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Diffeomorphism-invariant covariant Hamiltonians of a pseudo-Riemannian metric and a linear connection

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Abstract

Let $M \to N$ (resp. $C \to N$) be the fibre bundle of pseudo-Riemannian metrics of a given signature (resp. the bundle of linear connections) on an orientable connected manifold N. A geometrically defined class of first-order Ehresmann connections on the product fibre bundle $M \times_N C$ is determined such that, for every connection γ belonging to this class and every Diff$N$-invariant Lagrangian density Λ on $J^1(M \times_N C)$, the corresponding covariant Hamiltonian Λ^γ is also Diff$N$-invariant. The case of Diff$N$-invariant second-order Lagrangian densities on J^2M is also studied and the results obtained are then applied to Palatini and Einstein–Hilbert Lagrangians.

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1 Introduction

In Mechanics, the Hamiltonian function attached to a Lagrangian density $\Lambda = L(t, q^i, \dot{q}^i)dt$ on $\mathbb{R} \times TQ$ is given by $H = \dot{q}^i \partial L/\partial \dot{q}^i - L$, but — as it was early observed in [16] — this is not an invariant definition if an arbitrary fibred manifold $t\colon E \to \mathbb{R}$ is considered (thus generalizing the notion of an absolute time) instead of the direct product bundle $\mathbb{R} \times Q \to \mathbb{R}$; e.g., see [7, 23, 25] for this point of view. In this case, an Ehresmann connection is needed in order to lift the vector field $\partial/\partial t$ from $\mathbb{R}$ to E, and the Hamiltonian is then defined by applying the Poincaré–Cartan form attached to Λ to the horizontal lift of $\partial/\partial t$.

In the field theory — where no distinguished vector field exists on the base manifold — the need of an Ehresmann connection is even greater, in order to attach a covariant Hamiltonian to each Lagrangian density; e.g., see [23, 24, 4.1], and the definitions below.

Let $p\colon E \to N$ be an arbitrary fibred manifold over a connected manifold N, $n = \dim N$, $\dim E = m + n$, oriented by $v_n = dx^1 \wedge \cdots \wedge dx^n$. Throughout this paper, Latin (resp. Greek) indices run from 1 to n (resp. m). An Ehresmann connection on a fibred manifold $p\colon E \to N$ is a differential 1-form γ on E taking values in the vertical sub-bundle $V(p)$ such that $\gamma(X) = X$ for every $X \in V(p)$ (e.g., see [23, 24, 32, 34]). Once an Ehresmann connection γ is given, a decomposition of vector bundles holds $T(E) = V(p) \oplus \ker \gamma$, where $\ker \gamma$ is called the horizontal sub-bundle determined by γ. In a fibred coordinate system (x^j, y^α) for p, an Ehresmann connection can be written as

$$\gamma = (dy^\alpha + \gamma_j^\alpha dx^j) \otimes \frac{\partial}{\partial y^\alpha}, \quad \gamma_j^\alpha \in C^\infty(E).$$

According to [24], the covariant Hamiltonian Λ^γ associated to a Lagrangian density on J^1E, $\Lambda = Lv_n$, $L \in C^\infty(J^1E)$, with respect to γ is the Lagrangian density defined by,

$$\Lambda^\gamma = \left((p_0^1)^*\gamma - \theta\right) \wedge \omega_\Lambda - \Lambda, \tag{1.1}$$

where, $p_0^1\colon J^1E \to J^0E = E$ is the projection mapping, $\theta = \theta^\alpha \otimes \partial/\partial y^\alpha$, $\theta^\alpha = dy^\alpha - y_i^\alpha dx^i$ is the $V(p)$-valued 1-form on J^1E associated with the contact structure, written on a fibred coordinate system (x^i, y^α), and ω_Λ

is the Legendre form attached to Λ, i.e., the $V^*(p)$-valued p^1-horizontal $(n-1)$-form on J^1E given by

$$\omega_\Lambda = (-1)^{i-1} \frac{\partial L}{\partial y_i^\alpha} i_{\partial/\partial x^i} v_n \otimes dy^\alpha,$$

where $(x^i, y^\alpha; y_i^\alpha)$ is the coordinate system induced from (x^i, y^α) on the 1-jet bundle and $p^1 \colon J^1E \to N$ is the projection on the base manifold. Locally,

$$\Lambda^\gamma = \left((\gamma_i^\alpha + y_i^\alpha) \frac{\partial L}{\partial y_i^\alpha} - L \right) dx^1 \wedge \cdots \wedge dx^n. \tag{1.2}$$

From (1.1) we obtain the following decomposition of the Poincaré–Cartan form attached to Λ (e.g., see [17, 23, 27]): $\Theta_\Lambda = \theta \wedge \omega_\Lambda + \Lambda = (p_0^1)^* \gamma \wedge \omega_\Lambda - \Lambda^\gamma$.

A diffeomorphism $\Phi \colon E \to E$ is said to be an automorphism of p if there exists $\phi \in \mathrm{Diff} N$ such that $p \circ \Phi = \phi \circ p$. The set of such automorphisms is denoted by $\mathrm{Aut}(p)$ and its Lie algebra is identified to the space $\mathrm{aut}(p) \subset \mathfrak{X}(E)$ of p-projectable vector fields on E. Given a subgroup $\mathcal{G} \subseteq \mathrm{Aut}(p)$, a Lagrangian density Λ is said to be $\mathcal{G}$-invariant if $(\Phi^{(1)})^* \Lambda = \Lambda$ for every $\Phi \in \mathcal{G}$, where $\Phi^{(1)} \colon J^1E \to J^1E$ denotes the 1-jet prolongation of Φ. Infinitesimally, the $\mathcal{G}$-invariance equation can be reformulated as $L_{X^{(1)}} \Lambda = 0$ for every $X \in \mathrm{Lie}(\mathcal{G})$, $X^{(1)}$ denoting the 1-jet prolongation of the vector field X.

When a group $\mathcal{G}$ of transformations of E is given, a natural question arises:

- Determine a class — as small as possible — of Ehresmann connections γ such that Λ^γ is $\mathcal{G}$-invariant for every $\mathcal{G}$-invariant Lagrangian density Λ.

Below we tackle this question in the framework of General Relativity, i.e., the group $\mathcal{G}$ is the group of all diffeomorphisms of the ground manifold N acting in a natural way either on the bundle of pseudo-Riemannian metrics $p_M \colon M = M(N) \to N$ of a given signature (n^+, n^-), $n^+ + n^- = n$, or on the product bundle $p \colon M \times_N C \to N$, where $p_C \colon C = C(N) \to N$ is the bundle of linear connections on N. Namely, we solve the following two problems:

(P): Determine a class — as small as possible — of Ehresmann connections γ such that for every $\mathrm{Diff} N$-invariant first-order Lagrangian density Λ on the bundle $J^1(M \times_N C)$, the corresponding covariant Hamiltonian Λ^γ is also $\mathrm{Diff} N$-invariant.

Similarly to the problem **(P)**, we formulate the corresponding problem on J^2M as follows:

> **(P2):** Determine a class of second-order Ehresmann connections γ^2 on M such that for every Diff$N$-invariant second-order Lagrangian density Λ on the bundle J^2M, the corresponding covariant Hamiltonian Λ^{γ^2} — defined in (4.9) — is also Diff$N$-invariant.

Essentially, a class of first-order Ehresmann connections on the bundle $M \times_N C$ is obtained, defined by the conditions (C_M) and (C_C) below (see Propositions 3.4 and 3.5), solving the problem **(P)**. This class of connections also helps to solve **(P2)** by means of a natural isomorphism between J^1M and $M \times_N C^{\mathrm{sym}}$, where C^{sym} denotes the sub-bundle of symmetric connections on N (cf. Theorem 4.1). Finally, this approach is applied to Palatini and Einstein–Hilbert Lagrangians [3, 4], obtaining results compatible with their usual Hamiltonian formalisms.

2 Invariance under diffeomorphisms

2.1 Preliminaries

2.1.1 Jet-bundle notations

Let $p^k \colon J^kE \to N$ be the k-jet bundle of local sections of an arbitrary fibred manifold $p \colon E \to N$, with projections $p^k_l \colon J^kE \to J^lE$, $p^k_l(j^k_x s) = j^l_x s$, for $k \geq l$, $j^k_x s$ denoting the k-jet at x of a section s of p defined on a neighbourhood of $x \in N$.

A fibred coordinate system (x^i, y^α) on V induces a coordinate system (x^i, y^α_I), $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$, $0 \leq |I| = i_1 + \cdots + i_n \leq r$, on $(p^r_0)^{-1}(V) = J^rV$ as follows: $y^\alpha_I(j^r_x s) = (\partial^{|I|}(y^\alpha \circ s)/\partial x^I)(x)$, with $y^\alpha_0 = y^\alpha$.

Every morphism $\Phi \colon E \to E'$ whose associated map $\phi \colon N \to N'$ is a diffeomorphism, induces a map

$$\begin{aligned} \Phi^{(r)} &\colon J^rE \to J^rE', \\ \Phi^{(r)}(j^r_x s) &= j^r_{\phi(x)}(\Phi \circ s \circ \phi^{-1}). \end{aligned} \tag{2.1}$$

If Φ_t is the flow of a vector field $X \in \mathrm{aut}(p)$, then $\Phi^{(r)}_t$ is the flow of a vector field $X^{(r)} \in \mathfrak{X}(J^rE)$, called the infinitesimal contact transformation of order

r associated to the vector field X. The mapping

$$\mathrm{aut}(p) \ni X \mapsto X^{(r)} \in \mathfrak{X}(J^r E)$$

is an injection of Lie algebras, namely, one has

$$\begin{aligned}
&(\lambda X + \mu Y)^{(r)} = \lambda X^{(r)} + \mu Y^{(r)}, \\
&[X, Y]^{(r)} = [X^{(r)}, Y^{(r)}], \\
&\forall \lambda, \mu \in \mathbb{R}, \ \forall X, Y \in \mathrm{aut}(p).
\end{aligned}$$

In particular, for $r = 1$,

$$\begin{aligned}
X &= u^i \frac{\partial}{\partial x^i} + v^\alpha \frac{\partial}{\partial y^\alpha}, \quad u^i \in C^\infty(N), v^\alpha \in C^\infty(E), \\
X^{(1)} &= u^i \frac{\partial}{\partial x^i} + v^\alpha \frac{\partial}{\partial y^\alpha} + v_i^\alpha \frac{\partial}{\partial y_i^\alpha}, \quad v_i^\alpha = \frac{\partial v^\alpha}{\partial x^i} + y_i^\beta \frac{\partial v^\alpha}{\partial y^\beta} - y_k^\alpha \frac{\partial u^k}{\partial x^i}.
\end{aligned}$$

2.1.2 Coordinates on *M(N)*, *F(N)*, and *C(N)*

Every coordinate system (x^i) on an open domain $U \subseteq N$ induces the following coordinate systems:

(1) (x^i, y_{jk}) on $(p_M)^{-1}(U)$, where $p_M \colon M \to N$ is the bundle of metrics of a given signature, and the functions $y_{jk} = y_{kj}$ are defined by,

$$g_x = \sum_{i \le j} y_{ij}(g_x)(dx^i)_x \otimes (dx^j)_x, \ \forall g_x \in (p_M)^{-1}(U). \tag{2.2}$$

(2) (x^i, x^i_j) on $(p_F)^{-1}(U)$, where $p_F \colon F(N) \to N$ is the bundle of linear frames on N, and the functions x^i_j are defined by,

$$u = \left((\partial/\partial x^1)_x, \ldots, (\partial/\partial x^n)_x\right) \cdot \left(x^i_j(u)\right), \quad x = p_F(u), \forall u \in (p_F)^{-1}(U),$$

or equivalently,

$$u = (X_1, \ldots, X_n) \in F_x(N), \ X_j = x^i_j(u) \left(\frac{\partial}{\partial x^i}\right)_x, \quad 1 \le j \le n. \tag{2.3}$$

(3) (x^i, A^j_{kl}) on $(p_C)^{-1}(U)$, where $p_C \colon C \to N$ is the bundle of linear connections on N, and the functions A^j_{kl} are defined as follows. We first

recall some basic facts. Connections on $F(N)$ (i.e., linear connections of N) are the splittings of the Atiyah sequence (cf. [2]),

$$0 \to \mathrm{ad}F(N) \to T_{Gl(n,\mathbb{R})}F(N) \xrightarrow{(p_F)_*} TN \to 0,$$

where
(a) $\mathrm{ad}F(N) = T^*N \otimes TN$ is the adjoint bundle;
(b) $T_{Gl(n,\mathbb{R})}(F(N)) = T(F(N))/Gl(n,\mathbb{R})$; and
(c) $\mathrm{gau}F(N) = \Gamma(N, \mathrm{ad}F(N))$ is the gauge algebra of $F(N)$.
We think of $\mathrm{gau}F(N)$ as the 'Lie algebra' of the gauge group $\mathrm{Gau}F(N)$. Moreover, $p_C\colon C \to N$ is an affine bundle modelled over the vector bundle $\otimes^2 T^*N \otimes TN$. The section of p_C induced tautologically by the linear connection Γ is denoted by $s_\Gamma\colon N \to C$. Every $B \in \mathfrak{gl}(n,\mathbb{R})$ defines a one-parameter group $\varphi_t^B\colon U \times Gl(n,\mathbb{R}) \to U \times Gl(n,\mathbb{R})$ of gauge transformations by setting (cf. [5]), $\varphi_t^B(x,\Lambda) = (x, \exp(tB)\cdot\Lambda)$. Let us denote by $\bar{B} \in \mathrm{gau}(p_F)^{-1}(U)$ the corresponding infinitesimal generator. If (E_j^i) is the standard basis of $\mathfrak{gl}(n,\mathbb{R})$, then $\bar{E}_j^i = \sum_{h=1}^n x_h^j \partial/\partial x_h^i$, for $i,j = 1,\dots,n$, is a basis of $\mathrm{gau}(p_F)^{-1}(U)$. Let $\tilde{E}_j^i = \bar{E}_j^i \bmod G$ be the class of $\bar{E}_j^i$ on $\mathrm{ad}F(N)$. Unique smooth functions A_{jk}^i on $(p_C)^{-1}(U)$ exist such that,

$$\begin{aligned} s_\Gamma\left(\frac{\partial}{\partial x^j}\right) &= \frac{\partial}{\partial x^j} - (A_{jk}^i \circ \Gamma)\tilde{E}_k^i \\ &= \frac{\partial}{\partial x^j} - (A_{jk}^i \circ \Gamma) x_h^k \frac{\partial}{\partial x_h^i}, \end{aligned} \tag{2.4}$$

for every s_Γ and $A_{jk}^i(\Gamma_x) = \Gamma_{jk}^i(x)$, where Γ_{jk}^i are the Christoffel symbols of the linear connection Γ in the coordinate system (x^i), see [20, III, Proposition 7.4].

