ALBANIAN JOURNAL
OF MATHEMATICS
Volume 4, Number 3, Pages 79–88
ISSN 1930-1235: (2010)

PRINCIPALLY SUPPLEMENTED MODULES

UMMAHAN ACAR AND ABDULLAH HARMANCI

ABSTRACT. In this paper, principally supplemented modules are defined as generalizations of lifting, principally lifting and supplemented modules. Several properties of these modules are proved. New characterizations of principally semiperfect rings are obtained using principally supplemented modules.

1. INTRODUCTION

Throughout this paper R denotes a ring with unity. Modules are unital right $R-$modules. Let M be a module and N, K be submodules of M. We call K *a supplement* of N in M if $M = K + N$ and $K \cap N$ is small in K. A module M is called *supplemented* if every submodule of M has a supplement in M. A module M is called *lifting* if, for all $N \leq M$, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B$ is small in M. Supplemented and lifting modules have been discussed by several authors(see [7], [8], [9], [13]) and these modules are useful in characterizing semiperfect rings(see [1]).

In this paper, principally supplemented modules are discovered as analogous of lifting and supplemented modules, and used to characterize principally semiperfect rings introduced in chapter 3 and discussed in [6].

Let M be a module and N a submodule module M. N is called a *small(or superfluous) submodule* if whenever M =N + X, we have $M = X$. A projective module P is called *a projective cover* of a module M if there exists an epimorphism $f : P \longrightarrow M$ with Ker(f) small in P, and a ring is called *semiperfect* if every simple R-module has a projective cover. For more detailed discussion on small submodules, semiperfect rings, we refer to [1].

In this paper, a module M is defined to be *principally supplemented* if for all cyclic submodule N of M, there exists a submodule X of M such that $M = N + X$ with $N \cap X$ is small X, and a module M is called *principally lifting* if, for all cyclic submodule N of M, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B$ is small in M. Principally lifting modules are considered as generalizations of lifting modules in [9].

In section 2, various properties of principally supplemented modules are obtained and in section 3 we study some applications our results. One of our main results can be stated as follows:
Let M be a projective module. Then M is principally semiperfect if and only if M is principally supplemented. Also we prove for a projective module M with

---

2000 *Mathematics Subject Classification.* 16L30; 16E50.

Key words and phrases. principally lifting module, principally supplemented module.

©2008 Aulona Press *(Albanian J. Math.)*

Rad(M) small in M, M is principally supplemented if and only if $M/\mathrm{Rad}(M)$ is principally semisimple.

In what follows, by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_n$ and $\mathbb{Z}/\mathbb{Z}n$ we denote, respectively, integers, rational numbers, the ring of integers modulo n and the $\mathbb{Z}$-module of integers modulo n. For unexplained concepts and notations, we refer the reader to [1, 11].

2. Small Submodules and Supplements

Let M be module. A submodule N of M is called a *small(or superfluous) submodule* if, whenever $M = N + X$, we have $M = X$. Small submodule is named *superfluous submodule* in [1]. We begin by stating the next lemma which is contained in context[1, 11].

Lemma 1. *Let M be a module. Then we have the following.*

(1). *If K is small in M and $f : M \to N$ is a homomorphism, then $f(K)$ is small in N. In particular, if K is small in $M \subseteq N$, then K is small in N.*

(2). *Let $K_1 \subseteq M_1 \subseteq M$, $K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2$ is small in $M_1 \oplus M_2$ if and only if K_1 is small in M_1 and $K_2$is small in M_2.*

(3). *Let N, K be submodules of M with K is small in M and $N \leq K$. Then N is also small in M.*

Lemma 2. *Let N and L be submodules of M. Then the following are equivalent:*
(1). *$M = N + L$ and $N \cap L$ is small in L.*
(2). *$M = N + L$ and for any proper submodule K of L, $M \neq N + K$.*

Proof. (1) $\Rightarrow$ (2) Let N and K be submodules of M with $M = N + K$. Then $L = (L \cap N) + K$. Since $L \cap N$ is small in L, $L = K$.
(2) $\Rightarrow$ (1) If $L = (N \cap L) + K$ where $K \leq L$, then $M = N + L = N + K$. By (2), $K = L$. So $N \cap L$ is small in L. □

Lemma 3. *If $M \xrightarrow{f} M'$ is a homomorphism and N is a supplement in M with $Ker(f) \leq N$, then $f(N)$ is a supplement in $f(M)$.*

Proof. Let $M = N + K$ with $N \cap K$ small in K. Then $f(M) = f(N + K) = f(N) + f(K)$. Since $\mathrm{Ker}(f) \leq N$, we have $f(N) \cap f(K) = f(N \cap K)$. By Lemma 1 and being $f(N \cap K)$ small in $f(M)$, $f(N)$ is a supplement of $f(K)$ in $f(M)$. □

