{L})| \ll e^{-(B'-162)R}, \tag{9-9}$$

where $\sum'$ is defined in Lemma 9.4. To see the claim, we need only show that the error term in Lemma 9.4 is absorbed by that of Lemma 9.5. For $\mathscr{L}$ sufficiently large, notice $T_\star^{40.5n_K} \le e^{\eta\mathscr{L}}$ as $n_K \log T_\star = o(\mathscr{L})$; hence, for our choices of $A$ and $\ell$, we have

$$\left(\frac{2}{AT_\star\mathscr{L}}\right)^{2\ell} T_\star^{40.5n_K} \le \left(\frac{1}{2T_\star}\right)^{2\eta\mathscr{L}} e^{\eta\mathscr{L}}.$$

This proves the claim. Now, to establish (9-9), define the multiset of zeros

$$\mathcal{R}_m(\chi) := \Big\{\rho : L(\rho,\chi)=0,\ 1-\frac{m+1}{\mathscr{L}} \le \mathrm{Re}\{\rho\} \le 1-\frac{m}{\mathscr{L}},\ |\mathrm{Im}(\rho\}| \le T_\star\Big\}$$

for $1 \le m \le \mathscr{L}$. By Theorem 8.5 and Lemma 9.1, it follows that

$$\sum_{\chi \pmod H} \sum_{\rho\in R_m(\chi)} |F((1-\rho)\mathscr{L})| \le e^{-B'm} \sum_{\chi \pmod H} \#\mathcal{R}_m(\chi) \ll e^{-(B'-162)m}$$

for $\mathscr{L}$ sufficiently large. Summing over $m \ge R$ yields the desired conclusion. □

## 10. Proof of Theorem 3.1: exceptional case

For this section, we assume $\lambda_1 < 0.0875$. By Theorem 8.1, $\rho_1$ is a simple real zero and $\chi_1$ is a real Hecke character. For fixed $\eta \in (0, 10^{-3})$ sufficiently small, assume $\mathscr{L}$ is sufficiently large and that

$$B \ge \max\Big\{163, \frac{\log \mathrm{N}\mathfrak{m}}{\mathscr{L}} + 8\eta\Big\}, \quad \ell = \lfloor \eta\mathscr{L} \rfloor, \quad \text{and} \quad A = \frac{4}{\mathscr{L}}.$$

Thus $B, \ell$, and $A$ satisfy (9-8) and $B' := B - 2\ell A > 162$. For the moment, we do not make any additional assumptions on the minimum size of $B$ and hence $B'$. To prove Theorem 3.1 when $\rho_1$ is an exceptional zero, it suffices to show, by Lemma 9.2, that $h_H \mathscr{L}^{-1} S \gg \min\{1, \lambda_1\}$ for $B \ge \max\{593, (\log \mathrm{N}\mathfrak{m})/\mathscr{L} + 8\eta\}$ and $\mathscr{L}$ sufficiently large.

For a nontrivial zero $\rho$ of a Hecke $L$-function, write $\rho = \beta + i\gamma = \big(1 - \frac{\lambda}{\mathscr{L}}\big) + i\gamma$, so that by Lemma 9.1, $|F((1-\rho)\mathscr{L})| \le e^{-B'\lambda}$. From Lemma 9.5, with $T_\star \ge 1$ fixed and $1 \le R \le \mathscr{L}$ arbitrary, it follows that if we define

$$\Delta = \begin{cases} \eta & \text{if } T_\star = 1 \\ & \quad \text{and } R = R(\eta) \text{ is sufficiently large,} \\ O(e^{-(B'-162)R} + e^{-78\mathscr{L}}) & \text{if } T_\star = T_\star(\eta) \text{ is sufficiently large} \\ & \quad \text{and } 1 \le R \le \mathscr{L}, \end{cases} \tag{10-1}$$

then

$$h_H \mathscr{L}^{-1} S \ge 1 - \chi_1(\mathcal{C}) e^{-B'\lambda_1} - \sum_{\chi \pmod H} \sideset{}{^\star}\sum_{\rho} e^{-B'\lambda} - \Delta, \tag{10-2}$$

where the restricted sum $\sum^\star$ is over zeros $\rho \ne \rho_1$, counted with multiplicity, satisfying $0 < \lambda \le R$ and $|\gamma| \le T_\star$.

Suppose the arbitrary parameter $\lambda^\star > 0$ satisfies

$$\lambda > \lambda^\star \quad \text{for every zero } \rho \text{ occurring in the restricted sum of (10-2).} \tag{10-3}$$

It remains for us to divide into cases according to the range of $\lambda_1$ and value of $\chi_1(\mathcal{C}) \in \{\pm 1\}$. In each case, we make a suitable choice for $\lambda^\star$.