2.2 Natural lifts

Let $f_M\colon M \to M$, cf. [30] (resp. $\tilde{f}\colon F(N) \to F(N)$, cf. [20, p. 226]) be the natural lift of $f \in \mathrm{Diff}N$ to the bundle of metrics (resp. linear frame bundle); namely $f_M(g_x) = (f^{-1})^* g_x$ (resp. $\tilde{f}(X_1,\dots,X_n) = (f_*X_1,\dots,f_*X_n)$, where $(X_1,\dots,X_n) \in F_x(N)$); hence $p_M \circ f_M = f \circ p_M$ (resp. $p_F \circ \tilde{f} = f \circ p_F$), and $f_M\colon M \to M$ (resp. $\tilde{f}\colon F(N) \to F(N)$) have a natural extension to jet bundles $f_M^{(r)}\colon J^r(M) \to J^r(M)$ (resp. $\tilde{f}^{(r)}\colon J^r(FN) \to J^r(FN)$) as defined in the formula (2.1), i.e.,

$$f_M^{(r)}(j_x^r g) = j_{f(x)}^r (f_M \circ g \circ f^{-1}) \quad (\text{resp. } \tilde{f}^{(r)}(j_x^r s) = j_{f(x)}^r (\tilde{f} \circ s \circ f^{-1})).$$

As $\tilde{f}$ is an automorphism of the principal $Gl(n, \mathbb{R})$-bundle $F(N)$, it acts on linear connections by pulling back connection forms, i.e., $\Gamma' = \tilde{f}(\Gamma)$ where $\omega_{\Gamma'} = (\tilde{f}^{-1})^* \omega_\Gamma$ (see [20, II, Proposition 6.2-(b)], [5, 3.3]). Hence, there exists a unique diffeomorphism $\tilde{f}_C \colon C \to C$ such that,

(1) $p_C \circ \tilde{f}_C = f \circ p_C$, and
(2) $\tilde{f}_C \circ s_\Gamma = s_{\tilde{f}(\Gamma)}$ for every linear connection Γ.

If f_t is the flow of a vector field $X \in \mathfrak{X}(N)$, then the infinitesimal generator of $(f_t)_M$ (resp. $\tilde{f}_t$, resp. $(\tilde{f}_t)_C$) in $\operatorname{Diff} M$ (resp. $\operatorname{Diff} F(N)$, resp. $\operatorname{Diff} C$) is denoted by X_M (resp. $\tilde{X}$, resp. $\tilde{X}_C$) and the following Lie-algebra homomorphisms are obtained:

$$\begin{cases} \mathfrak{X}(N) \to \mathfrak{X}(M), & X \mapsto X_M, \\ \mathfrak{X}(N) \to \mathfrak{X}(F(N)), & X \mapsto \tilde{X}, \\ \mathfrak{X}(N) \to \mathfrak{X}(C), & X \mapsto \tilde{X}_C. \end{cases}$$

If $X = u^i \partial / \partial x^i \in \mathfrak{X}(N)$ is the local expression for X, then

(1) From [30, equations (2) to (4)] we know that the natural lift of X to M is given by,

$$X_M = u^i \frac{\partial}{\partial x^i} - \sum_{i \le j} \left(\frac{\partial u^h}{\partial x^i} y_{hj} + \frac{\partial u^h}{\partial x^j} y_{ih} \right) \frac{\partial}{\partial y_{ij}} \in \mathfrak{X}(M).$$

and its 1-jet prolongation,

$$\begin{aligned} X_M^{(1)} = u^i \frac{\partial}{\partial x^i} &- \sum_{i \le j} \left(\frac{\partial u^h}{\partial x^i} y_{hj} + \frac{\partial u^h}{\partial x^j} y_{hi} \right) \frac{\partial}{\partial y_{ij}} \\ &- \sum_{i \le j} \left(\frac{\partial^2 u^h}{\partial x^i \partial x^k} y_{hj} + \frac{\partial^2 u^h}{\partial x^j \partial x^k} y_{hi} + \frac{\partial u^h}{\partial x^i} y_{hj,k} + \frac{\partial u^h}{\partial x^j} y_{hi,k} + \frac{\partial u^h}{\partial x^k} y_{ij,h} \right) \\ &\times \frac{\partial}{\partial y_{ij,k}}. \end{aligned}$$

(2) From [10, Proposition 3] (also see [20, VI, Proposition 21.1]) we know that the natural lift of X to $F(N)$ is given by

$$\tilde{X} = u^i \frac{\partial}{\partial x^i} + \frac{\partial u^i}{\partial x^l} x^l_j \frac{\partial}{\partial x^i_j},$$

and its 1-jet prolongation

$$\begin{aligned}\tilde{X}^{(1)} &= u^i \frac{\partial}{\partial x^i} + \frac{\partial u^i}{\partial x^l} x^l_j \frac{\partial}{\partial x^i_j} + v^i_{jk} \frac{\partial}{\partial x^i_{j,k}},\\ v^i_{jk} &= \frac{\partial u^i}{\partial x^l} x^l_{j,k} - \frac{\partial u^l}{\partial x^k} x^i_{j,l} + \frac{\partial^2 u^i}{\partial x^k \partial x^l} x^l_j.\end{aligned}$$

(3) Finally,

$$\begin{aligned}\tilde{X}_C &= u^i \frac{\partial}{\partial x^i} - \left(\frac{\partial^2 u^i}{\partial x^j \partial x^k} - \frac{\partial u^i}{\partial x^l} A^l_{jk} + \frac{\partial u^l}{\partial x^k} A^i_{jl} + \frac{\partial u^l}{\partial x^j} A^i_{lk} \right) \frac{\partial}{\partial A^i_{jk}},\\ \tilde{X}^{(1)}_C &= u^i \frac{\partial}{\partial x^i} + w^i_{jk} \frac{\partial}{\partial A^i_{jk}} + w^i_{jkh} \frac{\partial}{\partial A^i_{jk,h}},\\ w^i_{jk} &= -\frac{\partial^2 u^i}{\partial x^j \partial x^k} + \frac{\partial u^i}{\partial x^l} A^l_{jk} - \frac{\partial u^l}{\partial x^k} A^i_{jl} - \frac{\partial u^l}{\partial x^j} A^i_{lk}, \qquad (2.5)\\ w^i_{jkh} &= -\frac{\partial^3 u^i}{\partial x^h \partial x^j \partial x^k} + \frac{\partial^2 u^i}{\partial x^h \partial x^l} A^l_{jk} - \frac{\partial^2 u^l}{\partial x^h \partial x^k} A^i_{jl} - \frac{\partial^2 u^l}{\partial x^h \partial x^j} A^i_{lk} \qquad (2.6)\\ &\quad + \frac{\partial u^i}{\partial x^l} A^l_{jk,h} - \frac{\partial u^l}{\partial x^k} A^i_{jl,h} - \frac{\partial u^l}{\partial x^j} A^i_{lk,h} - \frac{\partial u^l}{\partial x^h} A^i_{jk,l}.\end{aligned}$$

Let $p\colon M \times_N C \to N$ be the natural projection.

We denote by $\bar{f} = (f_M, \tilde{f}_C)$ (resp. $\bar{X} = (X_M, \tilde{X}_C) \in \mathfrak{X}(M \times_N C)$) the natural lift of f (resp. X) to $M \times_N C$. The prolongation to the bundle $J^1(M \times_N C)$ of $\bar{X}$ is as follows:

$$\begin{aligned}\bar{X}^{(1)} = \left(X^{(1)}_M, \tilde{X}^{(1)}_C \right) &= u^i \frac{\partial}{\partial x^i}\\ &\quad + \sum_{i \le j} v_{ij} \frac{\partial}{\partial y_{ij}} + \sum_{i \le j} v_{ijk} \frac{\partial}{\partial y_{ij,k}} + w^i_{jk} \frac{\partial}{\partial A^i_{jk}} + w^i_{jkh} \frac{\partial}{\partial A^i_{jk,h}}, \qquad (2.7)\end{aligned}$$

where

$$v_{ij} = -\frac{\partial u^h}{\partial x^i} y_{hj} - \frac{\partial u^h}{\partial x^j} y_{hi}, \qquad (2.8)$$

$$v_{ijk} = -\frac{\partial^2 u^h}{\partial x^i \partial x^k} y_{hj} - \frac{\partial^2 u^h}{\partial x^j \partial x^k} y_{hi} - \frac{\partial u^h}{\partial x^i} y_{hj,k} - \frac{\partial u^h}{\partial x^j} y_{hi,k} - \frac{\partial u^h}{\partial x^k} y_{ij,h}, \qquad (2.9)$$

and w^i_{jk}, w^i_{jkh} are given in the formulas (2.5) and (2.6), respectively.

2.3 Diff N- and $\mathfrak{X}(N)$-invariance

A differential form $\omega_r \in \Omega^r(J^1(M \times_N C))$, $r \in \mathbb{N}$, is said to be Diff$N$-invariant — or invariant under diffeomorphisms — (resp. $\mathfrak{X}(N)$-invariant) if the following equation holds: $(\bar{f}^{(1)})^*\omega_r = \omega_r$, $\forall f \in \operatorname{Diff} N$ (resp. $L_{\bar{X}^{(1)}}\omega_r = 0$, $\forall X \in \mathfrak{X}(N)$). Obviously, "Diff$N$-invariance" implies "$\mathfrak{X}(N)$-invariance" and the converse is almost true (see [14, 28]). Because of this, below we consider $\mathfrak{X}(N)$-invariance only.

A linear frame $(X_1, \dots, X_n)$ at x is said to be orthonormal with respect to $g_x \in M_x(N)$ (or simply g_x-orthonormal) if $g_x(X_i, X_j) = 0$ for $1 \le i < j \le n$, $g(X_i, X_i) = 1$ for $1 \le i \le n^+$, $g(X_i, X_i) = -1$ for $n^+ + 1 \le i \le n$.

As N is an oriented manifold, there exists a unique p-horizontal n-form $\mathbf{v}$ on $M \times_N C$ such that, $\mathbf{v}_{(g_x,\Gamma_x)}(X_1, \dots, X_n) = 1$, for every g_x-orthonormal basis $(X_1, \dots, X_n)$ belonging to the orientation of N. Locally $\mathbf{v} = \rho v_n$, where $\rho = \sqrt{(-1)^{n^-}\det(y_{ij})}$ and $v_n = dx^1 \wedge \cdots \wedge dx^n$. As proved in [30, Proposition 7], the form $\mathbf{v}$ is Diff$N$-invariant and hence $\mathfrak{X}(N)$-invariant. A Lagrangian density Λ on $J^1(M \times_N C)$ can be globally written as $\Lambda = \mathcal{L}\mathbf{v}$ for a unique function $\mathcal{L} \in C^\infty(J^1(M \times_N C))$ and Λ is $\mathfrak{X}(N)$-invariant if and only if the function $\mathcal{L}$ is. Therefore, the invariance of Lagrangian densities is reduced to that of scalar functions.

