Magic coset decompositions

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Abstract

By exploiting a "mixed" non-symmetric Freudenthal–Rozenfeld–Tits
magic square, two types of coset decompositions are analyzed for the
non-compact special Kähler symmetric rank-3 coset $E_{7(-25)}/[(E_{6(-78)} \times U(1))/\mathbb{Z}_3]$, occurring in supergravity as the vector multiplets' scalar manifold in $\mathcal{N} = 2$, $D = 4$ *exceptional* Maxwell–Einstein theory. The first
decomposition exhibits maximal manifest covariance, whereas the second (*triality-symmetric*) one is of Iwasawa type, with maximal $SO(8)$
covariance. Generalizations to *conformal non-compact*, real forms of non-
degenerate, simple groups "of type E_7" are presented for both classes of
coset parametrizations, and relations to rank-3 simple Euclidean Jordan
algebras and normed trialities over division algebras are also discussed.

<div align="center">CONTENTS</div>

1   Introduction

The role of groups in Physics is at least 3-fold. First, they represent symmetries that, by definition, introduce elegance in all the equations which are manifestly symmetry invariant. If that was all, one may argue that this would be a poor advantage. But symmetries also arise as fundamental principles in constructing new theories, like, for example, gauge symmetries for the
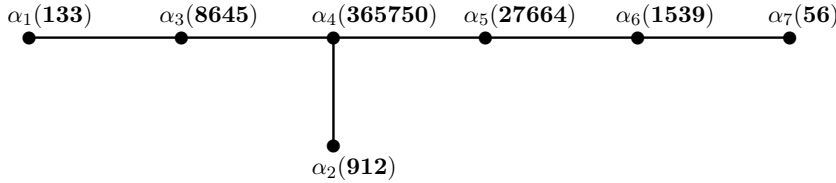
Standard Model of particle physics, conformal symmetry for string theory, or general covariance for the Einstein theory of relativity. Finally, symmetries, and then groups, play a key role in solving the equations of motion.

A particular class is represented by the (semi)simple Lie groups (and corresponding Lie algebras), which, once more, find application in a large number of mathematical and physical fields. All the finite dimensional complex Lie algebras have been classified by Wilhelm Killing, whose proofs have been made rigorous by Élie Cartan, who has also extended the classification to the non-compact, real cases. The well known result is that this classification has led to the discovery, beyond the famous classical series, of five exceptional algebras (of course together with the corresponding real forms): $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, and $\mathfrak{e}_8$.

Despite their sporadicity, the appearance of exceptional Lie groups (and algebras) in physics is anything but sporadic [1]. The importance of compact exceptional Lie groups in realizing grand unification gauge theories and consistent string theories is well recognized. Similarly, the relevance of non-compact real forms for the study of locally supersymmetric theories of gravity is well known. Other examples include sigma models based on quotients of exceptional Lie groups, which are of interest for string theory and conformal field theory applications as well. It is worth mentioning that the analysis of quantum criticality in Ising chains and the structure of magnetic materials such as cobalt niobate has also recently (and strikingly) turned out to be related to exceptional Lie groups of type E (see, e.g., [2,3], respectively).

Several properties of exceptional groups and algebras can be already inferred from abstract theoretical considerations. Nevertheless, it is often important to have explicit concrete realizations of the groups available in term of matrices, for both numerical or analytical calculations. For example, one could test conjectures related to confinement in non-Abelian gauge theories (see, e.g., [4]), and, more in general, perform explicit non-perturbative computations in exceptional lattice GUT theories and in random matrix theories.

In particular, among all the exceptional groups, there are specific motivations for physics to be interested in E_7: recently, a strict relation between cryptography and black hole physics based on E_7 (and E_6) exceptional supergravity has been discovered [5–8]. However, actual computation of entangled expectation values requires again an explicit determination of the Haar measure and of the range of the parameters. Moreover, fascinating group theoretical structures arise clearly in the description of the Attractor Mechanism for black holes in the Maxwell–Einstein supergravity [9], such as the so-called magic exceptional supergravity [10] we are focusing on in the

Figure 1: Dynkin diagram for $\mathfrak{e}_7$.

present investigation, which is related to the *minimally non-compact* real $E_{7(-25)}$ form [11] of E_7.

Before proceeding further, it is worth recalling some basic facts on the Lie algebra $\mathfrak{e}_7$ of E_7. Let us start by stating that $\mathfrak{e}_7$ is the unique exceptional Lie algebra of rank 7, and it is characterized by the Dynkin diagram drawn in figure 1, in which each dot corresponds to a simple root α_i. These are free generators of the root lattice $\Lambda_R = \sum_i \mathbb{Z}\alpha_i$. The space $H^* = \Lambda_R \otimes \mathbb{R}$ is endowed with a positive definite inner product $(\cdot|\cdot)$. The weight lattice Λ_W is the dual of Λ_R with respect to the hooked product, which means that it is freely generated over $\mathbb{Z}$ by the fundamental weights $\lambda^i \in H^*$, $i = 1, \ldots, 7$ defined by $\langle \alpha_i, \lambda^j \rangle = \delta_i^j$, with:

$$\langle \alpha, \lambda \rangle := 2 \frac{(\alpha|\lambda)}{(\alpha|\alpha)}. \tag{1.1}$$

There is a univocal correspondence between fundamental weights and fundamental representations, and all the irreducible finite dimensional representations can be generated from the basic ones, which are indicated in parenthesis in figure 1. Here, we are going to deal with the two lower dimensional, namely the fundamental **56** and the adjoint **133**.

The complex algebra $\mathfrak{e}_7$ is completely characterized by its Dynkin diagram, from which one can reconstruct the adjoint representation, that, being faithful, is isomorphic to the algebra itself. Since $\mathfrak{e}_7$ is a 133-dimensional complex algebra, it follows that such a representation is the aforementioned **133**.

The Lie algebra $\mathfrak{e}_7$ exhibits four distinct non-compact, real forms. This means that there are four inequivalent ways to select a 133-dimensional real subspace of the 266-dimensional real space underlying the complex algebra $\mathfrak{e}_7$, in such a way that the selected subspace endowed with the inherited Lie product is itself a (real) Lie algebra. For each simple Lie algebra $\mathfrak{g}$ there is a unique simply connected Lie group G (up to isomorphisms), such that

Table 1: The real forms of E_7.

Symbol	Real form	Maximal compact subgroup (mcs)
$E_{7(-133)}$	Compact	$E_{7(-133)}$
$E_{7(7)}$	Split	$SU(8)/\mathbb{Z}_2$
$E_{7(-5)}$	EVI	$(Spin(12) \times USp(2))/\mathbb{Z}_2$
$E_{7(-25)}$	EVII	$(E_{6(-78)} \times U(1))/\mathbb{Z}_3$

$\mathfrak{g}$ is the corresponding Lie algebra. The complex Lie group $E_{7(\mathbb{C})}$ contains a maximal compact subgroup, which is a 133-dimensional real Lie group (denoted as $E_{7(-133)}$), whose Lie algebra is then called the compact form (denoted[1] as $\mathfrak{e}_{7(-133)}$), where in parenthesis the signature of the Killing form (number of the positive eigenvalues minus number of the negative ones) is indicated.

The non-compact, real forms are in correspondence with the maximal compact subalgebras of $\mathfrak{e}_{7(-133)}$ (i.e., the compact Lie subalgebras that are not properly contained in a proper subalgebra of $\mathfrak{e}_{7(-133)}$ itself). The same holds at group level. There are four such subalgebras and therefore four corresponding real forms, which we collect in Table 1 (at Lie group level). For a recent treatment of E_7 groups (and algebras), see, e.g., [12].

The plan of the paper is as follows.

As anticipated, we are going to deal with the *minimally non-compact* real form of $\mathfrak{e}_7$ (E_7), namely with $\mathfrak{e}_{7(-25)}$ and its corresponding Lie group $E_{7(-25)}$, both denoted by EVII (see Table 1). In Section 2, by starting from its general construction through the Tits magic square, we study the Lie algebra $\mathfrak{e}_{7(-25)}$ itself, and we explicily construct a realization in the fundamental **56** representation embedded in $\mathfrak{usp}(28, 28)$. The matrix elements obtained with this technique turn out to be strictly related to the invariant totally symmetric rank-3 so-called d-tensor of the $E_{6(-78)}$ group, thus allowing for different expressions, depending on the choice of the basis for the relevant rank-3 (simple) Euclidean Jordan algebra.

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[1]The Killing form K on a complex Lie algebra is defined by $K(X, Y) :=$ $\text{Tr}(ad(X)ad(Y))$ and is non-degenerate for a simple algebra and on the corresponding real forms. In particular, for a non-compact form it is negative definite on the maximal compact subalgebra, namely on the maximal Lie subalgebra, whose exponentiation generates a compact Lie (sub)group.

In the present paper, we focus on two remarkable explicit parametrizations of the symmetric manifold[2]

$$\mathcal{M} := \frac{E_{7(-25)}}{K} = \frac{E_{7(-25)}}{\left(E_{6(-78)} \times U(1)\right) / \mathbb{Z}_3} \tag{1.2}$$

(obtained by suitably exponentiating the corresponding coset Lie algebra), which can be regarded as the classical vector multiplets' scalar manifold of the $\mathcal{N} = 2$, $D = 4$ Maxwell–Einstein so-called exceptional magic supergravity theory, based on the rank-3 Euclidean simple Jordan algebra $\mathfrak{J}_3(\mathbb{O})$ on the normed division algebra of the octonions $\mathbb{O}$ [10].

The first type of coset parametrization/decomposition, analyzed in Section 3, exhibits maximal manifest covariance with respect to the maximal compact subgroup (*mcs*) $E_{6(-78)} \times U(1)$ of $E_{7(-25)}$ (up to $\mathbb{Z}_3$; see (1.2)). Interestingly, such a coset parametrization, also exhibiting a manifest complex (actually, special Kähler) structure, can be generalized to encompass a more general class of Lie groups, which in Section 5 we identify *at least* as the *conformal non-compact* real forms of simple, non-degenerate Lie groups "of type E_7" [14], of which $E_{7(-25)}$ (in its **56** representation) can be considered as the generic representative. Groups "of type E_7" have recently appeared in Theoretical Physics, in the investigation of single- [15] and multi-centered [16–20] extremal black hole solutions in supergravity theories, as well as in the study of matter creation in the Universe [21].

The second coset parametrization, studied in Section 4, relies on the Iwasawa construction, already analyzed for the split form $E_{7(7)}$ e.g., in [22]. In this case, the maximal manifest covariance reduces down to an $SO(8)$ subgroup of $E_{7(7)}$, which will interestingly turn out to be related to the automorphism group $\text{Aut}(\mathbf{t}(\mathbb{O}))$ of the *normed triality* $\mathbf{t}(\mathbb{O})$ over the octonions $\mathbb{O}$ (entering the Tits' construction). The well known $SO(8)$ triality is manifest in such an approach, as detailed in the group theoretical analysis of Sections 4.1 and 4.2. As discussed in Section 5, also this construction of the Iwasawa decomposition can be generalized *at least* to the *conformal non-compact* real forms of simple, non-degenerate Lie groups "of type E_7"; the resulting manifest covariance is then given by an $SO(q) \times \mathcal{A}_q$ subgroup, which remarkably shares the same Lie algebra as the automorphism group $\text{Aut}(\mathbf{t}(\mathbb{A}))$ of the *normed triality* over the relevant normed division algebra (see, e.g., [23]) $\mathbb{A} = \mathbb{R}$ (reals), $\mathbb{C}$ (complex numbers), $\mathbb{H}$ (quaternions), and $\mathbb{O}$ (octonions).

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[2]For previous studies of exceptional cosets in supergravity, see, e.g., [13].

Table 2: The "mixed" magic square.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$SO(3)$	$SU(3)$	$USp(6)$	$F_{4(-52)}$
$\mathbb{C}$	$SU(3)$	$SU(3) \times SU(3)$	$SU(6)$	$E_{6(-78)}$
$\mathbb{H}_S$	$Sp(6,\mathbb{R})$	$SU(3,3)$	$SO^*(12)$	$E_{7(-25)}$
$\mathbb{O}_S$	$F_{4(4)}$	$E_{6(2)}$	$E_{7(-5)}$	$E_{8(-24)}$

Final remarks, comments and discussion of further possible developments are given in the concluding Section 6.

2   The Lie algebra $\mathfrak{e}_{7(-25)}$ in the 56

In order to construct the Lie algebra $\mathfrak{e}_{7(-25)}$, we are going to follow a procedure similar to the one outlined in Section 7 of Yokota [24], based on the non-symmetric "mixed" *magic square* [10, 25, 26] displayed in Table 2.

The rows and the columns contain the division algebras of the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$. Since at the group level we focus on the (minimally) *non-compact* form $E_{7(-25)}$, we need to start from the split form $\mathbb{H}_S$ of the quaternions in the third row. On the other hand, we are also interested in identifying explicitly its maximal compact subgroup $K := E_{6(-78)} \times U(1)/\mathbb{Z}_3$ [24], and therefore the usual form $\mathbb{C}$ of the complex field in the second row is to be considered.

The Tits' formula then yields the Lie algebra $\mathcal{L}$ corresponding to division algebras in row $\mathbb{A}$ and column $\mathbb{B}$ as follows [26]:

$$\mathcal{L}\left(\mathbb{A}, \mathbb{B}\right) = \operatorname{Der}\left(\mathbb{A}\right) \oplus \operatorname{Der}\left(\mathfrak{J}_3\left(\mathbb{B}\right)\right) \dotplus \left(\mathbb{A}' \otimes \mathfrak{J}_3'\left(\mathbb{B}\right)\right). \qquad (2.1)$$

The symbol $\oplus$ denotes the direct sum of algebras, whereas $\dotplus$ stands for the direct sum of vector spaces. Moreover, Der are the linear derivations, $\mathfrak{J}_3\left(\mathbb{B}\right)$ denotes the rank-3 Jordan algebra on $\mathbb{B}$, and the priming amounts to considering only traceless elements.

