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THE STRUCTURE OF FRACTIONAL SPACES GENERATED BY A TWO-DIMENSIONAL NEUTRON TRANSPORT OPERATOR AND ITS APPLICATIONS

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ABSTRACT. In this study, the structure of fractional spaces generated by the two-dimensional neutron transport operator A defined by formula $Au = \omega_1 \frac{\partial u}{\partial x} + \omega_2 \frac{\partial u}{\partial y}$ is investigated. The positivity of A in $C\left(\mathbb{R}^2\right)$ and $L_p\left(\mathbb{R}^2\right)$, $1 \le p < \infty$, is established. It is established that, for any $0 < \alpha < 1$ and $1 \le p < \infty$, the norms of spaces $E_{\alpha,p}\left(L_p\left(\mathbb{R}^2\right),\ A\right)$ and $E_\alpha\left(C\left(\mathbb{R}^2\right),\ A\right)$, $W_p^\alpha\left(\mathbb{R}^2\right)$ and $C^\alpha\left(\mathbb{R}^2\right)$ are equivalent, respectively. The positivity of the neutron transport operator in Hölder space $C^\alpha\left(\mathbb{R}^2\right)$ and Slobodeckij space $W_p^\alpha\left(\mathbb{R}^2\right)$ is proved. In applications, theorems on the stability of Cauchy problem for the neutron transport equation in Hölder and Slobodeckij spaces are provided.

1. INTRODUCTION

The neutron transport theory has a critical importance in nuclear engineering. It plays an important role in the design and safety of nuclear power stations. The most important equation in neutron transport theory is the neutron transport equation. We use the neutron transport equation in many physical applications, such as neutron transport, radiative transfer high frequency waves in heterogeneous, and random media and in many application of nuclear physics. The neutron transport equation describes the distribution of neutrons in terms of their positions in space and time, their energies, and their travel directions [12, 13, 14].

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It is well-known that the study of the various properties of partial differential equations is based on the positivity property of the differential operator in a Banach space. The positivity of a wider class of differential and difference operators in Banach spaces has been studied by many researchers [16, 1, 17, 3, 9, 8, 15, 10, 11, 19, 18, 6].

An operator A, densely defined in a Banach space E with the domain $D(A)$, is called positive in E, if its spectrum σ_A lies in the interior of the sector of angle φ, $0 < \varphi < \pi$, symmetric with respect to the real axis, and moreover on the edges of this sector $S_1(\varphi) = \{\rho e^{i\varphi} : 0 \le \rho \le \infty\}$ and $S_2(\varphi) = \{\rho e^{-i\varphi} : 0 \le \rho \le \infty\}$, and outside of the sector the resolvent $(\lambda I - A)^{-1}$ is subject to the bound [6]

$$\left\|(A-\lambda I)^{-1}\right\|_{E\to E} \le \frac{M}{1+|\lambda|}.$$

Throughout the present paper, M denotes positive constants, which may differ in time and thus is not a subject of precision. However, we will use the notation$M(\alpha,\, \omega_1,\, \omega_2, \dots)$ to stress the fact that the constant depends only on $\alpha,\, \omega_1,\, \omega_2, \dots$.

The infimum of all such angles φ is called the spectral angle of the positive operator A and is denoted by $\varphi(A) = \varphi(A, E)$. The operator A is said to be strongly positive in a Banach space E, if $\varphi(A, E) < \frac{\pi}{2}$.

The structure of fractional spaces generated by positive differential and difference operators and its applications to partial differential equations have been investigated by many researchers [5, 7]. Finally, a survey of results in fractional spaces generated by positive operators and their applications to partial differential equations are given in [2].

Nevertheless, the structure of fractional spaces generated by transport operators and its applications to partial differential equations have not been investigated sufficiently. In the present paper, the structure of fractional spaces generated by the two-dimensional neutron transport operator is studied. The positivity of the neutron transport operator in Hölder space $C^\alpha(\mathbb{R}^2)$ and Slobodeckij space $W_p^\alpha(\mathbb{R}^2), 1 \le p < \infty$, is proved. In applications, new theorems on the stability of Cauchy problem for the neutron transport equation in Hölder and Slobodeckij spaces are provided. Finally, some of these statements are formulated in [4] without proof.

2. Neutron transport operator

We consider the resolvent equation

$$\lambda u(x,y,\bar{\omega}) - Au(x,y,\bar{\omega}) = f(x,y,\bar{\omega}),\, (x,y) \in \mathbb{R}^2 \tag{2.1}$$

for the two-dimensional transport operator A defined by formula

$$Au(x,y,\bar{\omega}) = \omega_1 \frac{\partial u}{\partial x} + \omega_2 \frac{\partial u}{\partial y},\ (x,y) \in \mathbb{R}^2.$$

Here, the coefficients ω_1 and ω_2 are cosines of direction of neutrons on Ox and Oy. Denote $\bar{\omega} = (\omega_1, \omega_2)$, where ω_1 and ω_2 are constants. Then, for the solution of (2.1) the following formula holds:

$$(\lambda I - A)^{-1} f(x,y,\bar{\omega}) = \frac{1}{\sqrt{\omega_1^2+\omega_2^2}} \int_0^\infty e^{-\frac{\lambda s}{\sqrt{\omega_1^2+\omega_2^2}}} f\left(x+\frac{\omega_1 s}{\sqrt{\omega_1^2+\omega_2^2}}, y+\frac{\omega_2 s}{\sqrt{\omega_1^2+\omega_2^2}}, \bar{\omega}\right) ds. \tag{2.2}$$

Denote

$$B = -A.$$

Then, using formula (2.2), we get

$$(\lambda I + B)^{-1} f(x,y,\bar{\omega}) = \frac{1}{\sqrt{\omega_1^2+\omega_2^2}} \int_0^\infty e^{-\frac{\lambda s}{\sqrt{\omega_1^2+\omega_2^2}}} f\left(x+\frac{\omega_1 s}{\sqrt{\omega_1^2+\omega_2^2}}, y+\frac{\omega_2 s}{\sqrt{\omega_1^2+\omega_2^2}}, \bar{\omega}\right) ds. \tag{2.3}$$

3. The fractional space $E_\alpha\left(C\left(\mathbb{R}^2\right), B\right)$.

Recall that the space $C\left(\mathbb{R}^2\right)$ of all continuous functions $f(x,y,\bar{\omega})$ on $(x,y) \in \mathbb{R}^2$ defined by the norm (see [2]),

$$\|f\|_{C(\mathbb{R}^2)} = \sup_{(x,y)\in\mathbb{R}^2} |f(x,y,\bar{\omega})|.$$

Theorem 3.1. *For all $\lambda > 0$, the resolvent $(\lambda I + B)^{-1}$ satisfies the following estimate:*

$$\left\|(\lambda I + B)^{-1}\right\|_{C(\mathbb{R}^2)\to C(\mathbb{R}^2)} \le \frac{1}{\lambda}. \tag{3.1}$$