Proposition 2.1. *A function $\mathcal{L} \in C^\infty(J^1(M \times_N C))$ is $\mathfrak{X}(N)$-invariant if and only if the following system of partial differential equations hold:*

$$\begin{cases} 0 = X^i(\mathcal{L}), & \forall i, \\ 0 = X_h^i(\mathcal{L}), & \forall h, i, \\ 0 = X_h^{ik}(\mathcal{L}), & \forall h, i \le k, \\ 0 = X_i^{jkh}(\mathcal{L}), & \forall i, j \le k \le h, \end{cases} \tag{2.10}$$

where

$$\begin{aligned} X^i &= \frac{\partial}{\partial x^i}, \quad \forall i, \\ X_h^i &= -y_{hi}\frac{\partial}{\partial y_{ii}} - y_{hj}\frac{\partial}{\partial y_{ij}} - y_{ih,k}\frac{\partial}{\partial y_{ii,k}} - y_{hj,k}\frac{\partial}{\partial y_{ij,k}} - \sum_{s \le j} y_{sj,h}\frac{\partial}{\partial y_{sj,i}} \\ &\quad + A^i_{jk}\frac{\partial}{\partial A^h_{jk}} - A^r_{jh}\frac{\partial}{\partial A^r_{ji}} - A^r_{hk}\frac{\partial}{\partial A^r_{ik}} \\ &\quad + A^i_{jk,s}\frac{\partial}{\partial A^h_{jk,s}} - A^s_{jh,r}\frac{\partial}{\partial A^s_{ji,r}} - A^s_{hk,r}\frac{\partial}{\partial A^s_{ik,r}} - A^r_{jk,h}\frac{\partial}{\partial A^r_{jk,i}}, \ \forall h, i, \end{aligned}$$

$$\begin{aligned}
X_h^{ik} &= -y_{ih}\frac{\partial}{\partial y_{ii,k}} - y_{kh}\frac{\partial}{\partial y_{kk,i}} - y_{hj}\frac{\partial}{\partial y_{ij,k}} - y_{hj}\frac{\partial}{\partial y_{kj,i}} - \frac{\partial}{\partial A^h_{ik}} - \frac{\partial}{\partial A^h_{ki}} \\
&\quad + A^k_{js}\frac{\partial}{\partial A^h_{js,i}} - A^s_{jh}\frac{\partial}{\partial A^s_{jk,i}} - A^s_{hr}\frac{\partial}{\partial A^s_{kr,i}} \\
&\quad + A^i_{js}\frac{\partial}{\partial A^h_{js,k}} - A^s_{jh}\frac{\partial}{\partial A^s_{ji,k}} - A^s_{hr}\frac{\partial}{\partial A^s_{ir,k}}, \quad \forall h,\ i \le k, \qquad (2.11)\\
X_i^{jkh} &= \frac{\partial}{\partial A^i_{jk,h}} + \frac{\partial}{\partial A^i_{jh,k}} + \frac{\partial}{\partial A^i_{hk,j}} + \frac{\partial}{\partial A^i_{hj,k}} + \frac{\partial}{\partial A^i_{kj,h}} + \frac{\partial}{\partial A^i_{kh,j}}, \\
&\quad \forall i,\ h \le j \le k. \qquad (2.12)
\end{aligned}$$

Moreover, the vector fields $X^i, X_h^i, X_h^{ik}, X_i^{jkh}$ are linearly independent and they span an involutive distribution on $J^1(M \times_N C)$ of rank $n\binom{n+3}{3}$. Hence, the number of functionally invariant Lagrangians on $J^1(M \times_N C)$ is

$$\frac{1}{6}\left(5n^4 + 3n^3 - 5n^2 + 3n\right).$$

Proof. According to the formula (2.7), $\mathcal{L}$ is invariant if and only if,

$$\begin{aligned}
&u^i\frac{\partial \mathcal{L}}{\partial x^i} + \sum_{i\le j} v_{ij}\frac{\partial \mathcal{L}}{\partial y_{ij}} + \sum_{i\le j} v_{ijk}\frac{\partial \mathcal{L}}{\partial y_{ij,k}} + w^i_{jk}\frac{\partial \mathcal{L}}{\partial A^i_{jk}} + w^i_{jkh}\frac{\partial \mathcal{L}}{\partial A^i_{jk,h}} = 0,\\
&\forall u^i \in C^\infty(N),
\end{aligned}$$

and expanding on this equation by using the formulas (2.8), (2.9), (2.5) and (2.6), we obtain

$$\begin{aligned}
0 &= u^i\frac{\partial \mathcal{L}}{\partial x^i} \\
&\quad + \frac{\partial u^h}{\partial x^i}\Bigg(-y_{hi}\frac{\partial \mathcal{L}}{\partial y_{ii}} - y_{hj}\frac{\partial \mathcal{L}}{\partial y_{ij}} - y_{ih,k}\frac{\partial \mathcal{L}}{\partial y_{ii,k}} - y_{hj,k}\frac{\partial \mathcal{L}}{\partial y_{ij,k}} \\
&\quad - \sum_{s\le j} y_{sj,h}\frac{\partial \mathcal{L}}{\partial y_{sj,i}} + A^i_{jk}\frac{\partial \mathcal{L}}{\partial A^h_{jk}} - A^r_{jh}\frac{\partial \mathcal{L}}{\partial A^r_{ji}} - A^r_{hk}\frac{\partial \mathcal{L}}{\partial A^r_{ik}} \\
&\quad + A^i_{jk,s}\frac{\partial \mathcal{L}}{\partial A^h_{jk,s}} - A^s_{jh,r}\frac{\partial \mathcal{L}}{\partial A^s_{ji,r}} - A^s_{hk,r}\frac{\partial \mathcal{L}}{\partial A^s_{ik,r}} - A^r_{jk,h}\frac{\partial \mathcal{L}}{\partial A^r_{jk,i}}\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\partial^2 u^h}{\partial x^i \partial x^k}\left(-y_{ih}\frac{\partial \mathcal{L}}{\partial y_{ii,k}} - y_{hj}\frac{\partial \mathcal{L}}{\partial y_{ij,k}} - \frac{\partial \mathcal{L}}{\partial A^h_{ik}}\right.\\
&\left.+ A^k_{js}\frac{\partial \mathcal{L}}{\partial A^h_{js,i}} - A^s_{jh}\frac{\partial \mathcal{L}}{\partial A^s_{jk,i}} - A^r_{hr}\frac{\partial \mathcal{L}}{\partial A^r_{kr,i}}\right)\\
&- \frac{\partial^3 u^i}{\partial x^h \partial x^k \partial x^j}\frac{\partial \mathcal{L}}{\partial A^i_{jk,h}}.
\end{aligned}
$$

This equation is equivalent to the system of the statement as the values for u^h, $\partial u^h/\partial x^i$, $\partial^2 u^h/\partial x^i \partial x^j$ $(i \leq j)$, and $\partial^3 u^h/\partial x^i \partial x^j \partial x^k$ $(i \leq j \leq k)$ at a point $x \in N$ can be taken arbitrarily. Moreover, assume a linear combination holds

$$
\begin{aligned}
&\lambda_a X^a + \lambda^a_b X^b_a + \sum_{b\leq c} \lambda^a_{bc} X^{bc}_a + \sum_{b\leq c\leq d} \lambda^a_{bcd} X^{bcd}_a = 0,\\
&\lambda_a, \lambda^a_b, \lambda^a_{bc}, \lambda^a_{bcd} \in C^\infty(J^1(M \times_N C)).
\end{aligned}
\tag{2.13}
$$

By applying (2.13) to x^a (resp. y_{ab}) we obtain $\lambda_a = 0$ (resp. $\lambda^a_b = 0$); again by applying (2.13) to A^a_{bc}, $b \leq c$ (resp. A^a_{bc}, $c \leq b$) and taking the expressions of the vector fields (2.11) and (2.12) into account, we obtain $\lambda^a_{bc} = 0$, $b \leq c$ (resp. $\lambda^a_{bc} = 0$, $c \leq b$). Hence, (2.13) reads $\sum_{b\leq c\leq d} \lambda^a_{bcd} X^{bcd}_a = 0$, and by applying it to $A^a_{bc,d}$ and taking the expressions of the vector fields (2.12) into account, we finally obtain $\lambda^a_{bcd} = 0$. The distribution

$$
\mathcal{D}_{M\times_N C} = \left\{ \bar{X}^{(1)}_{(j^1_x g, j^1_x s_\Gamma)} : X \in \mathfrak{X}(N), \left(j^1_x g, j^1_x s_\Gamma\right) \in J^1(M \times_N C) \right\}
$$

in $T\left(J^1(M \times_N C)\right)$, where $\bar{X}^{(1)}$ is defined in (2.7), is involutive as

$$
\left[\bar{X}^{(1)}, \bar{Y}^{(1)}\right] = \overline{[X, Y]}^{(1)}, \quad \forall X, Y \in \mathfrak{X}(N),
$$

and it is spanned by $X^i, X^i_h, X^{ik}_h, X^{jkh}_i$, as proved by the formulas above. The rest of the statement follows from the following identities:

$$
\begin{aligned}
&\#\{X^i; X^i_h; X^{ik}_h, i \leq k; X^{jkh}_i, h \leq j \leq k : h, i, j, k = 1, \ldots, n\}\\
&\qquad = n + n^2 + n\tbinom{n+1}{2} + n\tbinom{n+2}{3} = n\tbinom{n+3}{3},\\
&\dim J^1\left(M \times_N C\right) - n\tbinom{n+3}{3} = \frac{1}{6}\left(5n^4 + 3n^3 - 5n^2 + 3n\right).
\end{aligned}
$$

□

3 Invariance of covariant Hamiltonians

3.1 Position of the problem

On the bundle $E = M \times_N C$, an Ehresmann connection can locally be written as follows:

$$\gamma = \sum_{i\leq j}\left(dy_{ij} + \gamma_{ijk}dx^k\right)\otimes\frac{\partial}{\partial y_{ij}} + \left(dA^i_{jk} + \gamma^i_{jkl}dx^l\right)\otimes\frac{\partial}{\partial A^i_{jk}},$$
$$\gamma_{ijk}, \gamma^i_{jkl} \in C^\infty(M \times_N C). \tag{3.1}$$

In particular, for a Lagrangian density Λ on $J^1(M \times_N C)$, we obtain

$$\Lambda^\gamma = \left(\sum_{i\leq j}\left(\gamma_{ijk} + y_{ij,k}\right)\frac{\partial L}{\partial y_{ij,k}} + \left(\gamma^i_{jkl} + A^i_{jk,l}\right)\frac{\partial L}{\partial A^i_{jk,l}} - L\right)$$
$$\times dx^1 \wedge \cdots \wedge dx^n,$$

or equivalently, $\mathcal{L}^\gamma = D^\gamma(\mathcal{L}) - \mathcal{L}$, where

$$D^\gamma = \sum_{i\leq j}\left(\gamma_{ijk} + y_{ij,k}\right)\frac{\partial}{\partial y_{ij,k}} + \left(\gamma^i_{jkl} + A^i_{jk,l}\right)\frac{\partial}{\partial A^i_{jk,l}}.$$

Remark 3.1. The horizontal form $(p^1_0)^*\gamma - \theta = (\gamma^\alpha_i + y^\alpha_i)\, dx^i \otimes \partial/\partial y^\alpha$ can also be viewed as the p^1_0-vertical vector field

$$D^\gamma = (\gamma^\alpha_i + y^\alpha_i)\,\frac{\partial}{\partial y^\alpha_i}, \tag{3.2}$$

taking the natural isomorphism $V(p^1_0) \cong (p^1_0)^*(p^*T^*N \otimes V(p))$ into account (cf. [23, 24, 32, 34]).

According to the previous formulas, this means: if the system (2.10) holds for a Lagrangian function $\mathcal{L}$, then it also holds for the covariant Hamiltonian $\mathcal{L}^\gamma$.