In particular, for the Lie algebra of $E_{7(-25)}$ the Tits' formula (2.1) reads:

$$\mathfrak{e}_{7(-25)} = \mathcal{L}\left(\mathbb{H}_S, \mathbb{O}\right) = \operatorname{Der}(\mathbb{H}_S) \oplus \operatorname{Der}(\mathfrak{J}_3(\mathbb{O})) \dotplus (\mathbb{H}_S' \otimes \mathfrak{J}_3'(\mathbb{O})). \qquad (2.2)$$

$\mathbb{H}'_S$ denotes the *imaginary* split quaternions, and the following multiplication rule holds for the units $i, j, k \in \mathbb{H}_S$ (cf. e.g. (A.18) of [27]):

$$i\,j = k = -j\,i, \quad j\,k = -i = -k\,j, \quad k\,i = j = -i\,k, \quad i^2 = -1, \quad j^2 = k^2 = 1. \tag{2.3}$$

An inner product can be defined on $\mathbb{H}_S$ as:

$$\langle h_1, h_2 \rangle := \mathrm{Re}(\bar{h}_1 h_2), \quad h_1, h_2 \in \mathbb{H}_S. \tag{2.4}$$

Also, notice that if L and R, respectively are the left and right translation in $\mathbb{H}_S$, then a derivation $D_{h_1, h_2} \in \mathrm{Der}(\mathbb{H}_S)$ can be constructed from h_1, $h_2 \in \mathbb{H}_S$ as:

$$D_{h_1, h_2} := [L_{h_1}, L_{h_2}] + [R_{h_1}, R_{h_2}]. \tag{2.5}$$

The rank-3 octonionic Jordan algebra $\mathfrak{J}_3(\mathbb{O})$ is defined as the algebra of the 3×3 Hermitian matrices of the form:

$$J = \begin{pmatrix} a_1 & o_1 & o_2 \\ o_1^* & a_2 & o_3 \\ o_2^* & o_3^* & a_3 \end{pmatrix} \tag{2.6}$$

with $a_i \in \mathbb{R}$, and $o_i \in \mathbb{O}$, $i = 1, 2, 3$. The Jordan product $\circ$ is thus realized as the symmetrized matrix multiplication:

$$j_1 \circ j_2 := \frac{1}{2}(j_1 j_2 + j_2 j_1), \quad j_1, j_2 \in \mathfrak{J}_3(\mathbb{O}). \tag{2.7}$$

It is then possible to introduce an inner product on the Jordan algebra:

$$\langle j_1, j_2 \rangle := \mathrm{Tr}(j_1 \circ j_2). \tag{2.8}$$

Furthermore, there is a cubic form, which is defined for any $j_1, j_2, j_3 \in \mathfrak{J}_3(\mathbb{O})$ as [28] (for a recent account, see, e.g., [29, 30]):

$$\begin{aligned} Det(j_1, j_2, j_3) := &\frac{1}{3}\mathrm{Tr}(j_1 \circ j_2 \circ j_3) - \frac{1}{6}(\mathrm{Tr}(j_1)\mathrm{Tr}(j_2 \circ j_3) \\ &+ \mathrm{Tr}(j_2)\mathrm{Tr}(j_1 \circ j_3) + \mathrm{Tr}(j_3)\mathrm{Tr}(j_1 \circ j_2)) \\ &+ \frac{1}{6}\mathrm{Tr}(j_1)\mathrm{Tr}(j_2)\mathrm{Tr}(j_3). \end{aligned} \tag{2.9}$$

In turn, this induces an action $\rhd$ of $\mathfrak{J}_3(\mathbb{O})$ on itself through $Det(j_1, j_2, j_3) := \frac{1}{3}\mathrm{Tr}\left((j_1 \rhd j_2) \circ j_3\right)$, which by definition (2.10) reads:

$$j_1 \rhd j_2 := j_1 \circ j_2 - \frac{1}{2}\mathrm{Tr}(j_1)j_2 - \frac{1}{2}\mathrm{Tr}(j_2)j_1 + \frac{1}{2}\mathrm{Tr}(j_1)\mathrm{Tr}(j_2)I_3$$
$$- \frac{1}{2}\mathrm{Tr}(j_1 \circ j_2)I_3, \tag{2.10}$$

with I_3 the 3×3 identity matrix.

An important ingredient entering (2.1) is the *Lie product* $[\cdot, \cdot]$, which in the case under consideration extends the multiplication structure also to $\mathbb{H}'_S \otimes \mathfrak{J}'_3(\mathbb{O})$; its general explicit expression can be found, e.g., in (2.5) of [12]:

$$[h_1 \otimes j_1, h_2 \otimes j_2] := \frac{1}{12}\langle j_1, j_2 \rangle D_{h_1, h_2} - \langle h_1, h_2 \rangle [L_{j_1}, L_{j_2}]$$
$$+ \frac{1}{2}[h_1, h_2] \otimes (j_1 \circ j_2 - \frac{1}{3}\langle j_1, j_2 \rangle I_3). \tag{2.11}$$

It is known (see, e.g., [27, 31]) that:

$$\mathrm{Der}\left(\mathfrak{J}_3(\mathbb{O})\right) \sim \mathfrak{f}_{4(-52)}; \tag{2.12}$$
$$\mathrm{Der}\left(\mathbb{H}_S\right) \sim \mathfrak{sl}\left(2, \mathbb{R}\right), \tag{2.13}$$

and therefore (2.2) can be recast as:

$$\mathfrak{e}_{7(-25)} = \mathfrak{sl}\left(2, \mathbb{R}\right) \oplus \mathfrak{f}_4 \dotplus (\mathbb{H}'_S \otimes \mathfrak{J}'_3(\mathbb{O})), \tag{2.14}$$

which implements the maximal *non-symmetric* embedding (whose compact form is given, e.g., by Table 15 of [32]; see also [33]):

$$E_{7(-25)} \supset SL(2, \mathbb{R}) \times F_{4(-52)};$$
$$\mathbf{56} = (\mathbf{4}, \mathbf{1}) + (\mathbf{2}, \mathbf{26}) ;$$
$$\mathbf{133} = (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{52}) + (\mathbf{3}, \mathbf{26}) . \tag{2.15}$$

We note in passing that, from the branching (2.15) of $\mathbf{56}$, this embedding is relevant for the *maximal* truncation of $\mathcal{N} = 2$, $D = 4$ magical exceptional theory (based on rank-3 simple Jordan algebra $\mathfrak{J}_3(\mathbb{O})$) to the smallest cubic $\mathcal{N} = 2$, $D = 4$ model, namely the so-called T^3 model, the truncation condition on the vectors (and their field strengths' fluxes, namely electric and magnetic charges) being given by $(\mathbf{2}, \mathbf{26}) = 0$.

As the next step, one needs to identify the subalgebra generating the maximal compact subgroup $K := E_{6(-78)} \times U(1)/\mathbb{Z}_3$ of $E_{7(-25)}$. By considering

the manifestly $\mathfrak{f}_{4(-52)}$-covariant decomposition of $\mathfrak{e}_{6(-78)}$ from Tits' formula (2.1):

$$\mathfrak{e}_{6(-78)} = \mathcal{L}(\mathbb{C}, \mathbb{O}) = \mathrm{Der}(\mathfrak{J}_3(\mathbb{O})) \dotplus \left(i \otimes \mathfrak{J}_3'(\mathbb{O}) \right), \qquad (2.16)$$

the Lie algebra $\mathfrak{K}$ of K can be identified as the subalgebra of $\mathfrak{e}_{6(-78)}$ defined by picking the only imaginary unit $i \in \mathbb{H}_S$ which satisfies $i^2 = -1$ and computing:

$$\mathfrak{K} = ad_i \oplus \mathrm{Der}(\mathfrak{J}_3(\mathbb{O})) \dotplus \left(i \otimes \mathfrak{J}_3'(\mathbb{O}) \right), \qquad (2.17)$$

where $ad_i \in \mathbb{H}_S$ denotes the *adjoint action* of i, generating the maximal compact subgroup $U(1)$ of $SL(2, \mathbb{R})$. It is worth remarking that, due to the following property of the Lie product:

$$[i \otimes j_1, i \otimes j_2] = -[L_{j_1}, L_{j_2}], \;\; j_1, j_2 \in \mathfrak{J}_3'(\mathbb{O}), \qquad (2.18)$$

the multiplication of $\mathfrak{J}_3'(\mathbb{O})$ by the imaginary unit i in the last summand of (2.16) and (2.17) is exactly what is needed to get the compact form of $E_{6(-78)}$ instead of the (minimally) non-compact real form $E_{6(-26)}$, when exponentiating the algebra.

As anticipated, by this procedure, inspired by the approach of Cacciatori Piazza and Scotti [12] and exploiting the methods explained in [24], one can construct the (smallest symplectic) fundamental irrep. **Fund = 56** of $E_{7(-25)}$ reproducing the structure constants of the **Adj = 133** irrep. (for whatever basis one chooses for the algebra).

Such an explicit symplectic realization reads as follows:

$$Y_I = \left(\begin{array}{c|c|c|c} \phi_I & \overrightarrow{0}_{27} & 0_{27} & \overrightarrow{0}_{27} \\ \hline \overrightarrow{0}_{27}^T & 0 & \overrightarrow{0}_{27}^T & 0 \\ \hline 0_{27} & \overrightarrow{0}_{27} & -\phi_I^T & \overrightarrow{0}_{27} \\ \hline \overrightarrow{0}_{27}^T & 0 & \overrightarrow{0}_{27}^T & 0 \end{array} \right), \quad I = 1, \ldots, 78; \qquad (2.19)$$

$$Y_{79} = \left(\begin{array}{c|c|c|c} \frac{i}{\sqrt{6}} I_{27} & \overrightarrow{0}_{27} & 0_{27} & \overrightarrow{0}_{27} \\ \hline \overrightarrow{0}_{27}^T & -i\sqrt{\frac{3}{2}} & \overrightarrow{0}_{27}^T & 0 \\ \hline 0_{27} & \overrightarrow{0}_{27} & -\frac{i}{\sqrt{6}} I_{27} & \overrightarrow{0}_{27} \\ \hline \overrightarrow{0}_{27}^T & 0 & \overrightarrow{0}_{27}^T & i\sqrt{\frac{3}{2}} \end{array} \right); \qquad (2.20)$$

$$Y_{\alpha+79} = \frac{1}{2} \left(\begin{array}{c|c|c|c} 0_{27} & \overrightarrow{0}_{27} & 2iA_\alpha & i\sqrt{2}\,\overrightarrow{e}_\alpha \\ \hline \overrightarrow{0}_{27}^{T} & 0 & i\sqrt{2}\,\overrightarrow{e}_\alpha^{T} & 0 \\ \hline -2iA_\alpha & -i\sqrt{2}\,\overrightarrow{e}_\alpha & 0_{27} & \overrightarrow{0}_{27} \\ \hline -i\sqrt{2}\,\overrightarrow{e}_\alpha^{T} & 0 & \overrightarrow{0}_{27}^{T} & 0 \end{array} \right), \quad \alpha = 1, \ldots, 27; \tag{2.21}$$

$$Y_{\alpha+106} = \frac{1}{2} \left(\begin{array}{c|c|c|c} 0_{27} & \overrightarrow{0}_{27} & -2A_\alpha & \sqrt{2}\,\overrightarrow{e}_\alpha \\ \hline \overrightarrow{0}_{27}^{T} & 0 & \sqrt{2}\,\overrightarrow{e}_\alpha^{T} & 0 \\ \hline -2A_\alpha & \sqrt{2}\,\overrightarrow{e}_\alpha & 0_{27} & \overrightarrow{0}_{27} \\ \hline \sqrt{2}\,\overrightarrow{e}_\alpha^{T} & 0 & \overrightarrow{0}_{27}^{T} & 0 \end{array} \right), \quad \alpha = 1, \ldots, 27, \tag{2.22}$$

where I_n is the $n \times n$ identity matrix, 0_{27} is the 27×27 null matrix, $\overrightarrow{0}_n$ is the zero vector in $\mathbb{R}^n$, and $\overrightarrow{e}_\alpha$, $\alpha = 1, \ldots, 27$, is the canonical basis of $\mathbb{R}^{27}$ throughout.

The 78 matrices ϕ_I realize a subalgebra $\mathfrak{e}_{6(-78)}$ in its irreducible representation **Fund** = **27**. An explicit expression can be found, e.g., in Section 2.1 of [12]:

$$\phi_I = \begin{cases} C_I & I = 1, \ldots, 52, \\ \tilde{C}_{I-52} & I = 53, \ldots, 78, \end{cases} \tag{2.23}$$

where, in turn, the matrices C_I realize a maximal $\mathfrak{f}_{4(-52)}$ subalgebra in its **Fund** = **26** irrep. (see, e.g., [34, 35]).

The 27 matrices A_α are related to the d-tensor of E_6, as explained in more detail in Section 2.1.

The first 79 matrices Y_I (2.19) and Y_{79} (2.20) generate the maximal compact subgroup K of $E_{7(-25)}$ and are anti-Hermitian, whereas the remaining ones $Y_{\alpha+79}$ (2.21) and $Y_{\alpha+106}$ (2.22) generate the non-compact symmetric coset $E_{7(-25)}/K$ and they are Hermitian.

By introducing (cf. [12]):

$$\widetilde{I} := \left(\begin{array}{c|c} I_{26} & \overrightarrow{0}_{26} \\ \hline \overrightarrow{0}_{26}^{T} & -2 \end{array} \right), \tag{2.24}$$

the two matrices Y_{106} and Y_{133} can be rewritten more explicitly as:

$$Y_{106} = \frac{1}{2} \left(\begin{array}{c|c|c|c} 0_{27} & \overrightarrow{0}_{27} & -i\sqrt{\frac{2}{3}}\widetilde{I} & i\sqrt{2}\overrightarrow{e}_{27} \\ \hline \overrightarrow{0}_{27}^{T} & 0 & i\sqrt{2}\overrightarrow{e}_{27}^{T} & 0 \\ \hline i\sqrt{\frac{2}{3}}\widetilde{I} & -i\sqrt{2}\overrightarrow{e}_{27} & 0_{27} & \overrightarrow{0}_{27} \\ \hline -i\sqrt{2}\overrightarrow{e}_{27}^{T} & 0 & \overrightarrow{0}_{27}^{T} & 0 \end{array} \right), \qquad (2.25)$$

$$Y_{133} = \frac{1}{2} \left(\begin{array}{c|c|c|c} 0_{27} & \overrightarrow{0}_{27} & \sqrt{\frac{2}{3}}\widetilde{I} & \sqrt{2}\overrightarrow{e}_{27} \\ \hline \overrightarrow{0}_{27}^{T} & 0 & \sqrt{2}\overrightarrow{e}_{27}^{T} & 0 \\ \hline \sqrt{\frac{2}{3}}\widetilde{I} & \sqrt{2}\overrightarrow{e}_{27} & 0_{27} & \overrightarrow{0}_{27} \\ \hline \sqrt{2}\overrightarrow{e}_{27}^{T} & 0 & \overrightarrow{0}_{27}^{T} & 0 \end{array} \right). \qquad (2.26)$$

Together with Y_{79} ($U(1)$ generator), Y_{106} and Y_{133} generate an $SL(2,\mathbb{R})$ subgroup, corresponding to the one appearing in (2.14) and (2.15).

2.1 The matrices A_α and the *d*-tensor of the 27 of $E_{6(-78)}$

By choosing a basis $\{j_a\}_{a=1,\ldots,26}$ of $\mathfrak{J}'_3(\mathbb{O})$ normalized as $\langle j_a, j_b \rangle = 2\delta_{ab}$, a completion to a basis for $\mathfrak{J}_3(\mathbb{O})$ can be obtained by adding $j_{27} = \sqrt{\frac{2}{3}}I_3$. The A_α's are 27×27 symmetric matrices representing, by means of the linear isomorphism $\mathfrak{J}_3(\mathbb{O}) \simeq \mathbb{R}^{27}$, $j_\alpha \mapsto \vec{e}_\alpha$, the action $\rhd$ of $\mathfrak{J}_3(\mathbb{O})$ on $\mathfrak{J}_3(\mathbb{O})$ itself. The components of A_α, explicitly computed in [12], satisfy the following relation [28]:

$$(A_\alpha)^\beta{}_\gamma = \frac{1}{2}\mathrm{Tr}\left((j_\alpha \rhd j_\gamma) \circ j_\beta\right) = \frac{3}{2}\, Det(j_\alpha, j_\gamma, j_\beta) =: \frac{1}{\sqrt{2}}\, d_{\alpha\gamma\beta}, \qquad (2.27)$$

where $d_{\alpha\gamma\beta} = d_{(\alpha\gamma\beta)}$ is the totally symmetric rank-3 invariant *d*-tensor of the **27** of of $E_{6(-78)}$, with a normalization suitable to match $Det(j_\alpha, j_\gamma, j_\beta)$ given by (2.10) (see below). We point out that the result (2.27) does not depend on the choice of the basis $\{j_\alpha\}$. Thus, the expressions of $Y_{\alpha+79}$ (2.21) and of $Y_{\alpha+106}$ (2.22) exhibit the maximal manifest compact $[(E_{6(-78)} \times U(1))/\mathbb{Z}_3]$-covariance. However, whenever the choice of the basis $\{j_\alpha\}$ is exploited in order to distinguish the identity matrix from the traceless ones, the $d_{\alpha\beta\gamma}$ of E_6 has a maximal manifestly $F_{4(-52)}$-invariance only. This also holds for the expressions of the Y_I (2.19), which are manifestly $F_{4(-52)}$-covariant only, due

to the splitting (2.23). Notice that the full $\left[(E_{6(-78)} \times U(1))/\mathbb{Z}_3\right]$-covariance can be recovered simply by picking a generic basis for the Jordan algebra.

