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A NOTE ON INEQUALITIES IN MULTIFUNCTIONAL ANALYTIC SPACES

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ABSTRACT. A general method "weighted method" will be presented which allows to extend various inequalities for one function case to inequalities for multifunctional case in the unit disk, unit ball and polydisk.

1. INTRODUCTION

Let $n \in \mathbb{N}$ and $\mathbb{C}^n = \{z = (z_1, ..., z_n) \,|z_k \in \mathbb{C}, 1 \le k \le n\}$ be the n-dimensional space of complex coordinates. Let U^n be the unit polydisk of $\mathbb{C}^n$, i.e. $U^n = \{z \in \mathbb{C}^n| \; |z_k| < 1, 1 \le k \le n\}$, T^n the distinguished boundary of U^n. We use m_{2n} to denote the volume measure on U^n given by $m_{2n}(U^n) = 1$. We use $m_{2n,\alpha} = \prod_{i=1}^n (1-|z_i|^2)^\alpha m_{2n}$ to denote the weighted measure on U^n. dm_1 is the standard Lebesgue measure on T. Let $H(U^n)$ be the space of all bounded holomorphic functions on U^n. We write as usual (see [1,10]) $z \cdot w = (z_1 w_1, \dots, z_n w_n)$, $z, w \in \mathbb{C}^n$; $e^{i\theta} = (e^{i\theta_1}, \dots, e^{i\theta_n})$, $d\theta = d\theta_1 \cdots d\theta_n$. When we write $0 \le \vec{r} < 1$, where $\vec{r} = (r_1, \dots, r_n)$, this means that $0 \le r_i < 1$ $(i = 1, \dots, n)$. The Hardy space $H^p(U^n)\,(0 < p < \infty)$ on U^n can be defined in a standard way as following:

$$H^p(U^n) = \{f \in H(U^n) : \; \frac{1}{(2\pi)^n} \sup_{0 \le \vec{r} < 1} \int_{[0,2\pi]^n} |f(\vec{r} \cdot e^{i\theta})|^p d\theta < \infty\}.$$

For $\vec{\alpha} > -1, 0 < p < \infty$, recall that the weighted Bergman space $A^p_{\vec{\alpha}}(U^n)$ consists of all holomorphic functions on the polydisk satisfying the condition

$$\|f\|^p_{A^p_{\vec{\alpha}}} = \int_{U^n} |f(z)|^p \prod_{i=1}^n (1-|z_i|^2)^{\alpha_i} dm_{2n} < \infty.$$

Let $\mathbb{B}_n$ be the unit ball in $\mathbb{C}^n$ and dv be the normalized Lebesgue measure of $\mathbb{B}_n$ (i.e. $v(\mathbb{B}_n) = 1$). The boundary of $\mathbb{B}_n$ will be denoted by S and is called the unit sphere in $\mathbb{C}^n$. The surface measure on S will be denoted by $d\sigma$. We denote the class of all holomorphic functions on the unit ball by $H(\mathbb{B}_n)$. Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in $\mathbb{C}^n$, we write

$$\langle z, w\rangle = z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n, \; |z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

The Hardy space $H^p(\mathbb{B}_n)\,(0 < p < \infty)$ on $\mathbb{B}_n$ is defined by (see [16])

$$H^p(\mathbb{B}_n) = \{f \in H(\mathbb{B}_n) : \|f\|_{H^p(\mathbb{B}_n)} = \sup_{0 \le r < 1} M_p(f, r) < \infty\},$$