**10A.** ***Moderate exceptional zero*** **($\eta \le \lambda_1 < 0.0875$ *or* $\chi_1(\mathcal{C}) = -1$).** For the moment, we do not make any assumptions on the size of $\lambda_1$ other than that $0 < \lambda_1 < 0.0875$. Select $T_\star = 1$ and $R = R(\eta)$ sufficiently large so $\Delta = \eta$ according to (10-1). By partial summation, our choice of $\lambda^\star$ in (10-2), and Theorem 8.6,

$$\sum_{\chi \pmod{H}} \sideset{}{^\star}\sum_{\rho} e^{-B'\lambda} \le \int_{\lambda^\star}^{R} e^{-B'\lambda}\, d\mathcal{N}(\lambda) \le e^{-(B'-162)R+188} + \int_{\lambda^\star}^{\infty} B' e^{-(B'-162)\lambda+188}\, d\lambda.$$

As $R = R(\eta)$ is sufficiently large and $B' > 162$, the above is

$$\le \left(1 - \frac{162}{B'}\right)^{-1} e^{188-(B'-162)\lambda^\star} + \eta.$$

Comparing with (10-2), we have

$$h_H \mathscr{L}^{-1} S \ge 1 - \chi_1(\mathcal{C}) e^{-B'\lambda_1} - \left(1 - \frac{162}{B'}\right)^{-1} e^{-(B'-162)\lambda^\star+188} - 2\eta. \tag{10-4}$$

Finally, we further subdivide into cases according to the size of $\lambda_1$ and value of $\chi_1(\mathcal{C}) \in \{\pm 1\}$. Recall $\eta > 0$ is sufficiently small.

*Case 1: $\lambda_1$ medium* ($10^{-3} \le \lambda_1 < 0.0875$). Here we also assume $B \ge 593$, in which case $B' \ge 592$. Select $\lambda^\star = 0.44$, which, by Theorem 8.2, satisfies (10-3) for the specified range of $\lambda_1$. Incorporating this estimate into (10-4) and noting $|\chi_1(\mathcal{C})| \le 1$, we deduce

$$h_H \mathscr{L}^{-1} S \ge 1 - e^{-592\times 10^{-3}} - \tfrac{592}{430} e^{-430\times 0.44+188} - 2\eta \ge 0.032 - 2\eta$$

for $\lambda \in [10^{-3}, 0.0875]$. Hence, for $\eta$ sufficiently small, $h_H \mathscr{L}^{-1} S \gg 1$ in this subcase, as desired.

*Case 2: $\lambda_1$ small* ($\eta \le \lambda_1 < 10^{-3}$). Here we also assume $B \ge 297$, in which case $B' \ge 296.5$. Select $\lambda^\star = 0.2103 \log(1/\lambda_1)$, which, by Theorem 8.2, satisfies (10-3). For $\lambda < 10^{-3}$, this implies $\lambda^\star > 1.45$. Applying both of these facts in (10-4) and noting $|\chi_1(\mathcal{C})| \le 1$, we see

$$\begin{aligned} h_H \mathscr{L}^{-1} S &\ge 1 - e^{-296.5\lambda_1} - \tfrac{296}{134} e^{-(134.5-188/1.45)\lambda^\star} - 2\eta \\ &\ge 1 - e^{-296.5\lambda_1} - \tfrac{296}{134}\lambda_1 - 2\eta \end{aligned}$$

since $4.84 \times 0.2103 = 1.017\cdots > 1$. As $1 - e^{-x} \ge x - \frac{x^2}{2}$ for $x \ge 0$, the above is

$$\ge 296.5\lambda_1 - \tfrac{(296.5)^2}{2}\lambda_1^2 - \tfrac{296}{134}\lambda_1 - 2\eta \ge 294.2\lambda_1(1 - 150\lambda_1) - 2\eta \ge 250\eta$$

because $\eta \le \lambda_1 < 10^{-3}$. Therefore, $h_H \mathscr{L}^{-1} S \gg 1$, completing the proof of this subcase.

*Case 3: $\lambda_1$ very small ($\lambda_1 < \eta$) and $\chi_1(\mathcal{C}) = -1$.* Here we also assume $B \geq 163$, in which case $B' > 162.5$. From (10-4), it follows that

$$h_H \mathscr{L}^{-1} S \geq 1 + e^{-162.5\lambda_1} - 325 e^{-0.5\lambda^\star + 188} - 2\eta \geq 2 - O(e^{-0.5\lambda^\star} + \eta + \lambda_1).$$

By Theorem 8.3, the choice $\lambda^\star = \frac{1}{81}\log(c_{11}/\lambda_1)$ satisfies (10-3) for some absolute constant $c_{11} > 0$. Since $\lambda_1 < \eta$, the above is therefore

$$\geq 2 - O(\eta^{0.5/81} + \eta) \geq 2 - O(\eta^{1/162}).$$

As $\eta$ is fixed and sufficiently small, we conclude $h_H \mathscr{L}^{-1} S \gg 1$ as desired. This completes the proof for a "moderate" exceptional zero.

**10B. *Truly exceptional zero* ($\lambda_1 < \eta$ *and* $\chi_1(\mathcal{C}) = +1$).** Select $T_\star = T_\star(\eta)$ sufficiently large and let $R = \frac{1}{80.1}\log(c_{12}/\lambda_1)$, where $c_{12} > 0$ is a sufficiently small absolute constant. By Theorem 8.3, it follows that the restricted sum over zeros $\rho$ in (10-2) is empty and therefore, by (10-2) and (10-1),

$$h_H \mathscr{L}^{-1} S \geq 1 - e^{-B'\lambda_1} - O(\lambda_1^{(B'-162)/80.1} + e^{-78\mathscr{L}})$$

as $\chi_1(\mathcal{C}) = 1$. Additionally assuming $B \geq 243$, in which case $B' \geq 242.2$, and noting $1 - e^{-x} \geq x - \frac{x^2}{2}$ for $x \geq 0$, we conclude that

$$\begin{aligned} h_H \mathscr{L}^{-1} S &\geq 242.2\lambda_1 - O(\lambda_1^2 + \lambda_1^{80.2/80.1} + e^{-78\mathscr{L}}) \\ &\geq \lambda_1(242.2 - O(\lambda_1^{0.001} + e^{-73\mathscr{L}})) \end{aligned}$$

since $\lambda_1 \gg e^{-4.8\mathscr{L}}$ by Theorem 8.4. As $\lambda_1 \leq \eta$ for fixed $\eta > 0$ sufficiently small, we conclude $h_H \mathscr{L}^{-1} S \gg \lambda_1$ as desired.