A manifestly $[SU(6) \times SU(2)]$-invariant expression of the d-tensor of the **27** of $E_{6(-78)}$ has been constructed in [36]. On the other hand, d-tensors for the non-compact real forms of E_6 have been more extensively considered in the literature, e.g., due to their appearance in the general form of the holomorphic prepotential F of cubic special Kähler geometry (see, e.g., [37]). For instance, in [38] the d-tensors of $E_{6(6)}$ (split) and $E_{6(-26)}$ (minimally non-compact) real forms have been explicitly constructed, with $USp(8)$ and $USp(6,2)$ maximal manifest invariance, respectively. By denoting with G_6 the U-duality[3] group of chiral supergravity theories with 8 supersymmetries in $D = 6$ space-time dimensions, and considering all U-duality groups G_5 of $\mathcal{N} = 2$, $D = 5$ supergravity theories with symmetric (vector multiplets') scalar manifold, manifestly $[G_6 \times SO(1,1)]$-invariant expressions of the G_5-invariant d-tensor have been derived, e.g., in [37, 41–45].

A necessary remark on the consistence of normalizations is in order. As a consequence of the choice (2.35) for the normalization of the matrices Y_A discussed in Section 2.2, the components $(A_\alpha)^\beta{}_\gamma := A_{\alpha\beta\gamma}$ are normalized as:

$$A_{\alpha\beta\gamma}A^{\eta\beta\gamma} = 5\delta_\alpha^\eta. \tag{2.28}$$

This is consistent with the normalization of the d-tensor (of $E_{6(-26)}$) given by the following expression of the Kähler-invariant $\left((X^0)^2\text{-rescaled}\right)$ holomorphic prepotential function characterizing special Kähler geometry (see, e.g., [37, 46, 47]):

$$f(z) := \frac{1}{3!}d_{\alpha\beta\gamma}z^\alpha z^\beta z^\gamma, \tag{2.29}$$

adopted, e.g., in [48]; in general, $\alpha = 1, \ldots, n_V$, where n_V denotes the number of Abelian vector multiplets coupled to the supergravity multiplet. Indeed, within the notation conventions adopted in [49], one can compute that (see also [50, 51]):

$$d_{\alpha\beta\gamma}d^{\eta\beta\gamma} = (q + 2)\,\delta_\alpha^\eta. \tag{2.30}$$

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[3]Here U-duality is referred to as the "continuous" symmetries of Cremmer and Julia [39]. Their discrete versions are the U-duality non-perturbative string theory symmetries introduced by Hull and Townsend [40].

For all the models reported in Table 3 below but the T^3 model, q can be defined as:

$$q \equiv \dim_\mathbb{R} \mathbb{A}, \qquad (2.31)$$

where $\mathbb{A}$ denotes the division algebra on which the corresponding rank-3 simple Jordan algebra $\mathfrak{J}_3(\mathbb{A})$ is constructed ($q = 8, 4, 2, 1$ for $\mathbb{A} = \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R}$, respectively). Furthermore, as observed in [52], in general q is related to the *inverse Coxeter number* λ by the relation:

$$\lambda = -\frac{2}{q+2}, \quad q = 0, 1, 2, 4, 8; \qquad (2.32)$$

$$\lambda = -\frac{1}{q+1}, \quad q = -2/3 \ (T^3 \text{ model}). \qquad (2.33)$$

The case $q = 0$ in (2.32) corresponds to the *triality symmetric* so-called $\mathcal{N} = 2$ *STU* model [53], based on $\mathfrak{J}_3 = \mathbb{R} \oplus \mathbf{\Gamma}_{1,1} \sim \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$; however, since the corresponding U-duality group G_4 is *semi-simple*, it will not be considered in the present investigation.

Coming back to the previous reasoning, by plugging $q = 8$ (corresponding to the octonionic theory considered above) into (2.30), one achieves the following result:

$$q = 8 : d_{\alpha\beta\gamma} d^{\eta\beta\gamma} = 10 \delta_\alpha^\eta, \qquad (2.34)$$

which matches (2.28) when taking (2.27) into account, and assuming for the d-tensor of $E_{6(-78)}$ the same normalization of the d-tensor of $E_{6(-26)}$.

2.2   Properties of the matrices Y_A

The Y_A's are orthonormalized (with signature $(-^{79}, +^{54})$) with respect to the product:

$$\langle Y, Y' \rangle_{\mathbf{56}} := \frac{1}{12} \text{Tr}(YY'). \qquad (2.35)$$

This normalization guarantees that the period of the maximal torus in the E_6 subgroup equals 4π, which is the standard choice for the period of the spin representations of the orthogonal subgroups [34, 35].

Furthermore, the complete symmetry of the d-tensor implies the matrices Y_A ($A = 1, \ldots, 133$) given by the expressions (2.19)–(2.22) to be *symplectic*

with respect to the canonical symplectic form:

$$\Omega := \begin{pmatrix} 0_{28} & -I_{28} \\ I_{28} & 0_{28} \end{pmatrix}, \tag{2.36}$$

namely (in a block-wise notation, and suppressing the index A):

$$Y := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{sp}\,(56, \mathbb{C}) \Leftrightarrow \Omega Y + Y^T \Omega = 0 \Leftrightarrow \begin{cases} A^T = -D \\ B^T = B \\ C^T = C \end{cases}. \tag{2.37}$$

Actually, it holds that:

$$Y \in \mathfrak{usp}\,(28, 28). \tag{2.38}$$

In order to show this, let us introduce:

$$\mathcal{H} := \begin{pmatrix} I_{28} & 0_{28} \\ 0_{28} & -I_{28} \end{pmatrix}, \tag{2.39}$$

and recall the infinitesimal condition:

$$Y \in \mathfrak{u}(28, 28) \Leftrightarrow \mathcal{H}Y + Y^\dagger \mathcal{H} = 0 \Leftrightarrow \begin{cases} A^\dagger = -A \\ B^\dagger = C \\ D^\dagger = -D \end{cases}. \tag{2.40}$$

Thus, by means of the isomorphism:

$$\mathfrak{sp}\,(2n, \mathbb{R}) \sim \mathfrak{usp}\,(n, n) \equiv \mathfrak{sp}\,(2n, \mathbb{C}) \cap \mathfrak{u}\,(n, n), \tag{2.41}$$

it follows that:

$$Y \in \mathfrak{usp}(28, 28) \Leftrightarrow \begin{cases} A = -A^\dagger = \overline{D}; \\ C = B^\dagger = \overline{B}. \end{cases} \tag{2.42}$$

It should be noted that, when considering n vector fields in presence of scalar fields, the isomorphism (2.41) has been exploited by Gaillard and Zumino [54] for the study of the generalization and non-compact nature of the electric-magnetic symmetry, naturally yielding in $D = 4$ a manifestly $USp\,(n, n)$-covariant basis of self-dual/anti-self-dual vector 2-form field strengths, rather than an $Sp\,(2n, \mathbb{R})$-covariant one; see also, e.g., the re-elaboration of such a treatment presented in [47].

Let us analyze the properties of the matrices Y_A (2.19)–(2.22) (following the notation of [12]):

(1) Y_I (2.19) with $I = 1, \ldots, 52$. According to (2.23), $\phi_I = C_I$. Up to a change of basis of the Jordan algebra, the matrices C_I are given in [34] (including the *Mathematica* routine used for their computation). As mentioned before, the C_I's realize a maximal $\mathfrak{f}_{4(-52)}$ subalgebra in its irreducible representation **Fund** $= \mathbf{26}$. In turn, this is embedded into the algebra $\mathfrak{e}_{6(-78)}$ (maximal compact subalgebra of $\mathfrak{e}_{7(-25)}$) in its **Fund** $= \mathbf{27}$ irrep., through the addition of an extra 27th row and column of 0's, according to the maximal and symmetric embedding: $E_6 \supset F_4$, $\mathbf{27} = \mathbf{26} + \mathbf{1}$. The symmetry properties are:

$$C_I = -C_I^T, \quad \overline{C_I} = C_I \Longrightarrow Y_I = -Y_I^\dagger \in \mathfrak{usp}\,(28, 28)\,. \tag{2.43}$$

(2) Y_I (2.19) with $I = 53, \ldots, 78$. According to (2.23), $\Phi_I = \tilde{C}_{I-52}$, as computed in [35], where the *Mathematica* routine to generate them is given, as well. The fact that the $\tilde{C}$'s are purely imaginary is a consequence of the presence of the factor i in the last summand of (2.16); they are defined in terms of the action (2.10) applied to the traceless part $\mathfrak{J}_3'(\mathbb{O})$ of the Jordan algebra. In turn, such an action of the Jordan algebra on itself is the one entering the cubic form and hence in the definition (2.27) of the matrices A_α's, implying that the $\tilde{C}_{I-52}$ coincide with the first 26 components of A_α, apart from an overall i. The symmetry properties are:

$$\tilde{C}_{I-52} = \tilde{C}_{I-52}^T, \quad \tilde{C}_{I-52}^\dagger = -\tilde{C}_{I-52} \Longrightarrow Y_I = -Y_I^\dagger \in \mathfrak{usp}(28, 28). \tag{2.44}$$

(3) Y_{79} (2.20). It generates a $U(1)$ subgroup, corresponding to the compact Cartan of the $SL(2, \mathbb{R})$ factor group, appearing in (2.14) and (2.15). The symmetry properties are:

$$Y_{79} = Y_{79}^T, \quad Y_{79}^\dagger = -Y_{79} \Longrightarrow Y_{79} \in \mathfrak{usp}\,(28, 28)\,. \tag{2.45}$$

(4) Y_I with $I = 80, \ldots, 106$, i.e., $Y_{\alpha+79}$ (2.21). The symmetry properties read as follows:

$$A_\alpha = A_\alpha^T, \quad A_\alpha^\dagger = A_\alpha \Longrightarrow Y_{\alpha+79}^\dagger = Y_{\alpha+79}, \quad Y_{\alpha+79} = -Y_{\alpha+79}^T,$$
$$Y_{\alpha+79} \in \mathfrak{usp}(28, 28). \tag{2.46}$$

(5) Y_I with $I = 107, \ldots, 133$, i.e., $Y_{\alpha+106}$ (2.22). The symmetry properties read as follows:

$$Y_{\alpha+106}^\dagger = Y_{\alpha+106}, \quad Y_{\alpha+106} = -Y_{\alpha+106}^T, \quad Y_{\alpha+106} \in \mathfrak{usp}(28, 28). \tag{2.47}$$

Thus, (2.38) results from (2.43) to (2.47).

As elucidated in the next section, the matrices Y_I, $I = 80, \ldots, 133$ given by (2.21) and (2.22) are the Hermitian generators of the symmetric maximal non-compact (special Kähler) Riemannian coset (1.2), which is the classical vector multiplets' scalar manifold of the magical $\mathcal{N} = 2$, $D = 4$ Maxwell–Einstein supergravity theory based on $\mathfrak{J}_3(\mathbb{O})$ [10]. As given by (2.27), the 27 matrices A_α are directly related to the invariant d-tensor of the **27** irrep. of $E_{6(-78)}$; they have been explicitly constructed in [12], to which the reader is addressed for further detail.

3 Manifestly $\left[(E_{6(-78)} \times U(1))/\mathbb{Z}_3\right]$-covariant coset construction

The quotient manifold $\mathcal{M}$ (1.2) has rank 3; this means that the maximal dimension of the intersection between a Cartan subalgebra of $E_{7(-25)}$ and the generators of $\mathcal{M}$ itself is 3. From the results reported above, the three generators of a Cartan subalgebra of $\mathcal{M}$ can be chosen to be the diagonal generators of the Jordan algebra $\mathfrak{J}_3(\mathbb{O})$ itself, namely Y_{123}, Y_{132}, and Y_{133}.

The coset $\mathcal{M}$ (1.2) is generated by the matrices Y_{79+I}, (2.21), and (2.22) with $I = 1, \ldots, 54$. Through the exponential mapping, it can be defined as follows:

$$\mathcal{M} := \exp\left(\sum_{\alpha=1}^{27} x_\alpha Y_{106+\alpha} + y_\alpha Y_{79+\alpha}\right), \tag{3.1}$$

with x_α, y_α, $\alpha = 1, \ldots 27$, real parameters.

From the commutation relations of the matrices Y's, which can be easily computed by means of the Mathematica program provided in [12], it holds that:

$$[Y_{51+\alpha}, Y_{106}] = -\sqrt{\frac{2}{3}} Y_{106+\alpha}, \quad [Y_{51+\alpha}, Y_{133}] = \sqrt{\frac{2}{3}} Y_{79+\alpha}. \tag{3.2}$$

The generators of $\mathfrak{e}_{6(-78)}$ which are not in $\mathfrak{f}_{4(-52)}$ mix the matrices $Y_{79+\alpha}$ with the $Y_{106+\alpha}$. Therefore, in order to make the complex structure of $\mathcal{M}$

manifest, it is necessary to introduce the following complex linear combinations of the matrices:

$$\zeta_\alpha := \frac{1}{\sqrt{2}} \left(Y_{79+\alpha} + \mathrm{i}\, Y_{106+\alpha} \right), \tag{3.3}$$

$$\bar{\zeta}_\alpha := \frac{1}{\sqrt{2}} \left(Y_{79+\alpha} - \mathrm{i}\, Y_{106+\alpha} \right).$$

This hints for the complex linear combinations of the parameters:

$$z_\alpha := \frac{1}{\sqrt{2}} (y_\alpha + \mathrm{i}\, x_\alpha), \tag{3.4}$$

$$\bar{z}_\alpha := \frac{1}{\sqrt{2}} (y_\alpha - \mathrm{i}\, x_\alpha),$$

which allows one to rewrite (3.1) as:

$$\mathcal{M} := \exp \left(\sum_{\alpha=1}^{27} \bar{z}_\alpha \zeta_\alpha + z_\alpha \bar{\zeta}_\alpha \right). \tag{3.5}$$

By introducing the 27-dimensional complex vector:

$$z := \sum_{\alpha=1}^{27} z_\alpha \vec{e}_\alpha, \tag{3.6}$$

and the 28×28 matrix:

$$\mathcal{A} := \left(\begin{array}{c|c} -\sqrt{2} \displaystyle\sum_{\alpha=1}^{27} \bar{z}_\alpha A_\alpha & z \\ \hline z^T & 0 \end{array} \right). \tag{3.7}$$

Equation (3.5) enjoys the simple form:

$$\mathcal{M} := \exp \left(\begin{array}{c|c} 0 & \mathcal{A} \\ \hline \mathcal{A}^\dagger & 0 \end{array} \right) = \left(\begin{array}{c|c} \mathrm{Ch}(\sqrt{\mathcal{A}\mathcal{A}^\dagger}) & \mathcal{A}\dfrac{\mathrm{Sh}(\sqrt{\mathcal{A}^\dagger\mathcal{A}})}{\sqrt{\mathcal{A}^\dagger\mathcal{A}}} \\ \hline \dfrac{\mathrm{Sh}(\sqrt{\mathcal{A}\mathcal{A}^\dagger})}{\sqrt{\mathcal{A}\mathcal{A}^\dagger}}\mathcal{A}^\dagger & \mathrm{Ch}(\sqrt{\mathcal{A}^\dagger\mathcal{A}}) \end{array} \right). \tag{3.8}$$

This is a Hermitian matrix, of the same form as the finite coset representative worked out [55] for the *split* (i.e., maximally non-compact) counterpart

$$\mathcal{M}_{\mathcal{N}=8} = \frac{E_{7(7)}}{SU\left(8\right)/\mathbb{Z}_2}, \tag{3.9}$$

which is the scalar manifold of *maximal* $\mathcal{N} = 8$, $D = 4$ supergravity, associated to $\mathfrak{J}_3\left(\mathbb{O}_S\right)$. On the other hand, as a consequence of (2.38), $\mathcal{M}$ also is an element of $USp\left(28, 28\right)$, whereas $\mathcal{M}_{\mathcal{N}=8}$ is real.

