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Non-commutative correspondences, duality and D-branes in bivariant K-theory

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Abstract

We describe a categorical framework for the classification of D-branes on non-commutative spaces using techniques from bivariant K-theory of C^*-algebras. We present a new description of bivariant K-theory in terms of non-commutative correspondences which is nicely adapted to the study of T-duality in open string theory. We systematically use the diagram calculus for bivariant K-theory as detailed in our previous paper [12]. We explicitly work out our theory for a number of examples of non-commutative manifolds.

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Introduction

Understanding the mathematical structures underpinning D-branes and Ramond–Ramond fields in superstring theory is important for systematically describing some of their novel properties. One of the best known examples of this is the classification of D-brane charges and RR-fields using techniques of K-theory [23, 27, 37, 39, 42, 54]. The K-theory framework naturally accounts for certain properties of D-branes and RR-fields that would not be realized if these objects were classified by ordinary cohomology or homology alone. For example, it explains the appearance of stable but non-BPS branes carrying torsion charges, and correctly incorporates both the self-duality and quantization conditions on RR-fields. It has also led to a variety of new predictions concerning the spectrum of superstring theory, such as the instability of D-branes wrapping non-contractible cycles in certain instances due to the fact that their cohomology classes do not "lift" to K-theory [21], and the obstruction to simultaneous measurement of electric and magnetic RR fluxes when torsion fluxes are included [24]. Moreover, certain properties of the string theory path integral, such as worldsheet anomalies and certain subtle phase factor contributions from the RR-fields, are most naturally formulated within the context of K-theory [21, 23].

A natural but very complicated problem is to determine the set of possible D-branes in a given closed string background. One of the difficulties is that many of the consistent conformally invariant boundary conditions for open strings have no geometrical description. Furthermore, in the presence of certain non-trivial background supergravity form fields, the worldvolume field theories of D-branes are best described in the language of non-commutative geometry. Previous work ([11, 33], among many other references) also showed that the formulation of T-duality in the presence of NS–NS flux requires the use of non-commutative manifolds. The simplest examples of non-commutative spacetimes result from applying T-duality to spacetimes compactified along directions which have non-trivial support of the NS–NS field.

In a previous paper [12], we took preliminary steps towards extending the K-theory classification of D-branes and RR-fields in the commutative case to spacetimes that are non-commutative manifolds. In particular, we obtained an explicit formula for the charges of D-branes in non-commutative spacetimes. The purpose of the present paper is to elaborate on these constructions somewhat and to flesh out many explicit examples, particularly some which arise naturally in string theory. The presentation here is hoped to be in a form that is more palatable to physicists. We describe the constructions of [12] by exploiting the diagram calculus developed there, which provides both a computationally useful tool as well as a heuristic guide to some of the more technical aspects of the formalism. The detailed proofs of many of our previous results are deferred to [12]. Instead, we focus our attention on presenting many new examples and relating them to standard constructions in the string theory literature.

There are various themes underlying this work, which we explain in detail in the following sections. In many applications it is useful to characterize the dynamics of D-branes in a categorical fashion. In this setting, D-branes are regarded, in a certain sense, as objects in some category. Such a point of view has been fruitful in topological string theory and applications to homological mirror symmetry [2], wherein B-model D-branes are regarded as objects in a derived category of coherent sheaves while A-model D-branes are objects in a Fukawa category. The collection of open string boundary conditions may also be regarded as a category in the context of two-dimensional open/closed topological field theory [38], and thereby used to explain the structure of boundary conformal field theory and its connections to K-theory. In this paper, we regard D-branes as objects in a certain category of separable C^*-algebras. This is the category underlying Kasparov's bivariant K-theory (also known as KK-theory). This characterization is closely related to the open string algebras for different boundary conditions that arise in open string field theory, which for boundary conditions of maximal support are Morita equivalent via open string bimodules.

The bivariant version of K-theory is useful for a variety of reasons. Firstly, it incorporates both the K-theory and K-homology descriptions of D-branes in a unified setting. But bivariant K-theory is richer than both K-theory and K-homology considered individually, as it carries an intersection product. Secondly, this product structure provides the correct mathematical framework in which to formulate notions of duality between generic separable C^*-algebras, such as Poincaré duality (PD) (see for example [40]). This can be used to explain the equivalence of the K-theory and K-homology descriptions of D-brane charges. It was also exploited in [12] to provide a categorical description of open string T-duality using an involutive type of KK-equivalence, which includes the more familiar notion of Morita duality [51] as a special case in some simpler situations. This characterization has the virtue of extending the usual geometric notions behind T-duality to examples involving "non-geometric" backgrounds. At least in some examples, Grange and Schäfer-Nameki [25] have proposed to view such noncommutative spacetimes as a globally defined, open string version of Hull's T-fold proposal [28]. Thirdly, bivariant theories are the appropriate venue for defining topological invariants of non-commutative spaces, an example being the Todd class defined in [12]. In the following, we will calculate the Todd class for some explicit examples of non-commutative manifolds. Finally, it is the setting that is used to define a generic notion of K-orientation, and hence to generalize the Freed–Witten anomaly cancellation condition [23] which selects the consistent sets of D-branes from our category. These formulations all culminate in a non-commutative version of the D-brane charge vector. In this paper we will work out explicit examples (both commutative and non-commutative) of all of these quantities.

A central result in this paper is a new description of bivariant K-theory in terms of equivalence classes of *non-commutative correspondences*. This description is nicely adapted to the study of dualities in open string theory such as T-duality, mirror symmetry and smooth analogues of the Fourier–Mukai transform. The commutative version of the correspondence picture nicely unifies the geometric construction of states of D-branes in terms of topological K-cycles and in terms of topological K-theory classes in spacetime. In the generic non-commutative setting, it therefore nicely captures the point of view of D-branes as objects in the KK-theory category with the morphisms between objects, which are the elements of the bivariant K-theory groups, providing generalizations of T-duality transformations. T-dual D-branes are themselves realized in terms of invertible elements of KK-theory which define order 2 T-duality actions up to Morita equivalence. As we discuss, via the universal coefficient theorem, bivariant K-theory provides a refinement of the K-theoretic notion of T-duality.

Let us now summarize the contents and structure of this paper. In Section 1 we review the definition of bivariant K-theory and bivariant cyclic theory for separable C^*-algebras. We also review the diagram calculus developed in an earlier paper [12], which is useful for manipulating the products in these groups. Section 2 is concerned with understanding Poincaré duality in K-theory and cyclic theory using the diagram calculus. In particular, we provide the construction of separable Poincaré duality algebras, yielding a large class of non-commutative examples. Section 3 reviews how the diagram calculus is used to formulate the Grothendieck–Riemann–Roch theorem that was proved in [12] for a class of separable C^*-algebras. Using this theorem, in Section 4 we analyze D-branes and their charges in non-commutative spaces, explaining how to generalize the commutative case in the context of the bivariant theories. Several non-commutative examples are included, where the D-brane charge formula is explicitly calculated. In Section 5, we give a definition of KK-theory in terms of certain equivalence classes of non-commutative correspondences, and relate it to T-duality in Type II superstring theory for non-commutative spacetime manifolds.

1 KK-theory and the diagram calculus

In this section, we will explain the basic aspects of Kasparov's KK-theory using the novel diagram calculus developed in [12], which greatly simplifies detailed calculations in this theory. These considerations apply in any bivariant theory with similar algebraic properties, such as Puschnigg's local bivariant cyclic cohomology which will be encountered later on in this section.

1.1 Axioms

For any pair of separable C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, Kasparov defines two abelian groups $\mathsf{KK}_0(\mathcal{A},\mathcal{B})$ and $\mathsf{KK}_1(\mathcal{A},\mathcal{B})$. They are constructed with the help of Kasparov's $\mathcal{A}$-$\mathcal{B}$ bimodules which generalize Fredholm modules. When $\mathcal{A} = \mathbb{C}$, the group $\mathsf{KK}_\bullet(\mathbb{C},\mathcal{A}) = \mathsf{K}_\bullet(\mathcal{A})$ is the K-theory of $\mathcal{A}$, while $\mathsf{KK}_\bullet(\mathcal{A},\mathbb{C}) = \mathsf{K}^\bullet(\mathcal{A})$ is the K-homology of $\mathcal{A}$. There is a canonical functor from the category of separable C^*-algebras with $*$-homomorphisms to an additive category $\underline{\mathsf{KK}}$, whose objects are separable C^*-algebras and the morphisms between any two objects $\mathcal{A},\mathcal{B}$ are given by $\mathrm{Mor}_{\underline{\mathsf{KK}}}(\mathcal{A},\mathcal{B}) = \mathsf{KK}(\mathcal{A},\mathcal{B})$ [26]. In particular, any algebra homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ determines an element $[\phi]_{\mathsf{KK}} \in \mathsf{KK}(\mathcal{A},\mathcal{B})$, represented by the bimodule $(\mathcal{B},\phi,0)$. Elements of $\mathsf{KK}(\mathcal{A},\mathcal{B})$ may in this way be regarded as generalized morphisms between the two algebras. The category $\underline{\mathsf{KK}}$ is not an abelian category, but it admits the structure of a triangulated category [36, 20].

The KK-theory groups may be axiomatically characterized by the following three properties [26]:

(1) *Homotopy*: The bifunctor $\mathsf{KK}(-,-)$ is homotopy invariant in both variables;
(2) *Stability*: $\mathsf{KK}(\mathcal{A}\otimes\mathcal{K},\mathcal{B}) \cong \mathsf{KK}(\mathcal{A},\mathcal{B}) \cong \mathsf{KK}(\mathcal{A},\mathcal{B}\otimes\mathcal{K})$, where $\mathcal{K}$ is the algebra of compact operators on a separable Hilbert space, and the isomorphisms are induced by the stabilization maps $\mathcal{A} \to \mathcal{A}\otimes\mathcal{K}$, $\mathcal{B} \to \mathcal{B}\otimes\mathcal{K}$ given by $a \mapsto a\otimes e$, $b \mapsto b\otimes e$ with e a projection of rank 1;
(3) *Split exactness*: If

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{D} \leftrightarrows \mathcal{D}/\mathcal{J} \longrightarrow 0$$

is a split exact sequence of separable C^*-algebras and $*$-homomorphisms, then there are split exact sequences of abelian groups given by

$$0 \longrightarrow \mathsf{KK}(\mathcal{A},\mathcal{J}) \longrightarrow \mathsf{KK}(\mathcal{A},\mathcal{D}) \leftrightarrows \mathsf{KK}(\mathcal{A},\mathcal{D}/\mathcal{J}) \longrightarrow 0\ ,$$

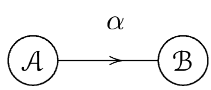
$$0 \longrightarrow \mathsf{KK}(\mathcal{D}/\mathcal{J},\mathcal{B}) \leftrightarrows \mathsf{KK}(\mathcal{D},\mathcal{B}) \longrightarrow \mathsf{KK}(\mathcal{J},\mathcal{B}) \longrightarrow 0.$$

These properties characterize $\underline{\mathsf{KK}}$ as a *universal* category which may be constructed by purely algebraic means. Hence, it is possible to construct the Kasparov groups as the unique formal bifunctor on the category of separable C^*-algebras satisfying the three axioms above. This point of view is extremely useful for obtaining alternative presentations of the KK-theory groups, such as the ones we consider in Section 5.

1.2 Diagrams

The realization above of elements of $\mathsf{KK}(\mathcal{A},\mathcal{B})$ in terms of generalized morphisms motivates the use of diagrams to represent classes in KK-theory which are constructed as follows. The first argument in $\mathsf{KK}(-,-)$ provides the input and the second argument provides the output of a diagram. Each tensor factor in the first argument will produce one input node, and each tensor factor in the second argument will produce one output node. It is sometimes convenient to add arrowheads that point from the inputs to the outputs.

Starting with the simplest situation, an element $\alpha \in \mathsf{KK}(\mathcal{A},\mathcal{B})$ is represented by the diagram

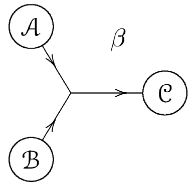


where the left node is regarded as the input node, while the node on the right is the output node. The distinction between the input and output nodes is very important in explicit computations. When $\mathcal{A} = \mathcal{B} = \mathbb{C}$, the corresponding group is $\mathsf{KK}(\mathbb{C},\mathbb{C}) \cong \mathbb{Z}$.

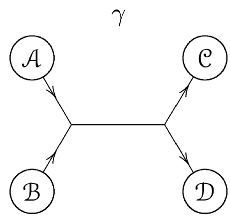
We will also need to treat more complicated situations. For example, an element

$$\beta \in \mathsf{KK}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C})$$

is represented as



while a class $\gamma \in \mathsf{KK}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C} \otimes \mathcal{D})$ is depicted by



The basic rule in treating these more complicated diagrams is that permutation of the input or output terminals may involve at most the switch of a sign, depending on the orientation of the Bott element.

1.3 Composition product

A key feature of Kasparov's KK-theory is the existence of the composition (or intersection) product

$$\otimes_{\mathcal{B}} : \mathsf{KK}_i(\mathcal{A}, \mathcal{B}) \times \mathsf{KK}_j(\mathcal{B}, \mathcal{C}) \longrightarrow \mathsf{KK}_{i+j}(\mathcal{A}, \mathcal{C}),$$

which may be illustrated with the diagram

$$\mathcal{A} \xrightarrow{\ \alpha\ } \mathcal{B} \rightsquigarrow \mathcal{B} \xrightarrow{\ \beta\ } \mathcal{C} \quad = \quad \mathcal{A} \xrightarrow{\ \alpha \otimes_{\mathcal{B}} \beta\ } \mathcal{C}$$

Here $\alpha \in \mathsf{KK}_i(\mathcal{A}, \mathcal{B}), \beta \in \mathsf{KK}_j(\mathcal{B}, \mathcal{C})$ and $\alpha \otimes_{\mathcal{B}} \beta \in \mathsf{KK}_{i+j}(\mathcal{A}, \mathcal{C})$. The wavy line joining the output node of α to the input node of β serves to stress that these two nodes, which have the same label $\mathcal{B}$, are annihilated in the process of taking the composition product. In particular, if $\phi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{C}$ are morphisms of C^*-algebras, then this product is compatible with composition of morphisms, giving

$$[\phi]_{\mathsf{KK}} \otimes_{\mathcal{B}} [\psi]_{\mathsf{KK}} = [\psi \circ \phi]_{\mathsf{KK}} \,.$$

Kasparov's composition product is bilinear and associative, and it defines the composition law in the additive category $\underline{\mathsf{KK}}$ [26]. It also makes $\mathsf{KK}_0(\mathcal{A}, \mathcal{A})$ into a unital ring whose unit $1_{\mathcal{A}}$ is the element of $\mathsf{KK}_0(\mathcal{A}, \mathcal{A})$ determined by the identity morphism $\mathrm{id}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ of the algebra $\mathcal{A}$. In particular, it now makes sense to talk about invertible elements of $\mathsf{KK}_0(\mathcal{A}, \mathcal{A})$. More generally, we say that $\alpha \in \mathsf{KK}_i(\mathcal{A}, \mathcal{B})$ is *invertible* if and only if there exists an element $\beta \in \mathsf{KK}_{-i}(\mathcal{B}, \mathcal{A})$ such that $\alpha \otimes_{\mathcal{B}} \beta = 1_{\mathcal{A}}$ and $\beta \otimes_{\mathcal{A}} \alpha = 1_{\mathcal{B}}$. The element β is then called the *inverse* of α and written $\beta = \alpha^{-1}$.

Elements of KK-theory determine, by means of the composition product, homomorphisms in K-theory and K-homology in the following way. Let $\alpha \in \mathsf{KK}_d(\mathcal{A}, \mathcal{B})$. Then

$$x \otimes_{\mathcal{A}} \alpha \ \in \ \mathsf{KK}_{j+d}(\mathbb{C}, \mathcal{B}) = \mathsf{K}_{j+d}(\mathcal{B})$$

for any $x \in \mathsf{KK}_j(\mathbb{C}, \mathcal{A}) = \mathsf{K}_j(\mathcal{A})$. Thus taking the product with α on the right gives a map $(-) \otimes_{\mathcal{A}} \alpha : \mathsf{KK}_j(\mathbb{C}, \mathcal{A}) \to \mathsf{KK}_{j+d}(\mathbb{C}, \mathcal{B})$, which in diagrammatic form is

$$\mathbb{C} \xrightarrow{x} \mathcal{A} \sim\!\sim\!\sim \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \quad = \quad \mathbb{C} \xrightarrow{x \otimes_{\mathcal{A}} \alpha} \mathcal{B}$$

Similarly, $\alpha \otimes_{\mathcal{B}} y \in \mathsf{KK}_{j+d}(\mathcal{A}, \mathbb{C}) = \mathsf{K}^{j+d}(\mathcal{A})$ for all $y \in \mathsf{KK}_j(\mathcal{B}, \mathbb{C}) = \mathsf{K}^j(\mathcal{B})$, and in this way multiplying by the element α on the left gives a map $\alpha \otimes_{\mathcal{B}} (-) : \mathsf{KK}_j(\mathcal{B}, \mathbb{C}) \to \mathsf{KK}_{j+d}(\mathcal{A}, \mathbb{C})$, which in diagrams is

$$\mathcal{A} \xrightarrow{\alpha} \mathcal{B} \sim\!\sim\!\sim \mathcal{B} \xrightarrow{y} \mathbb{C} \quad = \quad \mathcal{A} \xrightarrow{\alpha \otimes_{\mathcal{B}} y} \mathbb{C}$$

An invertible element $\alpha \in \mathsf{KK}(\mathcal{A}, \mathcal{B})$ determines an isomorphism between the K-theory of $\mathcal{A}$ and the K-theory of $\mathcal{B}$, as well as an isomorphism between the K-homology of $\mathcal{A}$ and the K-homology of $\mathcal{B}$. In this case we say that $\mathcal{A}$ and $\mathcal{B}$ are (*strongly*) KK-*equivalent*.