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where

$$M_p(f,r)=\left(\int_S |f(r\zeta)|^p d\sigma(\zeta)\right)^{1/p},\ r\in(0,1).$$

For real parameter $\alpha>-1$ we consider the weighted volume measure $dv_\alpha(z)=(1-|z|^2)^\alpha dv(z)$. Suppose $0<p<\infty$ and $\alpha>-1$, recall that the weighted Bergman space A^p_α on the unit ball consists of those functions $f\in H(\mathbb{B}_n)$ for which

$$\|f\|^p_{A^p_\alpha}=\int_{\mathbb{B}_n}|f(z)|^p dv_\alpha(z)<\infty.$$

For $f\in C^1(\mathbb{B}_n)$, the invariant gradient $\widetilde{\nabla} f$ is defined by $(\widetilde{\nabla} f)(z)=\nabla(f\circ\varphi_z)(0)$, where ∇f is the complex gradient of f, i.e.

$$\nabla f(z)=\left(\frac{\partial f}{\partial z_1}(z),\ldots,\frac{\partial f}{\partial z_n}(z)\right).$$

For $f\in H(\mathbb{B}_n)$ and $z\in B$, set

$$Q_f(z)=\sup_{w\in\mathbb{C}^n\setminus\{0\}}\frac{|\langle\nabla f(z),\bar{w}\rangle|}{(H_z(w,w))^{1/2}},$$

where $H_z(w,w)$ is the Bergman metric on $\mathbb{B}_n$, i.e.

$$H_z(w,w)=\frac{n+1}{2}\frac{(1-|z|^2)|w|^2+|\langle w,z\rangle|^2}{(1-|z|^2)^2}.$$

The Bloch space $\mathcal{B}$, which was introduced by Timoney (see [14, 15]), is the space of all $f\in H(\mathbb{B}_n)$ for which

$$\|f\|_{\mathcal{B}}=\sup_{z\in\mathbb{B}_n}Q_f(z)<\infty.$$

It is well known that $f\in\mathcal{B}$ if and only if $\sup_{z\in\mathbb{B}_n}(1-|z|^2)\,|\nabla f(z)|<\infty$.

For $1<p<\infty$, recall that the Möbius invariant Besov space B_p consists of those holomorphic functions f for which Q_f is p-integrable function with respect to the invariant measure $d\lambda(z)$. Here $d\lambda(z)=(1-|z|^2)^{-n-1}dv(z)$ is a Möbius invariant measure, that is for any $\psi\in Aut(\mathbb{B}_n)$ and $f\in L^1(\mathbb{B}_n)$,

$$\int_{\mathbb{B}_n}f(z)d\lambda(z)=\int_{\mathbb{B}_n}f\circ\psi(z)d\lambda(z).$$

From [2], we know that for $n\geq 2$, the Besov space is nontrivial if and only if $p>2n$.

The following inequality is a direct consequence of diagonal-mapping Theorem (see [1,12]) and the subharmonicity of $|f(z)|^p$,

$$\int_U |f(z,\cdot\cdot\cdot,z)|^p(1-|z|^2)^{\alpha_1+\cdots+\alpha_n+2n-2}dm_2(z)\leq C\|f\|^p_{A^p_{\vec{\alpha}}},\tag{1}$$

where $0<p<\infty$, $\vec{\alpha}=(\alpha_1,\cdot\cdot\cdot,\alpha_n)>-1$, $j=1,\cdot\cdot\cdot,n$, $f\in H(U^n)$.

If we put $f=f_1\cdots f_n$ in (1), then we get new inequality, i.e.

$$\int_U\prod_{i=1}^n|f_i(z)|^p(1-|z|^2)^{\alpha_1+\cdots+\alpha_n+2n-2}dm_2(z)$$

$$\leq\ C\int_{U^n}\prod_{i=1}^n|f_i(z_i)|^p\prod_{k=1}^n(1-|z_k|^2)^{\alpha_k}dm_{2n}(z).\tag{2}$$

Running from one function to n different functions in this simple example we see, the appearance of the certain weight. More concretely the additional weight $(1-$

$|z|^2)^{2n-2}$ appeared in our inequality with the addition of the amount of functions. The main goal of this note is to try to understand the connection of this weight with the structure of the Bergman space or other holomorphic function spaces, then try to generalize this effect and to find other cases where the similar change will occur during the very natural process of addition of the amount of functions in various inequalities for one holomorphic function, i.e. to generalize (2) and get various generalizations of the known theorem from one functional case to multifunctional case, that is to get estimate for more general expression of the type $|f_1|^{q_1}\cdots|f_k|^{q_k}, 0 < q_j < \infty, j = 1, \cdots, k$.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \le A \le CB$.