By using the machinery of *special Kähler geometry* (see, e.g., [37, 46, 47]), the symplectic sections defining the *symplectic frame* associated to the coset parametrization introduced above can be directly read from (3.7)and (3.8):

$$\mathcal{M} =: \left(\begin{array}{c|c} u_i^\Lambda\left(z, \overline{z}\right) & v_{i\Lambda}\left(z, \overline{z}\right) \\ \hline v^{i\Lambda}\left(z, \overline{z}\right) & u_\Lambda^i\left(z, \overline{z}\right) \end{array} \right), \tag{3.10}$$

where the symplectic index $\Lambda = 0, 1, \ldots 27$ (with 0 pertaining to the $\mathcal{N} = 2$, $D = 4$ graviphoton), and $i = \overline{\alpha}, 28$. Thus, the symplectic sections read (see, e.g., [47, 56]; subscript "28" omitted):

$$f_i^\Lambda := \frac{1}{\sqrt{2}} \left(u + v\right)_i^\Lambda = \left(\overline{f}_{\overline{\alpha}}^\Lambda, f^\Lambda\right) := \left(\overline{\mathcal{D}}_{\overline{\alpha}}\overline{L}^\Lambda, L^\Lambda\right)$$

$$= \exp\left(\frac{1}{2}K\right)\left(\overline{\mathcal{D}}_{\overline{\alpha}}\overline{X}^\Lambda, X^\Lambda\right); \tag{3.11}$$

$$h_{i\Lambda} := -\frac{i}{\sqrt{2}} \left(u - v\right)_{i\Lambda} = \left(\overline{h}_{\overline{\alpha}|\Lambda}, h_\Lambda\right) := \left(\overline{\mathcal{D}}_{\overline{\alpha}}\overline{M}_\Lambda, M_\Lambda\right)$$

$$= \exp\left(\frac{1}{2}K\right)\left(\overline{\mathcal{D}}_{\overline{\alpha}}\overline{F}_\Lambda, F_\Lambda\right), \tag{3.12}$$

where $\mathcal{D}$ is the Kähler-covariant differential operator,

$$\mathcal{V} := \left(L^\Lambda, M_\Lambda\right)^T = \exp\left(\frac{1}{2}K\right)\left(X^\Lambda, F_\Lambda\right)^T \tag{3.13}$$

is the symplectic vector of Kähler-covariantly holomorphic sections, and

$$K := -\ln\left[i\left(\overline{X}^\Lambda F_\Lambda - X^\Lambda\overline{F}_\Lambda\right)\right] \tag{3.14}$$

is the Kähler potential determining the corresponding geometry.

As announced, a key feature of (3.7) is that the matrix $\mathcal{A}$, generating the coset $\mathcal{M}$ (1.2)–(3.8), is written in terms of the invariant rank-3 d-tensor

of the **27** fundamental irrep. of $E_{6(-78)}$, thus yielding a formalism with manifest $\left[(E_{6(-78)} \times U(1))/\mathbb{Z}_3\right]$-covariance, which is the maximal compact possible symmetry of the framework under consideration. Within such a parametrization, the complex scalar fields of the corresponding $\mathcal{N} = 2$ magic theory, coordinatizing $\mathcal{M}$ (1.2), are defined by (3.4), and summarized in vector notation by (3.6).

Furthermore, attention should be paid not to confuse this symplectic frame with the so-called *"4D/5D special coordinates"* symplectic frame (see, e.g., [49]), in which the holomorphic prepotential function F exists and it is given by $\left(\left(X^0\right)^2$ times$\right)$ (2.29). Indeed, as commented below, F simply does not exist in the symplectic frame under consideration (namely, $2F = X^\Lambda F_\Lambda = 0$ [57]), and the d-tensor of the **27** of $E_{6(-26)}$ (appearing in (2.29)) is different from the d-tensor of **27** of $E_{6(-78)}$, appearing in the treatment of Section 2 and of the present section; such a difference is evident, e.g., when considering a manifestly $[G_6 \times SO(1,1)]$-invariant formalism, as done, e.g., in [38, 44].

As mentioned in Section 2, by exploiting the expressions (2.25) and (2.26) of the non-compact generators of the relevant $\mathfrak{sl}(2, \mathbb{R})$ subalgebra, the maximal manifest $\left[(E_{6(-78)} \times U(1))/\mathbb{Z}_3\right]$-covariance can be broken down to a manifest $\left[F_{4(-52)} \times U(1)\right]$-covariance (recall the maximal symmetric embedding (2.14) and (2.15)), in which (3.7) becomes:

$$\mathcal{A} = \left(\begin{array}{c|c} \frac{1}{\sqrt{3}} \bar{z}_{27} \tilde{I} - \sqrt{2} \sum_{a=1}^{26} \bar{z}_a A_a & z \\ \hline z^T & 0 \end{array} \right). \tag{3.15}$$

We note in passing that $F_{4(-52)}$ is particularly relevant, because it contains all the compact generators of $USp(6, 2)$, which is the maximal (non-compact) manifest covariance exhibited by the d-tensor of the **27** irrep. of $E_{6(-26)}$ constructed in [38]:

$$USp(6, 2) \cap USp(8) \cap F_{4(-52)} = USp(6) \times USp(2)$$
$$\sim USp(6) \times SU(2) = \text{mcs}\left[USp(6, 2)\right]. \tag{3.16}$$

3.1 Remarks

In order to gain more insight on the parametrization under consideration, it is useful to compare the infinitesimal element of the coset $\mathcal{M}$ (1.2), given

by the 28×28 matrix (recall (3.8) and (3.7)):

$$\ln \mathcal{M} := \left(\begin{array}{c|c} 0 & \mathcal{A} \\ \hline \mathcal{A}^\dagger & 0 \end{array} \right), \tag{3.17}$$

with an analogue expression, given by (6) of [48], which we recall here for ease of comparison:

$$\mathcal{B} := \left(\begin{array}{c|c|c|c} 0^\alpha_\beta & -t'^\alpha & d^{\alpha\beta\gamma}t_\gamma & 0^\alpha \\ \hline -t_\beta & 0 & 0_\beta & 0 \\ \hline d_{\alpha\beta\gamma}t'^\gamma & 0_\alpha & 0^\beta_\alpha & t_\alpha \\ \hline 0_\beta & 0 & t'^\beta & 0 \end{array} \right). \tag{3.18}$$

Following the treatment of [48], $\mathcal{B}$ is a real 28×28 matrix depending on $27 + 27 = 54$ parameters, parametrizing the generators of the maximal symmetric non-compact pseudo-Riemannian rank-3 coset

$$\widehat{\mathcal{M}} := \frac{E_{7(7)}}{E_{6(6)} \times SO\,(1,1)}, \tag{3.19}$$

with signature $(-^{27}, +^{27})$; in this case, the d-tensor appearing in (3.18) is the one pertaining to the **27** (or **27′**) irrep. of the split non-compact real form $E_{6(6)}$. On the other hand, by suitably replacing this latter by the d-tensor pertaining to the **27** (or **27′**) irrep. of the minimally non-compact real form $E_{6(-26)}$, the matrix $\mathcal{B}$ (3.18) can be regarded as parametrizing the generators of the maximal symmetric non-compact pseudo-Riemannian rank-3 coset

$$\widetilde{\mathcal{M}} := \frac{E_{7(-25)}}{E_{6(-26)} \times SO\,(1,1)}, \tag{3.20}$$

with signature $(-^{43}, +^{11})$; this pseudo-Riemannian counterpart of (1.2) can also be regarded as the classical vector multiplets' scalar manifold of the magical octonionic $\mathcal{N} = 2$ Maxwell–Einstein supergravity theory in $D = (4, 0)$ dimensions, obtained from its $D = (4, 1)$ uplift by timelike Kaluza–Klein reduction (see, e.g., [58]). Clearly, also other interpretations of $\mathcal{B}$ (3.18) are possible, within the maximal (symmetric) embeddings of (non-compact, real forms of) $E_6 \times U(1)$ into (non-compact, real forms of) E_7 (see, e.g., [59]), but they are not relevant for the present investigation. Notice that in the above expressions (3.19) for $\widehat{\mathcal{M}}$ and (3.20) for $\widetilde{\mathcal{M}}$ the issue of the presence of finite or discrete factors is not taken into account.

We will now relate the matrix $\mathcal{B}$ (3.18) (which, within the interpretation (3.20), provides a manifestly $\left[E_{6(-26)} \times SO(1,1)\right]$-covariant parametrization of the generators of the coset $\widetilde{\mathcal{M}}$) to the matrix $\ln \mathcal{M}$ (3.17) (which provides a manifestly $\left[(E_{6(-78)} \times U(1))/\mathbb{Z}_3\right]$-covariant parametrization of the generators of the coset $\mathcal{M}$ (1.2)).

(1) We start and move the vectors t' and t from the diagonal blocks of $\mathcal{B}$ to the off-diagonal ones. In order to achieve this, a symplectic automorphism generated by the following matrix has to be performed:

$$\mathcal{S} := \left(\begin{array}{c|c|c|c} I_{27} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ \hline 0 & 0 & I_{27} & 0 \\ \hline 0 & 1 & 0 & 0 \end{array}\right). \tag{3.21}$$

Thus, it follows that $(\mathcal{S}^T \Omega \mathcal{S} = \Omega)$

$$\mathcal{S}\mathcal{B}\mathcal{S}^{-1} = \left(\begin{array}{c|c|c|c} 0_\beta^\alpha & 0^\alpha & d^{\alpha\beta\gamma}t_\gamma & -t'^\alpha \\ \hline 0_\beta & 0 & -t'_\beta & 0 \\ \hline d_{\alpha\beta\gamma}t'^\gamma & -t_\alpha & 0_\alpha^\beta & 0_\alpha \\ \hline -t_\beta & 0 & 0_\beta & 0 \end{array}\right). \tag{3.22}$$

(2) Then, it is necessary to make the following identification between the 27-dimensional vectors t, t' of $\mathcal{B}$ (3.18) and z, $\bar{z}$ defined by (3.4) and (3.6):

$$\begin{cases} t = -\bar{z}; \\ t' = -z. \end{cases} \tag{3.23}$$

(3) By recalling the normalization of the d-tensor given by (2.27), it thus follows that $\mathcal{B}$ can be transformed into $\ln \mathcal{M}$ (3.17).

It is here worth remarking that an analytical direct exponentiation of the matrix $\mathcal{B}$ (3.8), which would yield an explicit symplectic frame, e.g., for the manifold (3.19) or (3.20), and which, through the 3-step procedure just mentioned, would provide a more explicit form of the treatment of Sections 2 and 3, does not seem to be feasible (in [60] the exponentiation of $\mathcal{B}$ (3.8) with $t = 0$ *or*, equivalently, $t' = 0$ has only been performed). It may be possible that a direct exponentiation of the matrix $\mathcal{B}$ (3.8) could be performed by exploiting the fundamental identity for the d-tensor of the symmetric coset. With the above normalization, such an identity can be derived from the

treatment given in [61] (*at least* for E_6):

$$d_{\alpha\beta\gamma}d^{\lambda\mu\gamma}d_{\mu\nu\rho} = d_{\nu\rho\alpha}\delta_\beta^\lambda + \frac{1}{3}d_{\nu\rho\beta}\delta_\alpha^\lambda - 6d_{\mu\nu\rho}T_{I|\alpha}^{\ \ \lambda}T_{\ \ \beta}^{I\ \mu}, \qquad (3.24)$$

where as in the explicit treatment of Sections 2 and 3, Greek indices run over the fundamental **27** (or $\overline{\mathbf{27}}$) irrep., and capital Latin indices run over the adjoint **78** irrep. of E_6; the $T_{I|\alpha\beta}$'s denote the realization of the generators of E_6 in its **27** irrep. (see, e.g., [61]), and they are, e.g., proportional to the ϕ_I's (2.23) appearing in the matrices Y_I (2.19). We note that the complete symmetrization of covariant indices of the identity (3.24) yields the well known identity:

$$d_{(\alpha\beta|\gamma}d^{\lambda\mu\gamma}d_{\mu|\nu\rho)} = \frac{4}{3}d_{(\nu\rho\beta}\delta_{\alpha)}^\lambda. \qquad (3.25)$$

We leave for the future the interesting task of exploiting the identity (3.24) *and/or* spectral techniques in order to perform the exponentiation of the matrix $\mathcal{B}$ (3.8), and thus to determine a more explicit expression of the maximally manifestly covariant symplectic frame introduced in Sections 2 and 3.

4 The Iwasawa decomposition

In this section, we are going to construct, along the lines of [22], another parametrization for the coset $\mathcal{M}$ (1.2), by exploiting the *Iwasawa decomposition*, which in this case turns out to be manifestly $SO(8)$-covariant, thus providing a manifestly *triality symmetric* description of the rank-3 coset $\mathcal{M}$. Within this treatment, we will denote by $\mathfrak{P}$ the Lie algebra of the coset $\mathcal{M}$, namely the complement in $\mathfrak{e}_{7(-25)}$ to its maximal compact subalgebra $\mathfrak{T} := \mathfrak{e}_{6(-78)} \oplus \mathfrak{u}(1)$.

As the first step, one needs to determine a maximal non-compact Cartan subalgebra $\mathfrak{H}_3$. As observed at the start of Section 3, a possible choice is:

$$\mathfrak{H}_3 := \langle Y_{123}, Y_{132}, Y_{133}\rangle_\mathbb{R} \subset \mathfrak{P}, \qquad (4.1)$$

generated by the diagonal elements of $\mathfrak{J}_3(\mathbb{O})$.

Next, a basis of $54 - 3 = 51$ positive roots of $\mathfrak{P}$ with respect to $\mathfrak{H}_3$ (4.1) is to be determined.

If the *adjoint action* of $\mathfrak{H}_3$ on $\mathfrak{e}_{7(-25)}$ is simultaneously diagonalized, we expect to be able to find only 102 non-vanishing vectors in $\mathbb{R}^3$. This follows

from the fact that, apart from $\mathfrak{H}_3$ itself, $\mathfrak{H}_3$ commutes with a 28-dimensional subalgebra $\mathfrak{S} \simeq \mathfrak{so}(8)$ of $\mathfrak{T}$; this can be easily understood by the following argument. By denoting with $\oplus_s$ the semi-direct sum of algebras, due to the *symmetric* nature of the embedding determining the coset $\mathcal{M}$ (1.2), the structure of the Cartan decomposition of $\mathfrak{e}_{7(-25)} = \mathfrak{T} \oplus_s \mathfrak{P}$ reads:

$$[\mathfrak{T}, \mathfrak{T}] \subseteq \mathfrak{T}, \quad [\mathfrak{P}, \mathfrak{P}] \subseteq \mathfrak{T}, \quad [\mathfrak{T}, \mathfrak{P}] \subseteq \mathfrak{P}. \tag{4.2}$$

As usual, the last relation implies that $\mathfrak{P}$ is a *representation space* for $\mathfrak{T}$, which in general will decompose in *irreducible* subspaces. In particular, $\mathfrak{P}$ is a representation space for the $\mathfrak{f}_{4(-52)}$ subalgebra of $\mathfrak{T}$. As it is well known, $\mathfrak{f}_{4(-52)}$ is the Lie algebra of the group $\mathrm{Aut}(\mathfrak{J}_3(\mathbb{O}))$; in turn the subalgebra of $\mathfrak{f}_{4(-52)}$ which keeps the diagonal elements of $\mathfrak{J}_3(\mathbb{O})$ fixed is precisely $\mathfrak{so}(8)$, namely the Lie algebra of the automorphism group $\mathrm{Aut}(\mathbf{t}(\mathbb{O}))$ of the *normed triality* on $\mathbb{O}$ (see, e.g., [62]). Therefore, since $\mathfrak{H}_3$ has been selected exactly as the subalgebra corresponding to the diagonal elements of $\mathfrak{J}_3(\mathbb{O})$, it has to commute with an $\mathfrak{so}(8)$ subalgebra of $\mathfrak{f}_{4(-52)}$; this can indeed be checked by inspecting the structure of the roots. Following [34,35], a Cartan subalgebra $\mathfrak{H}_7 \subset \mathfrak{e}_{7(-25)}$ can be obtained by adding the space $\mathfrak{H}_4$ generated by the four matrices Y_i, $i = 1, 6, 15, 36$ (recall (2.19)) to $\mathfrak{H}_3$. Then, the computation of the roots with respect to this system exactly yields 28 roots with vanishing components in the subspace $\mathfrak{H}_3$; these generate the $\mathfrak{H}_3$-preserving Lie algebra [34]:

$$\mathfrak{S} := \langle Y_1, \ldots Y_{21}, Y_{30}, \ldots, Y_{36} \rangle_{\mathbb{R}} = \mathfrak{so}(8), \tag{4.3}$$

whose $\langle Y_i \rangle_{i=1,6,15,36}$ is thus a Cartan subalgebra. Note that in [34, 35] a completion of $\langle Y_i \rangle_{i=1,6,15,36}$ to $\mathfrak{so}(8) \neq \mathfrak{S}$ (4.3) was worked out, but this is irrelevant for the present investigation.