Proof. Using formula (2.3) and triangle inequality, we obtain

$$\left|(\lambda I + B)^{-1} f(x,y,\bar{\omega})\right| \le \frac{1}{\sqrt{\omega_1^2+\omega_2^2}} \int_0^\infty e^{-\frac{\lambda s}{\sqrt{\omega_1^2+\omega_2^2}}} \left| f\left(x+\frac{\omega_1 s}{\sqrt{\omega_1^2+\omega_2^2}}, y+\frac{\omega_2 s}{\sqrt{\omega_1^2+\omega_2^2}}\bar{\omega}\right)\right| ds \le \|f\|_{C(\mathbb{R}^2)} \frac{1}{\lambda}$$

for any $(x,y) \in \mathbb{R}^2$ and $\lambda > 0$. Then, we have

$$\left\|(\lambda I + B)^{-1} f\right\|_{C(\mathbb{R}^2)} \le \frac{1}{\lambda} \|f\|_{C(\mathbb{R}^2)}$$

for any $\lambda > 0$. From that it follows (3.1). This completes the proof of Theorem 3.1. □

Let us study the structure of the fractional space $E_\alpha\left(C\left(\mathbb{R}^2\right), B\right)$, $0 < \alpha < 1$, of all functions $f(x,y,\bar{\omega})$ defined on $\mathbb{R}^2$ for which the following norm is finite; see [2].

$$\|f\|_{E_\alpha(C(\mathbb{R}^2),B)} = \|f\|_{C(\mathbb{R}^2)} + \sup_{(x,y)\in\mathbb{R}^2,\ \lambda\in\mathbb{R}^+} \left|\lambda^\alpha B(\lambda I + B)^{-1} f(x,y,\bar{\omega})\right|. \tag{3.2}$$

Recall that the Hölder space $C^{\alpha}\left(\mathbb{R}^2\right),\ 0<\alpha<1$, consists of all continuous functions $f\left(x,y,\bar{\omega}\right)$ defined on $\mathbb{R}^2$ for which the following norm is finite (see [6, page 15])

$$\|f\|_{C^{\alpha}(\mathbb{R}^2)}=\|f\|_{C(\mathbb{R}^2)}+\sup_{(x,y)\in\mathbb{R}^2,s\in\mathbb{R}^+}\frac{\left|f(x,y,\bar{\omega})-f\left(x+\frac{\omega_1 s}{\sqrt{\omega_1^2+\omega_2^2}},y+\frac{\omega_2 s}{\sqrt{\omega_1^2+\omega_2^2}},\bar{\omega}\right)\right|}{s^{\alpha}}. \tag{3.3}$$

Applying the definition of $E_{\alpha}\left(C\left(\mathbb{R}^2\right),B\right)$, we get the following estimate

$$\left\|(\lambda I+B)^{-1}\right\|_{E_{\alpha}(C(\mathbb{R}^2),B)\to\ E_{\alpha}(C(\mathbb{R}^2),B)}\leq\left\|(\lambda I+B)^{-1}\right\|_{C(\mathbb{R}^2)\to C(\mathbb{R}^2)}.$$

From (3.2) and Theorem 3.1 it follows the following theorem.

Theorem 3.2. *Let $0<\alpha<1$. Then, B is a positive operator in $E_{\alpha}\left(C\left(\mathbb{R}^2\right),B\right)$.*

Moreover, B is a positive operator in Hölder space $C^{\alpha}\left(\mathbb{R}^2\right)$. The proof of this statement is based on the following theorem.

Theorem 3.3. *Let $0<\alpha<1$. Then, the spaces $E_{\alpha}\left(C\left(\mathbb{R}^2\right),B\right)$ and $C^{\alpha}\left(\mathbb{R}^2\right)$ coincide and their respective norms are equivalent.*

Proof. Let us prove that

$$\left|\lambda^{\alpha}B\left(\lambda I+B\right)^{-1}f\left(x,y,\bar{\omega}\right)\right|\leq M_1\left(\alpha,\omega_1,\omega_2\right)\|f\|_{C^{\alpha}(\mathbb{R}^2)}$$

for $(x,y)\in\mathbb{R}^2$ and $\lambda>0$. For any $\lambda>0$, we have the equality

$$B\left(\lambda I+B\right)^{-1}f\left(x,y,\bar{\omega}\right)=\lambda\left[\tfrac{1}{\lambda}f\left(x,y,\bar{\omega}\right)-\left(\lambda I+B\right)^{-1}f\left(x,y,\bar{\omega}\right)\right].$$

Applying formulas (2.3) and

$$\int_0^{\infty}e^{-\frac{\lambda s}{\sqrt{\omega_1^2+\omega_2^2}}}ds=\frac{\sqrt{\omega_1^2+\omega_2^2}}{\lambda},$$

we get

$$\lambda^{\alpha}B\left(\lambda I+B\right)^{-1}f\left(x,y,\bar{\omega}\right)$$
$$=\frac{\lambda^{1+\alpha}}{\sqrt{\omega_1^2+\omega_2^2}}\int_0^{\infty}s^{\alpha}e^{-\frac{\lambda s}{\sqrt{\omega_1^2+\omega_2^2}}}\frac{\left[f\left(x,y,\bar{\omega}\right)-f\left(x+\frac{\omega_1 s}{\sqrt{\omega_1^2+\omega_2^2}},y+\frac{\omega_2 s}{\sqrt{\omega_1^2+\omega_2^2}},\bar{\omega}\right)\right]}{s^{\alpha}}ds. \tag{3.4}$$

Using formula (3.4) and the triangle inequality, we obtain

$$\left|\lambda^{\alpha}B\left(\lambda I+B\right)^{-1}f\left(x,y,\bar{\omega}\right)\right|$$

$$\leq \frac{\lambda^{1+\alpha}}{\sqrt{\omega_1^2+\omega_2^2}}\int\limits_0^{\infty}s^{\alpha}e^{-\frac{\lambda s}{\sqrt{\omega_1^2+\omega_2^2}}}\frac{\left|f\left(x,y,\bar{\omega}\right)-f\left(x+\frac{\omega_1 s}{\sqrt{\omega_1^2+\omega_2^2}},y+\frac{\omega_2 s}{\sqrt{\omega_1^2+\omega_2^2}},\bar{\omega}\right)\right|}{s^{\alpha}}ds$$

$$\leq \frac{\lambda^{1+\alpha}}{\sqrt{\omega_1^2+\omega_2^2}}\int\limits_0^{\infty}s^{\alpha}e^{-\frac{\lambda s}{\sqrt{\omega_1^2+\omega_2^2}}}ds\left\|f\right\|_{C^{\alpha}(\mathbb{R}^2)}.$$

Let

$$J=\frac{\lambda^{1+\alpha}}{\sqrt{\omega_1^2+\omega_2^2}}\int\limits_0^{\infty}s^{\alpha}e^{-\frac{\lambda s}{\sqrt{\omega_1^2+\omega_2^2}}}ds.$$

Then,

$$\left|\lambda^{\alpha}B\left(\lambda I+B\right)^{-1}f\left(x,y,\bar{\omega}\right)\right|\leq J\left\|f\right\|_{C^{\alpha}(\mathbb{R}^2)}.$$

Now, let us estimate J.

