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Gerbe-holonomy for surfaces with defect networks

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Abstract

We define the sigma-model action for world-sheets with embedded defect networks in the presence of a three-form field strength. We derive the defect gluing condition for the sigma-model fields and their derivatives, and use it to distinguish between conformal and topological defects. As an example, we treat the WZW model with defects labelled by elements of the centre $Z(\mathrm{G})$ of the target Lie group G; comparing the holonomy for different defect networks gives rise to a 3-cocycle on $Z(\mathrm{G})$. Next, we describe the factorization properties of two-dimensional quantum field theories in the presence of defects and compare the correlators for different defect networks in the quantum WZW model. This, again, results in a 3-cocycle on $Z(\mathrm{G})$. We observe that the cocycles obtained in the classical and in the quantum computation are cohomologous.

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Contents

1 Introduction

In this paper, we consider two-dimensional sigma models

$$S[X;\gamma] = \int_\Sigma \mathsf{d}\sigma^1 \wedge \mathsf{d}\sigma^2 \sqrt{\det\gamma}\,(\gamma^{-1})^{ab} G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu + S_{\text{top}}[X] \quad (1.1)$$

for maps X from a world-sheet Σ with metric γ to a target space M with metric G. The field variable X is allowed to be discontinuous across lines on the world-sheet. We shall refer to such lines of discontinuity as *defects*. The most familiar setting in which defects occur is provided by orbifold models, where the field has to be periodic only up to the action of the group of automorphisms of the target space. However, defect conditions much more general than a jump of the field by a target-space automorphism are possible. One of the main results of this paper is the formulation of the topological term $S_{\text{top}}[X]$ in the sigma-model action for world-sheets with defects, and, in particular, for situations in which the defect lines meet to form *defect junctions*. By varying the sigma-model action, we obtain the gluing condition to be imposed on the embedding field X and its derivatives at the defect. This allows us to analyze the world-sheet symmetries and, in particular, to distinguish between conformal and topological defects.

Circular defect lines can be treated by thinking of them as boundary conditions of a folded model [WA], but for defect junctions this is no longer possible. One can therefore expect that the study of defect junctions yields interesting information that cannot be obtained through the analysis of boundary conditions of the sigma model or some folded version thereof. We illustrate this on the example of the WZW model.

The topological term $S_{\text{top}}[X]$ of the sigma-model action can be understood as the logarithm of a U(1)-valued holonomy associated to an embedding of the world-sheet Σ in the target space M. The holonomy is computed in terms of the gauge potential B of a 3-form field strength H on M. Typically, the gauge potential cannot be defined globally and exists only patch-wise, which then leads to additional 1-forms A and functions g on two- and three-fold overlaps of these patches, respectively. These forms and functions enter the formulation of the world-sheet holonomy [Al]. It was realized in [Ga1] that the correct structure to capture the data composed of B, A and g on the target space is a class in the third (real) Deligne hypercohomology, which was subsequently identified in [Br, Mu, MSt] with a geometric object called a *gerbe with connection*. In Section 2.1, we give a brief recollection of the bits of the theory of gerbes that we shall need, and in Section 2.2, we review the holonomy formula for world-sheets with empty boundary and no defects. The notion of holonomy was generalized to world-sheets with boundaries in [Ga2, Ka, CJM, GR1, Ga4]. The boundary gets mapped to a D-brane which is a submanifold of the target space that supports a (global) curvature 2-form and a gerbe-twisted gauge bundle. The latter is described by a so-called gerbe module [CJM, GR1]. The holonomy in the presence of circular defect lines was first formulated in [FSW]. In this case, the defect circles get mapped to a submanifold $Q \subset M \times M$, termed a *bi-brane* in [FSW]. The bi-brane world-volume is equipped with a curvature 2-form and a gerbe bimodule, hence the name. We review this construction and the necessary background for gerbe bimodules in Sections 2.3 and 2.4.

In Sections 2.5 and 2.6, we extend the validity of the holonomy formula further to allow for defect junctions. There is, again, a corresponding target-space notion, which we call an *inter-bi-brane*. An inter-bi-brane consists of a collection $T = \bigsqcup_{n \geq 1} T_n$ of submanifolds $T_n \subset M \times M \times \cdots \times M$ of n copies of M, where n refers to the number of defect lines meeting at a junction. Each T_n is equipped with a twisted scalar field.

It turns out to be convenient not to restrict Q and T_n to be submanifolds of products of M, but, instead, to allow arbitrary manifolds endowed with projections to M and Q. We shall use this point of view in Section 2.

Defects in sigma models have also been investigated in the quantized theory. Most of the known results apply to the conformal régime, e.g., for free theories [Ba, Fu2, BB], for the WZW model [BG, AM, STs], or for rational conformal field theories in general [PZ, QS, Fr1, QRW]. The first systematic treatment of CFT correlators with defect junctions appeared in [Fr2]. Properties of defects were also studied in supersymmetric theories (see [BRo, BJR] for recent results), and in classical and quantized integrable models in $1+1$ dimensions (see, e.g., [BSi, Co] and the references therein).

There are at least two reasons why one should look at defect junctions once defect lines are allowed. The first reason is provided by the quantized sigma model and the factorization properties of the path integral, as explained in detail in Section 3.1. Consider, for example, the quantized sigma model on a world-sheet such as the one in figure 1. By the factorization of the path integral we mean that we can cut the world-sheet along any circle and express the original amplitude as a sum over intermediate states. If the circle along which we cut intersects the defect lines $D_1, D_2, \ldots, D_n$ then the states we sum over live in a Hilbert space $\mathcal{H}_{D_1 D_2 \ldots D_n}$ of "twisted" field configurations on the circle, cf., again, figure 1. That is, the field on the circle can have discontinuities where the defect lines $D_1, D_2, \ldots, D_n$ intersect the circle, and the allowed jumps in the value of the field are constrained by the defect condition. If the quantized sigma model is conformal — for example, if we are considering the WZW model — then there is a correspondence between states and fields. This correspondence works by starting with a boundary circle labelled by a state $|\phi\rangle$ and taking the radius of the circle to zero, using the scale transformations to transport $|\phi\rangle$ from one radius to another. What remains when the radius reaches zero is a field inserted at the centre of the circle. If several defect lines end on the boundary circle then the resulting field sits at a junction point of these defect lines. This is illustrated in figure 2 and discussed again in Section 3.2.

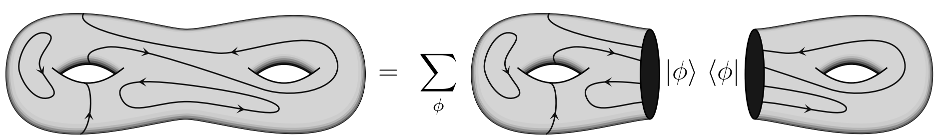


Figure 1: When a sum over intermediate states is inserted on a circle that intersects defect lines, the intermediate states lie in a twisted space of states.

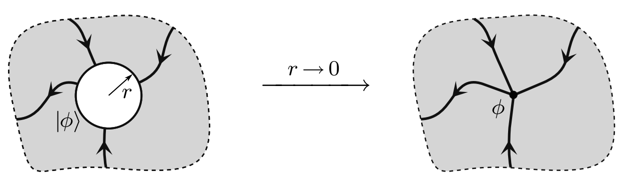


Figure 2: States $|\phi\rangle$ in a twisted space of states correspond to fields ϕ at defect junctions via the state-field correspondence.

The second reason to consider defect junctions is that they allow to extract interesting data from the classical theory, which one may next compare to the corresponding quantities in the quantized model. We illustrate this on the example of jump defects in the WZW model for a compact simple connected and simply connected Lie group G (with Lie algebra $\mathfrak{g}$). Let $\mathcal{G}$ be the gerbe on G with the curvature given by the Cartan 3-form

$$H(g) = \frac{1}{3}\mathrm{tr}_{\mathfrak{g}}\big(g^{-1}\,\mathsf{d}g \wedge g^{-1}\,\mathsf{d}g \wedge g^{-1}\,\mathsf{d}g\big), \quad g \in \mathrm{G}. \tag{1.2}$$

We shall use the gerbe $\mathcal{G}^{\star\mathsf{k}}$ for some integer $\mathsf{k} > 0$, which is given by the k-fold product of $\mathcal{G}$ with itself (cf. Section 2.1) and thus has curvature $\mathsf{k}H$. The jumps we allow are those by elements z of the centre $Z(\mathrm{G})$ of G. The corresponding defects have the property that they are *topological* in the sense that the defect line can be moved on the world-sheet without modifying the value of the action functional. This is described in more detail in Section 2.9. There also exist topological defect junctions for these jump defects, which can similarly be moved on the world-sheet without affecting the holonomy. Consider the world-sheets Σ_{L} and Σ_{R} which contain the respective networks Γ_{L} and Γ_{R} of defect lines. We take Γ_{L} and Γ_{R} to differ only in the subset of the world-sheet shown in figure 3. In this figure, a defect line is labelled by the element of $Z(\mathrm{G})$ by which the field jumps. Let $X_{\mathrm{L}}(\zeta)$ be the sigma-model field on Σ_{L}, and $X_{\mathrm{R}}(\zeta)$ the corresponding field on Σ_{R}. We choose X_{L} and X_{R} such that they are equal outside of the shaded region of the world-sheet shown in figure 3. In the shaded region, they are related as $X_{\mathrm{R}}(\zeta) = y \cdot X_{\mathrm{L}}(\zeta)$. In this way, X_{R} is uniquely determined by X_{L}. Let us denote by $S[(\Gamma, X); \gamma]$ the action functional for a field $X(\zeta)$ and a defect network Γ embedded in a world-sheet Σ with metric γ. One then finds that

$$\exp\big(-S[(\Gamma_{\mathrm{L}}, X_{\mathrm{L}}); \gamma]\big) = \psi_{\mathcal{G}^{\star\mathsf{k}}}(x, y, z) \cdot \exp\big(-S[(\Gamma_{\mathrm{R}}, X_{\mathrm{R}}); \gamma]\big) \tag{1.3}$$

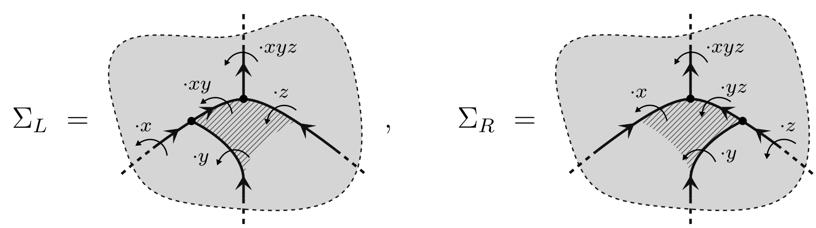


Figure 3: The relevant part of the two world-sheets Σ_{L} and Σ_{R} used in the definition of the 3-cocycle on $Z(\mathrm{G})$. The field jumps by multiplication with the indicated element of $Z(\mathrm{G})$ when crossing the defect line. The values of the field on Σ_{L} and Σ_{R} differ only in the shaded region.

holds for a U(1)-valued function $\psi_{\mathcal{G}^{\star k}}(x,y,z)$ which is *independent* of the choice of $X_{\rm L}$ (this choice then fixes $X_{\rm R}$), and which is invariant under deformations of the defect lines, provided that we do not move one vertex past another. We treat this example in detail in Section 2, where we also demonstrate that $\psi_{\mathcal{G}^{\star k}}$ is a 3-cocycle on $Z(\mathrm{G})$ and defines a class $[\psi_{\mathcal{G}^{\star k}}]$ in $H^3(Z(\mathrm{G}),\mathrm{U}(1))$, the third cohomology group of $Z(\mathrm{G})$ with values in U(1) (with trivial $Z(\mathrm{G})$-action). The configuration shown in figure 3 was studied in [JK] from the point of view of interacting orbifold string theories. There, figure 3 was used to show that the triviality of $[\psi_{\mathcal{G}^{\star k}}]$ is necessary to have a consistent interaction of closed strings in the orbifolded theory.

The comparison of (1.3) can also be carried out in the quantized WZW model for the affine Lie algebra $\widehat{\mathfrak{g}}_{\mathsf{k}}$, where the integer k is the one determining the gerbe $\mathcal{G}^{\star \mathsf{k}}$ used above. This is done in Section 3, with the following result. The topological defects of the quantum WZW model for the affine Lie algebra $\widehat{\mathfrak{g}}_{\mathsf{k}}$ which commute with the Kač–Moody currents are labelled by irreducible integrable highest-weight representations λ of $\widehat{\mathfrak{g}}_{\mathsf{k}}$. One can assign a representation λ_z to each element $z \in Z(\mathrm{G})$. The representations λ_z are precisely the simple currents of the WZW model (with the one exception of $\widehat{\mathfrak{e}(8)}_2$, which has a simple current even though $Z(\mathrm{E}(8)) = \{e\}$, see [Fu1]). Comparing correlators on the world-sheets $\Sigma_{\rm L}$ and $\Sigma_{\rm R}$ in figure 3 gives

$$\mathrm{Corr}(\Gamma_{\rm L},\Sigma_{\rm L}) = \psi_{\widehat{\mathfrak{g}}_{\mathsf{k}}}(x,y,z)\cdot \mathrm{Corr}(\Gamma_{\rm R},\Sigma_{\rm R}) \tag{1.4}$$

for a U(1)-valued function $\psi_{\widehat{\mathfrak{g}}_{\mathsf{k}}}(x,y,z)$ which is, again, a 3-cocycle on $Z(\mathrm{G})$ and which defines a cohomology class $[\psi_{\widehat{\mathfrak{g}}_{\mathsf{k}}}] \in H^3(Z(\mathrm{G}),\mathrm{U}(1))$. We compute $\psi_{\widehat{\mathfrak{g}}_{\mathsf{k}}}$ in Section 3.4.

The second main result of this paper is the observation that the cohomology classes obtained in the classical and quantum computations coincide,

$$[\psi_{\mathcal{G}^{\star k}}] = [\psi_{\widehat{\mathfrak{g}}_{\mathsf{k}}}]. \tag{1.5}$$

In the classical theory, the class $[\psi_{\mathcal{G}^{\star k}}]$ determines the obstruction to the existence of a $\mathcal{Z}$-equivariant gerbe, for $\mathcal{Z} \subset Z(\mathrm{G})$ a subgroup. The condition $[\psi_{\mathcal{G}^{\star k}}|_{\mathcal{Z}}] = 1$ imposes selection rules on k. If the condition holds $\mathcal{Z}$-equivariant gerbes exist and can be used to define the sigma model on the orbifold $\mathrm{G}/\mathcal{Z}$ [FGK,GR1,GR2]. Similarly, in the quantum WZW model, $[\psi_{\widehat{\mathfrak{g}}_{\mathsf{k}}}]$ is the obstruction to the existence of a simple-current orbifold [SY1, SY2,KS,FRS3]. Thus, one way to read (1.5) is that the classical obstruction to the orbifolding of the sigma model is preserved by quantization.

A related way to interpret (1.5) is as follows: If a discrete symmetry group S of a CFT is implemented by defects then this group automatically comes with the additional datum of a class $[\psi] \in H^3(S, \mathrm{U}(1))$ (this will be explained in Section 3.3). The same is true for the classical sigma model. Equation (1.5) states that, for the WZW model and for $S = Z(\mathrm{G})$, the class $[\psi]$ in $H^3(S, \mathrm{U}(1))$ is not changed when quantizing the model.

That for a given subgroup $\mathcal{Z} \subset Z(\mathrm{G})$ the values of k for which $[\psi_{\mathcal{G}^{\star\mathsf{k}}}|_{\mathcal{Z}}] = 1$ are precisely those for which $[\psi_{\widehat{\mathfrak{g}}_{\mathsf{k}}}|_{\mathcal{Z}}] = 1$ is already known from [GR1, GR2]. The novelty in (1.5) is the statement that the cohomology classes coincide for all k, and on all of $Z(\mathrm{G})$. Defect junctions thus give an explicit way to extract a non-perturbative CFT datum — the fusing matrix (6j-symbols) restricted to the simple-current sector — from a classical calculation with gerbes.

The paper is organized as follows: We start in Section 2 by reviewing the concept of the holonomy for world-sheets without defects and for those with circular defects. Then we give our construction of the holonomy in the presence of defect junctions and compute the 3-cocycle for the jump defects in the classical WZW model. The formulation of the quantum field theory in the presence of defect lines and the computation of the 3-cocycle in the quantum theory are given in Section 3. Finally, the results of the classical and quantum calculation are compared in Section 4.

2 Holonomy for world-sheets with defect networks

In this section, we give a prescription for the holonomy for a world-sheet with an embedded network of defect lines. We begin by collecting the necessary ingredients, starting with the definition of a gerbe in terms of its local data, and proceed to describe and justify the proposed holonomy formula.

2.1 Gerbes in terms of local data

Let M be a smooth manifold and let $\mathcal{O}^M = \{\mathcal{O}^M_i \mid i \in \mathcal{I}\}$ be a good open cover of M. Write the p-fold intersection of open sets as $\mathcal{O}^M_{i_1 i_2 \ldots i_p} = \bigcap_{k=1}^p \mathcal{O}^M_{i_k}$. The qualifier "good" means that each non-empty $\mathcal{O}^M_{i_1 i_2 \ldots i_p}$ is contractible.

For $p \geq 0$ and $r \geq 1$, let $\check{C}^{p,r}(\mathcal{O}^M)$ be the set whose elements ω are collections of smooth differential r-forms

$$\omega = \{\omega_{i_1 \ldots i_{p+1}} \in \Omega^r\big(\mathcal{O}^M_{i_1 \ldots i_{p+1}}\big) \mid i_k \in \mathcal{I} \text{ s.t. } \mathcal{O}^M_{i_1 \ldots i_{p+1}} \neq \emptyset\}, \qquad (2.1)$$

where $\omega_{i_1\dots i_{p+1}}$ is required to be antisymmetric in all indices. This is a Čech p-cochain with values in the sheaf of differential r-forms on M, but we shall not need this background in the present paper. Note that $\check{C}^{p,r}(\mathcal{O}^M)$ inherits the structure of a vector space from $\Omega^r(\mathcal{O}^M_{i_1\dots i_{p+1}})$. Below, we shall only use that $\Omega^r(\mathcal{O}^M_{i_1\dots i_{p+1}})$ is an abelian group, which will be written additively.

The sets $\check{C}^{p,0}(\mathcal{O}^M)$ are defined slightly differently. Namely, an element φ of $\check{C}^{p,0}(\mathcal{O}^M)$ is a collection $\varphi_{i_1\dots i_{p+1}}$, where each $\varphi_{i_1\dots i_{p+1}} \in \mathrm{U}(1)_{\mathcal{O}^M_{i_1\dots i_{p+1}}}$ is a smooth U(1)-valued function on $\mathcal{O}^M_{i_1\dots i_{p+1}}$ that is antisymmetric in all indices. The set $\check{C}^{p,0}(\mathcal{O}^M)$ inherits the structure of an abelian group from U(1), which will be written multiplicatively.

In order to describe a gerbe and its gauge transformations, one uses the first four components of a chain complex $A^\bullet_M(\mathcal{O}^M)$, given by (we drop $\mathcal{O}^M$ from the notation)

$$\begin{aligned} &A^0_M = \check{C}^{0,0}, \quad A^1_M = \check{C}^{0,1} \times \check{C}^{1,0}, \quad A^2_M = \check{C}^{0,2} \times \check{C}^{1,1} \times \check{C}^{2,0}, \\ &A^3_M = \check{C}^{0,3} \times \check{C}^{1,2} \times \check{C}^{2,1} \times \check{C}^{3,0}. \end{aligned} \tag{2.2}$$

Thus, for example, an element of A^1_M is a pair (Π, χ) where Π is a collection of smooth 1-forms Π_i on $\mathcal{O}^M_i$, and χ is a collection of smooth U(1)-valued functions χ_{ij} on the overlap $\mathcal{O}^M_{ij} = \mathcal{O}^M_i \cap \mathcal{O}^M_j$, which is antisymmetric in its Čech indices in the sense that $\chi_{ij}(x) = \chi_{ji}(x)^{-1}$. We shall also write elements of A^1_M as (Π_i, χ_{ij}), and similarly for the other components of $A^\bullet_M$. Each A^m_M forms an abelian group under the addition of the component r-forms and the multiplication of the U(1)-valued functions. For instance, the definition of the sum of elements of A^2_M reads

$$(B_i, A_{ij}, g_{ijk}) + (B'_i, A'_{ij}, g'_{ijk}) = (B_i + B'_i, A_{ij} + A'_{ij}, g_{ijk} \cdot g'_{ijk}). \tag{2.3}$$

The Deligne differential $D_{(r)} : A^r_M \to A^{r+1}_M$ is given by

$$\begin{aligned} D_{(0)}(f_i) &= (-\mathsf{i}\,\mathsf{d}\log f_i, f_j^{-1} \cdot f_i), \\ D_{(1)}(\Pi_i, \chi_{ij}) &= (\mathsf{d}\Pi_i, -\mathsf{i}\,\mathsf{d}\log \chi_{ij} + \Pi_j - \Pi_i, \chi_{jk}^{-1} \cdot \chi_{ik} \cdot \chi_{ij}^{-1}), \\ D_{(2)}(B_i, A_{ij}, g_{ijk}) &= (\mathsf{d}B_i, \mathsf{d}A_{ij} - B_j + B_i, -\mathsf{i}\,\mathsf{d}\log g_{ijk} + A_{jk} \\ &\qquad - A_{ik} + A_{ij}, g_{jkl}^{-1} \cdot g_{ikl} \cdot g_{ijl}^{-1} \cdot g_{ijk}), \end{aligned} \tag{2.4}$$

where we follow the conventions of [GSW1]. One verifies that $D_{(r+1)} \circ D_{(r)} = 0$. Below, we shall typically just write D instead of $D_{(r)}$.

Mathematically, the appropriate description of $A^\bullet_M$ is in terms of the Čech—Deligne double complex and the resulting Deligne hypercohomology.[1] We refer the reader to [Br] for a detailed exposition.

With these ingredients in hand, we can now define the notion of a gerbe in terms of its local data. A *gerbe with connection in terms of local data*, or a *gerbe* for short, on a smooth manifold M is a pair $\mathcal{G} = (\mathcal{O}^M, b)$ where $\mathcal{O}^M$ is a good open cover of M and $b \in A^2_M$ is such that

$$Db = (H_i, 0, 0, 1). \tag{2.5}$$

The objects H_i are 3-forms on $\mathcal{O}^M_i$ but it is not hard to see that they are, in fact, restrictions of a globally defined closed 3-form $H \in \Omega^3(M)$, called the *curvature* of $\mathcal{G}$.

Given two gerbes: $\mathcal{G} = (\mathcal{O}^M, b)$ and $\mathcal{H} = (\mathcal{O}^M, b')$ defined with respect to the same open cover of M, a *stable isomorphism* $\Phi : \mathcal{G} \to \mathcal{H}$ *in terms of local data* is an element $\Phi \in A^1_M$ such that $b' = b + D\Phi$. Given a third gerbe $\mathcal{K} = (\mathcal{O}^M, b'')$ and a stable isomorphism $\Psi : \mathcal{H} \to \mathcal{K}$, the *composition* $\Psi \circ \Phi$ is defined to be the stable isomorphism $\Psi + \Phi \in A^1_M$ from $\mathcal{G}$ to $\mathcal{K}$. Indeed, $b'' = b' + D\Psi = b + D\Phi + D\Psi$. The notion of a stable isomorphism can be extended to gerbes over the same base M but with two different open covers $\mathcal{O}^M$ and $\widetilde{\mathcal{O}}^M$ by passing to a common refinement.

Physically, a stable isomorphism is a gauge transformation of the gerbe data. Computations of observable quantities carried out with respect to distinct but gauge-equivalent (i.e., stably isomorphic) gerbes should give the same result. One distinguishing feature of gerbes is that the local data of gauge transformations between two given gerbes can themselves be related by another kind of a gauge transformation. This is captured by the notion of a 2-morphism, which is defined as follows: Let $\mathcal{G} = (\mathcal{O}^M, b)$ and $\mathcal{H} = (\mathcal{O}^M, b')$ be gerbes over a manifold M, and let $\Phi : \mathcal{G} \to \mathcal{H}$ and $\Psi : \mathcal{G} \to \mathcal{H}$ be two stable isomorphisms. A 2-*morphism* $\varphi : \Phi \Longrightarrow \Psi$ is an element of A^0_M such that

$$\Phi + D\varphi = \Psi. \tag{2.6}$$

[1] Strictly speaking, the definition of Deligne hypercohomology requires truncating the de Rham complex. In our paper, we could equivalently work with the full Čech—de Rham complex but since we want to adhere to the terminology of Deligne hypercohomology, we choose to truncate the underlying differential complex to $\underline{U(1)}_M \xrightarrow{\frac{1}{\mathsf{i}}\mathsf{d}\log} \underline{\Omega}^1(M) \xrightarrow{\mathsf{d}} \underline{\Omega}^2(M) \xrightarrow{\mathsf{d}} \underline{\Omega}^3(M)$, where the underlining indicates that we are dealing with the corresponding sheaf.

In terms of local data $\Phi = (P_i, K_{ij})$, $\Psi = (P'_i, K'_{ij})$ and $\varphi = (f_i)$, this means that

$$P_i - \mathsf{i}\,\mathsf{d}\log f_i = P'_i \quad \text{and} \quad K_{ij} \cdot f_i \cdot f_j^{-1} = K'_{ij}. \tag{2.7}$$

The word "2-morphism" derives from the 2-categorial interpretation of gerbes given in terms of local data, see [St, Wa1].

The presentation of the holonomy formulæ below requires some additional constructions for gerbes, namely trivial gerbes, the product of gerbes and pullback gerbes. Let us briefly review the local formulation of these notions.

- A *trivial gerbe* for a 2-form ω on M is the gerbe $I(\mathcal{O}^M, \omega) = (\mathcal{O}^M, (\omega_i, 0, 1))$, where $\mathcal{O}^M$ is a good open cover of M and ω_i is the restriction of ω to $\mathcal{O}^M_i$. The curvature 3-form of $I(\mathcal{O}^M, \omega)$ is exact, hence the name "trivial". We shall sometimes abbreviate the notation for the trivial gerbe as $I(\mathcal{O}^M, \omega) \equiv I(\omega)$.
- Given two gerbes: $\mathcal{G} = (\mathcal{O}^M, b)$ and $\mathcal{H} = (\mathcal{O}^M, b')$ defined with respect to the same good open cover of M, we take the *product gerbe* to be $\mathcal{G} \star \mathcal{H} = (\mathcal{O}^M, b + b')$. This is analogous to the tensor product of line bundles, hence the product notation. The curvature of $\mathcal{G} \star \mathcal{H}$ is $H_{\mathcal{G}} + H_{\mathcal{H}}$, where $H_{\mathcal{G}}$ and $H_{\mathcal{H}}$ are the curvatures of $\mathcal{G}$ and $\mathcal{H}$, respectively.
- Let $\mathcal{O}^M$ and $\mathcal{O}^N$ be good open covers of manifolds M and N. In order to define pullbacks of gerbes, stable isomorphisms and 2-morphisms, it is not enough to specify a smooth map from M to N — we also need to know how the Čech indices are related. To this end, we define a *Čech-extended map* $\check{f} : M \to N$ to be a pair (f, ϕ) where $f : M \to N$ is a smooth map and ϕ is an index map, $\phi : \mathcal{I}^M \to \mathcal{I}^N$, such that $f(\mathcal{O}^M_i) \subset \mathcal{O}^N_{\phi(i)}$. Since ϕ need not exist, not every map $f : M \to N$ can be turned into a Čech-extended map. Given another Čech-extended map $\check{g} = (g, \gamma) : N \to K$, their composition is defined component-wise as $\check{g} \circ \check{f} = (g \circ f, \gamma \circ \phi)$.
- Let $\mathcal{G} = (\mathcal{O}^N, b)$ be a gerbe on N and let $\check{f} : M \to N$ be a Čech-extended map. The *pullback gerbe* is $\check{f}^*\mathcal{G} = (\mathcal{O}^M, b')$ with $b' = (B'_i, A'_{ij}, g'_{ijk}) = (f^*B_{\phi(i)}, f^*A_{\phi(i)\phi(j)}, g_{\phi(i)\phi(j)\phi(k)} \circ f) \equiv \check{f}^*b$. The pullback $\check{f}^*C$ for $C = (c_{i_1}, c_{i_1 i_2}, \ldots, c_{i_1 i_2 \ldots i_{p+1}}) \in A^p_M$ is defined in the same way. To unclutter the notation, we shall frequently use the shorthand

$$f^* c_{i_1 i_2 \ldots i_m} \equiv f^* c_{\phi(i_1)\phi(i_2)\ldots\phi(i_m)}. \tag{2.8}$$

  If $\check{f}_1 = (f, \phi)$ and $\check{f}_2 = (f, \widetilde{\phi})$ are two Čech-extended maps that differ only in the choice of the index map, the resulting pullback gerbes are stably isomorphic. Indeed, one verifies that $\widetilde{b}' = b' + Dp$ for $p = (\Pi_i, \chi_{ij}) = (f^* A_{\phi(i)\widetilde{\phi}(i)}, (g_{\phi(i)\phi(j)\widetilde{\phi}(j)} \cdot g^{-1}_{\phi(i)\widetilde{\phi}(i)\widetilde{\phi}(j)}) \circ f)$.

A more geometric description of gerbes is given by bundle gerbes, see [Mu, MSt] and [St, Jh, Wa1, Wa2]. Their main advantage is that the choice of a good open cover of M is replaced by the more flexible concept of a surjective submersion $\pi : Y \to M$, which proves particularly convenient in the WZW setting, see [GR1, Me, GR2]. It deserves to be stressed that every bundle gerbe is stably isomorphic to a bundle gerbe whose surjective submersion comes from a good open cover of M, see [MSt]. The reason for us to use the local description is that the formalism is easier to set up and that we find the definition of the holonomy formula in terms of local data more intuitive.

2.1.1 The Lie-group example

As mentioned in the Introduction, the example that we consider in this paper is the WZW model for a compact simple connected and simply connected Lie group G at level $\mathsf{k} \in \mathbb{Z}_{>0}$ (geometrically, the value of the level sets the size of the group manifold). The Lie group comes equipped with the so-called basic gerbe [GR1, Me], which we denote by $\mathcal{G}$. It is the unique, up to a stable isomorphism, gerbe with the curvature given by the Cartan 3-form

$$H(g) = \frac{1}{3}\,\mathrm{tr}_{\mathfrak{g}}\big(g^{-1}\,\mathsf{d}g \wedge g^{-1}\,\mathsf{d}g \wedge g^{-1}\,\mathsf{d}g\big), \quad g \in \mathrm{G}, \tag{2.9}$$

the latter being fixed by the requirement of the vanishing of the Weyl anomaly in the quantized sigma model. Note that the works [GR1, Me] use the language of bundle gerbes; here, we have to decide once and for all on a good open cover $\mathcal{O}^{\mathrm{G}}$ of G and on local gerbe data $b = (B_i, A_{ij}, g_{ijk})$ such that $\mathcal{G} = (\mathcal{O}^{\mathrm{G}}, b)$. In fact, as we shall later want to consider translations by elements of the centre $Z(\mathrm{G})$ of G, we choose the good open cover $\mathcal{O}^{\mathrm{G}}$ to be invariant under such translations in the sense that there exists an action $Z(\mathrm{G}) \times \mathcal{I}^{\mathrm{G}} \to \mathcal{I}^{\mathrm{G}} : (z, i) \mapsto z.i$ such that

$$z\big(\mathcal{O}^{\mathrm{G}}_i\big) = \mathcal{O}^{\mathrm{G}}_{z.i} \quad \text{for all } z \in Z(\mathrm{G}), \tag{2.10}$$

where $z(\mathcal{O}^{\mathrm{G}}_i) = \{z \cdot g \mid g \in \mathcal{O}^{\mathrm{G}}_i\}$. We can use this action to turn z into a Čech-extended map from G to itself by setting

$$\check{z} : \mathrm{G} \to \mathrm{G}, \quad \check{z} = (g \mapsto z \cdot g, i \mapsto z.i). \tag{2.11}$$

Given a $Z(\mathrm{G})$-invariant cover of G, we obtain a natural definition of a left-regular action of $Z(\mathrm{G})$ on $A^r_{\mathrm{G}}, r \geq 0$ by the Čech-extended pullback

$$(z, \omega) \mapsto \big(\check{z^{-1}}\big)^* \omega = z.\omega \tag{2.12}$$

which we are going to encounter frequently in the sequel. Finally, for the WZW model at level k, one uses the k-fold product $\mathcal{G}^{\star\mathsf{k}}$ of the basic gerbe with itself.