Comparing all cases, we see that the most stringent condition is $B \geq 593$, thus completing the proof of Theorem 3.1 in the exceptional case. □

**Remark.** When $H \pmod{\mathfrak{q}}$ is primitive, the "truly exceptional" subcase considered in Section 10B is implied by a numerically much stronger result of Zaman [2016b, Theorem 1.1] using entirely different methods.

## 11. Proof of Theorem 3.1: nonexceptional case

For this section, we assume $\lambda_1 \geq 0.0875$. Thus, we no longer have any additional information as to whether $\rho_1$ is real or not, or whether $\chi_1$ is real or not. We proceed in a similar fashion as the exceptional case, but require a slightly more refined analysis due to the absence of the Deuring–Heilbronn phenomenon. Assume $\lambda^\star > 0$ satisfies $\lambda^\star < \min\{\lambda', \lambda_2\}$, where $\lambda'$ and $\lambda_2$ are defined in Section 8B. For $0 < \eta \leq 10^{-3}$ fixed, suppose $B \geq \max\{693.5, (\log \mathrm{N}\mathfrak{m})/\mathscr{L} + 8\eta\}$, $\ell = \lfloor \eta \mathscr{L} \rfloor$, and $A = 4/\mathscr{L}$. Thus $B, \ell$, and $A$ satisfy (9-8). By Lemma 9.2, it suffices to show $h_H \mathscr{L}^{-1} S \gg 1$. For simplicity, denote $B' = B - 2\ell A \geq 693$. For a nontrivial zero

$\rho$ of a Hecke $L$-function, as usual, write $\rho = \beta + i\gamma = (1 - \lambda/\mathscr{L}) + i\mu/\mathscr{L}$. From Lemma 9.5, as $F(0) = 1$, it follows that

$$h_H \mathscr{L}^{-1} S \geq 1 - |F(\lambda_1 + i\mu_1)| - |F(\lambda_1 - i\mu_1)| - \sum_{\chi \ (\mathrm{mod}\ H)} \sum_{\rho}{}^{\dagger} |F(\lambda + i\mu)| - \eta,$$

where the marked sum $\sum^{\dagger}$ runs over nontrivial zeros $\rho \neq \rho_1$ (or $\rho \neq \rho_1, \overline{\rho_1}$ if $\rho_1$ is complex) of $L(s,\chi)$, counted with multiplicity, satisfying $\lambda^\star \leq \lambda \leq R$ and $|\gamma| \leq 1$ for some $R = R(\eta) \geq 1$ sufficiently large. By Lemma 9.1, this implies

$$h_H \mathscr{L}^{-1} S \geq 1 - 2e^{-B'\lambda_1} - \sum_{\chi \ (\mathrm{mod}\ H)} \sum_{\substack{\lambda^\star \leq \lambda \leq R \\ |\gamma| \leq 1}} e^{-B'\lambda} - \eta. \tag{11-1}$$

Let $\Lambda > 0$ be a fixed parameter to be specified later. To bound the remaining sum over zeros, we apply partial summation using the quantity $\mathcal{N}(\lambda)$, defined in (8-5), over two different ranges: (i) $\lambda^\star \leq \lambda \leq \Lambda$ and (ii) $\Lambda < \lambda \leq R$.

For (i), partition the interval $[\lambda^\star, \Lambda]$ into $M$ subintervals with sample points

$$\lambda^\star = \Lambda_0 < \Lambda_1 < \Lambda_2 < \cdots < \Lambda_M = \Lambda.$$

By partial summation, we see

$$\begin{aligned} Z_1 := \sum_{\chi \ (\mathrm{mod}\ H)} \sum_{\substack{\lambda^\star < \lambda \leq \Lambda \\ |\gamma| \leq 1}} e^{-B'\lambda} &= \sum_{j=1}^{M} \sum_{\chi \ (\mathrm{mod}\ H)} \sum_{\Lambda_{j-1} < \lambda \leq \Lambda_j} e^{-B'\lambda} \\ &\leq e^{-B'\Lambda_{M-1}} \mathcal{N}(\Lambda_M) + \sum_{j=1}^{M-1} (e^{-B'\Lambda_{j-1}} - e^{-B'\Lambda_j}) \mathcal{N}(\Lambda_j). \end{aligned}$$

By Theorem 8.1, we may choose $\lambda^\star = 0.2866$. Furthermore, we select

$$\Lambda = 1, \qquad M = 32, \qquad \Lambda_r = \begin{cases} 0.286 + 0.001r, & 1 \leq r \leq 14, \\ 0.300 + 0.025(r - 14), & 15 \leq r \leq 22, \\ 0.5 + 0.05(r - 22), & 23 \leq r \leq 32, \end{cases}$$

and incorporate the estimates from Table 1 to bound $\mathcal{N}(\cdot)$, yielding $Z_1 \leq 0.9926$.