As a consequence, it holds:

$$ad_{\mathfrak{H}_3}|_{\mathfrak{S} \oplus \mathfrak{H}_3} = 0, \tag{4.4}$$

so that 31 eigenvalues vanish in $\mathbb{R}^3$, and thus only at most $133 - 31 = 102$ can be non-vanishing, as expected.

Let us show that actually all the remaining 102 eigenvalues of $ad_{\mathfrak{H}_3}$ on $\mathfrak{e}_{7(-25)}$ are non-vanishing. First, we can write:

$$\mathfrak{T} = \mathfrak{S} \oplus_s \mathfrak{T}', \tag{4.5}$$

$$\mathfrak{P} = \mathfrak{H}_3 \oplus_s \mathfrak{P}', \tag{4.6}$$

with $\dim_{\mathbb{R}} \mathfrak{T}' = \dim_{\mathbb{R}} \mathfrak{P}' = 51$. Now,

$$ad_{\mathfrak{H}_3} : \mathfrak{T}' \oplus_s \mathfrak{P}' \longrightarrow \mathfrak{T}' \oplus_s \mathfrak{P}'. \tag{4.7}$$

Indeed, $[\mathfrak{P}, \mathfrak{P}] \subseteq \mathfrak{T}$ (4.2) implies $[\mathfrak{H}_3, \mathfrak{P}'] \subseteq \mathfrak{T}$. Let $\langle , \rangle_{ck}$ be the Cartan–Killing product; then, its restriction to $\mathfrak{T}$ has a definite signature, usually chosen to be negative. The fact that $[\mathfrak{S}, \mathfrak{H}_3] = 0$ implies:

$$\langle \mathfrak{S}, [\mathfrak{H}_3, \mathfrak{P}'] \rangle_{ck} = -\langle [\mathfrak{H}_3, \mathfrak{S}], \mathfrak{P}' \rangle_{ck} = 0 \Rightarrow [\mathfrak{H}_3, \mathfrak{P}'] \in \mathfrak{T}'. \tag{4.8}$$

Next, from $[\mathfrak{T}, \mathfrak{P}] \subseteq \mathfrak{P}$ (4.2) it follows that $[\mathfrak{H}_3, \mathfrak{T}'] \subseteq \mathfrak{P}$. As the Cartan–Killing form is strictly positive on $\mathfrak{P}$, and $\mathfrak{H}_3$ is Abelian, one obtains:

$$\langle \mathfrak{H}_3, [\mathfrak{H}_3, \mathfrak{T}'] \rangle_{ck} = -\langle [\mathfrak{H}_3, \mathfrak{H}_3], \mathfrak{T}' \rangle_{ck} = 0 \Rightarrow [\mathfrak{H}_3, \mathfrak{T}'] \subseteq \mathfrak{P}'. \tag{4.9}$$

In this way, one can conclude that the set $\mathcal{W}$ of the remaining 102 roots of $\mathfrak{e}_{7(-25)}$ has eigenspaces in $\mathfrak{T}' \oplus_s \mathfrak{P}'$. Thus, each eigenvector has the form $\lambda_A := t_A + p_A$, $A = 1, \ldots, 102$, where $t_A \in \mathfrak{T}'$ and $p_A \in \mathfrak{P}'$ are both non-vanishing and uniquely determined by λ_A. Let us suppose that one of the roots $r_{A_0} \in \mathcal{W}$ vanishes: $r_{A_0} = 0$. This would imply that $ad_{\mathfrak{H}_3}(p_{A_0}) = 0$. But, in turn, this would also mean $p_{A_0} \in \mathfrak{H}_3$ (as $\mathfrak{H}_3$ is a *maximal* Cartan subalgebra in $\mathfrak{P}$), which cannot be the case, since $\mathfrak{H}_3 \cap \mathfrak{P}' = 0$.

Keeping in mind that we are considering the roots of $\mathfrak{e}_{7(-25)}$ relative to the choice (4.1) of $\mathfrak{H}_3$, we can thus conclude that all 102 roots in $\mathcal{W}$ are non-vanishing. Let us now fix a choice of 51 positive roots $\mathcal{W}_+$ so that $\mathcal{W} = \mathcal{W}_+ \cup \mathcal{W}_-$. The corresponding eigenspaces are one-dimensional, and they are generated by the eigenvectors λ_i^+, $i = 1, \ldots, 51$, with eigenvalues $r_i \in \mathcal{W}_+$. We can write in a unique way:

$$\lambda_i^+ = p_i + t_i, \tag{4.10}$$

implying that:

$$\lambda_i^- := p_i - t_i \tag{4.11}$$

are eigenvalues of $-r_i \in \mathcal{W}_-$.

Finally, by renaming $h_1 := Y_{123}$, $h_2 := Y_{132}$ and $h_3 := Y_{133}$, the *Iwasawa decomposition* of the coset $\mathcal{M}$ (1.2) can be written as:

$$\mathcal{M} := \exp(x_1 h_1 + x_2 h_2 + x_3 h_3) \exp\left(\sum_{i=1}^{51} y_i \lambda_i^+\right). \qquad (4.12)$$

which exhibits a manifest $SO(8)$-covariance. We anticipate that $\mathfrak{so}(8)$ is the Lie algebra of $\mathrm{Aut}(\mathbf{t}(\mathbb{O})) = Spin(8)$, namely the automorphism group of the *normed triality* $\mathbf{t}(\mathbb{O})$ over the division algebra of octonions $\mathbb{O}$ (see, e.g., [62]):

$$\mathfrak{so}(8) = \mathfrak{Aut}(\mathbf{t}(\mathbb{O})) =: \mathfrak{tri}(\mathbb{O}); \qquad (4.13)$$

see the discussion in Section 5.

4.1 *SO*(8)-*Triality*

Now, we want study the $SO(8)$-covariance of the Iwasawa parametrization (4.12) in more detail.

First, as pointed out above, the elements h_1, h_2, h_3 of the Cartan subalgebra $\mathfrak{H}_3$ commute with $SO(8)$, and it follows that they are three $SO(8)$-singlets. Thus, the 51-dimensional linear space Λ_+ generated by the positive roots $\mathcal{W}_+$ is invariant under the (adjoint) action of $SO(8)$, and it decomposes into irreps. of $SO(8)$ as:

$$\Lambda_+ = \mathbf{1}^3 + \mathbf{8}_v^2 + \mathbf{8}_c^2 + \mathbf{8}_s^2. \qquad (4.14)$$

The manifestly *triality-symmetric* decomposition (4.14) can be proven by means of the following general argument. Let us fix an orthonormal basis $L_1, \ldots, L_7$ of $\mathbb{R}^7$. Then, the $(133 - 7)/2 = 63$ positive roots of $E_{7(-25)}$ can be represented as (see, e.g., [63], p. 333):

$$L_m \pm L_n, \qquad 1 \le n < m \le 6; \qquad \sqrt{2}L_7;$$
$$\frac{\pm L_1 \pm L_2 \pm L_3 \pm L_4 \pm L_5 \pm L_6 + \sqrt{2}L_7}{2}, \qquad \text{odd number of} - \text{signs.} \quad (4.15)$$

Among these, the $(28 - 4)/2 = 12$ roots:

$$\mu_{mn}^\pm := L_m \pm L_n, \quad 1 \le n < m \le 4 \qquad (4.16)$$

are the positive roots of $\mathfrak{so}(8)$. $\mu_{mn}^{\pm}$ (4.16) provide a representation of the algebra $\mathfrak{so}(8)$ over the linear space generated by the remaining 51 roots in the usual way: if, consistent with (4.10), we call λ_i^+ the 51 complementary roots, then the linear operators $\mu_{mn}^{\pm}$ and their corresponding *adjoint* $\tilde{\mu}_{mn}^{\pm}$ are defined by:

$$\mu_{mn}^{\pm}(\lambda_i^+) := \lambda_i^+ + (L_m \pm L_n), \tag{4.17}$$

$$\tilde{\mu}_{mn}^{\pm}(\lambda_i^+) := \lambda_i^+ - (L_m \pm L_n), \tag{4.18}$$

where the result is intended to be zero when the vectors on the right-hand side are not roots.

This procedure allows to identify exactly 9 invariant subspaces of Λ_+:

(1) The three spaces respectively generated by $L_6 + L_5$, $L_6 - L_5$ and $\sqrt{2}L_7$ are one-dimensional invariant subspaces defining a $\mathbf{1}^3$ representation (sum of 3 $SO(8)$-singlets).

(2) The two 8-dimensional spaces V_5 and V_6, respectively generated by the basis:

$$\{L_5 \pm L_n\}_{n=1}^4; \tag{4.19}$$

$$\{L_6 \pm L_n\}_{n=1}^4; \tag{4.20}$$

are both representations with weights $\pm L_n$, and thus correspond to two copies of the *vector* representation $\mathbf{8}_v$.

(3) The two 8-dimensional spaces C_+ and C_-, respectively generated by the basis:

$$\frac{\pm L_1 \pm L_2 \pm L_3 \pm L_4 + L_5 - L_6 + \sqrt{2}L_7}{2}; \tag{4.21}$$

$$\frac{\pm L_1 \pm L_2 \pm L_3 \pm L_4 - L_5 + L_6 + \sqrt{2}L_7}{2}; \tag{4.22}$$

(with an *even* number of $-$ signs) are both representations with weights $\frac{\pm L_1 \pm L_2 \pm L_3 \pm L_4}{2}$ (with an *even* number of $-$ signs), thus providing two copies of the *chiral spinor* representation $\mathbf{8}_c$.

(4) The two 8-dimensional spaces S_+ and S_-, respectively generated by the basis:

$$\frac{\pm L_1 \pm L_2 \pm L_3 \pm L_4 + L_5 + L_6 + \sqrt{2}L_7}{2}; \tag{4.23}$$

$$\frac{\pm L_1 \pm L_2 \pm L_3 \pm L_4 - L_5 - L_6 + \sqrt{2}L_7}{2}; \tag{4.24}$$

(with an *odd* number of − signs) are both representations with weights $\frac{\pm L_1 \pm L_2 \pm L_3 \pm L_4}{2}$ (with an *odd* number of − signs), thus providing two copies of the *chiral spinor* representation $\mathbf{8}_s$ (conjugate of $\mathbf{8}_c$).

This implies (4.14), in which the $SO(8)$-*triality* is manifest. It is worth remarking that the appearance of the square for the three $\mathbf{8}$ irreps. in (4.14) is a consequence of the complex (in particular, *special Kähler*, as mentioned in previous sections) structure of the coset $\mathcal{M}$ (1.2).

4.2 Group theory

In the *Iwasawa parametrization* of $\mathcal{M}$ (1.2) worked out in Section 4, the resulting maximal manifest covariance group is nothing but the $SO(8)$ group (uniquely determined in $E_{7(-25)}$; see Sections 4.2.1 and 4.2.2) preserving the diagonal elements in the rank-3 simple Jordan algebra $\mathfrak{J}_3(\mathbb{O})$. As clearly evident form the chain (4.26) of embeddings, such an $SO(8)$ is placed as follows:

$$SO(8) \subset \left[(SO(10) \times U(1)) \cap F_{4(-52)} \right]. \tag{4.25}$$

As given by (4.13), it shares the same algebra $\mathfrak{so}(8) = \mathfrak{tri}(\mathbb{O})$ with the automorphism group $\mathrm{Aut}(\mathbf{t}(\mathbb{O})) = Spin(8)$ of the *normed triality* over the octonions $\mathbb{O}$ [62]. Furthermore, it is worth remarking that such an $SO(8)$ recently appeared as the stabilizer of the BPS generic charge orbit in the two-centered extremal black hole solutions of $\mathcal{N} = 2$, $D = 4$ exceptional supergravity; see Table 7 [17].

We also note that, at the level of (manifest) covariance, the Iwasawa parametrization of $\mathcal{M}$ (1.2) worked out in Section 4 differs from the Iwasawa parametrization of $\mathcal{M}_{\mathcal{N}=8}$ (3.9) studied in [22], whose manifest maximal (non-compact) covariance is $SL(7, \mathbb{R})$, with maximal compact subgroup $SO(7)$.

4.2.1 A first chain of embeddings

The chain of maximal symmetric embeddings relevant for the study of the maximal manifest covariance of the Iwasawa parametrization (4.12) of the irreducible Riemannian globally symmetric rank-3 symmetric special Kähler coset $\mathcal{M}$ (1.2) reads as follows (see, e.g., [59]):

(1) in the *compact* case:

$$
\begin{aligned}
E_{7(-25)} &\supset E_{6(-78)} \times U\left(1\right)' \\
&\supset SO\left(10\right) \times U\left(1\right)' \times U\left(1\right)'' \\
&\supset SO\left(8\right) \times U(1)' \times U(1)'' \times U(1)''';
\end{aligned} \tag{4.26}
$$

(2) in the relevant (namely, *minimally*) *non-compact* case:

$$
\begin{aligned}
E_{7(-25)} &\supset E_{6(-26)} \times SO\left(1,1\right)' \\
&\supset SO\left(9,1\right) \times SO\left(1,1\right)' \times SO\left(1,1\right)'' \\
&\supset SO\left(8\right) \times SO\left(1,1\right)' \times SO\left(1,1\right)'' \times SO(1,1)'''.
\end{aligned} \tag{4.27}
$$

In the last line of (4.27) the first two $SO\left(1,1\right)$ factors have the physical meaning of "extra" T-dualities generated by the Kaluza–Klein reductions, respectively $D = 5 \rightarrow D = 4$, and $D = 6 \rightarrow D = 5$.