2.2 Surface holonomy in the absence of defects

Here, we review the definition of the surface holonomy in the absence of defects [Al, Ga1]. In this case, the world-sheet is an oriented smooth compact two-manifold Σ with empty boundary and the target space is a (not necessarily connected) smooth manifold M with gerbe $\mathcal{G}$.

Write the local data for $\mathcal{G} = (\mathcal{O}^M, b)$ as $b = (B_i, A_{ij}, g_{ijk})$. Given a once differentiable map[2] $X : \Sigma \to M$, the holonomy $\mathrm{Hol}_{\mathcal{G}}(X)$ is an element of U(1) defined by the following formula:

$$\mathrm{Hol}_{\mathcal{G}}(X) = \prod_{t\in\triangle(\Sigma)} \left\{ \exp\Big(\mathsf{i}\int_t \widehat{B}_t\Big) \prod_{e\subset t} \left[\exp\Big(\mathsf{i}\int_e \widehat{A}_{te}\Big) \prod_{v\in e} \widehat{g}_{tev}(v) \right] \right\}, \tag{2.13}$$

whose ingredients we proceed to explain:

- $\triangle(\Sigma)$ is a triangulation of Σ which is subordinate to $\mathcal{O}^M$ with respect to X in the sense that for each triangle $t \in \triangle(\Sigma)$ there is an index $i_t \in \mathcal{I}$ such that $X(t) \subset \mathcal{O}^M_{i_t}$. This implies that also the image under X of an edge e lies in one of the open sets of the good cover. For each edge e and vertex v, we pick an assignment $e \mapsto i_e$ and $v \mapsto i_v$ such that $X(e) \subset \mathcal{O}^M_{i_e}$ and $X(v) \in \mathcal{O}^M_{i_v}$.
- $\widehat{B}_t = X^* B_{i_t}$ is the pullback of B_{i_t} to the triangle t along X. Here, X is understood as a map from t to $\mathcal{O}^M_{i_t}$. The 2-form $\widehat{B}_t$ is integrated over t using the orientation of the world-sheet Σ.
- $\widehat{A}_{te} = X^* A_{i_t i_e}$ is the pullback of $A_{i_t i_e}$ to the edge e along X. Here, X is understood as a map from e to $\mathcal{O}^M_{i_t i_e}$. The 1-form $\widehat{A}_{te}$ is integrated over e, where the orientation of e is the one induced by the triangle t via the inward-pointing normal. (For example, the orientation of the edges of a triangle embedded in $\mathbb{R}^2$ is counter-clockwise.)
- $\widehat{g}_{tev} = (X^* g_{i_t i_e i_v})^{\varepsilon_{tev}}$, where X maps v to $\mathcal{O}^M_{i_t i_e i_v}$ and $\varepsilon_{tev} = \pm 1$ is determined as follows: The edge e inherits an orientation from the triangle t. If the vertex sits at the end of e with respect to this orientation we set $\varepsilon_{tev} = 1$. Otherwise, $\varepsilon_{tev} = -1$. (For example, in the interval $[0, 1] \subset \mathbb{R}$ with the standard orientation, the point 0 has $\varepsilon = -1$ and the point 1 has $\varepsilon = 1$.)

[2]By this we mean a C^1-function, i.e., a function that is once continuously differentiable.

The reason why it is sufficient to require X to be once differentiable, rather than smooth, is that the holonomy $\mathrm{Hol}_{\mathcal{G}}(X)$ and the kinetic term in the action only depend on the first derivatives of the map X. This will play a role when discussing topological defects in Section 2.9 below.

The holonomy remains unchanged if we perform a gauge transformation on the gerbe data, and the relation $Db = (H_i, 0, 0, 1)$ ensures that the holonomy is independent of the choice of triangulation. We shall return to these points in Section 2.7.

2.3 Abelian bi-branes

In words, an abelian bi-brane of [FSW] is a submanifold Q of the product $M \times M$, together with a 2-form ω on Q such that the pullbacks of the gerbe $\mathcal{G}$ by the canonical projections to the two factors in $M \times M$ differ only by the trivial gerbe $I(\omega)$ when restricted to Q. Since we are working with local data, some extra choices are involved, making the actual definition lengthier. We shall also be slightly more general by not restricting ourselves to the case that Q is a submanifold of $M \times M$.

Let $\mathcal{G} = (\mathcal{O}^M, b)$ be a gerbe on M. An *abelian $\mathcal{G}$-bi-brane $\mathcal{B}$ in terms of local data* is a tuple

$$\mathcal{B} = \big(Q, \omega, \mathcal{O}^Q, \check{\iota}_1, \check{\iota}_2, \Phi\big), \tag{2.14}$$

where

(B.i) Q is a smooth manifold;
(B.ii) ω is a smooth 2-form on Q, called the curvature of the $\mathcal{G}$-bi-brane;
(B.iii) $\mathcal{O}^Q = \{\mathcal{O}^Q_i \mid i \in \mathcal{I}^Q\}$ is a good open cover of Q;
(B.iv) $\check{\iota}_1 = (\iota_1, \phi_1)$ and $\check{\iota}_2 = (\iota_2, \phi_2)$ are Čech-extended maps from Q to M;
(B.v) $\Phi : \check{\iota}_1^*\mathcal{G} \to \check{\iota}_2^*\mathcal{G} \star I(\mathcal{O}^Q, \omega)$ is a stable isomorphism.

By (B.v), the two pullbacks of the gerbe $\mathcal{G}$ differ by a trivial gerbe on Q. For the curvature of $\mathcal{G}$, this implies $\iota_1^* H - \iota_2^* H = \mathsf{d}\omega$. Thus, the difference of the pullbacks of the curvature has to be exact on Q.

Let the local data for the gerbe and the stable isomorphism be given by $b = (B_i, A_{ij}, g_{ijk}) \in A^2_M$ and $\Phi = (P_i, K_{ij}) \in A^1_Q$, respectively. The stable isomorphism now gives the condition (recall the shorthand notation (2.8))

$$\check{\iota}_1^*(B_i, A_{ij}, g_{ijk}) + D(P_i, K_{ij}) = \check{\iota}_2^*(B_i, A_{ij}, g_{ijk}) + (\omega, 0, 1). \tag{2.15}$$

There are a number of differences with respect to the original definition in [FSW, Wa2] which we would like to point out and justify. Namely, in [FSW, Wa2],

- a bi-brane is defined in terms of more general morphisms [Wa1] between the pullback gerbes, not just stable isomorphisms. In a nutshell, the data (2.14) can be understood as a gerbe-twisted line bundle over Q, while in the case of a general bi-brane, one also allows gerbe-twisted vector bundles of higher rank. In this paper, we shall restrict our attention to the case of abelian bi-branes. This is done for simplicity.
- Q is taken to be a submanifold of $M \times M$. This is recovered in the present definition as a special case upon choosing $\check{\imath}_1$ and $\check{\imath}_2$ to be the canonical projections onto the two factors. The reason why we use a more general definition is that it will allow us to treat several bi-branes using only one manifold Q, even if the worldvolumes of the individual bi-branes would intersect as submanifolds in $M \times M$.
- bi-branes between different target spaces M_1 and M_2 are allowed, in which case Q is a submanifold of $M_1 \times M_2$. This situation is covered by our approach because we can take M to be the disjoint union $M_1 \sqcup M_2$ and choose $\check{\imath}_1, \check{\imath}_2 : Q \to M$ to be the projections for a submanifold $Q \subset M_1 \times M_2 \subset M \times M$.

Two simple examples illustrate the data describing a bi-brane: the trivial $\mathcal{G}$-bi-brane and D-branes (the latter were dubbed $\mathcal{G}$-branes in the gerbe-theoretic context of [Ga4]). The trivial $\mathcal{G}$-bi-brane for the gerbe $\mathcal{G}$ over the target space M is given by

$$\mathcal{B}_{\rm triv} = \big(M, 0, \mathcal{O}^M, \check{\rm id}, \check{\rm id}, {\rm id}_{\mathcal{G}}\big). \tag{2.16}$$

This corresponds to the diagonal embedding of M into $M \times M$ with the 2-form ω set to zero. For a trivial $\mathcal{G}$-bi-brane, the holonomy for the world-sheet with an embedded defect network given below reduces to the form (2.13).

In order to describe a D-brane in a target space N, one takes the manifold M to be the disjoint union of N and a single point, $M = N \sqcup \{\bullet\}$. Let D be a submanifold of N with a 2-form field ω. Consider the bi-brane

$$\mathcal{B}_{\rm bnd} = (D, \omega, \mathcal{O}^D, \check{\imath}, \check{\bullet}, \Phi), \tag{2.17}$$

where $\check{\imath} = (\iota, \phi_1)$ is just the embedding of D into $N \subset M$ and $\check{\bullet} = (\bullet, \phi_2)$ is the constant map $D \to \{\bullet\}$. For simplicity, we take the open cover $\mathcal{O}^D$ of D to be the intersection of the open sets in $\mathcal{O}^N$ with D and assume that the open sets in $\mathcal{O}^N$ are small enough for the resulting cover $\mathcal{O}^D$ to be good.

Thus, $\phi_1 = \mathrm{id}$ in this case. The cover of $\{\bullet\}$ consists of one element, which we also label $\bullet$, and ϕ_2 is the constant map $\mathcal{I}^N \to \{\bullet\}$. Condition (B.v) on Φ and ω now reads $\Phi : \mathcal{G}|_D \to I(\omega)$, which is precisely the local data for a D-brane as defined in [GR1, Ga4].

2.3.1 The Lie-group example (cont'd)

Continuing the Lie group example from Section 2.1, we now want to define a $\mathcal{G}^{\star \mathsf{k}}$-bi-brane $\mathcal{B}_{Z(\mathrm{G})}$ for jump defects. The underlying manifold is

$$Q_{Z(\mathrm{G})} = \mathrm{G} \times Z(\mathrm{G}), \tag{2.18}$$

and the 2-form ω on $Q_{Z(\mathrm{G})}$ is zero. The open cover $\mathcal{O}^{Q_{Z(\mathrm{G})}}$ is indexed by pairs from $\mathcal{I}^{Q_{Z(\mathrm{G})}} = \mathcal{I}^{\mathrm{G}} \times Z(\mathrm{G})$, and the corresponding open sets are $\mathcal{O}^{Q_{Z(\mathrm{G})}}_{i,x} = \mathcal{O}^{\mathrm{G}}_i \times \{x\}$. The two Čech-extended maps $\check{\iota}_1 = (\iota_1, \phi_1)$ and $\check{\iota}_2 = (\iota_2, \phi_2)$ from $Q_{Z(\mathrm{G})}$ to G are given by the formulæ

$$\iota_1(g,x) = g, \quad \phi_1(i,x) = i; \quad \iota_2(g,x) = x^{-1} \cdot g, \quad \phi_2(i,x) = x^{-1}.i. \tag{2.19}$$

The pullback of $\mathcal{G}$ along the translation by $x \in Z(\mathrm{G})$ yields a gerbe stably isomorphic to $\mathcal{G}$. This follows since the left-regular action of $Z(\mathrm{G})$ induced by the pullback as in (2.12) commutes with the Deligne differential D, and translations by elements of $Z(\mathrm{G})$ preserve the Cartan 3-form. This conclusion also holds for all powers of $\mathcal{G}$, and so one can find a set of 1-morphisms

$$\mathcal{A}_{Z(\mathrm{G})} = \big\{ \mathcal{A}_x : \mathcal{G}^{\star \mathsf{k}} \to x.\mathcal{G}^{\star \mathsf{k}} \mid x \in Z(\mathrm{G}) \big\}, \tag{2.20}$$

cf. (2.12), constructed explicitly in [GR2, Section 3] in the framework of bundle gerbes (the present notation conforms with the unified treatment laid out in [GSW2, Sections 1 and 3]). The local data of $\Phi : \check{\iota}_1^* \mathcal{G}^{\star \mathsf{k}} \to \check{\iota}_2^* \mathcal{G}^{\star \mathsf{k}}$ on $\mathcal{O}^{Q_{Z(\mathrm{G})}}_{i,x}$ is given by the local data of $\mathcal{A}_x$ on $\mathcal{O}^{\mathrm{G}}_i$, where we use $\mathcal{O}^{\mathrm{G}}_i \cong \mathcal{O}^{Q_{Z(\mathrm{G})}}_{i,x}$.

The existence of the stable isomorphisms $\mathcal{A}_{Z(\mathrm{G})}$ is ensured by the triviality of the cohomology group $H^2(\mathrm{G}, \mathrm{U}(1))$. To see how this comes about, let us have a brief look at a general target space M with metric G and 3-form field H. Let S be a finite subgroup of the isometry group of M which also preserves H. Pick a good open cover $\mathcal{O}^M$ of M which is invariant under S in the sense that, for all $x \in S$, we have $x(\mathcal{O}^M_i) = \mathcal{O}^M_{x.i}$ for some index $x.i$. This turns x into a Čech-extended map as in (2.11). Let $\mathcal{G} = (\mathcal{O}^M, b)$ be a gerbe on M with curvature H. Clearly, the stable isomorphisms $\mathcal{A}_x : \mathcal{G} \to x.\mathcal{G}$ exist if and only if the orbit $\{x.\mathcal{G} \mid x \in S\}$ lies entirely within one stable-isomorphism class. In terms of local data, $\mathcal{A}_x$ obey $(\delta_S b)_x = D\mathcal{A}_x$, where $(\delta_S b)_x = x.b - b$ (cf. Appendix A.1 for a basic reminder on the finite-group cohomology). We shall call the collection $\mathcal{A}_S = \{\mathcal{A}_x \mid x \in S\}$ such that $(\delta_S b)_x = D\mathcal{A}_x$ an *element-wise presentation of S on b*. The obstructions to the existence of an element-wise presentation are contained in $\mathbb{H}^2(M) = \ker D_{(2)}/\mathrm{im}\, D_{(1)}$, the set of stable-isomorphism classes of gerbes over M with curvature $H = 0$. Indeed, while $D(\delta_S b)_x = (0, 0, 0, 1)$ holds true in consequence of $x^*H = H$, that $(\delta_S b)_x$ is a D-coboundary is *guaranteed* only if $\mathbb{H}^2(M)$ consists of just one element. The latter cohomology group satisfies $\mathbb{H}^2(M) \cong H^2(M, U(1))$, see [Gj] and also [Jh, Section 2.2], and so it trivializes for $M = \mathrm{G}$ a compact simple connected and simply connected Lie group.

2.4 Holonomy for world-sheets with circular defect lines

We now turn to the holonomy formula for world-sheets with circular defect lines, but without defect junctions. This is the situation treated in [FSW, Wa2]. As in Section 2.2, we are given a manifold M with gerbe $\mathcal{G}$, and, in addition, we now have a $\mathcal{G}$-bi-brane $\mathcal{B}$ with local data as in (2.14).

While the formulation of the holonomy itself does not require any further structure, the conditions to be imposed on the sigma-model fields at the defect do. The extra structure is a metric G on the target space M and a metric γ on the world-sheet Σ, both of which are part of the sigma-model data, entering explicitly the kinetic term in the action functional (1.1).

By a *circle-field configuration* on the world-sheet Σ we mean a pair (Λ, X), where Λ is an oriented one-dimensional submanifold of Σ with empty boundary, i.e., a collection of oriented circles in Σ, and

$$X : \Sigma \to M \sqcup Q \tag{2.21}$$

is a map from the world-sheet into the disjoint union of the target space M and the $\mathcal{G}$-bi-brane world-volume Q, with the following properties:

(L1) X maps $\Sigma - \Lambda$ to M and is once differentiable on $\Sigma - \Lambda$. Furthermore, it maps Λ to Q and is once differentiable on Λ.

(L2) Let $p \in \Lambda$ and let U be a small neighbourhood of p. As Σ and Λ are oriented, Λ splits U into two open sets U_1 and U_2. For example, if U is the open unit disc in $\mathbb{R}^2$ and Λ is the real line, both with the standard orientation, then U_1 is the upper open half-disc and U_2 the lower open half-disc. We demand that, for $\alpha = 1, 2$, the restriction $X|_{U_\alpha}$ has a differentiable extension $X_{|\alpha} : \overline{U}_\alpha \to M$ to the closure $\overline{U}_\alpha$ of U_α, such that $X_{|\alpha}(p) = \iota_\alpha(X(p))$.

(L3) With p, U_α and $X_{|\alpha}$ as in (L2), let $\widehat{t} \in T_p\Sigma$ be the unit vector tangent to Λ in the direction given by the orientation of Λ, and let $\widehat{n}_\alpha \in T_p\Sigma, \alpha = 1, 2$ be the unit vectors normal to Λ and pointing, each, to the side of U_α (so that, in particular, $\widehat{n}_1 = -\widehat{n}_2$). Then, for all $v \in T_{X(p)}Q$, we require that the constraint

$$G_{X_{|1}(p)}(\iota_{1*}v, X_{|1*}\widehat{n}_1) + G_{X_{|2}(p)}(\iota_{2*}v, X_{|2*}\widehat{n}_2) - \frac{\mathrm{i}}{2}\omega_{X(p)}(v, X_*\widehat{t}) = 0 \quad (2.22)$$

be satisfied by the tangent (pushforward) maps $X_{|\alpha*} : T\overline{U}_\alpha \to TM$. A derivation of the above defect constraint is presented in Appendix A.2.

Conditions (L2) and (L3) merit some comment. If Q is a submanifold of $M \times M$ then (L2) just means that the maps to the left and to the right of Λ have, each, a differentiable extension to Λ, and that Λ gets mapped to $Q \subset M \times M$ under these two extensions. Condition (L2) is a straightforward generalization of this requirement to the case when Q is not necessarily a submanifold. Condition (L3) is new; it forms part of the dynamical data of the sigma model, and is therefore not needed in [FSW]; it will play an important role in the discussion of topological defect lines in Section 2.9 below. It constrains the variation of the derivatives of the embedding map X across the defect in a manner dictated by the principle of least action applied to the sigma-model action functional.

For instance, if $\mathcal{B} = \mathcal{B}_{\text{triv}}$ is the trivial defect (2.16) then condition (L2) enforces the equality $X_{|1}(p) = X(p) = X_{|2}(p)$ for any $p \in \Lambda$. Thus, X is a continuous map $\Sigma \to M$, which — so far — is only required to be *differentiable* on $\Sigma - \Lambda$ and along Λ. Condition (2.22) now reads $G_{X(p)}(v, X_{|1*}\widehat{n}_1) + G_{X(p)}(v, X_{|2*}\widehat{n}_2) = 0$ for all $v \in T_{X(p)}M$. If we remember that $X_{|1}$ is the differentiable extension of X to the left of Λ, that $X_{|2}$ is the differentiable extension of X to the right of Λ, that $\widehat{n}_1 = -\widehat{n}_2$, and that G is a non-degenerate pairing, we see that (2.22) forces the normal derivative to be continuous across Λ.

As another example, consider the case in which $\mathcal{B} = \mathcal{B}_{\text{bnd}}$ describes a D-brane as in (2.17). Condition (L2) forces the neighbourhood of Λ in Σ to its right to be mapped to $\{\bullet\} \subset M$, while the neighbourhood of Λ in Σ to its left is mapped to N such that the extension reads $X_{|1} = X$ on Λ. Since $X_{|2}$ is constant, $X_{|2*} = 0$, and so (2.22) becomes $2G_{X(p)}(v, X_*\widehat{n}_1) - \mathrm{i}\omega_{X(p)}(v, X_*\widehat{t}) = 0$ for all $v \in T_{X(p)}D$. When written in terms of local coordinates X^μ for M, with $v = \delta X^\mu\partial_\mu$ tangent to D, this yields

$$\delta X^\mu\big(G_{\mu\nu}(X)\partial_n X^\nu - \mathrm{i}\omega_{\mu\nu}(X)\partial_t X^\nu\big)\big|_\Lambda = 0, \quad (2.23)$$

where ∂_t is the tangent derivative at the boundary Λ of $\Sigma - X^{-1}(\{\bullet\})$, $\partial_n = \frac{\gamma(\partial_t,\partial_a)}{\sqrt{\det\gamma}}\epsilon^{ab}\partial_a$ is the (inward-)normal one, and $\omega = \omega_{\mu\nu}\,\mathsf{d}X^\mu \wedge \mathsf{d}X^\nu$ with $\omega_{\mu\nu}$ antisymmetric in its indices. The above are just the standard mixed Dirichlet–Neumann boundary conditions for a world-sheet with boundary Λ, where the boundary gets mapped to the D-brane world-volume D endowed with the global twisted-gauge invariant 2-form $\omega = F_i + B_i$ (F_i being the "curvature" of the gerbe-twisted gauge field), see [Cl, Equation (3.3)].

One way to think of a configuration (Λ, X) is that it describes a string moving in a possibly disconnected target space, where the string is allowed to "tunnel" from one component into another by passing through Q.

From a category-theory perspective, M could be viewed as the set of objects and Q as the set of arrows, with ι_1 and ι_2 designating the source and the target of a given arrow, respectively. Unfortunately, we only know how to formulate composition in very special cases, so the analogy stops here.

Note that, in our formulation, the entire Λ gets mapped to the *same* $\mathcal{G}$-bi-brane manifold Q. In this sense, every defect circle, i.e., every connected component of Λ, carries the *same* defect condition. This may seem a restriction, but it is really just a convenient way to absorb the possibility of having different $\mathcal{G}$-bi-branes for different defect circles into the map X.

More specifically, note, first of all, that Λ divides the world-sheet Σ into connected components, which we shall call *patches*. The situation in which different patches get mapped to different target spaces $M_1, M_2, \ldots$ is accommodated by taking $M = M_1 \sqcup M_2 \sqcup \ldots$. Since X maps $\Sigma - \Lambda$ to M, each patch will sit entirely in one of the components M_k by the continuity of X. Next, suppose that we have several $\mathcal{G}$-bi-branes $\mathcal{B}_1, \mathcal{B}_2, \ldots$ on M, and that we want to label each of the defect circles by one of the $\mathcal{B}_k$. This is accounted for by setting $\mathcal{B} = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \ldots$, and choosing the map $X|_\Lambda$ accordingly. Since $X|_\Lambda$ is continuous, the image of each defect circle has to lie in one of the components $\mathcal{B}_k$.

The holonomy for a circle-field configuration (Λ, X) is a modification of (2.13) which includes an additional term associated to Λ,

$$\mathrm{Hol}(\Lambda, X) = \mathrm{Hol}_{\mathcal{G}}(X) \cdot \mathrm{Hol}_{\mathcal{B}}(X|_\Lambda), \tag{2.24}$$

where

$$\mathrm{Hol}_{\mathcal{B}}(X|_\Lambda) = \prod_{e \in \triangle(\Lambda)} \left[\exp\left(\mathsf{i}\int_e \widehat{P}_e\right) \prod_{v \in e} \widehat{K}_{ev}(v) \right]. \tag{2.25}$$

Above, $\mathrm{Hol}_{\mathcal{G}}(X)$ is given by the same expression[3] (2.13), together with a prescription as to how to deal with the jumps of X across Λ, which we give shortly. The novel term $\mathrm{Hol}_{\mathcal{B}}(X|_\Lambda)$ can be understood as the holonomy of a gerbe-twisted line bundle over Λ, see [FSW, Wa2], and also [CJM, GR1, Ga4] for the corresponding observation for boundary circles instead of defect circles. In more detail, $\mathrm{Hol}_{\mathcal{G}}(X)$ and $\mathrm{Hol}_{\mathcal{B}}(X|_\Lambda)$ are defined as follows:

- Let $\triangle(\Sigma)$ be a triangulation subordinate to (Λ, X) in the following sense. For each triangle $t \in \triangle(\Sigma)$, there must exist an index $i_t \in \mathcal{I}^M$ such that the interior of t gets mapped to $\mathcal{O}^M_{i_t}$. We require that Λ be covered by edges of $\triangle(\Sigma)$, and we denote by $\triangle(\Lambda)$ the resulting one-dimensional triangulation of Λ. For each edge $e \in \triangle(\Lambda)$, there must be an index $i_e \in \mathcal{I}^Q$ such that X maps e to $\mathcal{O}^Q_{i_e}$. It is understood that the assignments $t \mapsto i_t$ and $e \mapsto i_e$ are made once and for all.
- For each edge e and each vertex v of $\triangle(\Sigma)$ which do not lie on Λ, we fix indices $i_e \in \mathcal{I}^M$ and $i_v \in \mathcal{I}^M$ such that X maps e to $\mathcal{O}^M_{i_e}$ and v to $\mathcal{O}^M_{i_v}$. For vertices v that lie on Λ, we pick an assignment $v \mapsto i_v \in \mathcal{I}^Q$. As in Section 2.2, these maps are guaranteed to exist by the continuity properties of X.
- In expression (2.13) for $\mathrm{Hol}_{\mathcal{G}}(X)$, we still have $\widehat{B}_t = X^*B_{i_t}, \widehat{A}_{te} = X^*A_{i_t i_e}$ and $\widehat{g}_{tev} = (X^*g_{i_t i_e i_v})^{\varepsilon_{tev}}$. If one of the edges of t lies in Λ then the pullbacks use the differentiable extension of X from the interior of t to all of t (which exists by condition (L2) on X). If $e \subset \Lambda$ then $i_e \in \mathcal{I}^Q$, and it is understood that $A_{i_t i_e}$ and $g_{i_t i_e i_v}$ stand for $A_{i_t \phi_1(i_e)}$ and $g_{i_t \phi_1(i_e)\phi_1(i_v)}$, respectively, if the orientation of e, as induced from t, agrees with that of Λ, or for $A_{i_t \phi_2(i_e)}$ and $g_{i_t \phi_2(i_e)\phi_2(i_v)}$ otherwise, cf. figure 4.
- $\widehat{P}_e = X^*P_{i_e}$, where X is understood as a map from $e \subset \Lambda$ to $\mathcal{O}^Q_{i_e}$. The resulting 1-form on e is integrated using the orientation of Λ.
- $\widehat{K}_{ev} = (X^*K_{i_e i_v})^{-\varepsilon_{ev}}$ (note the minus sign), where X is understood as a map from $v \in \Lambda$ to $\mathcal{O}^Q_{i_e i_v}$. The edge e inherits an orientation from Λ. The sign convention reads: $\varepsilon_{ev} = +1$ if, with respect to this orientation, v sits at the end of e, and $\varepsilon_{ev} = -1$ otherwise.

Again, the expression for the holonomy $\mathrm{Hol}(\Lambda, X)$ is basically dictated by requiring invariance with respect to the choice of the triangulation, and with

[3]We emphasize that expression (2.13) loses its fundamental property — namely, the invariance under changes of the world-sheet triangulation and gauge transformations — in the presence of a defect network, and hence it now defines a collection of transport operators for the transgression bundle of [Ga1] instead of surface holonomy. It is with this understanding that we choose to denote it by the same symbol $\mathrm{Hol}_{\mathcal{G}}(X)$ for the sake of brevity. Analogous remarks apply to the defect-vertex corrections to the holonomy.

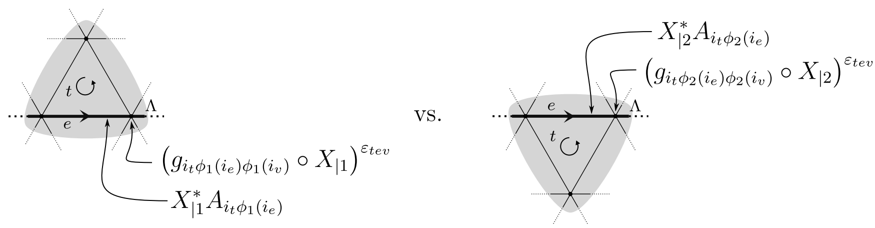


Figure 4: When a triangle t shares an edge e with a defect line, the orientation of the defect either agrees with that of ∂t or not. This decides which pullback map to apply to the connection 1-forms A_{ij} and the transition functions g_{ijk} on M.

respect to gauge transformations of the gerbe. This is discussed at length in Section 2.7.

For the trivial $\mathcal{G}$-bi-brane $\mathcal{B} = \mathcal{B}_{\text{triv}}$, it is easy to check that $\text{Hol}_{\mathcal{B}}(X|_\Lambda) = 1$ so that the holonomy in the presence of defects (2.24) reduces to the holonomy in the absence of defects (2.13).

In the case when the bi-brane describes a D-brane $\mathcal{B} = \mathcal{B}_{\text{bnd}}$, one can verify that (2.24) reproduces the holonomy for world-sheets with boundary given in [CJM, GR1, Ga4]. The world-sheet with boundary is obtained as follows: Given a circle-field configuration (Λ, X) on Σ for the target $M = N \sqcup \{\bullet\}$ with bi-brane $\mathcal{B}_{\text{bnd}}$, some parts of the world-sheet will be mapped to $\{\bullet\}$. On these components of Σ, there are no degrees of freedom, and so both the kinetic and the topological term of the action vanish. Hence, we may as well remove these parts of the world-sheet, which yields a new world-sheet Σ' with Λ as its boundary.

2.4.1 The Lie-group example (cont'd)

Let us inspect what a circle-field configuration (Λ, X) for the $\mathcal{G}^{\star\mathsf{k}}$-bi-brane $\mathcal{B}_{Z(\mathrm{G})}$ with world-volume (2.18) looks like. Condition (L2) means that whenever $p \in \Lambda$ gets mapped to $X(p) = (g, z) \in G \times Z(\mathrm{G})$ then $X_{|1}(p) = \iota_1(g, z) = g$ and $X_{|2}(p) = \iota_2(g, z) = z^{-1} \cdot g$. Thus,

$$\lim_{\zeta \searrow p} X(\zeta) = z \cdot \lim_{\zeta \nearrow p} X(\zeta), \tag{2.26}$$

where by $\zeta \searrow p$ and $\zeta \nearrow p$ we mean that $\zeta \in \Sigma$ approaches p in the neighbourhood U_1 and U_2, respectively, so that, e.g., $\lim_{\zeta \searrow p} X(\zeta) = X_{|1}(p)$. Condition (L3) now reads $G_g(v, X_{|1*}\widehat{n}_1) + G_{z^{-1}\cdot g}(z_*^{-1} v, X_{|2*}\widehat{n}_2) = 0$ for all $v \in T_g\mathrm{G}$. Since the (Cartan–Killing) metric on G is G-invariant, this implies $X_{|1*}\widehat{n}_1 + (z \cdot X_{|2})_*\widehat{n}_2 = 0$. Together with $X_{|1*}\widehat{t} = (z \cdot X_{|2})_*\widehat{t}$, which follows from the identity $X_{|1}(p) = z \cdot X_{|2}(p)$ valid for all $p \in \Lambda$, this yields an equality of the tangent maps

$$X_{|1*} = (z \cdot X_{|2})_* : T_p\Sigma \longrightarrow T_g\mathrm{G}. \tag{2.27}$$

In other words, also the first derivative of the field X has to jump in a controlled manner across Λ. Altogether, we see that this $\mathcal{G}^{\star\mathsf{k}}$-bi-brane describes jump defects with the value of the jump dictated by the value of the field on the defect circle.

2.5 Abelian inter-bi-branes

The introduction of a target-space structure for defect junctions on the world-sheet calls for the notion of a 2-morphism, as introduced in Section 2.1, as well as for that of the dual of a stable isomorphism and the dual of a $\mathcal{G}$-bi-brane, which we now define.