Performing the change of variables $\frac{\lambda s}{\sqrt{\omega_1^2+\omega_2^2}}=\tau$ or $s^{\alpha}=\frac{\tau^{\alpha}\left(\sqrt{\omega_1^2+\omega_2^2}\right)^{\alpha}}{\lambda^{a}}$, we get

$$\begin{aligned} J &= \frac{\lambda^{1+\alpha}}{\sqrt{\omega_1^2+\omega_2^2}}\int\limits_0^{\infty}\frac{\left(\sqrt{\omega_1^2+\omega_2^2}\right)^{\alpha}}{\lambda^{a}}\tau^{\alpha}e^{-\tau}\frac{\sqrt{\omega_1^2+\omega_2^2}}{\lambda}d\tau \\ &= \left(\sqrt{\omega_1^2+\omega_2^2}\right)^{\alpha}\int\limits_0^{\infty}\tau^{\alpha}e^{-\tau}d\tau=\left(\sqrt{\omega_1^2+\omega_2^2}\right)^{\alpha}\Gamma\left(1+\alpha\right)=M_1\left(\alpha,\omega_1,\omega_2\right). \end{aligned}$$

Thus, for any $\lambda>0$ and $(x,y)\in\mathbb{R}^2$, we establish the inequality

$$\left|\lambda^{\alpha}B\left(\lambda I+B\right)^{-1}f\left(x,y,\bar{\omega}\right)\right|\leq M_1\left(\alpha,\omega_1,\omega_2\right)\left\|f\right\|_{C^{\alpha}(\mathbb{R}^2)}.$$

Here, $M_1\left(\alpha,\omega_1,\omega_2\right)=\left(\sqrt{\omega_1^2+\omega_2^2}\right)^{\alpha}\Gamma\left(1+\alpha\right)$.

This means that

$$E_{\alpha}\left(C\left(\mathbb{R}^2\right),B\right)\subset C^{\alpha}\left(\mathbb{R}^2\right). \tag{3.5}$$

Let us prove the reverse inclusion. We consider two cases $\omega_1\neq 0$ and $\omega_2\neq 0$, separately. Let $\omega_1\neq 0$. Then, using identity (2.3) and putting $x+\frac{\omega_1 s}{\sqrt{\omega_1^2+\omega_2^2}}=\mu$, we obtain

$$\left(\lambda I+B\right)^{-1}f\left(x,y,\bar{\omega}\right)=\frac{1}{\omega_1}\int\limits_x^{\infty}e^{-\lambda\frac{\mu-x}{\omega_1}}f\left(\mu,y+\frac{\omega_2}{\omega_1}\left(\mu-x\right),\bar{\omega}\right)d\mu. \tag{3.6}$$

Moreover, let $\omega_2\neq 0$. Then, using the identity (2.3) and putting $y+\frac{\omega_2 s}{\sqrt{\omega_1^2+\omega_2^2}}=\nu$, we get

$$\left(\lambda I+B\right)^{-1}f\left(x,y,\bar{\omega}\right)=\frac{1}{\omega_2}\int\limits_y^{\infty}e^{-\lambda\frac{\nu-y}{\omega_2}}f\left(x+\frac{\omega_1}{\omega_2}\left(\nu-y\right),\nu,\bar{\omega}\right)d\nu.$$

For any positive operator B, we have that

$$\int_0^\infty (\lambda I + B)^{-2} f(x,y,\bar{\omega})\, d\lambda = B^{-1} f(x,y,\bar{\omega}).$$

Then, we can write

$$f(x,y,\bar{\omega}) = \int_0^\infty B(\lambda I + B)^{-2} f(x,y,\bar{\omega})\, d\lambda,$$

where the notation $(\lambda I + B)^{-2}$ means the square of the inverse of $(\lambda I + B)$.

Let $\omega_1 \neq 0$. Then, from this relation, formulas (3.3) and (3.6) imply that

$$\begin{aligned} & f(x,y,\bar{\omega}) - f(x+\omega_1\rho, y+\omega_2\rho, \bar{\omega}) \qquad (3.7) \\ & = \frac{1}{\omega_1}\int_0^\infty \int_x^{x+\omega_1\rho} e^{-\lambda\frac{\mu-x}{\omega_1}} B(\lambda I + B)^{-1} f\left(\mu, y + \frac{\omega_2}{\omega_1}(\mu - x), \bar{\omega}\right) d\mu d\lambda \\ & + \left[\frac{1}{\omega_1}\int_0^\infty \int_{x+\omega_1\rho}^\infty \left(e^{-\lambda\frac{\mu-x}{\omega_1}} - e^{-\lambda\frac{\mu-x-\omega_1\rho}{\omega_1}}\right)\right. \\ & \left. \times B(\lambda I + B)^{-1} f\left(\mu, y + \frac{\omega_2}{\omega_1}(\mu - x), \bar{\omega}\right) d\mu d\lambda\right]. \end{aligned}$$

Here, $\rho = \frac{s}{\sqrt{\omega_1^2+\omega_2^2}}$. Using the triangle inequality and definition of norm of spaces $E_\alpha(C(\mathbb{R}^2), B)$, we obtain

$$\begin{aligned} & \left| f(x,y,\bar{\omega}) - f(x+\omega_1\rho, y+\omega_2\rho, \bar{\omega}) \right| \\ & \leq \frac{1}{\omega_1}\int_0^\infty \int_x^{x+\omega_1\rho} \lambda^{-\alpha} e^{-\lambda\frac{\mu-x}{\omega_1}} d\mu d\lambda \, \|f\|_{E_\alpha(C(\mathbb{R}^2),B)} \\ & + \frac{1}{\omega_1}\int_0^\infty \int_{x+\omega_1\rho}^\infty \lambda^{-\alpha} \left| e^{-\lambda\frac{\mu-x}{\omega_1}} - e^{-\lambda\frac{\mu-x-\omega_1\rho}{\omega_1}} \right| d\mu d\lambda \, \|f\|_{E_\alpha(C(\mathbb{R}^2),B)} \\ & = (J_1 + J_2) \|f\|_{E_\alpha(C(\mathbb{R}^2),B)}, \end{aligned}$$

where

$$J_1 = \frac{1}{\omega_1}\int_0^\infty \int_x^{x+\omega_1\rho} \lambda^{-\alpha} e^{-\lambda\frac{\mu-x}{\omega_1}} d\mu d\lambda$$

and

$$J_2 = \frac{1}{\omega_1} \int\limits_0^\infty \int\limits_{x+\omega_1\rho}^\infty \lambda^{-\alpha} \left| e^{-\lambda \frac{\mu - x}{\omega_1}} - e^{-\lambda \frac{\mu - x - \omega_1 \rho}{\omega_1}} \right| d\mu d\lambda.$$