1.4 Exterior product

A very important computational tool in KK-theory is the bilinear exterior product

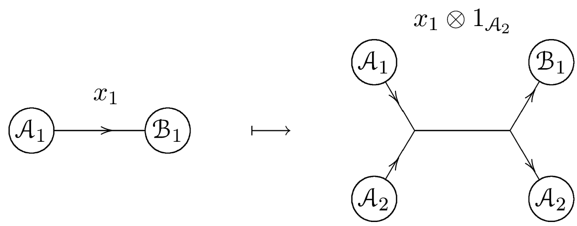
$$\otimes : \mathsf{KK}_i(\mathcal{A}_1, \mathcal{B}_1) \times \mathsf{KK}_j(\mathcal{A}_2, \mathcal{B}_2) \longrightarrow \mathsf{KK}_{i+j}(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{B}_1 \otimes \mathcal{B}_2),$$

which is defined in terms of the composition product by

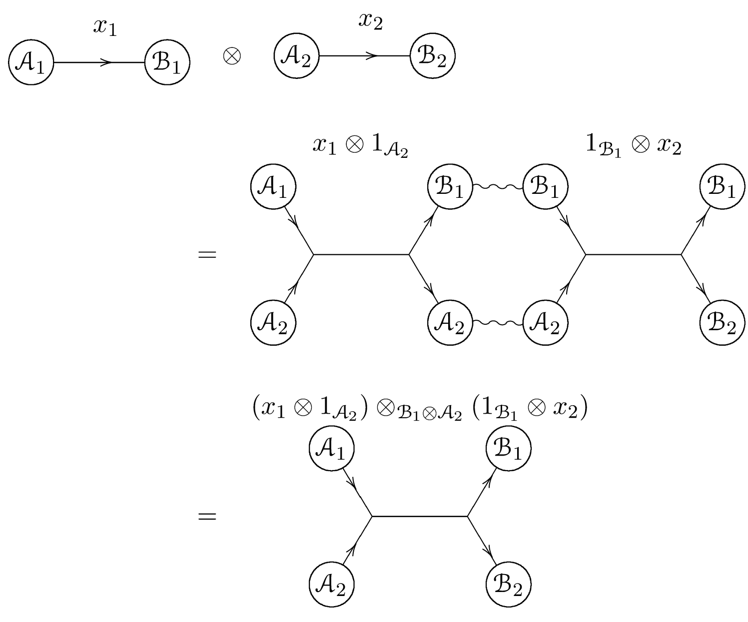
$$x_1 \otimes x_2 = (x_1 \otimes 1_{\mathcal{A}_2}) \otimes_{\mathcal{B}_1 \otimes \mathcal{A}_2} (1_{\mathcal{B}_1} \otimes x_2).$$

The exterior product with the class of the identity morphism, called *dilation*, is defined as follows. If $x \in \mathsf{KK}_j(\mathcal{A}, \mathcal{B})$ is represented by a Kasparov $\mathcal{A}$-$\mathcal{B}$ bimodule $(\mathcal{E}, F)$, then $x \otimes 1_{\mathcal{C}}$ is the element of $\mathsf{KK}_j(\mathcal{A} \otimes \mathcal{C}, \mathcal{B} \otimes \mathcal{C})$

represented by the $(\mathcal{A} \otimes \mathcal{C})$-$(\mathcal{B} \otimes \mathcal{C})$ bimodule $(\mathcal{E} \otimes \mathcal{C}, F \otimes \mathrm{id}_{\mathcal{C}})$. In the simple case where $x_1 = [\phi]_{\mathsf{KK}}$ is the class of an algebra morphism $\phi : \mathcal{A}_1 \to \mathcal{B}_1$, then $x_1 \otimes 1_{\mathcal{A}_2} = [\phi \otimes \mathrm{id}_{\mathcal{A}_2}]_{\mathsf{KK}}$ is the class of the morphism $\phi \otimes \mathrm{id}_{\mathcal{A}_2} : \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{B}_1 \otimes \mathcal{A}_2$. Diagrammatically, dilation is illustrated as



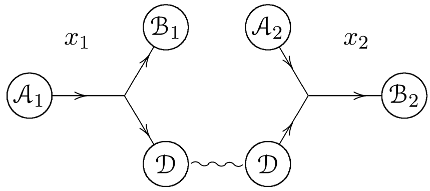
When taking the exterior product of elements $x_1 \in \mathsf{KK}(\mathcal{A}_1, \mathcal{B}_1)$ and $x_2 \in \mathsf{KK}(\mathcal{A}_2, \mathcal{B}_2)$ we must use dilation to establish a common output–input node to which we can apply the Kasparov composition product. This is described in the diagram



(1.1)

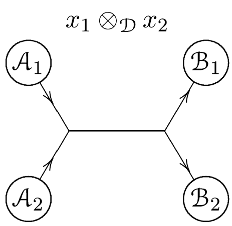
The diagram calculus allows one to keep track of the exterior product in more complicated situations. For example, if $\mathcal{A}_i, \mathcal{B}_i, i = 1, 2$, and $\mathcal{D}$ are all separable C^*-algebras, then the exterior product $x_1 \otimes_{\mathcal{D}} x_2$ of two elements

$x_1 \in \mathsf{KK}(\mathcal{A}_1, \mathcal{B}_1 \otimes \mathcal{D})$ and $x_2 \in \mathsf{KK}(\mathcal{D} \otimes \mathcal{A}_2, \mathcal{B}_2)$ can be illustrated as



(1.2)

While this product is defined by means of the dilation method as before, the diagrammatic calculus simplifies the process. To obtain the diagram describing the result, we first remove the common nodes joined by the wavy line, and then assemble input and output nodes together. In the diagram (1.2), the input nodes are labelled $\mathcal{A}_1$ and $\mathcal{A}_2$ while the output nodes are $\mathcal{B}_1$ and $\mathcal{B}_2$. Thus the result looks like this:



$\in \ \mathsf{KK}(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{B}_1 \otimes \mathcal{B}_2)$.

When $\mathcal{D} = \mathbb{C}$ this calculation is the same as that in equation (1.1).

1.5 Associativity

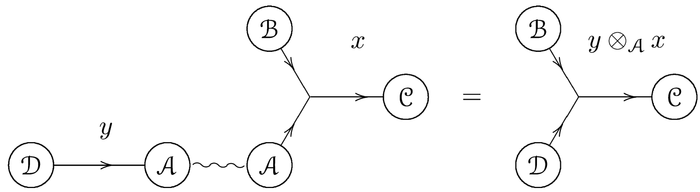
The exterior product is associative, and this highly non-trivial property may be stated as follows. We assume that all algebras involved below are separable.

Theorem 1.1. *Let* $x \in \mathsf{KK}(\mathcal{A}_1, \mathcal{B}_1 \otimes \mathcal{D}_1)$*,* $y \in \mathsf{KK}(\mathcal{D}_1 \otimes \mathcal{A}_2, \mathcal{B}_2 \otimes \mathcal{D}_2)$ *and* $z \in \mathsf{KK}(\mathcal{D}_2 \otimes \mathcal{A}_3, \mathcal{B}_3)$*. Then*

$$(x \otimes_{\mathcal{D}_1} y) \otimes_{\mathcal{D}_2} z = x \otimes_{\mathcal{D}_1} (y \otimes_{\mathcal{D}_2} z) \ .$$

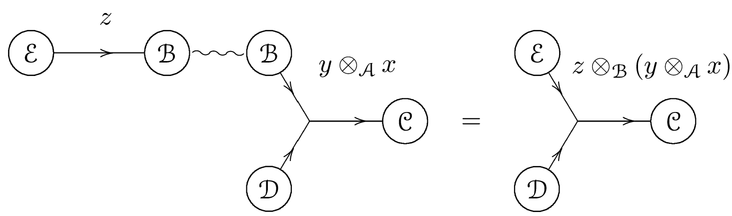
The nice aspect of the diagrammatic formalism is that one can show that all associativity formulae in Kasparov theory correspond to the fact that one can perform the concatenations that form part of computing the products in any order, except perhaps for signs (which disappear in KK_0). To illustrate this point, consider the following example. Let us take three elements $z \in \mathsf{KK}(\mathcal{E}, \mathcal{B})$, $y \in \mathsf{KK}(\mathcal{D}, \mathcal{A})$ and $x \in \mathsf{KK}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C})$. We would like to compute the product $z \otimes_{\mathcal{B}} (y \otimes_{\mathcal{A}} x)$. First we compute $y \otimes_{\mathcal{A}} x$ to get

$y \otimes_{\mathcal{A}} x =$

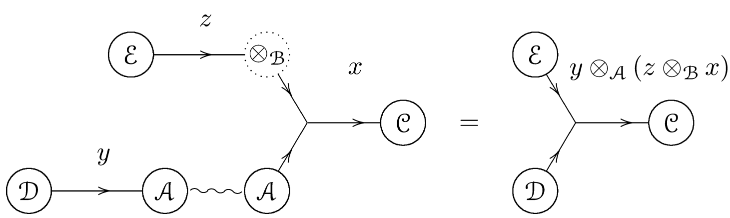


so that

$z \otimes_{\mathcal{B}} (y \otimes_{\mathcal{A}} x) =$



However, since the concatenations involved in this calculation can be performed in any order, the same computation may be carried out as



where the dashed circle indicates a node that is removed in the process of computing the Kasparov product with respect to the algebra marked in the circle. The two results are the same, which leads to the formula

$$z \otimes_{\mathcal{B}} (y \otimes_{\mathcal{A}} x) = \pm\, y \otimes_{\mathcal{A}} (z \otimes_{\mathcal{B}} x).$$

A rigorous proof of this formula can be found in [12, Appendix B].

1.6 Local bivariant cyclic cohomology

To make the Chern character possible, one needs a target group for a map defined on Kasparov's bivariant KK-theory, preferably one that is defined on a similar class of algebras and which transports the key algebraic properties of the theory. Several theories of such kind have been proposed, which work for large classes of topological and bornological algebras. For our purposes, the most convenient theory is Puschnigg's local bivariant cyclic cohomology [45], which we shall denote HL. This theory, which can be defined for separable C^*-algebras, has the same formal properties as Kasparov's KK-theory. In particular, if all algebras below are separable C^*-algebras then

(1) There exists a bilinear, associative composition product

$$\otimes_{\mathcal{B}} : \mathsf{HL}_i(\mathcal{A}, \mathcal{B}) \times \mathsf{HL}_j(\mathcal{B}, \mathcal{C}) \longrightarrow \mathsf{HL}_{i+j}(\mathcal{A}, \mathcal{C});$$

(2) The bifunctor $\mathsf{HL}(-,-)$ is homotopy invariant, split exact and satisfies excision in each variable;
(3) There exists an associative, bilinear exterior product

$$\otimes : \mathsf{HL}_i(\mathcal{A}_1, \mathcal{B}_1) \times \mathsf{HL}_j(\mathcal{A}_2, \mathcal{B}_2) \longrightarrow \mathsf{HL}_{i+j}(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathcal{B}_1 \otimes \mathcal{B}_2).$$

Thus essentially everything in our description of KK-theory above applies also to the bivariant cyclic homology.

This theory is also defined for more general topological algebras, such as smooth dense subalgebras of C^*-algebras.

Theorem 1.2 ([19, §23.4]). *Let $\mathcal{A}$ be a Banach algebra with the metric approximation property. Let $\mathcal{A}^\infty$ be a smooth subalgebra of $\mathcal{A}$. Then the inclusion map $\mathcal{A}^\infty \hookrightarrow \mathcal{A}$ induces an invertible element of $\mathsf{HL}_0(\mathcal{A}^\infty, \mathcal{A})$, hence $\mathcal{A}^\infty$ and $\mathcal{A}$ are HL-equivalent.*

All nuclear C^*-algebras have the metric approximation property [15]. The example of the Fréchet algebra $C^\infty(X)$ of smooth functions on a compact manifold X is of key importance. This algebra is a smooth subalgebra of the C^*-algebra of continuous functions $C(X)$. By Theorem 1.2, the inclusion $C^\infty(X) \hookrightarrow C(X)$ is an invertible element in $\mathsf{HL}(C^\infty(X), C(X))$. In particular, both the local cyclic homology and cohomology of these two algebras are isomorphic. Puschnigg proves that $\mathsf{HL}_\bullet(C^\infty(X)) \cong \mathsf{HP}_\bullet(C^\infty(X))$, so that the local cyclic homology coincides with the standard periodic cyclic homology. Combined with a fundamental result of Connes, this establishes an isomorphism between Puschnigg's local cyclic homology of $C(X)$ and the periodic de Rham cohomology of X, $\mathsf{HL}_\bullet(C(X)) \cong \mathsf{H}^\bullet(\Omega^\#(X), \mathrm{d})$ (with the standard grading on differential forms by form degree).

The local bivariant cyclic homology admits a Chern character with the required properties.

Theorem 1.3 ([19, §23.5]). *Let $\mathcal{A}$ and $\mathcal{B}$ be separable C^*-algebras. Then there exists a natural bivariant Chern character*

$$\mathsf{ch} : \mathsf{KK}_\bullet(\mathcal{A}, \mathcal{B}) \longrightarrow \mathsf{HL}_\bullet(\mathcal{A}, \mathcal{B})$$

such that

(1) ch *is multiplicative, i.e., if* $\alpha \in \mathsf{KK}_i(\mathcal{A}, \mathcal{B})$ *and* $\beta \in \mathsf{KK}_j(\mathcal{B}, \mathcal{C})$ *then*

$$\mathsf{ch}(\alpha \otimes_{\mathcal{B}} \beta) = \mathsf{ch}(\alpha) \otimes_{\mathcal{B}} \mathsf{ch}(\beta);$$

(2) ch *is compatible with the exterior product;*
(3) $\mathsf{ch}([\phi]_{\mathsf{KK}}) = [\phi]_{\mathsf{HL}}$ *for any algebra homomorphism* $\phi : \mathcal{A} \to \mathcal{B}$.

The last property implies that the Chern character sends invertible elements of KK-theory to invertible elements of bivariant local cyclic cohomology. In particular, KK-equivalence of algebras implies HL-equivalence, but not conversely. Moreover, if $\mathcal{A}$ and $\mathcal{B}$ are in the class $\underline{\mathfrak{N}}$ of C^*-algebras for which the universal coefficient theorem holds in KK-theory, i.e., those which are KK-equivalent to commutative C^*-algebras, then

$$\mathsf{HL}_\bullet(\mathcal{A}, \mathcal{B}) \cong \mathrm{Hom}_{\mathbb{C}}\big(\mathsf{K}_\bullet(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{C}, \mathsf{K}_\bullet(\mathcal{B}) \otimes_{\mathbb{Z}} \mathbb{C}\big). \tag{1.3}$$

If the K-theory $\mathsf{K}_\bullet(\mathcal{A})$ is finitely generated, this is also equal to $\mathsf{KK}_\bullet(\mathcal{A}, \mathcal{B}) \otimes_{\mathbb{Z}} \mathbb{C}$.

2 Non-commutative Poincaré duality

Following Connes [17], in this section we will introduce the notion of Poincaré duality suitable to generic separable C^*-algebras.

2.1 Fundamental classes

Given an algebra $\mathcal{A}$, let $\mathcal{A}^{\mathrm{o}}$ denote the opposite algebra of $\mathcal{A}$, i.e., the algebra with the same underlying vector space as $\mathcal{A}$ but with the product reversed. The stable homotopy category of C^*-algebras has an involution which is defined by the mapping

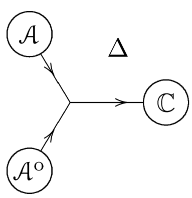
$$\big(\mathcal{A} \xrightarrow{f} \mathcal{B}\big) \longmapsto \big(\mathcal{A}^{\mathrm{o}} \xrightarrow{f^{\mathrm{o}}} \mathcal{B}^{\mathrm{o}}\big),$$

and this involution passes to the category $\underline{\mathsf{KK}}$. Thus given $x \in \mathsf{KK}_d(\mathcal{A}, \mathcal{B})$, there is a corresponding element $x^{\mathrm{o}} \in \mathsf{KK}_d(\mathcal{A}^{\mathrm{o}}, \mathcal{B}^{\mathrm{o}})$ with $(x^{\mathrm{o}})^{\mathrm{o}} = x$.

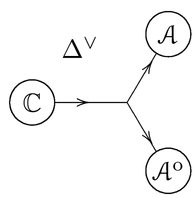
Definition 2.1. We say that a separable C^*-algebra $\mathcal{A}$ is a (*strong*) *PD algebra* if and only if there exists a *fundamental class* for $\mathcal{A}$ in K-homology, i.e., an element $\Delta \in \mathsf{KK}_d(\mathcal{A} \otimes \mathcal{A}^{\mathrm{o}}, \mathbb{C}) = \mathsf{K}^d(\mathcal{A} \otimes \mathcal{A}^{\mathrm{o}})$ with an *inverse* class $\Delta^\vee \in \mathsf{KK}_{-d}(\mathbb{C}, \mathcal{A} \otimes \mathcal{A}^{\mathrm{o}}) = \mathsf{K}_{-d}(\mathcal{A} \otimes \mathcal{A}^{\mathrm{o}})$ such that

$$\begin{aligned} \Delta^\vee \otimes_{\mathcal{A}^{\mathrm{o}}} \Delta &= 1_{\mathcal{A}} \in \mathsf{KK}_0(\mathcal{A}, \mathcal{A}) \\ \Delta^\vee \otimes_{\mathcal{A}} \Delta &= (-1)^d \, 1_{\mathcal{A}^{\mathrm{o}}} \in \mathsf{KK}_0(\mathcal{A}^{\mathrm{o}}, \mathcal{A}^{\mathrm{o}}). \end{aligned} \tag{2.1}$$

The use of the opposite algebra in this definition is to describe $\mathcal{A}$-bimodules as $(\mathcal{A} \otimes \mathcal{A}^{\mathrm{o}})$-modules. The classes Δ and $\Delta^\vee$ can be illustrated with the help of the diagrams

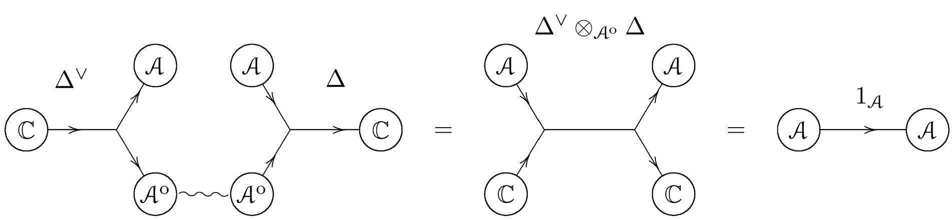


$\in \mathsf{KK}(\mathcal{A} \otimes \mathcal{A}^{\mathrm{o}}, \mathbb{C})$ and

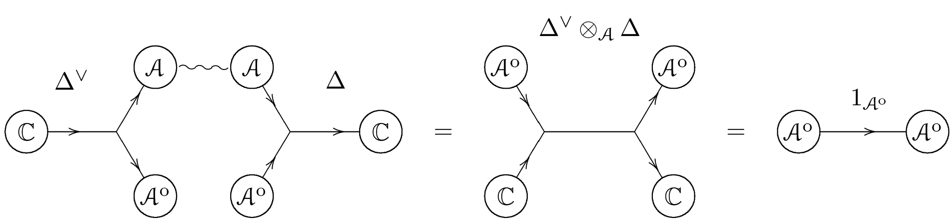


$\in \mathsf{KK}(\mathbb{C}, \mathcal{A} \otimes \mathcal{A}^{\mathrm{o}})$

The Poincaré conditions (2.1) can then be illustrated as follows. The first condition gives rise to the diagram



which represents the identity element $1_{\mathcal{A}} \in \mathsf{KK}_0(\mathcal{A}, \mathcal{A})$. The second condition (in the case d even) is described by the diagram



Definition 2.2. A fundamental class Δ of a PD algebra $\mathcal{A}$ is said to be *symmetric* if $\sigma(\Delta)^{\mathrm{o}} = \Delta$ in $\mathsf{K}^d(\mathcal{A} \otimes \mathcal{A}^{\mathrm{o}})$, where

$$\sigma : \mathcal{A} \otimes \mathcal{A}^{\mathrm{o}} \longrightarrow \mathcal{A}^{\mathrm{o}} \otimes \mathcal{A}$$

is the flip involution $x \otimes y^{\mathrm{o}} \mapsto y^{\mathrm{o}} \otimes x$ and σ also denotes the induced map on K-homology. In terms of the diagram calculus, Δ being symmetric implies that

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "x"] & \mathcal{A} \arrow[dr] & & \\
& & \cdot \arrow[r, "\Delta"] & \mathbb{C} \\
\mathcal{A}^{\mathrm{o}} \arrow[r, "y^{\mathrm{o}}"] & \mathcal{A}^{\mathrm{o}} \arrow[ur] & &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{A} \arrow[r, "y"] & \mathcal{A} \arrow[dr] & & \\
& & \cdot \arrow[r, "\Delta"] & \mathbb{C} \\
\mathcal{A}^{\mathrm{o}} \arrow[r, "x^{\mathrm{o}}"] & \mathcal{A}^{\mathrm{o}} \arrow[ur] & &
\end{tikzcd}$$

for all $x, y \in \mathsf{KK}(\mathcal{A}, \mathcal{A})$.

A more general form of Poincaré duality gives the notion of Poincaré duality pairs of algebras.