Suppose that we are given a target space M with gerbes $\mathcal{G}$ and $\mathcal{H}$, and a stable isomorphism $\Phi = (\Pi_i, \chi_{ij}) : \mathcal{G} \to \mathcal{H}$. We define the *dual stable isomorphism* $\Phi^\vee : \mathcal{H} \to \mathcal{G}$ by the local data $(-\Pi_i, \chi_{ij}^{-1})$. We also introduce 2-morphisms (the *death 2-morphisms*)

$$d_\Phi : \Phi^\vee \circ \Phi \Longrightarrow \mathrm{id}_{\mathcal{G}}, \tag{2.28}$$

with local data $d_\Phi = (1)$. One may wonder why ever we should give a special name to a 2-morphism with trivial data. The reason is that we have made a specific choice for the dual morphisms here; other choices, differing by 2-isomorphisms (gauge transformations), are possible which would lead to death 2-morphisms with non-trivial data. Furthermore, a generic d_Φ cannot be avoided in the framework of bundle gerbes [Wa2].

Let $\mathcal{B}$ be a $\mathcal{G}$-bi-brane of the form (2.14). The $\mathcal{G}$-bi-brane *dual* to $\mathcal{B}$ is defined as

$$\mathcal{B}^\vee = \big(Q, -\omega, \mathcal{O}^Q, \check{\iota}_2, \check{\iota}_1, \Phi^\vee\big). \tag{2.29}$$

Below, we shall often use the convenient notation $\mathcal{B}^+ \equiv \mathcal{B}$ and $\mathcal{B}^- \equiv \mathcal{B}^\vee$, and we shall refer to the data in $\mathcal{B}^\pm$ by a superscript $(\ )^\pm$. One can check that the holonomy (2.24) does not change if we simultaneously reverse the orientation of Λ and replace $\mathcal{B}$ by $\mathcal{B}^\vee$.

A $(\mathcal{G},\mathcal{B})$-*inter-bi-brane* $\mathcal{J}$ for a gerbe $\mathcal{G}$ over a manifold M and a $\mathcal{G}$-bi-brane $\mathcal{B}$ is an infinite tuple

$$\mathcal{J} = (T_n, \mathcal{O}^{T_n}, \varphi_n, \tau_n \mid n \in \mathbb{Z}_{>0}), \tag{2.30}$$

where, for every $n \in \mathbb{Z}_{>0}$,

(I.i) T_n is a smooth manifold;
(I.ii) $\mathcal{O}^{T_n} = \{\mathcal{O}_i^{T_n} \mid i \in \mathcal{I}^{T_n}\}$ is a good open cover of T_n;
(I.iii) φ_n is a 2-morphism;
(I.iv) $\tau_n = \big(\varepsilon_n^{k,k+1}, \check{\pi}_n^{k,k+1} \mid k = 1, 2, \ldots, n\big)$ are collections of maps.

The detailed description of the infinite sequence $(\varphi_n, \tau_n \mid n \in \mathbb{Z}_{>0})$, and of the conditions which the data have to obey is somewhat lengthy but straightforward. First of all, each T_n carries the data needed to formulate the holonomy for an n-fold junction of defect lines on the world-sheet. We allow the possibility that T_n is the empty set. For example, if one wants to consider world-sheets with three-valent defect vertices exclusively one could choose $T_n = \emptyset$ for $n \neq 3$. In what follows, we shall frequently refer to the manifold T given by the disjoint union

$$T = \bigsqcup_{n=1}^{\infty} T_n, \tag{2.31}$$

which we call the world-volume of the $(\mathcal{G},\mathcal{B})$-inter-bi-brane.

The maps $\varepsilon_n^{k,k+1}$ in the definition of τ_n are continuous functions

$$\varepsilon_n^{k,k+1} : T_n \to \{+1, -1\}, \quad \varepsilon_n^{n,n+1} \equiv \varepsilon_n^{n,1}. \tag{2.32}$$

They give a decomposition of T_n into up to 2^n disconnected pieces (not all combinations of the n signs $\varepsilon_n^{1,2}, \varepsilon_n^{2,3}, \ldots, \varepsilon_n^{n,n+1}$ need occur). This decomposition of T_n will be needed to accommodate the different orientations with which n edges can end on an n-valent defect vertex in the world-sheet. The objects $\check{\pi}_n^{k,k+1}$ are Čech-extended maps

$$\check{\pi}_n^{k,k+1} = \big(\pi_n^{k,k+1}, \psi_n^{k,k+1}\big) : T_n \to Q, \quad \check{\pi}_n^{n,n+1} \equiv \check{\pi}_n^{n,1}, \tag{2.33}$$

composed of smooth manifold maps $\pi_n^{k,k+1} : T_n \to Q$ and the attendant index maps $\psi_n^{k,k+1} : \mathcal{I}^{T_n} \to \mathcal{I}^Q$, subject to the condition

$$\check{\iota}_2^{\varepsilon_n^{k-1,k}} \circ \check{\pi}_n^{k-1,k} = \check{\iota}_1^{\varepsilon_n^{k,k+1}} \circ \check{\pi}_n^{k,k+1} \quad \text{for } k = 1, 2, \ldots, n, \tag{2.34}$$

in which the identifications $\varepsilon_n^{0,1} \equiv \varepsilon_n^{n,1}$ and $\check{\pi}_n^{0,1} \equiv \check{\pi}_n^{n,1}$ are implicit. Here, the manifold map from T_n to M appearing on the left-hand side is given by $p \mapsto \iota_2^{\varepsilon_n^{k-1,k}(p)} \circ \pi_n^{k-1,k}(p)$, and similarly for the right-hand side. Recall that the notation $\iota_1^{\pm}$ and $\iota_2^{\pm}$ refers to the maps from the definition of the $\mathcal{G}$-bi-brane $\mathcal{B}^+ = \mathcal{B}$ and its dual $\mathcal{B}^- = \mathcal{B}^\vee$. Put together with (B.iv), condition (2.34) enables us to induce another family of Čech-extended maps

$$\check{\pi}_n^k = \check{\iota}_1^{\varepsilon_n^{k,k+1}} \circ \check{\pi}_n^{k,k+1} = (\pi_n^k, \psi_n^k) : T_n \to M \tag{2.35}$$

from $\check{\pi}_n^{k,k+1}$. Just as for $\mathcal{G}$-bi-branes, we shall not — for the sake of transparency — spell out ψ_n^k and $\psi_n^{k,k+1}$ explicitly in formulæ involving pullbacks from M or Q to T_n.

Using $\check{\pi}_n^k$ and $\check{\pi}_n^{k,k+1}$, we can pull back data to T_n from M and Q, respectively. Thus, in particular, we obtain a family of 2-forms on T_n,

$$\omega_n^{k,k+1} = \left(\pi_n^{k,k+1}\right)^* \omega^{\varepsilon_n^{k,k+1}}. \tag{2.36}$$

We demand the sum of all these 2-forms to vanish for each $n \in \mathbb{Z}_{>0}$,

$$\sum_{k=1}^{n} \omega_n^{k,k+1} = 0. \tag{2.37}$$

In the light of the invariance arguments to be presented in Section 2.7, we could — more generally — have postulated the existence of 1-forms θ_n on T_n such that the above sum is equal to $\mathsf{d}\theta_n$ instead of being zero. However, the analysis of the defect conditions for the fields of the underlying sigma model, derived in Appendix A.2 through the application of the variational principle, shows that only those parts of T_n can be probed by the sigma-model field in which θ_n vanishes. Therefore, we may as well set $\theta_n = 0$ for all $n \in \mathbb{Z}_{>0}$ from the start.

In addition to the above, we also obtain, on each T_n, a family of gerbes and stable isomorphisms

$$\mathcal{G}_n^k = \left(\check{\pi}_n^k\right)^* \mathcal{G}, \quad \Phi_n^{k,k+1} = \left(\check{\pi}_n^{k,k+1}\right)^* \Phi^{\varepsilon_n^{k,k+1}}, \tag{2.38}$$

where $\Phi_n^{k,k+1}$ is readily verified to be a stable isomorphism $\mathcal{G}_n^k \to \mathcal{G}_n^{k+1} \star I(\omega_n^{k,k+1})$. Here, we have used the identification $\mathcal{G}_n^{n+1} \equiv \mathcal{G}_n^1$ and abbreviated $I(\omega_n^{k,k+1}) \equiv I(\mathcal{O}^{T_n}, \omega_n^{k,k+1})$, and we shall adhere to these conventions below.

Define the stable isomorphisms $\Xi_n : \mathcal{G}_n^1 \to \mathcal{G}_n^1$ on each component T_n by the cyclic composition of the stable isomorphisms $\Phi_n^{k,k+1}$

$$\Xi_n : \mathcal{G}_n^1 \xrightarrow{\Phi_n^{1,2}} \mathcal{G}_n^2 \star I\big(\omega_n^{1,2}\big) \xrightarrow{\Phi_n^{2,3} \star \mathrm{id}_{I(\omega_n^{1,2})}} \mathcal{G}_n^3 \star I\big(\omega_n^{2,3}\big) \star I\big(\omega_n^{1,2}\big) \longrightarrow \cdots$$
$$\xrightarrow{\Phi_n^{n,1} \star \mathrm{id}_{I(\omega_n^{n-1,n})} \star \cdots \star \mathrm{id}_{I(\omega_n^{1,2})}} \mathcal{G}_n^1 \star I\big(\omega_n^{n,1}\big) \star I\big(\omega_n^{n-1,n}\big) \star \cdots \star I\big(\omega_n^{1,2}\big) \equiv \mathcal{G}_n^1. \tag{2.39}$$

The last identity follows from the composition rule $I(\omega) \star I(\omega') = I(\omega + \omega')$, cf. Section 2.1, and condition (2.37). We may finally introduce a 2-morphism φ_n as

$$\varphi_n : \Xi_n \Longrightarrow \mathrm{id}_{\mathcal{G}_n^1}. \tag{2.40}$$

This completes the description of the data of an abelian $(\mathcal{G}, \mathcal{B})$-inter-bi-brane and of the conditions it has to satisfy.

Upon rewriting (2.40) in terms of the relevant local data $\Phi_n^{k,k+1} = (P_{n,i}^{k,k+1}, K_{n,ij}^{k,k+1}) \in A_{T_n}^1$ and $\varphi_n = (f_{n,i}) \in A_{T_n}^0$ on $\mathcal{O}^{T_n}$, we obtain the relation

$$\sum_{k=1}^{n} \big(P_{n,i}^{k,k+1}, K_{n,ij}^{k,k+1}\big) + \big(-\mathsf{i}\, \mathsf{d} \log f_{n,i}, f_{n,i}\, f_{n,j}^{-1}\big) = (0, 1). \tag{2.41}$$

In the sequel, we shall often employ the composite 2-morphism $\varphi = (f_i) \in A_T^0$ on the total world-volume T glued from the 2-morphisms φ_n as per

$$\varphi|_{T_n} = \varphi_n. \tag{2.42}$$

The simplest example of an inter-bi-brane is the trivial $(\mathcal{G}, \mathcal{B})$-inter-bi-brane $\mathcal{J}_{\mathrm{triv}}$ which is defined for the trivial $\mathcal{G}$-bi-brane $\mathcal{B} = \mathcal{B}_{\mathrm{triv}}$. In this case, one takes T_n in the form of 2^n copies of M, one for each possible set of values of the maps $\varepsilon_n^{k,k+1}$. The projections $\check{\pi}_n^{k,k+1}$ all coincide with the identity map on M. One then finds that Ξ_n is the identity stable isomorphism from $\mathcal{G}$ to itself, and one chooses for φ_n the identity 2-morphism with local data $\varphi_n = (1)$.

2.5.1 The Lie-group example (cont'd)

We shall now tailor down the exposition of the various bi-brane and inter-bi-brane structures associated with $Z(\mathrm{G})$-jump defects of the WZW model for G to the main task at hand which consists in obtaining the 3-cocycle of (1.3). The minimal set of inter-bi-brane data is provided by the two sets

$$T_3 = \mathrm{G} \times Z(\mathrm{G}) \times Z(\mathrm{G}) \times \{\pm 1\}^3, \quad T_4 = \mathrm{G} \times Z(\mathrm{G}) \times Z(\mathrm{G}) \times Z(\mathrm{G}) \times \{\pm 1\}^4, \tag{2.43}$$

describing three- and four-valent vertices, respectively. As the notation clearly suggests, the sign maps (2.32) are fixed in the form

$$\begin{aligned} \varepsilon_3^{k,k+1}(g,x,y,\varepsilon_{1,2},\varepsilon_{2,3},\varepsilon_{3,1}) &= \varepsilon_{k,k+1}, \\ \varepsilon_4^{k,k+1}(g,x,y,z,\varepsilon_{1,2},\varepsilon_{2,3},\varepsilon_{3,4},\varepsilon_{4,1}) &= \varepsilon_{k,k+1}. \end{aligned} \tag{2.44}$$

Since we shall only need for our purposes the distinguished connected components $T_{2+1} \equiv T_{3,++-}$ and $T_{3+1} \equiv T_{4,+++-}$ of T_3 and T_4, we fix the signs as

$$\varepsilon_3^{1,2} = +1 = \varepsilon_3^{2,3}, \quad \varepsilon_3^{3,1} = -1, \quad \varepsilon_4^{1,2} = +1 = \varepsilon_4^{2,3} = \varepsilon_4^{3,4}, \quad \varepsilon_4^{4,1} = -1 \tag{2.45}$$

for the remainder of the discussion. Below, we detail the remaining elements of the description solely for T_{2+1}, postponing the construction of T_{3+1} to Section 2.8.

The good open cover of T_{2+1} is obtained in the same way as for the $\mathcal{G}^{\star\mathsf{k}}$-bi-brane $\mathcal{B}_{Z(\mathrm{G})}$ of (2.18), that is we choose the open sets $\mathcal{O}^{T_{2+1}}_{i,x,y} = \mathcal{O}^{\mathrm{G}}_i \times \{(x,y)\}$ with $i \in \mathcal{I}^{\mathrm{G}}$ and $x, y \in Z(\mathrm{G})$ (and the redundant signs dropped from the notation, which is also what we do below). The Čech-extended maps $\breve{\pi}^{k,k+1}_{2+1} = \left(\pi_3^{k,k+1}, \psi_3^{k,k+1}\right)\big|_{T_{2+1}}$ for the edges then evaluate on points $(g,x,y) \in \mathcal{O}^{T_{2+1}}_{i,x,y}$ as

$$\begin{aligned} \pi^{1,2}_{2+1}(g,x,y) &= (g,x), & \psi^{1,2}_{2+1}(i,x,y) &= (i,x), \\ \pi^{2,3}_{2+1}(g,x,y) &= (x^{-1}\cdot g, y), & \psi^{2,3}_{2+1}(i,x,y) &= (x^{-1}.i, y), \\ \pi^{3,1}_{2+1}(g,x,y) &= (g, x\cdot y), & \psi^{3,1}_{2+1}(i,x,y) &= (i, x\cdot y). \end{aligned} \tag{2.46}$$

These manifestly obey condition (2.34), for example $\iota_2(\pi^{1,2}_{2+1}(g,x,y)) = x^{-1} \cdot g = \iota_1(\pi^{2,3}_{2+1}(g,x,y))$. The corresponding Čech-extended maps (2.35) for the patches are

$$\pi^1_{2+1}(g,x,y) = g, \quad \pi^2_{2+1}(g,x,y) = x^{-1}\cdot g, \quad \pi^3_{2+1}(g,x,y) = (x\cdot y)^{-1}\cdot g, \tag{2.47}$$

and similarly for ψ^k_{2+1}.

At this stage, we still have to fix the 2-morphisms φ_n from the collection (2.30). We shall only describe those supported by the subspace $T^{x,y}_{2+1} = \mathrm{G} \times \{(x,y)\} \subset T_{2+1}$, which we identify with G. Using (2.47) we get the three pullback gerbes on $T^{x,y}_{2+1}$ defined in (2.38),

$$\begin{aligned} \mathcal{G}^1_{2+1} &= \big(\check{\pi}^1_{2+1}\big)^* \mathcal{G}^{\star\mathsf{k}} = \mathcal{G}^{\star\mathsf{k}}, \\ \mathcal{G}^2_{2+1} &= \big(\check{\pi}^2_{2+1}\big)^* \mathcal{G}^{\star\mathsf{k}} = x.\mathcal{G}^{\star\mathsf{k}}, \\ \mathcal{G}^3_{2+1} &= \big(\check{\pi}^3_{2+1}\big)^* \mathcal{G}^{\star\mathsf{k}} = (x \cdot y).\mathcal{G}^{\star\mathsf{k}}, \end{aligned} \tag{2.48}$$

as well as the pullback 1-morphisms

$$\begin{aligned} \Phi^{1,2}_{2+1} &= \big(\check{\pi}^{1,2}_{2+1}\big)^* \Phi = \mathcal{A}_x : \mathcal{G}^1_{2+1} \longrightarrow \mathcal{G}^2_{2+1}, \\ \Phi^{2,3}_{2+1} &= \big(\check{\pi}^{2,3}_{2+1}\big)^* \Phi = x.\mathcal{A}_y : \mathcal{G}^2_{2+1} \longrightarrow \mathcal{G}^3_{2+1}, \\ \Phi^{3,1}_{2+1} &= \big(\check{\pi}^{3,1}_{2+1}\big)^* \Phi^\vee = \mathcal{A}^\vee_{x\cdot y} : \mathcal{G}^3_{2+1} \longrightarrow \mathcal{G}^1_{2+1}. \end{aligned} \tag{2.49}$$

We next fix 2-morphisms

$$\widetilde{\varphi}_{x,y} : (x.\mathcal{A}_y) \circ \mathcal{A}_x \Longrightarrow \mathcal{A}_{x\cdot y}, \tag{2.50}$$

where both sides are stable isomorphisms $\mathcal{G}^{\star\mathsf{k}} \to (x \cdot y).\mathcal{G}^{\star\mathsf{k}}$. These 2-morphisms can be read off from [GR2, Section 3] upon consulting [GSW2, Sections 1 and 3] whose conventions have been adopted in our discussion. Finally, we define the 2-morphism $\varphi_{2+1} : \Phi^{3,1}_{2+1} \circ \Phi^{2,3}_{2+1} \circ \Phi^{1,2}_{2+1} \Rightarrow \mathrm{id}_{\mathcal{G}^1_{2+1}}$ on $T^{x,y}_{2+1}$ as

$$\varphi_{2+1} : \mathcal{A}^\vee_{x\cdot y} \circ (x.\mathcal{A}_y) \circ \mathcal{A}_x \xRightarrow{\mathrm{id}_{\mathcal{A}^\vee_{x\cdot y}} \circ \widetilde{\varphi}_{x,y}} \mathcal{A}^\vee_{x\cdot y} \circ \mathcal{A}_{x\cdot y} \xRightarrow{d_{\mathcal{A}_{x\cdot y}}} \mathrm{id}_{\mathcal{G}^{\star\mathsf{k}}}. \tag{2.51}$$

The composition of 2-morphisms represented by the superposition of the corresponding double arrows is called 'vertical' in the 2-categorial language and denoted with the symbol $\bullet$, e.g., $\varphi_{2+1} = d_{\mathcal{A}_{x\cdot y}} \bullet \big(\mathrm{id}_{\mathcal{A}^\vee_{x\cdot y}} \circ \widetilde{\varphi}_{x,y}\big)$. We shall use the composition symbol in the reminder of the paper in order to shorten some formulæ.

The existence of the 2-morphisms $\widetilde{\varphi}_{x,y}$ follows form the triviality of the cohomology group $H^1(\mathrm{G}, \mathrm{U}(1))$. In order to see this, let us look at a general symmetry group S again, as we did at the end of Section 2.1. Suppose that an element-wise presentation $\mathcal{A}_S$ of S on b exists. In terms of local data, the 2-morphisms $\widetilde{\varphi}_{x,y} : (x.\mathcal{A}_y) \circ \mathcal{A}_x \Longrightarrow \mathcal{A}_{x\cdot y}$ have to solve $-D\widetilde{\varphi}_{x,y} = (\delta_S \mathcal{A})_{x,y}$ for all $x, y \in S$, where $(\delta_S \mathcal{A})_{x,y} = x.\mathcal{A}_y - \mathcal{A}_{x\cdot y} + \mathcal{A}_x$. We shall collect the 2-morphisms into a set $\widetilde{\varphi}_S = \{\widetilde{\varphi}_{x,y} \mid x, y \in S\}$ and call the pair $(\mathcal{A}_S, \widetilde{\varphi}_S)$ a *homomorphic presentation of S on b*. Assuming the existence of the element-wise presentation $\mathcal{A}_S$, the obstruction to the existence of a homomorphic presentation is contained in $\mathbb{H}^1(M) = \ker D_{(1)}/\mathrm{im}\, D_{(0)}$, the set of isomorphism classes of flat line bundles over M. Indeed, the equality $D\delta_S\mathcal{A} \equiv \delta_S D\mathcal{A} = \delta_S^2 b = (0,0,1)$ always holds due to $\delta_S^2 = 0$, but the existence of $\widetilde{\varphi}_S$ requires $(\delta_S\mathcal{A})_{x,y}$ to lie in the image of $D_{(0)}$ for all $x, y \in S$. The cohomology group $\mathbb{H}^1(M)$ satisfies $\mathbb{H}^1(M) \cong H^1(M, U(1))$, see [Gj, Jh], and so it trivializes for $M = \mathrm{G}$ a compact simple connected and simply connected Lie group.

2.6 Holonomy for world-sheets with defect networks

After all the preparations, we can, at last, describe our construction of the holonomy for world-sheets with defect networks.

A *defect network* Γ on a world-sheet Σ is an oriented graph embedded in Σ, together with an ordering of the edges around each vertex. By this we mean that the edges of Γ are oriented submanifolds of Σ, and that, for each vertex of Γ, the edges emanating from this vertex have been labelled in the counter-clockwise order as $e_{1,2}, e_{2,3}, \ldots, e_{n,1}$. (Since the world-sheet is oriented, this is equivalent to marking one of the edges attached to the vertex.) We allow, in particular, circular edges that are not attached to any vertex. The set of edges in Γ is denoted by E_Γ, and the set of its vertices by V_Γ.

A *network-field configuration* on Σ for the target space M with the $\mathcal{G}$-bi-brane $\mathcal{B}$ and the $(\mathcal{G},\mathcal{B})$-inter-bi-brane $\mathcal{J}$ is a pair (Γ, X), where Γ is a defect network and

$$X : \Sigma \to M \sqcup Q \sqcup T \qquad (2.52)$$

is a map from the world-sheet into the disjoint union of the target space M, the $\mathcal{G}$-bi-brane world-volume Q, and the $(\mathcal{G},\mathcal{B})$-inter-bi-brane world-volume T, with the following properties:

(N1) X restricts to a once differentiable map $\Sigma - \Gamma \to M$, and to a once differentiable map $\Gamma - V_\Gamma \to Q$, and it maps V_Γ to T. Furthermore, we have $X(v) \in T_{n_v}$ for a vertex $v \in V_\Gamma$ of valence n_v.

(N2) In a neighbourhood of a point $p \in \Gamma - V_\Gamma$, the map X obeys conditions (L1)–(L3) for a circle-field configuration from Section 2.4.

(N3) Let $v \in V_\Gamma$ be an n_v-valent vertex and let $e_{k,k+1}$ be an edge converging at v. If the edge is oriented towards v we demand that $\varepsilon_{n_v}^{k,k+1}(X(v)) = +1$, and otherwise that $\varepsilon_{n_v}^{k,k+1}(X(v)) = -1$.

(N4) Let v and $e_{k,k+1}$ be as in (N3). The map X sends $e_{k,k+1}$ with its endpoints removed to Q. We demand that X have a differentiable extension $X_{k,k+1} : e_{k,k+1} \to Q$, and that $X_{k,k+1}(v) = \pi_{n_v}^{k,k+1}(X(v))$ hold.

Condition (N3) ensures that a vertex gets mapped to the correct component of T_{n_v} according to the orientation of the edges converging at v, and condition (N4) restricts the jump of X at the vertex itself. There are two implications of (N4) that we wish to emphasize.

First, let $U \subset \Sigma$ be a small neighbourhood of a vertex $v \in V_\Gamma$ of valence n_v. The defect network Γ divides U into n_v open sets $U_1, U_2, \ldots, U_{n_v}$, labelled counter-clockwise around v such that U_k sits between the edges $e_{k-1,k}$ and $e_{k,k+1}$. The map X sends U_k to M. Condition (N2) implies that it has a differentiable extension to $\overline{U}_k - \{v\}$, and condition (N4) ensures that, in fact, X has a differentiable extension $X_k : \overline{U}_k \to M$, and that $X_k(v) = \pi_{n_v}^{k}(X(v))$.

Second, if $\mathcal{B} = \mathcal{B}_{\text{triv}}$ is the trivial $\mathcal{G}$-bi-brane and $\mathcal{J} = \mathcal{J}_{\text{triv}}$ is the trivial $(\mathcal{G}, \mathcal{B})$-inter-bi-brane then — as we have already seen in Section 2.4 — X has a differentiable extension to all of Σ for Γ composed solely of circles. By the same argument, one finds that, for a general defect network, X has a differentiable extension to $X - V_\Gamma$. However, by the previous remark, it has a differentiable extension to $\overline{U}_k$ for each of the sectors U_k around a vertex v. Thus, it is differentiable on all of Σ.

The holonomy for a network-field configuration (Γ, X) is a modification of (2.25) which includes an additional U(1)-factor associated to the vertices of Γ,

$$\mathrm{Hol}(\Gamma, X) = \mathrm{Hol}_{\mathcal{G}}(X) \cdot \mathrm{Hol}_{\mathcal{B}}(X|_{E_\Gamma}) \cdot \mathrm{Hol}_{\mathcal{J}}(X|_{V_\Gamma}), \tag{2.53}$$

where

$$\mathrm{Hol}_{\mathcal{J}}(X|_{V_\Gamma}) = \prod_{v \in V_\Gamma} \widehat{f}_v(v). \tag{2.54}$$

$\mathrm{Hol}_{\mathcal{G}}(X)$ is given by the same expression (2.13) and $\mathrm{Hol}_{\mathcal{B}}(X|_{E_\Gamma})$ by the expression (2.25), together with a prescription as to how to treat the vertices of Γ. Here are the details:

- The expressions $\mathrm{Hol}_{\mathcal{G}}(X)$ and $\mathrm{Hol}_{\mathcal{B}}(X|_{E_\Gamma})$ are evaluated with respect to a triangulation $\triangle(\Sigma)$ subordinate to (Γ, X). Such a triangulation is defined in the same way as the triangulation subordinate to (Λ, X) from Section 2.4, with the additional requirement that V_Γ is a subset of the set of the vertices of $\triangle(\Sigma)$, and that we have chosen, for each vertex $v \in V_\Gamma$, an index $i_v \in \mathcal{I}^{T_{n_v}}$ such that $X(v) \in \mathcal{O}^{T_{n_v}}_{i_v}$, where n_v is the valency of v.

- $\mathrm{Hol}_\mathcal{G}(X)$ is computed as described below (2.24), except when a vertex v of a triangle t lies in V_Γ. Suppose that t lies between the defect edges $e_{k-1,k}$ and $e_{k,k+1}$. Then, in the U(1)-factor $g_{i_t i_e i_v}$, the index i_v stands for $\psi^k_{n_v}(i_v)$. If e is an edge of Γ then i_e stands for $\phi_1(i_e)$ or $\phi_2(i_e)$, depending on the relative orientation of e and ∂t, as explained below (2.24).
- $\mathrm{Hol}_\mathcal{B}(X|_{E_\Gamma})$ is computed as described below (2.24), except when a vertex v of an edge e lies in V_Γ. Suppose that the edge e is the edge $e_{k,k+1}$ for the vertex v. Then

$$\widehat{K}_{ev} = \left(X^*_{k,k+1} K_{i_e \psi^{k,k+1}_{n_v}(i_v)}\right)^{-\varepsilon_{ev}}, \tag{2.55}$$

  where the sign ε_{ev} is as detailed below (2.24). The definition of $\widehat{P}_e = X^* P_{i_e}$ is not affected.
- Finally, $\widehat{f}_v = X^* f_{i_v}$, with $f_{i_v}|_{X(v)} = f_{n_v, i_v}$ at an n_v-valent vertex v.

We shall discuss in the next section how $\mathrm{Hol}(\Gamma, X)$ is determined from the requirement of its independence of the diverse choices made.

2.7 Holonomy formulæ from invariance analysis — a derivation

In the previous sections, we introduced a host of target-space structures associated with the gerbe, and used them to postulate the sigma-model action functional in the presence of defect networks embedded in the world-sheet. At this stage, we could perform an *a posteriori* verification of the invariance of the holonomy formulæ thus obtained under allowed changes of the arbitrary choices made: the choice of representatives of local data of the gerbe, those of the stable isomorphisms and 2-morphisms, as well as of the Čech cover of the target space and of the world-sheet triangulation subordinate to it. This was the route taken in [Al, Ga1] for world-sheets without defects, in [GR1, Ga4] for world-sheet boundaries, and in [FSW, Wa2] for circular defects, and it could readily be adapted to the study of defect junctions. However, this would leave us with the question as to how canonical our choices for the specific target-space structures — that of a bi-brane and that of an inter-bi-brane — are. Therefore, we choose to take essentially the reverse route in the present section in which we successively *derive* all components of the postulated description from some elementary invariance considerations. In so doing, we reveal certain twisted gauge symmetries associated intrinsically with $\mathcal{G}$-bi-branes and $(\mathcal{G},\mathcal{B})$-inter-bi-branes.

Let us first look for a modification of the bulk holonomy formula (2.13) necessary to accommodate the embedding of a collection Λ of non-intersecting defect circles in the world-sheet Σ. To this end, we compare the value $\mathrm{Hol}^p_{\mathcal{G}}(X)$ of the bulk holonomy attained on the gauge-transformed local data

$$b^p = b + Dp, \quad p = (\Pi_i, \chi_{ij}) \in A^1_M \tag{2.56}$$

of the bulk gerbe with that obtained for the original data $b = (B_i, A_{ij}, g_{ijk})$,

$$\mathrm{Hol}^p_{\mathcal{G}}(X) = \mathrm{Hol}_{\mathcal{G}}(X) \cdot \prod_{e\in\triangle(\Lambda)} \left[\exp\left(\mathsf{i} \int_e \big(\widehat{\Pi}_{1,e} - \widehat{\Pi}_{2,e} \big) \right) \prod_{v\in e} \big(\widehat{\chi}_{1,ev} \big)^{-1} \cdot \widehat{\chi}_{2,ev}(v) \right], \tag{2.57}$$

where, in the conventions of Section 2.4,

- the triangulation $\triangle(\Lambda)$ is induced by $\triangle(\Sigma)$;
- $\widehat{\Pi}_{\alpha,e} = X^*_{|\alpha} \Pi_{\phi_\alpha(i_e)}, \alpha = 1,2,$ with the extensions $X_{|\alpha}$ understood as maps from $e \subset \Lambda$ to $\mathcal{O}^M_{\phi_\alpha(i_e)}$;
- $\widehat{\chi}_{\alpha,ev} = X^*_{|\alpha} \chi_{\phi_\alpha(i_e)\phi_\alpha(i_v)}, \alpha = 1,2,$ with $X_{|\alpha}$ understood as maps from $v \in \Lambda$ to $\mathcal{O}^M_{\phi_\alpha(i_e)\phi_\alpha(i_v)}$.