Now, let us estimate J_1 and J_2, separately. First, let us estimate J_1.

$$J_1 = \frac{1}{\omega_1} \int\limits_0^\infty \int\limits_x^{x+\omega_1\rho} \lambda^{-\alpha} e^{-\lambda \frac{\mu - x}{\omega_1}} d\mu d\lambda = \int\limits_0^\infty \frac{1 - e^{-\lambda\rho}}{\lambda^{\alpha+1}} d\lambda.$$

The change of variable $\lambda\rho = \tau$ yields, $J_1 = \int\limits_0^\infty \frac{1-e^{-\lambda\rho}}{\lambda^{\alpha+1}} d\lambda = \rho^\alpha \int\limits_0^\infty \frac{1-e^{-\tau}}{\tau^{\alpha+1}} d\tau$.

Applying the inequality $e^{-\tau} \geq 1 - \tau$ we obtain

$$\int\limits_0^\infty \frac{1 - e^{-\tau}}{\tau^{\alpha+1}} d\tau \leq \frac{1}{\alpha(1-\alpha)}.$$

Actually,

$$\int\limits_0^\infty \frac{1 - e^{-\tau}}{\tau^{\alpha+1}} d\tau \leq \int\limits_0^1 \tau^{-\alpha} d\tau + \int\limits_1^\infty \tau^{-\alpha-1} d\tau = \frac{1}{\alpha} + \frac{1}{1-\alpha} = \frac{1}{\alpha(1-\alpha)}.$$

It follows that

$$J_1 \leq \rho^\alpha \frac{1}{\alpha(1-\alpha)}. \tag{3.8}$$

Now, let us estimate J_2.

$$J_2 = \frac{1}{\omega_1} \int\limits_0^\infty \int\limits_{x+\omega_1\rho}^\infty \lambda^{-\alpha} e^{-\lambda \frac{u-x}{\omega_1}} \left(e^{\lambda\rho} - 1 \right) du d\lambda = \int\limits_0^\infty \frac{1 - e^{-\lambda\rho}}{\lambda^{\alpha+1}} d\lambda.$$

Thus, we have that $\int\limits_0^\infty \frac{1-e^{-\lambda\rho}}{\lambda^{\alpha+1}} d\lambda \leq \rho^\alpha \frac{1}{\alpha(1-\alpha)}$. Therefore,

$$J_2 \leq \rho^\alpha \frac{1}{\alpha(1-\alpha)}. \tag{3.9}$$

Finally, by combining estimates (3.8) and (3.9), we obtain

$$|f(x,y,\bar{\omega}) - f(x + \omega_1\rho, y + \omega_2\rho, \bar{\omega})| \leq \rho^\alpha \frac{2}{\alpha(1-\alpha)} \|f\|_{E_\alpha(C(\mathbb{R}^2),B)}$$

for any $(x,y) \in \mathbb{R}^2$ and $\lambda > 0$. Thus, for any $(x,y) \in \mathbb{R}^2$, we get the inequality

$$\sup_{(x,y)\in\mathbb{R}^2, s\in\mathbb{R}^+} \frac{\left| f(x,y,\bar{\omega}) - f\left(x + \frac{\omega_1 s}{\sqrt{\omega_1^2+\omega_2^2}}, y + \frac{\omega_2 s}{\sqrt{\omega_1^2+\omega_2^2}}, \bar{\omega} \right) \right|}{s^\alpha}$$

$$\leq \frac{2\left(\sqrt{\omega_1^2+\omega_2^2}\right)^{-\alpha}}{\alpha(1-\alpha)} \|f\|_{E_\alpha(C(\mathbb{R}^2),B)} = M_2(\alpha, \omega_1, \omega_2) \|f\|_{E_\alpha(C(\mathbb{R}^2),B)}.$$

Here, $M_2(\alpha,\omega_1,\omega_2)=\frac{2\left(\sqrt{\omega_1^2+\omega_2^2}\right)^{-\alpha}}{\alpha(1-\alpha)}$.

Hence, we have proved that $C^\alpha(\mathbb{R}^2)\subset E_\alpha\left(C(\mathbb{R}^2),B\right)$. Also for $\omega_2\neq 0$ the inequality can be proved analogously. Thus, we prove (3.5). This completes the proof of Theorem 3.3. $\square$

Theorem 3.4. *The operator B is a positive operator in $C^\alpha(\mathbb{R}^2)$, $0<\alpha<1$, and the following estimate, for $\lambda>0$, holds:*

$$\left\|(\lambda I+B)^{-1}\right\|_{C^\alpha(\mathbb{R}^2)\to C^\alpha(\mathbb{R}^2)}\leq\frac{M_3(\alpha,\omega_1,\omega_2)}{\lambda}.$$

Proof. Actually, applying Theorem 3.3, we can write

$$\left\|(\lambda I+B)^{-1}\right\|_{C^\alpha(\mathbb{R}^2)\to C^\alpha(\mathbb{R}^2)}\leq M_3(\alpha,\omega_1,\omega_2)\left\|(\lambda I+B)^{-1}\right\|_{E_\alpha\to E_\alpha}.$$

Then, from estimate 3 it follows

$$\left\|(\lambda I+B)^{-1}\right\|_{C^\alpha(\mathbb{R}^2)\to C^\alpha(\mathbb{R}^2)}\leq M_3(\alpha,\omega_1,\omega_2)\left\|(\lambda I+B)^{-1}\right\|_{C(\mathbb{R}^2)\to C(\mathbb{R}^2)}.$$

Finally, using this estimate and Theorem 3.1, we complete the proof of Theorem 3.4. $\square$

4. The fractional space $E_{\alpha,p}\left(L_p(\mathbb{R}^2),\ B\right)$.

Recall that the space $L_p(\mathbb{R}^2)$, $1\leq p<\infty$, of all Lebesgue measurable functions $f(x,y,\bar{\omega})$ defined on $\mathbb{R}^2$ for which the following norm is finite; see [2].