If $X \in \{X^i, X^i_h, X^{ik}_h, X^{jkh}_i\}$, then $X(\mathcal{L}^\gamma) = X(D^\gamma(\mathcal{L}))$, as $\mathcal{L}$ is assumed to be invariant and hence $X(\mathcal{L}) = 0$. Therefore

$$\begin{aligned} X(\mathcal{L}^\gamma) &= X(D^\gamma(\mathcal{L})) \\ &= [X, D^\gamma](\mathcal{L}), \end{aligned}$$

and we conclude the following:

Proposition 3.2. *The property **(P)** holds for an Ehresmann connection γ on $M \times_N C$ if and only if the vector field D^γ transforms the sections of the distribution $\mathcal{D}_{M\times_N C}$ into themselves, namely, $[D^\gamma, \Gamma(\mathcal{D}_{M\times_N C})] \subseteq \Gamma(\mathcal{D}_{M\times_N C})$.*

The problem thus reduces to compute the brackets $\left[X^i, D^\gamma\right]$, $\left[X^i_h, D^\gamma\right]$, $\left[X^{ik}_h, D^\gamma\right]$, and $[X^{jkh}_i, D^\gamma]$. We have

$$\left[X^h, D^\gamma\right] = \sum_{i\le j} \frac{\partial \gamma_{ijk}}{\partial x^h}\frac{\partial}{\partial y_{ij,k}} + \frac{\partial \gamma^i_{jkl}}{\partial x^h}\frac{\partial}{\partial A^i_{jk,l}}, \tag{3.3}$$

$$\left[X^{cda}_b, D^\gamma\right] = X^{cda}_b, \quad \forall b, c \le d \le a,$$

$$\begin{aligned}\left[X^i_h, D^\gamma\right] &= \sum_{a\le b} Y^i_h\left(\gamma_{abk}\right)\frac{\partial}{\partial y_{ab,k}} + \sum_{i\le h}\gamma_{ihk}\frac{\partial}{\partial y_{ii,k}} + \sum_{h<i}\gamma_{hik}\frac{\partial}{\partial y_{ii,k}} \\ &+ \sum_{h\le j}\gamma_{hjk}\frac{\partial}{\partial y_{ij,k}} + \sum_{j<h}\gamma_{jhk}\frac{\partial}{\partial y_{ij,k}} + \sum_{a\le b}\gamma_{abh}\frac{\partial}{\partial y_{ab,i}} \\ &+ \left(Y^i_h\left(\gamma^a_{bcr}\right) - \delta^h_a\gamma^i_{bcr} + \delta^c_i\gamma^a_{bhr} + \delta^b_i\gamma^a_{hcr} + \delta^r_i\gamma^a_{bch}\right)\frac{\partial}{\partial A^a_{bc,r}},\end{aligned} \tag{3.4}$$

$$\left[X^{ik}_h, D^\gamma\right] = \sum_{a\le b} Y^{ik}_h\left(\gamma_{abc}\right)\frac{\partial}{\partial y_{ab,c}} + Y^{ik}_h\left(\gamma^d_{abc}\right)\frac{\partial}{\partial A^d_{ab,c}} + X^{ik}_h - Y^{ik}_h, \tag{3.5}$$

where

$$\begin{aligned}Y^i_h &= -y_{hi}\frac{\partial}{\partial y_{ii}} - y_{hj}\frac{\partial}{\partial y_{ij}} + A^i_{jk}\frac{\partial}{\partial A^h_{jk}} - A^r_{jh}\frac{\partial}{\partial A^r_{ji}} - A^r_{hk}\frac{\partial}{\partial A^r_{ik}}, \\ Y^{ik}_h &= -\frac{\partial}{\partial A^h_{ik}} - \frac{\partial}{\partial A^h_{ki}},\end{aligned}$$

and the following formula has been used:

$$\frac{\partial y_{rs,k}}{\partial y_{ij,h}} = \delta^k_h\left(\delta^r_i\delta^s_j + \delta^r_j\delta^s_i - \delta^i_j\delta^i_r\delta^j_s\right).$$

3.2 The class of the Ehresmann connections defined

Let $p\colon M\times_N C \to N$, $\mathrm{pr}_1\colon M\times_N C\to M$, $\mathrm{pr}_2\colon M\times_N C\to C$ be the natural projections. By taking the differential of pr_1 and pr_2, a natural identification is obtained $T(M \times_N C) = TM \times_{TN} TC$. Hence

$$\begin{aligned} V(p) &= V(p_M) \times_N V(p_C) \\ &= \mathrm{pr}_1^* V(p_M) \oplus \mathrm{pr}_2^* V(p_C) \end{aligned}$$

and two unique vector-bundle homomorphisms exist

$$\gamma_M \colon \mathrm{pr}_1^* TM \to \mathrm{pr}_1^* V(p_M), \quad \gamma_C \colon \mathrm{pr}_2^* TC \to \mathrm{pr}_2^* V(p_C),$$

such that,

$$\begin{aligned} \gamma(X) &= \left(\gamma_M\left(\mathrm{pr}_{1*}X\right), \gamma_C\left(\mathrm{pr}_{2*}X\right)\right), \quad \forall X \in T(M \times_N C), \\ \gamma_M(Y) &= Y, \quad \forall Y \in \mathrm{pr}_1^* V(p_M), \\ \gamma_C(Z) &= Z, \quad \forall Z \in \mathrm{pr}_2^* V(p_C). \end{aligned}$$

If γ is given by the local expression of formula (3.1), then

$$\gamma_M = \sum_{i \le j} \left(dy_{ij} + \gamma_{ijk} dx^k\right) \otimes \frac{\partial}{\partial y_{ij}}, \quad \gamma_C = \left(dA^i_{jk} + \gamma^i_{jkl} dx^l\right) \otimes \frac{\partial}{\partial A^i_{jk}},$$
$$\gamma_{ijk}, \gamma^i_{jkl} \in C^\infty(M \times_N C).$$

3.2.1 The first geometric condition on γ

Let $q \colon F(N) \to M$ be the projection given by

$$\begin{aligned} q(X_1, \ldots, X_n) &= g_x \\ &= \varepsilon_h w^h \otimes w^h, \end{aligned} \tag{3.6}$$

where $(w^1, \ldots, w^n)$ is the dual coframe of $(X_1, \ldots, X_n) \in F_x(N)$, i.e., g_x is the metric for which $(X_1, \ldots, X_n)$ is a g_x-orthonormal basis and $\varepsilon_h = 1$ for $1 \le h \le n^+$, $\varepsilon_h = -1$ for $n^+ + 1 \le h \le n$. As readily seen, q is a principal G-bundle with $G = O(n^+, n^-)$.

Given a linear connection Γ and a tangent vector $X \in T_x N$, for every u in $p^{-1}(x)$ there exists a unique Γ-horizontal tangent vector $X_u^{h_\Gamma} \in T_u(FN)$ such that, $(p_F)_* X_u^{h_\Gamma} = X$. The local expression for the horizontal lift is known to be ([20, Chapter III, Proposition 7.4]),

$$\left(\frac{\partial}{\partial x^j}\right)^{h_\Gamma} = \frac{\partial}{\partial x^j} - \Gamma^i_{jk} x^k_l \frac{\partial}{\partial x^i_l}. \tag{3.7}$$

Lemma 3.3. *Given a metric $g_x \in p_M^{-1}(x)$, let $u \in p_F^{-1}(x)$ be a linear frame such that $q(u) = g_x$. The projection $q_*(X_u^{h_{\Gamma_x}})$ does not depend on the linear frame u chosen over g_x.*

Proof. In fact, any other linear frame projecting onto g_x can be written as $u \cdot A$, $A \in G$. As the horizontal distribution is invariant under right translations (see [20, II, Proposition 1.2]), the following equation holds: $(R_A)_* \left(X_u^{h_\Gamma}\right) = X_{u\cdot A}^{h_\Gamma}$. Hence

$$\begin{aligned} q_* \left(X_{u\cdot A}^{h_\Gamma}\right) &= q_* \left((R_A)_* \left(X_u^{h_\Gamma}\right)\right) \\ &= (q \circ R_A)_* \left(X_u^{h_\Gamma}\right) \\ &= q_* \left(X_u^{h_\Gamma}\right). \end{aligned}$$

□

Proposition 3.4. *An Ehresmann connection γ on $M \times_N C$ satisfies the following condition:*

$$(C_M)\text{:}\ \gamma_M\left((g_x, \Gamma_x), X\right) = X - q_*\left(\left((p_M)_*(X)\right)_u^{h_{\Gamma_x}}\right),$$

$\forall X \in T_{g_x}M$, $u \in q^{-1}(g_x)$, (which does not depend on the linear frame $u \in q^{-1}(g_x)$ chosen, according to Lemma 3.3*) if and only if the following equations hold:*

$$\gamma_{klj} = -\left(y_{al}A_{jk}^a + y_{ak}A_{jl}^a\right), \tag{3.8}$$

where the functions γ_{klj} (resp. y_{ij}, resp. A_{jk}^i) are defined in the formula (3.1) *(resp.* (2.2)*, resp.* (2.4)*).*

Proof. Letting $(\chi_j^i)_{i,j=1}^n = \left((x_j^i)_{i,j=1}^n\right)^{-1}$, the dual coframe of the linear frame $u = (X_1, \ldots, X_n) \in F_x(N)$ given in (2.3) is $(w^1, \ldots, w^n)$, $w^h = \chi_k^h(u) \left(dx^k\right)_x$, $1 \le h \le n$, and the projection q is given by

$$\begin{aligned} q(u) &= g_x \\ &= \sum\nolimits_{h=1}^n \varepsilon_h \chi_k^h(u)\chi_l^h(u) \left(dx^k\right)_x \otimes \left(dx^l\right)_x. \end{aligned}$$

Therefore the equations of the projection (3.6) are as follows:

$$x^i \circ q = x^i,$$
$$y_{kl} \circ q = \sum\nolimits_{h=1}^{n} \varepsilon_h \chi_k^h \chi_l^h.$$

Hence

$$q_* \left(\frac{\partial}{\partial x_b^a} \right)_u = \sum_{k \le l} \varepsilon_h \left\{ \frac{\partial \chi_k^h}{\partial x_b^a} \chi_l^h + \chi_k^h \frac{\partial \chi_l^h}{\partial x_b^a} \right\} (u) \left(\frac{\partial}{\partial y_{kl}} \right)_{g_x}.$$

Taking derivatives with respect to x_b^a on the identity $\chi_r^h x_i^r = \delta_i^h$, multiplying the outcome by χ_k^i, and summing up over the index i, the following formula is obtained: $\partial \chi_k^h / \partial x_b^a = -\chi_a^h \chi_k^b$. Replacing this equation into the expression for $q_* \left(\partial / \partial x_b^a \right)_u$ above, we have

$$q_* \left(\frac{\partial}{\partial x_b^a} \right)_u = -\sum_{k \le l} \left\{ \chi_k^b(u) y_{al}\left(g_x\right) + \chi_l^b(u) y_{ak}\left(g_x\right) \right\} \left(\frac{\partial}{\partial y_{kl}} \right)_{g_x}.$$

From (3.7), evaluated at $u \in q^{-1}(g_x)$, we deduce

$$\begin{aligned} q_* \left(\frac{\partial}{\partial x^j} \right)_u^{h_\Gamma} &= \left(\frac{\partial}{\partial x^j} \right)_{g_x} - \Gamma_{jc}^a(x) x_b^c(u) q_* \left(\frac{\partial}{\partial x_b^a} \right)_{g_x} \\ &= \left(\frac{\partial}{\partial x^j} \right)_{g_x} \\ &\quad + \sum\nolimits_{k \le l} \Gamma_{jc}^a(x) x_b^c(u) \left\{ \chi_k^b(u) y_{al}\left(g_x\right) + \chi_l^b(u) y_{ak}\left(g_x\right) \right\} \\ &\quad \times \left(\frac{\partial}{\partial y_{kl}} \right)_{g_x} \\ &= \left(\frac{\partial}{\partial x^j} \right)_{g_x} + \sum_{k \le l} \left\{ \Gamma_{jk}^a(x) y_{al}\left(g_x\right) + \Gamma_{jl}^a(x) y_{ak}\left(g_x\right) \right\} \left(\frac{\partial}{\partial y_{kl}} \right)_{g_x}. \end{aligned}$$

The condition (C_M) holds automatically whenever $X \in V(p_M)$. Hence, (C_M) holds if and only if it holds for $X = (\partial / \partial x^j)_{g_x}$, namely,

$$\begin{aligned} \sum_{k \le l} \gamma_{klj}(g_x, \Gamma_x) \left(\frac{\partial}{\partial y_{kl}} \right)_{g_x} &= \gamma_M \left((g_x, \Gamma_x), \left(\frac{\partial}{\partial x^j} \right)_{g_x} \right) \\ &= \left(\frac{\partial}{\partial x^j} \right)_{g_x} - q_* \left(\frac{\partial}{\partial x^j} \right)_u^{h_{\Gamma_x}} \end{aligned}$$

$$= -\sum_{k\leq l}\left\{\Gamma^a_{jk}(x)y_{al}\left(g_x\right)+\Gamma^a_{jl}(x)y_{ak}\left(g_x\right)\right\} \times\left(\frac{\partial}{\partial y_{kl}}\right)_{g_x},$$

thus proving formula (3.8) in the statement. □

3.2.2 The canonical covariant derivative

As is known (e.g., see [20, III, Section 1], [23, pp. 157–158]) every connection Γ on a principal G-bundle $P\to N$ induces a covariant derivative ∇^Γ on the vector bundle associated to P under a linear representation $\rho\colon G\to Gl(m,\mathbb{R})$ with standard fibre $\mathbb{R}^m$. In particular, this applies to the principal bundle of linear frames, thus proving that every linear connection Γ on N induces a covariant derivative ∇^Γ on every tensorial vector bundle $E\to N$.

The bundles $(p_C)^*E$, where E is a tensorial vector bundle, are endowed with a canonical covariant derivative ∇^E completely determined by the formula:

$$\left(\left(\nabla^E\right)_X(f\xi)\right)(\Gamma_x)=\left((Xf)\,\xi\right)(\Gamma_x)+f\left(\Gamma_x\right)\left(\nabla^{\Gamma_x}_{(p_C)_*X}\xi\right)(x), \tag{3.9}$$

for all $X\in T_{\Gamma_x}C$, $f\in C^\infty(C)$, and every local section ξ of E defined on a neighbourhood of x. The uniqueness of ∇^E follows from (3.9) as the sections of E span the sections of $(p_C)^*E$ over $C^\infty(C)$, see [8, 0.3.6]. Below, we are specially concerned with the cases $E=TN$ and $E=\wedge^2T^*N\otimes TN$.

