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Inequalities for the Growth and Derivatives of a Polynomial

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Abstract

In this paper, we present some inequalities for the growth and derivatives of a polynomial with zeros outside a circle of arbitrary radius $k > 0$. Our results provide improvements and generalizations of some well known polynomial inequalities.

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1 Introduction and statement of results

Let P_n be the class of polynomials $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ of degree n. For $P \in P_n$, define

$$M(P,r) := \max_{|z|=r} |P(z)| \text{ and } m := \min_{|z|=k} |P(z)|.$$

If $P \in P_n$, then it is known that

$$M(P',1) \le nM(P,1). \tag{1.1}$$

Further, if $P \in P_n$ and $P(z) \ne 0$ in $|z| < 1$, then

$$M(P',1) \le \frac{n}{2} M(P,1). \tag{1.2}$$

The inequality (1.1) is better known as S. Bernstein's inequality (for reference, see [12]), although it first appeared in a paper of M.Riesz [11] and the inequality (1.2) is a well-known result due to Lax [9] conjectured by Erdös.

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In the literature, there already exists some refinements and extensions of (1.2) (for example, see Malik [10], Bidkham and Dewan [2], Dewan and Mir [3], Jain [7]).

It was shown by Malik [10] that if $P \in P_n$ and $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, then

$$M(P', 1) \leq \frac{n}{1+k} M(P, 1). \tag{1.3}$$

As a generalization of (1.3), Dewan and Bidkham [2] proved that if $P \in P_n$ and $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, then for $0 \leq r \leq R \leq k$,

$$M(P', R) \leq \frac{n(R+k)^{n-1}}{(r+k)^n} M(P, r). \tag{1.4}$$

The above inequality (1.4) (for $r = 1$) was further generalized to the s^{th} derivative by Jain [[7], inequality (1.2)] by proving the following result.

Theorem A. If $P \in P_n$ and $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, then for $0 \leq s < n$ and $1 \leq R \leq k$,

$$M(P^{(s)}, R) \leq \Big(\frac{1}{R^s + k^s}\Big)\Big[\Big\{\frac{d^{(s)}}{dx^{(s)}}(1+x^n)\Big\}_{x=1}\Big]\Big(\frac{R+k}{1+k}\Big)^n M(P, 1). \tag{1.5}$$

Equality holds in (1.5) (with $s = 1$) for $P(z) = (z+k)^n$.

In this paper, we obtain certain extensions and refinements of (1.5) and hence of inequalities (1.2), (1.3) and (1.4) as well. More precisely, we prove

Theorem 1.1. *If $P \in P_n$ and $P(z) \neq 0$ in $|z| < k,\ k > 0$, then for $0 \leq s < n$ and $0 < r \leq R \leq k$, we have*

$$M(P^{(s)}, R) \leq \left\{\frac{c(n,s)R + \left|\frac{a_s}{a_0}\right| k^{s+1}}{c(n,s)(k^{s+1} + R^{s+1}) + \left|\frac{a_s}{a_0}\right|(k^{s+1}R^s + Rk^{2s})}\right\}\left[\left\{\frac{d^{(s)}}{dx^{(s)}}(1+x^n)\right\}_{x=1}\right] \times \left(\frac{R+k}{r+k}\right)^n M(P, r). \tag{1.6}$$

The result is best possible (with $s = 1$) and equality in (1.6) holds for $P(z) = (z+k)^n$.

