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THE MATRIX POWER MEANS AND INTERPOLATIONS

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Abstract. It is well-known that the Heron mean is a linear interpolation between the arithmetic and the geometric means while the matrix power mean $P_t(A,B) := A^{1/2}\left(\frac{I+(A^{-1/2}BA^{-1/2})^t}{2}\right)^{1/t}A^{1/2}$ interpolates between the harmonic, the geometric, and the arithmetic means. In this article, we establish several comparisons between the matrix power mean, the Heron mean, and the Heinz mean. Therefore, we have a deeper understanding about the distribution of these matrix means.

1. Introduction

The main result in Kubo and Ando theory of operator means [6] is that there exists an affine-isomorphism between operator means and operator monotone functions on $[0,\infty)$. More precisely, for each operator mean σ there exists a unique operator monotone functions f_σ on $[0,\infty)$ such that

$$A\sigma B = A^{1/2} f_\sigma(A^{-1/2}BA^{-1/2})A^{1/2}$$

whenever positive definite matrices A and B. The function f_σ is called the *representing function* of σ and satisfies $f_\sigma(x) = 1\sigma x$. In this fashion, for $(1+x)/2$, $\sqrt{x}$ and $2x/(1+x)$ we have the corresponding arithmetic, geometric, and harmonic

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means for matrices given by

$$A\nabla B = \frac{A+B}{2}, \quad A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}, \quad A!B = (A^{-1}\nabla B^{-1})^{-1},$$

respectively. Under this isomorphism, we can compare two Kubo-Ando means via their representing functions. For example, the well-known harmonic-arithmetic-geometric mean inequality (HAGM inequality) for positive numbers,

$$1!x \le 1\sharp x \le 1\nabla x$$

implies the matrix analog

$$A!B \le A\#B \le A\nabla B. \tag{1.1}$$

There is a vast amount of literature devoted to different refinements of the previous inequality on the cone of positive definite matrices (see [2, 3] and the references therein). For the arithmetic and the geometric means, an easy refinement can be obtained via a linear interpolation, namely the Heron means

$$H_t(A,B) = (1-t)A\nabla B + tA\sharp B, \quad t \in [0,1].$$

It follows immediately from (1.1) that,

$$A\#B \le H_t(A,B) \le A\nabla B.$$

Another well-known refinement of the AGM inequality is provided by the Heinz means

$$G_t(A,B) = \frac{A\sharp_t B + A\sharp_{1-t}B}{2} \quad (t \in [0,1]),$$

where $A\sharp_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$. In this case,

$$A\#B \le G_t(A,B) \le A\nabla B.$$

A less well-known example of such a refinement is provided by the power means of numbers whose representing function is $f_t(x) = (1\nabla x^t)^{1/t}$. And its corresponding Kubo-Ando extension is given by

$$P_t(A,B) := A^{1/2}\left(\frac{I + (A^{-1/2}BA^{-1/2})^t}{2}\right)^{1/t} A^{1/2}.$$

It is known [7] that P_t is increasing on t and $\lim_{t\to 0} P_t(A,B) = A\#B$. Hence the family of power means provides an interpolation of the harmonic, the geometric, and the arithmetic means.

The matrix power means and the matrix Heinz mean can be thought as parametric curves in the Riemannian manifold of positive definite matrices that pass through the geometric and the arithmetic means. In the case of the power means, the curve also contains the harmonic mean. Therefore, it is important and interesting to understand the behavior of these curves and the relations between one another. As such, in this paper, with respect to the Loewner order on this manifold we compare these families of means at each point t. In addition, we also consider linear interpolations between the harmonic, the geometric, and the arithmetic means.

2. Comparison of the Power Mean and Linear Interpolations

2.1. **Heron Means and Power Means.** It is worth to noting that, recently, the matrix Heron mean

$$H_t(A,B) = t\frac{A+B}{2} + (1-t)A\#B$$

and the matrix power means have been heavily studied [4, 5, 7]. In this section, we consider the linear interpolation between the arithmetic and the geometric means by the Heron mean.

Lemma 2.1. *For any $x \geq 0$ and $t \in [0,1/2]$,*

$$\left(\frac{1+x^t}{2}\right)^{1/t} \leq t\frac{1+x}{2} + (1-t)x^{1/2}. \tag{2.1}$$

The inequality is reversed if $t \in [1/2,1]$.