Correspondingly, the adjoint irrep. **133** of $E_{7(-25)}$ branches as (subscripts denote $U\left(1\right)$-charges or $SO\left(1,1\right)$-weights, for (4.26) and (4.27), respectively, throughout; see, e.g., [32]):

$$
\begin{aligned}
\mathbf{133} = {}& \mathbf{78}_0 + \mathbf{1}_0 + \mathbf{27}_{-2} + \mathbf{27}'_{+2} & (4.28) \\
= {}& \mathbf{1}_{0,0} + \mathbf{16}_{0,-3} + \mathbf{16}'_{0,+3} + \mathbf{45}_{0,0} + \mathbf{1}_{0,0} \\
& + \mathbf{1}_{-2,+4} + \mathbf{10}_{-2,-2} + \mathbf{16}_{-2,+1} \\
& + \mathbf{1}_{+2,-4} + \mathbf{10}_{+2,+2} + \mathbf{16}'_{+2,-1} \\
= {}& \mathbf{1}_{0,0,0} + \mathbf{8}_{c,0,-3,1} + \mathbf{8}_{s,0,-3,-1} + \mathbf{8}_{c,0,+3,-1} + \mathbf{8}_{s,0,+3,+1} & (4.29) \\
& + \mathbf{1}_{0,0,0} + \mathbf{8}_{v,0,0,+2} + \mathbf{8}_{v,0,0,-2} + \mathbf{28}_{0,0,0} + \mathbf{1}_{0,0,0} & (4.30) \\
& + \mathbf{1}_{-2,+4,0} + \mathbf{1}_{-2,-2,+2} + \mathbf{1}_{-2,-2,-2} + \mathbf{8}_{v,-2,-2,0} \\
& + \mathbf{8}_{c,-2,+1,+1} + \mathbf{8}_{s,-2,+1,-1} & (4.31) \\
& + \mathbf{1}_{+2,-4,0} + \mathbf{1}_{+2,+2,-2} + \mathbf{1}_{+2,+2,+2} + \mathbf{8}_{v,+2,+2,0} \\
& + \mathbf{8}_{c,+2,-1,-1} + \mathbf{8}_{s,+2,-1,+1}. & (4.32)
\end{aligned}
$$

Recalling the treatment Section 4, in line (4.28) one can recognize:

$$
\mathfrak{T} := \mathbf{Adj}\left(E_6 \times U\left(1\right)\right) = \mathbf{78}_0 + \mathbf{1}_0; \tag{4.33}
$$

$$
\mathfrak{P} := \mathbf{27}_{-2} + \mathbf{27}'_{+2}, \tag{4.34}
$$

where, by the definitions introduced in Section 4, $\mathfrak{P}$ denotes (the irreducible decomposition of) the Lie algebra of the coset $\mathcal{M}$ (1.2) (as representation

space of $\mathfrak{T}$). Furthermore, $\mathbf{27}_{-2} + \mathbf{27}'_{+2}$ manifestly shows the complex (*special Kähler*) structure of $\mathcal{M}$ itself, which is then spoiled by the further subsequent branchings needed for the Iwasawa parametrization (4.12).

Furthermore, the lines (4.29) and (4.30) give the $SO(8) \times [U(1)]^3$ (or $SO(8) \times [SO(1,1)]^3$) irreducible branching of the 79 compact generators of $E_{7(-25)}$, namely of the generators of its maximal compact subgroup $E_{6(-78)} \times U(1)$. On the other hand, the lines (4.31) and (4.32) give the $SO(8) \times [U(1)]^3$ (or $SO(8) \times [SO(1,1)]^3$) irreducible branching of the 54 non-compact generators of $E_{7(-25)}$, namely of the generators of $\mathcal{M}$ itself. In particular, recalling the definitions of Section 4:

$$\mathfrak{T}' := \mathbf{1}_{0,0,0} + \mathbf{8}_{c,0,-3,1} + \mathbf{8}_{s,0,-3,-1} + \mathbf{8}_{c,0,+3,-1} + \mathbf{8}_{s,0,+3,+1}$$
$$+ \mathbf{1}_{0,0,0} + \mathbf{8}_{v,0,0,+2} + \mathbf{8}_{v,0,0,-2} + \mathbf{1}_{0,0,0}; \tag{4.35}$$
$$\mathfrak{S} := \mathbf{Adj}(SO(8)) = \mathbf{28}_{0,0,0}. \tag{4.36}$$

$\mathfrak{T}'$ is the Lie algebra of the *non-maximal* (and *non-symmetric*) coset ($\dim_{\mathbb{R}} = 51$):

$$\frac{E_{6(-78)} \times U(1)'}{SO(8)} = \frac{E_{6(-78)}}{SO(8)} \times U(1)', \tag{4.37}$$

or, in the choice of chain (4.27), of its relevant (i.e., *minimally*) non-compact form:

$$\frac{E_{6(-26)} \times SO(1,1)'}{SO(8)} = \frac{E_{6(-26)}}{SO(8)} \times SO(1,1)'. \tag{4.38}$$

Out of the six $SO(8)$-singlets:

$$\mathbf{1}_{-2,+4,0}, \ \mathbf{1}_{-2,-2,+2}, \ \mathbf{1}_{-2,-2,-2}, \mathbf{1}_{+2,-4,0}, \ \mathbf{1}_{+2,+2,-2}, \ \mathbf{1}_{+2,+2,+2} \tag{4.39}$$

in lines (4.31) and (4.32), three linear combinations generate $\mathfrak{H}_3$, whereas the remaining linear combinations, orthogonal with respect to the Cartan–Killing form, together with the manifestly $SO(8)$ -triality-symmetric branching:

$$\mathbf{8}_{v,-2,-2,0} + \mathbf{8}_{c,-2,+1,+1} + \mathbf{8}_{s,-2,+1,-1}$$
$$+ \mathbf{8}_{v,+2,+2,0} + \mathbf{8}_{c,+2,-1,-1} + \mathbf{8}_{s,+2,-1,+1} \tag{4.40}$$

of lines (4.31) and (4.32), generate $\mathfrak{P}'$.

Analogously, the smallest non-trivial symplectic irrep., namely the fundamental **56** of $E_{7(-25)}$, branches as (see, e.g., [32]):

$$\mathbf{56} = \mathbf{27}_{+1} + \mathbf{27}'_{-1} + \mathbf{1}_{+3} + \mathbf{1}_{-3} \tag{4.41}$$

$$= \mathbf{1}_{+1,+4} + \mathbf{10}_{+1,-2} + \mathbf{16}_{+1,+1}$$

$$+ \mathbf{1}_{-1,-4} + \mathbf{10}_{-1,+2} + \mathbf{16}'_{-1,-1}$$

$$+ \mathbf{1}_{+3,0} + \mathbf{1}_{-3,0} \tag{4.42}$$

$$= \mathbf{1}_{+1,+4,0} + \mathbf{1}_{+1,-2,+2} + \mathbf{1}_{+1,-2,-2} + \mathbf{8}_{v,+1,-2,0} + \mathbf{8}_{c,+1,+1,+1}$$

$$+ \mathbf{8}_{s,+1,+1,-1} + \mathbf{1}_{-1,-4,0} + \mathbf{1}_{-1,+2,-2} + \mathbf{1}_{-1,+2,+2} + \mathbf{8}_{v,-1,+2,0}$$

$$+ \mathbf{8}_{c,-1,-1,-1} + \mathbf{8}_{s,-1,-1,+1} + \mathbf{1}_{+3,0,0} + \mathbf{1}_{-3,0,0}, \tag{4.43}$$

It is instructive to analyze the branchings (4.41)–(4.43) more in depth.

From the structure of the matrices Y_I (2.19) and (2.20), $I = 1, \ldots, 79$, the structure of the first branching (4.41) is evident, where the subscripts denote the charge (weight) with respect to $U(1)'$ ($SO(1,1)'$). Next, let us look at the **27** irrep. of E_6; which is realized over the 27-dimensional linear space of octonionic Hermitian matrices underlying the exceptional Jordan algebra $\mathfrak{J}_3(\mathbb{O})$. It decomposes as follows:

$$\begin{pmatrix} a & X & Y \\ X^* & b & Z \\ Y^* & Z^* & c \end{pmatrix}_{\mathbf{27}} = \begin{pmatrix} 0 & X & Y \\ X^* & 0 & 0 \\ Y^* & 0 & 0 \end{pmatrix}_{\mathbf{16}} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & Z \\ 0 & Z^* & c \end{pmatrix}_{\mathbf{10}} + \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\mathbf{1}}, \tag{4.44}$$

where a, b, c are the real numbers and X, Y, Z are the real (in the linear sense) octonions. As hinted in (4.44), this yields a decomposition $\mathbf{27} = \mathbf{16} + \mathbf{10} + \mathbf{1}$ of invariant spaces under the maximal symmetric subgroup $R := SO(10) \times U(1)''$ of E_6 (we consider, without loss of any generality, the compact chain (4.26) of embeddings). Indeed, the one-dimensional space **1** is easily seen to be invariant under R. As the spaces in the decomposition (4.44) are orthogonal with respect to the trace product (which is preserved by R), its complement is also R-invariant. On the other hand, the 16-dimensional subspace defines the largest subalgebra in $\mathfrak{J}_3(\mathbb{O})$ complementary to the one-dimensional space **1**. This proves our assertion.

The $U(1)''$-charges of the spaces in the right-hand side of (4.44) can be determined by noting that $U(1)'' \nsubseteq F_{4(-52)}$. From the treatment of Section 2, the Lie algebra $\mathfrak{e}_{6(-78)}$ is obtained by adding the left (or right) action of $\mathfrak{J}'_3(\mathbb{O})$ on $\mathfrak{J}_3(\mathbb{O})$ (where the prime here denotes the matrix tracelessness). This means that the generator of $U(1)''$ must be realized by a traceless

matrix $C_{U(1)''}$ in $\mathfrak{J}_3(\mathbb{O})$ that by left Jordan-multiplication acts proportionally to the identity on the three subspaces of the decomposition (4.44). This implies that:

$$C_{U(1)''} = \begin{pmatrix} 2\gamma & 0 & 0 \\ 0 & -\gamma & 0 \\ 0 & 0 & -\gamma \end{pmatrix}. \tag{4.45}$$

Writing (4.44) as $V_{27} = V_{16} + V_{10} + V_1$, we see that:

$$C_{U(1)''} \circ V_{16} = \frac{\gamma}{2} V_{16}, \quad C_{U(1)''} \circ V_{10} = -\gamma V_{10}, \quad C_{U(1)''} \circ V_1 = 2\gamma V_1. \tag{4.46}$$

By choosing the normalization of the charges in such the way that $\exp(xC_{U(1)''})$ has period 2π, one then obtains:

$$\mathbf{27} = \mathbf{16}_1 + \mathbf{10}_{-2} + \mathbf{1}_4, \tag{4.47}$$

which matches the convention, e.g., of [32]. Obviously, $U(1)''$ commutes with $U(1)'$; therefore (4.42) is obtained.

For the last branching (4.43), the decompositions of $\mathbf{10}$ and $\mathbf{16}$ have to be analyzed. As $SO(8)$ leaves the diagonal matrices of $\mathfrak{J}_3(\mathbb{O})$ invariant, it follows that under its action the space V_{10} decomposes as $V_{10} = V_8 + V_{1,I} + V_{1,II}$ in the following way:

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & b & Z \\ 0 & Z^* & c \end{pmatrix}}_{\mathbf{10}} = \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Z \\ 0 & Z^* & 0 \end{pmatrix}}_{\mathbf{8}_v} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\mathbf{1}_I} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{pmatrix}}_{\mathbf{1}_{II}}. \tag{4.48}$$

In order to determine the $U(1)'''$ charges, we again observe that $U(1)''' \subsetneq F_{4(-52)}$. Moreover, the $U(1)'''$ charge of $\mathbf{1}_{1,4}$ in (4.42) must be zero and, therefore, the $U(1)'''$ generator must be realized by a matrix of the form:

$$C_{U(1)'''} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma' & 0 \\ 0 & 0 & -\gamma' \end{pmatrix}. \tag{4.49}$$

By choosing the normalization as before in such a way that the period of $\exp(xC_{U(1)'''})$ is 2π, one can fix $\gamma' = 2$, and the charges turn out to be 0, 2 and -2 for V_8, $V_{1,I}$ and $V_{1,II}$, respectively. Since V_8 is contained in the

vector representation V_{10} of $SO(10)$, it has to correspond to the vector rep. $\mathbf{8}_v$ of $SO(8)$, so that:

$$\mathbf{10} = \mathbf{8}_{v,0} + \mathbf{1}_2 + \mathbf{1}_{-2}. \tag{4.50}$$

The charge operator $C_{U(1)'''}$ (4.49) splits V_{16} into eigenspaces V_8^+ and V_8^- with eigenvalues 1 and -1, respectively:

$$\underbrace{\begin{pmatrix} 0 & X & Y \\ X^* & 0 & 0 \\ Y^* & 0 & 0 \end{pmatrix}}_{\mathbf{16}} = \underbrace{\begin{pmatrix} 0 & X & 0 \\ X^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\mathbf{8}_c} + \underbrace{\begin{pmatrix} 0 & 0 & Y \\ 0 & 0 & 0 \\ Y^* & 0 & 0 \end{pmatrix}}_{\mathbf{8}_s}. \tag{4.51}$$

The weights of $\mathbf{16}$ are $\frac{1}{2}\{\epsilon_1, \ldots, \epsilon_8\}$ where the ϵ's can assume all possible signs; this means that $\mathbf{16}$ breaks into the direct sum of the conjugate irreducible spinor representations $\mathbf{8}_s$ and $\mathbf{8}_c$ of $SO(8)$, with $\frac{1}{2}C_{U(1)'''}$ measuring their chirality:

$$\mathbf{16} = \mathbf{8}_{s,-1} + \mathbf{8}_{c,1}, \tag{4.52}$$

which allows one to recover (4.43).

4.2.2 A second chain of embeddings

A second chain of maximal and symmetric embeddings, relevant in order to highlight the relation to the symmetry groups of the rank-3 Euclidean Jordan algebra $\mathfrak{J}_3(\mathbb{O})$ and also for a subsequent generalization at least for all *conformal non-compact* form of *non-degenerate* [64] groups of type E_7 [14] (see Section 5), reads as follows:

$$\begin{aligned} E_{7(-25)} &\supset E_{6(-26)} \times SO(1,1)' \\ &\supset F_{4(-52)} \times SO(1,1)' \\ &\supset SO(9) \times SO(1,1)' \\ &\supset SO(8) \times SO(1,1)', \end{aligned} \tag{4.53}$$

where $SO(8)$ in the fourth line of (4.53) coincides with the $SO(8)$ in the third line of (4.26) and (4.27). Moreover, (4.53) also clarifies (4.25). As

already mentioned above, it holds that (see, e.g., [29, 30, 65]):

$$E_{7(-25)} = \text{Conf}\left(\mathfrak{J}_3\left(\mathbb{O}\right)\right) = \text{Aut}\left[\mathfrak{M}\left(\mathfrak{J}_3\left(\mathbb{O}\right)\right)\right] = G_4; \tag{4.54}$$

$$E_{6(-26)} = \text{Str}_0\left(\mathfrak{J}_3\left(\mathbb{O}\right)\right) = G_5; \tag{4.55}$$

$$F_{4(-52)} = \text{Aut}\left(\mathfrak{J}_3\left(\mathbb{O}\right)\right) = \text{mcs}\left[\text{Str}_0\left(\mathfrak{J}_3\left(\mathbb{O}\right)\right)\right]; \tag{4.56}$$

$$\mathfrak{so}\left(8\right) = \mathfrak{Aut}\left(\mathbf{t}\left(\mathbb{O}\right)\right) =: \mathfrak{tri}(O), \tag{4.57}$$

where $\mathbf{t}\left(\mathbb{O}\right)$ denotes the *normed triality* over the octonions $\mathbb{O}$ (see, e.g., [62]), and (4.13) has been recalled. In (4.53), $SO\left(1,1\right)$ has the physical meaning of "extra" T-duality generated by the Kaluza–Klein reduction $D = 5 \rightarrow D = 4$.