Lemma 4. *Let M be an R-module and K; L; N be submodules of M. Then;*
(1) *If K is a supplement of N in M and T is small in M then K is a supplement of $N + T$ in M.*
(2) *If $M \xrightarrow{f} M'$ is an epimorphism with small kernel and L is a supplement of K in M, then the submodule $f(L)$ of M' is a supplement of $f(K)$ in M'.*

Proof. (1) Let K be a supplement of N in M. Then $M = N + K$ and $N \cap K$ is small in K. Then $M = N + K + T$. Let $K = K \cap (N + T) + L$ for some $L \leq K$. Then $M = N + L + T = N + L$ since T is small in M. Then $K = K \cap N + L$. It implies $K = L$ sine $K \cap N$ is small in K.
(2) Let L be a supplement of K in M. Then L is a supplement of $K + Ker(f)$ by (1). By Lemma 3, f(L) = f(L + Kerf) is also a supplement of $f(K)$ in M'. □

Note that the converse statement of Lemma 4 (2) need not be true in general. For if $\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z}$ denotes the canonical epimorphism, then the zero submodule $(\overline{0})$ of $\mathbb{Z}/2\mathbb{Z}$ is small in $\mathbb{Z}/2\mathbb{Z}$ but $\pi^{-1}(\overline{0}) = 2\mathbb{Z}$ is not small in $\mathbb{Z}$.

A module M is *distributive* if for all submodules K, L, and N, $N \cap (K + L) = N \cap K + N \cap L$ or $N + (K \cap L) = (N + K) \cap (N + L)$. Lemma 5 may be very well known and obvious but we prove it for the sake of easy reference.

Lemma 5. *Let $M = M_1 \oplus M_2 = K + N$ and $K \le M_1$. If M is distributive and $K \cap N$ is small in N, then $K \cap N$ is small in $M_1 \cap N$.*

Proof. Let $M_1 \cap N = (K \cap N) + L$. Since M is distributive, $N = M_1 \cap N \oplus M_2 \cap N$. We have $M = K + N = K + M_1 \cap N + M_2 \cap N = K + L + (M_2 \cap N)$ and $N = K \cap N + L + (M_2 \cap N)$. Since $K \cap N$ is small in N, $N = L \oplus (M_2 \cap N)$. This and $N = (N \cap M_1) \oplus (N \cap M_2)$ and $L \le M_1 \cap N$ imply $L = M_1 \cap N$. Hence $K \cap N$ is small in $M_1 \cap N$. □

3. Principally Supplemented Modules

In a semiregular module M, every cyclic submodule mR has a direct summand P such that $M = P \oplus K$, P is projective module and $(mR) \cap K$ is small in K[12, Theorem B.51]. In this note we introduce principally supplemented modules which generalizing semiregular modules, principally lifting modules, also supplemented modules.

Definition 6. Let N be a cyclic submodule of M. A submodule L is called *a principally supplement* of N in M if N and L satisfy the conditions in Lemma 2 and the module M is called *principally supplemented* if every cyclic submodule of M has a principally supplement in M.

Clearly, every supplemented module and every lifting module, therefore every principally lifting module is principally supplemented. There are principally supplemented modules but neither supplemented nor principally lifting.