Thus, the variation is pushed to the defect Λ — the (gerbe-)gauge symmetry remains unaffected by the presence of the defect away from it, and — accordingly — we should seek a cancellation of the defect variation through the introduction of degrees of freedom *localized at the defect*, with transformation properties dictated by the gauge transformations of the pullback gerbes on both patches welded by a particular defect circle. The defect being one-dimensional, we are led to take as the local data for the defect fields a Čech–Deligne cochain $\Phi = (P_i, K_{ij}) \in A^1_Q$ coupled to the defect as in the expression $\mathrm{Hol}_{\mathcal{B}}(X|_\Lambda)$ of (2.25) and transforming as

$$(P_i, K_{ij}) \mapsto (P_i, K_{ij}) + \check{\iota}^*_2 \big(\Pi_i, \chi_{ij}\big) - \check{\iota}^*_1 \big(\Pi_i, \chi_{ij}\big) - D(W_i). \tag{2.58}$$

Here, the second and third term on the right-hand side describe a twist induced by the bulk transformation p, and the last one, written in terms of a cochain $\eta = (W_i) \in A^0_Q$, is an independent gauge transformation of Φ allowed due to the emptiness of the boundary of Λ. The overall transformation displayed is that of a $\mathcal{G}$-(bi-)twisted gauge field over Q.

Having ensured the invariance of the corrected holonomy formula Hol (Λ, X) of (2.24) under gauge transformations of the bulk data, we should now demand that it be invariant under arbitrary changes of the (Λ, X)-subordinate triangulation of Σ, which turns out to constrain the defect data. The defining relation (2.5) of $\mathcal{G}$ protects the invariance of $\mathrm{Hol}(\Lambda, X)$ under all changes which do not affect the edges and the vertices of $\triangle(\Sigma)$ lying within the defect Λ, and so the remaining freedom of manœuvre consists in shifting the vertices of $\triangle(\Lambda)$ (with the bulk edges converging at them moved accordingly). In what follows, we consider a particularly simple example of the general move, which suffices for our purposes. Call $e_v^\pm$ the edge of $\triangle(\Lambda)$ for which v is an endpoint with $\varepsilon_{e_v^\pm v} = \pm 1$ and shift each vertex v of the original triangulation $\triangle(\Lambda)$ along the defect line to a nearby new location $v' \in \Lambda$ such that the segment $[v, v']$, starting at v and ending at v', has the same orientation as the defect line. Assume, furthermore, that $X([v, v']) \subset \mathcal{O}^Q_{i_{e_v^-} i_{e_v^+}}$. The shifted vertices define altogether a new triangulation $\triangle'(\Lambda)$ of the same defect, compatible with the new triangulation $\triangle'(\Sigma)$ by construction. The only (potential) change in the assignment of Čech indices to the elements of the triangulation comes from $v' \mapsto i_{v'} \in \mathcal{I}^Q$ replacing the former $v \mapsto i_v \in \mathcal{I}^Q$. Let us denote by $\mathrm{Hol}'(\Lambda, X)$ the holonomy calculated for the new triangulation $\triangle'(\Sigma)$. After a short calculation, one obtains the relation

$$\mathrm{Hol}'(\Lambda, X) = \mathrm{Hol}(\Lambda, X) \cdot \prod_{v \in \triangle(\Lambda)} \times \left[\exp\left(\mathsf{i} \int_{[v,v']} \widehat{\omega}^{(1)}_{e_v^- e_v^+} \right) \cdot \widehat{\omega}^{(0)}_{e_v^- e_v^+ v'}(v') \cdot \left(\widehat{\omega}^{(0)}_{e_v^- e_v^+ v} \right)^{-1}(v) \right], \quad (2.59)$$

where

- the 1-form in $\widehat{\omega}^{(1)}_{e_v^- e_v^+} = X^* \omega^{(1)}_{i_{e_v^-} i_{e_v^+}}$ pulled back by X, understood as a map from $[v, v']$ to $\mathcal{O}^Q_{i_{e_v^-} i_{e_v^+}}$, is defined as

$$\omega^{(1)}_{ij} = \iota_1^* A_{ij} - \iota_2^* A_{ij} + P_j - P_i - \mathsf{i}\, \mathsf{d} \log K_{ij} \in \Omega^1\big(\mathcal{O}^Q_{ij}\big); \quad (2.60)$$

- the U(1)-valued function in $\widehat{\omega}^{(0)}_{e_v^- e_v^+ v^{(\prime)}} = X^* \omega^{(0)}_{i_{e_v^-} i_{e_v^+} i_{v^{(\prime)}}}$ pulled back by X, understood as a map from $v^\pm$ to $\mathcal{O}^Q_{i_{e_v^-} i_{e_v^+} i_{v^{(\prime)}}}$, is defined as

$$\omega^{(0)}_{ijk} = \iota_1^* g_{ijk} \cdot \iota_2^* g_{ijk}^{-1} \cdot K_{jk}^{-1} \cdot K_{ik} \cdot K_{ij}^{-1} \in \mathrm{U}(1)_{\mathcal{O}^Q_{ijk}}. \quad (2.61)$$

The requirement that the unphysical change of the triangulation be unobservable translates into the constraints

$$\omega_{ij}^{(1)} = 0, \quad \omega_{ijk}^{(0)} = 1. \tag{2.62}$$

The inspection of (2.60) and (2.61) reveals that $\omega_{ij}^{(1)}$ and $\omega_{ijk}^{(0)}$ are, in fact, the lower-degree components of the Čech–Deligne 2-cochain $\Omega = \big(\omega_i^{(2)}, \omega_{ij}^{(1)}, \omega_{ijk}^{(0)}\big) \in A_Q^2$ given by the formula

$$\Omega = \check{\iota}_1^* b - \check{\iota}_2^* b + D\Phi. \tag{2.63}$$

We may now use the identity

$$D\big(\check{\iota}_1^* b - \check{\iota}_2^* b + D\Phi\big) = (\iota_1^* H - \iota_2^* H, 0, 0, 1), \tag{2.64}$$

following directly from (2.5), to rephrase the former requirement of invariance as

$$\check{\iota}_1^* b - \check{\iota}_2^* b + D\Phi = (\omega, 0, 1), \quad \mathsf{d}\omega = \iota_1^* H - \iota_2^* H \tag{2.65}$$

for a globally defined 2-form $\omega \in \Omega^2(Q), \omega|_{\mathcal{O}_i^Q} \equiv \omega_i^{(2)}$.

We have thus retrieved the structure of a 1-morphism of Section 2.3, with the definition of the global 2-form ω included, from elementary invariance considerations.

Now that we have identified the local degrees of freedom to be assigned to the defect line, we may incorporate vertices of a generic defect network Γ in our description. In analogy with the previous derivation, we take as the starting point the defect line-corrected holonomy $\mathrm{Hol}(\Lambda, X)$, calculated for $\Lambda = E_\Gamma$, and study its variation under gauge transformations replacing the gauge fields b and Φ with the new ones

$$\begin{cases} b^p = b + Dp, & p = (\Pi_i, \chi_{ij}) \in A_M^1 \\ \Phi^{p,\eta} = \Phi + \check{\iota}_2^* p - \check{\iota}_1^* p - D\eta, & \eta = (W_i) \in A_Q^0 \end{cases}. \tag{2.66}$$

Once again, the transformed holonomy, $\mathrm{Hol}^{p,\eta}(E_\Gamma, X)$, differs from the original one by terms evaluated at the newly introduced junction points exclusively,

$$\begin{gathered} \mathrm{Hol}^{p,\eta}(E_\Gamma, X) = \mathrm{Hol}(E_\Gamma, X) \cdot \prod_{v \in V_\Gamma} \prod_{k=1}^{n_v} \widehat{W}_{n_v,v}^{k,k+1}(v)^{-1}, \\ \widehat{W}_{n_v,v}^{k,k+1} = \big(X_{k,k+1}^* W_{\psi_{n_v}^{k,k+1}(i_v)}\big)^{\varepsilon_{e_{k,k+1}^v}}, \end{gathered} \tag{2.67}$$

and it is there that we should localize the new degrees of freedom $\varphi_n = (f_{n,i}) \in A^0_{T_n}$. They are to be coupled to the defect as in the expression $\mathrm{Hol}_{\mathcal{J}}(X|_{V_\Gamma})$ of (2.53) and to undergo twisted gauge transformations

$$\varphi_n \to \varphi_n + \sum_{k=1}^{n} \eta_n^{k,k+1}, \tag{2.68}$$

with

$$\eta_n^{k,k+1} = \big(\check{\pi}_n^{k,k+1}\big)^* \eta^{\varepsilon_n^{k,k+1}}. \tag{2.69}$$

The Φ-twisted scalar fields φ_n enjoy no proper gauge freedom for purely dimensional reasons. As we shall see in the next section, the admissible choices of φ_n turn out to be very restricted.

The vertex-corrected formula for the holonomy is now invariant with respect to arbitrary gauge transformations of the local data involved. What remains to be ascertained at this stage is that it does not alter under arbitrary changes of the world-sheet triangulation, taken together with the attendant Čech labels. Just as in the case of a circle-field configuration, we readily convince ourselves that the relevant changes are those which involve the vertices of the defect network, and even in this latter case the ambiguity is very restricted — the sole freedom that we have is in the choice of the Čech labels assigned to the vertices. Under a change $i_v \to i'_v$, the holonomy picks up a phase. The transformed one, $\mathrm{Hol}'(\Gamma, X)$, reads

$$\mathrm{Hol}'(\Gamma, X) = \mathrm{Hol}(\Gamma, X) \cdot \prod_{v \in V_\Gamma} \widehat{\theta}^{(0)}_{n_v, vv}(v)^{-1}, \tag{2.70}$$

where the U(1)-valued function in $\widehat{\theta}^{(0)}_{n_v,vv} = X^* \theta^{(0)}_{n_v, i_v i'_v}$ pulled back by X^*, understood as a map from the vertex v of valence n_v to $\mathcal{O}^{T_{n_v}}_{i_v i'_v}$, is given by

$$\theta^{(0)}_{n,ij} = f_{n,i} \cdot f^{-1}_{n,j} \cdot \prod_{k=1}^{n} K^{k,k+1}_{n,ij} \in \mathrm{U}(1)_{\mathcal{O}^{T_n}_{ij}}, \tag{2.71}$$

with $K^{k,k+1}_{n,ij}$ as defined above (2.41). We are led to impose the constraint

$$\theta^{(0)}_{n,ij} = 1. \tag{2.72}$$

The definition of the functions $\theta^{(0)}_{n,ij}$ identifies them, for every $n \in \mathbb{Z}_{>0}$, as the 0-degree component of the Čech–Deligne 1-cochain $\Theta_n = \big(\theta^{(1)}_{n,i}, \theta^{(0)}_{n,ij}\big) \in A^1_{T_n}$

defined as

$$\Theta_n = \sum_{k=1}^{n} \Phi_n^{k,k+1} + D\varphi_n \tag{2.73}$$

and, accordingly, satisfying

$$D\Theta_n = \sum_{k=1}^{n} \big(\omega_n^{k,k+1}, 0, 1\big), \tag{2.74}$$

with $\omega_n^{k,k+1}$ as in (2.36). The last identity, in conjunction with the requirement of invariance, produces the result

$$\sum_{k=1}^{n} \Phi_n^{k,k+1} + D\varphi_n = (\theta_n, 1), \quad \mathsf{d}\theta_n = \sum_{k=1}^{n} \omega_n^{k,k+1} \tag{2.75}$$

for globally defined 1-forms $\theta_n \in \Omega^1(T_n), \theta_n|_{\mathcal{O}_i^{T_n}} \equiv \theta_{n,i}^{(1)}$. The dynamical arguments of Appendix A.2 ultimately fix the vertex data by imposing the constraint

$$\theta_n = 0 \tag{2.76}$$

for all $n \in \mathbb{Z}_{>0}$.

We have thus recovered the structure of a 2-morphism of Section 2.5 from elementary invariance considerations.

2.8 Defect-vertex data via induction

The assignment of the holonomy $\mathrm{Hol}(\Gamma, X)$ to a given world-sheet with an embedded defect network involves a number of choices for the coupled target-space backgrounds (b, Φ, φ), reflecting the underlying twisted gauge symmetry. Besides the unphysical choice of the gauge, cf. (2.66) and (2.68), there is also the all-relevant choice of the gauge class which forms an integral part of the definition of the sigma model.

The Φ-twisted scalar field φ of (2.42) has no proper gauge symmetry but the possible choices of φ for b and Φ fixed are strongly constrained. To see this, note that any two such choices φ' and φ must be related via

$$\varphi' = \varphi + \gamma, \quad \gamma = (c_i) \in \ker D_{(0)} \subset A_T^0 \tag{2.77}$$

by untwisted U(1)-valued scalars. Thus, the freedom in the choice of the Φ-twisted scalar field is parameterized by (locally) constant phases,

$$\mathsf{d}c_i = 0, \quad \left(c_j \cdot c_i^{-1}\right)|_{\mathcal{O}_{ij}^T} = 1, \tag{2.78}$$

readily seen to compose the group

$$\ker D_{(0)} = \mathrm{U}(1)^{|\pi_0(T)|}, \tag{2.79}$$

for $\pi_0(T)$ the set of connected components of T.

The restricted character of the set of admissible Φ-twisted scalar fields motivates further investigation of special solutions to the defining equations (2.40). In conformal field theory, we can generate four-valent defect vertices (or n-valent vertices, for that matter) from three-valent ones as follows: Recall that a defect vertex corresponds to a defect-field insertion in CFT (cf. figure 2). Consider two three-valent defect fields joined by one common defect line of a small length ε. Taking the limit $\varepsilon \to 0$ and possibly compensating for the resulting divergence leads to a four-valent defect field. It turns out that we can mimic this procedure in the classical sigma model.

Recall from Section 2.5 that a $(\mathcal{G}, \mathcal{B})$-inter-bi-brane is defined in terms of a tower of component world-volumes $T = \bigsqcup_{n=1}^{\infty} T_n$, with a 2-morphism φ_n on each T_n. Below, we propose a method to construct the φ_n with $n > 3$ from $(T_3, \mathcal{O}^{T_3}, \varphi_3, \tau_3)$ and some extra data. For the sake of concreteness, we shall restrict our discussion to the special case of vertices of valence $n = 4$ with three incoming edges and one outgoing edge.

The point of departure in our construction are the data $(T_{2+1}, \mathcal{O}^{T_{2+1}}, \varphi_{2+1}, \tau_{2+1})$ for the three-valent vertex with two incoming edges and one outgoing edge. It consists of the $(\mathcal{G}, \mathcal{B})$-inter-bi-brane world-volume $T_{3,++-} \equiv T_{2+1}$, mapped to the $\mathcal{G}$-bi-brane world-volume Q by each of the three Čech-extended maps

$$\check{\pi}_{2+1}^{1,2}, \check{\pi}_{2+1}^{2,3}, \check{\pi}_{2+1}^{3,1} : T_{2+1} \to Q \tag{2.80}$$

satisfying the constraints

$$\check{\iota}_1 \circ \check{\pi}_{2+1}^{2,3} = \check{\iota}_2 \circ \check{\pi}_{2+1}^{1,2}, \quad \check{\iota}_1 \circ \check{\pi}_{2+1}^{3,1} = \check{\iota}_1 \circ \check{\pi}_{2+1}^{1,2}, \quad \check{\iota}_2 \circ \check{\pi}_{2+1}^{3,1} = \check{\iota}_2 \circ \check{\pi}_{2+1}^{2,3}, \tag{2.81}$$

and of a 2-morphism

$$\varphi_{2+1} : \big(\Phi^{3,1}_{2+1} \star \mathrm{id}_{I(\omega^{1,2}_{2+1}+\omega^{2,3}_{2+1})}\big) \circ \big(\Phi^{2,3}_{2+1} \star \mathrm{id}_{I(\omega^{1,2}_{2+1})}\big) \circ \Phi^{1,2}_{2+1} \Longrightarrow \mathrm{id}_{\mathcal{G}^1_{2+1}}, \tag{2.82}$$

defined for $\mathcal{G}^1_{2+1} = \big(\check{\iota}_1 \circ \check{\pi}^{1,2}_{2+1}\big)^* \mathcal{G}$. The latter canonically induces another 2-morphism

$$\widetilde{\varphi}_{2+1} : \big(\Phi^{2,3}_{2+1} \star \mathrm{id}_{I(\omega^{1,2}_{2+1})}\big) \circ \Phi^{1,2}_{2+1} \Longrightarrow \Phi^{1,3}_{2+1}, \tag{2.83}$$

with $\Phi^{3,1}_{2+1} = (\Phi^{1,3}_{2+1})^\vee$, giving a decomposition of φ_{2+1} of the form

$$\varphi_{2+1} = d_{\Phi^{1,3}_{2+1}} \bullet \big(\mathrm{id}_{\Phi^{3,1}_{2+1} \star \mathrm{id}} \circ \widetilde{\varphi}_{2+1}\big). \tag{2.84}$$

Next, we assume that we are given a manifold $T_{3+1} \equiv T_{4,+++-}$ together with four Čech-extended maps

$$\check{\mathrm{v}}_{I,J,K} = (\mathrm{v}_{I,J,K}, \nu_{I,J,K}) : T_{3+1} \to T_{2+1}, \quad 1 \le I < J < K \le 4 \tag{2.85}$$

subject to the conditions

$$\begin{aligned}
&\check{\pi}^{1,2}_{2+1} \circ \check{\mathrm{v}}_{1,3,4} = \check{\pi}^{3,1}_{2+1} \circ \check{\mathrm{v}}_{1,2,3}, \quad \check{\pi}^{2,3}_{2+1} \circ \check{\mathrm{v}}_{1,2,4} = \check{\pi}^{3,1}_{2+1} \circ \check{\mathrm{v}}_{2,3,4},\\
&\check{\pi}^{1,2}_{2+1} \circ \check{\mathrm{v}}_{1,2,3} = \check{\pi}^{1,2}_{2+1} \circ \check{\mathrm{v}}_{1,2,4}, \quad \check{\pi}^{2,3}_{2+1} \circ \check{\mathrm{v}}_{1,2,3} = \check{\pi}^{1,2}_{2+1} \circ \check{\mathrm{v}}_{2,3,4},\\
&\check{\pi}^{2,3}_{2+1} \circ \check{\mathrm{v}}_{1,3,4} = \check{\pi}^{2,3}_{2+1} \circ \check{\mathrm{v}}_{2,3,4}, \quad \check{\pi}^{3,1}_{2+1} \circ \check{\mathrm{v}}_{1,3,4} = \check{\pi}^{3,1}_{2+1} \circ \check{\mathrm{v}}_{1,2,4}.
\end{aligned} \tag{2.86}$$

Their existence is the basis of our construction, and we shall provide examples of such maps presently. In order to understand the index structure, one should have a look at figure 5. For example, the right-hand side of the last equation in (2.86) can be understood as passing from the image of $v \in \Sigma_{\mathrm{L|R}}$ in T_{3+1} to the image of $v \in \Sigma_{\mathrm{R}}$ in T_{2+1} (with adjacent patches 1, 2, 4), and subsequently to that of the endpoint of the edge between patches 1 and 4 (the edge $e^{3,1}$ with respect to the ordering for the vertex $v \in \Sigma_{\mathrm{R}}$). For the left-hand side of that equation, one uses Σ_{L} instead.

The maps $\check{\mathrm{v}}_{I,J,K}$ are readily seen to induce the inter-bi-brane structure T_{3+1} for the four-valent vertices. Indeed, first of all, they provide us with

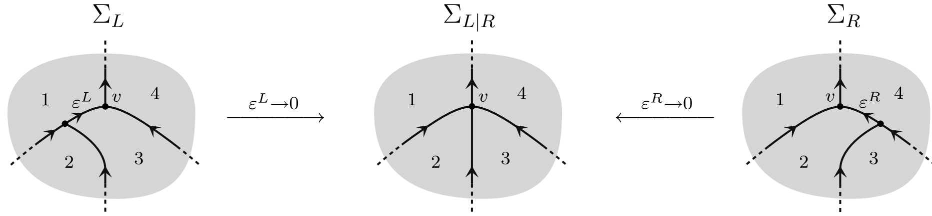


Figure 5: The four-valent defect vertex in $\Sigma_{\mathrm{L|R}}$ obtained as a result of collapsing a pair of three-valent vertices in two inequivalent ways, whereby the two 2-morphisms $\widetilde{\varphi}^{\mathrm{L}}$ and $\widetilde{\varphi}^{\mathrm{R}}$ are induced at the vertex.

the data of τ_{3+1} as per

$$\begin{aligned} \check{\pi}^{1,2}_{3+1} &= \check{\pi}^{1,2}_{2+1} \circ \check{\mathrm{v}}_{1,2,3}, & \check{\pi}^{2,3}_{3+1} &= \check{\pi}^{2,3}_{2+1} \circ \check{\mathrm{v}}_{1,2,3}, \\ \check{\pi}^{3,4}_{3+1} &= \check{\pi}^{2,3}_{2+1} \circ \check{\mathrm{v}}_{1,3,4}, & \check{\pi}^{4,1}_{3+1} &= \check{\pi}^{3,1}_{2+1} \circ \check{\mathrm{v}}_{1,3,4}, \end{aligned} \tag{2.87}$$

and hence also with the patch maps $\check{\pi}^k_{3+1}, k = 1,2,3,4$. The latter give us the pullback gerbes $\mathcal{G}^k_{3+1} = \big(\check{\pi}^k_{3+1}\big)^* \mathcal{G}$ on T_{3+1}. It is a simple exercise to verify that conditions (2.86) in conjunction with (2.81) ensure that the maps $\check{\pi}^{k,k+1}_{3+1} : T_{3+1} \to Q$ satisfy the constraints (2.34). We supplement the above collection with the extra definitions

$$\check{\pi}^{1,3}_{3+1} \equiv \check{\pi}^{3,1}_{3+1} = \check{\pi}^{3,1}_{2+1} \circ \check{\mathrm{v}}_{1,2,3}, \quad \check{\pi}^{2,4}_{3+1} \equiv \check{\pi}^{4,2}_{3+1} = \check{\pi}^{2,3}_{2+1} \circ \check{\mathrm{v}}_{1,2,4}, \tag{2.88}$$

allowing us to write down all the pullback 1-morphisms

$$\begin{aligned} \Phi^{I,J}_{3+1} &= \big(\check{\pi}^{I,J}_{3+1}\big)^* \Phi : \mathcal{G}^I_{3+1} \to \mathcal{G}^J_{3+1}, \quad I < J, \quad (I,J) \neq (1,4), \\ \Phi^{4,1}_{3+1} &= \big(\check{\pi}^{4,1}_{3+1}\big)^* \Phi = \big(\Phi^{1,4}_{3+1}\big)^\vee : \mathcal{G}^4_{3+1} \to \mathcal{G}^1_{3+1}. \end{aligned} \tag{2.89}$$

We can use these to give the two different definitions of the defect-vertex 2-morphism

$$\begin{aligned} \varphi^L_{3+1} &= d_{\Phi^{1,4}_{3+1}} \bullet \big(\mathrm{id} \circ \widetilde{\varphi}^{1,3,4}\big) \bullet \big(\mathrm{id} \circ \widetilde{\varphi}^{1,2,3}\big), \\ \varphi^R_{3+1} &= d_{\Phi^{1,4}_{3+1}} \bullet \big(\mathrm{id} \circ \widetilde{\varphi}^{1,2,4}\big) \bullet \big(\mathrm{id} \circ (\widetilde{\varphi}^{2,3,4} \star \mathrm{id}_{\mathrm{id}_{I(\omega^{1,2}_{3+1})}}) \circ \mathrm{id}\big), \end{aligned} \tag{2.90}$$

acting as

$$\begin{aligned} &\big(\Phi^{4,1}_{3+1} \star \mathrm{id}_{I(\omega^{1,2}_{3+1} + \omega^{2,3}_{3+1} + \omega^{3,4}_{3+1})}\big) \circ \big(\Phi^{3,4}_{3+1} \star \mathrm{id}_{I(\omega^{1,2}_{3+1} + \omega^{2,3}_{3+1})}\big) \circ \big(\Phi^{2,3}_{3+1} \star \mathrm{id}_{I(\omega^{1,2}_{3+1})}\big) \\ &\quad \circ \Phi^{1,2}_{3+1} \xLongrightarrow{\varphi^{L,R}_{3+1}} \mathrm{id}_{\mathcal{G}^1_{3+1}}. \end{aligned} \tag{2.91}$$

They are expressed in terms of the corresponding pullback 2-morphisms

$$\widetilde{\varphi}^{I,J,K} = \check{\mathrm{v}}_{I,J,K}^{*}\widetilde{\varphi}_{2+1} : \big(\Phi_{3+1}^{J,K} \star \mathrm{id}_{I(\omega_{3+1}^{I,J})}\big) \circ \Phi_{3+1}^{I,J} \Longrightarrow \Phi_{3+1}^{I,K}, \tag{2.92}$$

and the death 2-morphism $d_{\Phi_{3+1}^{1,4}}$. Clearly, the two definitions, φ_{3+1}^{L} and φ_{3+1}^{R}, correspond to the two inequivalent ways of clustering the incoming defect-lines converging at a given four-valent defect vertex. It is worth underlining that while each of the two definitions in (2.90) requires only two of the four maps $\check{\mathrm{v}}_{I,J,K}$, the verification of the constraints (2.34) for the induced maps $\check{\pi}_{3+1}^{k,k+1}$ uses *all four* $\check{\mathrm{v}}_{I,J,K}$.

A generic example of an induced $(\mathcal{G},\mathcal{B})$-inter-bi-brane structure can be obtained from the $\mathcal{G}$-bi-brane world-volume $Q \subset M \times M$ and the $(\mathcal{G},\mathcal{B})$-inter-bi-brane world-volumes $T_n \subset M \times M \times \cdots \times M$ embedded as submanifolds in the respective multiple direct products of the target space M with itself, with $\pi_n^{k,k+1} : M_{(1)} \times M_{(2)} \times \cdots \times M_{(n)} \to M_{(k)} \times M_{(k+1)}$ given by the canonical projections $(M_{(l)} \equiv M, l = 1, 2, \ldots, n)$. In this setting, given the world-volume $T_{2+1} \subset M \times M \times M$ of the $(\mathcal{G},\mathcal{B})$-inter-bi-brane, we choose for the world-volume $T_{3+1} \subset M \times M \times M \times M$ the common intersection of the preimages $\mathrm{v}_{I,J,K}^{-1}(T_{2+1})$ of T_{2+1} under the canonical projections $\mathrm{v}_{I,J,K} \equiv \pi_{3+1}^{I,J,K} : M_{(1)} \times M_{(2)} \times M_{(3)} \times M_{(4)} \to M_{(I)} \times M_{(J)} \times M_{(K)}$. The conditions (2.86) are trivially satisfied.

2.8.1 The Lie-group example (cont'd)

We now proceed to demonstrate how the data $(T_{2+1}, \mathcal{O}^{T_{2+1}}, \varphi_{2+1}, \tau_{2+1})$ for three-valent vertices with signature $(+1,+1,-1)$, introduced in Section 2.5, can be used to induce the data $(T_{3+1}, \mathcal{O}^{T_{3+1}}, \varphi_{3+1}, \tau_{3+1})$ for four-valent vertices with signature $(+1,+1,+1,-1)$ in accord with the general scheme discussed above. We start with the definition of the Čech-extended maps $\check{\mathrm{v}}_{I,J,K} : T_{3+1} \to T_{2+1}$, which — for $(g,x,y,z) \in \mathcal{O}_{i,x,y,z}^{T_{3+1}}$, written in the previously adopted shorthand notation with the redundant signs dropped — reads

$$\begin{aligned}
&\mathrm{v}_{1,2,3}(g,x,y,z) = (g,x,y), \quad \nu_{1,2,3}(i,x,y,z) = (i,x,y),\\
&\mathrm{v}_{1,3,4}(g,x,y,z) = (g,x\cdot y,z), \quad \nu_{1,3,4}(i,x,y,z) = (i,x\cdot y,z),\\
&\mathrm{v}_{2,3,4}(g,x,y,z) = (x^{-1}\cdot g,y,z), \quad \nu_{2,3,4}(i,x,y,z) = (x^{-1}.i,y,z),\\
&\mathrm{v}_{1,2,4}(g,x,y,z) = (g,x,y\cdot z), \quad \nu_{1,2,4}(i,x,y,z) = (i,x,y\cdot z).
\end{aligned} \tag{2.93}$$

One readily verifies that $\check{\mathrm{v}}_{I,J,K}$ obey condition (2.86), and so they can be used to pull back the data $(T_{2+1}, \mathcal{O}^{T_{2+1}}, \varphi_{2+1}, \tau_{2+1})$ to T_{3+1}. Thus, we induce the relevant Čech-extended maps $\check{\pi}^{I,J}_{3+1}$ in the form

$$\begin{aligned} &\pi^{1,2}_{3+1}(g,x,y,z)=(g,x), \quad \pi^{2,3}_{3+1}(g,x,y,z)=(x^{-1}\cdot g,y),\\ &\pi^{3,4}_{3+1}(g,x,y,z)=((x\cdot y)^{-1}\cdot g,z), \quad \pi^{4,1}_{3+1}(g,x,y,z)=(g,x\cdot y\cdot z),\\ &\pi^{3,1}_{3+1}(g,x,y,z)=(g,x\cdot y), \quad \pi^{4,2}_{3+1}(g,x,y,z)=(x^{-1}\cdot g,y\cdot z), \end{aligned} \tag{2.94}$$

and similarly for $\psi^{I,J}_{3+1}$. These, in turn, give us the Čech-extended maps $\check{\pi}^{k}_{3+1}$ for the patches

$$\begin{aligned} &\pi^{1}_{3+1}(g,x,y,z)=g, \quad \pi^{2}_{3+1}(g,x,y,z)=x^{-1}\cdot g,\\ &\pi^{3}_{3+1}(g,x,y,z)=(x\cdot y)^{-1}\cdot g, \quad \pi^{4}_{3+1}(g,x,y,z)=(x\cdot y\cdot z)^{-1}\cdot g, \end{aligned} \tag{2.95}$$

and similarly for ψ^{k}_{3+1}. With the help of the induced maps, we then obtain on $T^{x,y,z}_{3+1}=\mathrm{G}\times\{(x,y,z)\}\subset T_{3+1}$ (again, identified with G) the pullback gerbes

$$\begin{aligned} \mathcal{G}^1_{3+1}=\mathcal{G}^{\star\mathsf{k}}, \quad \mathcal{G}^2_{3+1}=x.\mathcal{G}^{\star\mathsf{k}},\\ \mathcal{G}^3_{3+1}=(x\cdot y).\mathcal{G}^{\star\mathsf{k}}, \quad \mathcal{G}^4_{3+1}=(x\cdot y\cdot z).\mathcal{G}^{\star\mathsf{k}}, \end{aligned} \tag{2.96}$$

the pullback 1-morphisms

$$\begin{aligned} &\Phi^{1,2}_{3+1}=\mathcal{A}_x, \quad \Phi^{2,3}_{3+1}=x.\mathcal{A}_y, \quad \Phi^{3,4}_{3+1}=(x\cdot y).\mathcal{A}_z, \quad \Phi^{4,1}_{3+1}=\mathcal{A}^{\vee}_{x\cdot y\cdot z},\\ &\quad\Phi^{1,3}_{3+1}=\mathcal{A}_{x\cdot y}, \quad \Phi^{2,4}_{3+1}=x.\mathcal{A}_{y\cdot z} \end{aligned} \tag{2.97}$$

and the pullback 2-morphisms

$$\widetilde{\varphi}^{1,2,3}=\widetilde{\varphi}_{x,y}, \quad \widetilde{\varphi}^{1,3,4}=\widetilde{\varphi}_{x\cdot y,z}, \quad \widetilde{\varphi}^{2,3,4}=x.\widetilde{\varphi}_{y,z}, \quad \widetilde{\varphi}^{1,2,4}=\widetilde{\varphi}_{x,y\cdot z}, \tag{2.98}$$

where we have used the action $x.\widetilde{\varphi}_{y,z}\equiv\big(x^{\check{-}1}\big)^*\widetilde{\varphi}_{y,z}$ of $Z(\mathrm{G})$ given in (2.12). Putting all the pieces together, we arrive at the two definitions of the 2-morphism on T_{3+1}

$$\begin{aligned} \varphi^{L}_{3+1}\big|_{T^{x,y,z}_{3+1}}&=d_{\mathcal{A}_{x\cdot y\cdot z}}\bullet\big(\mathrm{id}\circ\widetilde{\varphi}_{x\cdot y,z}\big)\bullet\big(\mathrm{id}\circ\widetilde{\varphi}_{x,y}\big),\\ \varphi^{R}_{3+1}\big|_{T^{x,y,z}_{3+1}}&=d_{\mathcal{A}_{x\cdot y\cdot z}}\bullet\big(\mathrm{id}\circ\widetilde{\varphi}_{x,y\cdot z}\big)\bullet\big(\mathrm{id}\circ x.\widetilde{\varphi}_{y,z}\circ\mathrm{id}\big), \end{aligned} \tag{2.99}$$

acting as

$$\varphi^{L,R}_{3+1}\big|_{T^{x,y,z}_{3+1}}:\mathcal{A}^{\vee}_{x\cdot y\cdot z}\circ(x\cdot y).\mathcal{A}_z\circ x.\mathcal{A}_y\circ\mathcal{A}_x\Longrightarrow\mathrm{id}_{\mathcal{G}^{\star\mathsf{k}}} \tag{2.100}$$

and differing at most by a constant on each connected component $T^{x,y,z}_{3+1}$ of the world-volume T_{3+1} (recall that the Lie group G was assumed connected).