$$\|f\|_{L_p(\mathbb{R}^2)}=\left(\int\limits_{(x,y)\in\mathbb{R}^2}|f(x,y,\bar{\omega})|^p\,dxdy\right)^{\frac{1}{p}}.$$

Theorem 4.1. *For all $\lambda>0$ and $1\leq p<\infty$, the resolvent $(\lambda I+B)^{-1}$ satisfies the following estimate:*

$$\left\|(\lambda I+B)^{-1}\right\|_{L_p(\mathbb{R}^2)\to L_p(\mathbb{R}^2)}\leq\tfrac{1}{\lambda}. \tag{4.1}$$

Proof. Using formula (2.3) and the Minkowski's inequality for the integral, we obtain

$$\begin{aligned}\left\|(\lambda I+B)^{-1}\right.&\left.f(x,y,\bar{\omega})\right\|_{L_p(\mathbb{R}^2)}\\ \leq&\ \frac{1}{\sqrt{\omega_1^2+\omega_2^2}}\int\limits_0^\infty e^{-\frac{\lambda s}{\sqrt{\omega_1^2+\omega_2^2}}}\\ &\left(\int\limits_{(x,y)\in\mathbb{R}^2}\left|f\left(x+\frac{\omega_1 s}{\sqrt{\omega_1^2+\omega_2^2}},y+\frac{\omega_2 s}{\sqrt{\omega_1^2+\omega_2^2}},\bar{\omega}\right)\right|^p dxdy\right)^{\frac{1}{p}}ds.\end{aligned}$$

Making the change of variables $\bar{x}=x+\frac{\omega_1 s}{\sqrt{\omega_1^2+\omega_2^2}}$ and $\bar{y}=y+\frac{\omega_2 s}{\sqrt{\omega_1^2+\omega_2^2}}$ we have

$$\begin{aligned}&\left\|(\lambda I+B)^{-1}f(x,y,\bar{\omega})\right\|_{L_p(\mathbb{R}^2)}\\ &\leq\frac{1}{\sqrt{\omega_1^2+\omega_2^2}}\int\limits_0^\infty e^{-\frac{\lambda s}{\sqrt{\omega_1^2+\omega_2^2}}}ds\left(\int\limits_{(x,y)\in\mathbb{R}^2}|f(\bar{x},\bar{y},\bar{\omega})|^p\,dxdy\right)^{\frac{1}{p}}=\|f\|_{L_p(\mathbb{R}^2)}\frac{1}{\lambda}\end{aligned}$$

for any $\lambda > 0$. From that it follows (4.1). This completes the proof of Theorem 4.1. $\square$

In this section, we study the structure of the fractional space $E_{\alpha,p}\left(L_p\left(\mathbb{R}^2\right),\ B\right)$, $0<\alpha<1$ and $1 \le p < \infty$, of all functions $f\left(x,y,\bar{\omega}\right)$ defined on $\mathbb{R}^2$ for which the following norm is finite; see [2].

$$\|f\|_{E_{\alpha,p}\left(L_p\left(\mathbb{R}^2\right),\ B\right)} = \left[\int\limits_{(x,y)\in\mathbb{R}^2} \left|f\left(x,y,\bar{\omega}\right)\right|^p dxdy \right.$$
$$\left. + \int\limits_0^\infty \int\limits_{(x,y)\ \in\mathbb{R}^2} \left|\lambda^\alpha B\left(\lambda I + B\right)^{-1} f\left(x,y,\bar{\omega}\right)\right|^p dxdy \frac{d\lambda}{\lambda} \right]^{\frac{1}{p}}.$$

Recall that (see [2]) the Slobodeckij space $W_p^\alpha\left(\mathbb{R}^2\right)$, $0<\alpha<1$ and $1 \le p < \infty$, of all continuous functions $f\left(x,y,\bar{\omega}\right)$ defined on $\mathbb{R}^2$ for which the following norm is finite

$$\|f\|_{W_p^\alpha\left(\mathbb{R}^2\right)}$$
$$= \left[\int\limits_{(x,y)\in\mathbb{R}^2} \left|f\left(x,y,\bar{\omega}\right)\right|^p dxdy \right.$$
$$\left. + \int\limits_0^\infty \int\limits_{(x,y)\in\mathbb{R}^2} \frac{\left|f\left(x,y,\bar{\omega}\right) - f\left(x + \frac{\omega_1 s}{\sqrt{\omega_1^2+\omega_2^2}}, y + \frac{\omega_2 s}{\sqrt{\omega_1^2+\omega_2^2}}, \bar{\omega}\right)\right|^p}{s^{\alpha p+1}} dxdyds \right]^{\frac{1}{p}}.$$

Note that the first term $\left[\int\limits_{(x,y)\in\mathbb{R}^2} \left|f\left(x,y,\bar{\omega}\right)\right|^p dxdy \right]^{\frac{1}{p}}$ in these norms can be discarded, since this leads to equivalent norms. Applying the definition of $E_{\alpha,p}\left(L_p\left(\mathbb{R}^2\right),\ B\right)$, we get the following estimate

$$\left\|\left(\lambda I + B\right)^{-1}\right\|_{E_{\alpha,p}\left(L_p\left(\mathbb{R}^2\right),\ B\right)\to\ E_{\alpha,p}\left(L_p\left(\mathbb{R}^2\right),\ B\right)} \le \left\|\left(\lambda I + B\right)^{-1}\right\|_{L_p\left(\mathbb{R}^2\right)\to L_p\left(\mathbb{R}^2\right)}. \quad (4.2)$$

From (4.2) and Theorem 4.1, we obtain the following theorem.

Theorem 4.2. *Let $0<\alpha<1$ and $1 \le p < \infty$. Then, B is the positive operator in $E_{\alpha,p}\left(L_p\left(\mathbb{R}^2\right),\ B\right)$.*

Moreover, B is the positive operator in fractional space $W_p^\alpha\left(\mathbb{R}^2\right)$. The proof of this statement is based on the following theorem.