Examples 7. (1). The $\mathbb{Z}$-module $\mathbb{Q}$ of rational numbers has no maximal submodules. Every cyclic submodule of $\mathbb{Q}$ is small, therefore $\mathbb{Q}$ is principally supplemented $\mathbb{Z}$-module. But $\mathbb{Q}$ is not supplemented.
(2). Consider the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus (Z/\mathbb{Z}2)$. We prove M is principally supplemented module but not supplemented. Let $(u, \overline{v}) \in M$. We first prove that $(u, \overline{v})\mathbb{Z}$ has a supplement in M. We divide the proof in some cases :
Case (i) $u = 1$ and $\overline{v} = \overline{1}$. It is rutin to show that $M = (1, \overline{1})\mathbb{Z} + (\mathbb{Q} \oplus (\overline{0}))$ and $(1, \overline{1})\mathbb{Z} \cap (\mathbb{Q} \oplus (\overline{0})) = (1, \overline{0})\mathbb{Z}$ is small in $(\mathbb{Q} \oplus (\overline{0}))$.
Case (ii) $u = 1$ and $\overline{v} = \overline{0}$. Then $(u, \overline{v})\mathbb{Z} = (1, \overline{0})\mathbb{Z}$ is small in $\mathbb{Q} \oplus (\overline{0}))$.
Case (iii) $u = 0$ and $\overline{v} = \overline{1}$. Then $(u, \overline{v})\mathbb{Z} = (1, \overline{0})\mathbb{Z}$ is direct summand of M.
Case (iv) $u \ne 1, 0$ and $\overline{v} = \overline{1}$. Let $(x, \overline{y}) \in M$. We prove $(x, \overline{y}) \in (u, \overline{1})\mathbb{Z} + (\mathbb{Q} \oplus (\overline{0}))$. For if $\overline{y} = \overline{1}$, then $(x, \overline{y}) = (x, \overline{1}) = (u, \overline{1}) + (x - u, \overline{0}) \in (u, \overline{1})\mathbb{Z} + (\mathbb{Q} \oplus (\overline{0}))$. Assume that $\overline{y} = \overline{0}$. Then $(x, \overline{y}) = (x, \overline{0}) = (u, \overline{1})0 + (x, \overline{0}) \in (u, \overline{1})\mathbb{Z} + (\mathbb{Q} \oplus (\overline{0}))$. Hence $(x, \overline{y}) \in (u, \overline{1})\mathbb{Z} + (\mathbb{Q} \oplus (\overline{0}))$ and so $M = (u, \overline{1})\mathbb{Z} + (\mathbb{Q} \oplus (\overline{0}))$. Since $((u, \overline{1})\mathbb{Z}) \cap (\mathbb{Q} \oplus (\overline{0})) = (2u, \overline{0})\mathbb{Z}$ and $(2u, \overline{0})\mathbb{Z}$ is small in $\mathbb{Q} \oplus (\overline{0})$. It follows that, in either cases, $(u, \overline{v})\mathbb{Z}$ has a supplement in M and M is principally supplemented $\mathbb{Z}$-module.
If M were supplemented $\mathbb{Z}$-module, its direct summand $\mathbb{Q}$ would be a supplemented $\mathbb{Z}$-module. A contradiction. So M is not supplemented.
(3). Consider the $\mathbb{Z}$-modules $M_1 = \mathbb{Z}/\mathbb{Z}2$ and $M_2 = \mathbb{Z}/\mathbb{Z}8$. It is clear that M_1 and M_2 are principally supplemented. Let $M = M_1 \oplus M_2$. Then M is a principally supplemented module $\mathbb{Z}$-module but not principally lifting. Let $N_1 = (\overline{1}, \overline{2})\mathbb{Z}$, $N_2 = (\overline{1}, \overline{1})\mathbb{Z}$, $N_3 = (\overline{0}, \overline{2})\mathbb{Z}$, $N_4 = (\overline{0}, \overline{4})\mathbb{Z}$, $N_5 = (\overline{1}, \overline{4})\mathbb{Z}$, M_1 and M_2 are proper

cyclic submodules of M. $M = M_1 \oplus M_2 = N_2 \oplus N_5$ and N_3, N_4 are small submodules of M. $M = N_1 + N_2$ and $N_1 \cap N_2 = N_4$ small in N_2. Hence M is principally supplemented module. Since $M = N_1 + N_2$, N_1 is not small in M and it is not a direct summand of M and does not contain any nonzero direct summand of M. Hence M is not principally lifting.

Let M be a module. A submodule N is called *fully invariant* if for each endomorphism f of M, $f(N) \leq N$. Let $S = End(M_R)$, the ring of R-endomorphisms of M. Then M is a left S-, right R-bimodule and a principal submodule N of the right R-module M is fully invariant if and only if N is a sub-bimodule of M. Clearly 0 and M are fully invariant submodules of M. The right R-module M is called a *duo module* provided every submodule of M is fully invariant. For the readers' convenience we state and prove Lemma 8 which is proved in [14].

Lemma 8. *Let a module $M = \bigoplus_{i \in I} M_i$ be a direct sum of submodules M_i $(i \in I)$ and let N be a fully invariant submodule of M. Then $N = \bigoplus_{i \in I}(N \cap M_i)$.*

Proof. For each $j \in I$, let $p_j : M \to M_j$ denote the canonical projection and let $i_j : M_j \to M$ denote inclusion. Then $i_j p_j$ is an endomorphism of M and hence $i_j p_j(N) \subseteq N$ for each $j \in I$. It follows that $N \subseteq \bigoplus_{j \in I} \; i_j p_j(N) \subseteq \bigoplus_{j \in I}(N \cap M_j) \subseteq N$, so that $N = \bigoplus_{j \in I}(N \cap M_j)$. □

It is easily proved that finite direct sum of supplemented modules is again supplemented. But this is not the case for principally supplemented modules. But it is the case for some classes of modules.