2. Main result

We propose a general method which we call "weight method". The main tool is the following vital theorem.

Theorem A. *Let μ be a positive Borel measure on Y, X_i, Y be any quasi normed spaces, $\beta, q_i \in (0,\infty)$, $i = 1, \cdots, k$. If*

$$\sup_{z\in Y} |f_i|^{q_i}(1-|z|^2)^{\beta} \le C\|f_i\|^{q_i}_{X_i}, i = 1, 2, \cdots, k. \tag{3}$$

and

$$\int_Y |f_1(z)|^{q_1} d\mu(z) \le C\|f_1\|^{q_1}_{X_1},$$

then

$$\int_Y \prod_{i=1}^{k} |f_i|^{q_i}(1-|z|^2)^{\beta k-\beta} d\mu(z) \le C\|f_1\|^{q_1}_{X_1} \cdots \|f_k\|^{q_k}_{X_k}. \tag{4}$$

Proof. We use induction. For $k = 1$, we are lead to have the estimate

$$\int_Y |f_1|^{q_1} d\mu(z) \le C\|f_1\|^{q_1}_{X_1}.$$

This is obvious. Assume that (4) is true for k, let us prove that (4) is also true for $k+1$. We have

$$\begin{aligned}
&\int_Y \prod_{i=1}^{k+1} |f_i(z)|^{q_i}(1-|z|^2)^{\beta k-\beta}(1-|z|^2)^{\beta} d\mu(z)\\
\le\ & \Big(\sup_{z\in Y} |f_{k+1}(z)|^{q_{k+1}}(1-|z|^2)^{\beta}\Big)\Big(\int_Y \prod_{i=1}^{k} |f_i|^{q_i}(1-|z|^2)^{\beta k-\beta} d\mu(z)\Big)\\
\le\ & C\Big(\sup_{z\in Y} |f_{k+1}(z)|^{q_{k+1}}(1-|z|^2)^{\beta}\Big)\prod_{i=1}^{k} \|f_i\|^{q_i}_{X_i}\\
\le\ & C\prod_{i=1}^{k+1} \|f_i\|^{q_i}_{X_i}.
\end{aligned}$$

Remark 1. Uniform estimates are known for functions from many holomorphic spaces in the unit disk, polydisk, unit ball (see [1, 3, 10, 16] and references therein).

Hence we can put instead of X_i various spaces including Bergman, Hardy, BMOA, Q_p, mixed norm spaces, Lipschitz and holomorphic Lizorkin Triebel classes (see [7, 8, 9, 11, 13]). Using these uniform estimates(analogues of (3)) and the one functional result we will get the multifunctional generalization of many concrete one functional inequalities (for example from recent Zhu's book [16]) by simple induction as we did in Theorem A. On that way a certain weight of the type $(1-|z|^2)^t$, with some fixed t, depending on the structure of the quasi norm of the space, will appear. We will give below two simple concrete examples for Hardy and weighted Bergman spaces in the unit disk, then turning our attention to the case of the unit ball $\mathbb{B}_n$.

For $0<p<\infty,\ \alpha>-1,\ f\in A^p_\alpha,\ z\in U$, we have (see [3])

$$|f(z)|\le \frac{C\|f\|_{A^p_\alpha}}{(1-|z|^2)^{(2+\alpha)/p}}.$$

Let $Y=U$. From Theorem A, we obtain the following corollaries immediately.