Proof. In [8], it is shown that

$$\begin{aligned} \inf_{x>0} \{t \mid P_t(1,x) \text{ is concave on } t\} &= \frac{\ln 2}{2} \\ \sup_{x>0} \{t \mid P_t(1,x) \text{ is convex on } t\} &= \frac{1}{2}. \end{aligned}$$

Moreover, their proof also shows that the inflection point is unique. Note that (2.1) can be rephrased in terms of the power means for numbers as

$$P_t(1,x) \geq tP_1(1,x) + (1-t)P_0(1,x)$$

for $0 \leq t \leq 1/2$ and reversed for $1/2 \leq t \leq 1$. Since equality occurs for $t = 0, 1/2, 1$ and right-hand-side is linear, the result follows from the following facts: $P_t(1,x)$ is a concave function of t on $[1/2,1]$ for any $x > 0$, $P_t(1,x)$ is monotonically increasing on t for any $x > 0$, $P_t(1,x)$ has a unique inflection point, and $P_t(1,x)$ is convex for some subset of $[0,1/2]$. □

Theorem 2.2. *For $t \in [0,1/2]$ and positive matrices A and B,*

$$H_t(A,B) \geq P_t(A,B). \tag{2.2}$$

For $t \in [1/2,1]$ the inequality is reversed. Furthermore, equality occurs only for $t \in \{0,1/2,1\}$.

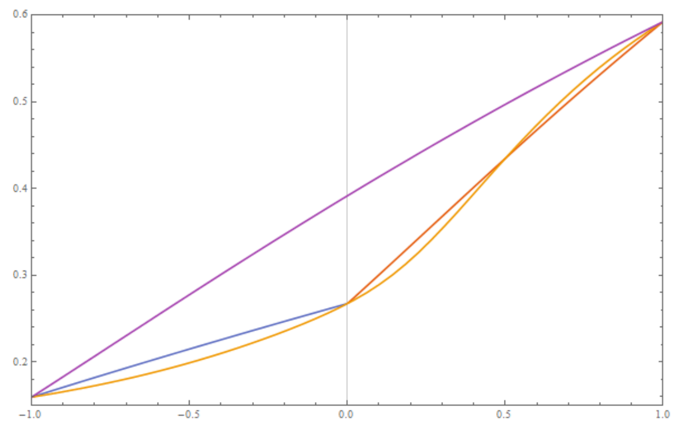
Proof. For $t \in [0,1/2]$, applying Lemma 2.1 for $X = A^{-1/2}BA^{-1/2}$ and multiplying on the left and the right of the inequality (2.1) by $A^{1/2}$ we obtain (2.2). □

2.2. **Harmonic-Geometric Interpolation and Power Means.** In this section we consider the harmonic-geometric linear interpolation

$$F_t(A,B) = t\frac{A!B}{2} + (1-t)A\#B.$$

We obtain a similar result as in Theorem 2.2 relating this interpolation and the power means.

FIGURE 1. Graphs of $\lambda_2(K_t(A,B))$ (purple) and $\lambda_2(P_t(A,B))$ (light orange) on $t \in [-1,1]$, $\lambda_2(F_{-t}(A,B))$ (blue) on $[-1,0]$, and $\lambda_2(H_t(A,B))$ (dark orange) on $t \in [0,1]$ for two 3×3 positive definite matrices A and B.



Theorem 2.3. *For $t \in [0,1]$ and positive matrices A and B,*

$$F_t(A,B) \geq P_{-t}(A,B).$$

Proof. It suffices to show that

$$t(1+x^{-1})^{-1} + (1-t)\sqrt{x} \geq \left(\frac{1+x^t}{2}\right)^{1/t}. \tag{2.3}$$

For $a, b \geq 0$, the power mean $P_s(a,b)$ is a convex function of s on $\mathbb{R}^-$ (see, for example, [1]); that is, for any $p, q \leq 0$ and for any $t \in [0,1]$,

$$tP_p(a,b) + (1-t)P_q(a,b) \geq P_{tp+(1-t)q}(a,b). \tag{2.4}$$

The inequality (2.3) follows from (2.4) by setting $p = -1$ and letting $q \to 0^-$. □