Theorem 9. *Let $M = M_1 \oplus M_2$ be a decomposition of M with M_1 and M_2 principally supplemented modules. If M is a duo module, then M is principally supplemented.*

Proof. Let $M = M_1 \oplus M_2$ be a duo module and mR be a submodule of M. By Lemma 8, $mR = ((mR) \cap M_1) \oplus ((mR) \cap M_2)$. Let $m = m_1 + m_2$ where $m_1 \in M_1$, $m_2 \in M_2$. Then $m_1 R = (mR) \cap M_1$ and $m_2 R = (mR) \cap M_2$. Since $(mR) \cap M_1$ and $(mR) \cap M_2$ are principal submodules of M_1 and M_2 respectively, there exist $A_1 \leq M_1$ such that $M_1 = m_1 R + A_1$, $(m_1 R) \cap A_1$ is small in A_1 and $A_2 \leq M_2$ such that $M_2 = (m_2 R) + A_2$ and $(m_2 R) \cap A_2$ is small A_2. Then $M = (m_1 R) + (m_2 R) + A_1 + A_2 = (mR) + A_1 + A_2$. We prove $(mR) \cap (A_1 + A_2)$ is small in $A_1 + A_2$.

$$\begin{aligned}(mR) \cap (A_1 + A_2) &= ((mR) \cap M_1 + (mR) \cap M_2) \cap (A_1 + A_2) \\ &\leq (A_1 \cap (((mR) \cap M_1) + M_2)) + (A_2 \cap (((mR) \cap M_2) + M_1)) \\ &\leq ((mR) \cap M_1) \cap (A_1 + M_2) + ((mR) \cap M_2) \cap (A_2 + M_1).\end{aligned}$$

On the other hand
$((mR) \cap M_1) \cap (A_1 + M_2) = (m_1 R) \cap (A_1 + M_2) \leq A_1 \cap ((m_1 R) + M_2) \leq (m_1 R) \cap (A_1 + M_2)$ implies $(m_1 R) \cap (A_1 + M_2) = A_1 \cap ((m_1 R) + M_2) = (m_1 R) \cap A_1$.
Similarly $(m_2 R) \cap (A_2 + M_1) = A_2 \cap ((m_2 R) + M_1) = (m_2 R) \cap A_2$. Since $(m_1 R) \cap A_1$ and $(m_2 R) \cap A_2$ are small in A_1 and A_2 respectively, by Lemma 1 (2)
$(m_1 R) \cap A_1 + (m_2 R) \cap A_2$ is small in $A_1 + A_2$. Again by Lemma 1 (3) $(mR) \cap (A_1 + A_2)$ is small in $A_1 + A_2$. □

Theorem 10. *Let M be a principally supplemented duo module. Then every direct summand of M is a principally supplemented module.*

Proof. Let $M = M_1 \oplus M_2$ and $m \in M_1$. There exists A a submodule such that $M = mR + A$ and $(mR) \cap A$ is small A. Then $M_1 = mR + (M_1 \cap A)$. By Lemma 8, $A = (A \cap M_1) \oplus (A \cap M_2)$. We prove that $(mR) \cap (A \cap M_1)$ is small in $A \cap M_1$. Let T be a submodule of $A \cap M_1$ with $A \cap M_1 = (mR) \cap (A \cap M_1) + T$. Then $A = (mR) \cap (A \cap M_1) + T + (A \cap M_2) = ((mR) \cap A) + T + (A \cap M_2)$. Since $(mR) \cap A$ is small in A, $A = T \oplus (A \cap M_2)$. It follows that $T = A \cap M_1$ that is what we have to prove. □

Theorem 11. *Let M be a principally supplemented distributive module. Then every direct summand of M is a principally supplemented module.*

Proof. Let $M = M_1 \oplus M_2$ and $m \in M_1$. There exists a submodule A of M such that $M = mR + A$ and $(mR) \cap A$ is small in A. Then $M_1 = (mR) + (M_1 \cap A)$. By Lemma 5, $(mR) \cap A$ is small in $M_1 \cap A$. □

For a module M, let $\mathrm{Rad}(M)$ denote the radical of M. A module M is said to be *a principally semisimple* if every cyclic submodule is a direct summand of M. Every semisimple module is principally semisimple. Every principally semisimple module is principally supplemented.

Lemma 12. *Let M be a principally supplemented distributive module. Then $M/\mathrm{Rad}(M)$ is a principally semisimple module.*

Proof. Let $m \in M$. There exists a submodule M_1 such that $M = mR + M_1$ and $(mR) \cap M_1$ is small in M_1. Then $M/\mathrm{Rad}(M) = [(mR + \mathrm{Rad}(M))/\mathrm{Rad}(M)] + [(M_1 + \mathrm{Rad}(M))/\mathrm{Rad}(M)]$. Now we prove that $(mR + \mathrm{Rad}(M)) \cap (M_1 + \mathrm{Rad}(M)) = \mathrm{Rad}(M)$. The distributivity of M implies $(mR + \mathrm{Rad}(M)) \cap (M_1 + \mathrm{Rad}(M)) = (mR) \cap M_1 + \mathrm{Rad}M$. Since $(mR) \cap M_1$ is small in M_1, therefore small in M, $(mR) \cap M_1 \leq \mathrm{RadM}$. Hence $M/\mathrm{Rad}(M) = [(mR + \mathrm{Rad}(M))/\mathrm{Rad}(M)] \oplus [(M_1 + \mathrm{Rad}(M))/\mathrm{Rad}(M)]$ and so every principal submodule of $M/\mathrm{Rad}(M)$ is a direct summand. □

Theorem 13 may be proved easily by making use of Lemma 12 for distributive modules. But we prove it in another way in general.