Corollary 1. *Assume that $f_i\in A^{q_i}_\alpha,\ q_i\in(0,\infty),\ \alpha\in(-1,\infty), i=1,\cdots,k, i\in\mathbb{N}$. Then the following inequality holds.*

$$\int_U\Big(\prod_{i=1}^k|f_i(z)|^{q_i}\Big)(1-|z|^2)^{2k-2}(1-|z|^2)^{k\alpha}dm_2(z)\le C\prod_{i=1}^k\|f_i\|^{q_i}_{A^{q_i}_\alpha}. \tag{5}$$

Corollary 2. *Assume that $f_i\in H^{p_i}$, $p_i\in(0,\infty)$, $i=1,\cdots,k, i\in\mathbb{N}$. Then the following inequality holds.*

$$\int_U\prod_{i=1}^k|f_k(z)|^{p_i}(1-|z|^2)^{k-1}dm_1(z)\le C\prod_{i=1}^k\|f_i\|^{p_i}_{H^{p_i}}. \tag{6}$$

Remark 2. As we noticed in Corollary 1 for Bergman spaces the transferring from one function case to k function case needs the addition of $(1-|z|^2)^{2k-2}$, for Hardy space as the Corollary 2 shows the weight is $(1-|z|^2)^{k-1}$.

3. Multifunctional inequalities in higher dimension

The same approach can be developed much further, clearly we can easily note that the main part of our method is based on uniform estimates of function $|f(z)|,\ f\in X\subset H(U)$ (or even in more general form $X\subset H(\Omega)$, where Ω is the unit ball or the polydisk in $\mathbb{C}^n$). The next aim is to show that Corollaries 1 and 2 are also true for the unit ball and polydisk, uniform estimates that we used in the disk for Hardy space H^p and Bergman space A^p_α should be transferred to unit ball. The following inequalities are well known.

$$|f(z)|\le\frac{C\|f\|_{A^p_\alpha}}{(1-|z|^2)^{(n+1+\alpha)/p}},\ 0<p<\infty,\ \alpha>-1,\ f\in A^p_\alpha,\ z\in\mathbb{B}_n$$

and

$$|f(z)|\le\frac{C\|f\|_{H^p}}{(1-|z|)^{n/p}},\ 0<p<\infty,\ f\in H^p,\ z\in\mathbb{B}_n.$$

A big amount of results in the unit ball from [16] can be extended from one functional case to multifunctional case using addition of some weight and induction and some simple multifunction of ideas that we used above. For $f \in H(\mathbb{B}_n)$, $0 < p < \infty$, $\alpha > -1$, we have (see, e.g. [16])

$$\int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) \le C \int_{\mathbb{B}_n} |\widetilde{\nabla} f(z)|^p dv_\alpha(z) = A_1(f);$$

$$\int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) \le C \int_{\mathbb{B}_n} |\nabla f(z)|^p (1-|z|^2)^p dv_\alpha(z) = A_2(f);$$

$$\int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) \le C \int_{\mathbb{B}_n} |\mathcal{R} f(z)|^p (1-|z|^2)^p dv_\alpha(z) = A_3(f).$$

Here $\mathcal{R}f$ denotes the radial derivative of f, that is, $\mathcal{R}f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$. Now we have the following generalization.

Theorem 1. *The following equalities hold.*

$$\int_{\mathbb{B}_n} |f_1|^{p_1} \cdots |f_k|^{p_k} (1-|z|^2)^{k(n+1)-(n+1)} \times (1-|z|^2)^{\alpha_1} \cdots (1-|z|^2)^{\alpha_k} dv(z)$$

$$(7) \le \quad C \prod_{i=1}^{k} A_j(f_i), \; j = 1, 2, 3,$$

where $0 < p_i < \infty$, $\alpha_i > -1$, $f_i \in A^{p_i}_{\alpha_i}, i = 1, \cdots, n$.

