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EIGENVALUE PROBLEMS INVOLVING THE FRACTIONAL $p(x)$-LAPLACIAN OPERATOR

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Abstract. In this paper, we study a nonlocal eigenvalue problem involving variable exponent growth conditions, on a bounded domain $\Omega \subset \mathbb{R}^n$. Using adequate variational techniques, mainly based on Ekeland's variational principle, we establish the existence of a continuous family of eigenvalues lying in a neighborhood at the right of the origin.

1. Introduction

A very interesting area of nonlinear analysis lies in the study of elliptic equations involving fractional operators. Recently, great attention has been focused on these problems, both for pure mathematical research and in view of concrete real-world applications. Indeed, this type of operator arises in a quite natural way in different contexts, such as the description of several physical phenomena, optimization, population dynamics, and mathematical finance. The fractional Laplacian operator $(-\Delta)^s$, $0 < s < 1$, also provides a simple model to describe some jump Lévy processes in probability theory (see [1, 4, 5, 6] and the references therein).

In last years, a large number of papers are written on fractional Sobolev spaces and nonlocal problems driven by this operator (see, for instance, [5, 6, 8, 20, 21, 22] for further details). Specifically, we refer to Di Nezza, Palatucci, and Valdinoci [8], for a full introduction to the study of the fractional Sobolev spaces and the

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fractional Laplacian operators.
On the other hand, attention has been paid to the study of partial differential equations involving the $p(x)$-Laplacian operators; see [9, 10, 12, 11, 14, 19] and the references therein.
So the natural question that arises is to see which result we will obtain, if we replace the $p(x)$-Laplacian operator by its fractional version (the fractional $p(x)$-Laplacian operator).
Currently, as far as we know, the only results for fractional Sobolev spaces with variable exponents and fractional $p(x)$-Laplacian operator are obtained by [2, 3, 7, 13, 24]. In particular, the authors generalized the last operator to the fractional case. Then, they introduced an appropriate functional space to study problems in which a fractional variable exponent operator is present.

Now, let us introduce the fractional Sobolev space with the variable exponent as it is defined in [7].
Let Ω be a smooth bounded open set in $\mathbb{R}^N$ and let $p : \overline{\Omega} \times \overline{\Omega} \longrightarrow]1, +\infty[$ be a continuous bounded function. We assume that

$$1 < p^- = \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) \leqslant p(x,y) \leqslant p^+ = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) < +\infty \tag{1.1}$$

and

$$p \text{ is symmetric, that is, } p(x,y) = p(y,x) \ \text{ for all } (x,y) \in \overline{\Omega} \times \overline{\Omega}. \tag{1.2}$$

We set

$$\bar{p}(x) = p(x,x) \qquad \text{for all } x \in \overline{\Omega}.$$

Throughout this paper, s is a fixed real number such that $0 < s < 1$.
We define the fractional Sobolev space with variable exponent via the Gagliardo approach as follows:

$$\begin{aligned} W &= W^{s,p(x,y)}(\Omega) \\ &= \left\{ u \in L^{\bar{p}(x)}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{sp(x,y)+N}}\, dxdy < +\infty, \text{ for some } \lambda > 0 \right\}, \end{aligned}$$

where $L^{\bar{p}(x)}(\Omega)$ is the Lebesgue space with variable exponent (see section 2).
The space $W^{s,p(x,y)}(\Omega)$ is a Banach space (see [13]) if it is equipped with the norm

$$\|u\|_W = \|u\|_{L^{\bar{p}(x)}(\Omega)} + [u]_{s,p(x,y)},$$

where $[.]_{s,p(x,y)}$ is a Gagliardo seminorm with variable exponent, which is defined by

$$[u]_{s,p(x,y)} = [u]_{s,p(x,y)}(\Omega) = inf \left\{ \lambda > 0 : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{sp(x,y)+N}}\, dxdy \leqslant 1 \right\}.$$

The space$(W, \|.\|_W)$ is separable reflexive ; see [3, Lemma 3.1].

The fractional $p(x)$-Laplacian operator is given by

$$(-\Delta_{p(x)})^s u(x) = p.v. \int_{\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}\, dy, \qquad \text{for all } x \in \Omega,$$

where $p.v.$ is a commonly used abbreviation in the principal value sense.

Remark 1.1. *Note that $(-\Delta_{p(x)})^s$ is a generalized operator of the fractional p-Laplacian operator $(-\Delta_p)^s$ (i.e., when $p(x,y) = p =$ constant) and is the fractional version of the $p(x)$-Laplacian operator $\Delta_{p(x)}u(x) = div\big(|\nabla u(x)|^{p(x)-2}u(x)\big)$, which is associated with the variable exponent Sobolev space.*

In this paper, we are concerned with the study of the eigenvalue problem,

$$(\mathcal{P}_s)\begin{cases} (-\Delta_{p(x)})^s u(x) + |u(x)|^{\bar{p}(x)-2}u(x) = \lambda |u(x)|^{r(x)-2}u(x) & in\ \Omega, \\ u = 0 & in\ \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is a smooth open and bounded set in $\mathbb{R}^N$ ($N \geqslant 3$), $\lambda > 0$ is a real number, $p : \overline{\Omega} \times \overline{\Omega} \longrightarrow]1, +\infty[$ is a continuous function satisfying (1.1) and (1.2) and $r : \overline{\Omega} \longrightarrow]1, +\infty[$ is a continuous function such that

$$1 < r^- = \min_{x\in\overline{\Omega}} r(x) \leqslant r(x) \leqslant r^+ = \max_{x\in\overline{\Omega}} r(x) < p^- \quad \text{for all } x \in \overline{\Omega}. \tag{1.3}$$

We will show that any $\lambda > 0$ sufficiently small is an eigenvalue of the above nonlocal nonhomogeneous problem. The proof relies on simple variational arguments based on Ekeland's variational principle.
Our main result generalizes the work of Mihăilescu and Rădulescu [16], in the fractional case. More precisely, we replace $\Delta_{p(x)}$, which is a local operator, by the nonlocal operator $(-\Delta_{p(x)})^s$.