3.2.3 The 2-form associated with γ_C

As $p_C\colon C\to N$ is an affine bundle modelled over $\otimes^2T^*N\otimes TN$, there is a natural identification

$$V(p_C)\cong(p_C)^*\left(\otimes^2T^*N\otimes TN\right)$$

and consequently, an Ehresmann connection γ_C on C can also be viewed as a homomorphism $\gamma_C\colon TC\to\otimes^2T^*N\otimes TN$. If γ_C is locally given by

$$\gamma_C=\left(dA^i_{jk}+\gamma^i_{jkl}dx^l\right)\otimes\frac{\partial}{\partial A^i_{jk}},\quad \gamma^i_{jkl}\in C^\infty(C), \tag{3.10}$$

then

$$\gamma_C=(dA^i_{jk}+\gamma^i_{jkl}dx^l)\otimes dx^j\otimes dx^k\otimes\frac{\partial}{\partial x^i},$$

and γ_C induces a 2-form $\tilde{\gamma}_C$ taking values in $(p_C)^*(T^*N \otimes TN)$ as follows:

$$\tilde{\gamma}_C(X,Y) = c_1^1\left((p_C)_*(Y) \otimes \gamma_C(X)\right) - c_1^1\left((p_C)_*(X) \otimes \gamma_C(Y)\right),$$
$$\forall X, Y \in T_{\Gamma_x}C,$$

where

$$c_1^1 \colon TN \otimes T^*N \otimes T^*N \otimes TN \to T^*N \otimes TN,$$
$$c_1^1(X_1 \otimes w_1 \otimes w_2 \otimes X_2) = w_1(X_1) w_2 \otimes X_2,$$
$$X_1, X_2 \in T_xN, \ w_1, w_2 \in T_x^*N.$$

If γ_C is given by (3.10), then from the very definition of $\tilde{\gamma}_C$ the following local expression is obtained:

$$\tilde{\gamma}_C = (dA^c_{lh} + (\gamma^c_{lha} - \gamma^c_{ahl})\, dx^a) \wedge dx^l \otimes dx^h \otimes \frac{\partial}{\partial x^c}.$$

3.2.4 The second geometric condition on γ

Let $\mathrm{alt}_{12} \colon \otimes^2 T^*N \otimes TN \to \wedge^2 T^*N \otimes TN$ be the operator alternating the two covariant arguments.

The vector bundle $(p_C)^*\left(\wedge^2 T^*N \otimes TN\right)$ admits a canonical section

$$\tau_N \colon C \to \wedge^2 T^*N \otimes TN,$$
$$\tau_N(\Gamma_x) = T^{\Gamma_x}, \quad \forall \Gamma_x \in C,$$

where T^{Γ_x} is the torsion of Γ_x. Locally,

$$\tau_N = \sum_{j<k} (A^i_{jk} - A^i_{kj}) dx^j \wedge dx^k \otimes \frac{\partial}{\partial x^i}.$$

From the previous formulas the next result follows:

Proposition 3.5. *Let γ be an Ehresmann connection on $M \times_N C$, let $\nabla^{(1)} = \nabla^{E_1}$ with $E_1 = TN$, let $R^{\nabla^{(1)}}$ be its curvature form, and finally, let $\nabla^{(2)} = \nabla^{E_2}$ with $E_2 = \wedge^2 T^*N \otimes TN$.*

(C_C) *Assume the component γ_C of γ is defined on C. Then, the equations*

$$\tilde{\gamma}_C = R^{\nabla^{(1)}}, \tag{3.11}$$

$$\mathrm{alt}_{12} \circ \gamma_C = \nabla^{(2)} \tau_N, \tag{3.12}$$

are locally equivalent to the following ones:

$$\gamma^h_{str} - \gamma^h_{rts} = A^h_{rm} A^m_{st} - A^h_{sm} A^m_{rt}, \tag{3.13}$$

$$\begin{aligned} \gamma^h_{rst} - \gamma^h_{srt} &= A^h_{tm}\left(A^m_{rs} - A^m_{sr}\right) + A^m_{ts}\left(A^h_{mr} - A^h_{rm}\right) \\ &\quad + A^m_{tr}\left(A^h_{sm} - A^h_{ms}\right). \end{aligned} \tag{3.14}$$

3.3 Solution to the problem (P)

Theorem 3.6. *If the connection γ on $M \times_N C$ satisfies the conditions (C_M) and (C_C) introduced above, then the vector field D^γ satisfies the property stated in Proposition 3.2 and, accordingly the covariant Hamiltonian with respect to γ of every $\mathfrak{X}(N)$-invariant Lagrangian is also $\mathfrak{X}(N)$-invariant.*

Proof. When γ_M satisfies the condition (C_M) the brackets (3.3), (3.4), and (3.5) are respectively given by

$$\left[X^h, D^\gamma\right] = \frac{\partial \gamma^i_{jkl}}{\partial x^h} \frac{\partial}{\partial A^i_{jk,l}}, \tag{3.15}$$

$$\left[X^i_h, D^\gamma\right] = \left(Y^i_h\left(\gamma^a_{bcr}\right) - \delta^h_a \gamma^i_{bcr} + \delta^c_i \gamma^a_{bhr} + \delta^b_i \gamma^a_{hcr} + \delta^r_i \gamma^a_{bch}\right) \frac{\partial}{\partial A^a_{bc,r}}, \tag{3.16}$$

$$\begin{aligned} \left[X^{ik}_h, D^\gamma\right] = \Bigg(&-\frac{\partial \gamma^d_{abc}}{\partial A^h_{ik}} + \delta^c_i \left(\delta^h_d A^k_{ab} - \delta^k_b A^d_{ah} - \delta^k_a A^d_{hb}\right) \\ &- \frac{\partial \gamma^d_{abc}}{\partial A^h_{ki}} + \delta^c_k \left(\delta^h_d A^i_{ab} - \delta^i_b A^d_{ah} - \delta^i_a A^d_{hb}\right)\Bigg) \frac{\partial}{\partial A^d_{ab,c}}. \end{aligned}$$

In addition, if γ_C satisfies the condition (C_C), then taking derivatives with respect to x^h in (3.13) and (3.14), we obtain

$$\frac{\partial \gamma^i_{klj}}{\partial x^h} = \frac{\partial \gamma^i_{jlk}}{\partial x^h}, \quad \frac{\partial \gamma^i_{jkl}}{\partial x^h} = \frac{\partial \gamma^i_{kjl}}{\partial x^h},$$

and renaming indices we deduce

$$\frac{\partial \gamma^i_{jjk}}{\partial x^h} = \frac{\partial \gamma^i_{jkj}}{\partial x^h} = \frac{\partial \gamma^i_{kjj}}{\partial x^h} \ (j < k),$$
$$\frac{\partial \gamma^i_{kkj}}{\partial x^h} = \frac{\partial \gamma^i_{kjk}}{\partial x^h} = \frac{\partial \gamma^i_{jkk}}{\partial x^h} \ (j < k),$$
$$\frac{\partial \gamma^i_{jkl}}{\partial x^h} = \frac{\partial \gamma^i_{klj}}{\partial x^h} = \frac{\partial \gamma^i_{ljk}}{\partial x^h} = \frac{\partial \gamma^i_{kjl}}{\partial x^h} = \frac{\partial \gamma^i_{lkj}}{\partial x^h} = \frac{\partial \gamma^i_{jlk}}{\partial x^h} \ (j < k < l).$$

From (3.15) we obtain

$$\left[X^h, D^\gamma\right] = \sum_{j<k<l} \frac{\partial \gamma^i_{jkl}}{\partial x^h} X^{jkl}_i + \frac{1}{2} \sum_{j<k} \frac{\partial \gamma^i_{jjk}}{\partial x^h} X^{jjk}_i + \frac{1}{2} \sum_{j<k} \frac{\partial \gamma^i_{kkj}}{\partial x^h} X^{kkj}_i + \frac{1}{6} \frac{\partial \gamma^i_{jjj}}{\partial x^h} X^{jjj}_i,$$

and consequently the values of $\left[X^h, D^\gamma\right]$ belong to the distribution $\mathcal{D}_{M\times_N C}$.

Moreover, as γ_C is assumed to be defined on C, we have

$$Y^i_h \left(\gamma^a_{bcr}\right) = \left(\delta^s_h A^i_{jk} - \delta^i_k A^s_{jh} - \delta^i_j A^s_{hk}\right) \frac{\partial \gamma^a_{bcr}}{\partial A^s_{jk}}.$$

For the sake of simplicity, below we set

$$\left(T^i_h\right)^a_{bcr} = A^i_{jk} \frac{\partial \gamma^a_{bcr}}{\partial A^h_{jk}} - A^s_{jh} \frac{\partial \gamma^a_{bcr}}{\partial A^s_{ji}} - A^s_{hk} \frac{\partial \gamma^a_{bcr}}{\partial A^s_{ik}} - \delta^h_a \gamma^i_{bcr} + \delta^b_i \gamma^a_{hcr} + \delta^c_i \gamma^a_{bhr} + \delta^r_i \gamma^a_{bch}.$$

Taking derivatives with respect to A^s_{jk}, equations (3.13) and (3.14) yield

$$\frac{\partial \gamma^a_{bcr}}{\partial A^s_{jk}} - \frac{\partial \gamma^a_{rcb}}{\partial A^s_{jk}} = \delta^j_r \delta^a_s A^k_{bc} - \delta^j_b \delta^a_s A^k_{rc} + \delta^j_b \delta^k_C A^a_{rs} - \delta^j_r \delta^k_C A^a_{bs},$$
$$\frac{\partial \gamma^a_{rbc}}{\partial A^s_{jk}} - \frac{\partial \gamma^a_{brc}}{\partial A^s_{jk}} = \delta^j_c \delta^a_s A^k_{rb} - \delta^a_s \delta^j_c A^k_{br} - \delta^a_s \delta^k_b A^j_{cr} - \delta^a_s \delta^j_r A^k_{cb} + \delta^a_s \delta^k_r A^j_{cb} + \delta^a_s \delta^j_b A^k_{cr}$$
$$+ \delta^j_c \delta^k_b A^a_{sr} - \delta^j_c \delta^k_r A^a_{sb} + \delta^j_r \delta^k_b A^a_{cs} - \delta^j_b \delta^k_r A^a_{cs} + \delta^j_c \delta^k_r A^a_{bs}$$
$$- \delta^j_c \delta^k_b A^a_{rs}.$$

From these expressions, the following symmetries of indices are obtained:

$$\begin{aligned}
&\left(T_h^i\right)_{bbc}^a = \left(T_h^i\right)_{bcb}^a = \left(T_h^i\right)_{cbb}^a \ (b<c),\\
&\left(T_h^i\right)_{bcc}^a = \left(T_h^i\right)_{cbc}^a = \left(T_h^i\right)_{ccb}^a \ (b<c),\\
&\left(T_h^i\right)_{bcd}^a = \left(T_h^i\right)_{dbc}^a = \left(T_h^i\right)_{cdb}^a = \left(T_h^i\right)_{bdc}^a = \left(T_h^i\right)_{dcb}^a = \left(T_h^i\right)_{cbd}^a \ (b<c<d),
\end{aligned}$$

and from (3.16) we obtain

$$\begin{aligned}
\left[X_h^i, D^\gamma\right] = &\sum_{b<c<d} \left(T_h^i\right)_{bcd}^a X_a^{bcd} + \frac{1}{2}\sum_{b<c} \left(T_h^i\right)_{bbc}^a X_a^{bbc}\\
&+ \frac{1}{2}\sum_{b<c} \left(T_h^i\right)_{ccb}^a X_a^{ccb} + \frac{1}{6}\left(T_h^i\right)_{bbb}^a X_a^{bbb}.
\end{aligned}$$

Hence $\left[X_h^i, D^\gamma\right]$ also takes values into the distribution $\mathcal{D}_{M\times_N C}$.