We may now compare the two induced 2-morphisms φ^L_{3+1} and φ^R_{3+1} on each $T^{x,y,z}_{3+1}$, identified with G itself, by applying (2.77) and (2.79) to the setting under consideration. Let $g \in \mathcal{O}^{\mathrm{G}}_i$ be an arbitrary point in $T^{x,y,z}_{3+1}$, and let $\big(f^{L,R}_i\big) \in A^0_{T^{x,y,z}_{3+1}}$ be the local data of $\varphi^{L,R}_{3+1}\big|_{T^{x,y,z}_{3+1}}$. We have the identity

$$f^R_i(g) = \psi(x,y,z) \cdot f^L_i(g) \tag{2.101}$$

for the U(1)-valued constant

$$\begin{aligned}\psi(x,y,z) = \big[\big(x^{-1}\big)^* f_{x^{-1}.i}(y,z) \cdot \big(f_i(x\cdot y,z)\big)^{-1}\\ \cdot f_i(x, y\cdot z) \cdot \big(f_i(x,y)\big)^{-1}\big](g)\end{aligned} \tag{2.102}$$

written in terms of the local data of the 2-morphism $\widetilde{\varphi}_{x,y} = (f_i(x,y)) \in A^0_{T^{x,y,z}_{3+1}}$. By virtue of (2.77), the expression $\psi(x,y,z)$ depends neither on the specific point $g \in \mathrm{G}$, nor on the attendant Čech index $i \in \mathcal{I}^{\mathrm{G}}$. This permitted us to drop both g and i when writing $\psi(x,y,z)$ in (2.101) and (2.102). We emphasize that only the particular combination $\psi(x,y,z)$ of the locally smooth functions $f_i(x,y)$ is constant on G — in general, none of the component terms has this property.

Note that $\psi(x,y,z)$ rewrites as

$$\psi(x,y,z) = \big[\big(\delta_{Z(\mathrm{G})} f_i\big)(x,y,z)\big](g), \tag{2.103}$$

where we consider the local data of the 2-morphisms $\widetilde{\varphi}_{x,y}$ as elements of the (left) $Z(\mathrm{G})$-module $\underline{\mathrm{U}(1)}_{T_{3+1}}$ of (the sheaf of) locally smooth U(1)-valued functions on T_{3+1}. The centre $Z(\mathrm{G})$ acts on $\underline{\mathrm{U}(1)}_{T_{3+1}}$ by the Čech-extended pullbacks

$$(x.f)_i(y,z) = \big(x^{-1}\big)^* f_{x^{-1}.i}(y,z). \tag{2.104}$$

Despite the form of (2.103), the object $(\psi(x,y,z) \mid x,y,z \in Z(\mathrm{G}))$, regarded as a 3-cochain on $Z(\mathrm{G})$ with values in the trivial $Z(\mathrm{G})$-module U(1), is *not* a 3-coboundary — it is not in the image of $\delta_{Z(\mathrm{G})} : C^2(Z(\mathrm{G}), \mathrm{U}(1)) \to C^3(Z(\mathrm{G}), \mathrm{U}(1))$. Being an element of the kernel of the Deligne differential

$D_{(0)}$ on the connected Lie group G, it is, on the other hand, $\delta_{Z(\mathrm{G})}$-closed,

$$\begin{aligned}\big(\delta_{Z(\mathrm{G})}\psi\big)(x,y,z,w) &= \frac{\psi(y,z,w)\cdot\psi(x,y\cdot z,w)\cdot\psi(x,y,z)}{\psi(x\cdot y,z,w)\cdot\psi(x,y,z\cdot w)} \\ &\equiv \frac{\Big(\big(\check{x^{-1}}\big)^*\psi(y,z,w)\Big)\cdot\psi(x,y\cdot z,w)\cdot\psi(x,y,z)}{\psi(x\cdot y,z,w)\cdot\psi(x,y,z\cdot w)} \\ &= \big[\big(\delta^2_{Z(\mathrm{G})}f_i\big)(x,y,z)\big](g) = 1. \end{aligned} \tag{2.105}$$

Above, the passage to the second line exploits the stated independence of $\psi(x,y,z)$ of the choice of the argument and of the Čech index of the constituent functions $f_i(x,y)$ by simply replacing the original expression with the pullback

$$\begin{aligned}\big(\check{x^{-1}}\big)^*\psi(y,z,w) = \big((x\cdot y)^{-1}\big)^* f_{(x\cdot y)^{-1}.i}(z,w)\cdot\Big(\big(x^{-1}\big)^* f_{x^{-1}.i}(y\cdot z,w)\Big)^{-1} \\ \cdot\big(x^{-1}\big)^* f_{x^{-1}.i}(y,z\cdot w)\cdot\Big(\big(x^{-1}\big)^* f_{x^{-1}.i}(y,z)\Big)^{-1}(g).\end{aligned} \tag{2.106}$$

Thus, $(\psi(x,y,z) \mid x,y,z\in Z(\mathrm{G}))$ is a U(1)-valued 3-cocycle on $Z(\mathrm{G})$. As shall become clear in the next section, it is the very associator 3-cocycle that we have been after all along.

2.9 Conformal and topological defects

Having specified a sigma-model description (2.53) of the coupling of target-space structures $\mathcal{G}, \Phi$ and φ to a world-sheet Σ with an embedded defect network Γ, we are now ready to discuss the local symmetries of the thus established two-dimensional field theory. They descend from the local-symmetry group of the sigma model without defects, which is the semidirect product $\mathrm{Diff}(\Sigma)\ltimes\mathrm{Weyl}(\gamma)$ of the group $\mathrm{Diff}(\Sigma)$ of orientation-preserving diffeomorphisms $\sigma\mapsto f(\sigma)$ of the world-sheet and the group $\mathrm{Weyl}(\gamma)$ of Weyl rescalings $\gamma(\sigma)\mapsto\exp(2w(\sigma))\cdot\gamma(\sigma)$ of the world-sheet metric tensor γ. Weyl rescalings remain a symmetry in the presence of defects as the holonomy formula does not involve the world-sheet metric at all. As a consequence, the energy-momentum tensor

$$T^{ab} = -\frac{1}{\sqrt{\det\gamma}}\frac{\delta S}{\delta\gamma_{ab}} \tag{2.107}$$

is traceless. Let $f:\Sigma\to\Sigma$ be an orientation-preserving diffeomorphism. Given a network-field configuration (Γ, X), we obtain a new network-field configuration $(f(\Gamma), X \circ f^{-1})$. Clearly, for $S[(\Gamma, X); \gamma] = S_{\mathrm{kin}}[X, \gamma] + \log \mathrm{Hol}(\Gamma, X)$, we find

$$S[(\Gamma, X); \gamma] = S[(f(\Gamma), X \circ f^{-1}); (f^{-1})^* \gamma]. \tag{2.108}$$

In this sense, the sigma model for the world-sheet with the defect network possesses diffeomorphism invariance. In particular, we may fix a metric γ_0 on Σ and take $f_{\mathrm{c}} : \Sigma \to \Sigma$ to be a conformal transformation. Due to the diffeomorphism invariance, and owing to the Weyl symmetry, the action obeys

$$S[(\Gamma, X); \gamma_0] = S[(f_{\mathrm{c}}(\Gamma), X \circ f_{\mathrm{c}}^{-1}); \gamma_0]. \tag{2.109}$$

If f_{c} maps the defect network Γ to itself, it is a symmetry of the model. The defects we describe are therefore classically conformally invariant.

It is convenient to pass to local complex coordinates $z = \sigma^1 + \mathrm{i}\sigma^2$ close to a defect line, such that the defect line coincides with the real axis and such that we can choose a gauge in which γ_0 is the unital metric $\delta_{ab} \mathrm{d}\sigma^a \otimes \mathrm{d}\sigma^b$. We shall use the complex derivatives $\partial = \frac{1}{2}(\partial_1 - \mathrm{i}\partial_2)$ and $\overline{\partial} = \frac{1}{2}(\partial_1 + \mathrm{i}\partial_2)$. The holomorphic and anti-holomorphic components of the energy-momentum tensor are then given by

$$T = G_X(\partial X, \partial X), \quad \overline{T} = G_X(\overline{\partial} X, \overline{\partial} X). \tag{2.110}$$

Inserting the choice $v = X_* \widehat{t}$ in the defect condition (2.22) yields $G_{X_{|1}(p)}(\partial_1 X_{|1}, \partial_2 X_{|1}) - G_{X_{|2}(p)}(\partial_1 X_{|2}, \partial_2 X_{|2}) = 0$, or, equivalently,

$$T_1(p) - \overline{T}_1(p) = T_2(p) - \overline{T}_2(p), \tag{2.111}$$

where p is a point on the real axis and T_α, for $\alpha = 1, 2$, stands for (2.110) with X replaced by the extension $X_{|\alpha}$. Thus, the classical energy-momentum tensor indeed obeys the defining equation of a conformal defect as given in [OA].

Ultimately, we are interested in topological defects, i.e., defects which one can move freely on the world-sheet. For simplicity, we restrict the following discussion to circle-field configurations. Let (Λ, X) be a circle-field configuration. If we deform the embedded defect circles from Λ to Λ_ε then we need to extend the map X to the domain swept during the deformation in order to obtain a new circle-field configuration. We shall now describe how this can be achieved.

Let U be a tubular neighbourhood of Λ. An *extension of X on U* is a map $\widehat{X} : U \to Q$ with the following properties. The defect circles Λ split U

into U_1 and U_2. We demand

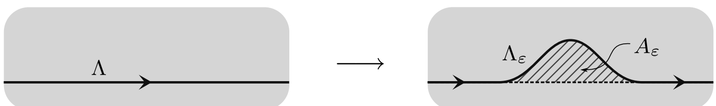
$$\widehat{X} = X \text{ on } \Lambda, \quad \iota_1 \circ \widehat{X} = X \text{ on } U_1, \quad \iota_2 \circ \widehat{X} = X \text{ on } U_2, \tag{2.112}$$

as well as, for all $p \in U$ and $v \in T_{\widehat{X}(p)}Q$,

$$\Delta G_{\widehat{X}(p)}\big(v, \widehat{X}_* \widehat{u}_2\big) = \tfrac{\mathsf{i}}{2}\omega_{\widehat{X}(p)}\big(v, \widehat{X}_* \widehat{u}_1\big), \quad \Delta G = \iota_1^* G - \iota_2^* G, \tag{2.113}$$

where $(\widehat{u}_1, \widehat{u}_2)$ form a right-handed orthonormal basis of $T_p\Sigma$.

Consider a deformation Λ_ε of a segment of the defect circles Λ as depicted below:



. (2.114)

If we are given a circle-field configuration (Λ, X) and an extension $\widehat{X}$ of X on a neighbourhood U of Λ then we can define a new circle-field configuration $(\Lambda_\varepsilon, X_\varepsilon)$ by setting $X_\varepsilon = \widehat{X}$ on Λ_ε, and $X_\varepsilon = \iota_2 \circ \widehat{X}$ in the shaded region A_ε. Outside of Λ and A_ε, we choose $X_\varepsilon = X$. Conditions (2.112) and (2.113) guarantee that $(\Lambda_\varepsilon, X_\varepsilon)$ is, again, a valid network-field configuration. In particular, it obeys the defect condition (2.22), which can be seen by rewriting (2.113) in the form

$$\begin{aligned} G_{\iota_1 \circ \widehat{X}(p)}\big(\iota_{1*}v, \big(\iota_1 \circ \widehat{X}\big)_* \widehat{u}_2\big) - G_{\iota_2 \circ \widehat{X}(p)}\big(\iota_{2*}v, \big(\iota_2 \circ \widehat{X}\big)_* \widehat{u}_2\big) \\ - \frac{\mathsf{i}}{2}\,\omega_{\widehat{X}(p)}\big(v, \widehat{X}_* \widehat{u}_1\big) = 0. \end{aligned} \tag{2.115}$$

In this way, an extension of X on U enables us to deform defect lines. We shall now address the questions of the uniqueness of an extension and of the behaviour of the sigma-model action under the replacement of (Λ, X) by $(\Lambda_\varepsilon, X_\varepsilon)$.

Suppose that $(\iota_1, \iota_2) : Q \to M \times M$ is an immersion (i.e., the tangent map is everywhere injective). This is, in particular, the case if Q is a submanifold of $M \times M$. Then, if an extension of X on U exists it is *unique*. To see this, use the local coordinates introduced above, such that defect Λ lies on the real line and such that their orientations agree. On the real line itself, $\widehat{X}$ is fixed by X. Set $\widehat{e}_a = \partial/\partial\sigma^a$, $a = 1, 2$ and consider (2.115) for $\widehat{u}_a = \widehat{e}_a$. For a point $p = (\sigma^1, \sigma^2)$ with $\sigma^2 > 0$ (say), we have $\iota_1 \circ \widehat{X} = X$, and so $(\iota_1 \circ \widehat{X})_* \widehat{e}_2$ is fixed. The metric $G_{\iota_2 \circ \widehat{X}(p)}$ is still non-degenerate when restricted to the image of ι_{2*}, hence condition (2.115) determines $(\iota_2 \circ \widehat{X})_* \widehat{e}_2$ uniquely in terms of $(\iota_1 \circ \widehat{X})_* \widehat{e}_2$ and $\widehat{X}_* \widehat{e}_1$. Since (ι_1, ι_2) is an immersion, this — in turn — determines $\widehat{X}_* \widehat{e}_2$. If it exists the solution to the resulting Cauchy problem is unique.

We do not have much to say regarding the existence of an extension $\widehat{X}$. We merely point out that an extension typically does not exist in the special case of D-branes, as condition (2.113) would imply that the classical energy-momentum tensor vanishes identically on the boundary (this follows from (2.111) and (2.117) below), while for the jump defects treated in the Lie group example, we shall see below that an extensions always exists.

Next, we compute the difference between the values of the action for the original circle-field configuration (Λ, X) and its deformation $(\Lambda_\varepsilon, X_\varepsilon)$ illustrated in (2.114). By a straightforward specialization of the calculation from Appendix A.3, the change in the holonomy term of the action is given by the integral of $\widehat{X}^*\omega$ over the shaded region A_ε. Together with a computation of the change in the kinetic term, this leads to

$$\begin{aligned} &S[(\Gamma_\varepsilon, X_\varepsilon); \gamma_0] - S[(\Gamma, X); \gamma_0] \\ &\quad = \int_{A_\varepsilon} \mathsf{d}\sigma_1 \wedge \mathsf{d}\sigma_2 \Big(\sum_{a=1,2} \big[G_{X_2}(\partial_a X_2, \partial_a X_2) - G_{X_1}(\partial_a X_1, \partial_a X_1) \big] \\ &\qquad - \mathsf{i}\omega_{\widehat{X}}(\partial_1 \widehat{X}, \partial_2 \widehat{X}) \Big), \end{aligned} \tag{2.116}$$

where we have abbreviated $X_\alpha = \iota_\alpha \circ \widehat{X}$, $\alpha = 1, 2$. Let D_1 be the left-hand side of condition (2.115) for $\widehat{u}_1 = \widehat{e}_1$, $\widehat{u}_2 = \widehat{e}_2$, $v = \widehat{X}_* \widehat{e}_2$, and let D_2 be the same expression for the choice $\widehat{u}_1 = \widehat{e}_2$, $\widehat{u}_2 = -\widehat{e}_1$, $v = \widehat{X}_* \widehat{e}_1$. Then, $D_2 - D_1$ is equal to the integrand in (2.116), and hence the difference between the values of the action vanishes. Thus, given a circle-field configuration for which an extension exists, we can shift the position of the defect line without modifying the value of the action. This is the hallmark of a topological defect. Indeed, computing $D_1 + D_2$ results in the identity

$$T_1(p) + \overline{T}_1(p) = T_2(p) + \overline{T}_2(p) \tag{2.117}$$

at a point $p \in \Lambda$. Together with (2.111), this implies that both T and $\overline{T}$ are continuous across the defect line, which is the defining property of a topological defect as given in [PZ].

If the defect under consideration is topological, the symmetry of the sigma model on a world-sheet with defect circles Λ is enhanced to include conformal transformations which do not obey $f(\Lambda) = \Lambda$. Indeed, if $\Lambda' = f_\varepsilon(\Lambda)$ and

$X' = X \circ f_\varepsilon^{-1}$ for an infinitesimal conformal transformation f_ε then — as we saw at the beginning of the section — the action for (Λ, X) is the same as that for (Λ', X'), and we know from the preceeding discussion that we can move the defect Λ' back to its original position Λ. In this manner, we have produced a new field configuration (Λ, X') with the same value of the action, where outside of a small neighbourhood of Λ, X' is related to X via $X' = X \circ f_\varepsilon^{-1}$.

Consider a pair of network-field configurations $(\Gamma_{\rm L}, X_{\rm L})$ and $(\Gamma_{\rm R}, X_{\rm R})$ with topological defect conditions at $\Gamma_{\rm L}$ and $\Gamma_{\rm R}$, differing exclusively within the region $\Sigma_{\rm L}$ resp. $\Sigma_{\rm R}$ of the world-sheet shown in the left — resp. rightmost drawing of figure 5. Since the defects are topological, we can take the limits $\varepsilon_{\rm L}, \varepsilon_{\rm R} \to 0$ without modifying the value of the action. Under the assumption of the existence of suitable Čech-extended maps $\check{\rm v}^{I,J,K} : T_{3+1} \to T_{2+1}$ with the properties detailed in Section 2.8, we may readily compare the values attained by the exponentiated sigma-model action functional $\exp(-S[(\Gamma, X); \gamma_0])$ on the two network-field configurations. After a little thought, one finds that the value for $(\Gamma_{\rm L}, X_{\rm L})$ is equal to the value that $\exp(-S[(\Gamma, X); \gamma_0])$ takes on the network-field configuration displayed in the middle drawing of figure 5 in which the four-valent defect vertex in $\Sigma_{\rm L|R}$ is understood to carry the pullback data of the 2-morphism $\varphi_{3+1}^{\rm L}$ defined in (2.90). By the same token, that for $(\Gamma_{\rm R}, X_{\rm R})$ is equal to the value that $\exp(-S[(\Gamma, X); \gamma_0])$ takes on the network-field configuration from the middle drawing but, this time, with the four-valent defect vertex taken to carry the pullback data of the 2-morphism $\varphi_{3+1}^{\rm R}$. Adducing the reasoning of Section 2.8, we conclude that the two values are related by a phase as per

$$\exp\bigl(-S\bigl[(\Gamma_{\rm L}, X_{\rm L}); \gamma_0\bigr]\bigr) = u(X(v)) \cdot \exp\bigl(-S\bigl[(\Gamma_{\rm R}, X_{\rm R}); \gamma_0\bigr]\bigr), \qquad (2.118)$$

with the function

$$u = f_{i_v}^{2,3,4} \cdot \bigl(f_{i_v}^{1,3,4}\bigr)^{-1} \cdot f_{i_v}^{1,2,4} \cdot \bigl(f_{i_v}^{1,2,3}\bigr)^{-1}, \qquad (2.119)$$

expressed in terms of the local data $(f_i^{I,J,K}) \in A^0_{T_{3+1}}$ of the induced 2-morphisms $\widetilde{\varphi}^{I,J,K}$. As argued before, u is constant on each connected component of T_{3+1}.

Thus, for classical topological defects with induced data on T_{3+1}, the operation of pulling one three-valent defect vertex past another changes the exponentiated action by a phase determined by the underlying local data $\bigl(T_3, \mathcal{O}^{T_3}, \varphi_3, \tau_3\bigr)$.

2.9.1 The Lie-group example (cont'd)

It is easy to convince oneself that the jump defects introduced previously satisfy the conditions listed in the preceding section and hence give us an example of topological defects for the WZW model. Indeed, this is an immediate consequence of the following facts: First of all, the extension is fixed as

$$(\iota_1, \iota_2) \circ \widehat{X}|_{U_1} = (X, z^{-1} \cdot X), \quad (\iota_1, \iota_2) \circ \widehat{X}|_{U_2} = (z \cdot X, X) \tag{2.120}$$

at the defect line associated with the jump of the embedding field by $z \in Z(\mathrm{G})$. Secondly, the curvature ω of the $\mathcal{G}^{\star \mathsf{k}}$-bi-brane $\mathcal{B}_{Z(\mathrm{G})}$ vanishes and the Cartan–Killing metric on the Lie group is G-invariant so that $\Delta G_{\widehat{X}(p)} = 0$ in (2.113). Let us now consider the pair of world-sheets with network-field configurations $(\Gamma_{\mathrm{L}}, X_{\mathrm{L}})$ and $(\Gamma_{\mathrm{R}}, X_{\mathrm{R}})$ and jumps across the defect lines as indicated in figure 3. Since the data of the $\big(\mathcal{G}^{\star \mathsf{k}}, \mathcal{B}_{Z(\mathrm{G})}\big)$-inter-bi-brane can be induced from that for three-valent defect vertices, we derive — as a corollary to the general statement (2.118)–(2.119), and using the explicit results for the induced data $\big(T_{3+1}, \mathcal{O}^{T_{3+1}}, \varphi_{3+1}^{\mathrm{L,R}}, \tau_{3+1}\big)$ from Section 2.8 — the compact relation

$$\exp\big(-S[(\Gamma_{\mathrm{L}}, X_{\mathrm{L}}); \gamma_0]\big) = \psi_{\mathcal{G}^{\star \mathsf{k}}}(x, y, z) \cdot \exp\big(-S[(\Gamma_{\mathrm{R}}, X_{\mathrm{R}}); \gamma_0]\big) \tag{2.121}$$

advertised in the Introduction, in which we may now identify the associator 3-cocycle as the one given by (2.102). In the path-integral approach to the quantization of the WZW model, an analogous statement could be inferred for the correlators.

In fact, the 3-cocycle has already appeared in the literature, to wit, in [GR1, GR2] in order to define $\mathcal{Z}$-equivariant gerbes, and in [JK], where it was employed in the path-integral quantization of the orbifold string theory. Let us now elaborate on the former point.

Consider a symmetry group S as at the end of Section 2.5 and assume in addition that M is connected. We call a homomorphic presentation $(\mathcal{A}_S, \widetilde{\varphi}_S)$ of S on b *associative* if the two 2-morphisms from $((x \cdot y).\mathcal{A}_z) \circ (x.\mathcal{A}_y) \circ \mathcal{A}_x$ to $\mathcal{A}_{x \cdot y \cdot z}$ constructed from $\tilde{\varphi}_S$ are equal, or, equivalently, if $(\delta_S \widetilde{\varphi})_{x,y,z} = 1$ for all $x, y, z \in S$. Note that, because of $(\delta_S \mathcal{A})_{x,y} = -D\widetilde{\varphi}_{x,y}$, any two homomorphic presentations $(\mathcal{A}_S, \tilde{\varphi}_S)$ and $(\mathcal{A}_S, \tilde{\varphi}'_S)$ (with the same underlying element-wise presentation) are related by a 2-cochain[4] $v \in C^2(S, \mathrm{U}(1))$ via $\tilde{\varphi}_{x,y} = v(x, y) \cdot \tilde{\varphi}'_{x,y}$. Thus, an associative homomorphic presentation for a given element-wise presentation $\mathcal{A}_S$ exists if and only if $(\delta_S \widetilde{\varphi})_{x,y,z} = \delta_S v(x, y, z)$ for some $v \in C^2(S, \mathrm{U}(1))$, where S acts by the Čech-extended pullback on the local data of $\widetilde{\varphi}_{x,y}$, and trivially on $v(x, y)$. Since $D\delta_S \widetilde{\varphi} =$

[4] If we had not assumed M connected then v would, instead, take values in $C^2(S, \mathrm{U}(1)^{|\pi_0(M)|})$.

$-\delta_S^2\mathcal{A} = (0,1)$, we readily see how the cohomology class of $\psi = \delta_S\widetilde{\varphi}$ determines the obstruction to associativity. Finally, an S-equivariant structure on the gerbe $\mathcal{G} = (\mathcal{O}^M, b)$ is an associative homomorphic presentation of S on b. It is a prerequisite of projecting the sigma model on M to the quotient target space M/S (the orbifold) by dividing out the action of the symmetry group S, see [GR1, GR2].

From the present point of view, the data needed to define a classical orbifold consists of a topological bi-brane and an inter-bi-brane with world-volume $T = T_3 \sqcup T_4$ which is associative in the sense that the two limits in figure 5 agree. This quite beautifully matches the construction in [FRS1, Fr2] of a general rational conformal field theory starting from the Cardy case. There, one equally fixes a topological defect B and endows it with an associative 3-valent vertex. In both cases, the orbifold amplitudes are obtained by embedding sufficiently fine defect networks into the world-sheet. In fact, for the CFT one can obtain *all* theories with a given chiral symmetry in this way, including the exceptional modular invariants [FRS1].

The intermediate steps leading to the explicit form of $\psi_{\mathcal{G}^{\star\mathsf{k}}}$ are rather involved technically (in particular, the geometric description of the WZW gerbe as a particular bundle gerbe of [Me] is used heavily), which is why we only cite the result that can be read off from [GR2, Section 3] and [GSW2, Section 2]. It is given by

$$\psi_{\mathcal{G}^{\star\mathsf{k}}}(x,y,z) = \exp\bigl(-2\pi\mathsf{i}\,\mathsf{k}\,\langle\tau_{x^{-1}0}, \mathrm{b}_{y,z}\rangle\bigr), \quad x,y,z \in Z(\mathrm{G}) \tag{2.122}$$

for $\langle\cdot,\cdot\rangle$ the standard scalar product on the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ (normalized as in [GR2] and employed to identify $\mathfrak{t}^*$ with $\mathfrak{t}$), $\tau_{x^{-1}0} \in \mathfrak{t}$ the simple coweight of $\mathfrak{g}$ determined, up to an irrelevant element of the coroot lattice, by the condition[5]

$$x = \exp\bigl(-2\pi\mathsf{i}\,\tau_{x^{-1}0}\bigr), \tag{2.123}$$

and $\mathrm{b}_{y,z}$ a particular 2-cocycle on $Z(\mathrm{G})$ defined (modulo $Q^\vee(\mathfrak{g})$) as follows: Let us denote by α_i the simple roots of $\mathfrak{g}$, by θ its highest root, and by $\mathcal{A}_W(\mathfrak{g})$ its fundamental Weyl alcove,

$$\mathcal{A}_W(\mathfrak{g}) = \bigl\{\lambda \in \mathfrak{t} \quad | \quad \langle\lambda,\theta\rangle \leq 1 \quad \wedge \quad \langle\lambda,\alpha_i\rangle \geq 0, i = 1,2,\ldots,\mathrm{rank}\,\mathfrak{g}\ \bigr\}. \tag{2.124}$$

[5]The condition realizes the isomorphism $Z(\mathrm{G}) \cong P^\vee(\mathfrak{g})/Q^\vee(\mathfrak{g})$, in which $P^\vee(\mathfrak{g})$ and $Q^\vee(\mathfrak{g})$ are the coweight lattice and the coroot lattice of $\mathfrak{g}$, respectively.

The action of the centre $Z(\mathrm{G})$ on the group G by multiplication maps conjugacy classes into conjugacy classes, and every conjugacy class C can be represented by a unique element $\tau \in \mathcal{A}_W(\mathfrak{g}) \subset \mathfrak{t}$ of the fundamental Weyl alcove of $\mathfrak{g}$ such that $\exp(2\pi\mathsf{i}\,\tau) \in C$. Accordingly, the action of $Z(\mathrm{G})$ induces an affine map $\tau \mapsto x.\tau$ of $\mathcal{A}_W(\mathfrak{g})$ to itself, determined by the relation

$$x \cdot \exp\bigl(2\pi\mathsf{i}\,\tau\bigr) = w_x^{-1} \cdot \exp\bigl(2\pi\mathsf{i}\,(x.\tau)\bigr) \cdot w_x \tag{2.125}$$

satisfied by a certain element w_x of the normalizer $N(T)$ of the Cartan subgroup $T \subset \mathrm{G}$. In particular, $\tau_{x^{-1}.0}$ is the preimage of the weight $\tau = 0$ under this action. The element w_x is fixed only up to the multiplication $w_x \mapsto t \cdot w_x$ by an arbitrary element $t \in T$, hence it is only the class $[w_x] \in N(T)/T$ of w_x in the Weyl group $N(T)/T$ of G that is determined uniquely. The assignment $x \mapsto [w_x]$ is an injective homomorphism, however, w_x itself cannot — in general — be chosen to depend multiplicatively on x, that is we cannot set $w_{x\cdot y}$ equal to $w_x \cdot w_y$ for all $x, y \in Z(\mathrm{G})$. Nevertheless, the condition $w_x \cdot w_y \cdot w_{x\cdot y}^{-1} \in T$ is always satisfied, which leads us to the definition

$$w_x \cdot w_y \cdot w_{x\cdot y}^{-1} = \exp\bigl(2\pi\mathsf{i}\,\mathrm{b}_{x,y}\bigr) \tag{2.126}$$

of the 2-cocycle $\mathrm{b}_{x,y}$, defined modulo $Q^\vee(\mathfrak{g})$. The action of $Z(\mathrm{G})$ on $\mathcal{A}_W(\mathfrak{g})$ and the elements $\tau_{x^{-1}0}, \mathrm{b}_{x,y}$ for all simple Lie groups with a non-trivial centre were listed in [GR2, Section 4]. These data were subsequently used to compute the 3-cocycles $\psi_{\mathcal{G}^{\star\mathsf{k}}}$, see also [GR1, GSW1] (we use the conventions of [GSW1], in terms of which $u_{x,y,z} = \psi_{\mathcal{G}^{\star\mathsf{k}}}(x, y, z)$).

3 World-sheets with defect networks in CFT

In "constructive" conformal field theory, one tries to determine the correlation functions of the theory from their symmetries and from a set of consistency relations known as sewing constraints [FS, Va, So]. For oriented closed conformal field theories, this approach was given a mathematical framework in [Se]. In this section, we describe its straightforward generalization to surfaces with defect lines and outline the simplifications that occur for topological defects. We shall show that if a discrete symmetry group of a CFT is implemented by defects it gets equipped with an associator 3-cocycle.

3.1 Sewing constraints for world-sheets with defects

From [Se], we know that a convenient way to encode the sewing constraints is to use the language of functors. We shall describe a symmetric monoidal category *WD* of "world-sheets with defect lines" and define a two-dimensional euclidean quantum field theory in the presence of defect lines as a symmetric monoidal functor from *WD* to *TV*, the symmetric monoidal category of locally convex topological vector spaces (see, e.g., the foreword to [Se], and Section 2 in [StT]).