Sketch of Proof. All inequalities in theorem 1 can be proved similarly. We use induction and the estimate

$$|f(z)| \le \frac{C\|f\|_{A^{p_i}_{\alpha_i}}}{(1-|z|^2)^{(n+1+\alpha_i)/p_i}}, \; 0 < p_i < \infty, \; \alpha_i > -1, \; f_i \in A^{p_i}_{\alpha_i}, i = 1, \cdots, n$$

and proceed similarly as in the proof of Theorem A. Note that in the unit disk we have weight $2k-2$ and in ball $k(n+1)-(n+1)$.

The following inequality is contained in [16]. For every $p \in (1, \infty)$ there exists a positive constant C such that

$$(8) \int_S |f(\tau\xi)|^p d\sigma(\xi) \le C \int_S |f(\xi)|^p d\sigma(\xi) \le C \int_S |Ref(\xi)|^p d\sigma(\xi), \; \tau \in (0,1)$$

for all $f \in H^p(\mathbb{B}_n)$ with $f(0) = 0$. Again based on induction and the one functional result we have the following extension of (8).

Theorem 2. *For every* $p_i \in (1, \infty)$, *there exists a positive constant* C *such that*

$$(9) \qquad \Big(\int_S \prod_{i=1}^{k} |f_i(\tau\xi)|^{p_i} d\sigma(\xi) \Big) (1-\tau)^{n(k-1)} \le C \prod_{i=1}^{k} \int_S |Ref_i(\xi)|^{p_i} d\sigma(\xi),$$

for all $f_i \in H^{p_i}$ *with* $f_i(0) = 0, i = 1, \cdots, k$, *where* $\tau \in (0,1)$.

Proof. Let $k = 1$. Then we have one functional result, i.e. (8). Suppose the result is true for the case of k, let us prove the case of $k+1$. Since for any $f \in H^{p_i}, i = 1, \cdots, k+1$,

$$\sup_{z \in \mathbb{B}_n} |f(z)|(1-|z|^2)^{n/p_i} \le C\|f\|_{H^{p_i}} \asymp C \int_S |f(\xi)|^p d\sigma(\xi),$$

we have

$$\begin{aligned}
&\Big(\int_S \prod_{i=1}^{k+1} |f_i(\tau\xi)|^{p_i} d\sigma(\xi)\Big)(1-\tau)^{nk} \\
\le\ & C\Big(\sup_{\xi\in S} |f_{k+1}(\tau\xi)|^{p_{k+1}}(1-|\tau\xi|^2)^n\Big)\cdot\Big(\int_S \prod_{i=1}^{k} |f_i(\tau\xi)|^{p_i} d\sigma(\xi)\Big)(1-\tau)^{n(k-1)} \\
\le\ & C\Big(\sup_{z\in\mathbb{B}_n} |f_{k+1}(z)|^{p_{k+1}}(1-|z|^2)^n\Big)\cdot\Big(\prod_{i=1}^{k}\int_S |Ref_i|^{p_i} d\sigma(\xi)\Big) \\
\le\ & C\int_S |f_{k+1}(\xi)|^{p_{k+1}} d\sigma(\xi)\cdot\Big(\prod_{i=1}^{k}\int_S |Ref_i|^{p_i} d\sigma(\xi)\Big) \\
\le\ & C\prod_{i=1}^{k+1}\int_S |Ref_i|^{p_i} d\sigma(\xi),\ \ \tau\in(0,1).
\end{aligned}$$

The following result is also contained in [16]. For every $p \in [1,\infty)$ and $\alpha > -1$, there exists a positive constant C such that

$$\int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) \le C\int_{\mathbb{B}_n} |Ref(z)|^p dv_\alpha(z) \tag{10}$$

for all $f \in H(\mathbb{B}_n)$ with $f(0) = 0$. Again based on induction and the one functional result we have the following extension of (10).