The proof for the third bracket is similar to the previous two cases but longer. Letting

$$\begin{aligned}
\left(T_h^{ik}\right)_{rbc}^a = &-\frac{\partial \gamma_{rbc}^a}{\partial A_{ik}^h} - \frac{\partial \gamma_{rbc}^a}{\partial A_{ki}^h} + \delta_i^c\left(\delta_a^h A_{rb}^k - \delta_b^k A_{rh}^a - \delta_r^k A_{hb}^a\right)\\
&+ \delta_k^c\left(\delta_a^h A_{rb}^i - \delta_b^i A_{rh}^a - \delta_r^i A_{hb}^a\right),
\end{aligned}$$

the following symmetries are obtained:

$$\begin{aligned}
&\left(T_h^{ik}\right)_{bbc}^a = \left(T_h^{ik}\right)_{bcb}^a = \left(T_h^{ik}\right)_{cbb}^a \ (b<c),\\
&\left(T_h^{ik}\right)_{bcc}^a = \left(T_h^{ik}\right)_{cbc}^a = \left(T_h^{ik}\right)_{ccb}^a \ (b<c),\\
&\left(T_h^{ik}\right)_{bcd}^a = \left(T_h^{ik}\right)_{dbc}^a = \left(T_h^{ik}\right)_{cdb}^a = \left(T_h^{ik}\right)_{bdc}^a = \left(T_h^{ik}\right)_{dcb}^a = \left(T_h^{ik}\right)_{cbd}^a\\
&\qquad (b<c<d).
\end{aligned}$$

Hence

$$\begin{aligned}
\left[X_h^{ik}, D^\gamma\right] = &\sum_{b<c<d} \left(T_h^{ik}\right)_{bcd}^a X_a^{bcd} + \frac{1}{2}\sum_{b<c} \left(T_h^{ik}\right)_{bbc}^a X_a^{bbc}\\
&+ \frac{1}{2}\sum_{b<c} \left(T_h^{ik}\right)_{ccb}^a X_a^{ccb} + \frac{1}{6}\left(T_h^{ik}\right)_{bbb}^a X_a^{bbb},
\end{aligned}$$

and the proof is complete. □

Theorem 3.7. *The Ehresmann connections on C satisfying equations* (3.11) *and* (3.12) *are the sections of an affine bundle over C modelled over the vector bundle $(p_C)^*\left(S^3T^*N \otimes TN\right)$. Consequently, there always exist Ehresmann connections on $M \times_N C$ fulfilling the conditions (C_M) and (C_C) introduced above.*

Proof. If two Ehresmann connections γ_C, γ'_C satisfy equations (3.11) and (3.12), then the difference tensor field $t = \gamma'_C - \gamma_C$, which is a section of the bundle $(p_C)^*\left(\otimes^3T^*N \otimes TN\right)$, satisfies the following symmetries:

$$t(X_1, X_2, X_3) = t(X_3, X_2, X_1), \tag{3.17}$$

$$t(X_1, X_2, X_3) = t(X_2, X_1, X_3), \tag{3.18}$$

according to (3.13) and (3.14), respectively, for all $X_1, X_2, X_3 \in T_xN$, $\Gamma_x \in C_x(N)$. Hence

$$t(X_1, X_3, X_2) \overset{(3.17)}{=} t(X_2, X_3, X_1) \overset{(3.18)}{=} t(X_3, X_2, X_1) \overset{(3.17)}{=} t(X_1, X_2, X_3),$$

thus proving that t is totally symmetric. The second part of the statement thus follows from the fact that an affine bundle always admits global sections, e.g., see [20, I, Theorem 5.7]. □

Remark 3.8. The results obtained above also hold if the bundle of linear connections is replaced by the subbundle $C^{\mathrm{sym}} = C^{\mathrm{sym}}(N) \subset C$ of symmetric linear connections; the only difference to be observed between both bundles is that in the symmetric cases equation (3.12), or equivalently (3.14), holds automatically.

4 The second-order formalism

In this section we consider the problem of invariance of covariant Hamiltonians for second-order Lagrangians defined on the bundle of metrics, i.e., for functions $\mathcal{L} \in C^\infty(J^2M)$, where M denotes, as throughout this paper, the bundle of pseudo-Riemannian metrics of a given signature (n^+, n^-) on N.

4.1 Second-order Ehresmann connections

A second-order Ehresmann connection on $p\colon E \to N$ is a differential 1-form γ^2 on J^1E taking values in the vertical sub-bundle $V(p^1)$ such that $\gamma^2(X) = X$ for every $X \in V(p^1)$. (We refer the reader to [29] for the basics on Ehresmann connections of arbitrary order.) Once a connection γ^2 is given, we have a decomposition of vector bundles $T(J^1E) = V(p^1) \oplus \ker \gamma^2$, where $\ker \gamma^2$ is called the horizontal sub-bundle determined by γ^2. In the coordinate system on J^1E induced from a fibred coordinate system (x^j, y^α) for p, a connection form can be written as

$$\gamma^2 = (dy^\alpha + \gamma^\alpha_j dx^j) \otimes \frac{\partial}{\partial y^\alpha} + (dy^\alpha_i + \gamma^\alpha_{ij} dx^j) \otimes \frac{\partial}{\partial y^\alpha_i}, \quad \gamma^\alpha_j, \gamma^\alpha_{ij} \in C^\infty(J^1E). \tag{4.1}$$

As in the first-order case, the action of the group $\mathrm{Aut}(p)$ on the space of second-order connections is defined by the formula

$$\Phi \cdot \gamma^2 = (\Phi^{(1)})_* \circ \gamma^2 \circ (\Phi^{(1)})^{-1}_*, \quad \forall \Phi \in \mathrm{Aut}(p).$$

As $\Phi^{(1)} \colon J^1M \to J^1M$ is a morphism of fibred manifolds over N, $(\Phi^{(1)})_*$ transforms the vertical subbundle $V(p^1)$ into itself; hence the previous definition makes sense.

4.2 A remarkable isomorphism

Theorem 4.1. *Let Γ^g be the Levi–Civita connection of a pseudo-Riemannian metric g on N. The mapping $\zeta_N \colon J^1M \to M \times_N C^{\mathrm{sym}}$, $\zeta_N(j^1_x g) = (g_x, \Gamma^g_x)$ is a diffeomorphism. There is a natural one-to-one correspondence between first-order Ehresmann connections on the bundle $p \colon M \times_N C^{\mathrm{sym}} \to N$ and second-order Ehresmann connections on the bundle $p_M \colon M \to N$, which is explicitly given by,*

$$\gamma^2 = ((\zeta^v_N)_*)^{-1} \circ \gamma \circ (\zeta_N)_*, \tag{4.2}$$

where $\gamma \colon T(M \times_N C^{\mathrm{sym}}) \to V(p)$ is a first-order Ehresmann connection,

$$(\zeta_N)_* : T(J^1M) \to T(M \times_N C^{\mathrm{sym}})$$

is the Jacobian mapping induced by ζ_N, and $(\zeta^v_N)_* : V(p^1_M) \to V(p)$ is its restriction to the vertical bundles.

Proof. As a computation shows, the equations of ζ_N in the coordinate systems introduced in Section 2.1.2, are as follows:

$$\begin{aligned} x^i \circ \zeta_N &= x^i, \\ y_{ij} \circ \zeta_N &= y_{ij}, \\ A^h_{ij} \circ \zeta_N &= \frac{1}{2} y^{hk}(y_{ik,j} + y_{jk,i} - y_{ij,k}), \quad i \le j, \end{aligned} \tag{4.3}$$

where $(y^{ij})^n_{i,j=1}$ is the inverse mapping of the matrix $(y_{ij})^n_{i,j=1}$ and the functions y_{ij} are defined in (2.2). Hence

$$\begin{aligned} x^i \circ \zeta_N^{-1} &= x^i, \\ y_{ij} \circ \zeta_N^{-1} &= y_{ij}, \\ y_{ij,k} \circ \zeta_N^{-1} &= y_{hi} A^h_{jk} + y_{hj} A^h_{ik}, \quad i \le j. \end{aligned} \tag{4.4}$$

As the diffeomorphism ζ_N induces the identity on the ground manifold N, it follows that the definition of γ^2 in (4.2) makes sense and the following formulas are obtained:

$$\begin{aligned} \gamma^2\left(\frac{\partial}{\partial x^r}\right) &= \sum_{a \le b} (\gamma_{abr} \circ \zeta_N) \frac{\partial}{\partial y_{ab}} + \sum_{i \le j} \gamma_{ijkr} \frac{\partial}{\partial y_{ij,k}}, \\ \gamma_{ijkr} &= \frac{1}{2} \sum_{a \le b} \frac{\delta_{ah}\delta_{bi} + \delta_{ai}\delta_{bh}}{1 + \delta_{hi}} (\gamma_{abr} \circ \zeta_N) y^{hl}(y_{jl,k} + y_{kl,j} - y_{jk,l}) \\ &\quad + \frac{1}{2} \sum_{a \le b} \frac{\delta_{ah}\delta_{bj} + \delta_{aj}\delta_{bh}}{1 + \delta_{hj}} (\gamma_{abr} \circ \zeta_N) y^{hl}(y_{il,k} + y_{kl,i} - y_{ik,l}) \\ &\quad + \sum_{j \le a} \frac{\delta_{ak}}{1 + \delta_{jk}} \left(\gamma^h_{jar} \circ \zeta_N\right) y_{hi} + \sum_{a \le j} \frac{\delta_{ak}}{1 + \delta_{jk}} \left(\gamma^h_{ajr} \circ \zeta_N\right) y_{hi} \\ &\quad + \sum_{i \le a} \frac{\delta_{ak}}{1 + \delta_{ik}} \left(\gamma^h_{iar} \circ \zeta_N\right) y_{hj} + \sum_{a \le i} \frac{\delta_{ak}}{1 + \delta_{ik}} \left(\gamma^h_{air} \circ \zeta_N\right) y_{hj}, \end{aligned}$$

where

$$\gamma = \sum_{i \le j} \left(dy_{ij} + \gamma_{ijk} dx^k\right) \otimes \frac{\partial}{\partial y_{ij}} + \sum_{j \le k} \left(dA^i_{jk} + \gamma^i_{jkl} dx^l\right) \otimes \frac{\partial}{\partial A^i_{jk}},$$

or equivalently,

$$\gamma = \frac{1}{2 - \delta_{ij}} \left(dy_{ij} + \gamma_{ijk} dx^k\right) \otimes \frac{\partial}{\partial y_{ij}} + \frac{1}{2 - \delta_{jk}} \left(dA^i_{jk} + \gamma^i_{jkl} dx^l\right) \otimes \frac{\partial}{\partial A^i_{jk}},$$

assuming $\gamma_{hir} = \gamma_{ihr}$ for $h > i$, and $\gamma^h_{jkr} = \gamma^h_{kjr}$ for $j > k$. Taking the symmetry $A^i_{jk} = A^i_{kj}$ into account, we obtain

$$\begin{aligned}\gamma_{ijkr} &= \frac{1}{2}(\gamma_{hir} \circ \zeta_N)\, y^{hl}(y_{jl,k} + y_{kl,j} - y_{jk,l}) \\ &\quad + \frac{1}{2}(\gamma_{hjr} \circ \zeta_N)\, y^{hl}(y_{il,k} + y_{kl,i} - y_{ik,l}) \\ &\quad + \left(\gamma^h_{jkr} \circ \zeta_N\right) y_{hi} + \left(\gamma^h_{ikr} \circ \zeta_N\right) y_{hj}.\end{aligned}$$

Hence

$$\gamma_{ijkr} \circ \zeta_N^{-1} = \gamma_{hir} A^h_{jk} + \gamma_{hjr} A^h_{ik} + \gamma^h_{jkr} y_{hi} + \gamma^h_{ikr} y_{hj}, \ i \leq j. \tag{4.5}$$

Permuting the indices i, j, k cyclically on the previous equation, we have

$$\gamma^s_{ijr} = -\gamma_{hkr} A^h_{ij} y^{ks} - \frac{1}{2}\left(\gamma_{ijkr} \circ \zeta_N^{-1} - \gamma_{jkir} \circ \zeta_N^{-1} - \gamma_{kijr} \circ \zeta_N^{-1}\right) y^{ks}, \tag{4.6}$$

thus proving that the mapping $\gamma \mapsto \gamma^2$ defined in the statement, is bijective. □

4.3 Covariant Hamiltonians for second-order Lagrangians

The Legendre form of a second-order Lagrangian density $\Lambda = L v_n$ on the bundle $p\colon E \to N$ is the $V^*(p^1)$-valued p^3-horizontal $(n-1)$-form ω_Λ on J^3E locally given by (e.g., see [17, 26, 35]),

$$\omega_\Lambda = i_{\partial/\partial x^i} v_n \otimes \left(L^{i0}_\alpha dy^\alpha + L^{ij}_\alpha dy^\alpha_j\right),$$

where

$$L^{ij}_\alpha = \frac{1}{2 - \delta_{ij}} \frac{\partial L}{\partial y^\alpha_{ij}}, \tag{4.7}$$

$$L^i_\alpha = \frac{\partial L}{\partial y^\alpha_i} - \sum_j \frac{1}{2 - \delta_{ij}} D_j\left(\frac{\partial L}{\partial y^\alpha_{ij}}\right), \tag{4.8}$$

and

$$D_j = \frac{\partial}{\partial x^j} + \sum_{I \in \mathbb{N}^n, |I| = 0}^{\infty} y^\alpha_{I+(j)} \frac{\partial}{\partial y^\alpha_I}$$

denotes the total derivative with respect to the variable x^j.

The Poincaré–Cartan form attached to Λ is then defined to be the ordinary n-form on J^3E given by, $\Theta_\Lambda = (p^3_2)^*\theta^2 \wedge \omega_\Lambda + \Lambda$, where θ^2 is the second-order structure form (cf. [33, (0.36)]) and the exterior product of $(p^3_2)^*\theta^2$ and the Legendre form, is taken with respect to the pairing induced by duality, $V(p^1)\times_{J^1E} V^*(p^1)\to\mathbb{R}$. The most outstanding difference with the first-order case is that the Legendre and Poincaré–Cartan forms associated with a second-order Lagrangian density are generally defined on J^3E, thus increasing by one the order of the density.