An *annulus with arcs* O is a triple (r, σ, L) with the following constituents (cf. figure 6):

(A.i) $0 < r < 1$ is a real number. It defines the annulus $A_r = \{z \in \mathbb{C} \mid r < |z| < r^{-1}\}$.
(A.ii) $\sigma : A_r \to \mathbb{R}$ is a smooth function. It defines a metric in conformal gauge on A_r via $g_{ij}(x) = \mathrm{e}^{2\sigma(x)}\,\delta_{ij}$.
(A.iii) L is a smooth oriented one-dimensional submanifold of A_r. If L has n connected components then, for each concentric circle $C \subset A_r$, the intersection $C \cap L$ is demanded to consist of n points.

Note that we obtain an ordering of the connected components of L upon labelling them by $1, 2, \ldots, n$ in the order in which they intersect the circle $|z| = 1$ starting from the point $z = 1$.

Given an annulus with arcs O, by O^+ we mean the subset $\{z \in \mathbb{C} \mid 1 \leq |z| < 1/r\}$ endowed with the metric and the one-dimensional submanifold inherited from O, and by O^- we denote the analogous restriction to $\{z \in \mathbb{C} \mid r < |z| \leq 1\}$. By $O_{(m)}$ we mean an ordered list $(O_1, O_2, \ldots, O_m)$ of a finite number of annuli with arcs.

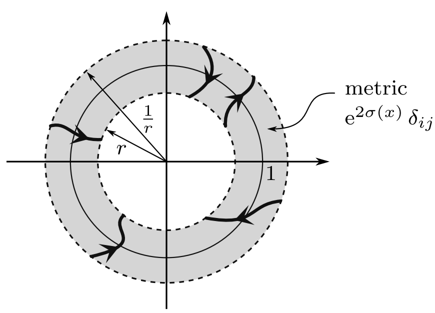


Figure 6: An annulus with arcs $O = (r, \sigma, L)$. Indicated are the inner and outer radius r and $1/r$, the metric defined in terms of the function σ, and the oriented submanifold L which describes the defect lines.

A *world-sheet with defect lines* Σ *from* $O_{(m)}$ *to* $O'_{(n)}$, to be denoted as $O_{(m)} \xrightarrow{\Sigma} O'_{(n)}$ in what follows, is a tuple $(W, L, \phi_{\text{in}}, \phi_{\text{out}})$, where (cf. figure 7):

(W.i) W is a smooth oriented two-dimensional manifold with riemannian metric, possibly with a non-empty boundary.
(W.ii) L is a smooth oriented one-dimensional submanifold of W.
(W.iii) ϕ_{in} is a smooth injective isometry from the disjoint union $O_1^+ \sqcup O_2^+ \sqcup \cdots \sqcup O_m^+$ to W which preserves the orientation, the boundaries and the one-dimensional submanifolds with their orientation.
(W.iv) ϕ_{out} is a smooth injective isometry from the disjoint union $O_1'^- \sqcup O_2'^- \sqcup \cdots \sqcup O_n'^-$ to W with the same properties as in (W.iii).

We refer to the boundary components of W in the image of ϕ_{in} as *in-going* and to those in the image of ϕ_{out} as *out-going*. A *defect line* is a connected component of L. Note that ϕ_{in} induces a numbering of the in-going boundary components by assigning the number k to the component which lies in $\phi_{\text{in}}(O_k^+)$. Similarly, out-going boundary components are numbered by ϕ_{out}.

Given world-sheets $O_{(k)} \xrightarrow{\Sigma_1} O'_{(l)}$ and $O'_{(l)} \xrightarrow{\Sigma_2} O''_{(m)}$, we can obtain the glued world-sheet $\Sigma_2 \circ \Sigma_1$ by identifying the boundaries parameterized by $O'_{(l)}$. The fact that we work with annuli and arcs instead of just circles and marked points ensures that the gluing results again in a smooth manifold with a smooth metric, and a smooth submanifold.

Two world-sheets with defect lines are *equivalent* if there is a smooth orientation-preserving isometry between them that is compatible with the

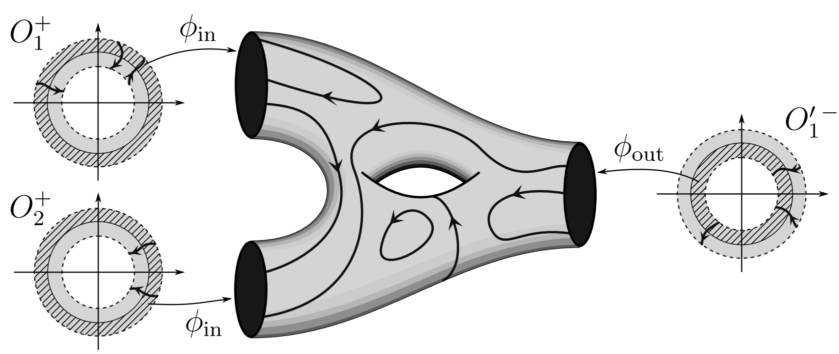


Figure 7: A world-sheet with defect lines from (O_1, O_2) to (O_1'). The shaded regions of the annuli indicate the subsets O_1^+, O_2^+ and $O_1'^-$. The maps ϕ_{in} and ϕ_{out} are defined in the shaded regions and map the solid circle $|z| = 1$ to the boundary of the world-sheet.

parameterizations $\phi_{\text{in/out}}$ and preserves the one-dimensional submanifolds with their orientation.

We can now describe the category WD. The objects[6] of WD are ordered lists $O_{(m)}$. The morphisms from $O_{(m)}$ to $O'_{(n)}$ are equivalence classes $[\Sigma]$ of world-sheets from $O_{(m)}$ to $O'_{(n)}$, and, if $m = n$, all $\pi \in S_m$ (the group of permutations of m objects) for which $O_i = O'_{\pi(i)}, i = 1, 2, \dots, m$. The permutations account for the freedom to choose a different numbering of the boundary components of a world-sheet Σ. The four possible compositions are defined as follows:

$O_{(k)} \xrightarrow{[\Sigma_1]} O'_{(l)} \xrightarrow{[\Sigma_2]} O''_{(m)}$ is the equivalence class of the glued world-sheet $[\Sigma_2 \circ \Sigma_1]$;
$O_{(k)} \xrightarrow{\pi} O'_{(k)} \xrightarrow{[\Sigma_2]} O''_{(m)}$ is defined by precomposing the parameterization ϕ_{in} with π;
$O_{(k)} \xrightarrow{[\Sigma_1]} O'_{(l)} \xrightarrow{\pi} O''_{(l)}$ is defined by precomposing the parameterization ϕ_{out} with π^{-1};
$O_{(k)} \xrightarrow{\pi_1} O'_{(k)} \xrightarrow{\pi_2} O''_{(k)}$ is the composition of permutations $\pi_2 \circ \pi_1$.

Since we are using equivalence classes of world-sheets, the composition is strictly associative. The identity morphism of $O_{(m)}$ is the identity permutation. The tensor product is the concatenation of lists on objects and disjoint union on morphisms. Both will be written as $\sqcup$. The symmetry isomorphism $O_{(m)} \sqcup O'_{(n)} \to O'_{(n)} \sqcup O_{(m)}$ is the obvious permutation $\pi \in S_{m+n}$.

Having said all this, we define a *euclidean quantum field theory with defect lines* as a symmetric monoidal functor $C : WD \to TV$, which depends continuously on the world-sheet metric and on the position of the defect lines.

Let us unpack this definition. To each annulus with arcs O, the functor assigns a space of states $C(O) = H(O)$. Since C is monoidal, we have $C(O_{(m)}) = H(O_1) \otimes H(O_2) \otimes \cdots \otimes H(O_m)$. The empty list $O = ()$ is the tensor unit of WD, and, accordingly, we have $C(O) = \mathbb{C}$, the tensor unit of TV. Given a morphism $[\Sigma] : O_{(m)} \to O'_{(n)}$, the functor provides a linear map

$$\begin{aligned} C(\Sigma) : H(O_1) \otimes H(O_2) \otimes \cdots \otimes H(O_m) \\ \longrightarrow H(O'_1) \otimes H(O'_2) \otimes \cdots \otimes H(O'_n), \end{aligned} \tag{3.1}$$

[6]We should really define the objects to be germs of annuli with arcs because we can always restrict an annulus $O = (r, \sigma, L)$ to one with a smaller radius $r' < r$, and this should not affect the amplitude of the QFT. We have avoided this point to make the exposition less technical.

the amplitude for the world-sheet Σ. As the morphisms are equivalence classes of world-sheets, equivalent world-sheets have to give the same amplitude. That C is monoidal on morphisms implies that $C(\Sigma \sqcup \Sigma') = C(\Sigma) \otimes C(\Sigma')$, and the symmetry of C implies that changing the numbering of the boundary components of Σ translates into the corresponding relabelling of the arguments of the linear map $C(\Sigma)$. The most non-trivial condition is the compatibility with composition, which amounts to the insertion of a sum over intermediate states in the path-integral language,

$$O_{(k)} \xrightarrow{[\Sigma_1]} O'_{(l)} \xrightarrow{[\Sigma_2]} O''_{(m)} \Longrightarrow C(\Sigma_2 \circ \Sigma_1) = C(\Sigma_2) \circ C(\Sigma_1). \tag{3.2}$$

In general, it will be very difficult to construct examples of such a symmetric monoidal functor $C : \mathit{WD} \to \mathit{TV}$. However, for a special subclass of defects in conformal field theories, the so-called topological defects, further progress can be made. This is the topic of the next section.

3.2 Topological defects in conformal field theory

Let $\Sigma = (W, L, \phi_{\text{in}}, \phi_{\text{out}})$ and $\Sigma' = (W, L', \phi_{\text{in}}, \phi_{\text{out}})$ be two world-sheets which differ only in the choice of defect lines. We say that Σ and Σ' have *homotopic defect lines* if L and L' are homotopic (as oriented paths) via a homotopy that is constant on the image of ϕ_{in} and on that of ϕ_{out}. We call the defects in a 2d-QFT *topological* if $C(\Sigma) = C(\Sigma')$ whenever Σ and Σ' have homotopic defect lines.

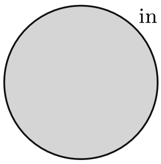
Recall that a 2d-QFT is conformal if an amplitude $C(\Sigma)$ changes only by an overall factor upon applying a Weyl transformation $\gamma(x) \mapsto \gamma'(x) = \Omega(x) \cdot \gamma(x)$ to the metric (where $\Omega \equiv 1$ on the image of ϕ_{in} and on that of ϕ_{out}). The factor is computed in terms of the Liouville action and the central charge, see, e.g., [Ga3] for more details.

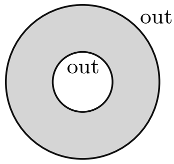
For a 2d-CFT with topological defects, the functor C simplifies in two significant ways. First, if $O = (r, \sigma, L)$ then $H(O)$ does not depend on r or σ, and it depends on L only through the number n of points in the intersection of L with the unit circle, and on n signs $\varepsilon^{k,k+1}, k = 1, 2, \ldots, n$. The sign $\varepsilon^{k,k+1}$ is $+1$ if the kth connected component of L is oriented from the outside of the unit circle to the inside. Otherwise, $\varepsilon^{k,k+1} = -1$. To specify C on objects of WD, it is thus enough to give vector spaces

$$H_{n,\vec{\varepsilon}}, \quad n \in \mathbb{Z}_{\geq 0}, \quad \vec{\varepsilon} = \left\{ \varepsilon^{k,k+1} \in \{\pm 1\} \mid k = 1, \ldots, n \right\}. \tag{3.3}$$

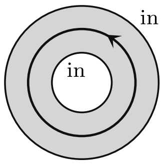
Elements of $H_{n>0,\vec{\varepsilon}}$ will be called *twisted states*, and those of $H_{n=0}$ *untwisted states*, in conformity with the physical jargon.

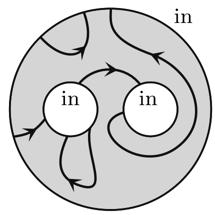
To fix C on world-sheets, it is enough to give it on a set of fundamental world-sheets from which all others can be obtained via gluing. As opposed to the theory without defects, we now need an infinite set of fundamental world-sheets. One possible choice is

$D_i =$ 

, $A_{oo} =$ 

,

$A^D_{ii} =$ 

, $P(n,m,k;L) =$ 

. (3.4)

In $P(n,m,k;L)$, the integers $n,m,k \in \mathbb{Z}_{\geq 0}$ designate how many defect lines end on each of the three boundary circles, and L is the corresponding set of defect lines. The defect lines are not allowed to contain closed loops (these are already generated by the A^D_{ii}).

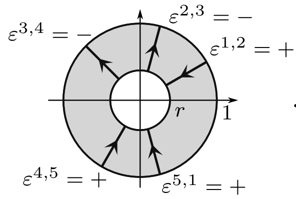
In [So, Le], a generators-and-relations approach to closed and open/closed conformal field theory is given. In both cases, a finite number of generators and relations are sufficient. In the presence of defects, already the number of amplitudes one needs to fix for the fundamental world-sheets (3.4) is infinite, and a concrete set of sufficient sewing constraints has not been worked out to date.

However, there exists an alternative approach to determine the functor $C : WD \to TV$ for a conformal field theory with topological defect lines [FRS1, Fr2]. This approach applies to rational conformal field theories and uses an associated three-dimensional topological field theory. In the case of the WZW model, this is just the three-dimensional Chern–Simons theory [Wi, FKi]. In the TFT-approach, one makes a proposal for all $C(\Sigma)$ simultaneously and then verifies that this, indeed, defines a symmetric monoidal functor. (Admittedly, a complete proof of this statement along the lines of [Fj] is not yet available.) The data that determine C are a rational vertex-operator algebra $\mathcal{V}$, a symmetric special Frobenius algebra A in the category $\mathrm{Rep}(\mathcal{V})$ of representations of $\mathcal{V}$, and an A-A-bimodule Q in $\mathrm{Rep}(\mathcal{V})$. We refer to [Fr2] for details; we do not need this general approach in the present paper. However, let us point out that in the special case of $\mathcal{V} = \mathbb{C}$, i.e., for a two-dimensional *topological* field theory with topological defect lines, the resulting algebraic structure is very similar to that of a planar algebra [Jo, KPS].

In order to prepare the subsequent discussion of symmetries implemented by defects, we need to assume some further properties of C. These are satisfied in the WZW model studied in Section 3.4.

Consider the world-sheet $A^r_{n,\vec{\varepsilon}}$ given by an annulus of outer radius one and inner radius r with n rays of defect lines, having orientations given by a list $\vec{\varepsilon} = (\varepsilon^{1,2}, \varepsilon^{2,3}, \ldots, \varepsilon^{n,1})$, e.g.,

$A^r_{5,+--++} =$



. (3.5)

We assume that the "propagator"

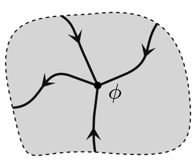
$$C(A^r_{n,\vec{\varepsilon}}) : H_{n,\vec{\varepsilon}} \to H_{n,\vec{\varepsilon}} \tag{3.6}$$

is invertible. If we are given an eigenvector ϕ of $C(A^r_{n,\vec{\varepsilon}})$ such that

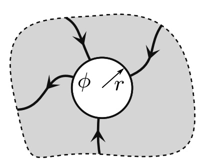
$$C(A^r_{n,\vec{\varepsilon}})\,\phi = r^{\Delta_\phi}\,\phi, \tag{3.7}$$

with Δ_ϕ the conformal weight of ϕ, we can define a *field insertion* ϕ to mean

$C\Big($



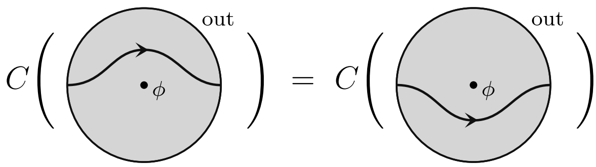
$\Big) \;:=\; r^{-\Delta_\phi} \cdot C\Big($



$\Big)$. (3.8)

The left-hand side shows a fragment of a world-sheet with the insertion, and the right-hand side, in which we have drawn a world-sheet Σ_ϕ with the corresponding hole marked by ϕ, means that the argument of the linear operator $C(\Sigma_\phi)$ corresponding to the (marked) in-going boundary shown in the figure is set to ϕ. The gluing properties of C imply that this definition is independent of r. Even if not made explicit in the notation, a field insertion by definition carries a local coordinate system since it corresponds to a small parameterized hole.

Denote by $T, \overline{T} \in H_0$ the holomorphic and anti-holomorphic components of the energy-momentum tensor. We demand that topological defects commute with T and $\overline{T}$ in the sense that



for $\phi = T$ or $\phi = \bar{T}$. (3.9)

This is, in fact, the original definition of topological defects [PZ] (the name itself was introduced in [BG]). For the more general *conformal* defects, treated, e.g., in [Ba, QRW, BB], condition (3.9) does not have to hold. The topological defects in the WZW model we shall be interested in satisfying property (3.9) also for the KaČ–Moody currents $\phi = J^a, \overline{J}^a$.

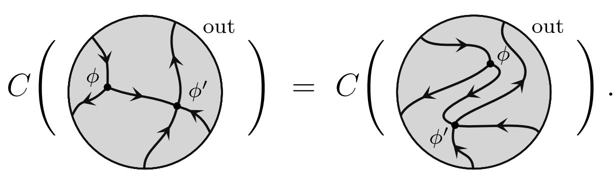
By virtue of (3.9), we have an action of $\mathrm{Vir} \oplus \mathrm{Vir}$ on each of the state spaces $H_{n,\vec{\varepsilon}}$. We can, in particular, use the operators L_0 and $\overline{L}_0$ to make (3.6) explicit,

$$C(A^r_{n,\vec{\varepsilon}}) = r^{L_0 + \overline{L}_0}. \tag{3.10}$$

We shall be interested in the subspace $H^{(0)}_{n,\vec{\varepsilon}}$ of $H_{n,\vec{\varepsilon}}$ consisting of the $\mathfrak{sl}(2,\mathbb{C})$-invariant states,

$$H^{(0)}_{n,\vec{\varepsilon}} = \Big(\bigcap_{m=0,\pm 1} \ker L_m\big|_{H_{n,\vec{\varepsilon}}} \Big) \cap \Big(\bigcap_{m=0,\pm 1} \ker \overline{L}_m\big|_{H_{n,\vec{\varepsilon}}} \Big). \tag{3.11}$$

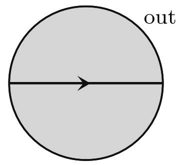
Since an element of $H^{(0)}_{n,\vec{\varepsilon}}$ is annihilated by the generators of translations, L_{-1} and $\overline{L}_{-1}$, an amplitude with an insertion of $\phi \in H^{(0)}_{n,\vec{\varepsilon}}$ is independent of the insertion point, e.g., for $\phi \in H^{(0)}_{3,+--}$ and $\phi' \in H^{(0)}_{4,-+++}$,



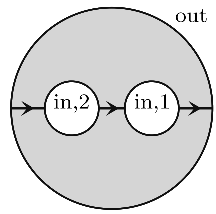
(3.12)

Consider the following two world-sheets:

$$D^D = $$



$$, \qquad M^D = $$



$$. \tag{3.13}$$

Let us abbreviate $A^D = H^{(0)}_{2,-+}$. Define, for $a, b \in A^D$,

$$\mathbf{1}^D = C(D^D)\,1, \quad m^D(a,b) = C(M^D)(a \otimes b). \tag{3.14}$$

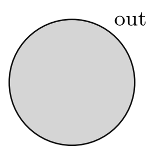
The notation $C(D^D)\,1$ refers to the fact that $[D^D]$ is a morphism from the empty list to $O_{(1)}$, which the functor C takes to a linear map $C(D^D) : \mathbb{C} \to H_{2,-+}$. We evaluate the map on 1 to get an element of $H_{2,-+}$.

By (3.9), we have $\mathbf{1}^D \in A^D$ and also $m^D(a,b) \in A^D$. Using the gluing property and the fact that the elements of A^D are $\mathfrak{sl}(2,\mathbb{C})$-invariant, one verifies that $\mathbf{1}^D$ and m^D turn A^D into an associative unital algebra. That is, for $a, b, c \in A^D$, we have

$$m^D(\mathbf{1}^D, a) = a = m^D(a, \mathbf{1}^D), \quad m^D(a, m^D(b,c)) = a = m^D(m^D(a,b), c). \tag{3.15}$$

The vector $\mathbf{1}^D$ can be understood as a twisted vacuum state, or as the identity field on the defect D. We also define the untwisted vacuum to be simply the correlator of the unit disc without defect lines, evaluated on $1 \in \mathbb{C}$,

$$\mathbf{1} = C\Bigg($$



$$\Bigg)\,1 \;\in\; H^{(0)}_0\,. \tag{3.16}$$

3.3 Symmetries implemented by defects

Topological defects can implement symmetries of the CFT. This leads to the notion of “group-like defects” [Fr1, Fr2], where one has one such defect for each element of the symmetry group. In the approach taken here, we would only have a single type of the defect line, which in the language of [Fr2] would be a superposition of all group-like defects.

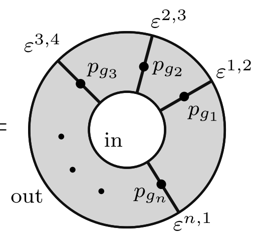
In the remainder of this section, we explain the notion of a symmetry that is implemented by defects using the framework developed in the previous two sections.

Let S be a finite group. We demand that the space $A^D = H^{(0)}_{2,-+}$ has a basis $\{p_g \mid g \in S\}$ such that

$$\sum_{g\in S} p_g = \mathbf{1}^D \quad \text{and} \quad m^D(p_g, p_h) = \delta_{g,h}\, p_g. \tag{3.17}$$

In the approach of [Fr2], p_g can be understood as projectors onto the individual group-like defects. Consider the annulus $A^r_{n,\vec{\varepsilon}}$ with projectors $p_{g_1}, p_{g_2}, \ldots, p_{g_n}$ inserted on the defect lines,

$$A^r_{n,\vec{\varepsilon}}(g_1, \ldots, g_n) \;=\; \tag{3.18}$$



We define the linear maps

$$P_{n,\vec{\varepsilon}}(g_1, g_2, \ldots, g_n) = C(A^r_{n,\vec{\varepsilon}}(g_1, g_2, \ldots, g_n))\, r^{-L_0 - \overline{L}_0}. \tag{3.19}$$

One verifies, using the gluing properties and (3.17), that $P_{n,\vec{\varepsilon}}(g_1, g_2, \ldots, g_n)$ are independent of r and obey

$$\begin{aligned} &P_{n,\vec{\varepsilon}}(g_1, g_2, \ldots, g_n) P_{n,\vec{\varepsilon}}(h_1, h_2, \ldots, h_n) \\ &\quad = \delta_{g_1,h_1}\, \delta_{g_2,h_2} \cdots \delta_{g_n,h_n}\, P_{n,\vec{\varepsilon}}(g_1, g_2, \ldots, g_n). \end{aligned} \tag{3.20}$$

We now impose the condition that a twisted state space contain an $\mathfrak{sl}(2,\mathbb{C})$-invariant vacuum state only if the overall twist is trivial, and that the vacuum is unique in this case,

$$\dim \operatorname{im} \left(P_{n,\vec{\varepsilon}}(g_1, g_2, \ldots, g_n)\big|_{H^{(0)}_{n,\vec{\varepsilon}}}\right) = \begin{cases} 1, & \text{if} \quad \prod_{i=1}^n g_i^{\varepsilon^{i,i+1}} = e, \\ 0, & \text{otherwise.} \end{cases} \tag{3.21}$$

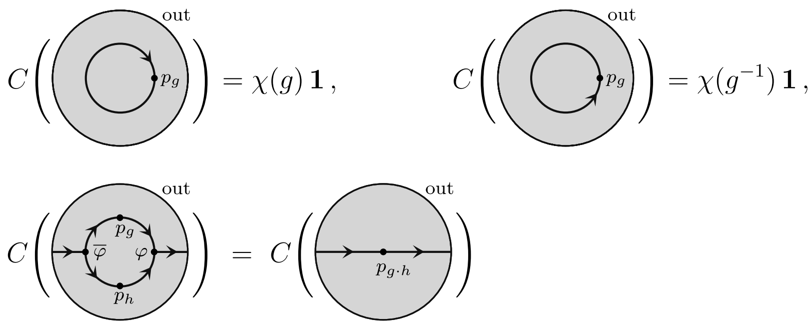
Choose non-zero vectors $\varphi_{g,h}$ in the image of $P_{3,-++}(g \cdot h, g, h)$ applied to $H^{(0)}_{3,-++}$. Then, the sum

$$\varphi = \sum_{g,h\in S} \varphi_{g,h} \tag{3.22}$$

obeys the condition

$$P_{3,-++}(g \cdot h, g, h)\, \varphi = \varphi_{g,h} \neq 0 \tag{3.23}$$

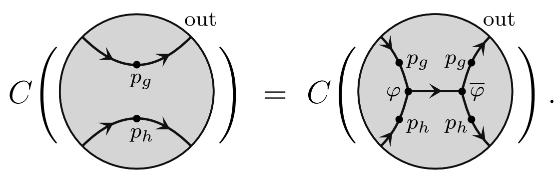
for all $g, h \in S$. We shall use φ to label all three-valent junctions with two incoming defect lines and one outgoing defect line. We demand that there exist a vector $\overline{\varphi} \in H^{(0)}_{3,+--}$ such that the following two non-degeneracy conditions for the defect correlators are satisfied (only the third one involves $\overline{\varphi}$)



(3.24)

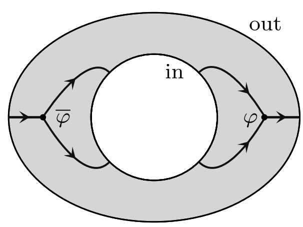
for some values $\chi(g) \in \mathbb{C}^{\times}$. This completes the list of properties that we demand of a symmetry implemented by defects.

Let us now look at some consequences of these properties. First, we shall demonstrate the identity



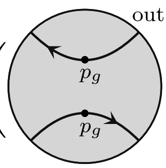
(3.25)

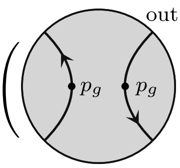
Both sides are in the image of $P_{4,-++-}(g, g, h, h)$, and the image is one-dimensional, hence they are proportional. Gluing both sides into the larger world-sheet



(3.26)

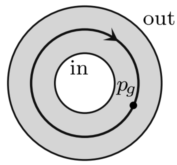
and applying (3.17) and (3.24), one obtains $p_{g \cdot h}$ in both cases. The proportionality constant is thus equal to one. This establishes (3.25). Along the same lines, one can verify the identity

$C\Big($ 

$\Big) = \chi(g) \cdot C\Big($ 

$\Big)$. (3.27)

Finally, consider the world-sheet

$A^r(g) =$ 

. (3.28)

We define the linear map $D_g : H_0 \to H_0$ as

$$D_g = C(A^r(g))\, r^{-L_0-\overline{L}_0}. \tag{3.29}$$

This is, again, independent of r, and it follows from the gluing properties and (3.25) that

$$D_g\, D_h = D_{g \cdot h}, \tag{3.30}$$

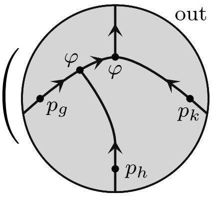
i.e., we obtain a representation of S on the untwisted state space H_0. If we apply that identity to $\mathbf{1} \in H_0$ we obtain $\chi(g)\,\chi(h) = \chi(g \cdot h)$, i.e., χ is a character of S.

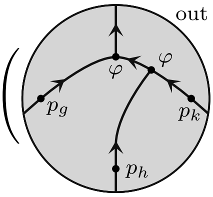
The operators D_g implement S as a symmetry of the CFT on world-sheets without defect lines. Let $O_{(m)} \xrightarrow{\Sigma} O'_{(n)}$ be a world-sheet without defect lines (i.e., the submanifold L is empty) but of arbitrary genus. Then,

$$(D_g)^{\otimes n} \circ C(\Sigma) = \chi(g)^{n-m}\, C(\Sigma) \circ (D_g)^{\otimes m}. \tag{3.31}$$

This follows from repeated application of (3.27) by the same arguments as those used in [Fr2, Section 3.1].

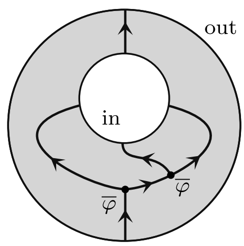
Finally, the associator 3-cocycle on S is obtained as follows: The two vectors

$v^L = C\Big($ 

$\Big)$, $\quad v^R = C\Big($ 

$\Big)$ (3.32)

lie in the image of $P_{4,-+++}(g \cdot h \cdot k, g, h, k)$ and are therefore linearly dependent. They are also both non-zero. To see this, embed each of (3.32) into a 'mirrored' picture, e.g.,

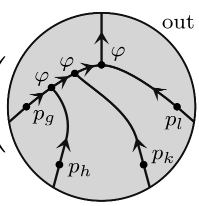


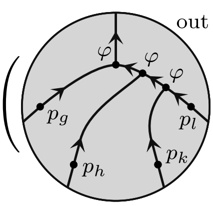
(3.33)

for v^{R}, and then use (3.24) twice. Define a $\mathbb{C}^{\times}$-valued 3-cochain ψ on S via

$$v^{\mathrm{L}} = \psi(g,h,k)\, v^{\mathrm{R}}. \tag{3.34}$$

The usual pentagon relation obtained from the two ways of relating

$C\Big($ 

$\Big)$ and $C\Big($ 

$\Big)$ (3.35)

shows that $\delta_S \psi = 1$, i.e., ψ is a cocycle. Furthermore, modifying the choice of vectors $\varphi_{g,h}$ in (3.22) amounts to replacing φ by $\varphi' = \sum_{g,h \in S} \lambda(g,h)\, \varphi_{g,h}$ for some 2-cochain $\lambda \in C^2(S, \mathbb{C}^{\times})$. The resulting change in ψ is $\psi = \psi' \cdot \delta_S \lambda$.

We can find[7] a cocycle ψ' cohomologous to ψ which is a normalized cochain and takes values in $\mathrm{U}(1) \subset \mathbb{C}^\times$.

Altogether, we see that an implementation of the symmetry group S by defects provides a cohomology class

$$[\psi] \in H^3(S, \mathrm{U}(1)). \tag{3.36}$$

3.4 3-cocycle from CFT description of the WZW model

The charge-conjugation modular invariant CFT constructed from the affine Lie algebra $\widehat{\mathfrak{g}}_{\mathsf{k}}$ is the WZW model for the compact simple connected and simply connected Lie group G of $\mathfrak{g}$ at level k. Let $\mathcal{O}_{\mathfrak{g},\mathsf{k}}$ be the category of direct sums of integrable highest-weight representations of $\widehat{\mathfrak{g}}_{\mathsf{k}}$. It is a semi-simple abelian braided monoidal category (in fact, it is even modular). The irreducible representations in $\mathcal{O}_{\mathfrak{g},\mathsf{k}}$ are labelled by integrable dominant weights $\lambda \in P_+^{\mathsf{k}}(\mathfrak{g})$ from the fundamental affine Weyl alcove

$$P_+^{\mathsf{k}}(\mathfrak{g}) = \big\{ \lambda \in P(\mathfrak{g}) \quad \big| \quad \langle \lambda, \theta \rangle \leq \mathsf{k} \quad \wedge \quad \langle \lambda, \alpha_i \rangle \geq 0,\ i = 1, 2, \ldots, \operatorname{rank} \mathfrak{g} \ \big\}. \tag{3.37}$$

We denote the corresponding representation by $\widehat{V}_\lambda$.