Remark 1.2. Since if $P(z) \neq 0$ in $|z| < k,\ k > 0$, the polynomial $P(tz) \neq 0$ in $|z| < \frac{k}{t}, \frac{k}{t} \geq 1, 0 < t \leq k$. Hence applying inequality (2.2) of Lemma (2.1) to $P(tz)$, we get for $0 \leq s < n$,

$$\frac{1}{c(n,s)}\left|\frac{a_s}{a_0}\right| t^s \Big(\frac{k}{t}\Big)^s \leq 1,$$

or

$$\frac{1}{c(n,s)}\left|\frac{a_s}{a_0}\right| k^s \leq 1. \tag{1.7}$$

The above inequality (1.7) gives

$$\frac{c(n,s)t^{s+1} + \left|\frac{a_s}{a_0}\right| k^{s+1} t^s}{c(n,s)(k^{s+1} + t^{s+1}) + \left|\frac{a_s}{a_0}\right|(k^{s+1}t^s + tk^{2s})} \leq \frac{t^s}{t^s + k^s}, \text{ for } 0 < t \leq k. \tag{1.8}$$

Since $R \leq k$, if we take $t = R$ in (1.8), we get

$$\frac{c(n,s)R + \left|\frac{a_s}{a_0}\right| k^{s+1}}{c(n,s)(k^{s+1} + R^{s+1}) + \left|\frac{a_s}{a_0}\right|(k^{s+1}R^s + Rk^{2s})} \leq \frac{1}{R^s + k^s}. \tag{1.9}$$

Using (1.9) in (1.6), the following result immediately follows from Theorem (1.1).

Corollary 1.3. *If $P \in P_n$ and $P(z) \neq 0$ in $|z| < k,\ k > 0$, then for $0 \leq s < n$ and $0 < r \leq R \leq k$, we have*

$$M(P^{(s)}, R) \leq \Big(\frac{1}{R^s + k^s}\Big)\Big[\Big\{\frac{d^{(s)}}{dx^{(s)}}(1+x^n)\Big\}_{x=1}\Big]\Big(\frac{R+k}{r+k}\Big)^n M(P, r). \tag{1.10}$$

The result is best possible (with $s = 1$) and equality in (1.10) *holds for $P(z) = (z+k)^n$.*

Remark 1.4. For $r = 1$, Corollary (1.3) reduces to Theorem A and for $s = 1$ it gives (1.4).

Next we prove the following theorem which gives an improvement of Corollary (1.3) (for $1 \leq s < n$), which in turn as a special case provides an improvement and extension of Theorem A. In fact, we prove

Theorem 1.5. *If $P \in P_n$ and $P(z) \neq 0$ in $|z| < k,\ k > 0$, then for $1 \leq s < n$ and $0 < r \leq R \leq k$, we have*

$$M(P^{(s)}, R) \leq \Bigg\{\frac{c(n,s)R + \frac{|a_s|}{|a_0|-m}k^{s+1}}{c(n,s)(k^{s+1} + R^{s+1}) + \frac{|a_s|}{|a_0|-m}(k^{s+1}R^s + Rk^{2s})}\Bigg\}\Bigg[\Bigg\{\frac{d^{(s)}}{dx^{(s)}}(1+x^n)\Bigg\}_{x=1}\Bigg]$$
$$\times\Big(\frac{R+k}{r+k}\Big)^n\Big(M(P,r) - m\Big). \tag{1.11}$$

The result is best possible (with $s = 1$) and equality in (1.11) holds for $P(z) = (z+k)^n$.

Remark 1.6. Since $P(z) \neq 0$ in $|z| < k,\ k > 0$, therefore, for every λ with $|\lambda| < 1$, it follows by Rouche's theorem that the polynomial $P(z) - \lambda m$, has no zeros in $|z| < k,\ k > 0$ and hence applying inequality (1.7) of Remark (1.2), we get for $1 \leq s < n$,

$$c(n,s)|a_0 - \lambda m| \geq |a_s|k^s. \tag{1.12}$$

If in (1.12), we choose the argument of λ suitably and note $|a_0| > m$, from Lemma (2.4), we get

$$c(n,s)(|a_0| - |\lambda|m) \geq |a_s|k^s. \tag{1.13}$$

If we let $|\lambda| \to 1$ in (1.13), we get

$$\frac{1}{c(n,s)}\frac{|a_s|}{|a_0| - m}k^s \leq 1,$$

which further implies by using the same arguments as in Remark (1.2), that

$$\frac{c(n,s)R + \frac{|a_s|}{|a_0|-m}k^{s+1}}{c(n,s)(k^{s+1} + R^{s+1}) + \frac{|a_s|}{|a_0|-m}(k^{s+1}R^s + Rk^{2s})} \leq \frac{1}{R^s + k^s}. \tag{1.14}$$

Now, using (1.14) in (1.11), the following improvement of Corollary (1.3) (for $1 \leq s < n$) and hence of Theorem A immediately follows from Theorem (1.5).