Theorem 4.3. *The spaces $E_{\alpha,p}\left(L_p\left(\mathbb{R}^2\right), B\right)$ and $W_p^\alpha\left(\mathbb{R}^2\right), 1 \le p < \infty$, are identical for $0<\alpha<1$, and their norms are equivalent.*

Proof. First, let us prove that

$$E_{\alpha,p}\left(L_p\left(\mathbb{R}^2\right),\ B\right)\subset W_p^{\alpha}\left(\mathbb{R}^2\right).$$

Using formula (3.4) and performing the change of variables $\frac{\lambda s}{\sqrt{\omega_1^2+\omega_2^2}}=\xi$ and $\frac{\lambda ds}{\sqrt{\omega_1^2+\omega_2^2}}=d\xi$, we get

$$I:=\left[\int_0^{\infty}\int_{(x,y)\ \in R^2}\left|\lambda^{\alpha}B\left(\lambda I+B\right)^{-1}f\left(x,y,\bar{\omega}\right)\right|^p dxdy\frac{d\lambda}{\lambda}\right]^{\frac{1}{p}}$$
$$=\left[\int_0^{\infty}\int_{(x,y)\in\mathbb{R}^2}\left|\frac{\lambda^{\alpha+1}}{\sqrt{\omega_1^2+\omega_2^2}}\int_0^{\infty}e^{-\xi}\left(f\left(x,y,\bar{\omega}\right)-f\left(x+\omega_1\frac{\xi}{\lambda},y+\omega_2\frac{\xi}{\lambda},\bar{\omega}\right)\right)\right.\right.$$
$$\left.\left.\frac{\sqrt{\omega_1^2+\omega_2^2}d\xi}{\lambda}\right|^p dxdy\frac{d\lambda}{\lambda}\right]^{\frac{1}{p}}.$$

Using the Minkowski's inequality for the integral, we obtain

$$I\leq\int_0^{\infty}e^{-\xi}d\xi\left[\int_0^{\infty}\int_{(x,y)\in\mathbb{R}^2}\left|\lambda^{\alpha}\bigg(f\left(x,y,\bar{\omega}\right)-f\big(x+\omega_1\frac{\xi}{\lambda},y+\omega_2\frac{\xi}{\lambda},\bar{\omega}\big)\bigg)\right|^p dxdy\frac{d\lambda}{\lambda}\right]^{\frac{1}{p}}.$$

The change of variables $\rho=\frac{\xi}{\lambda}$, yields

$$I\leq\int_0^{\infty}e^{-\xi}\xi^{\alpha}d\xi\left[\int_0^{\infty}\int_{(x,y)\in\mathbb{R}^2}\frac{\left|\left(f\left(x,y,\bar{\omega}\right)-f\left(x+\omega_1\rho,y+\omega_2\rho,\bar{\omega}\right)\right)\right|^p}{\rho^{p\alpha}}dxdy\frac{d\rho}{\rho}\right]^{\frac{1}{p}}.$$

Since $\int_0^{\infty}e^{-\xi}\xi^{\alpha}d\xi=\Gamma\left(1+\alpha\right)$, and making the change of variables $\rho=\frac{s}{\sqrt{\omega_1^2+\omega_2^2}}$, we obtain

$$I\leq\Gamma(1+\alpha)\left(\sqrt{\omega_1^2+\omega_2^2}\right)^{\alpha+\frac{1}{p}}$$
$$\times\left[\int_0^{\infty}\int_{(x,y)\in\mathbb{R}^2}\frac{\left|f\left(x,y,\bar{\omega}\right)-f\left(x+\frac{\omega_1 s}{\sqrt{\omega_1^2+\omega_2^2}},y+\frac{\omega_2 s}{\sqrt{\omega_1^2+\omega_2^2}},\bar{\omega}\right)\right|^p}{s^{p\alpha+1}}dxdyds\right]^{\frac{1}{p}}.$$

This means that the following inequality holds

$$\|f\|_{E_{\alpha,p}(L_p(\mathbb{R}^2),\ B)}\leq M_4\left(\alpha,\omega_1,\omega_2\right)\|f\|_{W_p^{\alpha}(\mathbb{R}^2)},$$

where $M_4(\alpha,\omega_1,\omega_2)=\Gamma(1+\alpha)\left(\sqrt{\omega_1^2+\omega_2^2}\right)^{\alpha+1}$.
Let us prove the reverse inclusion

$$W_p^\alpha\left(\mathbb{R}^2\right)\subset E_{\alpha,p}\left(L_p\left(\mathbb{R}^2\right),B\right).$$

For $\omega_1\neq 0$, using formula (3.7), and putting $\frac{\mu-x}{\omega_1}=u_1$, and performing the change of variables $\lambda=\frac{\xi}{\rho}$ and $s=\rho\eta$, we obtain

$$f(x,y,\bar{\omega})-f(x+\omega_1\rho,y+\omega_2\rho,\bar{\omega})$$

$$=\int_0^\infty\int_0^1 e^{-\xi\eta}B\left(\tfrac{\xi}{\rho}I+B\right)^{-1}f(x+\omega_1\rho\eta,y+\omega_2\rho\eta,\bar{\omega})\,d\xi d\eta$$

$$+\int_0^\infty\int_1^\infty e^{-\xi\eta}\left(e^{\xi}-1\right)B\left(\tfrac{\xi}{\rho}I+B\right)^{-1}f(x+\omega_1\rho\eta,y+\omega_2\rho\eta,\bar{\omega})\,d\xi d\eta.$$

Taking the integral, we get

$$II:=\left[\frac{1}{\sqrt{\omega_1^2+\omega_2^2}^{p\alpha+1}}\int_0^\infty\int_{(x,y)\in\mathbb{R}^2}\frac{|f(x,y,\bar{\omega})-f(x+\omega_1\rho,y+\omega_2\rho,\bar{\omega})|^p}{\rho^{p\alpha+1}}dxdyd\rho\right]^{\frac{1}{p}}$$

$$=\Bigg[\frac{1}{\sqrt{\omega_1^2+\omega_2^2}^{p\alpha+1}}\int_0^\infty\int_{(x,y)\in\mathbb{R}^2}\Bigg|\int_0^\infty\int_0^1 e^{-\xi\eta}B\left(\frac{\xi}{\rho}I+B\right)^{-1}f(x+\omega_1\rho\eta,y+\omega_2\rho\eta,\bar{\omega})\,d\xi d\eta$$

$$\int_0^\infty\int_1^\infty e^{-\xi\eta}\left(e^{\xi}-1\right)B\left(\frac{\xi}{\rho}I+B\right)^{-1}$$

$$\times f(x+\omega_1\rho\eta,y+\omega_2\rho\eta,\bar{\omega})\,d\xi d\eta\Bigg|^p dxdyd\rho\Bigg]^{\frac{1}{p}}.$$