Theorem 13. *Let M be a principally supplemented module. Then $M = M_1 \oplus M_2$, where M_1 is semisimple module and M_2 is a module with $\mathrm{Rad}(M_2)$ small in M_2.*

Proof. By Zorn's Lemma we may find a submodule M_1 of M such that $Rad(M) \oplus M_1$ is small in M. We prove M_1 is semisimple. Let $m \in M_1$. Since M is principally supplemented, there exists a submodule A of M such that $M = mR + A$ and $(mR) \cap A$ is small in A. Then $(mR) \cap A = 0$. Let K be a maximal submodule of mR. If K is unique maximal submodule in mR, then it is small, therefore small in mR and so in M. This is not possible since $(mR) \cap Rad(M) = 0$. Hence there exists $x \in mR$ such that $mR = K + xR$. We claim that $K \cap (xR) = 0$. Otherwise let $0 \neq x_1 \in K \cap (xR)$. By hypothesis there exists C_1 such that $M = x_1R + C_1$ with $(x_1R) \cap C_1$ is small in M. So $M = x_1R \oplus C_1$ since $(x_1R) \cap C_1 \leq K \cap Rad(M) = 0$. Hence $mR = x_1R \oplus ((mR) \cap C_1)$ and $K = x_1R \oplus (K \cap C_1)$. If $K \cap C_1$ is nonzero, let $0 \neq x_2 \in K \cap C_1$. By hypothesis there exists C_2 such that $M = x_2R + C_2$ with $(x_2R) \cap C_2$ is small in M. So $M = x_2R \oplus C_2$ since $(x_2R) \cap C_2 \leq K \cap Rad(M) = 0$. Then $K \cap C_1 = (x_2R) \oplus (K \cap C_1 \cap C_2)$. Hence $mR = x_1R \oplus x_2R \oplus ((mR) \cap C_1 \cap C_2)$ and $K = x_1R \oplus x_2R \oplus (K \cap C_1 \cap C_2)$. If $K \cap C_1 \cap C_2$ is nonzero, similarly there exists $0 \neq x_3 \in K \cap C_1 \cap C_2$ and $C_3 \leq M$ such that $M = x_3R \oplus C_3$. Then $mR = x_1R \oplus x_2R \oplus x_3R \oplus ((mR) \cap C_1 \cap C_2 \cap C_3)$ and $K = x_1R \oplus x_2R \oplus x_3R \oplus$

$(K \cap C_1 \cap C_2 \cap C_3)$. This process must terminate at a finite step, say t. At this step $mR = x_1R \oplus x_2R \oplus x_3R \oplus ... \oplus x_tR$ and so $mR = K$ since at t^{th} step we must have $K \cap C_1 \cap C_2 \cap ... \cap C_t \leq (mR) \cap C_1 \cap C_2 \cap ... \cap C_t = 0$. This is a contradiction. There exists $x \in mR$ such that $mR = K \oplus (xR)$. Then xR is simple module. Hence every cyclic submodule of M_1 contains a simple submodule. As in the proof of [1, Lemma 9.2], we may prove M_1 is semisimple. □

Principally lifting modules and principally hollow modules are defined and investigated in [9]. A module M is called *principally lifting* if for all $m \in M$, M has a decomposition $M = N \oplus S$ with $N \leq mR$ and $(mR) \cap S$ is small in S, while M is said to be *principally hollow* if every proper cyclic submodule of M is small in M.

Lemma 14. *Let M be an indecomposable module. Consider following conditions :*

(1) *M is a principally lifting module.*
(2) *M is a principally hollow module.*
(3) *M is a principally supplemented module.*

Then (1)⇔(2) and (2)⇒ (3).