Corollary 1.7. *If $P \in P_n$ and $P(z) \neq 0$ in $|z| < k,\ k > 0$, then for $1 \leq s < n$ and $0 < r \leq R \leq k$, we have*

$$M(P^{(s)}, R) \leq \Big(\frac{1}{R^s + k^s}\Big)\Big[\Big\{\frac{d^{(s)}}{dx^{(s)}}(1+x^n)\Big\}_{x=1}\Big]\Big(\frac{R+k}{r+k}\Big)^n\Big(M(P,r) - m\Big). \tag{1.15}$$

The result is best possible (with $s = 1$) and equality in (1.15) holds for $P(z) = (z+k)^n$.

2 Lemmas

For the proof of these theorems, we need the following lemmas.

Lemma 2.1. *If $P \in P_n$ and $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, and $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$, then for $1 \leq s < n$ and $|z| = 1$,*

$$k^{s+1}\left\{\frac{1+\frac{1}{c(n,s)}\left(\left|\frac{a_s}{a_0}\right|\right)k^{s-1}}{1+\frac{1}{c(n,s)}\left(\left|\frac{a_s}{a_0}\right|\right)k^{s+1}}\right\}\left|P^{(s)}(z)\right| \leq \left|Q^{(s)}(z)\right| \tag{2.1}$$

and

$$\frac{1}{c(n,s)}\left|\frac{a_s}{a_0}\right|k^s \leq 1. \tag{2.2}$$

The above Lemma is due to Aziz and Rather [1]. It is easy to see that (2.1) and (2.2) holds for $s = 0$ as well.

In the same paper, Aziz and Rather also proved

Lemma 2.2. *If $P \in P_n$ and $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, then for $1 \leq s < n$,*

$$M(P^{(s)},1) \leq n(n-1)\cdots(n-s+1)\left\{\frac{c(n,s)+\left|\frac{a_s}{a_0}\right|k^{s+1}}{c(n,s)(1+k^{s+1})+\left|\frac{a_s}{a_0}\right|(k^{s+1}+k^{2s})}\right\}M(P,1). \tag{2.3}$$

From Lemma (2.2), we easily get

Lemma 2.3. *If $P \in P_n$ and $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, then for $0 \leq s < n$,*

$$M(P^{(s)},1) \leq \left\{\frac{c(n,s)+\left|\frac{a_s}{a_0}\right|k^{s+1}}{c(n,s)(1+k^{s+1})+\left|\frac{a_s}{a_0}\right|(k^{s+1}+k^{2s})}\right\}\left[\left\{\frac{d^{(s)}}{dx^{(s)}}(1+x^n)\right\}_{x=1}\right]M(P,1). \tag{2.4}$$

Lemma 2.4. *If $P \in P_n$ and $P(z) \neq 0$ in $|z| < k, k > 0$, then $|P(z)| > m$ for $|z| < k$, and in particular*

$$|a_0| > m.$$

The above Lemma is due to Gardner, Govil and Musukula [5].

Lemma 2.5. *If $P \in P_n$ and $P(z) \neq 0$ in $|z| < k, k > 0$, then for $0 < r \leq R \leq k$,*

$$M(P,r) \geq \left(\frac{r+k}{R+k}\right)^n M(P,R). \tag{2.5}$$

The above Lemma is due to Jain [8].