Similarly to the first-order case (see [11, 24]), given a second-order Lagrangian density Λ on $p\colon E\to N$ and a second-order connection γ^2 on $p\colon E\to N$, by subtracting $(p^3_2)^*\theta^2$ from $(p^3_1)^*\gamma^2$ we obtain a p^3-horizontal form, and we can define the corresponding covariant Hamiltonian to be the Lagrangian density Λ^{γ^2} of third order,

$$\Lambda^{\gamma^2} = \left((p^3_1)^*\gamma^2 - (p^3_2)^*\theta^2\right)\wedge\omega_\Lambda - \Lambda. \tag{4.9}$$

Expanding on the right-hand side of the previous equation, we obtain a decomposition of Θ_Λ that generalizes the classical formula for the Hamiltonian in Mechanics; namely, $\Theta_\Lambda = (p^3_1)^*\gamma^2\wedge\omega_\Lambda - \Lambda^{\gamma^2}$. With the same notations as in the formulas (4.1), (4.7), and (4.8) the following formula is deduced:

$$L^{\gamma^2} = (\gamma^\alpha_i + y^\alpha_i)L^{i0}_\alpha + (\gamma^\alpha_{hi} + y^\alpha_{hi})L^{ih}_\alpha - L. \tag{4.10}$$

Because of equation (4.8), Θ_Λ and L^{γ^2} are generally defined on J^3E.

4.4 Invariant covariant Hamiltonians on J^2M

Lemma 4.2. *If γ is a first-order Ehresmann connection on $M\times_N C^{\mathrm{sym}}$ satisfying the conditions (C_M), then the following equation holds for the second-order Ehresmann connection γ^2 on M given in the formula* (4.2)*:*

$$\gamma_{abr}\circ\zeta_N = -y_{ab,r}.$$

Proof. Actually, from the formulas (3.8) and (4.3) we obtain

$$\begin{aligned}\gamma_{abr}\circ\zeta_N &= -\left(y_{mb}\left(A^m_{ra}\circ\zeta_N\right) + y_{ma}\left(A^m_{rb}\circ\zeta_N\right)\right)\\ &= -\frac{1}{2}\Big\{y_{mb}y^{mk}(y_{rk,a}+y_{ak,r}-y_{ra,k})\\ &\quad + y_{ma}y^{mk}(y_{rk,b}+y_{bk,r}-y_{rb,k})\Big\}\\ &= -y_{ab,r}.\end{aligned}$$

□

Lemma 4.3. *If a first-order connection γ on $M \times_N C^{\mathrm{sym}}$ satisfies the condition (C_C) introduced above, then the following formulas for its components hold:*

$$\gamma^h_{rts} - \gamma^h_{rst} = A^h_{sm}A^m_{rt} - A^h_{tm}A^m_{rs}. \tag{4.11}$$

Proof. As the bundle under consideration is that of symmetric connections, the following symmetry holds: $\gamma^h_{abc} = \gamma^h_{bac}$, and we have

$$\begin{aligned}
\gamma^h_{rts} &= \gamma^h_{str} - \left(A^h_{rm}A^m_{st} - A^h_{sm}A^m_{rt}\right) \quad \text{[by virtue of (3.13)]}\\
&= \gamma^h_{tsr} - \left(A^h_{rm}A^m_{st} - A^h_{sm}A^m_{rt}\right)\\
&= \left(\gamma^h_{rst} + A^h_{rm}A^m_{st} - A^h_{tm}A^m_{rs}\right) \quad \text{[by virtue of (3.13)]}\\
&\quad - \left(A^h_{rm}A^m_{st} - A^h_{sm}A^m_{rt}\right)\\
&= \gamma^h_{rst} + \left(A^h_{sm}A^m_{rt} - A^h_{tm}A^m_{rs}\right).
\end{aligned}$$

□

Proposition 4.4. *Let*

$$\zeta^2_N = \zeta^{(1)}_N\Big|_{J^2M} : J^2M \to J^1(M \times_N C^{\mathrm{sym}})$$

be the restriction to the closed submanifold $J^2M \subset J^1(J^1M)$ of the prolongation $\zeta^{(1)}_N : J^1(J^1M) \to J^1(M \times_N C^{\mathrm{sym}})$ of the mapping ζ_N defined in Theorem 4.1. For every $(j^1_x g, j^1_x\Gamma) \in J^1(M \times_N C^{\mathrm{sym}})$ there exists a unique $j^2_x g' \in J^2_x M$ such that, $j^1_x g' = j^1_x g$ and $j^1_x \Gamma^{g'} = j^1_x\Gamma$ and the mapping $\varkappa\colon J^1(M \times_N C^{\mathrm{sym}}) \to J^2M$ defined by $\varkappa(j^1_x g, j^1_x\Gamma) = j^2_x g'$ is a $\mathrm{Diff}\,N$-equivariant rectract of ζ^2_N.

Proof. From the formulas (4.3) and (4.4) we obtain

$$\begin{aligned}
\frac{\partial g'_{ij}}{\partial x^k} &= g'_{hi}(\Gamma^{g'})^h_{jk} + g'_{hj}(\Gamma^{g'})^h_{ik},\\
(\Gamma^{g'})^h_{ij} &= \frac{1}{2} g'^{hk}\left(\frac{\partial g'_{ik}}{\partial x^j} + \frac{\partial g'_{jk}}{\partial x^i} - \frac{\partial g'_{ij}}{\partial x^k}\right)
\end{aligned}$$

for every non-singular metric g' on N. Hence the second partial derivatives of g'_{ij} are completely determined, namely

$$\frac{\partial^2 g'_{ij}}{\partial x^k \partial x^l} = \frac{\partial g_{hi}}{\partial x^l}\Gamma^h_{jk} + g_{hi}\frac{\partial \Gamma^h_{jk}}{\partial x^l} + \frac{\partial g_{hj}}{\partial x^l}\Gamma^h_{ik} + g_{hj}\frac{\partial \Gamma^h_{ik}}{\partial x^l}.$$

Moreover, the Levi–Civita connection of a metric depends functorially on the metric, i.e., $\phi \cdot \Gamma^g = \Gamma^{\phi \cdot g}$ for every $\phi \in \mathrm{Diff}\, N$. Hence, by transforming the equations $j^1_x g' = j^1_x g$ and $j^1_x \Gamma^{g'} = j^1_x \Gamma^g$ by ϕ we can conclude. □

Theorem 4.5. *If a first-order Ehresmann connection γ on $M \times_N C^{\mathrm{sym}}$ satisfies the conditions (C_M) and (C_C) introduced above, then the covariant Hamiltonian Λ^{γ^2} attached to every* Diff N*-invariant second-order Lagrangian density Λ on M with respect to the second-order Ehresmann connection γ^2 on M defined in formula* (4.2)*, is defined on $J^2 M$ and it is also* Diff N*-invariant.*

Proof. Given a Diff N-invariant second-order Lagrangian density $\Lambda = \mathcal{L}\mathbf{v}$ on M, let $\Lambda' = \mathcal{L}'\mathbf{v}$ be the first-order Lagrangian density on $M \times_N C^{\mathrm{sym}}$ given by $\Lambda' = \varkappa^* \Lambda$, which is also Diff N-invariant as $\varkappa$ is a Diff N-equivariant mapping according to Proposition 4.4. Moreover, as $\varkappa$ is a retract of ζ^2_N, we have $\left(\zeta^2_N\right)^* \Lambda' = \left(\zeta^2_N\right)^* \varkappa^* \Lambda = (\varkappa \circ \zeta^2_N)^* \Lambda = \Lambda$, i.e., $\Lambda = \left(\zeta^2_N\right)^* \Lambda'$. This formula is equivalent to saying $\mathcal{L} = \mathcal{L}' \circ \zeta^2_N$, as the n-form $\mathbf{v}$ is Diff N-invariant, and it is even equivalent to $L = L' \circ \zeta^2_N$ because ζ^2_N induces the identity on N.

We claim $\mathcal{L}^{\gamma^2} = (\mathcal{L}')^\gamma \circ \zeta^2_N$. This formula will end the proof as the mapping ζ^2_N is Diff N-equivariant and $(\mathcal{L}')^\gamma$ is Diff N-invariant by virtue of Theorem 3.6.

To start with, we observe that formula (4.7) for Λ can be written, in the present case, as follows:

$$L^{abij} = \frac{1}{2 - \delta_{ij}} \frac{\partial L}{\partial y_{ab,ij}},$$

or equivalently, letting $\mathcal{L}^{abij} = \rho^{-1} L^{abij}$,

$$\mathcal{L}^{abij} = \frac{1}{2 - \delta_{ij}} \frac{\partial \mathcal{L}}{\partial y_{ab,ij}}. \tag{4.12}$$

Taking the formula in Lemma 4.2 into account, formula (4.10) for Λ reads as $L^{\gamma^2} = \sum_{a\leq b}(\gamma_{abij} + y_{ab,ij})L^{abij} - L$, or even

$$\mathcal{L}^{\gamma^2} = \sum_{a\leq b}(\gamma_{abij} + y_{ab,ij})\mathcal{L}^{abij} - \mathcal{L},$$

where $\mathcal{L}^{\gamma^2} = \rho^{-1}L^{\gamma^2}$. Hence $\mathcal{L}^{\gamma^2}$ is defined over J^2M. As $y_{ab,ij} = y_{ab,ji}$, we obtain

$$\begin{aligned}
\mathcal{L}^{\gamma^2} &= \sum_{a\leq b}\sum_{i\leq j}\left(\frac{1}{2}\left(\gamma_{abij} + \gamma_{abji}\right) + y_{ab,ij}\right)\frac{\partial\left(\mathcal{L}'\circ\zeta_N^2\right)}{\partial y_{ab,ij}} - \mathcal{L}'\circ\zeta_N^2 \\
&= \sum_{a\leq b}\sum_{i\leq j}\sum_{k\leq l}\left(\frac{1}{2}\left(\gamma_{abij} + \gamma_{abji}\right) + y_{ab,ij}\right) \\
&\quad\times\left(\frac{\partial\mathcal{L}'}{\partial A^h_{kl,q}}\circ\zeta_N^2\right)\frac{\partial(A^h_{kl,q}\circ\zeta_N^2)}{\partial y_{ab,ij}} - \mathcal{L}'\circ\zeta_N^2 \\
&= \sum_{k\leq l}\frac{1}{4}y^{hm}\left(\gamma_{kmql} + \gamma_{kmlq} + \gamma_{lmqk} + \gamma_{lmkq} - \gamma_{klqm} - \gamma_{klmq}\right) \\
&\quad\times\left(\frac{\partial\mathcal{L}'}{\partial A^h_{kl,q}}\circ\zeta_N^2\right) + \sum_{k\leq l}\frac{1}{2}y^{hm}\left(y_{km,ql} + y_{lm,qk} - y_{kl,qm}\right) \\
&\quad\times\left(\frac{\partial\mathcal{L}'}{\partial A^h_{kl,q}}\circ\zeta_N^2\right) - \mathcal{L}'\circ\zeta_N^2.
\end{aligned}$$

Moreover, we have

$$\left(\mathcal{L}'\right)^{\gamma} = \sum_{a\leq b}\left(\gamma_{abc} + y_{ab,c}\right)\frac{\partial\mathcal{L}'}{\partial y_{ab,c}} + \sum_{a\leq b}\left(\gamma^i_{abl} + A^i_{ab,l}\right)\frac{\partial\mathcal{L}'}{\partial A^i_{ab,l}} - \mathcal{L}'.$$

Hence

$$\begin{aligned}
\left(\mathcal{L}'\right)^{\gamma}\circ\zeta_N^2 &= \sum_{k\leq l}\left(\gamma^h_{klq}\circ\zeta_N + A^h_{kl,q}\circ\zeta_N\right)\left(\frac{\partial\mathcal{L}'}{\partial A^h_{kl,q}}\circ\zeta_N^2\right) - \mathcal{L}'\circ\zeta_N^2 \\
&= \sum_{k\leq l}\left\{-\frac{1}{2}\left(\gamma_{klrq} - \gamma_{lrkq} - \gamma_{rklq}\right)y^{rh}\right. \\
&\quad\left. + \frac{1}{2}\left(y_{kr,lq} + y_{lr,kq} - y_{kl,rq}\right)y^{hr}\right\}\left(\frac{\partial\mathcal{L}'}{\partial A^h_{kl,q}}\circ\zeta_N^2\right) - \mathcal{L}'\circ\zeta_N^2.
\end{aligned}$$

Consequently, the proof reduces to state that the following equation:

$$\frac{1}{4}\left(\gamma_{krql}+\gamma_{krlq}+\gamma_{lrqk}+\gamma_{lrkq}-\gamma_{klqr}-\gamma_{klrq}\right)=-\frac{1}{2}\left(\gamma_{klrq}-\gamma_{lrkq}-\gamma_{rklq}\right)$$

holds true, or equivalently,

$$0=\left(\gamma_{ijkr}-\gamma_{ijrk}\right)+\left(\gamma_{irjk}-\gamma_{irkj}\right)+\left(\gamma_{rjki}-\gamma_{rjik}\right). \tag{4.13}$$