In the context of eigenvalue, problems involving variable exponent represent a starting point in analyzing more complicated equations. A first contribution in this sense is the paper of X. L. Fan, Q. H. Zhang and D. Zhao [11], where the following eigenvalue problem has been considered,

$$(\mathcal{P}_1)\begin{cases} -div\big(|\nabla u(x)|^{p(x)-2}\nabla u(x)\big) = \lambda |u(x)|^{p(x)-2}u(x) & in \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geqslant 3$) is a bounded domain with smooth boundary $\partial\Omega$, the function$p : \overline{\Omega} \longrightarrow]1, +\infty[$ is continuous, and $\lambda > 0$ is a real number. The result obtained in [11] establishes the existence of infinitely many eigenvalues for problem $(\mathcal{P}_1)$ by using an argument based on the Ljusternik–Schnirelmann critical point theory. Denoting by Λ the set of all nonnegative eigenvalues, the authors showed that $sup\ \Lambda = +\infty$ and they pointed out that only under special conditions, which are somehow connected with a kind of monotony of the function $p(x)$, we have $inf\ \Lambda > 0$ (this is in contrast with the case when $p(x)$ is a constant; then, we always have $inf\ \Lambda > 0$).

Going further, another eigenvalue problem involving variable exponent growth conditions intensively studied is the following:

$$(\mathcal{P}_2)\begin{cases} -div\big(|\nabla u(x)|^{p(x)-2}\nabla u(x)\big) = \lambda|u(x)|^{q(x)-2}u(x) & in\ \Omega,\\ u=0 & on\ \partial\Omega,\end{cases}$$

where $p,q:\overline{\Omega}\longrightarrow]1,+\infty[$ are two continuous functions and $\lambda>0$ is a real number.
Note that when $p(x)\neq q(x)$, the competition between the growth rates involved in problem $(\mathcal{P}_2)$ is essential in describing the set of eigenvalues of this problem and we cite the following:

- In the case when $\min\limits_{x\in\overline{\Omega}} q(x) < \min\limits_{x\in\overline{\Omega}} p(x)$ and $q(x)$ has a subcritical growth, Mihăilescu and Rădulescu [16] used *Ekeland's variational principle* in order to prove the existence of a continuous family of eigenvalues which lies in a neighborhood of the origin. This result is later extended by Fan in [9].
- In the case when $\max\limits_{x\in\overline{\Omega}} p(x) < \min\limits_{x\in\overline{\Omega}} q(x)$ and $q(x)$ has a subcritical growth, a mountain pass argument, similar with that used by Fan and Zhang [10], can be applied in order to show that any $\lambda>0$ is an eigenvalue of problem $(\mathcal{P}_2)$.
- Finally, in the case when $\max\limits_{x\in\overline{\Omega}} q(x) < \min\limits_{x\in\overline{\Omega}} p(x)$, it can be proved that the energetic functional, which can be associated with the eigenvalue problem, has a nontrivial minimum for any positive λ large enough (see, [10]). Clearly, in this case, the result of Mihăilescu and Rădulescu [16] can be also applied. Consequently, in this situation there exist two positive constants λ^* and λ^{**} such that any $\lambda\in]0,\lambda^*[\cup]\lambda^{**},+\infty[$ is an eigenvalue of the problem.

In an appropriate context, we also point out the study of the eigenvalue problem,

$$(\mathcal{P}_3)\begin{cases} -div\big((|\nabla u(x)|^{p_1(x)-2}+|\nabla u(x)|^{p_2(x)-2})\nabla u(x)\big) = \lambda|u(x)|^{q(x)-2}u(x) & in\ \Omega,\\ u\ =\ 0 & on\ \partial\Omega,\end{cases}$$

where $p_1,p_2,q:\overline{\Omega}\longrightarrow]1,+\infty[$ are continuous functions satisfying

$$1<p_2(x)<\min_{x\in\overline{\Omega}} q(x)\leqslant \max_{x\in\overline{\Omega}} q(x) < p_1(x) < N \quad \text{for all } x\in\overline{\Omega}$$

and

$$\max_{x\in\overline{\Omega}} q(x) < \frac{Np_2(x)}{N-p_{2(x)}} \quad \text{for all } x\in\overline{\Omega}.$$

For this problem Mihăilescu and Rădulescu [15] proved the existence of two positive constants λ_0 and λ_1 with $\lambda_0\leqslant\lambda_1$ such that any $\lambda\in]\lambda_1,+\infty[$ is an eigenvalue of problem $(\mathcal{P}_3)$ while any $\lambda\in]0,\lambda_0[$ is not an eigenvalue of problem $(\mathcal{P}_3)$.

This paper is organized as follows. In section 2, we give some definitions and fundamental properties of the spaces $L^{q(x)}$ and W. In section 3, we introduce some important lemmas which show that the functional $\mathcal{J}_\lambda$ (see section 3) satisfies the geometrical conditions of the mountain pass theorem. Finally, using Ekeland's variational principle, we prove that the problem $(\mathcal{P}_s)$ has a continuous spectrum which concentrates around the origin.

2. Some preliminary results

In this section, we recall some necessary properties of variable exponent spaces. For more details, we refer the reader to [12, 14, 19], and the references therein. Consider the set

$$C_+(\overline{\Omega}) = \{q \in C(\overline{\Omega}) : q(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For all $q \in C_+(\overline{\Omega})$, we define

$$q^+ = \underset{x\in\overline{\Omega}}{sup}\ q(x) \ and\ q^- = \underset{x\in\overline{\Omega}}{inf}\ q(x).$$

For any $q \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space as

$$L^{q(x)}(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable} : \int_\Omega |u(x)|^{q(x)}dx < +\infty\right\}.$$

This vector space endowed with the *Luxemburg norm*, which is defined by

$$\|u\|_{L^{q(x)}(\Omega)} = inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{q(x)} dx \leqslant 1\right\},$$

is a separable reflexive Banach space.
Let $\hat{q} \in C_+(\overline{\Omega})$ be the conjugate exponent of q, that is, $\frac{1}{q(x)} + \frac{1}{\hat{q}(x)} = 1$. Then we have the following Hölder-type inequality]:

Lemma 2.1 (Hölder inequality)**.** *If $u \in L^{q(x)}(\Omega)$ and $v \in L^{\hat{q}(x)}(\Omega)$, then*

$$\left|\int_\Omega uvdx\right| \leqslant \left(\frac{1}{q^-} + \frac{1}{\hat{q}^-}\right)\|u\|_{L^{q(x)}(\Omega)}\|v\|_{L^{\hat{q}(x)}(\Omega)} \leqslant 2\|u\|_{L^{q(x)}(\Omega)}\|v\|_{L^{\hat{q}(x)}(\Omega)}.$$