Lemma 2.6. *If $P(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j,\ 1 \leq \mu \leq n$ is a polynomial of degree n having no zeros in $|z| < k,\ k > 0$, then for $0 < r \leq R \leq k$,*

$$M(P,r) \geq \left(\frac{r^\mu + k^\mu}{R^\mu + k^\mu}\right)^{\frac{n}{\mu}} M(P,R) + \left[1 - \left(\frac{r^\mu + k^\mu}{R^\mu + k^\mu}\right)^{\frac{n}{\mu}}\right]m. \tag{2.6}$$

The above Lemma is due to Dewan, Yadav and Pukhta [4].

Lemma 2.7. *The function*

$$T(x) = k^{s+1}\left\{\frac{1+\dfrac{1}{c(n,s)}\left(\frac{|a_s|}{x}\right)k^{s-1}}{1+\dfrac{1}{c(n,s)}\left(\frac{|a_s|}{x}\right)k^{s+1}}\right\}$$

is an increasing function of x.

Proof. The proof follows by considering the first derivative test of $T(x)$. □

Lemma 2.8. *If $P \in P_n$ and $P(z) \neq 0$ in $|z| < k, k \geq 1$ and $Q(z) = z^n\overline{P(\frac{1}{\bar{z}})}$, then for $|z| \geq 1/k$,*

$$\left|Q^{(s)}(z)\right| \geq mn(n-1)\dots(n-s+1)|z|^{n-s}. \tag{2.7}$$

The above Lemma is due to Govil [6].

Lemma 2.9. *If $P \in P_n$ and $P(z) \neq 0$ in $|z| < k,\ k \geq 1$, then for $1 \leq s < n$,*

$$\begin{aligned} M(P^{(s)},1) &\leq n(n-1)\cdots(n-s+1) \\ &\times\left\{\frac{c(n,s)+\frac{|a_s|k^{s+1}}{|a_0|-m}}{c(n,s)(1+k^{s+1})+\frac{|a_s|}{|a_0|-m}(k^{s+1}+k^{2s})}\right\}\Big(M(P,1)-m\Big). \end{aligned} \tag{2.8}$$

Proof. Since $P(z)$ has all its zeros in $|z| \geq k \geq 1$ and $m = \min\limits_{|z|=k}|P(z)|$, therefore,

$$m \leq |P(z)| \text{ for } |z| = k.$$

Hence it follows by Rouche's theorem that for $m > 0$ and for every real or complex number λ with $|\lambda| < 1$, the polynomial $P(z) - \lambda m$ does not vanish in $|z| < k, k \geq 1$. Applying inequality (2.1) of Lemma (2.1) to the polynomial $P(z) - \lambda m$, we get on $|z| = 1$ that

$$\begin{aligned} k^{s+1}&\left\{\frac{1+\dfrac{1}{c(n,s)}\left(\frac{|a_s|}{|a_0-\lambda m|}\right)k^{s-1}}{1+\dfrac{1}{c(n,s)}\left(\frac{|a_s|}{|a_0-\lambda m|}\right)k^{s+1}}\right\}\left|P^{(s)}(z)\right| \\ &\leq \left|Q^{(s)}(z) - \overline{\lambda}mn(n-1)\dots(n-s+1)z^{n-s}\right|. \end{aligned} \tag{2.9}$$

Since for every λ with $|\lambda| < 1$, we have

$$|a_0 - \lambda m| \geq |a_0| - |\lambda|m \geq |a_0| - m, \tag{2.10}$$

and $|a_0| > m$ by Lemma (2.4), we get on combining (2.9), (2.10) and Lemma (2.7) that for every λ with $|\lambda| < 1$,

$$\begin{aligned} k^{s+1}&\left\{\frac{1+\dfrac{1}{c(n,s)}\left(\frac{|a_s|}{|a_0|-m}\right)k^{s-1}}{1+\dfrac{1}{c(n,s)}\left(\frac{|a_s|}{|a_0|-m}\right)k^{s+1}}\right\}\left|P^{(s)}(z)\right| \\ &\leq \left|Q^{(s)}(z) - \overline{\lambda}mn(n-1)\dots(n-s+1)z^{n-s}\right|, \text{ for } |z| = 1. \end{aligned} \tag{2.11}$$