Theorem 3. *For every $p_i \in [1,\infty)$, there exists a positive constant C such that*

$$\int_{\mathbb{B}_n} \prod_{i=1}^{k} |f_i(z)|^{p_i}(1-|z|^2)^{\frac{n+1+\alpha}{p}(k-1)} dv(z) \le C\prod_{i=1}^{k}\int_{\mathbb{B}_n} |Ref_i(z)|^{p_i} dv_\alpha(z), \tag{11}$$

for all $f_i \in H(\mathbb{B}_n)$ with $f_i(0) = 0, i = 1,\cdots,k$.

Let

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}$$

be the Bergman metric between two points z and w in $\mathbb{B}_n$. The following results can be found in [16].

Lemma 1. *Let $f \in H(\mathbb{B}_n)$ and $1 \le p \le \infty$. Then $f \in B_p$ if and only if*

$$|f(z) - f(w)| \le C_p(\beta(z,w))^{1/q},$$

where $1/p + 1/q = 1$.

Using Lemma 1 and ideas we used on Theorem A we can get various multifunctional generalizations of theorems from [16]. We give several examples for Besov spaces.

Theorem 4. *Let*

$$\lambda_n = \begin{cases} 1 & , \quad n = 1 \\ 2n & , \quad n > 1 \end{cases}.$$

Let $f_i \in H(\mathbb{B}_n), i = 1,\dots,k,\ k \in \mathbb{N},\ \lambda_n < p < \infty,\ 0 < q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then the following statements holds.

(1)

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{\prod_{i=1}^{k}|f_i(z)-f_i(w)|^p(1-|z|^2)^{p/2}(1-|w|^2)^{p/2}(\beta(z,w))^{(pk-p)/q}}{|w-P_w(z)-(1-|w|^2)^{1/2}Q_w(z)|^p}d\lambda(z)d\lambda(w)$$
$$\leq C\prod_{i=1}^{k}\|f_i\|_{B_p}^p,$$

where P_w is the orthogonal projection into the space spanned by w and $Q_w = I - P_w$.

(2)

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{\prod_{i=1}^{k}|f_i(z)-f_i(w)|^p(1-|z|^2)^{p/2}(1-|w|^2)^{p/2}(\beta(z,w))^{(pk-p)/q}}{|w-z|^p}d\lambda(z)d\lambda(w)$$
$$\leq C\prod_{i=1}^{k}\|f_i\|_{B_p}^p.$$

(3)

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} \frac{\prod_{i=1}^{k}|f_i(z)-f_i(w)|^p(1-|z|)^{\alpha}(1-|w|)^{\alpha}(\beta(z,w))^{(pk-p)/q}}{|1-\langle z,w\rangle|^{2(n+1+\alpha)}}dv(z)dv(w)$$
$$\leq C\prod_{i=1}^{k}\|f_i\|_{B_p}^p,$$

where $\alpha > -1$.

(4) Let $w_r(f)(z) = \sup\{|f(z)-f(w)| : w \in D(z,r)\}$, where $D(z,r) = \{w \in \mathbb{B}_n : \beta(w,z) < r\}$. If $r > 0$, then

$$w_r(f_1\cdots f_k)(z) = \sup_{w\in D(z,r)} |f_1(z)-f_1(w)|\cdots|f_k(z)-f_k(w)| \times (\beta(z,w))^{(k-1)/q}$$
$$\leq C\prod_{i=1}^{k}\|f_i\|_{B_p}.$$

Remark 3. When $k = 1$, the results in (1), (3) and (4) were proved in [6] (or see [16]) and the result in (2) was proved in [5].

It should be noted that many so called multifunctional inequalities can be delivered in a prepared form from various inequalities from polydisk function theory (see, e.g. [1]). The simple idea is to cut one analytic function in the polydisk to n pieces, $f(z_1,\cdots,z_n) = f_1(z_1)\cdots f_n(z_n)$. But in this case the amount of functions will always be equal to the dimension and all q_i in $|f_1|^{q_1}\cdots|f_n|^{q_n}$ will be equal to each other. We will give two examples.