Correspondingly, the adjoint irrep. $\mathbf{133}$ of $E_{7(-25)}$ branches as (see, e.g., [32]):

$$\begin{aligned}
\mathbf{133} \rightarrow\ & \mathbf{78}_0 + \mathbf{1}_0 + \mathbf{27}_{-2} + \mathbf{27}'_{+2} \\
\rightarrow\ & \mathbf{26}_0 + \mathbf{52}_0 + \mathbf{1}_0 + \mathbf{1}_{-2} + \mathbf{26}_{-2} + \mathbf{1}_{+2} + \mathbf{26}_{+2} \\
\rightarrow\ & \mathbf{1}_0 + \mathbf{9}_0 + \mathbf{16}_0 + \mathbf{16}_0 + \mathbf{36}_0 + \mathbf{1}_0 \\
& + \mathbf{1}_{-2} + \mathbf{1}_{-2} + \mathbf{9}_{-2} + \mathbf{16}_{-2} \\
& + \mathbf{1}_{+2} + \mathbf{1}_{+2} + \mathbf{9}_{+2} + \mathbf{16}_{+2} \\
\rightarrow\ & \mathbf{1}_0 + \mathbf{1}_0 + \mathbf{8}_{v,0} + \mathbf{8}_{c,0} + \mathbf{8}_{s,0} + \mathbf{8}_{c,0} + \mathbf{8}_{s,0} + \mathbf{8}_{v,0} + \mathbf{28}_0 + \mathbf{1}_0 \\
& + \mathbf{1}_{-2} + \mathbf{1}_{-2} + \mathbf{1}_{-2} + \mathbf{8}_{v,-2} + \mathbf{8}_{c,-2} + \mathbf{8}_{s,-2} \\
& + \mathbf{1}_{+2} + \mathbf{1}_{+2} + \mathbf{1}_{+2} + \mathbf{8}_{v,+2} + \mathbf{8}_{c,+2} + \mathbf{8}_{s,+2}. \tag{4.58}
\end{aligned}$$

Analogously, the symplectic fundamental irrep. $\mathbf{56}$ of $E_{7(-25)}$ branches as (see, e.g., [32]):

$$\begin{aligned}
\mathbf{56} \rightarrow\ & \mathbf{27}_{+1} + \mathbf{27}'_{-1} + \mathbf{1}_{+3} + \mathbf{1}_{-3} \\
\rightarrow\ & \mathbf{1}_{+1} + \mathbf{26}_{+1} + \mathbf{1}_{-1} + \mathbf{26}_{-1} + \mathbf{1}_{+3} + \mathbf{1}_{-3} \\
\rightarrow\ & \mathbf{1}_{+1} + \mathbf{1}_{+1} + \mathbf{9}_{+1} + \mathbf{16}_{+1} + \mathbf{1}_{-1} + \mathbf{1}_{-1} + \mathbf{9}_{-1} + \mathbf{16}_{-1} + \mathbf{1}_{+3} + \mathbf{1}_{-3} \\
\rightarrow\ & \mathbf{1}_{+1} + \mathbf{1}_{+1} + \mathbf{1}_{+1} + \mathbf{8}_{v,+1} + \mathbf{8}_{c,+1} + \mathbf{8}_{s,+1} \\
& + \mathbf{1}_{-1} + \mathbf{1}_{-1} + \mathbf{1}_{-1} + \mathbf{8}_{v,-1} + \mathbf{8}_{c,-1} + \mathbf{8}_{s,-1} + \mathbf{1}_{+3} + \mathbf{1}_{-3}. \tag{4.59}
\end{aligned}$$

4.2.3 Comments on cartan subalgebras

In the analysis made in Sections 4.2.1 and 4.2.2, the 3-dimensional non-compact Cartan subalgebra $\mathfrak{H}_3$ (4.1) of $\mathcal{M}$ (1.2) is generated by a suitable

linear combination of the six $SO(8)$-singlets (4.39). Thus, $\mathfrak{H}_3$ is not the Lie algebra of the group factor

$$\widetilde{H}_3 := [SO(1,1)]^3 \subsetneq E_{6(-26)} \times SO(1,1) \tag{4.60}$$

commuting with $SO(8)$ in the branching (4.27), because by definition for the Lie group H_3 generated by $\mathfrak{H}_3$ it holds that (recall definition (3.20)):

$$H_3 \subsetneq \frac{E_{7(-25)}}{E_{6(-26)} \times SO(1,1)} =: \widetilde{\mathcal{M}}, \tag{4.61}$$

and, by definition:

$$\mathfrak{T} \cap \mathfrak{P} = 0. \tag{4.62}$$

As stated in the previous treatment, $\mathfrak{H}_3$ can be extended to a 7-dimensional maximal Cartan subalgebra $\mathfrak{H}$ of $\mathfrak{e}_{7(-25)}$ by adding a 4-dimensional maximal Cartan subalgebra of $\mathfrak{so}(8)$, which is clearly compact:

$$\mathfrak{H}_7 := \mathfrak{H}_3 \oplus \mathfrak{H}_4, \tag{4.63}$$

with signature $(+^3, -^4)$ (indeed, as mentioned above, compact generators are conventionally chosen with negative signature).

On the other hand, the factor $\widetilde{H}_3 \equiv [SO(1,1)]^3$ in (4.60) which commutes with $SO(8)$ in the branching (4.27) can be extended to $[SL(2,\mathbb{R})]^3$, which with further branchings gives rise to the (not maximal nor symmetric) embedding:

$$E_{6(-26)} \supsetneq [SU(2)]^4 \times [SL(2,\mathbb{R})]^3, \tag{4.64}$$

recently considered in [7] within the quantum-informational interpretation of $\mathcal{N} = 2$, $D = 4$ exceptional magic supergravity. As done above for $\mathfrak{H}_3$, the Lie algebra $\widetilde{\mathfrak{H}}_3$ of $\widetilde{H}_3$ can be extended to another 7-dimensional maximal Cartan subalgebra $\widetilde{\mathfrak{H}}_7$ of $\mathfrak{e}_{7(-25)}$ by adding a 4-dimensional maximal Cartan subalgebra of $\mathfrak{so}(8)$, which is clearly compact:

$$\widetilde{\mathfrak{H}}_7 := \widetilde{\mathfrak{H}}_3 \oplus \mathfrak{H}_4, \tag{4.65}$$

once again with signature $(+^3, -^4)$.

5   Generalizations to groups of type E$_7$

The results derived until now hold *at least* for the *conformal non-compact* real forms of (*non-degenerate* [64]) *simple* groups *"of type E$_7$"* [14, 15, 19, 20]. The first axiomatic characterization of groups *"of type E$_7$"* through a module (irreducible representation) was given in 1967 by Brown [14]. A group *G "of type E$_7$"* is a Lie group endowed with a representation **R** such that:

- **R** is *symplectic*, i.e., (the subscripts "*s*" and "*a*" stand for symmetric and skew-symmetric throughout):

$$\exists! \mathbb{C}_{[MN]} \equiv \mathbf{1} \in \mathbf{R} \times_a \mathbf{R}; \tag{5.1}$$

  $\mathbb{C}_{[MN]}$ defines a non-degenerate skew-symmetric bilinear form (*symplectic product*); given two different charge vectors $\mathcal{Q}_x$ and $\mathcal{Q}_y$ in **R**, such a bilinear form is defined as:

$$\langle \mathcal{Q}_x, \mathcal{Q}_y \rangle \equiv \mathcal{Q}_x^M \mathcal{Q}_y^N \mathbb{C}_{MN} = - \langle \mathcal{Q}_y, \mathcal{Q}_x \rangle. \tag{5.2}$$

- **R** admits a unique rank-4 completely symmetric primitive *G*-invariant structure, usually named *K*-tensor:

$$\exists! \mathbb{K}_{(MNPQ)} \equiv \mathbf{1} \in [\mathbf{R} \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}]_s; \tag{5.3}$$

  thus, by contracting the *K*-tensor with the same charge vector $\mathcal{Q}$ in **R**, one can construct a rank-4 homogeneous *G*-invariant polynomial (whose ς is the normalization constant):

$$\mathbf{q}(\mathcal{Q}) \equiv \varsigma \mathbb{K}_{MNPQ} \mathcal{Q}^M \mathcal{Q}^N \mathcal{Q}^P \mathcal{Q}^Q, \tag{5.4}$$

  which corresponds to the evaluation of the rank-4 symmetric invariant **q**-structure induced by the *K*-tensor on four identical modules **R**:

$$\begin{aligned} \mathbf{q}(Q) &\equiv \mathbf{q}(\mathcal{Q}_x, \mathcal{Q}_y, \mathcal{Q}_z, \mathcal{Q}_w)|_{\mathcal{Q}_x = \mathcal{Q}_y = \mathcal{Q}_z = \mathcal{Q}_w \equiv \mathcal{Q}} \\ &\equiv \varsigma \left[\mathbb{K}_{MNPQ} \mathcal{Q}_x^M \mathcal{Q}_y^N \mathcal{Q}_z^P \mathcal{Q}_w^Q \right]_{\mathcal{Q}_x = \mathcal{Q}_y = \mathcal{Q}_z = \mathcal{Q}_w \equiv \mathcal{Q}}. \end{aligned} \tag{5.5}$$

  A famous example of *quartic* invariant in $G = E_7$ is the *Cartan–Cremmer–Julia* invariant[4] ([68], p. 274), constructed out of the fundamental representation **R = 56**.

---

[4] As also mentioned in [66], it should be noted that the quartic form is given incorrectly by Cartan; the error seems to have been first observed by Freudenthal [67].

- If a trilinear map $T\colon \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is defined such that:

$$\langle T\left(\mathcal{Q}_x, \mathcal{Q}_y, \mathcal{Q}_z\right), \mathcal{Q}_w\rangle = \mathbf{q}\left(\mathcal{Q}_x, \mathcal{Q}_y, \mathcal{Q}_z, \mathcal{Q}_w\right), \tag{5.6}$$

then it holds that:

$$\langle T\left(\mathcal{Q}_x, \mathcal{Q}_x, \mathcal{Q}_y\right), T\left(\mathcal{Q}_y, \mathcal{Q}_y, \mathcal{Q}_y\right)\rangle = -2\left\langle \mathcal{Q}_x, \mathcal{Q}_y\right\rangle \mathbf{q}\left(\mathcal{Q}_x, \mathcal{Q}_y, \mathcal{Q}_y, \mathcal{Q}_y\right). \tag{5.7}$$

This last property makes the group of type E_7 amenable to a treatment in terms of (rank-3) Jordan algebras and related Freudenthal triple systems.

Remarkably, groups of type E_7, appearing in $D = 4$ supergravity as U-duality groups, admit a $D = 5$ uplift to groups of type E_6, as well as a $D = 3$ downlift to groups of type E_8. It should also be recalled that split forms of exceptional Lie groups of type E appear in the exceptional Cremmer–Julia [39] sequence $E_{11-D,\,(11-D)}$ of U-duality groups of M-theory compactified on a D-dimensional torus, in $D = 3, 4, 5$. Other sequences, composed by non-split, non-compact real forms of exceptional groups, are also relevant to non-maximal supergravity in various dimensions (see, e.g., the treatment in [61], also for a list of related references).

The connection of groups of type E_7 to supergravity can be summarized by stating that all $2 \leqslant \mathcal{N} \leqslant 8$-extended supergravities in $D = 4$ with symmetric scalar manifolds $\frac{G_4}{H_4}$ have G_4 of type E_7 [15, 19]. It is intriguing to notice that the first paper on groups of type E_7 was written about a decade before the discovery of of extended ($\mathcal{N} = 2$) supergravity [69], in which electromagnetic duality symmetry was observed [70].

In particular, *simple U*-duality groups of $\mathcal{N} = 2$, $D = 4$ theories with symmetric (vector multiplets') scalar manifolds (listed in Table 3) are *conformal non-compact*, real forms of simple non-degenerate groups of type E_7, which are the conformal symmetry group of simple Euclidean Jordan algebras of rank 3 [25].

Furthermore, the results of Sections 2 and 3 also hold for the relevant non-compact, real forms of (*non-degenerate* [64]) *semi-simple* groups *of type E_7* [14, 15, 19, 20], appearing in supergravity as *semi-simple U*- duality group of the infinite sequence of $\mathcal{N} = 2$ theories, with scalar manifold given by:

$$\frac{SL\left(2, \mathbb{R}\right)}{U\left(1\right)} \times \frac{SO\left(2, n\right)}{SO\left(2\right) \times SO\left(n\right)}, \quad n \geqslant 1, \quad \text{rank} = 1 + \min(2, n), \tag{5.8}$$

based on the *semi-simple* rank-3 Jordan algebra $\mathbb{R} \oplus \mathbf{\Gamma}_{1,n-1}$, where $\mathbf{\Gamma}_{1,n-1}$ stands for the Jordan algebra of degree two with a quadratic form of

Table 3: Vector multiplets' *symmetric* scalar manifolds (5.9) (up to possible finite factors in the stabiliser) of $\mathcal{N} = 2$, $D = 4$ supergravity models with *simple U*-duality groups (*alias conformal non-compact* real forms of *non-degenerate* [64], *simple* group *of type E_7* [14, 15, 19, 20]), with related *simple* rank-3 Jordan algebra. The relevant symplectic irrep. **R** of G_4 is also reported. $\mathbb{O}$, $\mathbb{H}$, $\mathbb{C}$, and $\mathbb{R}$, respectively denote the four division algebras of octonions, quaternions, complex, and real numbers. Note that, with the exception of the *triality symmetric STU* model [53], these models are all the ones for which the treatment of [61] holds (see, e.g., Table 1 therein). The $D = 5$ uplift of the T^3 model based on $\mathfrak{J}_3 = \mathbb{R}$ is the *pure $\mathcal{N} = 2$, $D = 5$* supergravity. $\mathfrak{J}_3(\mathbb{H})$ is related to both 8 and 24 supersymmetries, because the corresponding supergravity theories are *"twin"*, namely they share the very same bosonic sector [72].

$\mathfrak{J}_3$	G_4/H_4	$\mathbf{R}$	q
$\mathfrak{J}_3(\mathbb{O})$	$\dfrac{E_{7(-25)}}{E_{6(-78)} \times U(1)}$	$\mathbf{56}$	8
$\mathfrak{J}_3(\mathbb{H})$	$\dfrac{SO^*(12)}{SU(6) \times U(1)}$	$\mathbf{32}$	4
$\mathfrak{J}_3(\mathbb{C})$	$\dfrac{SU(3,3)}{SU(3) \times SU(3) \times U(1)}$	$\mathbf{20}$	2
$\mathfrak{J}_3(\mathbb{R})$	$\dfrac{Sp(6,\mathbb{R})}{SU(3) \times U(1)}$	$\mathbf{14'}$	1
$\mathbb{R}(T^3 \text{ model})$	$\dfrac{SL(2,\mathbb{R})}{U(1)}$	$\mathbf{4}$	$-2/3$

Lorentzian signature $(1, n-1)$, which is nothing but the Clifford algebra of $O(1, n-1)$ [71].

In other words, at the group level, the results of Sections 2 and 3 provide a manifestly [mcs $(\text{Conf}(\mathfrak{J}_3))$]-covariant symplectic frame for the *special Kähler geometry* of the corresponding symmetric, non-compact, vector multiplet's scalar manifold (of Riemannian nature), whose coset structure reads (up to possible finite factors in the stabilizer; see, e.g., [29, 65]; for a comprehensive list of manifolds, see, e.g., [73]):

$$\mathcal{M}_{\mathcal{N}=2} = \frac{\text{Conf}(\mathfrak{J}_3)}{\text{mcs}(\text{Conf}(\mathfrak{J}_3))}. \tag{5.9}$$

Here $\text{Conf}(\mathfrak{J}_3) = \text{Aut}(\mathfrak{M}(\mathfrak{J}_3))$ stands for the *conformal group* of $\mathfrak{J}_3$, which is nothing but the automorphism group of the *Freudenthal triple system*

$\mathfrak{M}$ [28] constructed on $\mathfrak{J}_3$ itself. The relevant non-compact, real forms of mcs[Conf $(\mathfrak{J}_3)$] $/U(1)$ (namely, the U-duality symmetries in $D = 5$) are the reduced structure algebras of the corresponding (q-parametrized) simple, rank-3 Euclidean Jordan algebras.