A very important role in manipulating the generalized Lebesgue spaces with variable exponent is played by the modular of the $L^{q(x)}(\Omega)$ space, which defined by

$$\begin{aligned} \rho_{q(.)} : L^{q(x)}(\Omega) &\longrightarrow \mathbb{R} \\ u &\longrightarrow \rho_{q(.)}(u) = \textstyle\int_\Omega |u(x)|^{q(x)}dx. \end{aligned}$$

Proposition 2.1. *Let $u \in L^{q(x)}(\Omega)$; then we have*

- (i) $\|u\|_{L^{q(x)}(\Omega)} < 1(resp. = 1, > 1) \Leftrightarrow \rho_{q(.)}(u) < 1(resp.= 1, > 1)$,
- (ii) $\|u\|_{L^{q(x)}(\Omega)} < 1 \Rightarrow \|u\|^{q+}_{L^{q(x)}(\Omega)} \leqslant \rho_{q(.)}(u) \leqslant \|u\|^{q-}_{L^{q(x)}(\Omega)}$,
- (iii) $\|u\|_{L^{q(x)}(\Omega)} > 1 \Rightarrow \|u\|^{q-}_{L^{q(x)}(\Omega)} \leqslant \rho_{q(.)}(u) \leqslant \|u\|^{q+}_{L^{q(x)}(\Omega)}$.

Proposition 2.2. *If $u, u_k \in L^{q(x)}(\Omega)$ and $k \in \mathbb{N}$, then the following assertions are equivalent:*

- (i) $\lim_{k\to+\infty} \|u_k - u\|_{L^{q(x)}(\Omega)} = 0$,

(*ii*) $\lim_{k\to+\infty} \rho_{q(.)}(u_k - u) = 0$,
(*iii*) $u_k \longrightarrow u$ *in measure in* Ω *and* $\lim_{k\to+\infty} \rho_{q(.)}(u_k) = \rho_{q(.)}(u)$.

In [13], the authors introduced the variable exponent Sobolev fractional space as follows:

$$E = W^{s,q(x),p(x,y)}(\Omega)$$
$$= \left\{ u \in L^{q(x)}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{sp(x,y)+N}}\, dxdy < +\infty, \text{ for some } \lambda > 0 \right\},$$

where $q : \overline{\Omega} \longrightarrow]1,+\infty[$ is a continuous function, such that

$$1 < q^- = \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} q(x) \leqslant q(x) \leqslant q^+ = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} q(x) < +\infty.$$

We would like to mention that the continuous and compact embedding theorem is proved in [13] under the assumption $q(x) > \bar{p}(x) = p(x,x)$. Here, we give a slightly different version of compact embedding theorem assuming that $q(x) = \bar{p}(x)$, which can be obtained by following the same discussions in [13].

Theorem 2.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and let $s \in]0,1[$. Let $p : \overline{\Omega}\times\overline{\Omega} \longrightarrow]1,+\infty[$ be a continuous variable exponent with $sp(x,y) < N$ for all $(x,y) \in \overline{\Omega}\times\overline{\Omega}$. Let (1.1) and (1.2) be satisfied. Let $r : \overline{\Omega} \longrightarrow]1,+\infty[$ be a continuous variable exponent such that*

$$p_s^*(x) = \frac{N\bar{p}(x)}{N - s\bar{p}(x)} > r(x) \geqslant r^- = \min_{x\in\overline{\Omega}} r(x) > 1 \quad \textit{for all } x \in \overline{\Omega}.$$

Then, there exists a constant $C = C(N,s,p,r,\Omega) > 0$ such that, for any $u \in W$,

$$\|u\|_{L^{r(x)}(\Omega)} \leqslant C\|u\|_W$$

Thus, the space W is continuously embedded in $L^{r(x)}(\Omega)$ for any $r \in]1, p_s^*[$. Moreover, this embedding is compact.

Remark 2.1. *Let W_0 denote the closure of $C_0^\infty(\Omega)$ in W, that is,*

$$W_0 = \overline{C_0^\infty(\Omega)}^{||.||_W}.$$

(1) *Theorem 2.1 remains true if we replace W by W_0.*
(2) *Since $p_s^*(x) > \bar{p}(x) \geqslant p^- > 1$, then Theorem 2.1 implies that $[.]_{s,p(x,y)}$ is a norm on W_0, which is equivalent to the norm $\|.\|_W$. So $(W_0, [.]_{s,p(x,y)})$ is a Banach space (see, for instance, [24]).*

Definition 2.1. *Let $p : \overline{\Omega}\times\overline{\Omega} \longrightarrow]1,+\infty[$, be a continuous variable exponent and let $s \in]0,1[$. For any $u \in W$, we define the modular $\rho_{p(.,.)} : W \longrightarrow \mathbb{R}$, by*

$$\rho_{p(.,.)}(u) = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dxdy + \int_\Omega |u(x)|^{\bar{p}(x)} dx$$

and

$$\|u\|_{\rho_{p(.,.)}} = inf\left\{ \lambda > 0 : \rho_{p(.,.)}\left(\frac{u}{\lambda}\right) \leqslant 1\right.$$

Remark 2.2.

(1) *It is easy to see that* $\|.\|_{\rho_{p(.,.)}}$ *is a norm, which is equivalent to the norm* $\|.\|_W$.

(2) $\rho_{p(.,.)}$ *also check the results of Propositions 2.1 and 2.2.*

We could also get the following properties:

Lemma 2.2. (*see* [24, Lemma 2.1])
Let $p : \overline{\Omega}\times\overline{\Omega} \longrightarrow]1,+\infty[$, *be a continuous variable exponent and let* $s \in]0,1[$. *For any* $u \in W_0$, *we have*

(*i*) $1 \leqslant [u]_{s,p(x,y)} \Rightarrow [u]^{p^-}_{s,p(x,y)} \leqslant \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dxdy \leqslant [u]^{p^+}_{s,p(x,y)}$,

(*ii*) $[u]_{s,p(x,y)} \leqslant 1 \Rightarrow [u]^{p^+}_{s,p(x,y)} \leqslant \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dxdy \leqslant [u]^{p^-}_{s,p(x,y)}$.

Let denote by $\mathcal{L}$ the operator associated to the $(-\Delta_{p(x)})^s$ defined as

$$\mathcal{L} : W_0 \longrightarrow W_0^*$$

$$u \longrightarrow \mathcal{L}(u) : W_0 \longrightarrow \mathbb{R}$$

$$\varphi \longrightarrow <\mathcal{L}(u), \varphi>$$

such that

$$< \mathcal{L}(u), \varphi >= \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}}\, dxdy,$$

where W_0^* is the dual space of W_0.