According to formulas (4.5) and (3.8), we obtain

$$\begin{aligned}\gamma_{ijkr}\circ\zeta_N^{-1}=&\left(\gamma^h_{jkr}-A^h_{ra}A^a_{jk}\right)y_{hi}+\left(\gamma^h_{ikr}-A^h_{ra}A^a_{ik}\right)y_{hj}\\&-\left(A^h_{rj}A^a_{ik}+A^h_{ri}A^a_{jk}\right)y_{ah}.\end{aligned}$$

The third term on the right-hand side of this equation is symmetric in the indices k and r, as $A^a_{bc}=A^a_{cb}$. Hence,

$$\begin{aligned}\left(\gamma_{ijkr}-\gamma_{ijrk}\right)\circ\zeta_N^{-1}=&\left(\gamma^h_{jkr}-\gamma^h_{jrk}-A^h_{ra}A^a_{jk}+A^h_{ka}A^a_{jr}\right)y_{hi}\\&+\left(\gamma^h_{ikr}-\gamma^h_{irk}-A^h_{ra}A^a_{ik}+A^h_{ka}A^a_{ir}\right)y_{hj}.\end{aligned}$$

By composing the right-hand side of equation (4.13) and ζ_N^{-1}, and taking the previous formula and formulas (3.13) and (4.11) into account, we conclude that this expression vanishes indeed. □

5 Palatini and Einstein–Hilbert Lagrangians

Let us compute the covariant Hamiltonian density attached to the Palatini Lagrangian. Following the notations in [20], the Ricci tensor field attached to the symmetric connection Γ is given by $S^\Gamma(X,Y)=\mathrm{tr}(Z\mapsto R^\Gamma(Z,X)Y)$, where R^Γ denotes the curvature tensor field of the covariant derivative ∇^Γ associated to Γ on the tangent bundle; hence $S^\Gamma=(R^\Gamma)_{jl}dx^l\otimes dx^j$, where

$$\begin{aligned}(R^\Gamma)_{jl}&=(R^\Gamma)^k_{jkl},\\(R^\Gamma)^i_{jkl}&=\partial\Gamma^i_{jl}/\partial x^k-\partial\Gamma^i_{jk}/\partial x^l+\Gamma^m_{jl}\Gamma^i_{km}-\Gamma^m_{jk}\Gamma^i_{lm}.\end{aligned}$$

The Lagrangian is the function on $J^1(M\times_N C^{\mathrm{sym}})$ thus given by,

$$\mathcal{L}_P(j^1_xg,j^1_x\Gamma)=g^{ij}(x)(R^\Gamma)_{ij}(x)$$

and local expression

$$\mathcal{L}_P = y^{ij}(A^k_{ij,k} - A^k_{ik,j} + A^m_{ij}A^k_{km} - A^m_{ik}A^k_{jm}).$$

As a computation shows, for every first-order connection γ on $M \times_N C^{\mathrm{sym}}$ satisfying (4.11) and taking the formula (1.2) into account, we obtain $\mathcal{L}^\gamma_P = 0$. This result is essentially due to the fact that the P–C form of the P density $\Lambda_P = \mathcal{L}_P \mathbf{v} = L_P v_n$ projects onto $M \times_N C^{\mathrm{sym}}$. In fact, the following general characterization holds:

Proposition 5.1. *Let $p\colon E \to N$ be an arbitrary fibred manifold and let γ be a first-order Ehresmann connection on E. The equation $L^\gamma = 0$ holds true for a Lagrangian $L \in C^\infty(J^1E)$ if and only if, (i) the Poincaré-Cartan form of the density $\Lambda = Lv_n$ projects onto J^0E and, (ii) $L = \langle (p^1_0)^*\gamma - \theta, dL|_{V(p^1_0)}\rangle$.*

Proof. The equation $L^\gamma = 0$ is equivalent to the equation $D^\gamma L = L$, where D^γ is the p^1_0-vertical vector field defined in the formula (3.2), and the general solution to the latter is $L = f(x^i, y^\alpha, \gamma^\alpha_i + y^\alpha_i)$, $f(x^i, y^\alpha, y^\alpha_i)$ being a homogeneous smooth function of degree one in the variables (y^α_i), $1 \le \alpha \le m$, $1 \le i \le n$, according to Euler's homogeneous function theorem. As f is defined for all values of the variables (y^α_i), $1 \le \alpha \le m$, $1 \le i \le n$, we conclude that the functions $L^i_\alpha = \partial L/\partial y^\alpha_i$ must be defined on E. Hence L is written as $L = L^i_\alpha(x^j, y^\beta)y^\alpha_i + L_0(x^j, y^\beta)$, but this is exactly the condition for the P-C form of Λ to be projectable onto $J^0E = E$, as follows from the local expression of this form, namely,

$$\begin{aligned}\Theta_\Lambda &= \frac{\partial L}{\partial y^\alpha_i}\theta^\alpha \wedge i_{\partial/\partial x^i} v_n + Lv_n \\ &= \frac{\partial L}{\partial y^\alpha_i} dy^\alpha \wedge i_{\partial/\partial x^i} v_n + \left(L - y^\alpha_i \frac{\partial L}{\partial y^\alpha_i}\right) v_n.\end{aligned}$$

Moreover, by imposing the condition $D^\gamma L = L$ we obtain $L_0 = L^i_\alpha \gamma^\alpha_i$, or in other words $L = (\gamma^\alpha_i + y^\alpha_i)\partial L/\partial y^\alpha_i$, which is equivalent to equation (ii) in the statement. □

The corresponding result for the second-order formalism is similar but the computations are more cumbersome. Let us compute the covariant Hamiltonian density attached to the Einstein–Hilbert Lagrangian. As a matter of notation, we set $S^g(X, Y) = S^{\Gamma^g}(X, Y)$ for the metric g, Γ^g being its Levi–Civita connection, and similarly, $(R^g)^i_{jkl} = (R^{\Gamma^g})^i_{jkl}$.

The E-H Lagrangian is thus given by $\mathcal{L}_{\mathrm{EH}} \circ j^2 g = (y^{ij} \circ g)(R^g)^h_{ihj}$. As the Levi–Civita connection Γ^g depends functorially on g, $\mathcal{L}_{\mathrm{EH}}$ is readily seen to be Diff N-invariant; it is in addition linear in the second-order variables $y_{ij,kl}$. By using the third formula in (4.3) the following local expression for $\mathcal{L}_{\mathrm{EH}}$ is obtained:

$$\begin{aligned}
\mathcal{L}_{\mathrm{EH}} &= \frac{1}{2} y^{ij} y^{hd} \left(y_{dj,hi} - y_{ij,dh} - y_{dh,ij} + y_{hi,dj} \right) + \mathcal{L}'_{\mathrm{EH}}, \\
\mathcal{L}'_{\mathrm{EH}} &= \frac{1}{2} y^{ij} \Big\{ y^{hm} y_{mr,j} y^{rd} \left(y_{id,h} + y_{hd,i} - y_{ih,d} \right) \\
&\quad - y^{hm} y_{mr,h} y^{rd} \left(y_{id,j} + y_{jd,i} - y_{ij,d} \right) \\
&\quad + \frac{1}{2} y^{hr} y^{md} \left(y_{id,j} + y_{jd,i} - y_{ij,d} \right) \left(y_{hr,m} + y_{mr,h} - y_{hm,r} \right) \\
&\quad - \frac{1}{2} y^{hr} y^{md} \left(y_{id,h} + y_{hd,i} - y_{ih,d} \right) \left(y_{jr,m} + y_{mr,j} - y_{jm,r} \right) \Big\}.
\end{aligned}$$

According to (4.12), for every first-order connection form γ on $M \times_N C^{\mathrm{sym}}$ satisfying the conditions (C_M) and (C_C) above, we have

$$\mathcal{L}^{\gamma^2}_{\mathrm{EH}} = \sum_{a \le b} \frac{1}{2 - \delta_{ij}} (\gamma_{abij} + y_{ab,ij}) \frac{\partial \mathcal{L}_{\mathrm{EH}}}{\partial y_{ab,ij}} - \mathcal{L}_{\mathrm{EH}},$$

and as a computation shows,

$$\begin{aligned}
\mathcal{L}^{\gamma^2}_{\mathrm{EH}} &= \frac{1}{2} y^{ij} \left(\gamma_{idjh} + \gamma_{jdih} - \gamma_{ijdh} - \gamma_{idhj} - \gamma_{hdij} + \gamma_{ihdj} \right) y^{hd} \\
&\quad + \frac{1}{2} y^{ij} \Big\{ y^{hm} y_{mr,h} y^{rd} \left(y_{id,j} + y_{jd,i} - y_{ij,d} \right) \\
&\quad - y^{hm} y_{mr,j} y^{rd} \left(y_{id,h} + y_{hd,i} - y_{ih,d} \right) \\
&\quad - \frac{1}{2} y^{hr} y^{md} \left(y_{id,j} + y_{jd,i} - y_{ij,d} \right) \left(y_{hr,m} + y_{mr,h} - y_{hm,r} \right) \\
&\quad + \frac{1}{2} y^{hr} y^{md} \left(y_{id,h} + y_{hd,i} - y_{ih,d} \right) \left(y_{jr,m} + y_{mr,j} - y_{jm,r} \right) \Big\} \\
&= 0,
\end{aligned}$$

where the formulas (4.6), (4.11), (4.3), and Lemma 4.3 have been used. In this case, the P–C form of the E–H density $\Lambda_{\mathrm{EH}} = \mathcal{L}_{\mathrm{EH}} \mathbf{v} = L_{\mathrm{EH}} v_n$,

$$\Theta_{\Lambda_{\mathrm{EH}}} = \sum_{k \le l} \left(L^{i,kl}_{\mathrm{EH}} dy_{kl} + L^{ij,kl}_{\mathrm{EH}} dy_{kl,j} \right) \wedge i_{\partial/\partial x^i} v_n + H v_n, \tag{5.1}$$

$$H = L'_{\mathrm{EH}} - \sum_{k \le l} L^{i,kl}_{\mathrm{EH}} y_{kl,i},$$

$$L_{\mathrm{EH}}^{i,kl} = \frac{\partial L'_{\mathrm{EH}}}{\partial y_{kl,i}} - \frac{1}{2-\delta_{ij}} y_{ab,j} \frac{\partial^2 L_{\mathrm{EH}}}{\partial y_{ab} \partial y_{kl,ij}},$$
$$L_{\mathrm{EH}}^{ij,kl} = \frac{1}{2-\delta_{ij}} \frac{\partial L_{\mathrm{EH}}}{\partial y_{kl,ij}},$$

(cf. (4.7), (4.8)) is not only projectable onto J^2M but also on J^1M (e.g., see [13]), although there is no first-order Lagrangian on J^1M admitting (5.1) as its P-C form. This fact is strongly related to a classical result by Hermann Weyl ([39, Appendix II], also see [18, 22]) according to which the only Diff$N$-invariant Lagrangians on J^2M depending linearly on the second-order coordinates $y_{ab,ij}$ are of the form $\lambda \mathcal{L}_{\mathrm{EH}} + \mu$, for scalars λ, μ. This also explains why a true first-order Hamiltonian formalism exists in the Einstein-Cartan gravitation theory, e.g., see [37, 38]. In fact, if

$$L_{\mathrm{EH}}^{i} = \frac{1}{2-\delta_{ij}} \frac{\partial L_{\mathrm{EH}}}{\partial y_{kl,ij}} y_{kl,j} \quad \left(\text{hence } L_{\mathrm{EH}}^{ij,kl} = \frac{\partial L_{\mathrm{EH}}^{i}}{\partial y_{kl,j}} \right)$$

and the momentum functions are defined as follows:

$$p_{kl,i} = L_{\mathrm{EH}}^{i,kl} - \frac{\partial L_{\mathrm{EH}}^{i}}{\partial y_{kl}},$$

then

$$d\Theta_{\Lambda_{\mathrm{EH}}} = dp_{kl,i} \wedge dy_{kl} \wedge i_{\partial/\partial x^i} v_n + dH \wedge v_n,$$

and from the Hamilton-Cartan equation (e.g., see [13, (1)]) we conclude that a metric g is an extremal for Λ_{EH} if and only if,

$$0 = \frac{\partial (p_{ab,i} \circ j^1 g)}{\partial x^i} - \frac{\partial H}{\partial y_{ab}} \circ j^1 g,$$
$$0 = \frac{\partial (y_{ab} \circ g)}{\partial x^i} + \frac{\partial H}{\partial y_{ab,i}} \circ j^1 g.$$

On the other hand, it is no longer true that the covariant Hamiltonians of the non-linear Lagrangians of the form $f(\mathcal{L}_{\mathrm{EH}})$, $f'' \neq 0$, considered in some cosmological models (e.g., see [1, 6, 9, 12, 19, 21, 31]) and those in higher dimensions (e.g., see [15, 36]) vanish. In fact, as a computation shows, one has $f(\mathcal{L}_{\mathrm{EH}})^{\gamma^2} = f'(\mathcal{L}_{\mathrm{EH}})\mathcal{L}_{\mathrm{EH}} - f(\mathcal{L}_{\mathrm{EH}})$, $\forall f \in C^\infty(\mathbb{R})$.

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