For (ii), apply partial summation along with Theorem 8.6. Since $B' \geq 693 > 162$ and $R = R(\eta)$ is sufficiently large, it follows that

$$Z_2 := \sum_{\chi \ (\mathrm{mod}\ H)} \sum_{\substack{\Lambda < \lambda \leq R \\ |\gamma| \leq 1}} e^{-B'\lambda} \leq e^{188 - (B' - 162)R} + B' \int_{\Lambda}^{\infty} e^{188 - (B' - 162)\lambda}\, d\lambda$$

for $\mathscr{L}$ sufficiently large depending on $\eta$. Evaluating the right-hand side with $B' \geq 693$ and $\Lambda = 1$, we deduce $Z_2 \leq 10^{-400}$.

Incorporating (i) and (ii) into (11-1), we conclude

$$h_H \mathscr{L}^{-1} S \geq 1 - 2e^{-B'\lambda_1} - 0.9926 - 10^{-400} - 2\eta \geq 0.0073 - 2\eta$$

as $\lambda_1 > 0.0875$ and $B' \geq 693$. Since $\eta \in (0, 10^{-3}]$ is fixed and sufficiently small, we conclude $h_H \mathscr{L}^{-1} S \gg 1$. This completes the proof of Theorem 3.1. □

## 12. Proofs of Theorems 1.2–1.5

*Proof of Theorem 1.2.* Let $Q(x, y) \in \mathbb{Z}[x, y]$ be a positive-definite primitive binary quadratic form of discriminant $D$. Let $K = \mathbb{Q}(\sqrt{D})$, and let $L$ be the ring class field of the order of discriminant $D$ in $K$. By Theorem 9.12 of [Cox 1989], the rational primes $p \nmid D$ represented by $Q$ are the primes which split in $K$ that satisfy a certain Chebotarev condition in $L$. We have that $D_K \mathcal{Q} \leq |D|$. The result follows. □

We now state a slightly weaker version of (3-2) and Theorem 1.1 which will be convenient for the remaining proofs. For positive integers $n$, let $\omega(n) = \#\{p : p \mid n\}$ and $\mathrm{rad}(n) = \prod_{p \mid n} p$.

**Theorem 12.1.** *Let $L/F$ be a Galois extension of number fields with Galois group $G$ and $L \neq \mathbb{Q}$, and let $C$ be any conjugacy class of $G$. Let $H$ be an abelian subgroup of $G$ such that $H \cap C$ is nonempty, and let $K \neq \mathbb{Q}$ be the subfield of $L$ fixed by $H$. Define*

$$M(L/K) := [L : K]^{3/2} n_K^{\omega(D_L)} \mathrm{rad}(D_L)^{5/2}.$$

*If $(M(L/K) n_K)^{n_K}$ is sufficiently large and*

$$x \gg [L : K]^{n_K} \mathrm{rad}(D_L)^{n_K - 694} M(L/K)^{694 n_K},$$

*then*

$$\pi_C(x, L/F) \gg \frac{(M(L/K) n_K)^{-15 n_K/2}}{[L : K]} \frac{x}{\log x}.$$

*Consequently, for all $L/F$, we have that*

$$P(C, L/F) \ll [L : K]^{n_K} \mathrm{rad}(D_L)^{n_K - 694} M(L/K)^{694 n_K}.$$

*Proof.* Let $\mathcal{P}(L/K)$ be the set of rational primes $p$ such that there is a prime ideal $\mathfrak{p}$ of $K$ such that $\mathfrak{p} \mid p$ and $\mathfrak{p}$ ramifies in $L$. By [Serre 1981, Proposition 6], $D_K \leq (n_K)^{n_K \omega(D_K)} \mathrm{rad}(D_K)^{n_K - 1}$. Since $L/K$ is abelian, we have by [Murty et al. 1988, Proposition 2.5] that

$$\mathcal{Q} \leq \Bigl( [L : K] \prod_{p \in \mathcal{P}(L/K)} p \Bigr)^{2 n_K}.$$

The primes in $\mathcal{P}(L/K)$ and the primes dividing $D_K$ all divide $D_L$. Since $K \neq \mathbb{Q}$, we have $\omega(D_K) \geq 1$ and $n_K \geq 2$. Thus the result follows from Theorem 1.1, and in particular (3-2). □

**Remark.** For comparison, if one uses [Serre 1981, Proposition 6] to bound $D_L$, then (1-5) implies that $P(C, L/F) \ll (n_L^{\omega(D_L)} \operatorname{rad}(D_L))^{40 n_L}$. We can replace $\omega(D_L)$ with 1 if $L/\mathbb{Q}$ is Galois.

**12A. $\mathrm{GL}_2$ *extensions.*** We now review some facts about $\mathrm{GL}_2$ extensions of $\mathbb{Q}$ and class functions to prove Theorems 1.3–1.5. Let

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z} \in \mathbb{Z}[\![e^{2\pi i z}]\!]$$

be a non-CM newform of even weight $k \geq 2$ and level $N \geq 1$. Let $\ell$ be a prime, and let $\mathbb{F}_\ell$ be the finite field of $\ell$ elements. By [Deligne 1971], there exists a representation

$$\rho_{f,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_\ell)$$

with the property that if $p \nmid \ell N$ and $\sigma_p$ is a Frobenius element at $p$ in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then $\rho_{f,\ell}$ is unramified at $p$, $\operatorname{tr} \rho_{f,\ell}(\sigma_p) \equiv a_f(p) \pmod{\ell}$, and $\det \rho_{f,\ell}(\sigma_p) \equiv p^{k-1} \pmod{\ell}$. Let $L = L_{f,\ell}$ be the subfield of $\overline{\mathbb{Q}}$ fixed by the kernel of $\rho_{f,\ell}$. Then $L/\mathbb{Q}$ is a Galois extension unramified outside $\ell N$ whose Galois group $\mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to a subgroup of

$$G = G_{k,\ell} = \{A \in \mathrm{GL}_2(\mathbb{F}_\ell) : \det A \text{ is a } (k-1)\text{-th power in } \mathbb{F}_\ell^\times\}.$$