Now choosing the argument of λ on the right hand side of (2.11) so that on $|z| = 1$,

$$\left|Q^{(s)}(z) - \overline{\lambda} mn(n-1)\dots(n-s+1)z^{n-s}\right| = \left|Q^{(s)}(z)\right| - |\lambda| mn(n-1)\dots(n-s+1), \tag{2.12}$$

which is possible by inequality (2.7) of Lemma (2.8). Hence we conclude from (2.11) that on $|z| = 1$,

$$\phi_{k,s}\left|P^{(s)}(z)\right| \leq \left|Q^{(s)}(z)\right| - |\lambda| mn(n-1)\dots(n-s+1), \tag{2.13}$$

where $\phi_{k,s} = k^{s+1}\left\{\dfrac{1+\frac{1}{c(n,s)}\left(\frac{|a_s|}{|a_0|-m}\right)k^{s-1}}{1+\frac{1}{c(n,s)}\left(\frac{|a_s|}{|a_0|-m}\right)k^{s+1}}\right\}$.

Letting $|\lambda| \to 1$ in (2.13), we obtain

$$\phi_{k,s}\left|P^{(s)}(z)\right| \leq \left|Q^{(s)}(z)\right| - mn(n-1)\dots(n-s+1). \tag{2.14}$$

Now, if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then $g(z) = z^n\overline{p(\frac{1}{\overline{z}})}$ has no zero in $|z| < 1$. Hence by inequality (2.1) of Lemma (2.1) with $k = 1$, we have for $|z| = 1$,

$$\left|g^{(s)}(z)\right| \leq \left|p^{(s)}(z)\right|. \tag{2.15}$$

Let $M = \max\limits_{|z|=1}|P(z)|$, then for every γ with $|\gamma| > 1$, it follows by Rouche's theorem that the polynomial $T(z) = P(z) - \gamma M z^n$ has all zeros in $|z| < 1$. Taking $S(z) = z^n\overline{T(\frac{1}{\overline{z}})} = Q(z) - \overline{\gamma}M$ and apply inequality (2.15) to $T(z)$, we get for $1 \leq s < n$ and for $|z| = 1$,

$$\left|S^{(s)}(z)\right| \leq \left|T^{(s)}(z)\right|,$$

which implies

$$\left|Q^{(s)}(z)\right| \leq \left|P^{(s)}(z) - \gamma M n(n-1)\cdots(n-s+1)z^{n-s}\right| \text{ for } |z| = 1. \tag{2.16}$$

Since $P(z)$ is of degree n, it follows for every $1 \leq s < n$,that the polynomial $P^{(s)}(z)$ is of degree $(n-s)$. By the repeated application of (1.1), we obtain for $|z| = 1$,

$$\left|P^{(s)}(z)\right| \leq n(n-1)\cdots(n-s+1)M. \tag{2.17}$$

Choose argument of γ suitably and note inequality (2.17), we obtain from inequality (2.16) for $|z| = 1$,

$$\left|Q^{(s)}(z)\right| \leq M|\gamma| n(n-1)\cdots(n-s+1) - \left|P^{(s)}(z)\right|. \tag{2.18}$$

Letting $|\gamma| \to 1$ in (2.18), we get

$$\left|P^{(s)}(z)\right| + \left|Q^{(s)}(z)\right| \leq Mn(n-1)\cdots(n-s+1). \tag{2.19}$$