Proof. (1)⇒(2) Let $m \in M$. By (1) there exists a submodule A of mR such that $M = A \oplus B$ and $(mR) \cap B$ is small in B. By hypothesis $A = 0$ or $A = M$. If $A = 0$ then mR is small in M. Otherwise $mR = M$. Let K be a maximal submodule of M. Let $k \in K$. Then kR is small in M; for there exists a submodule C of kR such that $M = C \oplus D$ and $(kR) \cap D$ is small in D. By hypothesis C must be zero since K is maximal. Every cyclic submodule of K is small. Let $x \in M \setminus K$. Then $M = K + xR$. Let X be a direct summand of M with $X \leq xR$ with $M = X \oplus Y$ for some $Y \leq M$ and $(xR) \cap Y$ small in Y. Again by hypothesis X is zero or $X = M$. If X is zero then xR is small in M and so $K = M$. A contradiction. Assume $X = M$ then $xR = M$ and so K is small in M. Thus every cyclic submodule of M is small in M.

(2)⇔(1) Let $m \in M$. Then mR is small in M. In this case we take $A = 0$ and $B = M$ to show that $M = A \oplus B$, $A \leq mR$ and $(mR) \cap B$ is small in B.

(2)⇔(3) Let $m \in M$. By (2) each cyclic submodule is hollow. Then $M = (mR) + M$ and $(mR) \cap M$ is small in M. So M is a principally supplemented. □

Note that Lemma 14 (3)⇒ (2) does not hold in general. There exists an indecomposable principally supplemented module but not principally hollow.

Example 15. Let F be a field and x and y commuting indeterminates over F. Consider the polynomial ring $R = F[x, y]$, the ideals $I_1 = (x^2)$ and $I_2 = (y^2)$ of R, and the ring $S = R/(x^2, y^2)$. Let $M = \overline{x}S + \overline{y}S$. Then M is an indecomposable S-module, principally supplemented but not principally hollow.

A module M is called *refinable* if for any submodule U, V of M with $M = U + V$ there is a direct summand U' of M such that $U' \subseteq U$ and $M = U' + V$(See namely [**?**]).

Let M be a module. M is called *a weakly principally supplemented module* if for each $m \in M$ there exists a submodule A such that $M = mR + A$ and $(mR) \cap A$ is small in M. Every weakly supplemented module is weakly principally supplemented. The module M is called *a ⊕-principally supplemented* if for each $m \in M$ there exists a direct summand A of M such that $M = mR + A$ and $(mR) \cap A$ is small in A. ⊕-supplemented modules are studied in[4]. Every ⊕-supplemented module is $\oplus$-principally supplemented and it is evident that every $\oplus$-principally supplemented is weakly principally supplemented. In a subsequent paper the authors investigates the interconnections between principally supplemented modules, weakly principally supplemented modules and $\oplus$-principally supplemented modules in detail. Recall that a module M is said to have the summand sum property if the sum of any two direct summands of M is again a direct summand of M. The summand sum property was studied by J. L. Garcia [2], who characterized modules with the summand sum property.

Theorem 16. *Let M be a refinable module. Consider following conditions*
(1) *M is principally lifting.*
(2) *M is principally $\oplus$-supplemented.*
(3) *M is principally supplemented.*
(4) *M is principally weak supplemented.*
Then $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (2)$.
If M has the summand sum property then $(4) \Rightarrow (1)$.

Proof. By definitions $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ always hold.

$(4) \Rightarrow (2)$ Let M be a principally weak supplemented module and $m \in M$. By (4) there exists a submodule A of M such that $M = mR + A$ and $(mR) \cap A$ is small in M. By hypothesis there exists a direct summand U of M with $U \leq A$ and $M = mR + U = U' \oplus U$ for some submodule U' of M. We claim that $(mR) \cap U$ is small in U. For if $(mR) \cap U + L = U$ for some submodule L of U, then $M = U' + ((mR) \cap U) + L = U' \oplus L$ as $(mR) \cap U$ is small in M. Hence $L = U$. Hence M is principally $\oplus$-supplemented.

$(4) \Rightarrow (1)$ Assume that M has the summand sum property and let $m \in M$. By (4) there exists a submodule A such that $M = mR + A$ and $(mR) \cap A$ is small in M. By hypothesis there exists a direct summand U_1 of M such that U_1 is contained in A and $M = mR + U_1 = U_1' \oplus U_1$. Since U_1 is direct summand and $(mR) \cap A$ is small in M, $(mR) \cap U_1$ is small in U_1. Again by hypothesis there exists a direct summand U_2 of M such that U_2 is contained in mR and $M = U_2 + U_1 = U_2 \oplus U_2'$. By the summand sum property $U_2 \cap U_1$ is a direct summand of M, $M = (U_2 \cap U_1) \oplus K$ for some submodule K of M. Then $U_1 = (U_2 \cap U_1) \oplus (K \cap U_1)$ and $M = U_2 \oplus (K \cap U_1)$. It is evident that $(mR) \cap (K \cap U_1)$ is small in $K \cap U_1$ since $(mR) \cap (K \cap U_1) \leq (mR) \cap U_1 \leq U_1$ and $(mR) \cap U_1$ is small in U_1, $(mR) \cap (K \cap U_1)$ is small in U_1 and so small in $K \cap U_1$ as $K \cap U_1$ is direct summand of M. □