Remark 2.4. From the convexity of the power mean in (2.4) we have the following inequality: for positive definite matrices A and B, for $p, q \leq 0$ and $t \in [0,1]$,

$$P_{tp+(1-t)q}(A,B) \leq tP_p(A,B) + (1-t)P_q(A,B).$$

Up to this point we have compared the power means with the linear interpolations between the harmonic and geometric means and between the geometric and arithmetic mean. Now, we define the linear interpolation between the harmonic and arithmetic mean with the parameter $t \in [-1,1]$

$$K_t(A,B) := \frac{t+1}{2}\left(\frac{A+B}{2}\right) + \frac{1-t}{2}A!B.$$

Figure 1 shows the fact (Theorem 2.5) that this interpolation is greater than the rest, pictorially.

Theorem 2.5. *Let A and B be positive definite matrices. Then,*

(i) *For $t \in [1/2, 1]$,*

$$K_t(A,B) \geq P_t(A,B).$$

(ii) *For* $t \in [-1, 0]$,

$$K_t(A, B) \geq F_{-t}(A, B).$$

(iii) *For* $t \in [0, 1/2]$,

$$K_t(A, B) \geq H_t(A, B).$$

Proof. (i). It suffices to show that for $x > 0$ and $t \in [1/2, 1]$,

$$\frac{t+1}{2}\left(\frac{1+x}{2}\right) + (1-t)\frac{x}{x+1} \geq \left(\frac{1+x^t}{2}\right)^{1/t}. \tag{2.5}$$

The function on right-hand-side is concave for $t \in [1/2, 1]$ (see, for example, [8]). Therefore, it is bounded above by its tangent line at $t = 1$,

$$y = \frac{1}{2}\left(x \ln x - (x+1)\ln\frac{x+1}{2}\right)(t-1) + \frac{x+1}{2}.$$

Hence, (2.5) follows from the following inequality

$$\begin{aligned} &\frac{t+1}{2}\left(\frac{1+x}{2}\right) + (1-t)\frac{x}{x+1} \\ &\qquad\qquad \geq \frac{1}{2}\left(x \ln x - (x+1)\ln\frac{x+1}{2}\right)(t-1) + \frac{x+1}{2}, \end{aligned}$$

or, equivalently,

$$\frac{1+x}{2} - \frac{2x}{x+1} \leq x \ln x - (x+1)\ln\frac{x+1}{2}.$$

Notice at $x = 1$ both left-hand-side and right-hand-side are equal. So, it suffices to show that

$$\frac{1}{2} - \frac{2}{(1+x)^2} \geq \ln\frac{2x}{x+1} \quad (0 < x \leq 1)$$

and

$$\frac{1}{2} - \frac{2}{(1+x)^2} \leq \ln\frac{2x}{x+1} \quad (x \geq 1).$$

Since the functions agree at $x = 1$, it suffices to show that the derivatives satisfy

$$\frac{4}{(1+x)^3} \leq \frac{1}{x(x+1)},$$

which is obvious.

Now we show (ii). The inequality in (iii) can be established by the same arguments. It suffices to show that for $x > 0$ and $-1 \leq t \leq 0$,

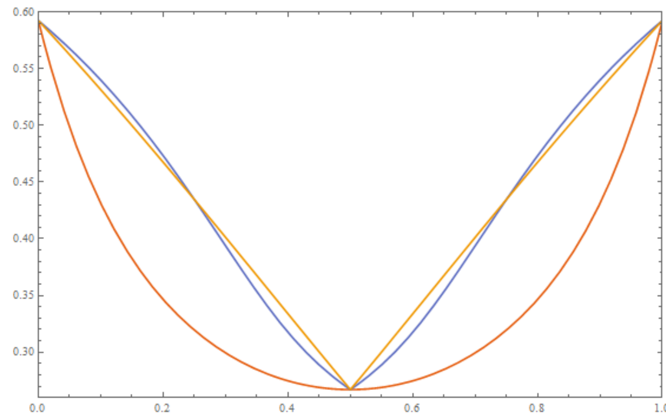
$$\frac{t+1}{2}\left(\frac{1+x}{2}\right) + (1-t)\left(\frac{x}{x+1}\right) \geq -2t\left(\frac{x}{x+1}\right) + (1+t)\sqrt{x}.$$