If $\ell$ is sufficiently large, then the representation is surjective, in which case

$$\mathrm{Gal}(L/\mathbb{Q}) \cong G. \tag{12-1}$$

When $k = 2$ and the level is $N$, $f$ is necessarily the newform of a non-CM elliptic curve $E/\mathbb{Q}$ of conductor $N$. In this case, we write $\rho_{f,\ell} = \rho_{E,\ell}$, and $L$ is the $\ell$-division field $\mathbb{Q}(E[\ell])$. It is conjectured that $\mathrm{Gal}(L/\mathbb{Q}) \cong \mathrm{GL}_2(\mathbb{F}_\ell)$ for all $\ell > 37$. When $E/\mathbb{Q}$ is non-CM and has squarefree level, it follows from the work of Mazur [1978] that $\ker \tilde{\rho}_{E,\ell} \cong \mathrm{GL}_2(\mathbb{F}_\ell)$ for all $\ell \geq 11$.

**Lemma 12.2.** *Let $L/\mathbb{Q}$ be a $\mathrm{GL}_2(\mathbb{F}_\ell)$ extension which is unramified outside of $\ell N$ for some $N \geq 1$. Let $C \subset \mathrm{GL}_2(\mathbb{F}_\ell)$ be a conjugacy class intersecting the subgroup $D$ of diagonal matrices. There exists a prime $p \nmid \ell N$ such that*

$$p \ll \ell^{(5209+1542\omega(N))\ell^2} \operatorname{rad}(N)^{1737\ell(\ell+1)} \quad \textit{and} \quad \left[\frac{L/\mathbb{Q}}{p}\right] = C.$$

*Proof.* If $K = L^D$ is the subfield of $L$ fixed by $D$, then $[L : K] = (\ell - 1)^2$ and $[K : \mathbb{Q}] = \ell(\ell + 1)$. Moreover, $\operatorname{rad}(D_L) \mid \ell \operatorname{rad}(N)$. The result now follows immediately from Theorem 12.1. □

*Proof of Theorem 1.3.* It follows from the proof of [Murty 1994, Theorem 4] and Mazur's torsion theorem [1978] that it suffices to consider $\ell \geq 11$. Let $L = \mathbb{Q}(E[\ell])$

be the $\ell$-division field of $E/\mathbb{Q}$. For $p \nmid \ell N_E$, we have that $E(\mathbb{F}_p)$ has an element of order $\ell$ if and only if

$$\operatorname{tr}\rho_{\ell,E}(\sigma_p) \equiv \det\rho_{\ell,E}(\sigma_p) + 1 \pmod{\ell}, \tag{12-2}$$

where $\sigma_p$ is a Frobenius automorphism at $p$ in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. If $\mathrm{Gal}(L/\mathbb{Q}) \cong \mathrm{GL}_2(\mathbb{F}_\ell)$, then the $\rho_{\ell,E}(\sigma_p) \in \mathrm{GL}_2(\mathbb{F}_\ell)$ which satisfy (12-2) form a union of conjugacy classes in $\mathrm{GL}_2(\mathbb{F}_\ell)$ which includes the identity element. The subgroup $D$ of diagonal matrices is a maximal abelian subgroup of $\mathrm{GL}_2(\mathbb{F}_\ell)$. Thus $\pi_{\{\mathrm{id}\}}(x, L/\mathbb{Q})$ is a lower bound for the function that counts the primes $p \le x$ such that $p \nmid \ell N_E$ and $\ell \mid \#E(\mathbb{F}_p)$. Since $\mathrm{rad}(D_L) \mid \ell\,\mathrm{rad}(N)$, Lemma 12.2 implies the claimed result.

Suppose now that $\mathrm{Gal}(L/\mathbb{Q})$ is not isomorphic to $\mathrm{GL}_2(\mathbb{F}_\ell)$. The possible cases are described in the proof of [Murty 1994, Theorem 4]. Applying similar analysis to all of these cases, one sees that the above case gives the largest upper bound for the least prime $p$ such that $\ell \mid \#E(\mathbb{F}_p)$. □

We require some basic results on class functions (see [Serre 1981]) for the proof of Theorem 1.5. Let $L/F$ be a Galois extension of number fields with Galois group $G$, and let $\phi : G \to \mathbb{C}$ be a class function. For each prime ideal $\mathfrak{p}$ of $F$, choose any prime ideal $\mathfrak{P}$ of $L$ dividing $\mathfrak{p}$. Let $D_{\mathfrak{P}}$ and $I_{\mathfrak{P}}$ be the decomposition and inertia subgroups of $G$ at $\mathfrak{p}$, respectively. We then have a distinguished Frobenius element $\sigma_{\mathfrak{P}} \in D_{\mathfrak{P}}/I_{\mathfrak{P}}$. For each $m \ge 1$, let

$$\phi(\mathrm{Frob}_{\mathfrak{p}}^m) = \frac{1}{|I_{\mathfrak{P}}|} \sum_{\substack{g \in D_{\mathfrak{P}} \\ gI_{\mathfrak{P}} = \sigma_{\mathfrak{P}}^m \in D_{\mathfrak{P}}/I_{\mathfrak{P}}}} \phi(g).$$