We are interested in the simple-current sector of the model. To each element in the centre $Z(\mathrm{G})$ of G, one can assign a weight $\lambda_z \in P_+^{\mathsf{k}}(\mathfrak{g})$ such that $\widehat{V}_{\lambda_z}$ is a simple current, see [SY2]. The weights λ_z for all $\widehat{\mathfrak{g}}_{\mathsf{k}}$ are listed in Section 4. The assignment $z \mapsto \lambda_z$ is injective, and it gives all simple currents except for the case of $\widehat{\mathfrak{e}(8)}_2$, already discussed in the Introduction. It is also compatible with the group structure in the sense that for all $z, w \in Z(\mathrm{G})$,

$$\widehat{V}_{\lambda_z} \otimes \widehat{V}_{\lambda_w} \cong \widehat{V}_{\lambda_{z\cdot w}}. \tag{3.38}$$

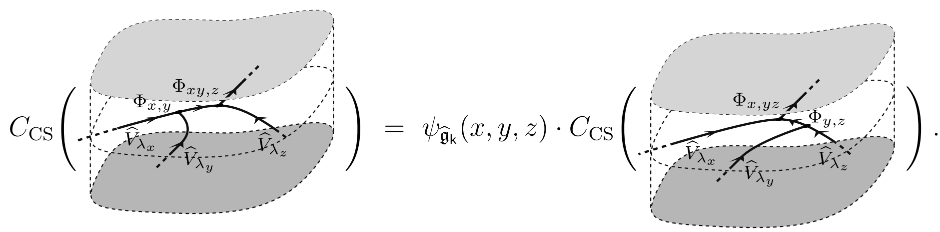
The different possible topological defects in the WZW model for $\widehat{\mathfrak{g}}_{\mathsf{k}}$ which commute with the Kač–Moody currents are in a one-to-one correspondence

[7] The argument is as follows (see, e.g., [NSW, Chapter I], specifically exercises 4, 5 of §2 and Proposition 1.6.1 of §6): Since $\mathbb{C}^\times \cong \mathrm{U}(1) \times \mathbb{R}_{>0}$ as multiplicative groups, we have $H^n(S, \mathbb{C}^\times) \cong H^n(S, \mathrm{U}(1)) \times H^n(S, \mathbb{R}_{>0})$. The isomorphism is provided by the decomposition $\psi = \psi_\theta\, \psi_r$, where $|\psi_\theta| = 1$ and $\psi_r \in \mathbb{R}_{>0}$. However, $H^n(S, \mathbb{R}_{>0}) = 1$ so that $\psi_r = \delta_S \chi$ for some χ. Thus, ψ is cohomologous to ψ_θ. Finally, every class in $H^n(S, A)$ (for A an abelian group) can be represented by a normalized cochain.

with objects of $\mathcal{O}_{\mathfrak{g},\mathsf{k}}$ [PZ, Fr2]. We choose the object

$$B = \bigoplus_{z \in Z(\mathrm{G})} \widehat{V}_{\lambda_z}. \tag{3.39}$$

The 3-cocycle associated to the $Z(\mathrm{G})$-symmetry can be computed within the TFT-approach. There, the CFT correlator is evaluated as the amplitude of the three-dimensional Chern–Simons theory at level k with the gauge group G, where the relevant three-manifold is a direct product $\Sigma \times I$ of the world-sheet and an interval, and the defect lines get replaced by a Wilson graph inside the three-manifold [Fr2]. For (3.34), one thus obtains



(3.40)

The Wilson lines are labelled by objects of $\mathcal{O}_{\mathfrak{g},\mathsf{k}}$, and the junction points by non-zero morphisms $\Phi_{g,h} \in \mathrm{Hom}(\widehat{V}_{\lambda_g} \otimes \widehat{V}_{\lambda_h}, \widehat{V}_{\lambda_{g\cdot h}})$. The choice of the morphisms $\Phi_{g,h}$ corresponds to the choice of the states $\varphi_{g,h}$ in (3.22).

By the definition of the Chern–Simons theory, the objects $\psi_{\mathfrak{g}\mathsf{k}}(x,y,z)$ in (3.40) are then entries of the fusing matrix (or 6j-symbols) of the category $\mathcal{O}_{\mathfrak{g},\mathsf{k}}$ restricted to the simple-current sector. The tensor product and the braiding in the simple-current sector of $\mathcal{O}_{\mathfrak{g},\mathsf{k}}$ can be described by abelian-group cohomology [JS, Proposition 3.1] (cf. Appendix A.1 for a brief overview of some pertinent facts about abelian-group cohomology). In fact, once we have chosen the basis $\Phi_{g,h}$, we obtain an abelian 3-cocycle (ψ, Ω) on $Z(\mathrm{G})$ with values in U(1) (a $Z(\mathrm{G})$-module with the trivial $Z(\mathrm{G})$-action), see [FRS3, Section 2]. The element ψ is an ordinary 3-cocycle on $Z(\mathrm{G})$ with values in U(1), and Ω is a 2-cochain on $Z(\mathrm{G})$. Together, they satisfy the hexagon condition, cf. (A.7). Furthermore, the diagonal elements of Ω are determined by the conformal weights via (see, e.g., [FRS3, Section 2])

$$\Omega(z,z) = \exp\bigl(2\pi \mathrm{i}\, h(\lambda_z)\bigr), \quad h(\lambda_z) = \frac{\langle \lambda_z, \lambda_z + 2\rho \rangle}{2(\mathsf{k} + g^\vee)}, \tag{3.41}$$

where ρ is the Weyl vector of $\mathfrak{g}$ and $g^\vee$ is its dual Coxeter number. If one chooses a different basis $\Phi_{g,h}$ the abelian 3-cocycle changes by a coboundary. The basis-independent information describing the tensor product and the braiding in the simple-current sector is therefore provided by a class $[\psi, \Omega] \in H^3_{\rm ab}(Z(\mathrm{G}), \mathrm{U}(1))$.

Given an abelian 3-cocycle (ψ, Ω), we obtain the function $q_{\psi,\Omega}(z) = \Omega(z, z)$ on $Z(\mathrm{G})$. It is proved in [EM, Ma] that $q_{\psi,\Omega}$ depends only on the class $[\psi, \Omega]$, and that it determines this class uniquely, cf. (A.9). This fact, together with (3.41), makes it feasible to compute a representative for the 3-cocycle $\psi_{\mathfrak{g}k}$ in (3.40), and therefore also in (3.34). We list the results in Table 1.

Table 1: The comparison data.

Algebra	$\mathbf{A_r} = \mathfrak{su}(r+1)$
Centre	$\mathbb{Z}_{r+1} = \{e, z, z^2, \ldots, z^r\}, \quad z = \mathrm{e}^{-2\pi \mathrm{i}\, \Lambda_r^\vee}$
Simple currents	$\lambda_{z^n} = \mathsf{k}\, \Lambda_{r+1-n}, \quad n \in \overline{1, r}, \quad h(\lambda_{z^n}) = \dfrac{\mathsf{k}\, n\, (r+1-n)}{2(r+1)}$
Abelian	$\psi_{\widehat{\mathfrak{su}(r+1)}_{\mathsf{k}}}\big(z^n, z^{n'}, z^{n''}\big) = (-1)^{\mathsf{k}\, r\, n\, (n'+n''-[n'+n'']_{r+1})\, /\, (r+1)}$
3-Cocycle	$\Omega_{\widehat{\mathfrak{su}(r+1)}_{\mathsf{k}}}\big(z^n, z^{n'}\big) = \mathrm{e}^{\pi \mathrm{i}\, \mathsf{k}\, r\, n\, n'\, /\, (r+1)}$
Algebra	$\mathbf{B_r} = \mathfrak{spin}(2r+1)$
Centre	$\mathbb{Z}_2 = \{e, z\}, \quad z = \mathrm{e}^{-2\pi \mathrm{i}\, \Lambda_1^\vee}$
Simple current	$\lambda_z = \mathsf{k}\, \Lambda_1, \quad h(\lambda_z) = \dfrac{\mathsf{k}}{2}$
Abelian	$\psi_{\widehat{\mathfrak{spin}(2r+1)}_{\mathsf{k}}}\big(z^n, z^{n'}, z^{n''}\big) = 1$
3-Cocycle	$\Omega_{\widehat{\mathfrak{spin}(2r+1)}_{\mathsf{k}}}\big(z^n, z^{n'}\big) = (-1)^{\mathsf{k}\, n\, n'}$
Algebra	$\mathbf{C_r} = \mathfrak{sp}(2r)$
Centre	$\mathbb{Z}_2 = \{e, z\}, \quad z = \mathrm{e}^{-2\pi \mathrm{i}\, \Lambda_r^\vee}$
Simple current	$\lambda_z = \mathsf{k}\, \Lambda_r, \quad h(\lambda_z) = \dfrac{\mathsf{k}\, r}{4}$
Abelian	$\psi_{\widehat{\mathfrak{sp}(2r)}_{\mathsf{k}}}\big(z^n, z^{n'}, z^{n''}\big) = (-1)^{\mathsf{k}\, r\, n\, n'\, n''}$
3-Cocycle	$\Omega_{\widehat{\mathfrak{sp}(2r)}_{\mathsf{k}}}\big(z^n, z^{n'}\big) = \mathrm{e}^{\pi \mathrm{i}\, \mathsf{k}\, r\, n\, n'\, /\, 2}$
Algebra	$\mathbf{D_{2s+1}} = \mathfrak{spin}(4s+2)$
Centre	$\mathbb{Z}_4 = \{e, z, z^2, z^3\}, \quad z = \mathrm{e}^{-2\pi \mathrm{i}\, \Lambda_{2s+1}^\vee}$

(Continued)

Table 1: Continued.

Simple currents	$\begin{cases} \lambda_z = \mathsf{k}\,\Lambda_{2s}, & h(\lambda_z) = \frac{\mathsf{k}\,(2s+1)}{8} \\ \lambda_{z^2} = \mathsf{k}\,\Lambda_1, & h(\lambda_{z^2}) = \frac{\mathsf{k}}{2} \\ \lambda_{z^3} = \mathsf{k}\,\Lambda_{2s+1}, & h(\lambda_{z^3}) = \frac{\mathsf{k}\,(2s+1)}{8} \end{cases}$
Abelian	$\psi_{\widehat{\mathfrak{spin}(4s+2)}_{\mathsf{k}}}\big(z^n, z^{n'}, z^{n''}\big) = (-1)^{\mathsf{k}\,n\,(n'+n''-[n'+n'']_4)\,/\,4}$
3-Cocycle	$\Omega_{\widehat{\mathfrak{spin}(4s+2)}_{\mathsf{k}}}\big(z^n, z^{n'}\big) = \mathrm{e}^{\pi\mathsf{i}\,\mathsf{k}\,(2s+1)\,n\,n'\,/\,4}$
Algebra	$\mathbf{D_{2s}} = \mathfrak{spin}(4s)$
Centre	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{e, z_1\} \times \{e, z_2\}$, $z_1 = \mathrm{e}^{-2\pi\mathsf{i}\,\Lambda_{2s}^\vee}, \quad z_2 = \mathrm{e}^{-2\pi\mathsf{i}\,\Lambda_1^\vee}$
Simple currents	$\begin{cases} \lambda_{z_1} = \mathsf{k}\,\Lambda_{2s}, & h(\lambda_{z_1}) = \frac{\mathsf{k}\,s}{4} \\ \lambda_{z_2} = \mathsf{k}\,\Lambda_1, & h(\lambda_{z_2}) = \frac{\mathsf{k}}{2} \\ \lambda_{z_1 z_2} = \mathsf{k}\,\Lambda_{2s-1}, & h(\lambda_{z_1 z_2}) = \frac{\mathsf{k}\,s}{4} \end{cases}$
Abelian	$\psi_{\widehat{\mathfrak{spin}(4s)}_{\mathsf{k}}}\big(z_1^{n_1} z_2^{n_2}, z_1^{n_1'} z_2^{n_2'}, z_1^{n_1''} z_2^{n_2''}\big) = (-1)^{\mathsf{k}\,(s\,n_1\,n_1'\,n_1''+n_1\,n_2'\,n_2''+n_2\,n_1'\,n_1'')}$
3-Cocycle	$\Omega_{\widehat{\mathfrak{spin}(4s)}_{\mathsf{k}}}\big(z_1^{n_1} z_2^{n_2}, z_1^{n_1'} z_2^{n_2'}\big) = \mathrm{e}^{\pi\mathsf{i}\,\mathsf{k}\,(s\,n_1\,n_1'+2n_2\,n_2'+n_1\,n_2'+n_2\,n_1')\,/\,2}$
Algebra	$\mathbf{E_6}$
Centre	$\mathbb{Z}_3 = \{e, z, z^2\}, \qquad z = \mathrm{e}^{-2\pi\mathsf{i}\,\Lambda_5^\vee}$
Simple currents	$\begin{cases} \lambda_z = \mathsf{k}\,\Lambda_1, & h(\lambda_z) = \frac{2\mathsf{k}}{3} \\ \lambda_{z^2} = \mathsf{k}\,\Lambda_5, & h(\lambda_{z^2}) = \frac{2\mathsf{k}}{3} \end{cases}$
Abelian	$\psi_{\widehat{\mathfrak{e}(6)}_{\mathsf{k}}}\big(z^n, z^{n'}, z^{n''}\big) = 1$
3-Cocycle	$\Omega_{\widehat{\mathfrak{e}(6)}_{\mathsf{k}}}\big(z^n, z^{n'}\big) = \mathrm{e}^{-2\pi\mathsf{i}\,\mathsf{k}\,n\,n'/3}$
Algebra	$\mathbf{E_7}$
Centre	$\mathbb{Z}_2 = \{e, z\}, \quad z = \mathrm{e}^{-2\pi\mathsf{i}\,\Lambda_1^\vee}$
Simple current	$\lambda_z = \mathsf{k}\,\Lambda_6, \quad h(\lambda_z) = \frac{3\mathsf{k}}{4}$
Abelian	$\psi_{\widehat{\mathfrak{e}(7)}_{\mathsf{k}}}\big(z^n, z^{n'}, z^{n''}\big) = (-1)^{\mathsf{k}\,n\,n'\,n''}$
3-Cocycle	$\Omega_{\widehat{\mathfrak{e}(7)}_{\mathsf{k}}}\big(z^n, z^{n'}\big) = \mathrm{e}^{-\pi\mathsf{i}\,\mathsf{k}\,n\,n'\,/\,2}$

4 The classical 3-cocycle vs the quantum 3-cocycle

In this final section of our paper, we bring to completion the discussion of the correspondence between the classical, i.e., gerbe-theoretic, and the quantum, i.e., conformal-field-theoretic, description of world-sheets related by an associator move of figure 3 in the setting of the WZW sigma model on a compact simple connected and simply connected Lie group G. We do so by demonstrating, through a case-by-case comparison, that the 3-cocycle component $\psi_{\hat{\mathfrak{g}}_{\mathsf{k}}} \in Z^3(Z(\mathrm{G}),\mathrm{U}(1))$ of a representative $(\psi_{\hat{\mathfrak{g}}_{\mathsf{k}}},\Omega_{\hat{\mathfrak{g}}_{\mathsf{k}}})$ of the class $[\psi_{\hat{\mathfrak{g}}_{\mathsf{k}}},\Omega_{\hat{\mathfrak{g}}_{\mathsf{k}}}] \in H^3_{\mathrm{ab}}(Z(\mathrm{G}),\mathrm{U}(1))$ fixed by (3.41) via the Eilenberg–MacLane map (cf. Appendix A.1), coincides with the associator 3-cocycle $\psi_{\mathcal{G}^{\star\mathsf{k}}}$ obtained from the analysis of the variation of the action functional of the WZW model under the associator move of the embedded defect network.

Here are the details of the comparison. The starting point is the computation of the diagonal components of the 2-cochain $\Omega_{\hat{\mathfrak{g}}_{\mathsf{k}}}$ from (3.41), using the data for λ_x given in table 1, and that for the metric on $P(\mathfrak{g})$ (the quadratic-form matrix F) taken, e.g., from [DMS, Chapter 13]. Having found $\Omega_{\hat{\mathfrak{g}}_{\mathsf{k}}}(x,x)$ for all $x \in Z(\mathrm{G})$, we then proceed according to the type of $Z(\mathrm{G})$ at hand, to wit:

- The non-cyclic centre $Z(\mathrm{G}) = \mathbb{Z}_2 \times \mathbb{Z}_2$ of $\mathrm{G} = \mathrm{Spin}(4s)$. In this case, we simply solve the coupled pentagon and hexagon equations explicitly for $\psi_{\widehat{\mathfrak{spin}(4s)}_{\mathsf{k}}}$, employing the definition (A.6) in the end (that is, dividing out an appropriate trivial 3-cocycle from the general solution) in order to get the specific representative from table 1. The latter is precisely the gerbe-theoretic 3-cocycle $u_{z_1^{n_1} z_2^{n_2}, z_1^{n_1'} z_2^{n_2'}, z_1^{n_1''} z_2^{n_2''}}$ for $\mathrm{G} = \mathrm{Spin}(4s)$ given in [GR2, Section 4].
- A cyclic centre of an even order, $Z(\mathrm{G}) = \mathbb{Z}_{2s}$ with generator z — this covers the cases of $\mathrm{SU}(2r+2)$ (with $s = r+1$) and $\mathrm{Spin}(4r+2)$ (with $s = 2$), as well as $\mathrm{Spin}(2r+1), \mathrm{Sp}(2r)$ and $\mathrm{E}(7)$ (all three with $s = 1$). We start by considering an auxiliary object, namely the CFT of the free boson compactified on the circle of a rational radius squared, $R^2 = \frac{p}{q}$, where p and q are two positive coprime integers (and where we use units in which the self-dual radius is 1). At these radii, the free-boson CFT has an enhanced chiral symmetry. The fusion ring of its representations is given by $\mathbb{Z}_{2pq}$ with generator ξ, and the relevant abelian 3-cocycle is [BSc, Fu2]

$$\begin{aligned}\psi_{\mathrm{FB}(p,q)}\big(\xi^n,\xi^{n'},\xi^{n''}\big) &= (-1)^{n\,\frac{n'+n''-[n'+n'']_{2pq}}{2pq}}\,,\\ \Omega_{\mathrm{FB}(p,q)}\big(\xi^n,\xi^{n'}\big) &= \mathrm{e}^{\frac{\pi\,\mathrm{i}\,n\,n'}{2pq}}\,,\end{aligned} \tag{4.1}$$

where $0 \leq [m]_{2pq} < 2pq$ is the unique integer such that $[m]_{2pq} = m \mod 2pq$. One can now check, for all the above-mentioned G, that $\Omega_{\hat{\mathfrak{g}}_{\mathsf{k}}}\big(z^n, z^n\big)$ obeys, for every $n \in \mathbb{Z}_{2s}$,

$$\Omega_{\hat{\mathfrak{g}}_{\mathsf{k}}}\big(z^n, z^n\big) = \Omega_{\mathrm{FB}(s,1)}\big(\xi^n, \xi^n\big)^{P(\mathfrak{g},\mathsf{k})}, \quad P(\mathfrak{g},\mathsf{k}) \in \mathbb{N}, \tag{4.2}$$

for an integer $P(\mathfrak{g},\mathsf{k})$ independent of n. At this stage, we may adduce the theorem of Eilenberg and MacLane cited in Appendix A.1 to conclude that the entire abelian 3-cocycle of interest can be written as

$$\big(\psi_{\hat{\mathfrak{g}}_{\mathsf{k}}}, \Omega_{\hat{\mathfrak{g}}_{\mathsf{k}}}\big) = \Big(\big(\psi_{\mathrm{FB}(s,1)}\big)^{P(\mathfrak{g},\mathsf{k})}, \big(\Omega_{\mathrm{FB}(s,1)}\big)^{P(\mathfrak{g},\mathsf{k})}\Big). \tag{4.3}$$

We now readily verify that the 3-cocycle $\big(\psi_{\mathrm{FB}(s,1)}\big)^{P(\mathfrak{g},\mathsf{k})}$ coincides, in each of the cases of interest, with the corresponding gerbe-theoretic 3-cocycle from [GR2, Section 4].

- A cyclic centre of an odd order, $Z(\mathrm{G}) = \mathbb{Z}_{2s+1}$ with generator z — this accounts for the remaining cases of $\mathrm{SU}(2r+1)$ (with $s = r$) and E(6) (with $s = 1$). For each of these groups, we first check that the diagonal components of $\Omega_{\hat{\mathfrak{g}}_{\mathsf{k}}}$ obey the identity

$$\Omega_{\hat{\mathfrak{g}}_{\mathsf{k}}}\big(z^n, z^n\big)^{N_n} = 1, \quad N_n = \frac{\mathrm{LCM}(2s+1, n)}{n} \tag{4.4}$$

for $\mathrm{LCM}(2s+1, n)$ the least common multiple of $2s+1$ and n. The number N_n thus defined is exactly the order of the element z^n of the centre, and so we see that $\Omega_{\hat{\mathfrak{g}}_{\mathsf{k}}}$ satisfies the assumptions of Lemma 2.17 of [FRS3], stated in Appendix A.1. Consequently, the abelian 3-cocycle $(\psi_{\hat{\mathfrak{g}}_{\mathsf{k}}}, \Omega_{\hat{\mathfrak{g}}_{\mathsf{k}}})$ has a representative with $\psi_{\hat{\mathfrak{g}}_{\mathsf{k}}} = 1$, in accord with the gerbe-theoretic result of [GR2, Section 4]. The corresponding 2-cochain $\Omega_{\hat{\mathfrak{g}}_{\mathsf{k}}}$ is then fixed by the hexagon equation to be a bihomomorphism, whence

$$\Omega_{\hat{\mathfrak{g}}_{\mathsf{k}}}\big(z^n, z^{n'}\big) = \Omega_{\hat{\mathfrak{g}}_{\mathsf{k}}}\big(z, z\big)^{n\,n'}, \tag{4.5}$$

which is the form of the 2-cochain given in table 1.

We shall now list the relevant algebraic data and the representatives of the abelian 3-cocycles obtained in the procedure detailed above. In so doing, we use the symbol $\Lambda_i^{(\vee)}$ to denote the ith fundamental (co)weight of $\mathfrak{g}$ (we follow the labelling conventions of [DMS]), and the shorthand notation $[m]_k$ for the unique integer $0 \leq [m]_k < k$ such that $[m]_k = m \mod k$ for $k \in \mathbb{Z}_{>0}$.

Appendix A Appendix

A.1 Some background on group cohomology

In general, group cohomology is defined for a group S and an S-module A, see, e.g., [NSW, Chapter I, §2]. We shall only need the case of a finite group S and the S-module given either by $A = \mathrm{U}(1)$, understood as an S-module with trivial S-action, or by $A = \check{C}^{p,r}(\mathcal{O})$, understood as an S-module with an S-action by pullback,

$$(g.\omega)_{i_1 i_2 \ldots i_{p+1}} = \left(g^{-1}\right)^* \omega_{g^{-1}.i_1 \; g^{-1}.i_2 \; \ldots \; g^{-1}.i_{p+1}}, \tag{A.1}$$

where we have assumed the cover $\mathcal{O}$ to be S-invariant as in (2.10).

An *n-cochain* on S is a function $S^n \to A$, and the set of n-cochains is denoted as $C^n(S,A)$. The coboundary operator $\delta_{(n)}$ is a map $C^n(S,A) \to C^{n+1}(S,A)$ which obeys $\delta_{(n+1)} \circ \delta_{(n)} = 1$. For $n = 1,2,3,4$, it is given by the formulæ

$$\begin{aligned}
(\delta_{(0)}\psi_{(0)})(a) &= a.\psi_{(0)} - \psi_{(0)}, \\
(\delta_{(1)}\psi_{(1)})(a,b) &= a.\psi_{(1)}(b) - \psi_{(1)}(a\cdot b) + \psi_{(1)}(a), \\
(\delta_{(2)}\psi_{(2)})(a,b,c) &= a.\psi_{(2)}(b,c) - \psi_{(2)}(a\cdot b,c) \\
&\quad + \psi_{(2)}(a,b\cdot c) - \psi_{(2)}(a,b), \\
(\delta_{(3)}\psi_{(3)})(a,b,c,d) &= a.\psi_{(3)}(b,c,d) - \psi_{(3)}(a\cdot b,c,d) + \psi_{(3)}(a,b\cdot c,d) \\
&\quad - \psi_{(3)}(a,b,c\cdot d) + \psi_{(3)}(a,b,c)
\end{aligned} \tag{A.2}$$

in the additive notation (e.g., for $A = \check{C}^{p,r}(\mathcal{O})$ with $r > 0$), and by the formulæ

$$\begin{aligned}
(\delta_{(0)}\psi_{(0)})(a) &= \frac{a.\psi_{(0)}}{\psi_{(0)}}, \\
(\delta_{(1)}\psi_{(1)})(a,b) &= \frac{a.\psi_{(1)}(b)\,\psi_{(1)}(a)}{\psi_{(1)}(a\cdot b)}, \\
(\delta_{(2)}\psi_{(2)})(a,b,c) &= \frac{a.\psi_{(2)}(b,c)\,\psi_{(2)}(a,b\cdot c)}{\psi_{(2)}(a\cdot b,c)\,\psi_{(2)}(a,b)}, \\
(\delta_{(3)}\psi_{(3)})(a,b,c,d) &= \frac{a.\psi_{(3)}(b,c,d)\,\psi_{(3)}(a,b\cdot c,d)\,\psi_{(3)}(a,b,c)}{\psi_{(3)}(a\cdot b,c,d)\,\psi_{(3)}(a,b,c\cdot d)}
\end{aligned} \tag{A.3}$$

in the multiplicative notation (e.g., for $A = \mathrm{U}(1)$ or $A = \check{C}^{p,0}(\mathcal{O})$), all written for $\psi_{(n)} \in C^n(S,A)$ and $a,b,c,d \in S$. The n-cocycles, the n-coboundaries and the nth cohomology group are denoted as

$$Z^n(S,A) = \ker \delta_{(n)}, \quad B^n(S,A) = \operatorname{im} \delta_{(n-1)}, \quad H^n(S,A) = \frac{Z^n(S,A)}{B^n(S,A)}, \tag{A.4}$$

respectively. We shall drop the subscript n from the coboundary operator henceforth, and we shall write δ_S whenever we want to emphasize that it is the coboundary operator for the cohomology of S.

For an abelian group S, one can introduce a different cohomology, namely abelian-group cohomology [EM, Ma]. We shall only need the third abelian cohomology group of S, with values in the trivial S-module $\mathrm{U}(1)$.

Abelian 2*-cochains on* S are just ordinary 2-cochains on the group, $C^2_{\mathrm{ab}}(S,\mathrm{U}(1)) = C^2(S,\mathrm{U}(1))$, and *abelian* 3*-cochains* are defined as

$$C^3_{\mathrm{ab}}(S,\mathrm{U}(1)) = \big\{(\psi,\Omega) \mid \psi \in C^3(S,\mathrm{U}(1)), \quad \Omega \in C^2(S,\mathrm{U}(1))\big\}. \tag{A.5}$$

The set $C^3_{\mathrm{ab}}(S,\mathrm{U}(1))$ is an abelian group under element-wise multiplication. The coboundary operator $\delta_{\mathrm{ab},(2)} : C^2_{\mathrm{ab}}(S,\mathrm{U}(1)) \to C^3_{\mathrm{ab}}(S,\mathrm{U}(1))$ is given by the formula

$$\delta_{\mathrm{ab},(2)}\varphi = \big(\, \delta_S\varphi,\ (a,b) \mapsto \varphi(a,b)/\varphi(b,a)\,\big). \tag{A.6}$$

The set of abelian 3-coboundaries $B^3_{\mathrm{ab}}(S,\mathrm{U}(1))$ is the image of $\delta_{\mathrm{ab},(2)}$. An element $(\psi,\Omega) \in C^3_{\mathrm{ab}}(S,\mathrm{U}(1))$ is an *abelian* 3*-cocycle on* S if the following conditions are satisfied for all $a,b,c,d \in S$:

$$\begin{aligned}
\text{Pentagon}:\quad & \psi(b,c,d)\,\psi(a,b\cdot c,d)\,\psi(a,b,c) \\
& \qquad = \psi(a\cdot b,c,d)\,\psi(a,b,c\cdot d), \\
\text{Hexagon}:\quad & \begin{cases} \psi(c,a,b)\,\Omega(a\cdot b,c)\,\psi(a,b,c) \\ \qquad = \Omega(a,c)\,\psi(a,c,b)\,\Omega(b,c) \\ \psi(b,c,a)^{-1}\,\Omega(a,b\cdot c)\,\psi(a,b,c)^{-1} \\ \qquad = \Omega(a,c)\,\psi(b,a,c)^{-1}\,\Omega(a,b). \end{cases}
\end{aligned} \tag{A.7}$$

(In the notation used in [Ma], $\psi(a,b,c) = f(a,b,c)$ and $\Omega(a,b) = d(a\,|\,b)$, see [Ma, Equations (17)–(19)].) Note that the pentagon condition just says that $\delta_S\psi \equiv 1$. The set of abelian 3-cocycles is denoted by $Z^3_{\mathrm{ab}}(S,\mathrm{U}(1))$.

The third abelian cohomology group of the abelian group S, with values in the trivial S-module U(1) is defined as

$$H^3_{\rm ab}(S,{\rm U}(1)) = Z^3_{\rm ab}(S,{\rm U}(1))/B^3_{\rm ab}(S,{\rm U}(1)). \tag{A.8}$$

The set $Q(S,{\rm U}(1))$ of *quadratic forms* on a group S, with values in U(1) is composed of all elements $q \in C^1(S,{\rm U}(1))$ such that $q(a) = q(a^{-1})$ and $\delta_S q : S \times S \to {\rm U}(1)$ is a bihomomorphism. The product of two quadratic forms is again a quadratic form, as is the function $q \equiv 1$, and so $Q(S,{\rm U}(1))$ is an abelian group. It is proved in [EM] (see [Ma, Theorem 3]) that the map

$$\begin{aligned} {\rm EM} : H^3_{\rm ab}(S,{\rm U}(1)) &\to Q(S,{\rm U}(1)) \\ [\psi,\Omega] &\mapsto q_{\psi,\Omega}(a) = \Omega(a,a) \end{aligned} \tag{A.9}$$

is an *isomorphism* of abelian groups. In particular, $q_{\psi,\Omega}$ depends only on the class $[\psi,\Omega]$ of the abelian 3-cocycle (ψ,Ω). Using this isomorphism, it was demonstrated in [FRS3, Lemma 2.17] that the class $[\psi] \in H^3(S,{\rm U}(1))$ of the 3-cocycle component of an abelian 3-cocycle (ψ,Ω) is trivial iff the identity

$$\Omega(a,a)^{N_a} = 1 \tag{A.10}$$

holds for every element $a \in S$, with N_a the order of a.

A.2 Field equations and defect conditions

In this Appendix, we perform a detailed derivation of the field equations and defect conditions in a generic non-linear sigma model with a topological term defined — as in Section 2.6 — on a world-sheet Σ with an embedded defect network Γ. The defect conditions, which characterize the defect in the very same manner as boundary conditions characterize a boundary state, are always to be imposed on the fields of the model, both in the classical régime (extremal field configurations) and in the quantum régime (the definition of the path integral for a world-sheet with a defect network on it).