Lemma 2.3. (*see*[3]). *Assume that assumptions* (1.1) *and* (1.2) *are satisfied and that*$0 < s < 1$. *Then, the following assertions hold:*

- $\mathcal{L}$ *is a bounded and strictly monotone operator.*
- $\mathcal{L}$ *is a mapping of type* (S_+), *that is, if* $u_k \rightharpoonup u$ *in* W_0 *and* $\limsup_{k\longrightarrow+\infty} < \mathcal{L}(u_k) - \mathcal{L}(u), u_k - u > \leqslant 0$, *then* $u_k \longrightarrow u$ *in* W_0.
- $\mathcal{L}$ *is a homeomorphism.*

3. Main results

Definition 3.1. *We say that* $u \in W_0$ *is a weak solution of problem* $(\mathcal{P}_s)$, *if, for all* $\varphi \in W_0$, *we have*

$$\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}\big(u(x)-u(y)\big)\big(\varphi(x)-\varphi(y)\big)}{|x-y|^{N+sp(x,y)}}\, dxdy$$

$$+\int_\Omega |u(x)|^{\bar{p}(x)-2}u(x)\varphi(x)dx - \lambda\int_\Omega |u(x)|^{r(x)-2}u(x)\varphi(x)dx = 0. \quad (3.1)$$

Moreover, we say that λ *is an eigenvalue of problem* $(\mathcal{P}_s)$, *if there exists* $u \in W_0 \setminus \{0\}$ *which satisfies* (3.1), *that is,* u *is the corresponding eigenfunction to* λ.

Let us consider the energy functional $\mathcal{J}_\lambda$ corresponding to the problem $(\mathcal{P}_s)$, defined by $\mathcal{J}_\lambda : W_0 \longrightarrow \mathbb{R}$

$$\mathcal{J}_\lambda(u) = \int_{\Omega\times\Omega} \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dxdy \;+\; \int_\Omega \frac{1}{\bar{p}(x)}|u(x)|^{\bar{p}(x)}dx$$

$$-\lambda \int_\Omega \frac{1}{r(x)}|u(x)|^{r(x)}dx$$

for any $\lambda > 0$.

3.1. **Some important lemmas.** Now, we introduce some important lemmas that show that the functional $\mathcal{J}_\lambda$ satisfies the geometrical conditions of the mountain pass theorem that are necessary to establish the proof of the existence result.

Lemma 3.1. *Let Ω be a smooth bounded open set in $\mathbb{R}^N$ and $s \in]0,1[$. Let $p : \overline{\Omega}\times\overline{\Omega} \longrightarrow]1,+\infty[$, be a continuous variable exponent satisfied* (1.1) *and* (1.2) *with $sp(x,y) < N$ for all $(x,y) \in \overline{\Omega}\times\overline{\Omega}$ and let $r : \overline{\Omega} \longrightarrow]1,+\infty[$ be a continuous variable exponent such that $1 < r(x) < p^-$ for all $x \in \overline{\Omega}$. Then,*

(1) $\mathcal{J}_\lambda$ is well defined,
(2) $\mathcal{J}_\lambda \in C^1(W_0,\mathbb{R})$ and for all $u,\varphi \in W_0$, its Gâteaux derivative is given by:

$$< \mathcal{J}'_\lambda(u), \varphi >= \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}}\, dxdy$$

$$+ \int_\Omega |u(x)|^{\bar{p}(x)-2}u(x)\varphi(x)dx - \lambda\int_\Omega |u(x)|^{r(x)-2}u(x)\varphi(x)dx.$$

Proof. (i) Let $u \in W_0$; then

$$\begin{aligned}
\mathcal{J}_\lambda(u) &= \int_{\Omega\times\Omega} \tfrac{1}{p(x,y)} \tfrac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dxdy \;+\; \int_\Omega \tfrac{1}{\bar{p}(x)}|u(x)|^{\bar{p}(x)}dx \\
&\quad - \lambda\int_\Omega \tfrac{1}{r(x)}|u(x)|^{r(x)}dx \\
&\leqslant \tfrac{1}{p^-}\left[\int_{\Omega\times\Omega} \tfrac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\, dxdy + \int_\Omega |u(x)|^{\bar{p}(x)}dx\right] \\
&\quad - \tfrac{\lambda}{r^+}\int_\Omega |u(x)|^{r(x)}dx \\
&= \tfrac{1}{p^-}\rho_{p(.,.)}(u) - \tfrac{\lambda}{r^+}\rho_{r(.)}(u)
\end{aligned}$$

By Proposition 2.1 and Remark 2.2-(ii) we get

$$\mathcal{J}_\lambda(u) \leqslant \frac{1}{p^-}\left[\|u\|_{W_0}^{p+} + \|u\|_{W_0}^{p-}\right] - \frac{\lambda}{r^+}\left[\|u\|^{r+}_{L^{r(x)}(\Omega)} + \|u\|^{r-}_{L^{r(x)}(\Omega)}\right].$$

Using Theorem 2.1, we obtain

$$\begin{aligned}
\mathcal{J}_\lambda(u) &\leqslant \tfrac{1}{p^-}\left[\|u\|_{W_0}^{p+}+\|u\|_{W_0}^{p-}\right]-\tfrac{\lambda}{r^+}\left[C^{r^+}\|u\|_{W_0}^{r+}+C^{r^-}\|u\|_{W_0}^{r-}\right]\\
&\leqslant \tfrac{1}{p^-}\left[\|u\|_{W_0}^{p+}+\|u\|_{W_0}^{p-}\right]-\tfrac{\lambda}{r^+}\ max\ \left\{C^{r^+},C^{r^-}\right\}\left[\|u\|_{W_0}^{r+}+\|u\|_{W_0}^{r-}\right]\\
&\leqslant \left(\tfrac{1}{p^-}-\tfrac{\lambda}{r^+}\ max\ \left\{C^{r^+},C^{r^-}\right\}\right)\left[\|u\|_{W_0}^{p+}+\|u\|_{W_0}^{r+}\right]<+\infty.
\end{aligned}$$

(ii)- **Existence of the Gâteaux derivative**. We define

$$\Psi(u)=\int_{\Omega\times\Omega}\frac{1}{p(x,y)}\frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\ dxdy\ ,\ \Phi(u)=\int_\Omega\frac{1}{\bar{p}(x)}|u(x)|^{\bar{p}(x)}dx$$

and

$$\Phi_\lambda(u)=\lambda\int_\Omega\frac{1}{r(x)}|u(x)|^{r(x)}dx.$$

Then

$$\mathcal{J}_\lambda(u)=\Psi(u)+\Phi(u)-\Phi_\lambda(u)\ and\ \mathcal{J}'_\lambda(u)=\Psi'(u)+\Phi'(u)-\Phi'_\lambda(u). \tag{3.2}$$