Definition 2.3. A pair of separable C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is said to be a (*strong*) *PD pair* if and only if there exists a fundamental class $\Delta \in \mathsf{KK}_d(\mathcal{A} \otimes \mathcal{B}, \mathbb{C}) = \mathsf{K}^d(\mathcal{A} \otimes \mathcal{B})$ and an inverse class $\Delta^\vee \in \mathsf{KK}_{-d}(\mathbb{C}, \mathcal{A} \otimes \mathcal{B}) = \mathsf{K}_{-d}(\mathcal{A} \otimes \mathcal{B})$ such that

$$\begin{aligned}
\Delta^\vee \otimes_{\mathcal{B}} \Delta &= 1_{\mathcal{A}} \in \mathsf{KK}_0(\mathcal{A}, \mathcal{A}), \\
\Delta^\vee \otimes_{\mathcal{A}} \Delta &= (-1)^d \, 1_{\mathcal{B}} \in \mathsf{KK}_0(\mathcal{B}, \mathcal{B}).
\end{aligned}$$

In analogy to Poincaré duality, we can illustrate the classes Δ and $\Delta^\vee$ with the help of the diagrams

$$\begin{tikzcd}
\mathcal{A} \arrow[dr] & & \\
& \cdot \arrow[r, "\Delta"] & \mathbb{C} \\
\mathcal{B} \arrow[ur] & &
\end{tikzcd}
\in \ \mathsf{KK}(\mathcal{A} \otimes \mathcal{B}, \mathbb{C}) \quad \text{and} \quad
\begin{tikzcd}
& & \mathcal{A} \\
\mathbb{C} \arrow[r, "\Delta^\vee"] & \cdot \arrow[ur] \arrow[dr] & \\
& & \mathcal{B}
\end{tikzcd}
\in \ \mathsf{KK}(\mathbb{C}, \mathcal{A} \otimes \mathcal{B})$$

The product on the right with Δ and the product on the left with $\Delta^\vee$ give inverse isomorphisms

$$\mathsf{K}_i(\mathcal{A}) \xrightarrow{(-)\otimes_{\mathcal{A}}\Delta} \mathsf{K}^{i+d}(\mathcal{B}) \quad \text{and} \quad \mathsf{K}^i(\mathcal{B}) \xrightarrow{\Delta^\vee \otimes_{\mathcal{B}} (-)} \mathsf{K}_{i-d}(\mathcal{A}).$$

As can be checked with the diagram calculus, one also gets Poincaré duality with coefficients in any auxiliary algebras $\mathcal{C}, \mathcal{D}$ given by

$$\mathsf{KK}_i(\mathcal{C}, \mathcal{A} \otimes \mathcal{D}) \cong \mathsf{KK}_{i-d}(\mathcal{C} \otimes \mathcal{B}, \mathcal{D}).$$

KK-equivalent algebras share the same duality properties [12].

Remark 2.4. The word "strong" in the definition of Poincaré duality above is sometimes used because there are weaker notions of the duality. The weaker conditions are described in detail in [12, §2.7].

All of these notions carry over easily to the local bivariant cyclic cohomology described in Section 1.6. Algebras $\mathcal{A}$ and $\mathcal{B}$ are a (*strong*) *cyclic Poincaré dual* (*C-PD*) *pair* if there exists a class

$$\Xi \in \mathsf{HL}_d(\mathcal{A} \otimes \mathcal{B}, \mathbb{C}) = \mathsf{HL}^d(\mathcal{A} \otimes \mathcal{B})$$

and a class $\Xi^\vee \in \mathsf{HL}_d(\mathbb{C}, \mathcal{A} \otimes \mathcal{B}) = \mathsf{HL}_d(\mathcal{A} \otimes \mathcal{B})$ such that

$$\begin{aligned} \Xi^\vee \otimes_{\mathcal{B}} \Xi &= 1_{\mathcal{A}} \in \mathsf{HL}_0(\mathcal{A}, \mathcal{A}), \\ \Xi^\vee \otimes_{\mathcal{A}} \Xi &= (-1)^d 1_{\mathcal{B}} \in \mathsf{HL}_0(\mathcal{B}, \mathcal{B}). \end{aligned}$$

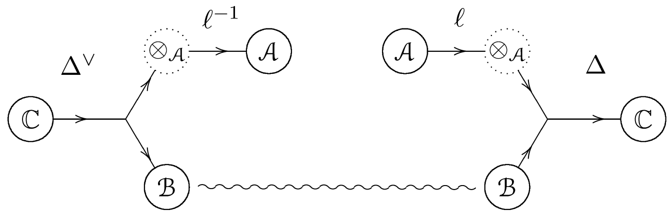
The element Ξ is called a *fundamental cyclic cohomology class* for the pair $(\mathcal{A}, \mathcal{B})$, and $\Xi^\vee$ is called its *inverse*. An algebra $\mathcal{A}$ is a (*strong*) *C-PD algebra* if $(\mathcal{A}, \mathcal{A}^{\circ})$ is a C-PD pair. By multiplicativity of the Chern character, each PD pair can also be made into a C-PD pair for HL. However, in many instances, one does not wish to take the HL fundamental class Ξ to be equal to $\mathsf{ch}(\Delta)$. We will see this explicitly later on.

2.2 Constructing PD algebras

An application of the diagram calculus answers the question of how big is the space of fundamental classes of an algebra.

Proposition 2.5. *Let $(\mathcal{A}, \mathcal{B})$ be a PD pair, and let $\Delta \in \mathsf{K}^d(\mathcal{A} \otimes \mathcal{B})$ be a fundamental class with inverse $\Delta^\vee \in \mathsf{K}_{-d}(\mathcal{A} \otimes \mathcal{B})$. Let $\ell \in \mathsf{KK}_0(\mathcal{A}, \mathcal{A})$ be an invertible element. Then $\ell \otimes_{\mathcal{A}} \Delta \in \mathsf{K}^d(\mathcal{A} \otimes \mathcal{B})$ is another fundamental class, with inverse $\Delta^\vee \otimes_{\mathcal{A}} \ell^{-1} \in \mathsf{K}_{-d}(\mathcal{A} \otimes \mathcal{B})$.*

Proof. The harder direction to prove is that the product $(\Delta^\vee \otimes_{\mathcal{A}} \ell^{-1}) \otimes_{\mathcal{B}} (\ell \otimes_{\mathcal{A}} \Delta)$ produces the identity element in $\mathsf{KK}(\mathcal{A}, \mathcal{A})$. This product is illustrated by the diagram



If we start tracing through the diagram at the incoming node labelled $\mathcal{A}$, which comes from the element $\ell \in \mathsf{KK}(\mathcal{A}, \mathcal{A})$, we obtain the expression $\ell \otimes_{\mathcal{A}} (\Delta^\vee \otimes_{\mathcal{B}} \Delta) \otimes_{\mathcal{A}} \ell^{-1}$. Note here that the element ℓ is attached to the product $\Delta^\vee \otimes_{\mathcal{B}} \Delta$, as we are not allowed to change the order of the elements $\Delta^\vee$ and Δ. The calculation is now completed by using associativity of the Kasparov product (see [12, Proposition 2.7] for details). $\square$

Similarly, one gets a converse.

Proposition 2.6. *Let $(\mathcal{A}, \mathcal{B})$ be a PD pair, and let $\Delta_1, \Delta_2 \in \mathsf{K}^d(\mathcal{A} \otimes \mathcal{B})$ be fundamental classes with inverses $\Delta_1^\vee, \Delta_2^\vee \in \mathsf{K}_{-d}(\mathcal{A} \otimes \mathcal{B})$. Then $\Delta_1^\vee \otimes_{\mathcal{B}} \Delta_2$ is an invertible element in $\mathsf{KK}_0(\mathcal{A}, \mathcal{A})$, with inverse given by $(-1)^d \, \Delta_2^\vee \otimes_{\mathcal{B}} \Delta_1 \in \mathsf{KK}_0(\mathcal{A}, \mathcal{A})$.*

Corollary 2.7. *Let $(\mathcal{A}, \mathcal{B})$ be a PD pair. Then the moduli space of fundamental classes for $(\mathcal{A}, \mathcal{B})$ is isomorphic to the group of invertible elements in the ring $\mathsf{KK}_0(\mathcal{A}, \mathcal{A})$.*

Remark 2.8. In the commutative case $\mathcal{A} = C(X)$, the abelian group of units of $\mathsf{KK}_0(C(X), C(X))$ is, by the universal coefficient theorem for KK-theory, an extension of the automorphism group $\mathrm{Aut}(\mathsf{K}^\bullet(X))$ by $\mathrm{Ext}_{\mathbb{Z}}(\mathsf{K}^\bullet(X), \mathsf{K}^{\bullet+1}(X))$.

If $\mathcal{A}$ is KK-equivalent to the algebra $C_0(X)$ for some locally compact space X, then $\mathcal{A}^{\mathrm{o}}$ is KK-equivalent to $C_0(X)^{\mathrm{o}} = C_0(X)$ as well, and hence we have the following result.

Theorem 2.9. *Let $\mathcal{A}$ be a separable C^*-algebra satisfying the universal coefficient theorem for KK-theory, and with finitely generated K-theory. Then $\mathcal{A}$ is always part of a PD pair, and $\mathcal{A}$ is in addition a PD algebra if and only if either $\mathrm{rk}(\mathsf{K}_0(\mathcal{A})) = \mathrm{rk}(\mathsf{K}_1(\mathcal{A}))$ (in which case we can take $d = 1$) or $\mathrm{Tors}(\mathsf{K}_0(\mathcal{A})) \cong \mathrm{Tors}(\mathsf{K}_1(\mathcal{A}))$ (in which case we can take $d = 0$).*

Proof. By hypothesis, $\mathcal{A}$ is KK-equivalent to a commutative C^*-algebra, hence we can assume $\mathcal{A}$ abelian without loss of generality. By the universal coefficient theorem, one has

$$\begin{aligned}\mathrm{rk}\left(\mathsf{K}^j(\mathcal{A})\right) &= \mathrm{rk}\left(\mathsf{K}_j(\mathcal{A})\right),\\ \mathrm{Tors}\left(\mathsf{K}_j(\mathcal{A})\right) &\cong \mathrm{Tors}\left(\mathsf{K}^{j+1}(\mathcal{A})\right)\end{aligned}$$

for $j = 0, 1 \bmod 2$. Thus the condition for $\mathcal{A}$ to be a PD algebra (i.e., we can take the other algebra of the PD pair to be $\mathcal{A}^{\mathrm{o}}$) is necessary to have an isomorphism $\mathsf{K}_j(\mathcal{A}) \to \mathsf{K}^{j+d}(\mathcal{A})$. It remains to show that for $\mathcal{A}$ and $\mathcal{B}$ commutative, an isomorphism $\mathsf{K}_j(\mathcal{A}) \to \mathsf{K}^{j+d}(\mathcal{B})$ can always be implemented by a suitable fundamental class Δ. By the Künneth theorem and the universal coefficient theorem, we can build Δ and $\Delta^\vee$ from knowledge of the K-theory groups $\mathsf{K}_\bullet(\mathcal{A})$, one cyclic summand at a time. Alternatively, realize $\mathcal{A}$ as $C_0(X)$ for some (possibly non-compact) manifold X, take $\mathcal{B} = C_0(T^*X)$ and construct Δ from the Dirac operator on the Clifford algebra bundle of T^*X. When X is spin^c, the algebra $\mathcal{B}$ is KK-equivalent to $\mathcal{A} = \mathcal{A}^{\mathrm{o}}$. □

We can conclude from this last result that PD pairs are quite common.

Lemma 2.10 ([12, §7.1])**.** *Let $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2$ be separable C^*-algebras such that $(\mathcal{A}, \mathcal{B}_1)$ and $(\mathcal{A}, \mathcal{B}_2)$ are both PD pairs. Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are KK-equivalent.*

3 Riemann–Roch theorem for non-commutative spaces

In the commutative case, the Grothendieck–Riemann–Roch theorem is a key ingredient in arriving at the Minasian–Moore formula for D-brane charge in Type II superstring theory. In this section, we will describe the non-commutative version of this theorem. Later on, we will connect this formalism with the construction of D-brane charges in non-commutative spaces.

3.1 Grothendieck–Riemann–Roch theorem in the smooth case

We begin by recalling the classical case, in particular the properties of K-theory that are required to define the Gysin map and to state the Grothendieck–Riemann–Roch theorem. Let $f : N \to M$ be a smooth map of smooth manifolds. It is said to be K-*oriented* if $TN \oplus f^*(TM)$ is a spin^c

vector bundle over N. In this case, there is a Gysin or "wrong way" map

$$f_! : \mathsf{K}^\bullet(N) \longrightarrow \mathsf{K}^{\bullet+d}(M), \tag{3.1}$$

where $d = \dim(M) - \dim(N)$, and we regard complex K-theory as being $\mathbb{Z}_2$-graded by even and odd degree. Under the same assumptions, there are also Gysin maps

$$f_* : \mathsf{H}_c^\bullet(N, \mathbb{Q}) \longrightarrow \mathsf{H}_c^{\bullet+d}(M, \mathbb{Q})$$

defined in the standard way using Poincaré duality in cohomology with compact supports.

Then the Grothendieck–Riemann–Roch theorem, in the version given by Atiyah and Hirzebruch [3], can be phrased as the commutativity of the diagram

$$\begin{tikzcd}
\mathsf{K}^\bullet(N) \arrow[r, "f_!"] \arrow[d, "\mathsf{Todd}(N)\smile\mathsf{ch}"'] & \mathsf{K}^{\bullet+d}(M) \arrow[d, "\mathsf{Todd}(M)\smile\mathsf{ch}"] \\
\mathsf{H}_c^\bullet(N, \mathbb{Q}) \arrow[r, "f_*"'] & \mathsf{H}_c^{\bullet+d}(M, \mathbb{Q})
\end{tikzcd}$$

with the bottom row $\mathbb{Z}_2$-graded by even and odd degree, giving

$$\mathsf{ch}\big(f_!(\xi)\big) \smile \mathsf{Todd}(M) = f_*\big(\mathsf{ch}(\xi) \smile \mathsf{Todd}(N)\big) \tag{3.2}$$

for all $\xi \in \mathsf{K}^\bullet(N)$. Here $\mathsf{Todd}(M) \in \mathsf{H}_c^\bullet(M, \mathbb{Q})$ denotes the Todd characteristic class of the tangent bundle of M. There are many useful variants of this beautiful formula. In the remainder of this section, we will develop a non-commutative version which encompasses both a cohomological and a homological Riemann–Roch theorem, along with other variants in a unified framework.

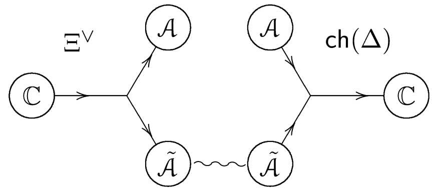
3.2 Todd classes

We will begin by recalling from [12] the definition of the Todd class of a PD algebra $\mathcal{A}$ as an element in bivariant cyclic cohomology. Let $\underline{\mathfrak{D}}$ be the class of all separable C^*-algebras $\mathcal{A}$ for which there exists another separable C^*-algebra $\mathcal{B}$ such that $(\mathcal{A}, \mathcal{B})$ is a PD pair. For any such algebra $\mathcal{A}$, we use Lemma 2.10 to fix a representative of the KK-equivalence class of $\mathcal{B}$ and denote it by $\tilde{\mathcal{A}}$. In general, there is no canonical choice for $\tilde{\mathcal{A}}$. If $\mathcal{A}$ is a PD algebra, then the canonical choice $\tilde{\mathcal{A}} = \mathcal{A}^\mathrm{o}$ will always be made.

Definition 3.1. Let $\mathcal{A} \in \underline{\mathfrak{D}}$, let $\Delta \in \mathsf{K}^d(\mathcal{A} \otimes \tilde{\mathcal{A}})$ be a fundamental K-homology class for the pair $(\mathcal{A}, \tilde{\mathcal{A}})$, and let $\Xi \in \mathsf{HL}^d(\mathcal{A} \otimes \tilde{\mathcal{A}})$ be a fundamental cyclic cohomology class. Then the *Todd class* of $\mathcal{A}$ is defined to be the class in the ring $\mathsf{HL}_0(\mathcal{A}, \mathcal{A})$ given by

$$\mathsf{Todd}(\mathcal{A}) \ := \ \Xi^\vee \otimes_{\tilde{\mathcal{A}}} \mathsf{ch}(\Delta)$$

$$=$$



The Todd class is invertible, with inverse given by

$$\mathsf{Todd}(\mathcal{A})^{-1} = (-1)^d \, \mathsf{ch}\big(\Delta^\vee\big) \otimes_{\tilde{\mathcal{A}}} \Xi.$$

The following example provides the motivation behind this definition.

Example 3.2. Let $\mathcal{A} = C(X)$ with X a compact complex manifold. Then $\mathcal{A}$ is a PD algebra with fundamental class Δ given by the Dolbeault operator on $X \times X$. After passage from $\mathcal{A}$ to the dense subalgebra $C^\infty(X)$, we can identify HL with the usual periodic cyclic homology HP. Thus $\mathsf{HL}_0(\mathcal{A}, \mathcal{A})$ can be identified with $\mathrm{End}(\mathsf{H}^\bullet(X, \mathbb{Q}))$ (see equation (1.3)). The natural choices of fundamental classes Ξ and $\Xi^\vee$ come from the usual Poincaré duality in rational cohomology using the orientation cycle $[X]$. Then $\mathsf{Todd}(\mathcal{A})$ is just cup product with the usual Todd cohomology class $\mathsf{Todd}(X) \in \mathsf{H}^\bullet(X, \mathbb{Q})$.

3.3 K-oriented morphisms

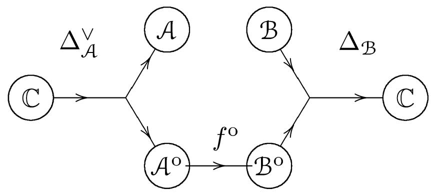
One of the most interesting applications of the present formalism is a definition of Gysin or "wrong way" homomorphisms, following Connes and Skandalis [18]. More generally, we can now study K-oriented morphisms. If $f\colon \mathcal{A} \to \mathcal{B}$ is a morphism of C^*-algebras in a suitable category, a K-orientation is a functorial way of assigning a corresponding element $f! \in \mathsf{KK}_d(\mathcal{B}, \mathcal{A})$. (Note that this element points in the *opposite* direction from f.) The Gysin homomorphism is then given by

$$f_! := (-) \otimes_{\mathcal{B}} f! : \mathsf{K}_\bullet(\mathcal{B}) \longrightarrow \mathsf{K}_{\bullet+d}(\mathcal{A}).$$

If $\mathcal{A}$ and $\mathcal{B}$ are both PD algebras, then any morphism $f\colon \mathcal{A} \to \mathcal{B}$ is K-oriented and the element $f!$ is determined as

$$f! \;=\; (-1)^{d_{\mathcal{A}}}\, \Delta^\vee_{\mathcal{A}} \otimes_{\mathcal{A}^{\mathrm{o}}} [f^{\mathrm{o}}]_{\mathsf{KK}} \otimes_{\mathcal{B}^{\mathrm{o}}} \Delta_{\mathcal{B}}$$

$= (-1)^{d_{\mathcal{A}}}$



with $d = d_{\mathcal{A}} - d_{\mathcal{B}}$. To check functoriality of this construction, observe that if $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are PD algebras, and if $f : \mathcal{A} \to \mathcal{B}, g : \mathcal{B} \to \mathcal{C}$ are morphisms of C^*-algebras, then

$$\begin{aligned}&\big((-1)^{d_{\mathcal{A}}}\, \Delta^\vee_{\mathcal{A}} \otimes_{\mathcal{A}^{\mathrm{o}}} [f^{\mathrm{o}}]_{\mathsf{KK}} \otimes_{\mathcal{B}^{\mathrm{o}}} \Delta_{\mathcal{B}}\big) \otimes_{\mathcal{B}} \big((-1)^{d_{\mathcal{B}}}\, \Delta^\vee_{\mathcal{B}} \otimes_{\mathcal{B}^{\mathrm{o}}} [g^{\mathrm{o}}]_{\mathsf{KK}} \otimes_{\mathcal{C}^{\mathrm{o}}} \Delta_{\mathcal{C}}\big)\\ &\quad = (-1)^{d_{\mathcal{A}}}\, \Delta^\vee_{\mathcal{A}} \otimes_{\mathcal{A}^{\mathrm{o}}} \big[(g \circ f)^{\mathrm{o}}\big]_{\mathsf{KK}} \otimes_{\mathcal{C}^{\mathrm{o}}} \Delta_{\mathcal{C}}\end{aligned}$$

by associativity of the Kasparov product and the basic relation

$$(-1)^{d_{\mathcal{B}}}\, \Delta^\vee \otimes_{\mathcal{B}} \Delta = 1_{\mathcal{B}^{\mathrm{o}}}.$$

This can again be checked using the diagram calculus.