Combining inequalities (2.14) and (2.19), we have for $|z| = 1$,

$$\begin{aligned}(1+\phi_{k,s})\left|P^{(s)}(z)\right| &\leq \left|P^{(s)}(z)\right| + \left|Q^{(s)}(z)\right| - mn(n-1)\dots(n-s+1)\\ &\leq Mn(n-1)\cdots(n-s+1) - mn(n-1)\dots(n-s+1)\\ &= n(n-1)\dots(n-s+1)(M-m),\end{aligned}$$

which is equivalent to (2.8) and this completes the proof of Lemma (2.9). □

3 Proofs of theorems

Proof of theorem (1.1). Since $P(z) \neq 0$ in $|z| < k,\ k > 0$, the polynomial $P(Rz)$ has no zero in $|z| < \frac{k}{R},\ \frac{k}{R} \geq 1$. Hence using Lemma (2.3), we have for $0 \leq s < n$,

$$R^s M(P^{(s)}, R) \leq \left\{ \frac{c(n,s) + \left|\frac{a_s}{a_0}\right| R^s (\frac{k}{R})^{s+1}}{c(n,s)(1 + (\frac{k}{R})^{s+1}) + \left|\frac{a_s}{a_0}\right| R^s ((\frac{k}{R})^{s+1} + (\frac{k}{R})^{2s})} \right\} \times \left[\left\{ \frac{d^{(s)}}{dx^{(s)}}(1+x^n) \right\}_{x=1} \right] M(P,R),$$

which gives

$$M(P^{(s)}, R) \leq \left\{ \frac{c(n,s)R + \left|\frac{a_s}{a_0}\right| k^{s+1}}{c(n,s)(k^{s+1} + R^{s+1}) + \left|\frac{a_s}{a_0}\right| (k^{s+1}R^s + Rk^{2s})} \right\} \times \left[\left\{ \frac{d^{(s)}}{dx^{(s)}}(1+x^n) \right\}_{x=1} \right] M(P,R). \tag{3.1}$$

Now, if $0 < r \leq R \leq k$, then by Lemma (2.5), we get,

$$M(P,R) \leq \left(\frac{R+k}{r+k} \right)^n M(P,r). \tag{3.2}$$

Combining (3.1) and (3.2), we obtain

$$M(P^{(s)}, R) \leq \left\{ \frac{c(n,s)R + \left|\frac{a_s}{a_0}\right| k^{s+1}}{c(n,s)(k^{s+1} + R^{s+1}) + \left|\frac{a_s}{a_0}\right| (k^{s+1}R^s + Rk^{2s})} \right\} \left[\left\{ \frac{d^{(s)}}{dx^{(s)}}(1+x^n) \right\}_{x=1} \right] \times \left(\frac{R+k}{r+k} \right)^n M(P,r),$$

which proves Theorem (1.1).

Proof of theorem (1.5). Since $P(z)$ has no zero in $|z| < k,\ k > 0$, the polynomial $P(Rz)$ has no zero in $|z| < \frac{k}{R},\ \frac{k}{R} \geq 1$. Hence using Lemma (2.9), we have for $1 \leq s < n$,

$$R^s M(P^{(s)}, R) \leq \left\{ \frac{c(n,s) + \frac{|a_s|}{|a_0|-m'} R^s (\frac{k}{R})^{s+1}}{c(n,s)(1 + (\frac{k}{R})^{s+1}) + \frac{|a_s|}{|a_0|-m'} R^s ((\frac{k}{R})^{s+1} + (\frac{k}{R})^{2s})} \right\} \times \left[\left\{ \frac{d^{(s)}}{dx^{(s)}}(1+x^n) \right\}_{x=1} \right] \Big(M(P,R) - m' \Big), \tag{3.3}$$

where $m' = \min\limits_{|z|=\frac{k}{R}} |P(Rz)| = \min\limits_{|z|=k} |P(z)| = m$.

This gives

$$M(P^{(s)}, R) \leq \left\{ \frac{c(n,s)R + \frac{|a_s|}{|a_0|-m} k^{s+1}}{c(n,s)(k^{s+1} + R^{s+1}) + \frac{|a_s|}{|a_0|-m} (k^{s+1}R^s + Rk^{2s})} \right\} \times \left[\left\{ \frac{d^{(s)}}{dx^{(s)}}(1+x^n) \right\}_{x=1} \right] \Big(M(P,R) - m \Big). \tag{3.4}$$

The above inequality when combined with Lemma (2.6) (for $\mu = 1$) gives inequality (1.11) and this completes the proof of Theorem (1.5).

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