• For any $u,\varphi\in W_0$, we have

$$<\Psi'(u),\varphi>=\int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}}\ dxdy. \tag{3.3}$$

Indeed,

$$\begin{aligned}
<\Psi'(u),\varphi> &= \lim_{t\longrightarrow 0}\tfrac{\Psi(u+t\varphi)-\Psi(u)}{t}\\
&= \lim_{t\longrightarrow 0}\left\{\int_{\Omega\times\Omega}\tfrac{|(u(x)+t\varphi(x))-(u(y)+t\varphi(y))|^{p(x,y)}-|u(x)-u(y)|^{p(x,y)}}{tp(x,y)|x-y|^{N+sp(x,y)}}\ dxdy.\right\}
\end{aligned} \tag{3.4}$$

Let us consider $M:[0,1]\longrightarrow\mathbb{R}$

$$\alpha\longmapsto\frac{\big|(u(x)-u(y))+\alpha t(\varphi(x)-\varphi(y))\big|^{p(x,y)}}{tp(x,y)|x-y|^{N+sp(x,y)}}.$$

The function M is continuous on $[1,0]$ and differentiable on $]0,1[$. Then by the mean value theorem, there exists $\theta\in]0,1[$ such that

$$M'(\alpha)(\theta)=M(1)-M(0).$$

Then

$$\frac{\big|(u(x)-(u(y))+\theta t(\varphi(x)-\varphi(y))\big|^{p(x,y)-2}\big[(u(x)-u(y))+t\theta(\varphi(x)-\varphi(y))\big](\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}}$$

$$=S_t(u,\varphi)=\frac{|(u(x)-u(y))+t(\varphi(x)-\varphi(y))|^{p(x,y)}-|u(x)-u(y)|^{p(x,y)}}{tp(x,y)|x-y|^{N+sp(x,y)}}. \tag{3.5}$$

Combining (3.4) and (3.5), we get,

$$< \Psi'(u),\varphi >= \lim_{t\longrightarrow 0}\int_{\Omega\times\Omega} S_t(u,\varphi)\ dxdy.$$

Since $t,\theta\in[0.1]$, so $t\theta\leqslant 1$, which implies

$$S_t(u,\varphi)\leqslant \frac{\big|(u(x)-(u(y))+(\varphi(x)-\varphi(y))\big|^{p(x,y)-2}\big[(u(x)-u(y))+(\varphi(x)-\varphi(y))\big](\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}}.$$

On the other hand,

$$S_t(u,\varphi)\underset{t\to 0}{\longrightarrow}\frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp(x,y)}}.$$

Hence, by the dominated convergence theorem, we obtain (3.3).
By the same argument, we have

$$< \Phi'(u),\varphi >=\int_\Omega |u(x)|^{\bar{p}(x)-2}u(x)\varphi(x)\ dx \ and\ < \Phi'_\lambda(u),\varphi > \lambda\int_\Omega |u(x)|^{r(x)-2}u(x)\varphi(x)dx.$$

Then by relation (3.2), the result holds.

Continuity of the Gâteaux derivative of $\mathcal{J}_\lambda$. Assume that $u_k\longrightarrow u$ in W_0, and we show that $\Psi'(u_k)\longrightarrow\Psi'(u)$ in W_0^*. Indeed,

$$< \Psi'(u_k)-\Psi'(u),\varphi >=$$

$$\int_{\Omega\times\Omega}\frac{\big[|u_k(x)-u_k(y)|^{p(x,y)-2}(u_k(x)-u_k(y))-|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))\big]}{|x-y|^{N+sp(x,y)}}$$

$$\times(\varphi(x)-\varphi(y))\ dxdy$$

$$=\int_{\Omega\times\Omega}\left[\frac{|u_k(x)-u_k(y)|^{p(x,y)-2}(u_k(x)-u_k(y))}{|x-y|^{(\frac{N}{p(x,y)}+s)(p(x,y)-1)}}-\frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{(\frac{N}{p(x,y)}+s)(p(x,y)-1)}}\right]$$

$$\times\frac{(\varphi(x)-\varphi(y))}{|x-y|^{\frac{N}{p(x,y)}+s}}dxdy\ .$$

Let us set

$$F_k(x,y)=\frac{|u_k(x)-u_k(y)|^{p(x,y)-2}(u_k(x)-u_k(y))}{|x-y|^{(\frac{N}{p(x,y)}+s)(p(x,y)-1)}}\in L^{\hat{p}(x,y)}(\Omega\times\Omega),$$

$$F(x,y)=\frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{(\frac{N}{p(x,y)}+s)(p(x,y)-1)}}\in L^{\hat{p}(x,y)}(\Omega\times\Omega),$$

$$\overline{\varphi}(x,y)=\frac{\varphi(x)-\varphi(y)}{|x-y|^{\frac{N}{p(x,y)}+s}}\in L^{p(x,y)}(\Omega\times\Omega),$$

where $\frac{1}{p(x,y)}+\frac{1}{\hat{p}(x,y)}=1$.
Hence, by the Hölder inequality (see Lemma 2.1), we obtain

$$< \Psi'(u_k)-\Psi'(u),\varphi >\leqslant 2\|F_k-F\|_{L^{\hat{p}(x,y)}(\Omega\times\Omega)}\|\overline{\varphi}\|_{L^{p(x,y)}(\Omega\times\Omega)}.$$

Thus

$$\|\Psi'(u_k)-\Psi'(u)\|_{W_0^*}\leqslant 2\|F_k-F\|_{L^{\hat{p}(x,y)}(\Omega\times\Omega)}.$$

Now, let

$$v_k(x,y)=\frac{u_k(x)-u_k(y)}{|x-y|^{\frac{N}{p(x,y)}+s}}\in L^{p(x,y)}(\Omega\times\Omega)\ and\ v(x,y)=\frac{u(x)-u(y)}{|x-y|^{\frac{N}{p(x,y)}+s}}\in L^{p(x,y)}(\Omega\times\Omega).$$

Since $u_k \longrightarrow u$ in W_0. Then $v_k \longrightarrow v$ in $L^{p(x,y)}(\Omega\times\Omega)$.
Hence, for a subsequence of $(v_k)_{k\geqslant 0}$, we get

$$v_k(x,y)\longrightarrow v(x,y)\ a.e.\ in\ \Omega\times\Omega\ and\ \exists\, h\in L^{p(x,y)}(\Omega\times\Omega)\text{ such that }|v_k(x,y)|\leqslant h(x,y).$$