Proposition 1. *(Extension of Riesz inequality) Let $2 \leq p \leq q < \infty$, $k = (k_1,\cdots,k_n)$, $k_j \in \mathbb{N}$, $j = 1,\cdots,n$. Then*

$$\int_0^1\cdots\int_0^1\prod_{i=1}^{n}(1-|z_i|)^{k_iq+qp-1}\left|\frac{\partial^{k_1}f_1(z_1)}{\partial z_1^{k_1}}\right|^q\cdots\left|\frac{\partial^{k_n}f_n(z_n)}{\partial z_n^{k_n}}\right|^q d|z_1|\cdots d|z_n|$$
$$\leq \prod_{i=1}^{n}\int_T|f_i(\xi)|^p dm_1(\xi).$$

The proof of Proposition 1 is a direct consequence of a polydisk version of M. Riesz inequality (see [1]).

Theorem 5. *Let μ be a Borel measure in U^n, $k=(k_1,k_2,\cdots,k_n)\in\mathbb{Z}_+^n$, $k_j\neq 0$, $1\leq j\leq n$, $\triangle_l(w)=\{z\in U^n: 1-l_j<|z_j|<1; |argw_j-argz_j|<l_j/2\}$, $l=(l_1,\cdots,l_n)$, $0<l_j<1$, $j=1,\cdots,n$, $w\in T^n$. Then the following two assertions are equivalent*

1)

$$\int_{U^n}\Big|\frac{\partial^{k_1}}{\partial z_1^{k_1}}f_1(z_1)\Big|^p\cdots\Big|\frac{\partial^{k_n}}{\partial z_n^{k_n}}f_n(z_n)\Big|^p d\mu(z)\leq C\prod_{k=1}^n\int_T|f_k(\xi)|^p dm_1(\xi),$$

if $f_k\in H^p(U), 2\leq p<\infty$, $k=1,\cdots,n$.

2) $\mu(\triangle_l(w))\leq Cl_1^{k_1p+1}\cdots l_n^{k_np+1}, 2\leq p<\infty$, $k_j\in\mathbb{Z}_+$, $j=1,\cdots,n$.

Proof. The implication *1)* $\Rightarrow$ *2)* is a direct consequence of using test function

$$f_j(z_j)=\Big(\frac{1-\tau_j^2}{1-\tau_jz_j}\Big)^{1/p},\ 0<|\tau_j|<1,\ z_j\in U,\ 1\leq j\leq n.$$

The reverse was proved in a book of Djrbashian and Shamoian (see [1]) even for all $f\in H^p(U^n)$(we need only those $f\in H^p(U^n)$ such that $f=f_1\cdots f_n$, where $f_k\in H^p(U), k=1,\cdots,n.$).

Remark 4. It should be point out that there are concrete cases when the addition of the amount of functions does not change the structure of the equalities and inequalities, so we just add functions without any additional weight. For example, since

$$\int_{\mathbb{B}_n}|\widetilde{\nabla}f(z)|^p dv_\alpha(z)\leq C\int_{\mathbb{B}_n}|f(z)|^p dv_\alpha(z),\ 0<p<\infty,\ \alpha>-1, f\in H(\mathbb{B}_n),$$

we get

$$\int_{\mathbb{B}_n}|\widetilde{\nabla}(f_1(z)\cdots f_n(z))|^p dv_\alpha(z)\leq C\int_{\mathbb{B}_n}\prod_{i=1}^n|f_i(z)|^p dv_\alpha(z),\ 0<p<\infty,\ \alpha>-1.$$

Hence the addition of the amount of functions does not mean that the addition weight will always appear.

Remark 5. Apparently(since these all ideals and proofs are not complicated) the similar "weight effects" will also appear in inequalities for several functions for Hardy, Bergman classes in various domains G in $\mathbb{C}^n$, the classical weight $(1-|z|^2)^t$ must be replaced in this case by $dist(z,\partial G)$, the distance from a point in G to the boundary of the domain, see [4].

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