Let us start by stating some conventions. We use the two-dimensional Levi-Civita symbols ϵ_{ab} and ϵ^{ab} such that $\epsilon_{12} = 1 = \epsilon^{12}$ and $\epsilon_{ab}\,\epsilon^{cb} = \delta_a^{\ c}$. In the component notation for differential forms, we use the following basis:

$$\mathsf{d}y^{\mu_1} \wedge \mathsf{d}y^{\mu_2} \wedge \cdots \wedge \mathsf{d}y^{\mu_p} = \sum_{\sigma \in S_p} (-1)^{{\rm sgn}(\sigma)}\, \mathsf{d}y^{\mu_{\sigma(1)}} \otimes \mathsf{d}y^{\mu_{\sigma(2)}} \otimes \cdots \otimes \mathsf{d}y^{\mu_{\sigma(p)}}. \tag{A.11}$$

The standard 'kinetic' term written in terms of the intrinsic world-sheet metric γ, the associated metric volume form $\mathrm{vol}_{\Sigma,\gamma} = \sqrt{\det\gamma}\,\mathsf{d}\sigma^1\wedge\mathsf{d}\sigma^2$ (σ^a are local coordinates on Σ) and the target-space metric G reads

$$\begin{aligned} S_{\mathrm{kin}}[X;\gamma] &= \int_{\Sigma-\Gamma} G_X(\mathsf{d}X \overset{\wedge}{,} \star_\gamma\,\mathsf{d}X) \\ &= \int_{\Sigma-\Gamma} \mathrm{vol}_{\Sigma,\gamma}\,\left(\gamma^{-1}\right)^{ab} G_{\mu\nu}(X)\,\partial_a X^\mu\,\partial_b X^\nu. \end{aligned} \tag{A.12}$$

The world-sheet metric defines, in particular, the Hodge operator $\star_\gamma$ on $\Omega^\bullet(\Sigma)$ as per

$$\star_\gamma 1 = \mathrm{vol}_{\Sigma,\gamma}, \quad \star_\gamma \mathsf{d}\sigma^a = \sqrt{\det\gamma}\,\left(\gamma^{-1}\right)^{ab}\varepsilon_{bc}\,\mathsf{d}\sigma^c, \quad \star_\gamma \mathrm{vol}_{\Sigma,\gamma} = 1. \tag{A.13}$$

We have also used the notation $\mathsf{d}X = \partial_a X^\mu\,\mathsf{d}\sigma^a\otimes\partial_\mu \in \mathsf{T}^*_\sigma\Sigma\otimes\mathsf{T}_{X(\sigma)}M$, hence the familiar local form of $S_{\mathrm{kin}}[X;\gamma]$. The integral in (A.12) splits into contributions from the patches into which the world-sheet is partitioned by the embedded defect network Γ. Whenever a functional variation of the integral produces a contribution from a component e of the boundary of the patch, we should use in the integrand the appropriate local extension $X_{|\alpha}$ described in Section 2.4, with the choice of $\alpha\in\{1,2\}$ depending on the relative orientation of e and that of the defect line covering e.

The variation of (A.12) in the direction of X reads

$$\begin{aligned} \delta_X S_{\mathrm{kin}}[X;\gamma] &= \int_{\Sigma-\Gamma} \mathrm{vol}_{\Sigma,\gamma}\left(\gamma^{-1}\right)^{ab}\left(2G_{\mu\nu}(X)\,\partial_a\delta X^\mu\,\partial_b X^\nu\right. \\ &\quad \left. + \delta X^\lambda\partial_\lambda G_{\mu\nu}(X)\,\partial_a X^\mu\,\partial_b X^\nu\right) \\ &= -2\int_{\Sigma-\Gamma} G_X\big(\delta X, \mathrm{vol}_{\Sigma,\gamma}\,\Delta_{(2)}X + \Gamma_{\text{L-C}}(\mathsf{d}X \overset{\wedge}{,} \star_\gamma\,\mathsf{d}X)\big) \\ &\quad + 2\int_{E_\Gamma} \big(G_{X_{|1}}(\delta X_{|1}, \star_\gamma \mathsf{d}X_{|1}) - G_{X_{|2}}(\delta X_{|2}, \star_\gamma \mathsf{d}X_{|2})\big), \end{aligned} \tag{A.14}$$

where $\delta X = \delta X^\mu\,\partial_\mu$ is the variation field, with the one-sided (local) extensions $\delta X_{|\alpha}$ to $\overline{U}_\alpha$. We have also used the notation

$$\begin{aligned} G_X\big(\delta X, \mathrm{vol}_{\Sigma,\gamma}\,\Delta_{(2)}X\big) &= \delta X^\mu\,G_{\mu\nu}(X)\,\big(\Delta_{(2)}X^\nu\big)\,\mathrm{vol}_{\Sigma,\gamma}, \\ G_X\big(\delta X, \Gamma_{\text{L-C}}(\mathsf{d}X \overset{\wedge}{,} \star_\gamma\,\mathsf{d}X)\big) &= \delta X^\mu\,G_{\mu\nu}(X)\,\left\{ {}^{\ \nu}_{\rho\sigma} \right\}(X)\,\mathsf{d}\,X^\rho\wedge\star_\gamma\mathsf{d}X^\sigma, \end{aligned} \tag{A.15}$$

with

$$\Delta_{(2)} = \frac{1}{\sqrt{\det\gamma}}\,\partial_a\left(\sqrt{\det\gamma}\,\left(\gamma^{-1}\right)^{ab}\partial_b\right) \tag{A.16}$$

the world-sheet Laplacian, and

$$\left\{ {}^{\ \nu}_{\rho\sigma} \right\} = \frac{1}{2}\left(G^{-1}\right)^{\nu\lambda}\left(\partial_\rho G_{\sigma\lambda} + \partial_\sigma G_{\rho\lambda} - \partial_\lambda G_{\rho\sigma}\right) \tag{A.17}$$

the Christoffel symbols of the target-space metric G. As usual, the boundary term in (A.14) comes from integration by parts and the application of Stokes' theorem. Its geometric interpretation becomes manifest upon introducing a coordinate $t \in \mathbb{R}$ along an edge $e \in E_\Gamma$ of Γ, together with the attendant normalized tangent vector field $\widehat{t} = \frac{1}{\sqrt{\gamma(\partial_t,\partial_t)}}\,\partial_t$. It is then straightforward to show that the two normalized vector fields $\widehat{n}_\alpha, \alpha = 1, 2$ normal to that edge which were described in Section 2.4 are given by $\widehat{n}_\alpha = (-1)^\alpha\,(\widehat{t}\lrcorner\,\star_\gamma\,\mathsf{d}\sigma^a)\,\partial_a$, and so the variation of the "kinetic" term rewrites as

$$\begin{aligned} \delta_X S_{\rm kin}[X;\gamma] = &-2\int_{\Sigma-\Gamma} G_X\big(\delta X, {\rm vol}_\Sigma\,\Delta_{(2)}X + \Gamma_{\text{L-C}}(\mathsf{d}X \overset{\wedge}{,} \star_\gamma\,\mathsf{d}X)\big) \\ &- 2\int_{E_\Gamma} {\rm vol}_{E_\Gamma,\gamma}\left(G_{X_{|1}}\left(\delta X_{|1}, X_{|1*}\widehat{n}_1\right) + G_{X_{|2}}\left(\delta X_{|2}, X_{|2*}\widehat{n}_2\right)\right), \end{aligned} \tag{A.18}$$

where ${\rm vol}_{E_\Gamma,\gamma}$ is the volume form for E_Γ (locally given by $\sqrt{\gamma(\partial_t,\partial_t)}\,\mathsf{d}t$) and $X_{|\alpha*} : \mathsf{T}\overline{U}_\alpha \to \mathsf{T}M$ are the tangent maps for $X_{|\alpha}$.

Passing, next, to the topological term

$$\begin{aligned} S_{\rm top}[X] = &\sum_{t\in\triangle(\Sigma)}\left[\mathsf{i}\int_t \widehat{B}_t + \sum_{e\subset t}\left(\mathsf{i}\int_e \widehat{A}_{te} + \sum_{v\in e}\log\widehat{g}_{tev}(v)\right)\right] \\ &+ \sum_{e\in\triangle(E_\Gamma)}\left(\mathsf{i}\int_e \widehat{P}_e + \sum_{v\in e}\log\widehat{K}_{ev}(v)\right) \\ &+ \sum_{v\in V_\Gamma}\log\widehat{f}_v(v), \end{aligned} \tag{A.19}$$

in which all triangulations have been correlated as discussed in Section 2.6, we find

$$
\begin{aligned}
\frac{1}{\mathsf{i}}\,\delta_X S_{\rm top}[X] = & \int_{\Sigma-\Gamma} X^*\big(\delta X \lrcorner\, H\big) \\
& + \sum_{t\in\triangle(\Sigma)} \sum_{e\subset t} \Big[\int_e X^*\big(\delta X \lrcorner\, (B_{i_t} + \mathsf{d}A_{i_t i_e})\big) \\
& + \sum_{v\in e} \varepsilon_{tev}\, X^*\big(\delta X \lrcorner\, \big(A_{i_t i_e} - \mathsf{i}\,\mathsf{d}\log g_{i_t i_e i_v}\big)\big)(v) \Big] \\
& + \sum_{e\in\triangle(E_\Gamma)} \Big[\int_e X^*(\delta X \lrcorner\, \mathsf{d}P_{i_e}) \\
& + \sum_{v\in e} \varepsilon_{ev}\, X^*\big(\delta X \lrcorner\, \big(P_{i_e} + \mathsf{i}\,\mathsf{d}\log K_{i_e i_v}\big)\big)(v) \Big] \\
& - \mathsf{i} \sum_{v\in V_\Gamma} X^*\big(\delta X \lrcorner\, \mathsf{d}\log f_{i_v}\big)(v) \\
= & \int_{\Sigma-\Gamma} X^*\big(\delta X \lrcorner\, H\big) + \sum_{t\in\triangle(\Sigma)} \sum_{e\subset t} \Big[\int_e X^*(\delta X \lrcorner\, B_{i_e}) \\
& - \sum_{v\in e} \varepsilon_{tev}\, X^*\big(\delta X \lrcorner\, A_{i_e i_v}\big)(v) \Big] \\
& + \sum_{e\in\triangle(E_\Gamma)} \Big[\int_e X^*(\delta X \lrcorner\, \mathsf{d}P_{i_e}) \\
& + \sum_{v\in e} \varepsilon_{ev}\, X^*\big(\delta X \lrcorner\, \big(P_{i_e} + \mathsf{i}\,\mathsf{d}\log K_{i_e i_v}\big)\big)(v) \Big] \\
& - \mathsf{i} \sum_{v\in V_\Gamma} X^*\big(\delta X \lrcorner\, \mathsf{d}\log f_{i_v}\big)(v),
\end{aligned}
\tag{A.20}
$$

where — so far — we have only used the defining relations of the local data of the gerbe, cf. (2.5), alongside the trivial relation $\sum_{e\subset t}\sum_{v\in e} \varepsilon_{tev}\,\delta X \lrcorner\, A_{i_t i_v} = 0$. The first line integral in the above formula reduces to a contribution from the embedded defect network, and so — upon recalling the indexing conventions of Section 2.6 and the definition (2.15) of the $\mathcal{G}$-bi-brane curvature ω — we readily see that it combines with the other line integral as

$$
\sum_{t\in\triangle(\Sigma)} \sum_{e\subset t} \int_e X^*(\delta X \lrcorner\, B_{i_e}) + \sum_{e\in\triangle(E_\Gamma)} \int_e X^*(\delta X \lrcorner\, \mathsf{d}P_{i_e})
$$

$$= \sum_{e\in\triangle(E_\Gamma)} \int_e \big[X_{|1}^*(\delta X_{|1} \lrcorner\, B_{\phi_1(i_e)}) - X_{|2}^*(\delta X_{|2} \lrcorner\, B_{\phi_2(i_e)}) + X^*(\delta X \lrcorner\, \mathsf{d}P_{i_e})\big]$$

$$= \sum_{e\in\triangle(E_\Gamma)} \int_e X^*(\delta X \lrcorner\, \omega_{i_e}) \equiv \int_{E_\Gamma} X^*(\delta X \lrcorner\, \omega), \tag{A.21}$$

where we have used that

$$\delta X_{|\alpha}|_{E_\Gamma} = \iota_{\alpha*}\delta X|_{E_\Gamma}, \tag{A.22}$$

which holds by (L2).

Next, we turn to the vertex contributions. The one coming from the internal vertices, $v \in \Sigma - V_\Gamma$, is easily checked to vanish,

$$\begin{aligned}
&\sum_{e\in\triangle(E_\Gamma)} \sum_{v\in e-V_\Gamma} \varepsilon_{ev}\, X^*\big(\delta X \lrcorner \big(P_{i_e} + \mathsf{i}\,\mathsf{d}\log K_{i_e i_v}\big)\big)(v) \\
&\quad - \sum_{\substack{t\in\triangle(\Sigma)\\ e\subset t}} \sum_{v\in e-V_\Gamma} \varepsilon_{tev}\, X^*\big(\delta X \lrcorner\, A_{i_e i_v}\big)(v) \\
&\qquad = \sum_{e\in\triangle(E_\Gamma)} \sum_{v\in e-V_\Gamma} \varepsilon_{ev}\, \big[X_{|2}^*\big(\delta X_{|2} \lrcorner\, A_{\phi_2(i_e)\phi_2(i_v)}\big) \\
&\qquad\quad - X_{|1}^*\big(\delta X_{|1} \lrcorner\, A_{\phi_1(i_e)\phi_1(i_v)}\big) \\
&\qquad\quad + X^*\big(\delta X \lrcorner \big(P_{i_e} - P_{i_v} + \mathsf{i}\,\mathsf{d}\log K_{i_e i_v}\big)\big)\big](v) = 0,
\end{aligned} \tag{A.23}$$

by virtue of (2.15) and (A.22). The one coming from the vertices of the defect network, on the other hand, does not vanish identically. At a given vertex $v \in V_\Gamma$ of, say, valence n_v, it splits into a sum of terms sourced by the defect lines converging at the vertex, completed with the vertex insertion of the 2-morphism data. We shall first focus on the defect-line terms, further separating the case of $\varepsilon_{n_v}^{k,k+1} = +1$ from that of $\varepsilon_{n_v}^{k,k+1} = -1$. In the former case, we obtain

$$\begin{aligned}
&X_{k+1}^*\big(\delta X_{k+1} \lrcorner\, A_{\phi_2(i_e)\psi_{n_v}^{k+1}(i_v)}\big)(v) - X_k^*\big(\delta X_k \lrcorner\, A_{\phi_1(i_e)\psi_{n_v}^k(i_v)}\big)(v) \\
&\quad + X_{k,k+1}^*\big(\delta X_{k,k+1} \lrcorner \big(P_{i_e} + \mathsf{i}\,\mathsf{d}\,\log K_{i_e\psi_{n_v}^{k,k+1}(i_v)}\big)\big)(v) \\
&\qquad = X_{k+1}^*\big(\delta X_{k+1} \lrcorner\, A_{\phi_2(i_e)\psi_{n_v}^{k+1}(i_v)}\big)(v) - X_k^*\big(\delta X_k \lrcorner\, A_{\phi_1(i_e)\psi_{n_v}^k(i_v)}\big)(v) \\
&\qquad\quad + X_{k,k+1}^*\big(\delta X_{k,k+1} \lrcorner \big(\iota_1^* A_{\phi_1(i_e)\phi_1\circ\psi_{n_v}^{k,k+1}(i_v)} \\
&\qquad\quad - \iota_2^* A_{\phi_2(i_e)\phi_2\circ\psi_{n_v}^{k,k+1}(i_v)} + P_{\psi_{n_v}^{k,k+1}(i_v)}\big)\big)(v) \\
&\qquad = X_{k,k+1}^*\big(\delta X_{k,k+1} \lrcorner\, P_{\psi_{n_v}^{k,k+1}(i_v)}\big)(v),
\end{aligned} \tag{A.24}$$

where we have used (2.15) and a counterpart of the consistency condition (A.22) for the vertex

$$\begin{aligned} \delta X_k|_{V_\Gamma} &= \iota_{1*}^{\varepsilon_{n_v}^{k,k+1}} \delta X_{k,k+1}|_{V_\Gamma}, \\ \iota_{1*}^{\varepsilon_{n_v}^{k,k+1}} \delta X_{k,k+1}|_{V_\Gamma} &= \iota_{2*}^{\varepsilon_{n_v}^{k-1,k}} \delta X_{k-1,k}|_{V_\Gamma}. \end{aligned} \tag{A.25}$$

Similarly in the second case, the defect-line terms reduce as

$$\begin{aligned} & X_{k+1}^*\big(\delta X_{k+1} \lrcorner\, A_{\phi_1(i_e)\psi_{n_v}^{k+1}(i_v)}\big)(v) - X_k^*\big(\delta X_k \lrcorner\, A_{\phi_2(i_e)\psi_{n_v}^{k}(i_v)}\big)(v) \\ &\quad - X_{k,k+1}^*\big(\delta X_{k,k+1} \lrcorner\, \big(P_{i_e} + \mathsf{i}\,\mathsf{d}\,\log K_{i_e\psi_{n_v}^{k,k+1}(i_v)}\big)\big)(v) \\ &\quad = X_{k+1}^*\big(\delta X_{k+1} \lrcorner\, A_{\phi_1(i_e)\psi_{n_v}^{k+1}(i_v)}\big)(v) - X_k^*\big(\delta X_k \lrcorner\, A_{\phi_2(i_e)\psi_{n_v}^{k}(i_v)}\big)(v) \\ &\qquad - X_{k,k+1}^*\big(\delta X_{k,k+1} \lrcorner\, \big(\iota_1^* A_{\phi_1(i_e)\phi_1\circ\psi_{n_v}^{k,k+1}(i_v)} \\ &\qquad - \iota_2^* A_{\phi_2(i_e)\phi_2\circ\psi_{n_v}^{k,k+1}(i_v)} + P_{\psi_{n_v}^{k,k+1}(i_v)}\big)\big)(v) \\ &\quad = -X_{k,k+1}^*\big(\delta X_{k,k+1} \lrcorner\, P_{\psi_{n_v}^{k,k+1}(i_v)}\big)(v). \end{aligned} \tag{A.26}$$

Combining the two with the defect insertion and using the remaining consistency condition

$$\delta X_{k,k+1}|_{V_\Gamma} = \pi_{n_v *}^{k,k+1} \delta X \tag{A.27}$$

for the vertex variations of the various maps involved yields

$$\begin{aligned} & \sum_{k=1}^{n_v} \varepsilon_{n_v}^{k,k+1}\, X_{k,k+1}^*\big(\delta X_{k,k+1} \lrcorner\, P_{\psi_{n_v}^{k,k+1}(i_v)}\big)(v) \\ &\quad - \mathsf{i}\, X^*\big(\delta X \lrcorner\, \mathsf{d}\log f_{i_v}\big)(v) = X^*(\delta X \lrcorner\, \theta_{n_v})(v), \end{aligned} \tag{A.28}$$

cf. (2.75).

Thus, at the end of the day, we find the neat result

$$\begin{aligned} \delta_X S[X;\gamma] = -2 \int_{\Sigma-\Gamma} & \Big[G_X\big(\delta X, \mathrm{vol}_{\Sigma,\gamma}\, \Delta_{(2)} X \\ & + \Gamma_{\text{L-C}}(\mathsf{d}X \overset{\wedge}{,} \star_\gamma\, \mathsf{d}X)\big) - \frac{\mathsf{i}}{2}\, X^*(\delta X \lrcorner\, H)\Big] \\ & - 2 \int_{E_\Gamma} \mathrm{vol}_{E_\Gamma} \Big[G_{X_{|1}}\big(\iota_{1*}\delta X, X_{|1*}\widehat{n}_1\big) \\ & + G_{X_{|2}}\big(\iota_{2*}\delta X, X_{|2*}\widehat{n}_2\big) - \frac{\mathsf{i}}{2}\, \omega(\delta X, X_*\widehat{t})\Big] \\ & + \mathsf{i} \sum_{v\in V_\Gamma} X^*(\delta X \lrcorner\, \theta_{n_v})(v), \end{aligned} \tag{A.29}$$

from which we read off the (dynamical) field equations

$$\begin{aligned}\Delta_{(2)} X^\lambda + \Big[\{ {}^{\lambda}_{\mu\nu} \}(X) \left(\gamma^{-1}\right)^{ab} \\ - \frac{3\mathsf{i}}{2\sqrt{\det\gamma}} \left(G^{-1}\right)^{\lambda\rho}(X)\, H_{\rho\mu\nu}(X)\, \varepsilon^{ab} \Big]\, \partial_a X^\mu\, \partial_b X^\nu = 0, \qquad \text{(A.30)}\end{aligned}$$

written in terms of the components of the curvature 3-form $H = H_{\lambda\mu\nu}\, \mathsf{d}X^\lambda \wedge \mathsf{d}X^\mu \wedge \mathsf{d}X^\nu$, which we take to be antisymmetric in their indices. The resulting defect gluing conditions are

$$\begin{aligned} &G_{X_{|1}}\left(\iota_{1*}\delta X, X_{|1*}\widehat{n}_1\right) + G_{X_{|2}}\left(\iota_{2*}\delta X, X_{|2*}\widehat{n}_2\right) \\ &\quad - \frac{\mathsf{i}}{2}\,\omega(\delta X, X_*\widehat{t}) = 0 \quad \text{at } E_\Gamma, \qquad \text{(A.31)} \\ &X^*(\delta X \lrcorner\, \theta_n) = 0 \quad \text{at } V_\Gamma. \end{aligned}$$

The latter of the two defect conditions forces us to set

$$\theta_n = 0, \quad n \in \mathbb{Z}_{>0} \qquad \text{(A.32)}$$

in the entire region of the $(\mathcal{G}, \mathcal{B})$-inter-bi-brane world-volume accessible to the string, and so it effectively eliminates θ_n from further analysis. This leaves us with only the first of the defect gluing conditions as a non-trivial constraint of the sigma-model dynamics.

A.3 A homotopy move of the vertex

Our aim is to derive the variation of the sigma-model action functional under a homotopy move of the defect network within the world-sheet depicted in figure A1. To this end, we demand that the initial network-field configuration (Γ, X) (for the drawing on the left-hand side) admit — in a sense to be specified below — an extension which determines the final network-field configuration $(\widetilde{\Gamma}, \widetilde{X})$ (for the drawing on the right-hand side) and thereby defines the homotopy move of the three-valent vertex of the embedded defect network along the edge e_3.

We begin our description of the extension by choosing, for the sake of simplicity, a sufficiently fine triangulation of the world-sheet, so that the various embedding maps $X_k, X_{k,k+1}$ associated with the vertex v_1 (as discussed in Section 2.6) are well defined in the entire region of the world-sheet shown in the left-hand side of figure A1. Furthermore, we mark the defect edges e_4, e_1 and e_3 converging at v_1 in the initial defect network Γ as $e_{1,2}, e_{2,3}$ and

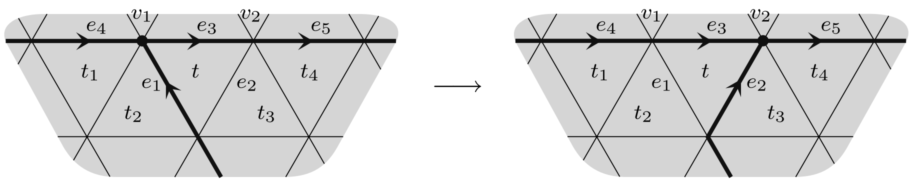


Figure A1: A homotopy move of a three-valent vertex of the defect network.

$e_{3,1}$, respectively. Similarly, the defect edges e_3, e_2 and e_5 converging at v_2 in the final defect network $\widetilde{\Gamma}$ are marked as $e_{1,2}, e_{2,3}$ and $e_{3,1}$, respectively. We may now define the extension of the map X for Γ to be a pair of maps

$$\widehat{X}_{e_1} : t \to Q, \quad \widehat{X}_{v_1} \; : \; e_3 \to T_3 \tag{A.33}$$

such that the following compatibility conditions are satisfied

$$\begin{aligned} &\widehat{X}_{e_1}|_{e_1} = X_{2,3}|_{e_1}, \quad \iota_2 \circ \widehat{X}_{e_1} = X_3|_t, \\ &\widehat{X}_{v_1}|_{v_1} = X|_{v_1}, \quad \pi_3^{3,1} \circ \widehat{X}_{v_1} = X_{3,1}|_{e_3}, \quad \pi_3^{2,3} \circ \widehat{X}_{v_1} = \widehat{X}_{e_1}|_{e_3}, \end{aligned} \tag{A.34}$$

alongside the gluing condition

$$\begin{aligned} &G_{\iota_1 \circ \widehat{X}_{e_1}(p)}\big(\iota_{1*}v, \big(\iota_1 \circ \widehat{X}_{e_1}\big)_* \widehat{u}_2\big) - G_{\iota_2 \circ \widehat{X}_{e_1}(p)}\big(\iota_{2*}v, \big(\iota_2 \circ \widehat{X}_{e_1}\big)_* \widehat{u}_2\big) \\ &\quad - \frac{\mathsf{i}}{2}\, \omega_{\widehat{X}_{e_1}(p)}\big(v, \widehat{X}_{e_1 *}\widehat{u}_1\big) = 0, \end{aligned} \tag{A.35}$$

to be satisfied at every $p \in t$ for all $v \in T_{\widehat{X}_{e_1}(p)}Q$ and for any right-handed orthonormal basis $\big(\widehat{u}_1, \widehat{u}_2\big)$ of $T_p\Sigma$, and the gluing condition

$$\begin{aligned} &G_{X_1(q)}\big(\iota_{1*}v, X_{1*}\widehat{n}_1\big) - G_{\iota_1 \circ \widehat{X}_{e_1}(q)}\big(\iota_{2*}v, \big(\iota_1 \circ \widehat{X}_{e_1}\big)_* \widehat{n}_2\big) \\ &\quad - \frac{\mathsf{i}}{2}\, \omega_{\pi_3^{1,2} \circ \widehat{X}_{v_1}(q)}\big(v, \big(\pi_3^{1,2} \circ \widehat{X}_{v_1}\big)_* \widehat{t}\big) = 0, \end{aligned} \tag{A.36}$$

to be satisfied at every $q \in e_3$ for all $v \in T_{\pi_3^{1,2} \circ \widehat{X}_{v_1}(q)}Q$ and for a triple of unit vectors $\widehat{t}, \widehat{n}_1, \widehat{n}_2 \in T_q\Sigma$ such that $\widehat{t}$ is tangent to e_3 and points from v_1 to v_2, and $\widehat{n}_1$ (resp. $\widehat{n}_2$) is normal to e_3 and points to the outside (resp. inside) of t. The upper line in (A.34) in conjunction with the gluing condition (A.35) identifies $\widehat{X}_{e_1}$ as an extension of X to t across e_1 in the sense of Section 2.9. The bottom line in (A.34), on the other hand, is a straightforward generalization of the there defined notion of an extension across a defect line to the setting of figure A1, and (A.36) ensures that the defect gluing condition for the defect edge marked as $e_{1,2}$ holds to the left of the three-valent defect vertex all along the way as the latter gets shifted from v_1 to v_2.

Keeping track of all the Čech indices involved quickly becomes rather cumbersome, and so we make certain simplifying assumptions which render our demonstration more tractable without any loss of generality of the final result. Thus, we presuppose that $\iota_1 \circ \widehat{X}_{e_1}$ embeds t in the same open set $\mathcal{O}^M_{i_1}$ as the one into which the map X sends the adjacent triangles t_1 and t_2. Analogously, we assume that all three triangles t, t_3 and t_4 are embedded in the same set $\mathcal{O}^M_{i_2}$ by the original map X. The map $\pi_3^{1,2} \circ \widehat{X}_{v_1}$ is taken to embed e_3 in a single set $\mathcal{O}^Q_{i_3}$, just as $X_{3,1}$ is taken to embed $e_3 \cup e_5$ in a single set $\mathcal{O}^Q_{i_4}$. Finally, the map $\widehat{X}_{e_1}$ sends the entire triangle t into $\mathcal{O}^Q_{i_5}$, which is also where X sends e_1, and the map $\widehat{X}_{v_1}$ takes the entire edge e_3 into $\mathcal{O}^{T_3}_{i_6}$. We have the obvious compatibility conditions for the index maps

$$\begin{gathered} i_1 = \phi_1(i_5), \quad i_2 = \phi_2(i_5), \\ i_3 = \psi_3^{1,2}(i_6), \quad i_4 = \psi_3^{3,1}(i_6), \quad i_5 = \psi_3^{2,3}(i_6). \end{gathered} \tag{A.37}$$

We may use $\widehat{X}_{e_1}$ and $\widehat{X}_{v_1}$ to construct a new network-field configuration $(\widetilde{\Gamma}, \widetilde{X})$ for the drawing on the right-hand side of figure A1 starting from the original one (Γ, X). This is achieved by setting

$$\begin{aligned} &\widetilde{X}|_{\Sigma - t} = X|_{\Sigma - t}, \quad \widetilde{X}|_{t-(e_2\cup e_3)} = \iota_1 \circ \widehat{X}_{e_1}|_{t-(e_2\cup e_3)}, \\ &\widetilde{X}|_{e_2 - v_2} = \widehat{X}_{e_1}|_{e_2 - v_2}, \quad \widetilde{X}|_{e_3 - v_2} = \pi_3^{1,2} \circ \widehat{X}_{v_1}|_{e_3 - v_2}, \\ &\widetilde{X}|_{v_2} = \widehat{X}_{v_1}|_{v_2}. \end{aligned} \tag{A.38}$$

We are now ready to compare the value of the holonomy for $(\widetilde{\Gamma}, \widetilde{X})$ with that attained on (Γ, X). Upon rewriting (2.53) in the simple setting described and taking into account all the compatibility conditions listed, alongside (2.15) and (2.41), we obtain

$$\begin{aligned} \frac{1}{\mathsf{i}} \log \frac{\mathrm{Hol}\,(\widetilde{\Gamma}, \widetilde{X})}{\mathrm{Hol}\,(\Gamma, X)} = &\int_t \big((\iota_1 \circ \widehat{X}_{e_1})^* B_{i_1} - X_3^* B_{i_2} \big) \\ &+ \int_{e_2} \widehat{X}_{e_1}^* P_{i_5} - \int_{e_1} X_{2,3}^* P_{i_5} \\ &+ \int_{e_3} \big((\pi_3^{1,2} \circ \widehat{X}_{v_1})^* P_{i_3} - X_{3,1}^* P_{i_4} \big) \\ &- \mathsf{i}\, \log f_{i_6}\big(\widehat{X}_{v_1}(v_2)\big) + \mathsf{i}\, \log f_{i_6}\big(X(v_1)\big) \\ \equiv &\int_t \widehat{X}_{e_1}^* \big(\iota_1^* B_{\phi_1(i_5)} - \iota_2^* B_{\phi_2(i_5)} \big) \end{aligned}$$

$$
\begin{aligned}
&+\int_{e_2} \widehat{X}_{e_1}^* P_{i_5} - \int_{e_1} \widehat{X}_{e_1}^* P_{i_5} \\
&+\int_{e_3} \widehat{X}_{v_1}^* \big((\pi_3^{1,2})^* P_{\psi_3^{1,2}(i_6)} - (\pi_3^{3,1})^* P_{\psi_3^{3,1}(i_6)} \big) \\
&- \mathsf{i} \log f_{i_6}\big(\widehat{X}_{v_1}(v_2)\big) + \mathsf{i} \log f_{i_6}\big(\widehat{X}_{v_1}(v_1)\big) \\
= &\int_t \widehat{X}_{e_1}^* \omega + \int_{e_3} \widehat{X}_{v_1}^* \big((\pi_3^{1,2})^* P_{\psi_3^{1,2}(i_6)} \\
&+ (\pi_3^{2,3})^* P_{\psi_3^{2,3}(i_6)} - (\pi_3^{3,1})^* P_{\psi_3^{3,1}(i_6)} \big) \\
&- \mathsf{i} \log f_{i_6}\big(\widehat{X}_{v_1}(v_2)\big) + \mathsf{i} \log f_{i_6}\big(\widehat{X}_{v_1}(v_1)\big) \\
= &\int_t \widehat{X}_{e_1}^* \omega. \qquad \text{(A.39)}
\end{aligned}
$$

A straightforward calculation of the difference of the kinetic terms of the sigma-model action functional evaluated on the two network-field configurations $(\widetilde{\Gamma}, \widetilde{X})$ and (Γ, X) completes the derivation, cf. (2.116). Thus, as explained below (2.116), we see, using (A.35), that the action functional remains invariant under the vertex move, $S[(\widetilde{\Gamma}, \widetilde{X}); \gamma_0] = S[(\Gamma, X); \gamma_0]$.

Note, in particular, that upon fixing the trivial defect condition at e_4 (whereby the relevant 2-morphism f_{i_6} reduces to the trivial death 2-isomorphism), we recover a result on the change of the holonomy under a homotopy move of the vertex-free segment of the defect network. It is unaffected by the presence of the defect vertex due to the equality $\theta_n = 0$, imposed on the basis of the analysis of Appendix A.2.

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