So we have

$$F_k(x,y)\longrightarrow F(x,y)\ a.e.\ in\ \Omega\times\Omega\ and\ |F_k(x,y)|=|v_k(x,y)|^{p(x,y)-1}\leqslant |h(x,y)|^{p(x,y)-1}.$$

Then, by the dominated convergence theorem, we deduce that

$$F_k\longrightarrow F\ in\ L^{\hat{p}(x,y)}(\Omega\times\Omega).$$

Consequently

$$\Psi'(u_k)\longrightarrow \Psi'(u)\ in\ W_0^*.$$

By the same argument, we show that

$$\Phi'(u_k)\longrightarrow \Phi'(u)\ in\ \left(L^{\bar{p}(x)}(\Omega)\right)^*\quad and\quad \Phi'_\lambda(u_k)\longrightarrow \Phi'_\lambda(u)\ in\ \left(L^{r(x)}(\Omega)\right)^*.$$

Then by relation (3.2), we deduce the continuity of $\mathcal{J}'_\lambda$. □

The following result shows that the functional $\mathcal{J}_\lambda$ satisfies the first geometrical condition of the mountain pass theorem;

Lemma 3.2. *Let Ω be a smooth bounded open set in $\mathbb{R}^N$ and let $s\in]0,1[$. Let $p:\overline{\Omega}\times\overline{\Omega}\longrightarrow]1,+\infty[$, be a continuous variable exponent satisfied (1.1) and (1.2) with $sp(x,y)<N$ for all $(x,y)\in\overline{\Omega}\times\overline{\Omega}$ and let $r:\overline{\Omega}\longrightarrow]1,+\infty[$ be a continuous variable exponent such that $1<r(x)<p^-$ for all $x\in\overline{\Omega}$. Then, there exists $\lambda^*>0$ such that, for any $\lambda\in]0,\lambda^*[$, there exist $R,a>0$ such that $\mathcal{J}_\lambda(u)\geqslant a>0$ for any $u\in W_0$ with $\|u\|_{W_0}=R$.*

Proof. Since $r(x)<p_s^*(x)$ for all $x\in\overline{\Omega}$, so by Remark 2.1-(i) W_0 is continuously embedded in $L^{r(x)}(\Omega)$. Then there exists a positive constant c_1 such that

$$\|u\|_{L^{r(x)}(\Omega)}\leqslant c_1\|u\|_{W_0}\qquad\text{for all } u\in W_0. \tag{3.6}$$

We fix $R\in]0,1[$ such that $R<\frac{1}{c_1}$. Then relation (3.6) implies

$$\|u\|_{L^{r(x)}(\Omega)}<1,\qquad\text{for all } u\in W_0\ with\ R=\|u\|_{W_0}.$$

By Proposition 2.1-(ii), we get

$$\int_\Omega |u(x)|^{r(x)}dx\leqslant \|u\|^{r^-}_{L^{r(x)}(\Omega)}\qquad\text{for all } u\in W_0\ with\ R=\|u\|_{W_0}. \tag{3.7}$$

Combining (3.6) and (3.7), we get

$$\int_\Omega |u(x)|^{r(x)}dx\leqslant c_1^{r^-}\|u\|^{r^-}_{W_0}\qquad\text{for all } u\in W_0\ with\ R=\|u\|_{W_0}. \tag{3.8}$$

Using the fact that $\|u\|_{W_0} < 1$ and (3.8), we deduce that, for any $u \in W_0$ with $R = \|u\|_{W_0}$, the following inequalities hold true:

$$\begin{aligned}
\mathcal{J}_\lambda(u) &\geqslant \frac{1}{p^+}\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dxdy + \frac{1}{p^+}\int_\Omega |u(x)|^{\bar{p}(x)}dx \\
&\quad - \frac{\lambda}{r^-}\int_\Omega |u(x)|^{r(x)}dx \\
&\geqslant \frac{1}{p^+}\left[\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dxdy + \int_\Omega |u(x)|^{\bar{p}(x)}dx\right] \\
&\quad - \frac{\lambda}{r^-}\int_\Omega |u(x)|^{r(x)}\,dx \\
&\geqslant \frac{1}{p^+}\|u\|_{W_0}^{p^+} - \frac{\lambda}{r^-}c_1^{r^-}\|u\|_{W_0}^{r^-} \\
&\geqslant \frac{1}{p^+}R^{p^+} - \frac{\lambda}{r^-}c_1^{r^-}R^{r^-} \\
&\geqslant R^{r^-}\left(\frac{1}{p^+}R^{p^+-r^-} - \frac{\lambda}{r^-}c_1^{r^-}\right).
\end{aligned} \tag{3.9}$$

By the inequality (3.9), we can choose λ^* in order to

$$\frac{1}{p^+}R^{p^+-r^-} - \frac{\lambda}{r^-}c_1^{r^-} > 0$$

Hence, if

$$\lambda^* = \frac{R^{p^+-r^-}}{2p^+}\cdot\frac{r^-}{c_1^{r^-}}, \tag{3.10}$$

then, for any $\lambda \in]0,\lambda^*[$ and any $u \in W_0$ with $\|u\|_{W_0} = R$, there exists $a = \frac{R^{p^+}}{2p^+} > 0$ such that

$$\mathcal{J}_\lambda(u) \geqslant a > 0,$$

which completes the proof. □

The following result shows that the functional $\mathcal{J}_\lambda$ satisfies the second geometrical condition of the mountain pass theorem;

Lemma 3.3. *Let Ω be a smooth bounded open set in $\mathbb{R}^N$ and $s \in]0,1[$. Let $p : \overline{\Omega}\times\overline{\Omega} \longrightarrow]1,+\infty[$, be a continuous variable exponent satisfied (1.1) and (1.2) with $sp(x,y) < N$ for all $(x,y) \in \overline{\Omega}\times\overline{\Omega}$ and let $r : \overline{\Omega} \longrightarrow]1,+\infty[$ be a continuous variable exponent such that $1 < r(x) < p^-$ for all $x \in \overline{\Omega}$. Then, there exists $\varphi \in W_0$ such that $\varphi \geqslant 0$, $\varphi \neq 0$ and $\mathcal{J}_\lambda(t\varphi) < 0$ for any t small enough.*