The following examples illustrate the connection with the classical notion of K-orientation given in Section 3.1 above.

Example 3.3. Let $h : X \hookrightarrow Y$ be a K-oriented smooth embedding of smooth compact manifolds. Since $TX \oplus TX$ has a canonical almost complex structure, the normal bundle

$$N_Y X = h^*(TY)/TX$$

is a spin^c vector bundle. In particular, the zero section embedding $\iota^X : X \hookrightarrow N_Y X$ is K-oriented. Let $S(N_Y X)$ denote the bundle of spinors associated to the spin^c structure on $N_Y X$. Let $\mathcal{E} = (\mathcal{E}_x)_{x\in X},\ \mathcal{E}_x = L^2(N_x, S_x)$ be the Hilbert bundle over $N = N_Y X$ obtained from the pullback $S = {\pi_N}^* S(N_Y X)$, where $\pi_N : N_Y X \to X$ is the bundle projection. For $\xi \in C(X)$, the composition $\xi \circ \pi_N$ acts by multiplication as an endomorphism of $\mathcal{E}$ and thus there is a natural homomorphism $C(X) \to \mathrm{End}(\mathcal{E})$. When the codimension $n = \dim(Y) - \dim(X)$ of X in Y is even, Clifford multiplication by the orientation of $N_Y X$ splits $S(N_Y X)$ into a Whitney sum of half-spin bundles $S(N_YX)^{\pm}$ which define a $\mathbb{Z}_2$-grading $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$. Clifford multiplication $c(v) : {\pi_N}^*S(N_YX)^+ \to {\pi_N}^*S(N_YX)^-$ by the tautological section v of the bundle

$$ {\pi_N}^*N_YX \longrightarrow N_YX, $$

which assigns to a vector in N_YX the same vector in ${\pi_N}^*N_YX$, defines a morphism

$$ F_{v(\nu)} \in \mathrm{Hom}\left(\mathcal{E}^+_{v(\nu)}, \mathcal{E}^-_{v(\nu)}\right) $$

for all $\nu \in N_YX$. The corresponding Kasparov bimodule $(\mathcal{E}, F)$ defines an invertible element $\iota^X! \in \mathsf{KK}_n(C(X), C_0(N_YX))$ associated to the classical Atiyah–Bott–Shapiro (ABS) representative of the Thom class of the complex vector bundle N_YX. Upon choosing a Riemannian metric on Y, there is a diffeomorphism Φ from a tubular neighbourhood U of $h(X)$ onto a neighbourhood of the zero section in the normal bundle $\iota^X(X)$, giving an invertible element $[\Phi]_{\mathsf{KK}} \in \mathsf{KK}_0(C_0(N_YX), C_0(U))$. For any open subset $j : U \hookrightarrow Y$, the extension by zero defines an element $j! \in \mathsf{KK}_0(C_0(U), C(Y))$. Then we may associate to the embedding h the class in KK-theory defined by

$$ h! = \iota^X! \otimes_{C_0(N_YX)} [\Phi]_{\mathsf{KK}} \otimes_{C_0(U)} j! \in \mathsf{KK}_n\big(C(X), C(Y)\big) $$

which, by homotopy invariance of KK-theory and functoriality of Gysin classes, is independent of the choices made. In the applications to string theory, this construction establishes that the charge of a D-brane supported on X takes values in the K-theory of spacetime Y [42, 54].

Example 3.4. Let $\pi : Y \to Z$ be a K-oriented, proper smooth submersion of smooth compact manifolds. Since every smooth compact manifold Y has a smooth embedding $h : Y \hookrightarrow \mathbb{R}^{2q}$ for q sufficiently large, a parametrized version yields a smooth embedding $\kappa = (\pi, h) : Y \hookrightarrow Z \times \mathbb{R}^{2q}$ which is K-oriented. The corresponding KK-theory class is $\kappa! \in \mathsf{KK}_a(C(Y), C_0(Z \times \mathbb{R}^{2q}))$, where $a = \dim(Z) - \dim(Y) + 2q$. Let $\iota^Z : Z \hookrightarrow Z \times \mathbb{R}^{2q}$ denote the zero section embedding, with invertible Thom class $\iota^Z! \in \mathsf{KK}_{2q}(C(Z), C_0(Z \times \mathbb{R}^{2q}))$. Then the KK-theory class corresponding to the submersion π is defined as

$$ \pi! = \kappa! \otimes_{C_0(Z\times\mathbb{R}^{2q})} \left(\iota^Z!\right)^{-1} \in \mathsf{KK}_b\big(C(Y), C(Z)\big), $$

where $b = \dim(Z) - \dim(Y)$, and is again independent of the choices made. When $Y = X \times Z$ is a product manifold, the corresponding Gysin map $\pi_!$ gives the analytic index for Z-families of elliptic operators on X.

Example 3.5. Let $f : N \to M$ be an arbitrary K-oriented smooth proper map, as in the setting of Section 3.1 above. The map f can be canonically factored as $f = p^M \circ \mathrm{gr}(f)$, firstly into the K-oriented smooth graph embedding $\mathrm{gr}(f) : N \hookrightarrow N \times M$ defined by $\mathrm{gr}(f)(x) = (x, f(x))$. The construction of Example 3.3 above then gives an element $\mathrm{gr}(f)! \in \mathsf{KK}_m(C(N), C(N \times M))$ with $m = \dim(M)$. Secondly, the projection $p^M : N \times M \to M$ is a K-oriented proper submersion when restricted to the image of $\mathrm{gr}(f)$. The corresponding KK-theory class is

$$p^M! \in \mathsf{KK}_b\big(C(M \times N), C(M)\big),$$

where $b = -\dim(N)$. The Gysin map of f in equation (3.1) is then defined via the KK-theory element

$$f! = \mathrm{gr}(f)! \otimes_{C(N\times M)} p^M!.$$

Example 3.6. Let $\mathcal{J}$, $\mathcal{A}$ and $\mathcal{B}$ be separable C^*-algebras, and suppose there is a split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{J} \arrow[r, "j"] & \mathcal{A} \arrow[r, shift right, "q"'] & \mathcal{B} \arrow[l, bend right, "s"'] \arrow[r] & 0. \end{tikzcd} \tag{3.3}$$

Then the morphism j is naturally K-oriented. Indeed, by the split exactness property of KK from Section 1.1, j and s induce an isomorphism

$$\Psi = (j_* \oplus s_*)\colon\ \mathsf{KK}(\mathcal{A}, \mathcal{J}) \oplus \mathsf{KK}(\mathcal{A}, \mathcal{B}) \xrightarrow{\approx} \mathsf{KK}(\mathcal{A}, \mathcal{A}).$$

We define $j! \in \mathsf{KK}(\mathcal{A}, \mathcal{J})$ to be the image of $1_{\mathcal{A}}$ in $\mathsf{KK}(\mathcal{A}, \mathcal{J})$ under Ψ^{-1} followed by projection onto the first summand in the direct sum and observe that it has the desired properties of a K-orientation. This is the same as the Kasparov element called π_s in [8, §17.8]. This choice of K-orientation depends on the splitting s, which is the reason for the notation.

3.4 The non-commutative Grothendieck–Riemann–Roch theorem

The Grothendieck–Riemann–Roch formula compares the two bivariant cyclic cohomology classes $\mathsf{ch}(f!)$ and $f*$.

Theorem 3.7. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are PD algebras with given HL fundamental classes. Let $f* \in \mathsf{HL}(\mathcal{B}, \mathcal{A})$ be the Gysin map defined using the HL fundamental classes. Then one has the Grothendieck–Riemann–Roch formula*

$$\mathsf{ch}(f!) = (-1)^{d_{\mathcal{B}}}\,\mathsf{Todd}(\mathcal{B}) \otimes_{\mathcal{B}} (f*) \otimes_{\mathcal{A}} \mathsf{Todd}(\mathcal{A})^{-1}. \tag{3.4}$$

Proof. We write out both sides of equation (3.4) using the various definitions and simplify using associativity of the Kasparov product and the functorial properties of the bivariant Chern character (see Theorem 1.3). After some algebra, one finds that the theorem then follows if

$$\begin{aligned} \big(\Xi^\vee_{\mathcal{B}} \otimes_{\mathcal{B}^{\rm o}} \mathsf{ch}(\Delta_{\mathcal{B}})\big) \otimes_{\mathcal{B}} \Xi_{\mathcal{B}} &= (-1)^{d_{\mathcal{B}}} \mathsf{ch}(\Delta_{\mathcal{B}}), \\ \Xi^\vee_{\mathcal{A}} \otimes_{\mathcal{A}} \big(\mathsf{ch}(\Delta^\vee_{\mathcal{A}}) \otimes_{\mathcal{A}^{\rm o}} \Xi_{\mathcal{A}}\big) &= (-1)^{d_{\mathcal{A}}} \mathsf{ch}(\Delta^\vee_{\mathcal{A}}). \end{aligned}$$

Both of these equalities follow easily from the diagram calculus; see [12, Theorem 7.10] for further details. □

Corollary 3.8. $\mathsf{ch}\big(f_!(\xi)\big) \otimes_{\mathcal{A}} \mathsf{Todd}(\mathcal{A}) = (-1)^{d_{\mathcal{B}}}\, f_*\big(\mathsf{ch}(\xi) \otimes_{\mathcal{B}} \mathsf{Todd}(\mathcal{B})\big)$ *for all* $\xi \in \mathsf{K}_\bullet(\mathcal{B})$.

Example 3.9. Let $\pi : X \to Z$ be a smooth fibration over a closed smooth manifold Z whose fibres X/Z are compact, closed spin^c manifolds of even dimension. Let $g^{X/Z}$ be a metric on the vertical tangent sub-bundle $T(X/Z)$ of TX. Let $S_{X/Z}$ be the spinor bundle associated to $(T(X/Z), g^{X/Z})$. Let $T^{\rm H}X$ be a horizontal vector sub-bundle of TX such that $TX = T^{\rm H}X \oplus T(X/Z)$. Then $(T^{\rm H}X, g^{X/Z})$ determines a canonical Euclidean connection $\nabla^{X/Z}$ [7, Theorem 1.9]. Clifford multiplication of $T^*(X/Z)$ on $S_{X/Z}$ and the connection $\nabla^{X/Z}$ define a fibrewise Dirac operator $\not{D}_z$ acting on the Hilbert space $\mathcal{E}_z = L^2(X/Z, S_{X/Z})$ along the fibre $\pi^{-1}(z) \cong X/Z$ for $z \in Z$. Then $\{\not{D}_z\}_{z\in Z}$ is a smooth family of elliptic operators parametrized by Z acting on an infinite-dimensional, $\mathbb{Z}_2$-graded Hilbert bundle $\mathcal{E} \to Z$ whose fibre at $z \in Z$ is $\mathcal{E}_z$. The corresponding Kasparov bimodule defines the longitudinal Dirac element $\pi! \in \mathsf{KK}(C_0(X), C_0(Z))$ [18]. Given a complex vector bundle $E \to X$, the Atiyah–Singer index theorem asserts that $\pi_!(E) = E \otimes_{C_0(X)} \pi!$ is equal to the analytic index of the family of Dirac operators on X/Z coupled to E. Corollary 3.8 then expresses the Chern character of the index in K-theory in terms of the Chern character of E as

$$\mathsf{ch}\big(\pi_!(E)\big) = \pi_*\big(\mathsf{Todd}(X/Z) \smile \mathsf{ch}(E)\big)\ .$$

This quantity is used to compute global worldsheet anomalies in string theory [23].

4 D-brane charge in non-commutative spaces

In this section, we will apply our formalism to the description of D-brane charges in very general non-commutative settings. After recalling the classical situation, we will describe the physical context in which the KK-theory of C^*-algebras classifies states of D-branes in superstring theory. We will then derive the non-commutative formula for Ramond–Ramond charges and present some non-commutative examples.

4.1 The Minasian–Moore formula

We begin by recalling the classical formula. Let X be a compact spin^c manifold of dimension d. Then Poincaré duality in ordinary cohomology of X is the statement that the pairing given by the cup product

$$(x,y)_{\mathsf{H}} = \big\langle x \smile y, [X]\big\rangle \tag{4.1}$$

for $x \in \mathsf{H}^\bullet(X,\mathbb{Z})$, $y \in \mathsf{H}^{d-\bullet}(X,\mathbb{Z})$ is non-degenerate. On the other hand, in K-theory the analytic index provides another pairing given by

$$(E,F)_{\mathsf{K}} = \mathrm{index}(\not{D}_{E\otimes F}),$$

where E, F are classes in $\mathsf{K}^0(X)$ represented by vector bundles over X, and $\not{D}$ is the Dirac operator associated with the spin^c structure on X.

The classical Chern character gives a ring isomorphism

$$\mathsf{ch} : \mathsf{K}^j(X) \otimes \mathbb{Q} \longrightarrow \mathsf{H}^j(X,\mathbb{Q}),$$

where $j = 0, 1$ and on the right-hand side we mean the periodized cohomology of X with rational coefficients. This isomorphism is not compatible with the two pairings. However, the Atiyah–Singer index theorem provides the formula

$$\mathrm{index}(\not{D}_{E\otimes F}) = \big\langle \mathsf{Todd}(X) \smile \mathsf{ch}(E\otimes F), [X]\big\rangle$$

from which it follows that the modified Chern character Ch defined by the generalized Mukai vector $\mathsf{Ch}(E) = \sqrt{\mathsf{Todd}(X)} \smile \mathsf{ch}(E)$ is an isometry with respect to the index pairing on K-theory and the topological pairing on cohomology. This simple observation is the starting point of [12] and plays an important role in this section.

Definition 4.1. An (*untwisted*) *D-brane* in X is a triple (W, E, f), where $f : W \hookrightarrow X$ is a closed, embedded spin^c submanifold and $E \in \mathsf{K}^0(W)$. The submanifold W is called the *worldvolume* and the class E the *Chan–Paton bundle* of the D-brane.

From Example 3.3 it follows that any D-brane (W, E, f) in X determines a canonically defined KK-theory class $f! \in \mathsf{KK}(C(W), C(X))$. In fact, every D-brane naturally determines a Fredholm module over the algebra $C(X)$

and we can think of D-branes (W, E, f) as providing K-homology classes on spacetime X, Poincaré duality to K-theory classes $f_!(E)$. In this context, Baum–Douglas equivalence [4] (or stable homotopy equivalence) is sometimes referred to as "gauge equivalence" between D-branes.

Definition 4.2. The *Ramond–Ramond charge* of a D-brane (W, E, f) in X is the modified Chern characteristic class

$$\mathsf{Q}(W, E) := \mathsf{Ch}\big(f_!(E)\big) = \mathsf{ch}\big(f_!(E)\big) \smile \sqrt{\mathsf{Todd}(X)} \in \mathsf{H}^\bullet(X, \mathbb{Q}).$$

This is the Minasian–Moore formula [37]. In the underlying boundary superconformal field theory, the charge vector $\mathsf{Q}(W, E)$ is the zero-mode part of the boundary state of the D-brane in the Ramond–Ramond sector. In the setting of the D-brane field theory on the worldvolume W, $\mathsf{Q}(W, E) = f_*(\mathsf{D}_{\mathrm{WZ}}(W, E))$ is the image of the *Wess–Zumino class* $\mathsf{D}_{\mathrm{WZ}}(W, E) \in \mathsf{H}^\bullet(W, \mathbb{Q})$ (for vanishing B-field) under the Gysin map in cohomology. From the Grothendieck–Riemann–Roch formula (3.2) and naturality of the Todd class, one has

$$\mathsf{D}_{\mathrm{WZ}}(W, E) = \mathsf{ch}(E) \smile \sqrt{\mathsf{Todd}(W)/\mathsf{Todd}(N_X W)}. \tag{4.2}$$

This formula characterizes the Ramond–Ramond charge as the anomaly inflow on the D-brane worldvolume, determined by the index theorem on W.

4.2 Algebraic characterization of D-branes

We will now describe a physical framework for KK-theory which will naturally apply to the construction of D-brane charges in more general situations. (See [1, 43] for related but somewhat differently motivated characterizations.) Let us fix a closed string background specified by a compact spin^c manifold X. In the absence of background supergravity form fields, this is equivalent to fixing the commutative C^*-algebra $\mathcal{A} = C(X)$. An open string in X may be regarded as an embedding of a compact Riemann surface $\Sigma \hookrightarrow X$, called the *worldsheet*, with boundary $\partial\Sigma$. A classical D-brane then corresponds to a choice of submanifold $W \subset X$ such that the open string fields define relative maps

$$(\Sigma, \partial\Sigma) \longrightarrow (X, W).$$

Alternatively, we may regard an open string as an oriented embedding of the interval $I = [0, 1]$ into X, with the boundaries $t = 0$ and $t = 1$ referring to the *endpoints* of the strings. The worldsheet is then $\Sigma = \mathbb{R} \times I$. In the associated boundary superconformal field theory, we require that the Cauchy problem for the Euler–Lagrange equations on Σ have a unique solution locally. This requires imposing suitable boundary conditions on the open string fields. A classical D-brane simply represents such a choice of boundary conditions. To fully specify the boundary conditions, one must also specify a Hermitean vector bundle E over W with connection. The requirement that the boundary conditions preserve superconformal invariance constrains the submanifold W. For example, in the absence of H-flux, the worldvolume W must be spin^c [23] and we recover Definition 4.1 above of a D-brane.

We would now like to extend this description to *quantum* D-branes, those which define consistent boundary conditions after quantization of the boundary superconformal field theory. Unfortunately, there is no known satisfactory way to define a fully quantum boundary condition in these generic instances. In the following, we will describe a conjectural C^*-algebraic framework for quantum D-branes in the context of open string field theory.

Let $\mathfrak{B}$ be the set of boundary conditions for open strings in X. For $a, b \in \mathfrak{B}$, we call an open string with a boundary conditions at its $t = 0$ end and b boundary conditions at its $t = 1$ end an *a-b open string*. One can glue two open strings together by the usual concatenation of paths $I \hookrightarrow X$, provided that the boundary condition at the initial endpoint of one string matches the boundary condition at the final endpoint of the other string. After quantization, this can be used to define a non-commutative algebra of observables $\mathcal{D}_a$ consisting of a-a open string fields [51, 55], thus generating a set of D-brane algebras $\mathcal{D}_a$, $a \in \mathfrak{B}$. These algebras are required to carry an action of the worldsheet superconformal algebra by automorphisms of $\mathcal{D}_a$. By concatenation, for any pair of boundary conditions $a, b \in \mathfrak{B}$, the a-b open string fields form a $\mathcal{D}_a$-$\mathcal{D}_b$ bimodule $\mathcal{E}_{a,b}$. Similarly, the b-a open string fields generate a $\mathcal{D}_b$-$\mathcal{D}_a$ bimodule $\mathcal{E}_{b,a}$. In particular, $\mathcal{E}_{a,a} = \mathcal{D}_a$ is the trivial $\mathcal{D}_a$-bimodule given by the natural actions of the algebra $\mathcal{D}_a$ on itself via multiplication from the left and from the right.