Up to symplectic re-parametrization, for the infinite sequence (5.8) of $\mathcal{N} = 2$ theories with semi-simple U-duality group $\text{Conf}(\mathfrak{J}_3) = SL(2, \mathbb{R}) \times SO(2, n)$, the results of Sections 2 and 3 match the so-called *Calabi-Vesentini* $\mathcal{N} = 2$ symplectic frame [57, 74] (see also [20] for a recent study), whose (compact) manifest covariance is the maximal one:

$$\text{mcs}(\text{Conf}(\mathfrak{J}_3)) = \text{mcs}(SL(2, \mathbb{R}) \times SO(2, n)) = U(1) \times SO(2) \times SO(n). \tag{5.10}$$

All the vector multiplets' scalar manifolds of the aforementioned $\mathcal{N} = 2$, $D = 4$ supergravity theories related to cubic Euclidean Jordan algebras are special Kähler, maximal, non-compact, symmetric cosets with structure (5.9), and have $rank^5$ 3 (except the rank-1 case of T^3 model). They also are Einstein spaces, with constant (negative) Ricci scalar curvature R:

$$R_{i\bar{j}} = \lambda g_{i\bar{j}} \Rightarrow R = \lambda n_V, \tag{5.11}$$

where $R_{i\bar{j}}$ is the special Kähler Ricci tensor, and the real parameter λ has been computed in [76] (see also [51]):

$$\lambda = \begin{cases} -\dfrac{2}{3} n_V \text{ for: } T^3 \text{ model } (n_V = 1), STU \text{ model } (n_V = 3), \\ \quad \text{ and } \mathfrak{J}_3(\mathbb{A})\text{-models } (n_V = 3q + 3); \\ -\dfrac{(n_V^2 - 2n_V + 3)}{n_V} \text{ for } \mathbb{R} \oplus \mathbf{\Gamma}_{1,n-1} \text{ models } (n_V = n + 1 \geqslant 2). \end{cases} \tag{5.12}$$

Similarly, also the results about the Iwasawa decomposition worked out in Section 4 can be generalized *at least* to the *conformal non-compact* real forms of (*non-degenerate* [64]) *simple* groups *of type* E_7 [14, 15, 19, 20], listed as $D = 4$ U-duality groups G_4's in Table 3.

---

[5]The *rank* of a manifold is defined as the maximal dimension (in $\mathbb{R}$) of a Riemann-flat, totally geodesic sub-manifold of the manifold itself (see, e.g., [75], p. 209).

Indeed, in light of (4.54)–(4.57), the chain of maximal and symmetric embeddings (4.53) enjoys the following generalization:

$$
\begin{aligned}
\mathrm{Conf}\left(\mathfrak{J}_3\left(\mathbb{A}\right)\right) \supset{} & \mathrm{Str}_0\left(\mathfrak{J}_3\left(\mathbb{A}\right)\right) \times SO\left(1,1\right)' \\
\supset{} & \mathrm{Aut}\left(\mathfrak{J}_3\left(\mathbb{A}\right)\right) \times SO\left(1,1\right)' \\
\supset{} & SO\left(q+1\right) \times \mathcal{A}_q \times SO\left(1,1\right)' \\
\supset{} & SO\left(q\right) \times \mathcal{A}_q \times SO\left(1,1\right)',
\end{aligned}
\tag{5.13}
$$

by recalling the definition introduced just below (2.30), $q := \dim_{\mathbb{R}}\mathbb{A} = 1, 2, 4$, and 8 for $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and $\mathbb{O}$, respectively. We report the symmetry groups of *simple* rank-3 Euclidean Jordan algebras in Table 4. *Mutatis mutandis* (also with the help of the tables), the treatment and the results of the whole Section 4 can be extended to all $\mathcal{N} = 2$, $D = 4$ *symmetric* supergravities reported in Table 3 (but the T^3 model).

The extra factor group $\mathcal{A}_q$, which exists only for $q = 2$ and $q = 4$, is reported in Table 5; in [77], its appearance was observed within the study of the charge orbits of asymptotically flat 0-(black holes) and 1-(black strings) branes in minimal magical Maxwell–Einstein supergravity theories in $D = 5$ space–time dimensions. We note that $\mathcal{A}_q$ is related to $\widehat{G}_{\mathrm{cent}}$ and G_{paint} (Lie groups usually introduced in the treatment of *supergravity billiards* and timelike Kaluza–Klein reductions; for recent treatment and set of related references, see, e.g., [58]; see also Table 5 therein, also for subtleties concerning the case $q = 8$ in $D = 5, 6$) as follows [77]:

$$
D = 5, 6 : \widehat{G}_{\mathrm{cent}} = SO\left(1,1\right) \times SO\left(q-1\right) \times \mathcal{A}_q;
\tag{5.14}
$$

$$
D = 3, 4 : \widehat{G}_{\mathrm{cent}} = G_{\mathrm{paint}} = SO\left(q\right) \times \mathcal{A}_q.
\tag{5.15}
$$

According to [78], $\mathcal{A}_q$ can be related to the structure of the Hopf maps, chiral Weyl spinors and division algebras; we hope to study this intriguing connection in future investigations, also along the lines of [79].

Extending the considerations made above on $SO\left(8\right)$, it can be observed that $SO\left(q\right) \times \mathcal{A}_q$ shares the same Lie algebra $\mathfrak{tri}\left(\mathbb{A}\right)$ of $\mathrm{Aut}(\mathbf{t}\left(\mathbb{A}\right))$, which is the automorphism group of the *normed triality* over the division algebra $\mathbb{A}$ (see, e.g., [62]); see Table 6.

Besides this fact, it is intriguing to notice that $SO\left(q\right) \times \mathcal{A}_q$ appears in *at least* three (apparently unrelated) contexts:

(1) As $\widehat{G}_{\mathrm{cent}} = G_{\mathrm{paint}}$ in $D = 3, 4$, as given by (5.15) (see, e.g., [58,77]).

Table 4: Invariance groups associated to *simple* rank-3 Euclidean Jordan algebras $\mathfrak{J}_3(\mathbb{A})$. Conf($\mathfrak{J}_3(\mathbb{A})$)'s are *conformal non-compact* real forms of (*non-degenerate* [64]) *simple* group *of type* E_7 [14,15,19,20]. G_5, G_4, and G_3, respectively denote the U-duality groups of the corresponding supergravity theories with 8 supersymmetries in $D = 5$, 4 and 3. The lower 4×4 part is known as the *"Magic Square"*, due to its symmetry along the diagonal (see, e.g., [10]).

$\mathfrak{J}_3(\mathbb{A})$	Aut$(\mathfrak{J}_3(\mathbb{A}))$ = mcs(G_5)	Str$_0(\mathfrak{J}_3(\mathbb{A}))$ = G_5	Conf$(\mathfrak{J}_3(\mathbb{A}))$ = G_4	QConf$(\mathfrak{J}_3(\mathbb{A}))$ = G_3
$\mathbb{R}$	Id	Id	$Sl(2,\mathbb{R})$	$G_{2(2)}$
$\mathfrak{J}_3^{\mathbb{R}}$	$SO(3)$	$SL(3,\mathbb{R})$	$Sp(6,\mathbb{R})$	$F_{4(4)}$
$\mathfrak{J}_3^{\mathbb{C}}$	$SU(3)$	$SL(3,\mathbb{C})$	$SU(3,3)$	$E_{6(2)}$
$\mathfrak{J}_3^{\mathbb{H}}$	$USp(6)$	$SU^*(6)$	$SO^*(12)$	$E_{7(-5)}$
$\mathfrak{J}_3^{\mathbb{O}}$	$F_{4(-52)}$	$E_{6(-26)}$	$E_{7(-25)}$	$E_{8(-24)}$

Table 5: The extra commuting group $\mathcal{A}_q$ (see, e.g., [77]).

q	$\mathcal{A}_q$
8	$-$
4	$SO(3)$
2	$SO(2)$
1	$-$

(2) As stabilizer group $\mathcal{G}_{p=2}\left(\mathfrak{J}_3\left(\mathbb{A}\right)\right)$ of BPS generic charge orbits of 2-centered extremal black holes in $\mathcal{N} = 2$, $D = 4$ magical models, as derived in [17], and reported in Table 7.

(3) According to (5.13), as group of maximal manifest covariance of the Iwasawa decomposition of the (vector multiplets') scalar manifold of $\mathcal{N} = 2$, $D = 4$ magical models, whose U-duality groups are (some instances of) *conformal non-compact* real forms of (*non-degenerate* [64]) *simple* groups *of type* E_7 [14, 15, 19, 20].

6   Conclusion

The present investigation, and in particular the generalizations discussed in Section 5, pave the way to a number of interesting further developments. We list a selection of them below.

Starting with the treatment given in Section 3, it should be pointed out that a more explicit expression of the symplectic frame determined by the comparison of (3.7) and (3.8) with the general formulae (3.10)–(3.14) of special Kähler geometry would be needed also in order to check that the prepotential F does *not* exist in the symplectic frame introduced in Sections 2 and 3, which can be considered the *analogue of the Calabi-Vesentini* one [57, 74] for *non-degenerate, conformal non-compact, and simple* groups of type E_7.

Furthermore, it would be interesting to extend the maximally manifestly-covariant symplectic frame and/or the Iwasawa symplectic frame, respectively introduced in Sections 2, 3 and in Section 4, to

- *compact* groups of type E_7;
- other *non-compact* real forms of groups of type E_7 (possibly related to $\mathcal{N} > 2$-extended supergravity theories), also in relation to rank-3 Jordan algebras on *split* forms of division algebras (for a recent treatment of the non-supersymmetric cases of $\mathfrak{J}_3(\mathbb{H}_s)$ and $\mathfrak{J}_3(\mathbb{C}_s)$, see, e.g., [30]);

Table 6: The Lie algebra $\mathfrak{tri}(\mathbb{A})$ of the automorphism group $\mathrm{Aut}(\mathbf{t}(\mathbb{A}))$ of the *normed triality* $\mathbf{t}(\mathbb{A})$ over the division algebra $\mathbb{A}$, and the group $SO(q) \times \mathcal{A}_q$, in terms of the parameter q. See, e.g., [62], in particular (5) and (21) therein.

q	$\mathfrak{tri}(\mathbb{A})$	$SO(q) \times \mathcal{A}_q$	$\mathrm{Aut}(\mathbf{t}(\mathbb{A}))$
1	$\{0\}$	Id	$\left\{ (g_1, g_2, g_3) \in [O(1)]^3 : g_1 g_2 g_3 = 1 \right\}$
2	$[\mathfrak{u}(1)]^2$	$SO(2) \times SO(2) \sim [U(1)]^2$	$\left\{ (g_1, g_2, g_3) \in [U(1)]^3 : g_1 g_2 g_3 = 1 \right\} \times \mathbb{Z}_2$
4	$[\mathfrak{usp}(2)]^3$	$SO(4) \times SO(3) \sim [SU(2)]^3$	$[USp(2)]^3 / \{\pm(1,1,1)\}$
8	$\mathfrak{so}(8)$	$SO(8)$	$Spin(8)$

Table 7: BPS generic charge orbits of 2-centered extremal black holes in $\mathcal{N} = 2$, $d = 4$ magical models. Conf($\mathfrak{J}_3(\mathbb{A})$) denotes the "conformal" group of $\mathfrak{J}_3(\mathbb{A})$ (see, e.g., [65]) [17].

A	$\mathcal{O}_{p=2,BPS} = \dfrac{\mathfrak{J}_3(\mathbb{A})}{\mathcal{G}_{p=2}(\mathfrak{J}_3(\mathbb{A}))}$
$\mathbb{O}$	$\dfrac{E_{7(-25)}}{SO(8)}$
$\mathbb{H}$	$\dfrac{SO^*(12)}{[SU(2)]^3}$
$\mathbb{C}$	$\dfrac{SU(3,3)}{[U(1)]^2}$
$\mathbb{R}$	$Sp(6,\mathbb{R})$

- other classes of supergravities, such as $\mathcal{N} = 2$, $D = 4$ with homogeneous *non-symmetric* scalar manifolds [37, 41].

Furthermore, considering the generalizations of the Iwasawa parametrization discussed in Section 5, it would be interesting to explore its extension also to theories related to *semi-simple* rank-3 Jordan algebras, such as $\mathbb{R} \oplus \mathbf{\Gamma}_{m,n}$ (for $m = 1$, recall (5.8)), where $\mathbf{\Gamma}_{m,n}$ stands for the rank-2 Jordan algebra with a quadratic form of Lorentzian signature (m, n), which is nothing but the Clifford algebra of $O(m, n)$ [71].

Concerning the generalization to $\mathcal{N} > 2$-extended supergravities, it would be interesting to compare the application of the Iwasawa decomposition under consideration to the case of $\mathbb{O}_S$ with the Iwasawa parametrization of $\mathcal{M}_{\mathcal{N}=8}$ (3.9) studied in [22], whose manifest maximal (non-compact) covariance is $SL(7, \mathbb{R})$, with mcs $SO(7)$. In this respect, the following remark made in [17] should be relevant: as it holds for the stabilizer of $\mathcal{O}_{\mathcal{N}=2,\mathfrak{J}_3(\mathbb{O}),\text{tab}7,p=2}$ (see Table 7), the Lie algebra $\mathfrak{so}(8)$ of the stabilizer of the 2-centered orbit [17]

$$\mathcal{O}_{\mathcal{N}=2,J_3^{\mathbb{O}},\text{nBPS},p=2,\mathbf{I}} = \frac{E_{7(-25)}}{SO(8)} \tag{6.1}$$

is nothing but the Lie algebra $\mathfrak{tri}(\mathbb{O})$ of the automorphism group $\text{Aut}(\mathbf{t}(\mathbb{O}))$ of the *normed triality* over $\mathbb{O}$ (see Table 6). It is here worth observing that the Lie algebra $\mathfrak{so}(4,4)$ of the stabilizer of the 2-centered orbit [17]

$$\mathcal{O}_{\mathcal{N}=8,\frac{1}{8}\text{-BPS},p=2,\mathbf{I}} = \frac{E_{7(7)}}{SO(4,4)} \tag{6.2}$$

enjoys an analogous interpretation as the Lie algebra $\mathfrak{tri}\,(\mathbb{O}_s)$ of the automorphism group $\text{Aut}(\mathbf{t}\,(\mathbb{O}_S))$ of the *normed triality* over $\mathbb{O}_S$. On the other hand, a similar interpretation does not seem to hold for the stabilizer of the 2-centered orbit [17]

$$\mathcal{O}_{\mathcal{N}=8,\frac{1}{8}\text{-BPS},p=2,\mathbf{II}} = \frac{E_{7(7)}}{SO\,(5,3)}, \tag{6.3}$$

as well as for the stabilizer of the 2-centered orbit [17]

$$\mathcal{O}_{\mathcal{N}=2,J_3^{\mathbb{O}},\text{nBPS},p=2,\mathbf{II}} = \frac{E_{7(-25)}}{SO\,(7,1)}. \tag{6.4}$$

However, it is intriguing to note that the maximal manifest compact covariance $SO\,(7) = \text{mcs}\,(SO\,(7,1)) = \text{mcs}\,(SL\,(7,\mathbb{R}))$ exhibited by the Iwasawa parametrization of $\mathcal{M}_{\mathcal{N}=8}$ (3.9) [22] may provide a clue for the stabilizer of $\mathcal{O}_{\mathcal{N}=2,J_3^{\mathbb{O}},\text{nBPS},p=2,\mathbf{II}}$ (6.4). We leave to future studies the in-depth investigation of these fascinating connections, here just briefly outlined.

Finally, we would like to put forward an hint[6] for a further physical application. It should be observed that the Iwasawa coset decomposition can yield a nilpotent algebra exhibiting the same symmetry of the systems recently discussed in [80] in the framework of supergravity theories timelike-reduced down to $D = 3^*$ dimensions. Thus, it would be interesting to investigate the possible Lax pair structures hidden in the Iwasawa formalism, which might allow for a more explicit integration procedure within the $D = 3^*$ nilpotent orbits formalism of [80]. We hope to report on this intriguing connection in future studies.

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