Note that $\phi(\mathrm{Frob}_{\mathfrak{p}}^m)$ is independent of the aforementioned choice of $\mathfrak{P}$. If $\mathfrak{p}$ is unramified in $L$, this definition agrees with the value of $\phi$ on the conjugacy class $\mathrm{Frob}_{\mathfrak{p}}^m$ of $G$. For $x \ge 2$, we define

$$\pi_\phi(x) = \sum_{\substack{\mathfrak{p} \text{ unramified in } L \\ \mathrm{N}_{F/\mathbb{Q}}\mathfrak{p} \le x}} \phi(\mathrm{Frob}_{\mathfrak{p}}), \qquad \tilde{\pi}_\phi(x) = \sum_{\substack{\mathfrak{p} \text{ unramified in } L \\ \mathrm{N}_{F/\mathbb{Q}}\mathfrak{p}^m \le x}} \frac{1}{m}\phi(\mathrm{Frob}_{\mathfrak{p}}^m).$$

Let $C \subset G$ be stable under conjugation, and let $\mathbf{1}_C : G \to \{0, 1\}$ be the class function given by the indicator function of $C$. Now, define $\pi_C(x, L/F) = \pi_{\mathbf{1}_C}(x)$ and $\tilde{\pi}_C(x, L/F) = \tilde{\pi}_{\mathbf{1}_C}(x)$. Serre [1981, Proposition 7] proved that if $x \ge 2$, then

$$|\pi_C(x, L/F) - \tilde{\pi}_C(x, L/F)| \le 4n_F((\log D_L)/n_L + \sqrt{x}). \tag{12-3}$$

By arguments similar to the proof of Theorem 1.1, we have that if $A$ is an abelian subgroup of $G$ such that $A \cap C$ is nonempty, then $\tilde{\pi}_C(x, L/F) = \tilde{\pi}_{\mathrm{Ind}_A^G C}(x, L/L^A)$.

*Proof of Theorem 1.5.* Let $\ell$ be an odd prime such that (12-1) is satisfied. Assuming $\gcd(k-1,\ell-1)=1$, we have $G\cong \mathrm{GL}_2(\mathbb{F}_\ell)$. To prove the theorem, we consider

$$\pi_f(x;\ell,a) := \#\{p\le x : p\nmid \ell N,\ a_f(p)\equiv a \pmod{\ell},\\ \ell \text{ splits in } \mathbb{Q}((a_f(p)^2-4p^{k-1})^{1/2})\}.$$

Note that for $p\nmid \ell N$, $a_f(p)^2-4p^{k-1}=\mathrm{tr}(\rho_{f,\ell}(\sigma_p))-4\det(\rho_{f,\ell}(\sigma_p))^2$, where $\sigma_p$ is Frobenius at $p$ in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The subset $C\subset G$ given by

$$C=\{A\in G : \mathrm{tr}(A)\equiv a \pmod{\ell},\ \mathrm{tr}(A)^2-4\det(A) \text{ is a square in } \mathbb{F}_\ell^\times\}$$

is a conjugacy-invariant subset of $G$, so we bound $\tilde{\pi}_C(x,L/\mathbb{Q})$. Let $B\subset G$ denote the subgroup of upper triangular matrices; the condition that $\mathrm{tr}(A)^2-4\det(A)$ is a square in $\mathbb{F}_\ell^\times$ means that $\sigma_p$ is conjugate to an element in $B$. If $\Gamma$ is a maximal set of elements $\gamma\in B$ which are nonconjugate in $G$ with $\mathrm{tr}(\gamma)\equiv a \pmod{q}$, then $C=\bigsqcup_{\gamma\in\Gamma} C_G(\gamma)$, where $C_G(\gamma)$ denotes the conjugacy class of $\gamma$ in $G$. Since $B$ is a subgroup of $G$ with the property that every element of $C$ is conjugate to an element of $B$, it follows from [Zywina 2015, Lemma 2.6] that

$$\tilde{\pi}_C(x,L/\mathbb{Q})=\sum_{\gamma\in\Gamma}\frac{\tilde{\pi}_{C_B(\gamma)}(x,L/L^B)}{[\mathrm{Cent}_G(\gamma):\mathrm{Cent}_B(\gamma)]},$$

where $\mathrm{Cent}_G(\gamma)$ denotes the centralizer of $\gamma$ in $G$ (and similarly for $B$). If $C_1=\bigsqcup_{\gamma\,\in\,\Gamma \text{ nonscalar}} C_B(\gamma)$, then it follows that $\tilde{\pi}_C(x;L/\mathbb{Q})\ge \frac{1}{|G|}\tilde{\pi}_{C_1}(x,L/L^B)$ for all $x\ge 2$.

*Case 1: $\ell N$ sufficiently large, $a\not\equiv 0 \pmod{\ell}$.* Let $U$ be the normal subgroup of $B$ consisting of the matrices whose diagonal entries are both 1. We observe that $U\cdot C_1\subset C_1$; therefore, using arguments from [Zywina 2015, Lemma 2.6], we have that $\tilde{\pi}_{C_1}(x,L/L^B)=\tilde{\pi}_{C_2}(x,L^U/L^B)$ for $x\ge 2$, where $C_2$ is the image of $C_1\cap B$ in $B/U$. It follows from (12-3) and Theorem 12.1 that if $\ell N$ is sufficiently large and $x$ is bounded below as in Theorem 12.1, then

$$\tilde{\pi}_{C_2}(x,L^U/L^B)>0 \quad\text{if and only if}\quad \pi_{C_2}(x,L^U/L^B)>0. \tag{12-4}$$

It is straightforward to compute $n_{L^B}=\ell+1$ and $[L^U:L^B]=(\ell-1)^2$. Since $L^U/L^B$ is abelian and all of the ramified primes divide $\ell N$, the theorem now follows from Theorem 12.1.