We would now like to associate to this data a $\mathbb{C}$-linear category that we may wish to call the "category of D-branes". The objects of this category are the elements of $\mathfrak{B}$, and the set of morphisms from a to b is the bimodule $\mathcal{E}_{a,b}$ (or more precisely the corresponding state space of the boundary superconformal field theory). Given some other boundary condition $c \in \mathfrak{B}$, we must then require that there is an associative bilinear composition map

$$\mathcal{E}_{a,b} \times \mathcal{E}_{b,c} \longrightarrow \mathcal{E}_{a,c}. \tag{4.3}$$

The naive guess for this map is the natural string vertex combining an a-b open string field and a b-c open string field to generate an a-c open string field. However, the problem is that this map need not be well-defined. Recall that elements of the open string bimodule $\mathcal{E}_{a,b}$ are the vertex operators $V_{a,b} : I \to \mathrm{End}(\mathcal{H}_{a,b})$, with $\mathcal{H}_{a,b}$ a separable Hilbert space. The product of $V_{a,b}$ and $V_{b,c}$ is encoded in a singular operator product expansion which for $t > t'$ takes the formal symbolic form

$$V_{a,b}(t) \cdot V_{b,c}(t') = \sum_{j=1}^{N} \frac{1}{(t-t')^{h_j}} W_{a,b,c|j}(t,t') \tag{4.4}$$

for some positive integer N, where $W_{a,b,c|j} : I \times I \to \mathrm{End}(\mathcal{H}_{a,c})$, the real numbers $h_j \in [0,\infty)$ are known as conformal dimensions, and the quantity $(t-t'\,)^{-h_j}$ is understood in terms of its formal power series expansion

$$\frac{1}{(t-t')^{h_j}} = \sum_{n=0}^{\infty} \frac{h_j\,(h_j+1)\cdots(h_j+n-1)}{n!}(t')^n t^{-n-h_j}$$

for $t > t'$. When the conformal dimensions are non-zero, the leading singularities of the operator product expansion do not give an associative algebra in the standard sense.

One resolution to this problem was pointed out by Seiberg and Witten [51]. They consider a particular limit of the boundary conformal field theory on X whereby the dimensions h_j vanish. Quantization of the point particle at the endpoint of an open string with boundary condition $a \in \mathfrak{B}$ gives a separable Hilbert space $\mathcal{H}_a$ which is acted upon by open string tachyon vertex operators $V_a(t)$, $t \in [0,1]$. In the Seiberg–Witten limit, these operators live in a separable C^*-algebra $\mathcal{D}_a$ representing a non-commutative spacetime. (This algebra is a deformation of $\mathcal{A} = C(X)$ when there is a constant B-field present on X.) The product $\mathcal{D}_a \otimes \mathcal{D}_b$ generates the full algebra of operators on the string ground states, acting irreducibly on the quantum mechanical Hilbert space $\mathcal{H}_{a,b} = \mathcal{H}_a \otimes \mathcal{H}_b^\vee$. In this case, the mapping (4.3) is well-defined and can be computed from equation (4.4), which takes an a-b vertex operator $V_{a,b}(t)$ and a b-c vertex operator $V_{b,c}(t'\,)$ to the a-c vertex operator $V_{a,c}(t'\,)$ given by the product

$$V_{a,c}(t') = \lim_{t\to t'} V_{a,b}(t) \cdot V_{b,c}(t').$$

Furthermore, by associativity of the operator product expansion in the limit,

$$(V_{a,b}\,V_b)\,V_{b,c} = V_{a,b}\,(V_b\,V_{b,c})$$

for any b-b vertex operator V_b, the assignment extends to a map

$$\mathcal{E}_{a,b} \otimes_{\mathcal{D}_b} \mathcal{E}_{b,c} \longrightarrow \mathcal{E}_{a,c}. \tag{4.5}$$

It follows that the bimodule $\mathcal{E}_{a,b}$ in this instance establishes a (strong) Morita equivalence between the C^*-algebras $\mathcal{D}_a$ and $\mathcal{D}_b$, with dual $\mathcal{E}^\vee_{a,b} \cong \mathcal{E}_{b,a}$. (The opposite algebra $\mathcal{D}^o_a$ is generated by reversing the orientations of the a-a open strings.) Furthermore, we can define elements

$$\alpha_{a,b} \in \mathsf{KK}(\mathcal{D}_a, \mathcal{D}_b) \quad \text{and} \quad \alpha_{b,a} \in \mathsf{KK}(\mathcal{D}_b, \mathcal{D}_a)$$

by the equivalence classes of the Kasparov bimodules $(\mathcal{E}_{a,b}, 0)$ and $(\mathcal{E}_{b,a}, 0)$, respectively. Since one can canonically identify the $\mathcal{D}_a$-bimodule

$$\mathcal{E}_{a,b} \otimes_{\mathcal{D}_b} \mathcal{E}_{b,a}$$

with $\mathcal{D}_a$ and the $\mathcal{D}_b$-bimodule

$$\mathcal{E}_{b,a} \otimes_{\mathcal{D}_a} \mathcal{E}_{a,b}$$

with $\mathcal{D}_b$, one has

$$\alpha_{a,b} \otimes_{\mathcal{D}_b} \alpha_{b,a} = 1_{\mathcal{D}_a} \quad \text{and} \quad \alpha_{b,a} \otimes_{\mathcal{D}_a} \alpha_{a,b} = 1_{\mathcal{D}_b}.$$

It follows that the D-brane algebras $\mathcal{D}_a$ and $\mathcal{D}_b$ in this instance are KK-equivalent. From the above arguments, it also follows that the KK-equivalence $\alpha_{a,c} = \alpha_{a,b} \otimes_{\mathcal{D}_b} \alpha_{b,c}$ is associated to the Morita equivalence between $\mathcal{D}_a$ and $\mathcal{D}_c$. This involutive sort of KK-equivalence is the categorical statement of T-duality of the non-commutative spacetimes represented by these algebras [12, 51].

In the general case, the operator product expansion (4.4) is not associative, and it need not even lead to a well-defined map (4.3). In particular, $\mathcal{E}_{a,b}$ need not be a Morita equivalence bimodule. There are special instances when the desired categorical construction goes through without the need of taking any limits. When the worldsheet superconformal field theory on $\Sigma = \mathbb{R} \times I$ is a two-dimensional topological field theory, then the set of D-branes has the structure of a category called the category of "topological D-branes" [2, 38]. To extend the construction to more generic situations, we will assume that generally there is an appropriate "completion" of $\mathcal{E}_{a,b}$ to a Kasparov bimodule $(\mathcal{E}_{a,b}, F_{a,b})$, defining an element in $\mathsf{KK}(\mathcal{D}_a, \mathcal{D}_b)$, with the mapping (4.5) given by the Kasparov composition product. The canonical duality above is generically lost, as the KK-theory classes of $(\mathcal{E}_{a,b}, F_{a,b})$

and $(\mathcal{E}_{b,a}, F_{b,a})$ need not even be inverses of each other. When they are, the D-brane algebras $\mathcal{D}_a$, $\mathcal{D}_b$ are KK-equivalent. If the additional application of this KK-equivalence is of order 2 up to Morita equivalence, then the D-brane algebras are said to be *T-dual*, in the spirit of [12]. In Section 5 we will encounter a number of examples of T-dual algebras in this sense. The correspondence picture there will relate these dualities to generalizations of the (smooth) Fourier–Mukai transform and show that the proper physical interpretation of KK-theory groups is that of generalized morphisms between open string algebras (boundary conditions) which provide dualities among the corresponding non-commutative spacetimes. It is not clear whether or not this fits into Witten's non-commutative $*$-algebras [53], as in our formulation the manifest background independence of open string field theory, represented by stable isomorphisms $\mathcal{D}_a \otimes \mathcal{K} \cong \mathcal{D}_b \otimes \mathcal{K}$ for $a, b \in \mathfrak{B}$, is generically lost.

In this formalism, it is more natural to think of the D-branes themselves as algebras $\mathcal{D}_a$, rather than as Fredholm modules over the spacetime algebra $\mathcal{A}$. The string theory then requires some extra structure. We assume that for each $a \in \mathfrak{B}$ there are canonical elements

$$\lambda_a \in \mathsf{KK}(\mathcal{D}_a, \mathcal{A}) \quad \text{and} \quad \xi_a \in \mathsf{KK}(\mathcal{A}, \mathcal{D}_a),$$

which determine maps between the K-theory and K-homology of $\mathcal{A}$ and of $\mathcal{D}_a$ via the composition product as explained in Section 1.3. Passing this structure over to HL-theory, this is simply the construction of a boundary state for the D-brane in local cyclic homology $\mathsf{HL}_\bullet(\mathcal{A})$ of the closed string background, now including all excited oscillator states of the strings and not just their zero modes. The boundary state is not arbitrary but satisfies conditions of modular invariance in the boundary conformal field theory, dictated through the Cardy conditions. Presumably this is the essence of the generalized structure of the morphisms between D-brane algebras in bivariant theories, although in this crude target space picture we have not been able to make this more precise in the general case. One instance where the Cardy condition can be written down is the case when a pair of D-brane algebras $\mathcal{D}_a, \mathcal{D}_b$ are KK-equivalent, implemented by a class

$$\alpha_{ab} \ \in \ \mathsf{KK}(\mathcal{D}_a, \mathcal{D}_b)$$

with inverse

$$\alpha_{ab}^{-1} \in \mathsf{KK}(\mathcal{D}_b, \mathcal{D}_a).$$

Then the Cardy condition in the present context can be stated as the requirement that the map $\mathsf{KK}(\mathcal{D}_a,\mathcal{D}_a) \to \mathsf{KK}(\mathcal{D}_b,\mathcal{D}_b)$ induced by the elements $\lambda_b \otimes_{\mathcal{A}} \xi_a$ and $\lambda_a \otimes_{\mathcal{A}} \xi_b$ is given by the "similarity transformation"

$$\big(\lambda_b \otimes_{\mathcal{A}} \xi_a\big) \otimes_{\mathcal{D}_a} \psi \otimes_{\mathcal{D}_a} \big(\lambda_a \otimes_{\mathcal{A}} \xi_b\big) = \alpha_{ab}^{-1} \otimes_{\mathcal{D}_a} \psi \otimes_{\mathcal{D}_a} \alpha_{ab}$$

for all $\psi \in \mathsf{KK}(\mathcal{D}_a,\mathcal{D}_a)$.

Below we will assume that this picture extends to more general situations in which spacetime is described by a non-commutative separable C^*-algebra $\mathcal{A}$. Our original definition of D-brane can be easily cast into this framework.

Example 4.3. Consider a D-brane (W,E,f) in X in the sense of Definition 4.1, and let

$$\mathcal{D} = C(W) \quad \text{and} \quad \mathcal{A} = C(X).$$

In the Ramond sector, the ground states of W-W open strings, with both ends on W, generate the $\mathcal{D}$-bimodule $\mathcal{E}_{W,W} = L^2(W, S(W) \otimes S(N_X W))$, where $S(W)$ is the bundle of spinors on W and $S(N_X W)$ is the bundle of spinors on the normal bundle $N_X W$ to W in X. Using the Dirac operator associated to $S(W) \otimes S(N_X W)$ and pushing forward with the group homomorphism induced by f, we obtain a K-homology class in $\mathsf{K}^\bullet(\mathcal{A}) = \mathsf{K}_\bullet(X)$. However, we can also consider a "spacetime-filling" D-brane $(X, \mathbb{1}_X^{\mathbb{C}}, \mathrm{id}_X)$, with $\mathbb{1}_X^{\mathbb{C}}$ the trivial complex line bundle over X, and open strings with one end on W and the other end on X. Then the Hilbert space of ground state Ramond sector X-W strings is the $\mathcal{A}$-$\mathcal{D}$ bimodule $\mathcal{E}_{X,W} = L^2(W, S(W) \otimes E)$, giving a class $\lambda_W \in \mathsf{KK}(\mathcal{D},\mathcal{A})$ in the manner explained above. When $E \cong \mathbb{1}_W^{\mathbb{C}}$, this is the element $f!$ constructed in Example 3.3, while the element $\xi_W \in \mathsf{KK}(\mathcal{A},\mathcal{D})$ is the class $[f^*]_{\mathsf{KK}}$ of the induced pullback morphism

$$f^* : \mathcal{A} \longrightarrow \mathcal{D}.$$

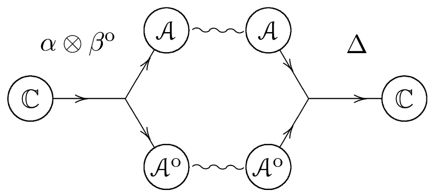
The equivalence between these two descriptions of the D-brane is essentially the process of tachyon condensation on spacetime-filling branes to stable states of lower-dimensional D-branes, as explained in [42, 47, 54], represented by the KK-theory class $f!$. Generic classes in $\mathsf{KK}(\mathcal{D},\mathcal{A})$ can be interpreted in terms of "generalized brane decays" as described in [48].

4.3 The isometric pairing formula

Just as in the commutative case, Poincaré duality leads to a non-degenerate pairing in K-theory (modulo torsion). Since Poincaré duality was formulated in terms of classes in bivariant KK-theory, there is a corresponding pairing in K-homology as well. Moreover, once we have the correct notion of Poincaré duality in bivariant cyclic homology, we are able to define a pairing on ordinary cyclic homology and cohomology. Below we assume that the algebra $\mathcal{A}$ is a PD algebra, but our results all generalize straightforwardly to more general algebras in the class $\underline{\mathfrak{D}}$ introduced in Section 3.2, with the opposite algebras $\mathcal{A}^{\mathrm{o}}$ replaced everywhere by $\tilde{\mathcal{A}}$.

Let $\alpha \in \mathsf{K}_i(\mathcal{A})$, $\beta \in \mathsf{K}_{-d-i}(\mathcal{A})$. Then there is a pairing $(-,-)_{\mathsf{K}} : \mathsf{K}_i(\mathcal{A}) \times \mathsf{K}_{-d-i}(\mathcal{A}) \to \mathbb{Z}$ given by

$$(\alpha,\beta)_{\mathsf{K}} \;=\; (\alpha\otimes\beta^{\mathrm{o}})\otimes_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{o}}}\Delta$$

$=$ 

$\in \mathsf{KK}_0(\mathbb{C},\mathbb{C}) = \mathbb{Z}$ (4.6)

If the fundamental class Δ is symmetric, then this defines a symmetric form. This formula generalizes the index pairing in K-theory. If we take $\mathcal{A} = \mathcal{A}^{\mathrm{o}} = C(X)$ for X a compact spin^c manifold and let $\Delta = \not{D}\otimes\not{D}$ be the Dirac class, then using the definition of Kasparov's product one computes for any $\alpha,\beta\in\mathsf{K}^0(X)$, $(\alpha,\beta)_{\mathsf{K}} = \not{D}_\alpha\otimes_{C(X)}\beta = \mathrm{index}(\not{D}_{\alpha\otimes\beta})$.

If $\mathcal{A}$ also satisfies Poincaré duality in bivariant local cyclic homology, then there is a non-degenerate pairing $(-,-)_{\mathsf{H}} : \mathsf{HL}_i(\mathcal{A})\otimes_{\mathbb{C}}\mathsf{HL}_{d-i}(\mathcal{A}) \to \mathbb{C}$ given by the explicit formula

$$(x,y)_{\mathsf{H}} = (x\otimes y^{\mathrm{o}})\otimes_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{o}}}\Xi \tag{4.7}$$

for $x\in\mathsf{HL}_i(\mathcal{A})$ and $y\in\mathsf{HL}_{d-i}(\mathcal{A})$. It is important to note that there is another way to define a pairing on cyclic homology. It arises when we replace Ξ in equation (4.7) by the image $\mathsf{ch}(\Delta)$ under the Chern character of the fundamental class Δ that provides Poincaré duality in K-theory. As in the commutative case, the Todd class links the two pairings. We then arrive at an analogue of the classical isometry result that we started this section with.

Theorem 4.4. *Suppose that the non-commutative spacetime $\mathcal{A}$ satisfies the universal coefficient theorem for local bivariant cyclic homology and that $\mathsf{HL}_\bullet(\mathcal{A})$ is a finite-dimensional vector space. If $\mathcal{A}$ has symmetric* (*even-dimensional*) *fundamental classes both in K-theory and in cyclic cohomology,* *then the modified Chern character*

$$\mathsf{ch} \otimes_{\mathcal{A}} \sqrt{\mathsf{Todd}(\mathcal{A})} \, : \, \mathsf{K}_\bullet(\mathcal{A}) \, \longrightarrow \, \mathsf{HL}_\bullet(\mathcal{A})$$

is an isometry with respect to the inner products (4.6) *and* (4.7).

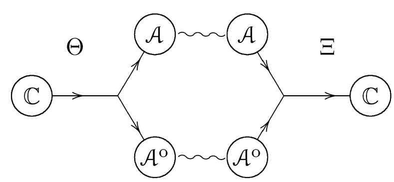
Proof. We have to prove the formula

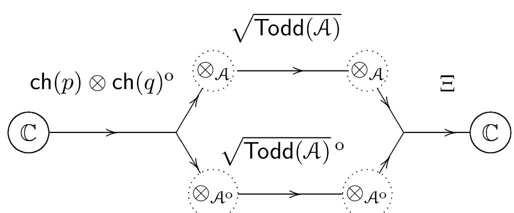
$$\big(p,q\big)_{\mathsf{K}} = \big(\mathsf{ch}(p) \otimes_{\mathcal{A}} \sqrt{\mathsf{Todd}(\mathcal{A})}, \quad \mathsf{ch}(q) \otimes_{\mathcal{A}} \sqrt{\mathsf{Todd}(\mathcal{A})}\big)_{\mathsf{H}}. \tag{4.8}$$

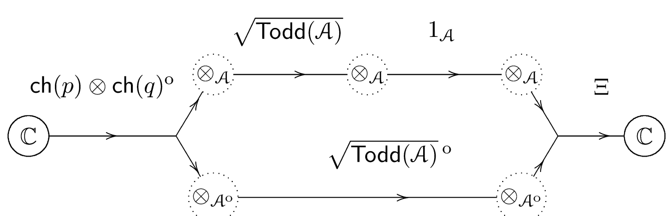
The left-hand side of equation (4.8) is given by $(p,q)_{\mathsf{K}} = (p \otimes q^{\mathrm{o}}) \otimes_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{o}}} \Delta$. Let us denote

$$\Theta = \big(\mathsf{ch}(p) \otimes_{\mathcal{A}} \sqrt{\mathsf{Todd}(\mathcal{A})}\big) \otimes \big(\mathsf{ch}(q)^{\mathrm{o}} \otimes_{\mathcal{A}} \sqrt{\mathsf{Todd}(\mathcal{A})}^{\mathrm{o}}\big).$$

Then the right-hand side of equation (4.8) is given by

$\big(\mathsf{ch}(p) \otimes_{\mathcal{A}} \sqrt{\mathsf{Todd}(\mathcal{A})}\,, \mathsf{ch}(q) \otimes_{\mathcal{A}} \sqrt{\mathsf{Todd}(\mathcal{A})}\,\big)_{\mathsf{H}} \; =$ 

$=$ 

$=$ 

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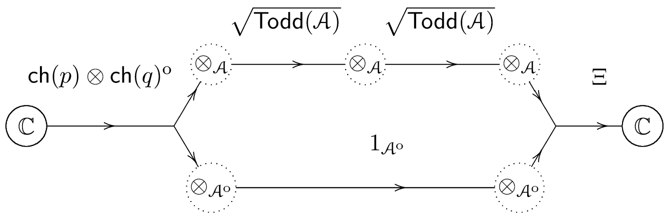

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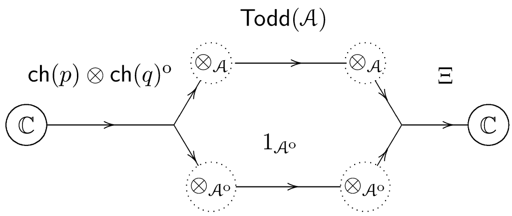


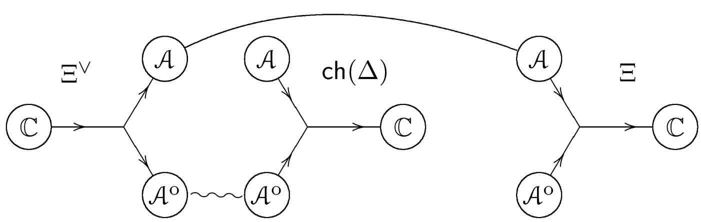
$$= \big(\mathsf{ch}(p) \otimes \mathsf{ch}(q)^{\mathrm{o}}\big) \otimes_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{o}}} \big(\mathsf{Todd}(\mathcal{A}) \otimes_{\mathcal{A}} \Xi\big) \tag{4.9}$$

The step from the third to the fourth line of equation (4.9) uses symmetry of the fundamental class Ξ.