4. Applications

In this section, we introduce and study some properties of principally semiperfect modules. A projective module P is called *a projective cover* of a module M if there exists an epimorphism $f : P \longrightarrow M$ with $Kerf$ is small in P, and a ring is called *perfect* (or *semiperfect*) if every R-module (or every simple R-module) has a projective cover. For more detailed discussion on small submodules, perfect and semiperfect rings. A module M is called *principally semiperfect* if every factor module of M by a cyclic submodule has a projective cover. A ring R is called *principally semiperfect* in case the right R-module R is principally semiperfect. Every semiperfect module is principally semiperfect.

Theorem 17. *Let M be a projective module. Then following conditions are equivalent.*

(1) *M is principally semiperfect.*
(2) *M is principally supplemented.*

Proof. (1)$\Rightarrow$ (2) Let $m \in M$. By (1) M/mR has a projective cover $P \stackrel{f}{\to} M/mR$. There exists $P \stackrel{g}{\to} M$ such that $f = \pi g$, where $M \stackrel{\pi}{\to} M/mR$ is the natural epimorphism. Let $m \in M$. There exists $x \in P$ such that $\pi(m) = f(x)$ since f is epimorphism. So $\pi(m) = f(x) = \pi(g(x))$ and then $m - g(x) \in Ker(\pi) = mR$. Hence $M = g(P) + mR$. We prove $g(P) \cap (mR)$ is small in $g(P)$. It suffices to show that $g(P) \cap (mR) = g(Ker(f)$ since $Ker(f)$ is small in P and any homomorphic image of small modules is small under epimorphic maps. Let $x \in Ker(f)$. Then $\pi g(x) = f(x) = 0$. So $g(x) \in Ker(\pi) = mR$. Hence $g(Ker(f) \leq g(P) \cap (mR)$. Let $mr \in g(P) \cap (mR)$ and $g(x) = mr$ for some $x \in P$. Then $f(x) = \pi(g(x)) = \pi(mr) = 0$. Hence $x \in Ker(f)$ and so $g(P) \cap (mR) \leq g(Ker(f)$. It follows that $g(P) \cap (mR) = g(Ker(f)$ and $g(P)$ is a complement of mR.

(2)$\Rightarrow$ (1) Let $m \in M$. By (2) there exists a submodule A such that $M = mR + A$ such that $(mR) \cap A$ is small in A. Let $M \stackrel{f}{\to} M/(mR)$ defined by $f(y) = a$ where $y = mr + a$ with $mr \in mR$, $a \in A$, and $M \stackrel{\pi}{\to} M/(mR)$ the natural epimorphism. There exists $M \stackrel{g}{\to} M$ such that $fg = \pi$. Then $M = g(M) + (mR) \cap A$. Hence $M = g(M) \cong M/\mathrm{Ker}(g)$. Since M is projective $M = \mathrm{Ker}(f) \oplus B$ and B is projective. Let $(fg)_B$ denote the restriction of fg on B. Then $\mathrm{Ker}(fg)_B = (mR) \cap A$ and so $B \stackrel{(fg)_B}{\to} M/(mR)$ is a projective cover of M. □

Let R be a module. R is called *semiregular ring* if every cyclicly presented R-module has a projective cover. We give a complete proof to Theorem 18 for the convenience of the reader.

Theorem 18. *Let R be a ring. The following conditions are equivalent :*

(1) *R is principally semiperfect.*
(2) *R is principally lifting.*
(3) *R is semiregular.*
(4) *R is principally supplemented.*

Proof. (1) $\Rightarrow$ (2) Let $x \in R$. By (1) R/xR has a projective cover $P \stackrel{f}{\to} R/xR$ so that $\mathrm{Ker}(f)$ is small in P. Let $R \stackrel{\pi}{\to} R/xR$ be the natural epimorphism. Then there exists a map g such that $f = \pi g$. Then $R = g(P) + xR$ and $g(P) \cap (xR) = g(\mathrm{Ker}(f))$ is small in $g(P)$ since homomorphic images of small submodules are small.

(2) $\Rightarrow$ (3) Assume that R is principally lifting. Let $x \in R$. Then there exists a direct summand right ideal A of R such that $R = A \oplus B$ and $(xR) \cap B$ is small in B. Then $xR = A \oplus (xR) \cap B$ and $(xR) \cap B$ is δ-small in M. By [**?**, Theorem 3.5] R is semiregular.