*Case 2: $\ell N$ sufficiently large, $a\equiv 0 \pmod{\ell}$.* Let $H$ be the normal subgroup of $B$ consisting of matrices whose eigenvalues are both equal. We have that $H\cdot C_1\subset C_1$ since multiplying a trace zero matrix by a scalar does not change the trace. Let $C_3$ be the image of $C_1\cap B$ in $B/H$. The arguments are now the same as in the previous case, with $L^H$ replacing $L^U$. In fact, since $B/H\cong \mathbb{F}_\ell^\times$ is abelian of order

$\ell - 1$ and $C_3$ is a singleton, we obtain a slightly better exponent than what is stated in Theorem 1.5 when $a \equiv 0 \pmod{\ell}$.

*Case 3: $\ell N$ not sufficiently large.* Let $A_2 = U$ and $A_3 = H$. The lower bound for $\pi_{C_i}(x, L^{A_i}/L^B)$ ($i = 2$ or $3$) given by Theorem 12.1 only holds when $\ell N$ is sufficiently large. Therefore, when $\ell N$ is not sufficiently large, we cannot verify (12-4) using Theorem 12.1. For these finitely many exceptional cases, we use Weiss' lower bound on $\pi_{C_i}(x, L^{A_i}/L^B)$ that follows from [Weiss 1983, Theorem 5.2], which holds uniformly for all choices of $N$ and $\ell$. Continuing the proof as in Case 1 (this requires us to take $c_{10}$ sufficiently small and $c_{11}$ to be sufficiently large in [Weiss 1983, Theorem 5.2]), we see that the least prime $p \nmid \ell N$ such that $a_f(p) \equiv a \pmod{\ell}$ is absolutely bounded in all of the finitely many exceptional cases. This proves the theorem. □

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## References

[Ahn and Kwon 2014] J.-H. Ahn and S.-H. Kwon, "Some explicit zero-free regions for Hecke $L$-functions", *J. Number Theory* **145** (2014), 433–473. MR Zbl

[Bach and Sorenson 1996] E. Bach and J. Sorenson, "Explicit bounds for primes in residue classes", *Math. Comp.* **65**:216 (1996), 1717–1735. MR Zbl

[Cox 1989] D. A. Cox, *Primes of the form $x^2 + ny^2$: Fermat, class field theory and complex multiplication*, Wiley, New York, 1989. MR Zbl

[Deligne 1971] P. Deligne, "Formes modulaires et représentations $l$-adiques", exposé 355, pp. 139–172 in *Séminaire Bourbaki,* 1968/1969, Lecture Notes in Math. **175**, Springer, Berlin, 1971. MR Zbl

[Ditchen 2013] J. Ditchen, "On the average distribution of primes represented by binary quadratic forms", preprint, 2013. arXiv

[DLM 2010] F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, and B. V. Saunders (editors), "Digital library of mathematical functions", electronic reference, National Institute of Standards and Technology, 2010, available at http://dlmf.nist.gov/.

[Duke 1989] W. Duke, "Some problems in multidimensional analytic number theory", *Acta Arith.* **52**:3 (1989), 203–228. MR Zbl

[Fogels 1962a] E. Fogels, "On the zeros of Hecke's $L$-functions, I", *Acta Arith.* **7**:2 (1962), 87–106. MR Zbl

[Fogels 1962b] E. Fogels, "On the zeros of Hecke's $L$-functions, II", *Acta Arith.* **7**:2 (1962), 131–147. MR Zbl

[Gallagher 1970] P. X. Gallagher, "A large sieve density estimate near $\sigma = 1$", *Invent. Math.* **11** (1970), 329–339. MR Zbl

[Golod and Šafarevič 1964] E. S. Golod and I. R. Šafarevič, "On the class field tower", *Izv. Akad. Nauk SSSR Ser. Mat.* **28** (1964), 261–272. In Russian. MR Zbl

[Graham 1977] S. W. Graham, *Applications of sieve methods*, Ph.D. thesis, University of Michigan, 1977, available at http://search.proquest.com/docview/302861551.

[Graham 1978] S. Graham, "An asymptotic estimate related to Selberg's sieve", *J. Number Theory* **10**:1 (1978), 83–94. MR Zbl

[Heath-Brown 1992] D. R. Heath-Brown, "Zero-free regions for Dirichlet $L$-functions, and the least prime in an arithmetic progression", *Proc. London Math. Soc.* (3) **64**:2 (1992), 265–338. MR Zbl

[Ihara 2006] Y. Ihara, "On the Euler–Kronecker constants of global fields and primes with small norms", pp. 407–451 in *Algebraic geometry and number theory*, edited by V. Ginzburg, Progr. Math. **253**, Birkhäuser, Boston, 2006. MR Zbl

[Iwaniec and Kowalski 2004] H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications **53**, American Mathematical Society, Providence, RI, 2004. MR Zbl

[Jutila 1970] M. Jutila, "A new estimate for Linnik's constant", *Ann. Acad. Sci. Fenn. Ser. A I Math.* **471** (1970), 1–8. MR Zbl

[Jutila 1977] M. Jutila, "On Linnik's constant", *Math. Scand.* **41**:1 (1977), 45–62. MR Zbl