We now use the definition of the Todd class to compute

$$\mathsf{Todd}(\mathcal{A}) \otimes_{\mathcal{A}} \Xi \;=\; \big(\Xi^{\vee} \otimes_{\mathcal{A}^{\mathrm{o}}} \mathsf{ch}(\Delta)\big) \otimes_{\mathcal{A}} \Xi$$

=



$$= \big(\Xi^{\vee} \otimes_{\mathcal{A}} \Xi\big) \otimes_{\mathcal{A}^{\mathrm{o}}} \mathsf{ch}(\Delta) \;=\; \mathsf{ch}(\Delta) \tag{4.10}$$

Inserting equation (4.10) into equation (4.9), we arrive at

$$\begin{aligned}
&\big(\mathsf{ch}(p) \otimes_{\mathcal{A}} \sqrt{\mathsf{Todd}(\mathcal{A})}, \mathsf{ch}(q) \otimes_{\mathcal{A}} \sqrt{\mathsf{Todd}(\mathcal{A})}\;\big)_{\mathsf{H}} \\
&\quad = \big(\mathsf{ch}(p) \otimes \mathsf{ch}(q^{\mathrm{o}})\big) \otimes_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{o}}} \mathsf{ch}(\Delta) = \mathsf{ch}\big((p \otimes q^{\mathrm{o}}) \otimes_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{o}}} \Delta\big) \\
&\quad = \mathsf{ch}\big((p,q)_{\mathsf{K}}\big).
\end{aligned}$$

The result now follows from the fact that the Chern character $\mathbb{Z} = \mathsf{KK}_0(\mathbb{C},\mathbb{C}) \xrightarrow{\mathsf{ch}} \mathsf{HL}_0(\mathbb{C},\mathbb{C}) = \mathbb{C}$ is injective. $\square$

This result has the following physical consequence, in which the K-orientation requirement provides the natural generalization of the Freed–Witten anomaly cancellation condition [23] to our non-commutative settings.

Corollary 4.5. *Let $\mathcal{A}, \mathcal{D}$ be non-commutative D-branes such that $\mathcal{A}$ satisfies the hypotheses of Theorem* 4.4*, and with given* K*-oriented morphism $f: \mathcal{A} \to \mathcal{D}$ and Chan–Paton bundle $\xi \in \mathsf{K}_\bullet(\mathcal{D})$. Then there is a non-commutative analogue of the Minasian–Moore formula for the Ramond–Ramond charge, valued in $\mathsf{HL}_\bullet(\mathcal{A})$ and given by*

$$\mathsf{Q}(\mathcal{D}, \xi) = \mathsf{ch}\big(f_!(\xi)\big) \otimes_{\mathcal{A}} \sqrt{\mathsf{Todd}(\mathcal{A})}.$$

Remark 4.6. If $\mathcal{D}$ is also a PD algebra, then by using the Grothendieck–Riemann–Roch formulas of Section 3.4 we may write the non-commutative charge vector in the form

$$\mathsf{Q}(\mathcal{D}, \xi) = \mathsf{ch}(\xi) \otimes_{\mathcal{D}} \mathsf{Todd}(\mathcal{D}) \otimes_{\mathcal{D}} (f*) \otimes_{\mathcal{A}} \sqrt{\mathsf{Todd}(\mathcal{A})}^{\,-1}.$$

If the HL-theory Gysin class $f* \in \mathsf{HL}(\mathcal{D}, \mathcal{A})$ induced by the morphism $f : \mathcal{A} \to \mathcal{D}$ obeys

$$(f*) \otimes_{\mathcal{A}} \sqrt{\mathsf{Todd}(\mathcal{A})}^{\,-1} = \Lambda \otimes_{\mathcal{D}} (f*)$$

for some element $\Lambda \in \mathsf{HL}(\mathcal{D}, \mathcal{D})$, then one has a non-commutative version of the Wess–Zumino class (4.2) valued in $\mathsf{HL}_\bullet(\mathcal{D})$, intrinsic to the D-brane $(\mathcal{D}, \xi)$ itself, and given by

$$\mathsf{D}_{\rm WZ}(\mathcal{D}, \xi) = \mathsf{ch}(\xi) \otimes_{\mathcal{D}} \mathsf{Todd}(\mathcal{D}) \otimes_{\mathcal{D}} \Lambda.$$

This formula enables one to derive a criterion under which the non-commutative charge vector is covariant under T-duality transformations. Suppose that the D-branes $(\mathcal{D}, \xi)$ are $(\mathcal{D}', \xi')$ are KK-equivalent, where the equivalence is implemented by inverse classes $\alpha \in \mathsf{KK}(\mathcal{D}, \mathcal{D}')$ and $\alpha^{-1} \in \mathsf{KK}(\mathcal{D}', \mathcal{D})$ with $\xi' = \xi \otimes_{\mathcal{D}} \alpha$. Then the Todd classes are related by [12, Theorem 7.4]

$$\mathsf{Todd}(\mathcal{D}') = \mathsf{ch}(\alpha)^{-1} \otimes_{\mathcal{D}} \mathsf{Todd}(\mathcal{D}) \otimes_{\mathcal{D}} \mathsf{ch}(\alpha). \tag{4.11}$$

If

$$\Lambda' = \mathsf{ch}(\alpha)^{-1} \otimes_{\mathcal{D}} \Lambda \otimes_{\mathcal{D}} \mathsf{ch}(\alpha),$$

then the Wess–Zumino classes are related covariantly as

$$\mathsf{D}_{\rm WZ}(\mathcal{D}', \xi') = \mathsf{D}_{\rm WZ}(\mathcal{D}, \xi) \otimes_{\mathcal{D}} \mathsf{ch}(\alpha).$$

Suppose that in an analogous way $(\mathcal{D}', \xi')$ is KK-equivalent to $(\mathcal{D}'', \xi'')$, where the equivalence is implemented by inverse classes $\alpha' \in \mathsf{KK}(\mathcal{D}', \mathcal{D}'')$

and $\alpha'^{-1} \in \mathsf{KK}(\mathcal{D}'', \mathcal{D}')$ with $\xi'' = \xi' \otimes_{\mathcal{D}'} \alpha'$, such that $\mathcal{D}$, $\mathcal{D}''$ are Morita equivalent with associated KK-equivalence $\alpha \otimes_{\mathcal{D}'} \alpha'$. Then by multiplicativity of the Chern character, one has

$$\mathsf{D}_{\mathrm{WZ}}(\mathcal{D}'', \xi'') = \mathsf{D}_{\mathrm{WZ}}(\mathcal{D}, \xi) \otimes_{\mathcal{D}} \mathsf{ch}(\alpha \otimes_{\mathcal{D}'} \alpha'),$$

and so the Wess–Zumino classes are related by the image of this KK-equivalence in

$$\mathsf{HL}_\bullet(\mathcal{D}'') \cong \mathsf{HL}_\bullet(\mathcal{D}).$$

These formulas all express the T-duality invariance of the non-commutative D-brane charge formula. The stated conditions above presumably reflect the need for addition of Myers terms to the Wess–Zumino class through appropriate modification of the morphism $f : \mathcal{A} \to \mathcal{D}$.

4.4 Twisted D-branes

A nice application of this formalism is to the example of D-branes in a compact, oriented even-dimensional manifold X with background H-flux. Let

$$\mathcal{A}_H := \mathsf{CT}(X, H) = C_0(X, \mathcal{E}_H)$$

be the C^*-algebra of sections of a locally trivial bundle of compact operators $\mathcal{E}_H \to X$ with Dixmier–Douady invariant $[H] \in \mathsf{H}^3(X, \mathbb{Z})$, whose image in real cohomology is represented by a closed three-form H on X. The opposite algebra is $\mathcal{A}_H^{\mathrm{o}} = \mathsf{CT}(X, -H)$. Recall that

$$\mathsf{CT}(X, H) = C_0(P_H, \mathcal{K})^{\mathrm{PU}}$$

consists of PU-equivariant $\mathcal{K}$-valued continuous maps from a principal PU-bundle $P_H \to X$ with Dixmier–Douady class $[H]$, where the projective unitary group $\mathrm{PU} = \mathrm{PU}(\mathcal{H})$ of a fixed separable Hilbert space $\mathcal{H}$ acts by the adjoint action on the algebra of compact operators $\mathcal{K} = \mathcal{K}(\mathcal{H})$. Let $\mathcal{L}^1_{P_H} = P_H \times_{\mathrm{PU}} \mathcal{L}^1$ be the algebra bundle of trace class operators $\mathcal{L}^1 = \mathcal{L}^1(\mathcal{H})$, where PU acts by the adjoint action on $\mathcal{L}^1$. Let $\mathcal{A}_H^\infty := \mathsf{CT}^\infty(X, H) = C^\infty(X, \mathcal{L}^1_{P_H})$ denote the locally convex $*$-algebra of smooth sections, which is a smooth subalgebra of $\mathcal{A}_H$. Then the inclusion map $\iota : \mathsf{CT}^\infty(X, H) \hookrightarrow$

$\mathsf{CT}(X,H)$ induces an isomorphism on K-theory [35]

$$\iota^! : \mathsf{K}_\bullet\big(\mathsf{CT}^\infty(X,H)\big) \xrightarrow{\approx} \mathsf{K}_\bullet\big(\mathsf{CT}(X,H)\big).$$

Upon choosing a bundle Gerbe connection [41], there are natural isomorphisms [10]

$$\mathsf{K}_\bullet\big(\mathsf{CT}^\infty(X,H)\big) \cong \mathsf{K}^\bullet(X,H)$$

where the right-hand side is the twisted K-theory of X expressed in terms of projective Hilbert bundles with fixed reduction of structure group to unitaries of the form $1+$ trace class operators. The same data also define an isomorphism [35]

$$\mathsf{HP}_\bullet\big(\mathsf{CT}^\infty(X,H)\big) \cong \mathsf{H}^\bullet(X,H),$$

where the right-hand side denotes the periodized twisted de Rham cohomology

$$\mathsf{H}^\bullet(X,H) = \mathsf{H}^\bullet\big(\Omega^\#(X), \mathrm{d} - H\wedge\big).$$

The canonical Connes–Chern character

$$\mathsf{ch} : \mathsf{K}_\bullet\big(\mathsf{CT}^\infty(X,H)\big) \longrightarrow \mathsf{HP}_\bullet\big(\mathsf{CT}^\infty(X,H)\big)$$

can be expressed in terms of differential forms on X [10, 34, 35], leading to the twisted Chern character

$$\mathsf{ch}_H : \mathsf{K}^\bullet(X,H) \longrightarrow \mathsf{H}^\bullet(X,H).$$

This can be viewed as the Chern–Weil representative of the Connes–Chern character on twisted K-theory.

The algebra $\mathcal{A}_H$ is not generally a PD algebra, but it is a member of the class $\underline{\mathfrak{D}}$ and a dual algebra $\tilde{\mathcal{A}}_H$ may be constructed as follows [12, 52]. Let $\mathsf{Cliff}(X)$ be the Clifford algebra bundle of the cotangent bundle of X with respect to a chosen Riemannian metric. Its Dixmier–Douady invariant is the third integral Stiefel–Whitney class $\mathsf{W}_3(X) \in \mathsf{H}^3(X,\mathbb{Z})$, which is the topological obstruction to the existence of a spin^c structure on X. The Dirac operator class $[\not{D}]$, constructed in the usual fashion from the Riemannian metric and the Levi–Civita connection on TX, naturally lives in $\mathsf{KK}(C_0(X,\mathsf{Cliff}(X)),\mathbb{C})$ and gives a fundamental class $\not{D}\otimes\not{D}$. Kasparov product with $\not{D}\otimes\not{D}$ establishes Poincaré duality between the algebras $C(X)$ and $C_0(X,\mathsf{Cliff}(X))$. Set $\tilde{\mathcal{A}}_H := \mathsf{CT}(X,\mathsf{W}_3(X)-H) \cong C_0(X,\mathcal{E}_{-H}\otimes$

$\mathsf{Cliff}(X)$). Let $p_i : X \times X \to X$, $i = 1, 2$, be the projection onto the ith factor, and $\delta : X \to X \times X$ the diagonal map. Then

$$\mathcal{A}_H \otimes \tilde{\mathcal{A}}_H \cong \mathsf{CT}\big(X \times X, p_1^*(H) + p_2^*(\mathsf{W}_3(X) - H)\big)$$

with $\delta^*(\mathcal{A}_H \otimes \tilde{\mathcal{A}}_H) \cong \mathsf{CT}(X, \mathsf{W}_3(X)) \otimes \mathcal{K}$, which is Morita equivalent to C_0 $(X, \mathsf{Cliff}(X))$. It follows that a fundamental class for the pair $(\mathcal{A}_H, \tilde{\mathcal{A}}_H)$ is

$$\Delta = \big[\delta^*\big]_{\mathsf{KK}} \otimes_{\mathsf{CT}(X, \mathsf{W}_3(X))} \not{D}.$$

When X is spin^c, one has $\mathsf{W}_3(X) = 0$ and the algebra $C_0(X, \mathsf{Cliff}(X))$ is Morita equivalent to $C(X)$ [44]. Moreover, in this case, $\tilde{\mathcal{A}}_H = \mathcal{A}_H^{\mathrm{o}}$ and the restriction of $\mathcal{A}_H \otimes \mathcal{A}_H^{\mathrm{o}}$ to the diagonal is stably isomorphic to $C(X)$.

Since $\mathsf{H}^\bullet(X, \mathsf{W}_3(X)) \cong \mathsf{H}^\bullet(X, \mathbb{R})$, a fundamental cyclic cohomology class Ξ for $\mathcal{A}_H$ is provided as usual by the homology orientation cycle $[X]$. In the spin case, the Todd class $\mathsf{Todd}(\mathcal{A}_H)$ is given by the Atiyah–Hirzebruch class $\widehat{A}(X)$ of the manifold X and the modified Chern character reduces to

$$\mathsf{Ch}_H(\xi) = \mathsf{ch}_H(\xi) \wedge \sqrt{\widehat{A}(X)}$$

for $\xi \in \mathsf{K}^\bullet(X, H)$. In general, the modified Chern character gives an isometry between the natural bilinear pairings on twisted K-theory and cohomology which are defined as follows. The Grothendieck group of the category of $\mathsf{Cliff}(X)$-modules is isomorphic to the twisted K-theory $\mathsf{K}^0(X, \mathsf{W}_3(X)) \cong \mathsf{K}_0(C_0(X, \mathsf{Cliff}(X)))$. Given any $\mathsf{Cliff}(X)$-module $\mathcal{C}$ over X and choosing a Clifford connection on $\mathcal{C}$, there is a Dirac operator $\not{D}_{\mathcal{C}}$ acting on $C_0(X, \mathcal{C})$. The index map $[\mathcal{C}] \mapsto \mathrm{index}(\not{D}_{\mathcal{C}})$ defines a homomorphism $\mathsf{K}^0(X, \mathsf{W}_3(X)) \to \mathbb{Z}$. This map can be computed via the standard Atiyah–Singer index theorem [6, Theorem 4.3]

$$\mathrm{index}(\not{D}_{\mathcal{C}}) = \int_X \mathsf{ch}(\mathcal{C}) \wedge \widehat{A}(X).$$

The tensor product of twisted bundles then defines a bilinear Poincaré pairing

$$\mathsf{K}^\bullet(X, H) \otimes \mathsf{K}^\bullet\big(X, \mathsf{W}_3(X) - H\big) \longrightarrow \mathsf{K}^0\big(X, \mathsf{W}_3(X)\big) \xrightarrow{\mathrm{index}} \mathbb{Z}.$$

Using the Poincaré pairing (4.1) with the cup product on twisted cohomology

$$\mathsf{H}^\bullet(X, H) \otimes \mathsf{H}^\bullet\big(X, \mathsf{W}_3(X) - H\big) \longrightarrow \mathsf{H}^{\mathrm{even}}(X, \mathbb{R}),$$

the isometric pairing formula follows.

Let us now consider a D-brane (W, E, f) in X, where $f : W \hookrightarrow X$ is the inclusion of the oriented submanifold W, and assume for simplicity that X is spin (as is the case when X is the spacetime of Type II superstring theory). In this case, the constructions of Section 3.3 determine a canonical element $f! \in \mathsf{KK}_d(\mathsf{CT}(W, f^*H + W_3(N_X W)), \mathsf{CT}(X, H))$ where $d = \dim(X) - \dim(W) \mod 2$ (see [13] for details). Since X is spin and W is oriented, one has $\mathsf{W}_3(N_X W) = \mathsf{W}_3(W)$. For the D-brane algebra, we thus take

$$\mathcal{D} = \mathsf{CT}\big(W, f^*H + \mathsf{W}_3(W)\big),$$

and the Chan–Paton bundle E is generically an infinite-dimensional twisted bundle [10] whose class lives in the twisted K-theory $\mathsf{K}^0(W, f^*H + \mathsf{W}_3(W))$. There are two special cases of interest. When the brane worldvolume is spin^c, i.e., $\mathsf{W}_3(W) = 0$, the algebra $\mathcal{D}$ is just the restriction of $\mathcal{A}_H$ to the submanifold W. The Chan–Paton bundle on the brane in this case is an infinite-dimensional twisted bundle $E \in \mathsf{K}^0(W, f^*H)$. When H is a torsion element in $\mathsf{H}^3(X, \mathbb{Z})$, the algebra $\mathcal{D}$ is an Azumaya algebra, and the bundle E is of finite rank and represents 't Hooft flux corresponding to a finite number of D-branes wrapping W [31, 55]. (The torsion condition guarantees anomaly cancellation on spacetime-filling brane–antibrane pairs.) But generically it is of infinite rank, corresponding to an infinite number of wrapped branes. Alternatively, we can use honest Chan–Paton vector bundles $E \in \mathsf{K}^0(W)$ and (stably) commutative algebras on the D-branes by replacing the spin^c condition with the Freed–Witten anomaly cancellation condition [23]

$$f^*H + \mathsf{W}_3(W) = 0$$

in $\mathsf{H}^3(W, \mathbb{Z})$, which guarantees the absence of global worldsheet anomalies for Type II superstrings. In any case, we obtain the twisted D-brane charge vector

$$\mathsf{Q}_H(W, E) = \mathsf{ch}_H\big(f_!(E)\big) \wedge \sqrt{\widehat{A}(X)}$$

in $\mathsf{H}^\bullet(X, H)$. Only Freed–Witten anomaly-free D-branes have the same Wess–Zumino class (4.2) valued in ordinary cohomology $\mathsf{H}^\bullet(W, \mathbb{Q})$ as in the untwisted case.