Using triangle inequality, we obtain

$$II\leq\frac{1}{\sqrt{\omega_1^2+\omega_2^2}^{p\alpha+1}}\left[\int_0^\infty\int_{(x,y)\in\mathbb{R}^2}\rho^{p\alpha+1}\left|\int_0^\infty\int_0^1 e^{-\xi\eta}B\left(\frac{\xi}{\rho}I+B\right)^{-1}f(x,y,\bar{\omega})\,d\xi d\eta\right|^p dxdyd\rho\right]^{\frac{1}{p}}$$

$$\leq\frac{1}{\sqrt{\omega_1^2+\omega_2^2}^{p\alpha+1}}$$

$$\left[\int_0^\infty\int_{(x,y)\in\mathbb{R}^2}\rho^{p\alpha+1}\left|\int_0^\infty\int_0^1 e^{-\xi\eta}B\left(\frac{\xi}{\rho}I+B\right)^{-1}f(x+\omega_1\rho\eta,y+\omega_2\rho\eta,\bar{\omega})\,d\xi d\eta\right|^p dxdyd\rho\right]^{\frac{1}{p}}$$

$$+\frac{1}{\sqrt{\omega_1^2+\omega_2^2}^{p\alpha+1}}\Bigg[\int_0^\infty\int_{(x,y)\in\mathbb{R}^2}\rho^{p\alpha+1}\Bigg|\int_0^\infty\int_1^\infty e^{-\xi\eta}\left(e^{\xi}-1\right)B\left(\frac{\xi}{\rho}I+B\right)^{-1}$$

$$\times f(x+\omega_1\rho\eta,y+\omega_2\rho\eta,\bar{\omega})\,d\xi d\eta\Bigg|^p dxdyd\rho\Bigg]^{\frac{1}{p}}.$$

The change of variables $\bar{x} = x + \omega_1 \rho\eta$ and $\bar{y} = y + \omega_2 \rho\eta$ results

$$II \leq \frac{1}{\sqrt{\omega_1^2+\omega_2^2}^{p\alpha+1}} \left[\int_0^\infty \int_{(x,y)\in\mathbb{R}^2} \rho^{p\alpha+1} \left| \int_0^\infty \int_0^1 e^{-\xi\eta} B\left(\frac{\xi}{\rho}I+B\right)^{-1} f(x,y,\bar{\omega})\, d\xi d\eta \right|^p dxdyd\rho \right]^{\frac{1}{p}}$$
$$+ \frac{1}{\sqrt{\omega_1^2+\omega_2^2}^{p\alpha+1}}$$
$$\times \left[\int_0^\infty \int_{(x,y)\in\mathbb{R}^2} \rho^{p\alpha+1} \int_0^\infty \int_1^\infty e^{-\xi\eta}\left(e^{\xi}-1\right) \left| B\left(\frac{\xi}{\rho}I+B\right)^{-1} f(x,y,\bar{\omega})\, d\xi d\eta \right|^p dxdyd\rho \right]^{\frac{1}{p}}.$$

Using the Minkowski's inequality for the integral, we obtain

$$II \leq \frac{1}{\sqrt{\omega_1^2+\omega_2^2}^{p\alpha+1}}$$
$$\left\{ \int_0^\infty \int_0^1 e^{-\xi\eta} d\xi d\eta \left[\int_0^\infty \int_{(x,y)\in\mathbb{R}^2} \rho^{p\alpha+1} \left| B\left(\frac{\xi}{\rho}I+B\right)^{-1} f(x,y,\bar{\omega}) \right|^p dxdyd\rho \right]^{\frac{1}{p}} \right.$$
$$+ \frac{1}{\sqrt{\omega_1^2+\omega_2^2}^{p\alpha+1}} \int_0^\infty \int_1^\infty e^{-\xi\eta}\left(e^{\xi}-1\right) d\xi d\eta$$
$$\left. + \left[\int_0^\infty \int_{(x,y)\in\mathbb{R}^2} \rho^{p\alpha+1} \left| B\left(\frac{\xi}{\rho}I+B\right)^{-1} f(x,y,\bar{\omega}) \right|^p dxdyd\rho \right]^{\frac{1}{p}} \right\}.$$

Performing the change of variables $\lambda = \frac{\xi}{\rho}$ and $d\rho = -\frac{\xi d\lambda}{\lambda^2}$, we get

$$II \leq \frac{1}{\sqrt{\omega_1^2+\omega_2^2}^{p\alpha+1}}$$
$$\left\{ \int_0^\infty \int_0^1 e^{-\xi\eta}\xi^{-\alpha} d\xi d\eta \left[\int_0^\infty \int_{(x,y)\in\mathbb{R}^2} \left| \lambda^\alpha B(\lambda I+B)^{-1} f(x,y,\bar{\omega}) \right|^p dxdyd\rho \right]^{\frac{1}{p}} \right.$$
$$+ \frac{1}{\sqrt{\omega_1^2+\omega_2^2}^{p\alpha+1}} \int_0^\infty \int_1^\infty e^{-\xi\eta}\left(e^{\xi}-1\right)\xi^{-\alpha} d\xi d\eta$$
$$\left. + \left[\int_0^\infty \int_{(x,y)\in\mathbb{R}^2} \left| \lambda^\alpha B(\lambda I+B)^{-1} f(x,y,\bar{\omega}) \right|^p dxdyd\rho \right]^{\frac{1}{p}} \right\}.$$

Since

$$\int_0^\infty \int_0^1 e^{-\xi\eta}\xi^{-\alpha}d\xi d\eta = \int_0^\infty \frac{1-e^{-\xi}}{\xi^{\alpha+1}}d\xi = \int_0^1 \frac{1-e^{-\xi}}{\xi^{\alpha+1}}d\xi + \int_1^\infty \frac{1-e^{-\xi}}{\xi^{\alpha+1}}d\xi$$

$$\leq \int_0^1 \frac{\xi}{\xi^{\alpha+1}}d\xi + \int_1^\infty \frac{1}{\xi^{\alpha+1}}d\xi = \frac{1}{\alpha\left(1-\alpha\right)}$$

and

$$\int_0^\infty \int_1^\infty e^{-\xi\eta}\left(e^{\xi}-1\right)\xi^{-\alpha}d\xi d\eta = \int_0^\infty \frac{1-e^{-\xi}}{\xi^{\alpha+1}}d\xi = \int_0^1 \frac{1-e^{-\xi}}{\xi^{\alpha+1}}d\xi + \int_1^\infty \frac{1-e^{-\xi}}{\xi^{\alpha+1}}d\xi$$

$$\leq \int_0^1 \frac{\xi}{\xi^{\alpha+1}}d\xi + \int_1^\infty \frac{1}{\xi^{\alpha+1}}d\xi = \frac{1}{\alpha\left(1-\alpha\right)},$$

we have the inequality

$$II \leq \frac{1}{\sqrt{\omega_1^2+\omega_2^2}^{\alpha+1}}\frac{1}{\alpha\left(1-\alpha\right)}\left[\int_0^\infty \int_{(x,y)\in\mathbb{R}^2} \left|\lambda^\alpha B\left(\lambda I+B\right)^{-1} f\left(x,y,\bar{\omega}\right)\right|^p dxdyd\rho\right]^{\frac{1}{p}}.$$

The last inequality yields

$$\|f\|_{W_p^\alpha(\mathbb{R}^2)} \leq \frac{M_5\left(\alpha,\omega_1,\omega_2\right)}{\alpha\left(1-\alpha\right)}\|f\|_{E_{\alpha,p}(L_p(\mathbb{R}^2),\ B)},$$

where $M_5\left(\alpha,\omega_1,\omega_2\right) = \frac{1}{\sqrt{\omega_1^2+\omega_2^2}^{\alpha+1}}\frac{1}{\alpha(1-\alpha)}$.
This estimate for $\omega_2 \neq 0$ can be proved analogously.