[Kadiri 2012] H. Kadiri, "Explicit zero-free regions for Dedekind zeta functions", *Int. J. Number Theory* **8**:1 (2012), 125–147. MR Zbl

[Kadiri and Ng 2012] H. Kadiri and N. Ng, "Explicit zero density theorems for Dedekind zeta functions", *J. Number Theory* **132**:4 (2012), 748–775. MR Zbl

[Kolesnik and Straus 1983] G. Kolesnik and E. G. Straus, "On the sum of powers of complex numbers", pp. 427–442 in *Studies in pure mathematics*, edited by P. Erdős, Birkhäuser, Basel, 1983. MR Zbl

[Kowalski and Michel 2002] E. Kowalski and P. Michel, "Zeros of families of automorphic $L$-functions close to 1", *Pacific J. Math.* **207**:2 (2002), 411–431. MR Zbl

[Lagarias and Odlyzko 1977] J. C. Lagarias and A. M. Odlyzko, "Effective versions of the Chebotarev density theorem", pp. 409–464 in *Algebraic number fields: $L$-functions and Galois properties* (Durham, UK, 1975), edited by A. Fröhlich, Academic Press, London, 1977. MR Zbl

[Lagarias et al. 1979] J. C. Lagarias, H. L. Montgomery, and A. M. Odlyzko, "A bound for the least prime ideal in the Chebotarev density theorem", *Invent. Math.* **54**:3 (1979), 271–296. MR Zbl

[Lamzouri et al. 2015] Y. Lamzouri, X. Li, and K. Soundararajan, "Conditional bounds for the least quadratic non-residue and related problems", *Math. Comp.* **84**:295 (2015), 2391–2412. MR Zbl

[Li 2012] X. Li, "The smallest prime that does not split completely in a number field", *Algebra Number Theory* **6**:6 (2012), 1061–1096. MR Zbl

[Linnik 1944a] U. V. Linnik, "On the least prime in an arithmetic progression, I: The basic theorem", *Rec. Math. [Mat. Sbornik] N.S.* **15(57)**:2 (1944), 139–178. MR Zbl

[Linnik 1944b] U. V. Linnik, "On the least prime in an arithmetic progression, II: The Deuring–Heilbronn phenomenon", *Rec. Math. [Mat. Sbornik] N.S.* **15(57)**:3 (1944), 347–368. MR Zbl

[Makai 1964] E. Makai, "On a minimum problem, II", *Acta Math. Acad. Sci. Hungar.* **15** (1964), 63–66. MR Zbl

[Mazur 1978] B. Mazur, "Rational isogenies of prime degree", *Invent. Math.* **44**:2 (1978), 129–162. MR Zbl

[Milne 2013] J. S. Milne, "Class field theory (version 4.02)", course notes, 2013, available at http://www.jmilne.org/math/CourseNotes/CFT.pdf.

[Murty 1994] V. K. Murty, "The least prime which does not split completely", *Forum Math.* **6**:5 (1994), 555–565. MR Zbl

[Murty et al. 1988] M. R. Murty, V. K. Murty, and N. Saradha, "Modular forms and the Chebotarev density theorem", *Amer. J. Math.* **110**:2 (1988), 253–281. MR Zbl

[Ono and Soundararajan 1997] K. Ono and K. Soundararajan, "Ramanujan's ternary quadratic form", *Invent. Math.* **130**:3 (1997), 415–454. MR Zbl

[Rademacher 1959] H. Rademacher, "On the Phragmén–Lindelöf theorem and some applications", *Math. Z* **72**:1 (1959), 192–204. MR Zbl

[Serre 1981] J.-P. Serre, "Quelques applications du théorème de densité de Chebotarev", *Inst. Hautes Études Sci. Publ. Math.* 54 (1981), 323–401. MR Zbl

[Stark 1974] H. M. Stark, "Some effective cases of the Brauer–Siegel theorem", *Invent. Math.* **23** (1974), 135–152. MR Zbl

[Weiss 1980] A. R. Weiss, *The least prime ideal with prescribed decomposition behaviour*, Ph.D. thesis, Ohio State University, 1980, available at http://search.proquest.com/docview/303045630.

[Weiss 1983] A. Weiss, "The least prime ideal", *J. Reine Angew. Math.* **338** (1983), 56–94. MR Zbl

[Xylouris 2011] T. Xylouris, "On the least prime in an arithmetic progression and estimates for the zeros of Dirichlet $L$-functions", *Acta Arith.* **150**:1 (2011), 65–91. MR Zbl

[Zaman 2016a] A. Zaman, "Explicit estimates for the zeros of Hecke $L$-functions", *J. Number Theory* **162** (2016), 312–375. MR Zbl

[Zaman 2016b] A. Zaman, "On the least prime ideal and Siegel zeros", *Int. J. Number Theory* **12**:8 (2016), 2201–2229. MR Zbl

[Zaman 2017a] A. Zaman, *Analytic estimates for the Chebotarev density theorem and their applications*, Ph.D. thesis, University of Toronto, 2017, available at http://tinyurl.com/zamanthesis.

[Zaman 2017b] A. Zaman, "Bounding the least prime ideal in the Chebotarev density theorem", *Funct. Approx. Comment. Math.* (online publication March 2017).

[Zywina 2015] D. Zywina, "Bounds for the Lang–Trotter conjectures", pp. 235–256 in *SCHOLAR: a scientific celebration highlighting open lines of arithmetic research*, edited by A. C. Cojocaru et al., Contemp. Math. **655**, American Mathematical Society, Providence, RI, 2015. MR Zbl

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