4.5 Non-commutative D2-branes

D2-branes can wrap two-dimensional manifolds, which in the presence of a constant B-field are described by non-commutative Riemann surfaces. Let Γ_g be the fundamental group of a compact, oriented Riemann surface Σ_g of genus $g \geq 1$ with the presentation

$$\Gamma_g = \Big\{ U_j, V_j, j = 1, \ldots, g \Big| \prod_{j=1}^{g} [U_j, V_j] = 1 \Big\}.$$

Since $\mathsf{H}^2(\Gamma_g, U(1)) \cong \mathbb{R}/\mathbb{Z}$, for each $\theta \in [0,1)$, there is a unique multiplier σ_θ (a two-cocycle with values in $U(1)$) on Γ_g up to isomorphism, representing the holonomy of a non-trivial B-field on Σ_g. Let $\mathcal{A}_\theta := C^*_r(\Gamma_g, \sigma_\theta)$ be the C^*-algebra generated by the unitaries U_j and V_j with the relation

$$\prod_{j=1}^{g} [U_j, V_j] = \exp(2\pi\,\mathrm{i}\,\theta).$$

In [14, 32], the K-theory of the algebra $\mathcal{A}_\theta$ is computed as follows:

- $\mathsf{K}_0(\mathcal{A}_\theta) \cong \mathsf{K}^0(\Sigma_g) = \mathbb{Z}^2$, and if $\theta \notin \mathbb{Q}$ the algebras $C^*_r(\Gamma_g, \sigma_\theta)$ are distinguished by the image of the canonical trace $\mathrm{tr} : C^*_r(\Gamma_g, \sigma_\theta) \to \mathbb{C}$, induced by evaluation at the identity element of Γ_g, on K-theory as

$$\mathrm{tr}^{\,!}(n\, e_0 + m\, e_1) = n + m\theta,$$

  with the K-theory generators chosen as $e_0 = [1]$, the class of the identity element, and e_1 such that $\mathrm{tr}^{\,!}(e_1) = \theta$; and
- $\mathsf{K}_1(\mathcal{A}_\theta) \cong \mathsf{K}^1(\Sigma_g) = \mathbb{Z}^{2g}$, with U_j and V_j forming a basis for $\mathsf{K}_1(\mathcal{A}_\theta)$.

The algebra $\mathcal{A}_\theta$ is a PD algebra, and the inverse of the fundamental class of $\mathcal{A}_\theta$ is the Bott class

$$\Delta^\vee = e_0 \otimes e_1^{\mathrm{o}} - e_1 \otimes e_0^{\mathrm{o}} + \sum_{j=1}^{g} \left(U_j \otimes V_j^{\mathrm{o}} - V_j \otimes U_j^{\mathrm{o}} \right).$$

Let

$$\mu_\theta : \mathsf{K}^\bullet(\Sigma_g) \xrightarrow{\approx} \mathsf{K}_\bullet\big(C^*_r(\Gamma_g, \sigma_\theta)\big)$$

be the twisted version of the Kasparov isomorphism [14] for the Lie group $\mathrm{SO}_0(2,1) \supset \Gamma_g$ with $\Gamma_g \setminus \mathrm{SO}_0(2,1)/\mathrm{SO}(2) \cong \Sigma_g$, and let ν_θ be its analogue in periodic cyclic homology. Then there is a commutative diagram

$$\begin{array}{ccc} \mathsf{K}^\bullet(\Sigma_g) & \xrightarrow{\ \mu_\theta\ } & \mathsf{K}_\bullet(\mathcal{A}_\theta) \\ {\scriptstyle \mathsf{ch}}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{ch}_{\Gamma_g}} \\ \mathsf{H}^\bullet(\Sigma_g, \mathbb{Z}) & \xrightarrow[\ \nu_\theta\ ]{} & \mathsf{HL}_\bullet(\mathcal{A}_\theta) \end{array} \tag{4.12}$$

whose maps are all isomorphisms. Using the diagram (4.12) one shows that the Todd class $\mathsf{Todd}(\mathcal{A}_\theta)$ is determined by $\nu_\theta(\mathsf{Todd}(\Sigma_g))$.

The modified Chern character in this case is $\mathsf{Ch}_\theta : \mathsf{K}_\bullet(\mathcal{A}_\theta) \to \mathsf{HL}_\bullet(\mathcal{A}_\theta)$, where

$$\mathsf{Ch}_\theta(\zeta) = \nu_\theta\Big(\mathsf{ch}\big(\mu_\theta^{-1}(\zeta)\big) \smile \sqrt{\mathsf{Todd}(\Sigma_g)}\ \Big)$$

for $\zeta \in \mathsf{K}_\bullet(\mathcal{A}_\theta)$ gives an isometry between the natural bilinear pairings in K-theory and in cohomology, lifted from those of Σ_g via the diagram (4.12). This is the charge vector for spacetime-filling non-commutative D2-branes on the Riemann surface Σ_g. More generally, the charge vector for a D-brane $(\mathcal{D}, \xi)$, with K-oriented morphism $f : \mathcal{A}_\theta \to \mathcal{D}$ and Chan–Paton bundle $\xi \in \mathsf{K}_\bullet(\mathcal{D})$, is given by

$$\mathsf{Q}_\theta(\mathcal{D}, \xi) = \nu_\theta\Big(\mathsf{ch}\big(\mu_\theta^{-1} \circ f_!(\xi)\big) \smile \sqrt{\mathsf{Todd}(\Sigma_g)}\Big).$$

This example also furnishes an explicit computation of the Todd class for the non-commutative manifold $C^*(\Gamma_g)$. Since the Baum–Connes conjecture holds for Γ_g and Γ_g has the Dirac-dual Dirac property, it follows that the algebras $C(\Sigma_g)$ and $C^*(\Gamma_g)$ are KK-equivalent. Explicitly, $(C(\Sigma_g), C^*(\Gamma_g))$ is a PD pair whose natural inverse fundamental class $\Delta_{\Gamma_g}^\vee = [F\Gamma_g]$ is the class of the universal flat $C^*(\Gamma_g)$-bundle in $\mathsf{KK}(\mathbb{C}, C(\Sigma_g) \otimes C^*(\Gamma_g))$. The corresponding twisting of the class of the Dirac operator $\not{\partial}$ on Σ_g defines the Baum–Connes assembly map

$$\alpha = [F\Gamma_g] \otimes_{C(\Sigma_g)} \not{\partial}$$

in K-theory, which is an invertible element in $\mathsf{KK}(C(\Sigma_g), C^*(\Gamma_g))$. Let α^{-1} be the dual Dirac class in $\mathsf{KK}(C^*(\Gamma_g), C(\Sigma_g))$. Then the fundamental K-homology class $\Delta_{\Gamma_g} \in \mathsf{KK}(C(\Sigma_g) \otimes C^*(\Gamma_g), \mathbb{C})$ is constructed as

$$\Delta_{\Gamma_g} = \alpha^{-1} \otimes_{C(\Sigma_g)} (\not{\partial} \otimes \not{\partial}\,).$$

The natural choices of fundamental classes in cyclic cohomology are constructed as follows. Consider the usual fundamental homology class of Σ_g, viewed as an element $[\Sigma_g] \in \mathsf{HL}(C(\Sigma_g), \mathbb{C})$, and the canonical trace tr on $C^*(\Gamma_g)$, viewed as an element of $\mathsf{HL}(C^*(\Gamma_g), \mathbb{C})$. Then

$$\Xi_{\Gamma_g} = [\Sigma_g] \otimes [\,\mathrm{tr}\,]_{\mathsf{HL}}.$$

A short calculation then shows that, with these natural choices, the Todd class of the algebra $C^*(\Gamma_g)$ is given by

$$\mathsf{Todd}\big(C^*(\Gamma_g)\big) = \mathsf{ch}(\alpha)^{-1} \otimes_{C(\Sigma_g)} \mathsf{Todd}\big(C(\Sigma_g)\big) \otimes_{C(\Sigma_g)} \mathsf{ch}(\alpha), \qquad (4.13)$$

where

$$\begin{aligned}\mathsf{Todd}\big(C(\Sigma_g)\big) &= \big([\Sigma_g]\otimes[\Sigma_g]\big)\otimes_{C(\Sigma_g)}\big(\mathsf{Todd}(\Sigma_g)\otimes\mathsf{Todd}(\Sigma_g)\big)\\ &\in \mathsf{HL}\big(C(\Sigma_g),C(\Sigma_g)\big)\end{aligned}$$

is given by cup product with the cohomology class $\mathsf{Todd}(\Sigma_g)$ dual to $\mathsf{ch}(\not{\partial}\,)$. This is a special case of [12, Corollary 7.5] (see equation (4.11)). There is a canonical K-oriented morphism $c:\mathbb{C}\to C^*(\Gamma_g)$, and the composition product

$$[c]_{\mathsf{HL}}\otimes_{C^*(\Gamma_g)}\mathsf{Todd}\big(C^*(\Gamma_g)\big)\otimes_{C^*(\Gamma_g)}[\,\mathrm{tr}\,]_{\mathsf{HL}}\in\mathsf{HL}(\mathbb{C},\mathbb{C})=\mathbb{C}$$

is the Todd *genus* of $C^*(\Gamma_g)$ [12, Remark 7.13], which works out simply to $1-g$, the Todd genus of the Riemann surface Σ_g.

Remark 4.7. A similar argument computes the Todd genus for the reduced C^*-algebra $C^*_r(\Gamma)$, where Γ is any discrete group satisfying the Baum–Connes conjecture, having the Dirac-dual Dirac property, and whose classifying space $B\Gamma$ is a spin^c manifold. Then the Todd genus of $C^*_r(\Gamma)$ is equal to the Todd genus of $B\Gamma$. We will revisit these examples in the next section in conjunction with non-commutative correspondences.

5 Correspondences and open string T-duality

There are several different but equivalent definitions of KK-theory, each adapted to different applications. As we discuss below, the one that is best suited for T-duality in open string theory is due to Connes and Skandalis [18], Baum and Block [5] in the case of manifolds. In this final section, we will propose a new description of elements of KK-theory based on the constructions of Section 3. This description generalizes the Baum–Connes–Skandalis description of elements of KK-theory to the setting of generic separable C^*-algebras, and it gives a more precise meaning to the point of view that Kasparov bimodules generalize $*$-homomorphisms of C^*-algebras. In this sense, it is very closely related to Cuntz's picture of KK-theory. In the special case of the Kasparov groups $\mathsf{KK}(C_0(X),C_0(Y)\otimes\mathcal{A})$, where $\mathcal{A}$ is a unital separable C^*-algebra, the correspondence description of [18] has been extended by Block and Weinberger [9].

5.1 Geometric correspondences and KK-theory for manifolds

We will first recall the description of [18] for the KK-theory groups $\mathsf{KK}(X,Y):=\mathsf{KK}(C_0(X),C_0(Y))$ of smooth manifolds X,Y. In this picture, elements of $\mathsf{KK}_d(X,Y)$ are represented by *geometric correspondences*

$$\begin{tikzcd} & (Z,E) \arrow[dl, "f_X"'] \arrow[dr, "g^Y"] & \\ X & & Y \end{tikzcd} \tag{5.1}$$

where E is a complex vector bundle over the smooth manifold Z, and f_X, g^Y are continuous maps from Z to X and Y, respectively, such that f_X is proper, g^Y is smooth and K-oriented and $d = \dim(Z) - \dim(Y) \mod 2$. Elements of $\mathsf{KK}_d(X,Y)$ define homomorphisms in $\mathrm{Hom}(\mathsf{K}^\bullet(X), \mathsf{K}^{\bullet+d}(Y))$. To each correspondence (5.1) there corresponds a morphism of K-theory groups given by the assignment

$$\big(Z, E, f_X, g^Y\big) \longmapsto {g^Y}_!\big({f_X}^*(-) \otimes E\big), \tag{5.2}$$

implemented by a KK-theory element $[f_X]_{\mathsf{KK}} \otimes_{C_0(Z)} [[E]] \otimes_{C_0(Z)} (g^Y)!$. Here $[f_X]_{\mathsf{KK}} \in \mathsf{KK}(X,Z)$ is defined by the pullback morphism ${f_X}^*\colon C_0(X) \to C_0(Z)$, $[[E]] \in \mathsf{KK}(Z,Z)$ is the *bivariant* K-theory class defined by the vector bundle E [8, §24.5] and $(g^Y)!$ is defined by the K-orientation as in Section 3.3. This is reminiscent of the smooth analogue of the Fourier–Mukai transform, and as such is well-suited to describe T-duality, as we discuss below. It is a geometric presentation of the analytic index for families of elliptic operators on X parametrized by Y.

Two correspondences

$$\begin{tikzcd} & (Z_j,E_j) \arrow[dl, "f_j"'] \arrow[dr, "g_j"] & \\ X & & Y \end{tikzcd}$$

with $j = 1,2$ are said to be *cobordant* if there is a “correspondence with boundary”

$$\begin{tikzcd} & (Z,E) \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

such that

(1) $\partial Z = Z_1 \sqcup Z_2$;
(2) $E|_{Z_j} = E_j, j = 1, 2$;
(3) $f|_{Z_j} = f_j, j = 1, 2$ and $g|_{Z_j} = (-1)^{j+1}\, g_j,\ \ j = 1, 2,$ where the minus sign indicates the same map with the opposite K-orientation.

We denote the collection of all cobordism classes of correspondences between X and Y by $\Omega(X, Y)$. Then the assignment (5.2) descends to a well-defined surjection

$$\Omega_{d \bmod 2}(X, Y) \longrightarrow \mathsf{KK}_d(X, Y), \tag{5.3}$$

and hence every element of $\mathsf{KK}_d(X, Y)$ can be represented by a cobordism class of geometric correspondences between X and Y. The image under the map (5.3) of the cobordism class represented by a correspondence (5.1) is denoted $[Z, E]$, with the summation rules

$$[Z, E_1 \oplus E_2] = [Z, E_1] + [Z, E_2] \quad \text{and} \quad [Z_1 \sqcup Z_2, E_1 \sqcup E_2] = [Z_1, E_1] + [Z_2, E_2].$$

Analytically, these are "cobordism classes" of families of elliptic operators on X that are parametrized by Y.

The special case $\mathsf{KK}(X, \mathrm{pt})$, with X compact, is the K-homology of X which can be represented analytically by "cobordism" classes of generalized elliptic operators on X. Since a K-oriented map $Z \to \mathrm{pt}$ is the same thing as a spin^c structure on Z, a correspondence in this case is simply a spin^c manifold Z equipped with a complex vector bundle E and a proper map to X. Since X is compact, properness just means Z is compact. Thus a correspondence in this case is just a geometric K-cycle (Z, E, f_X) over X [4]. The kernel of the map (5.3) is then the subgroup generated by the equivalence relation on geometric K-cycles given by Baum-Douglas vector bundle modification [4]. On the other hand, $\mathsf{KK}(\mathrm{pt}, Y)$ is just the K-theory of Y represented via an ABS-type construction using a spin^c structure on the bundle $TZ \oplus (g^Y)^*(TY)$ rather than on TZ. Both of these limiting cases naturally define the charge of a D-brane (Z, E), supported on Z with Chan–Paton bundle E, in spacetime X or Y.

In the general case, the kernel of the map (5.3) was described in [5]. Presumably, it could also be described using Jakob's approach [29, 30] by replacing actual vector bundles $E \to Z$ with K-theory classes, i.e., by considering instead the *virtual correspondences*

$$\begin{tikzcd} & (Z,\xi) \arrow[dl, "f_X"'] \arrow[dr, "g^Y"] & \\ X & & Y \end{tikzcd} \tag{5.4}$$

where now $\xi \in \mathsf{K}^\bullet(Z)$. Given a virtual correspondence (5.4), let $F \to Z$ be a smooth, real spin^c vector bundle of even rank. Let $1\!\!1_Z^{\mathbb{R}}$ denote the trivial real line bundle over Z. Let $\pi : S \to Z$ be the unit sphere bundle of $F \oplus 1\!\!1_Z^{\mathbb{R}}$. It is also a spin^c bundle with even-dimensional fibres which has a nowhere zero section $s : Z \to S$ defined by $z \mapsto (0_z, 1)$. Both π and s are proper maps endowed with a canonical K-orientation. Then the *vector bundle modification* of (5.4) is defined to be the virtual correspondence

$$\begin{tikzcd} & \big(S, s_!(\xi)\big) \arrow[dl, "f_X\circ\pi"'] \arrow[dr, "g^Y\circ\pi"] & \\ X & & Y. \end{tikzcd}$$

The map (5.3) is then made into an isomorphism by quotienting by the subgroup of cobordism classes of virtual correspondences under the equivalence relation generated by this generalized notion of vector bundle modification. This stabilization makes the assignment (5.2) into a well-defined natural transformation of bivariant theories which is an equivalence when X, Y are compact [5]. Note that the grading on $\mathsf{KK}_d(X, Y)$ in this case is given on homogeneous correspondences with $\xi \in \mathsf{K}^j(Z)$ by $d = \dim(Z) - \dim(Y) + j$ mod 2.

The composition product of KK-theory, which is notoriously difficult to define within the analytic framework, is particularly easy to describe in this setting. It is a morphism

$$\mathsf{KK}(X, M) \times \mathsf{KK}(M, Y) \longrightarrow \mathsf{KK}(X, Y)$$

described as the composition of correspondences, with some caveats that we explain below. Given two correspondences

$$
\begin{tikzcd}
 & (Z_1, E_1) \arrow[dl, "f_X"'] \arrow[dr, "g^M"] & & (Z_2, E_2) \arrow[dl, "f_M"'] \arrow[dr, "g^Y"] & \\
X & & M & & Y
\end{tikzcd}
$$

we define their *composition* to be the correspondence (5.1) with the fibred product $Z = Z_1 \times_M Z_2$, the bundle $E = E_1 \boxtimes E_2$ and the compositions $f_X : Z \to Z_1 \to X$, $g^Y : Z \to Z_2 \to Y$. The *caveat* is that in order for the space Z to be a manifold, f_M should be smooth and the maps g^M and f_M have to be transverse, i.e., for all $(z_1, z_2) \in Z$, one has

$$\mathrm{d}f_M(T_{z_2}Z_2) + \mathrm{d}g^M(T_{z_1}Z_1) = T_{f_M(z_2)}M \ .$$

Such choices can always be made by homotopy invariance of the pullback and Gysin maps, along with standard transversality theorems. The notation for the composition product is

$$[Z_1, E_1] \otimes_M [Z_2, E_2] = [Z, E].$$

It is manifestly associative. The unit element of the ring $\mathsf{KK}(X, X)$ is denoted $1_X = [\mathrm{id}_X]_{\mathsf{KK}}$.

Definition 5.1. Two manifolds X, Y are said to be KK-*equivalent* if there are elements

$$\alpha \in \mathsf{KK}(X, Y) \quad \text{and} \quad \beta \in \mathsf{KK}(Y, X)$$

such that $\alpha \otimes_Y \beta = 1_X \in \mathsf{KK}(X, X)$ and $\beta \otimes_X \alpha = 1_Y \in \mathsf{KK}(Y, Y)$.