Thus, we have proved that $W_p^\alpha\left(\mathbb{R}^2\right) \subset E_{\alpha,p}\left(L_p\left(\mathbb{R}^2\right),B\right)$. This completes the proof of Theorem 4.3. □

From that and Theorems 4.1 and 4.2, we have the following theorem.

Theorem 4.4. *The operator B is a positive operator in $W_p^\alpha\left(\mathbb{R}^2\right)$ and the following estimate, for $\lambda > 0$, holds*

$$\left\|\left(\lambda I+B\right)^{-1}\right\|_{W_p^a(\mathbb{R}^2)\to W_p^a(\mathbb{R}^2)} \leq \frac{M\left(\alpha\right)}{\lambda}.$$

5. Applications

In this section, we consider the application of results of sections 3 and 4. For a positive operator A in E the following result is established in [9].

We consider the initial value problem

$$\begin{cases} r\frac{\partial u(t,x,y)}{\partial t} - \omega_1\left(x,y\right)\frac{\partial u(t,x,y)}{\partial x} - \omega_2\left(x,y\right)\frac{\partial u(t,x,y)}{\partial y} = f\left(t,x,y\right) & \begin{pmatrix} (x,y)\in\mathbb{R}^2 \\ 0\leq t\leq T\end{pmatrix} \\ u(0,x,y) = \varphi\left(x,y\right), & (x,y)\in\mathbb{R}^2. \end{cases} \tag{5.1}$$

A function $u(t,x,y)$ is called a solution of problem (5.1) if the following conditions are satisfied:

(1) $u(t,x,y)$ is continuously differentiable with respect to t, x, and y on the region $[0,T]\times\mathbb{R}^2$. The derivative at the endpoints of the region are understood as the appropriate unilateral derivatives.
(2) $u(t,x,y)$ satisfies the equation and initial condition in (5.1).

A solution of problem (5.1) defined in this manner will from now be referred to as a solution of problem (5.1) in the space $C\left([0,T],C\left(\mathbb{R}^2\right)\right)$. Here $C\left([0,T],E\right)$ stands for the Banach space of the continuous functions $\varphi(t)$ defined on $[0,T]$ with values in E, equipped with the norm

$$\|\varphi\|_{C([0,T],E)}=\max_{0\le t\le T}\|\varphi\|_E .$$

If $u(t,x,y)$ is a solution in $C\left([0,T],C\left(\mathbb{R}^2\right)\right)$ of problem (5.1), then the data of the problem must satisfy the following conditions:

(1) $f(t,x,y)$ belongs to $C\left([0,T],C\left(\mathbb{R}^2\right)\right)$.
(2) $\varphi(x,y)$, $\varphi_x(x,y)$, $\varphi_y(x,y)$, $\omega_1(x,y)$, and $\omega_2(x,y)$ belong to $C\left(\mathbb{R}^2\right)$.

Assume that all these conditions hold, and here $\varphi(x,y)$ and $f(t,x,y)$ are sufficiently smooth functions so that which guarantee problem (5.1) has a solution $u(t,x,y)$ in $C\left([0,T],C\left(\mathbb{R}^2\right)\right)$.

Theorem 5.1. *Let $0<\alpha<1$. Then, for the solution of initial value problem (5.1), we have the following stability inequality*

$$\max_{0\le t\le T}\|u(t,.,.)\|_{C^\alpha(\mathbb{R}^2)}\le M(\alpha)\left[\|\varphi(.,.)\|_{C^\alpha(\mathbb{R}^2)}+\max_{0\le t\le T}\|f(t,.,.)\|_{C^\alpha(\mathbb{R}^2)}\right].$$

The proof of Theorem 5.1 is based on Theorem 3.1 on the positivity of the neutron transport operator B defined by the formula

$$Bu(t,x,y)=-\omega_1\frac{\partial u}{\partial x}-\omega_2\frac{\partial u}{\partial y},\ (x,y)\in\mathbb{R}^2 \tag{5.2}$$

and on Theorem 3.3 on the structure of fractional spaces $E_\alpha=E_\alpha\left(C\left(\mathbb{R}^2\right),B\right)$ on the following theorem on stability of the Cauchy problem for the abstract first order differential equation.

Theorem 5.2. *Suppose that A is a positive operator in a Banach space E and that $f\in C\left([0,T],E\right)$. Then, for the solution of the Cauchy problem*

$$u'(t)+Au(t)=f(t),\ 0\le t\le T,\ u(0)=\varphi$$

in a Banach space E with the positive operator A, the following stability inequality holds

$$\|u\|_{C([0,T],\ E)}\le M\left[\|\varphi\|_E+\|f\|_{C([0,T],\ E)}\right].$$

Theorem 5.3. *Let* $0 < \alpha < 1$, *and let* $1 \leq p < \infty$. *Then, for the solution of initial value problem* (5.1), *we have the following stability inequality*

$$\left(\int_0^T \|u(s,.,.)\|^p_{W_p^\alpha(\mathbb{R}^2)} \, ds \right)^{\frac{1}{p}} \leq M(\alpha) \left[\|\varphi(.,.)\|_{W_p^\alpha(\mathbb{R}^2)} + \left(\int_0^T \|f(s,.,.)\|^p_{W_p^\alpha(\mathbb{R}^2)} \, ds \right)^{\frac{1}{p}} \right].$$

The proof of theorem is based on Theorem 4.1 on the positivity of the neutron transport operator B defined by the formula (5.2) and on Theorem 4.3 on the structure of fractional spaces $E_{\alpha,p} = E_{\alpha,p}(L_p(\mathbb{R}^2), B)$ on the following theorem on stability of the Cauchy problem (5.1).

Theorem 5.4. *Let* A *be positive operator in a Banach space* E, *and let* $f \in L_p([0,T], E)$ $1 \leq p < \infty$. *Then, for the solution of* (5.1), *the following stability inequality holds*

$$\|u\|_{L_p([0,T],\ E)} \leq M \left[\|\varphi\|_E + \|f\|_{L_p([0,T],\ E)} \right].$$

Here, $L_p([0,T],\ E)$ *is the space of all abstract strongly measurable* E*-valued functions* $f(t)$ *defined on* $[0,T]$ *for which the norm*

$$\|u\|_{L_p([0,T],\ E)} = \left(\int_0^T \|u(s)\|^p_E \, ds \right)^{\frac{1}{p}}, \ 1 \leq p < \infty,$$

is finite.

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