Since both sides are linear on t and equal at $t = -1$, it suffices to show that

$$\frac{1}{2}\left(\frac{1+x}{2}\right) + \left(\frac{x}{x+1}\right) \geq \sqrt{x}.$$

This is just the arithmetic geometric mean inequality for $\frac{1+x}{2}$ and $\frac{2x}{x+1}$. $\square$

FIGURE 2. Graphs of $\lambda_2(G_t(A,B))$ (dark orange), $\lambda_2(H_{2t-1}(A,B))$ (light orange), and $\lambda_2(P_{2t-1}(A,B))$ (blue) on $t \in [0,1]$ for two 3×3 positive definite matrices A and B.



3. Comparison of the Power Means and NonLinear Interpolations

3.1. **Heinz Means and Power Means.** In this subsection, using similar ideas as the ones in the previous sections, we can derive the following relation between the Heinz means and the power means.

Theorem 3.1. *For positive definite matrices A and B and $t \in [1/2, 1]$,*

$$G_t(A,B) \le P_{2t-1}(A,B).$$

Proof. It suffices to show that for $x > 0$,

$$\frac{x^t + x^{1-t}}{2} \le \left(\frac{1 + x^{2t-1}}{2}\right)^{1/(2t-1)},$$

or, equivalently,

$$\frac{e^{(2t-1)y/2} + e^{-(2t-1)y/2}}{2} \le \left(\frac{e^{-(2t-1)y/2} + e^{(2t-1)y/2}}{2}\right)^{1/(2t-1)},$$

where $x = e^y$. That is,

$$\cosh((2t-1)y/2) \le \cosh((2t-1)y/2)^{1/(2t-1)}.$$

The last inequality is obviously true as $\cosh((2t-1)y/2) \ge 1$ and $1/(2t-1) \ge 1$ on $(1/2, 1]$. $\square$

3.2. **Heinz Means, Heron Means, and Power Means.** In this subsection we show how these three means relate to each other on the interval $[1/2, 1]$. Figure 2 shows a graphic representation of this relation for the second largest eigenvalues of the means of positive definite matrices.

Theorem 3.2. *For positive definite matrices A and B and for $t \in [1/4, 3/4]$,*

$$G_t(A,B) \le P_{|2t-1|}(A,B) \le H_{|2t-1|}(A,B)$$

and for $t \in [0, 1/4] \cup [3/4, 1]$,

$$G_t(A, B) \leq H_{|2t-1|}(A, B) \leq P_{|2t-1|}(A, B).$$

Proof. All is left to show is that for $t \in [1/2, 1]$,

$$G_t(A, B) \leq H_{2t-1}(A, B).$$

The proof for $t \in [0, 1/2]$ follows by symmetry. Using the simplification as before, this reduces to show the following inequality,

$$\frac{2t-1}{2}(1+x) + (2-2t)x^{1/2} \geq \frac{x^t + x^{1-t}}{2} \tag{3.1}$$

for $t \in [1/2, 1]$ and $x > 0$. By dividing by $x^{1/2}$ and substituting x by e^y, (3.1) becomes

$$(2t-1)\cosh\left(\frac{y}{2}\right) + (2-2t) \geq \cosh\left(\frac{y}{2}(2t-1)\right).$$

By the concavity of the function $x \mapsto x^{2t-1}$ on this interval,

$$\begin{aligned}\cosh\left(\frac{y}{2}\right)^{2t-1} &= \left(\frac{e^{y/2} + e^{-y/2}}{2}\right)^{2t-1} \\ &\geq \left(\frac{e^{(2t-1)y/2} + e^{-(2t-1)y/2}}{2}\right) = \cosh\left(\frac{y}{2}(2t-1)\right).\end{aligned}$$

So, the desired inequality follows if we show

$$(2t-1)\cosh\left(\frac{y}{2}\right) + (2-2t) \geq \cosh\left(\frac{y}{2}\right)^{2t-1}.$$

Equivalently,

$$za + (1-z) \geq a^z$$

or

$$z(a-1) + 1 \geq a^z$$

for any positive real a and $0 \leq z \leq 1$. However, this is just Bernoulli's inequality,

$$(x+1)^r \leq 1 + rx$$

for $x = a + 1$ and $z = r$. This completes the proof. □

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