Proof. Assumption (1.3) implies that $r^- < p^-$. Let $\varepsilon > 0$ be such that $r^- + \varepsilon \leqslant p^-$. Since $r \in C(\overline{\Omega})$, then we can find an open set $\Omega_0 \subset \Omega$ such that

$$|r(x) - r^-| \leqslant \varepsilon \quad \text{for all } x \in \Omega_0.$$

Consequently,

$$r(x) \leqslant r^- + \varepsilon \leqslant p^- \quad \text{for all } x \in \Omega_0.$$

Let $\varphi \in C_0^\infty(\Omega)$ be such that $supp\ \varphi \subset \overline{\Omega}_0$, $\varphi(x) = 1$ for all $x \in \Omega_0$, and $0 \leqslant \varphi \leqslant 1$ in $\in \Omega$. Then using the above information for any $t \in]0,1[$, we have

$$\begin{aligned}
\mathcal{J}_\lambda(t\varphi) &= \int_{\Omega\times\Omega} \frac{t^{p(x,y)}}{p(x,y)} \frac{|\varphi(x)-\varphi(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dxdy \;+\; \int_\Omega \frac{t^{\bar{p}(x)}}{\bar{p}(x)}|\varphi(x)|^{\bar{p}(x)}dx \\
&\quad - \;\lambda \int_\Omega \frac{t^{r(x)}}{r(x)}|\varphi(x)|^{r(x)}dx \\
&\leqslant \frac{t^{p-}}{p^-}\left[\int_{\Omega\times\Omega} \frac{|\varphi(x)-\varphi(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dxdy + \int_\Omega |\varphi(x)|^{\bar{p}(x)}dx\right] \\
&\quad - \;\lambda \int_{\Omega_0} \frac{t^{r(x)}}{r(x)}|\varphi(x)|^{r(x)}dx \\
&\leqslant \frac{t^{p^-}}{p^-}\rho_{p(.,.)}(u) - \frac{\lambda}{r^+}t^{r^-+\varepsilon}\int_{\Omega_0}|\varphi(x)|^{r(x)}dx \\
&\leqslant t^{r^-+\varepsilon}\left[\frac{\rho_{p(.,.)}(\varphi)}{p^-}t^{p^--r^--\varepsilon} - \frac{\lambda}{r^+}\int_{\Omega_0}|\varphi(x)|^{r(x)}dx\right]
\end{aligned}$$

Thus

$$\mathcal{J}_\lambda(t\varphi) < 0 \quad \text{for any } t < \xi^{\frac{1}{p^--r^--\varepsilon}},$$

where

$$0 < \xi < min\left\{1, \frac{\frac{\lambda p^-}{r^+}\int_{\Omega_0}|\varphi(x)|^{r(x)}dx}{\rho_{p(.,.)}(\varphi)}\right\}$$

Finally, we point out that $\rho_{p(.,.)}(\varphi) > 0$ (this fact implies that $\varphi \neq 0$). Indeed, since $supp\ \varphi \subset \Omega_0 \subset \Omega$ and $0 \leqslant \varphi \leqslant 1$ in Ω, so we get

$$0 < \int_{\Omega_0} |\varphi(x)|^{r(x)}dx \leqslant \int_\Omega |\varphi(x)|^{r(x)}dx \leqslant \int_\Omega |\varphi(x)|^{r^-}dx. \tag{3.11}$$

On the other hand, since $1 < r^- < p_s^*(x)$ for all $x \in \overline{\Omega}$, then W_0 is continuously embedded in $L^{r^-}(\Omega)$, so there exists $c_2 > 0$ such that

$$\|\varphi\|_{L^{r^-}(\Omega)} \leqslant c_2\|\varphi\|_{W_0}. \tag{3.12}$$

Combining (3.11) and (3.12), we get

$$0 < \frac{1}{c_2}\|\varphi\|_{L^{r^-}(\Omega)} \leqslant \|\varphi\|_{W_0}.$$

This fact and Proposition 2.1 ((ii) or (iii)) imply that

$$\rho_{p(.,.)}(\varphi) > 0.$$

Lemma 3.3 is proved □

3.2. **Existence result.** Our main result is given by the following theorem.

Theorem 3.1. *Let Ω be a smooth bounded open set in $\mathbb{R}^N$ and let $s \in]0,1[$. Let $p : \overline{\Omega}\times\overline{\Omega} \longrightarrow]1,+\infty[$, be a continuous variable exponent satisfied (1.1) and (1.2) with $sp(x,y) < N$ for all $(x,y) \in \overline{\Omega}\times\overline{\Omega}$ and let $r : \overline{\Omega} \longrightarrow]1,+\infty[$ be a continuous variable exponent such that $1 < r(x) < p^-$ for all $x \in \overline{\Omega}$. Then there exists $\lambda^* > 0$ such that every $\lambda \in]0,\lambda^*[$ is an eigenvalue of problem $(\mathcal{P}_s)$.*

The proof of Theorem 3.1 is based on Ekeland's variational principle and the mountain pass theorem, and it is divided to two steps.

Proof of Theorem 3.1. **Step 1**: Let $\lambda^* > 0$ be defined as in (3.10) and let $\lambda \in]0, \lambda^*[$. By Lemma 3.2, it follows that

$$\underset{\partial B_R(0)}{inf}\ \mathcal{J}_\lambda > 0, \tag{3.13}$$

where $\partial B_R(0) = \big\{u \in B_R(0) : \|u\|_{W_0} = R\big\}$ and $B_R(0)$ is the ball centered at the origin and of radius R in W_0.
On the other hand, by Lemma 3.3, there exists $\varphi \in W_0$ such that $\mathcal{J}_\lambda(t\varphi) < 0$ for any t small enough. Moreover, by (3.9), for all $u \in B_R(0)$, we get

$$\mathcal{J}_\lambda(u) \geqslant \frac{1}{p^+}\|u\|_{W_0}^{p^+} - \frac{\lambda}{r^-}c_1^{r^-}\|u\|_{W_0}^{r^-}. \tag{3.14}$$

Then we have

$$-\infty < \bar{c} = \underset{\overline{B_R(0)}}{inf}\ \mathcal{J}_\lambda < 0. \tag{3.15}$$

Combining (3.13) and (3.15), then we can assume that

$$0 < \varepsilon < \underset{\partial B_R(0)}{inf}\ \mathcal{J}_\lambda - \underset{B_R(0)}{inf}\ \mathcal{J}_\lambda.$$