Example 5.2. The smooth analogue of the *Fourier–Mukai transform* is a correspondence

$$
\begin{tikzcd}
 & \big(M \times \mathbb{T}^n \times \widehat{\mathbb{T}}^n, \mathcal{P}\big) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
M \times \mathbb{T}^n & & M \times \widehat{\mathbb{T}}^n
\end{tikzcd}
$$

where $\mathbb{T}^n$ is an n-dimensional torus, $\widehat{\mathbb{T}}^n$ is the dual torus and $\mathcal{P}$ is the Poincaré line bundle defined initially on $\mathbb{T}^n \times \widehat{\mathbb{T}}^n$ and then pulled back via the projection map to the product manifold $M \times \mathbb{T}^n \times \widehat{\mathbb{T}}^n$. It defines an element α in $\mathsf{KK}_n(M \times \mathbb{T}^n, M \times \widehat{\mathbb{T}}^n)$ which is a KK-equivalence. The element α is interpreted analytically as the families Dirac operator, and its inverse β is the "dual Dirac" or Bott element. Viewed in this way, it is a refinement of the usual smooth Fourier–Mukai transform. In open string theory, this is the statement that (topological) T-duality is naturally an invertible element of the KK-theory of the dual pair of spacetimes and is the starting point for a general axiomatic description of T-duality for C^*-algebras [12].

5.2 Non-commutative correspondences and KK-theory for C^*-algebras

We will now give a new description of KK-theory via "non-commutative correspondences", which are well suited for describing the composition product. Let $\mathcal{A}, \mathcal{B}$ be separable C^*-algebras. We will represent elements of $\mathsf{KK}(\mathcal{A}, \mathcal{B})$ by *non-commutative correspondences*

$$\begin{tikzcd}[column sep=small] & (\mathcal{C}, \xi) & \\ \mathcal{A} \arrow[ur, "f^{\mathcal{A}}"] & & \mathcal{B} \arrow[ul, "g_{\mathcal{B}}"'] \end{tikzcd} \tag{5.5}$$

where $\xi \in \mathsf{KK}(\mathcal{C}, \mathcal{C})$ with $\mathcal{C}$ a separable C^*-algebra, and $f^{\mathcal{A}}, g_{\mathcal{B}}$ are homomorphisms $\mathcal{A} \to \mathcal{C}$ and $\mathcal{B} \to \mathcal{C}$, respectively, such that $g_{\mathcal{B}}$ is K-oriented. To see that this data defines an element in $\mathsf{KK}(\mathcal{A}, \mathcal{B})$, note that $[f^{\mathcal{A}}]_{\mathsf{KK}} \in \mathsf{KK}(\mathcal{A}, \mathcal{C})$ and the Gysin element corresponding to $g_{\mathcal{B}}$ as defined in Section 3.3 is $g_{\mathcal{B}}! \in \mathsf{KK}(\mathcal{C}, \mathcal{B})$. Thus the composition product gives an assignment

$$\big(\mathcal{C}, \xi, f^{\mathcal{A}}, g_{\mathcal{B}}\big) \longmapsto \big[f^{\mathcal{A}}\big]_{\mathsf{KK}} \otimes_{\mathcal{C}} \xi \otimes_{\mathcal{C}} g_{\mathcal{B}}! \in \mathsf{KK}(\mathcal{A}, \mathcal{B}). \tag{5.6}$$

The associated element in $\mathrm{Hom}(\mathsf{K}_\bullet(\mathcal{A}), \mathsf{K}_\bullet(\mathcal{B}))$ is $g_{\mathcal{B}}{}^!({f^{\mathcal{A}}}_*(-) \otimes_{\mathcal{C}} \xi)$. Note the analogy with equation (5.2). In the commutative case $\mathcal{C} = C_0(Z)$ of equation (5.1), we represent $\mathsf{KK}_0(\mathcal{C}, \mathcal{C})$ as $\mathrm{End}(\mathsf{K}^\bullet(Z))$ and take ξ to be tensor product with the vector bundle E.

Proposition 5.3. *Given separable C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, any class in KK_d $(\mathcal{A}, \mathcal{B})$ comes from a non-commutative correspondence as above, in fact from one with trivial $\xi = 1_{\mathcal{C}}$.*

Proof. This is basically a restatement of a theorem of Cuntz, found in [8, Corollary 17.8.4]. It suffices to take $d = 0$, since one can compose with the K-oriented map $C_0(\mathbb{R}) \to \mathbb{C}$ which switches parity of degree. The theorem just cited asserts that given $x \in \mathsf{KK}(\mathcal{A}, \mathcal{B})$, we can write it in the form $x = f^*(j!)$ where $f \colon \mathcal{A} \to \mathcal{C}$, j is the inclusion of the ideal in a split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{B} \otimes \mathcal{K} \arrow[r, "j"] & \mathcal{C} \arrow[r, "q"'] & \mathcal{D} \arrow[l, bend right, "s"'] \arrow[r] & 0, \end{tikzcd}$$

and $j!$ is defined as in Example 3.6. Then x is defined by the correspondence

$$\begin{tikzcd} & \mathcal{C} & \\ \mathcal{A} \arrow[ur, "f"] & & \mathcal{B}, \arrow[ul, "j\circ\iota"'] \end{tikzcd}$$

with $\iota \colon \mathcal{B} \to \mathcal{B} \otimes \mathcal{K}$ the usual stabilization map. ☐

Remark 5.4. Proposition 5.3 cannot be used as a way of giving a new definition of KK-theory groups, as we assumed we already knew what a K-oriented map is in the course of proving it. We could get around this by starting only with KK classes associated to $*$-homomorphisms and to split exact sequences. The effect would then be to define KK as in Section 1.1 as the universal quotient of the additive stable homotopy category of separable C^*-algebras satisfying split exactness [8, Theorem 22.2.1]. (See also [36] for another variant of this using triangulated categories.)

In this setting, we would like to describe the composition product in KK-theory. It is a morphism

$$\mathsf{KK}(\mathcal{A}, \mathcal{D}) \times \mathsf{KK}(\mathcal{D}, \mathcal{B}) \longrightarrow \mathsf{KK}(\mathcal{A}, \mathcal{B})$$

which should be described as the composition of correspondences, subject to some conditions as in the case of manifolds. Given the correspondences

$$\begin{tikzcd} & \mathcal{C}_1 & & \mathcal{C}_2 & \\ \mathcal{A} \arrow[ur, "f_1"] & & \mathcal{D} \arrow[ul, "g_1"'] \arrow[ur, "f_2"] & & \mathcal{B} \arrow[ul, "g_2"'] \end{tikzcd}$$

we would like to define their *composition* as the correspondence

$$\begin{tikzcd} & & \mathcal{C} & & \\ & \mathcal{C}_1 \arrow[ur, dotted] & & \mathcal{C}_2 \arrow[ul, dotted] & \\ \mathcal{A} \arrow[ur, "f_1"'] \arrow[uurr, dotted, bend left, "f_{\mathcal{A}}"] & & \mathcal{D} \arrow[ul, "g_1"] \arrow[ur, "f_2"'] & & \mathcal{B} \arrow[ul, "g_2"] \arrow[uull, dotted, bend right, "g_{\mathcal{B}}"'] \end{tikzcd}$$

where $\mathcal{C}$ is something like a suitable pushout or colimit of the diagram

$$\begin{tikzcd} \mathcal{C}_1 & & \mathcal{C}_2\ . \\ & \mathcal{D} \arrow[ul, "g_1"] \arrow[ur, "f_2"'] & \end{tikzcd} \tag{5.7}$$

Part of the problem is that the notion of pushout for arbitrary diagrams (5.7) in the category of C^*-algebras is the amalgamated C^*-algebraic free product $\mathcal{C}_1 *_{\mathcal{D}} \mathcal{C}_2$, the universal C^*-algebra generated by $*$-representations of $\mathcal{C}_1$ and $\mathcal{C}_2$ on the same Hilbert space such that representations agree on $g_1(d)$ and $f_2(d)$ for $d \in \mathcal{D}$. It is not obvious that the natural map $\mathcal{C}_2 \to \mathcal{C}_1 *_{\mathcal{D}} \mathcal{C}_2$ should be K-oriented just because $g_1 \colon \mathcal{D} \to \mathcal{C}_1$ is K-oriented, nor is it obvious that the composition $g_{\mathcal{B}}$ of the maps g_2 and $\mathcal{C}_2 \to \mathcal{C}$ should be K-oriented. Thus the commutative case of Section 5.1 above provides a guide. In that case, we needed to use transversality to define the composition of geometric correspondences, and the correct algebra to take for $\mathcal{C}$ was an amalgamated *tensor* product $\mathcal{C}_1 \otimes_{\mathcal{D}} \mathcal{C}_2$ which corresponds to the fibred product of spaces.

A partial answer to this problem is discussed in [20, §8.5], which addresses the problem of composing quasihomomorphisms. This corresponds to the case where f_2 is the identity above, and g_1 and g_2 are split injections of ideals. Some other cases of interest are as follows. If g_1 is the identity, i.e., the first correspondence is simply a $*$-homomorphism $f_1 : \mathcal{A} \to \mathcal{D}$, then we can compose f_1 and f_2 to get a composition correspondence

$$\begin{tikzcd} & \mathcal{D} & \\ \mathcal{A} \arrow[ur, "f_2 \circ f_1"] & & \mathcal{B}\ . \arrow[ul, "g_2"'] \end{tikzcd}$$

Example 5.5. Let α be an action of $\mathbb{R}$ on a C^*-algebra $\mathcal{A}$. Connes' "Thom isomorphism" [16, 22] (see also [8, Theorem 19.3.6] and [20, Chapter 10]) defines a KK-equivalence between the suspension of $\mathcal{A}$, $\mathcal{A} \otimes C_0(\mathbb{R})$

and the crossed product $\mathcal{A} \rtimes_\alpha \mathbb{R}$. This may be realized by non-commutative correspondences associated to two Kasparov elements. One of them, in $\mathsf{KK}_1(\mathcal{A}, \mathcal{A} \rtimes_\alpha \mathbb{R})$, is described in [8, §19.3]. The inverse element, basically described in [20, Chapter 10], is the Kasparov element in $\mathsf{KK}_1(\mathcal{A} \rtimes_\alpha \mathbb{R}, \mathcal{A})$ associated to the extension

$$0 \longrightarrow \mathcal{A} \otimes \mathcal{K} \longrightarrow \mathcal{W} \longrightarrow \mathcal{A} \rtimes_\alpha \mathbb{R} \longrightarrow 0,$$

where $\mathcal{W}$ is Rieffel's "Weiner–Hopf extension algebra" [49]. Combined with Takai duality, this provides a primitive example of T-duality in the context of a non-commutative orbifold spacetime.

Example 5.6. Other non-commutative correspondences come from continuous trace C^*-algebras, discussed in Section 4.4 in the context of D-branes in a background H-flux. They have been studied in [46], and in the context of T-duality in [11, 33] and other papers in the series by the same authors. We recall here the simplest of these examples. Let

$$\begin{array}{ccc} \mathbb{T} & \longrightarrow & E \\ & & \big\downarrow \pi \\ & & M \end{array}$$

be a principal circle bundle and H a closed, integral three-form on E. Then there is a continuous trace C^*-algebra $\mathsf{CT}(E, H)$ with spectrum equal to E and Dixmier–Douady invariant equal to $[H] \in \mathsf{H}^3(E, \mathbb{Z})$.

Define $\widehat{E}$ to be the T-dual of E, which is another oriented $\mathbb{T}$-bundle over M

$$\begin{array}{ccc} \widehat{\mathbb{T}} & \longrightarrow & \widehat{E} \\ & & \big\downarrow \widehat{\pi} \\ & & M \end{array} \tag{5.8}$$

with first Chern class $c_1(\widehat{E}) = \pi_*[H]$, where $\pi_* : \mathsf{H}^k(E, \mathbb{Z}) \to \mathsf{H}^{k-1}(M, \mathbb{Z})$ denotes the pushforward maps on cohomology. The Gysin sequence for E enables one to define a T-dual H-flux $[\widehat{H}] \in H^3(\widehat{E}, \mathbb{Z})$ satisfying $c_1(E) = \widehat{\pi}_*[\widehat{H}]$, and such that $[H] = [\widehat{H}]$ in $\mathsf{H}^3(E \times_M \widehat{E}, \mathbb{Z})$ where $E \times_M \widehat{E}$ is the fibred product. Then there is a non-commutative correspondence

$$
\begin{tikzcd}
 & \big(\mathsf{CT}(E \times_M \widehat{E}), \xi\big) & \\
\mathsf{CT}(E,H) \arrow[ur, "f"] & & \mathsf{CT}\big(\widehat{E}, \widehat{H}\,\big) \arrow[ul, "g"']
\end{tikzcd}
$$

where ξ is an analogue of the Poincaré line bundle, which determines an invertible element

$$
\alpha \in \mathsf{KK}_1\left(\mathsf{CT}(E,H), \mathsf{CT}(\widehat{E},\widehat{H})\right),
$$

and hence a KK-equivalence [12, §6.2].

Thus we see that bivariant K-theory is a convenient setting for T-duality. But it also yields a potentially more refined version of T-duality, which was originally presented as an isomorphism between the K-theory groups of the spacetime $\mathcal{A}$ and its T-dual $\mathcal{B}$. In Example 5.6 above, $\mathcal{A} = \mathsf{CT}(E,H)$ and $\mathcal{B} = \mathsf{CT}(\,\widehat{E},\widehat{H}\,)$. On the other hand, when viewed as a non-commutative correspondence, we have observed above and in [12, §6.2] that T-duality gives a KK-equivalence $\alpha \in \mathsf{KK}_1(\mathcal{A},\mathcal{B})$. We have seen that such a KK-equivalence determines an isomorphism of K-theory groups in $\mathsf{Isom}(\mathsf{K}_\bullet(\mathcal{A}), \mathsf{K}_{\bullet+1}(\mathcal{B}))$. However, α may contain more refined information, as the universal coefficient theorem of Rosenberg and Schochet [50] states that there is a split short exact sequence of abelian groups given by

$$
0 \to \mathrm{Ext}_{\mathbb{Z}}\big(\mathsf{K}_{\bullet+1}(\mathcal{A}), \mathsf{K}_\bullet(\mathcal{B})\big) \to \mathsf{KK}_\bullet\big(\mathcal{A},\mathcal{B}\big) \to \mathrm{Hom}_{\mathbb{Z}}\big(\mathsf{K}_\bullet(\mathcal{A}), \mathsf{K}_\bullet(\mathcal{B})\big) \to 0. \tag{5.9}
$$

This holds for a large class of C^*-algebras $\mathcal{A},\mathcal{B}$ that includes $\mathsf{CT}(E,H)$ and its T-duals. In the exact sequence (5.9), $\mathsf{KK}_\bullet(\mathcal{A},\mathcal{B})$ denotes the $\mathbb{Z}_2$-graded group defined as the direct sum $\mathsf{KK}_\bullet(\mathcal{A},\mathcal{B}) = \mathsf{KK}_0(\mathcal{A},\mathcal{B}) \oplus \mathsf{KK}_1(\mathcal{A},\mathcal{B})$, and $\mathsf{K}_\bullet(\mathcal{A})$ similarly denotes $\mathbb{Z}_2$-graded groups. The first map (on the left) in the exact sequence has degree 1 and the second map has degree 0. The refined information when T-duality is viewed as a KK-equivalence α is contained in the group $\mathrm{Ext}_{\mathbb{Z}}(\mathsf{K}_{\bullet+1}(\mathcal{A}), \mathsf{K}_\bullet(\mathcal{B}))$.

Example 5.7. Let X be a compact space, let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{V}$ be a finitely generated projective module over $C(X,\mathcal{A}) \cong C(X) \otimes \mathcal{A}$. By a slight modification of the Swan–Serre theorem (noticed by Mishchenko and Fomenko), $\mathcal{V}$ can be identified with the module of sections of an $\mathcal{A}$-vector bundle over X. (This is defined like an ordinary vector bundle, except that $\mathcal{A}$ plays the role of the scalars and the transition functions of the bundle are required to be $\mathcal{A}$-linear.) The projective module $\mathcal{V}$ gives a class $[\mathcal{V}] \in \mathsf{K}_0(C(X) \otimes \mathcal{A}) = \mathsf{KK}(\mathbb{C}, C(X) \otimes \mathcal{A})$. If in addition X is a closed spin^c manifold, then $C(X)$ is a strong PD algebra, and so with a chosen fundamental class $\Delta \in \mathsf{KK}(C(X) \otimes C(X), \mathbb{C})$ the composition product

$$[\mathcal{V}] \otimes_{C(X)} \Delta \in \mathsf{KK}\big(C(X), \mathcal{A}\big)$$

is defined. This element can be viewed as coming from a very special kind of non-commutative correspondence, essentially like that of equation (5.1) but with the ordinary bundle E replaced by the module $\mathcal{V}$. We can also view this non-commutative correspondence as

$$\begin{tikzcd} & \big(C(X) \otimes \mathcal{A}\,,\, \mathcal{V}\big) & \\ C(X) \arrow[ur, "\mathrm{id}_{C(X)}\otimes 1"] & & \mathcal{A}\ , \arrow[ul, "1\otimes \mathrm{id}_{\mathcal{A}}"'] \end{tikzcd} \tag{5.10}$$

where the K-orientation of the map on the right comes from the spin^c structure on X. Because the algebra $C(X)$ is commutative, we can let $C(X)$ act on both sides on $\mathcal{V}$ and thus view $\mathcal{V}$ not just as a class $[\mathcal{V}] \in \mathsf{KK}(\mathbb{C}, C(X) \otimes \mathcal{A})$ but also as a class $[[\mathcal{V}]] \in \mathsf{KK}(C(X), C(X) \otimes \mathcal{A})$, as required for a non-commutative correspondence.

This situation arises frequently in connection with the Baum–Connes conjecture. Let Γ be a torsion-free discrete group such that there is a model for its classifying space $B\Gamma$ which is a compact spin^c manifold. Then there is a canonical flat bundle $\Upsilon := E\Gamma \times_\Gamma C^*_r(\Gamma)$ over $B\Gamma$ such that the space of all continuous sections $\mathcal{V} := C_0(B\Gamma, \Upsilon)$ is a finitely generated projective $C(B\Gamma) \otimes C^*_r(\Gamma)$-module. Then

$$\begin{tikzcd} & \big(C(B\Gamma) \otimes C^*_r(\Gamma)\,,\, [[\mathcal{V}]]\big) & \\ C(B\Gamma) \arrow[ur] & & C^*_r(\Gamma) \arrow[ul] \end{tikzcd} \tag{5.11}$$

is a non-commutative correspondence of the sort (5.10). If Γ has the (strong) Dirac-dual Dirac property, then the non-commutative correspondence (5.11) determines a KK-equivalence between the algebras $C(B\Gamma)$ and $C^*_r(\Gamma)$, i.e., the Baum–Connes conjecture is true. Groups Γ that satisfy these hypotheses include cocompact torsion-free discrete subgroups of connected Lie groups whose non-compact semisimple part is $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$.

Remark 5.8. If algebras $\mathcal{A}$ and $\mathcal{B}$ are KK-equivalent, then by the diagram calculus for KK-theory we can produce, by inflation, a KK-equivalence between $\mathcal{C}\otimes\mathcal{A}$ and $\mathcal{C}\otimes\mathcal{B}$ for any other algebra $\mathcal{C}$. Starting from Examples 5.2 and 5.7 above, one can thus deduce the parametrized Fourier–Mukai transform and the parametrized Baum–Connes conjecture as further examples of non-commutative correspondences.

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