(3) $\Rightarrow$ (4) Assume that R is semiregular. Let $x \in R$ and $\pi : R \to R/xR$ natural epimorphism. By hypothesis R/xR has a projective cover $f : P \to R/xR$. There exists $g : P \to R$ such that $f = \pi g$. Then $R = g(P) + xR)$ and $g(P) \cap (xR)$ is small in $g(P)$ since $g(P) \cap (xR) = g(\mathrm{Ker}(f)$ and $\mathrm{Ker}(f)$ is small in P. Hence R is principally supplemented.

(4) $\Rightarrow$ (1) Clear from Theorem 17. □

Example 19. Let $R = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \mid x, y, z \in \mathbb{Z}_4 \right\}$ denote the ring of upper triangular matrices over integers. It is easy to check that principal right ideals of R are either small in R or direct summands of R. Hence R is principally supplemented right R-module. Let e_{12} denote the matrix unit having 1 at $(1, 2)$ and zero elsewhere. Let $I = e_{12}R$. Then I is small right ideal and Jacobson radical $J(R)$ of R is equal to I. Hence $R/J(R)$ is not semisimple. Therefore R is not semiperfect ring.

Theorem 20. *Let M be a projective module with* $\mathrm{Rad}(M)$ *is small in M. Consider following conditions :*

(1) *M is principally supplemented.*
(2) *$M/\mathrm{Rad}(M)$ is principally semisimple.*

Then (1)$\Rightarrow$ (2). *If M is refinable module then* (2)$\Rightarrow$ (1).

Proof. (1)$\Rightarrow$ (2) Since P is a principally supplemented module, $P/\mathrm{Rad}(P)$ is principally semisimple by Lemma 12. (2)$\Rightarrow$ (1) Let mR be any cyclic submodule of P. By (2) There exists a submodule U of P such that

$$P/\ \mathrm{Rad}(P) = [((mR){+}\mathrm{Rad}(P))/\mathrm{Rad}(P)] \oplus [U/\ \mathrm{Rad}(P)].$$

Then $P = (mR) + U$ and $((mR){+}\mathrm{Rad}(P)) \cap U = (mR) \cap U{+}\ \mathrm{Rad}(P)] = \mathrm{Rad}(P)$. Since $P = (mR) + U$, being M refinable there exists a direct summand A of M such that $A \leq U$ and $M = (mR) + U = (mR) + A = B \oplus A$. $(mR) \cap U$ is small in M so it is small in U since U is direct summand. this completes the proof. □

References

[1] Anderson F.W. and Fuller K.R.,1974, *Rings and Categories of Modules*, Springer-Verlag, New York.
[2] Garcia J. L., 1989, *Properties of direct summand of modules*, Comm. Algebra 17, 7392.
[3] Goodearl K.R.,1976, Ring Theory : Nonsingular Rings and Modules, Dekker, New York
[4] Harmanci A., Keskin D. and Smith P.F., 1999, *On ⊕-Supplemented Modules*, Acta Math. Hungar.,83(1/2): 161-169.
[5] Kamal M. A. and Yousef A., *On Principally Lifting Modules*, Int. Electron. J. Algebra, 2 (2007) 127-137.
[6] Inankil H., Halicioglu S. and Harmanci A., *Principally δ-lifting modules*, Accepted for publication in Vietnam J. Mathematics.
[7] Keskin D.,1998, *Finite Direct Sums Of* $(D1)$*-Modules*, Turkish J. Math. 22 , no. 1, 85–91.
[8] Keskin D.,2000, *On Lifting Modules*, Comm. Algebra 28, no. 7, 3427–3440.
[9] Kamal M. A. and Yousef A., 2(2007), *On Principally Lifting Modules* Int. Electron. J. Algebra, 127-137.
[10] Lomp C., *Regular and Biregular Module Algebras*, Arab. J. Sci. and Eng., 33(2008), 351-363.
[11] Mohamed S.H and Müller B.J., 1990, *Continuous and Discrete Modules*, London Math. Soc. LNS 147 Cambridge Univ. Press, Cambridge.
[12] Nicholson W. K. and Yousif M. F., Quasi-Frobenius Rings, Cambridge Tracts in Mathematics 158, Cambridge University Press, 2003.
[13] Oshiro K., 1984, *Lifting Modules, Extending Modules and Their Applications To Generalized Uniserial Rings*, Hokkaido Math. J., 13, 339-346.
[14] Ozcan A. C., Harmanci A. and Smith P. F., *Duo Modules*, Glasgow Math. J., 48(3)(2006), 533-545.

Ummahan Acar, Mugla University, Faculty of Science, Department of Mathematics, Mugla, Turkey
E-mail address: uacar@mu.edu.tr

Abdullah Harmanci, Hacettepe University, Department of Mathematics, Ankara - Turkey
E-mail address: harmanci@hacettepe.edu.tr