Applying Ekeland's variational principle to the functional
$\mathcal{J}_\lambda : \overline{B_R(0)} \longrightarrow \mathbb{R}$, we find $u_\varepsilon \in \overline{B_R(0)}$ such that

$$\begin{cases} \mathcal{J}_\lambda(u_\varepsilon) < \underset{\overline{B_R(0)}}{inf}\ \mathcal{J}_\lambda + \varepsilon, \\ \mathcal{J}_\lambda(u_\varepsilon) < \mathcal{J}_\lambda(u) + \varepsilon\|u - u_\varepsilon\|_{W_0} \quad \text{for all } u \neq u_\varepsilon. \end{cases} \tag{3.16}$$

So

$$\mathcal{J}_\lambda(u_\varepsilon) \leqslant \underset{\overline{B_R(0)}}{inf}\ \mathcal{J}_\lambda + \varepsilon \leqslant \underset{B_R(0)}{inf}\ \mathcal{J}_\lambda + \varepsilon < \underset{\partial B_R(0)}{inf}\ \mathcal{J}_\lambda.$$

It follows that $u_\varepsilon \in B_R(0)$.
Now, we consider $\mathcal{I}_\lambda^\varepsilon : \overline{B_R(0)} \longrightarrow \mathbb{R}$

$$u \longrightarrow \mathcal{J}_\lambda(u) + \varepsilon\|u - u_\varepsilon\|_{W_0}.$$

By (3.16), we get

$$\mathcal{I}_\lambda^\varepsilon(u_\varepsilon) = \mathcal{J}_\lambda(u) < \mathcal{I}_\lambda^\varepsilon(u) \qquad \text{for all } u \neq u_\varepsilon.$$

Thus u_ε is a minimum point of $\mathcal{I}_\lambda^\varepsilon$ on $\overline{B_R(0)}$. It follows that, for any $t > 0$ small enough and $v \in B_R(0)$,

$$\frac{\mathcal{I}_\lambda^\varepsilon(u_\varepsilon + tv) - \mathcal{I}_\lambda^\varepsilon(u_\varepsilon)}{t} \geqslant 0.$$

By this fact,we claim that

$$\frac{\mathcal{J}_\lambda(u_\varepsilon + tv) - \mathcal{J}_\lambda(u_\varepsilon)}{t} + \varepsilon\|v\|_{W_0} \geqslant 0.$$

When t tends to 0^+, we have that

$$< \mathcal{J}'_\lambda(u_\varepsilon), v > +\varepsilon\|v\|_{W_0} \geqslant 0.$$

This gives

$$\|\mathcal{J}_\lambda(u_\varepsilon)\|_{W_0^*} \leqslant \varepsilon. \tag{3.17}$$

Step 2 *(Palais–Smale condition).* From (3.17), we deduce that there exists a sequence $\{w_k\} \subset B_r(0)$ such that

$$\mathcal{J}_\lambda(w_k) \longrightarrow \bar{c} \qquad \textit{and} \qquad \mathcal{J}'_\lambda(w_k) \longrightarrow 0. \tag{3.18}$$

From (3.14) and (3.18), we have that $\{w_k\}$ is bounded in W_0. Thus there exists $w \in W_0$ such that $w_k \rightharpoonup w$ in W_0.
By (1.3), we have that $r(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$, so by Theorem 2.1 and Remark 2.1, we deduce that W_0 is compactly embedded in $L^{r(x)}(\Omega)$; then

$$w_k \longrightarrow w \text{ in } L^{r(x)}(\Omega). \tag{3.19}$$

Using Lemma 2.1, we have

$$\int_\Omega |w_k|^{r(x)-2} w_k (w_k - w)dx \leqslant 2\|w_k\|_{L^{r(x)}(\Omega)} \|w_k - w\|_{L^{r(x)}(\Omega)}.$$

So, by (3.19), we get

$$\lim_{k\longrightarrow+\infty} \int_\Omega |w_k|^{r(x)-2} w_k (w_k - w)dx = 0. \tag{3.20}$$

Since $\bar{p}(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$, by the same argument, we have

$$\lim_{k\longrightarrow+\infty} \int_\Omega |w_k|^{\bar{p}(x)-2} w_k (w_k - w)dx = 0. \tag{3.21}$$

On the other hand, from (3.18), we get

$$\lim_{k\longrightarrow+\infty} < \mathcal{J}'_\lambda(w_k), w_k - w >= 0.$$

Namely,

$$\lim_{k\longrightarrow+\infty} \Big\{ \int_{\Omega\times\Omega} \frac{|w_k(x) - w_k(y)|^{p(x,y)-2}\big(w_k(x) - w_k(y)\big)\big((w_k(x) - w_k(y)) - (w(x) - w(y))\big)}{|x-y|^{N+sp(x,y)}}\, dxdy$$
$$+ \int_\Omega |w_k(x)|^{\bar{p}(x)-2} w_k(x)\big(w_k(x) - w(x)\big)dx - \lambda \int_\Omega |w_k(x)|^{r(x)-2} w_k(x)\big(w_k(x) - w(x)\big)dx \Big\} = 0.$$

Hence, relations (3.20) and (3.21) yield

$$\lim_{k\longrightarrow+\infty} \int_{\Omega\times\Omega} \frac{|w_k(x) - w_k(y)|^{p(x,y)-2}\big(w_k(x) - w_k(y)\big)\big((w_k(x) - w_k(y)) - (w(x) - w(y))\big)}{|x-y|^{N+sp(x,y)}} dxdy = 0.$$

Using the above information, Lemma 2.3-(ii), and the fact that $w_k \rightharpoonup w$ in W_0, we get

$$\begin{cases} \lim sup < \mathcal{L}(w_k), w_k - w >\leqslant 0, \\ w_k \rightharpoonup w \text{ in } W_0, \\ \mathcal{L} \text{ is a mapping of type } (S_+). \end{cases} \Rightarrow w_k \longrightarrow w \textit{ in } W_0.$$

Then by (3.18), we obtain

$$\mathcal{J}_\lambda(w) = \lim_{k\longrightarrow+\infty} \mathcal{J}_\lambda(w_k) = \bar{c} < 0 \qquad \textit{and} \qquad \mathcal{J}'_\lambda(w) = 0.$$

We conclude that w is a nontrivial critical point of $\mathcal{J}_\lambda$. Thus w is a nontrivial weak solution for problem $(\mathcal{P}_s)$. Finally any $\lambda \in]0, \lambda^*[$ is an eigenvalue of problem $(\mathcal{P}_s)$.
The proof of Theorem 3.1 is complete. □

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