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COMPLEX ISOSYMMETRIC OPERATORS

MUNEO CHŌ[1], JI EUN LEE [2], T. PRASAD[3*], and KÔTARÔ TANAHASHI[4]

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Abstract. In this paper, we introduce complex isosymmetric and (m, n, C)-isosymmetric operators on a Hilbert space $\mathcal{H}$ and study properties of such operators. In particular, we prove that if $T \in \mathcal{B}(\mathcal{H})$ is an (m, n, C)-isosymmetric operator and N is a k-nilpotent operator such that T and N are C-doubly commuting, then $T + N$ is an $(m + 2k - 2, n + 2k - 1, C)$-isosymmetric operator. Moreover, we show that if T is (m, n, C)-isosymmetric and if S is (m', D)-isometric and n'-complex symmetric with a conjugation D, then $T \otimes S$ is $(m + m' - 1, n + n' - 1, C \otimes D)$-isosymmetric.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on a separable infinite-dimensional complex Hilbert space $\mathcal{H}$ with the inner product $\langle \cdot, \cdot \rangle$. A conjugate linear operator $C : \mathcal{H} \to \mathcal{H}$ is said to be a *conjugation* if it satisfies $\langle Cx, Cy \rangle = \langle y, x \rangle$, for all $x, y \in \mathcal{H}$, and $C^2 = I$. For a conjugation C, there exists an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $\mathcal{H}$ such that $Ce_n = e_n$ for all n (see [5] for more information). An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a *complex symmetric* operator if there exists a conjugation C on $\mathcal{H}$ such that $T = CT^*C$ (see [5, 6, 7]). Operators defined by Hankel matrices, binormal operators, all normal operators, compressed Toeplitz operators, algebraic operators of order two, and some Volterra integration operators are complex symmetric. We refer the reader

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*Corresponding author.
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to [5, 6, 7] for more details. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *skew complex symmetric* if there exists a conjugation C on $\mathcal{H}$ such that $CTC = -T^*$.

M. Stankus [8] introduced and studied isosymmetric operators. According to M. Stankus [8] or [9], an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be an *isosymmetry* if

$$T^{*2}T - T^*T^2 - T^* + T = 0.$$

Self-adjoint operators, isometric operators, and some classes of non-normal operators are isosymmetries (see [8] for more details). Recently the authors in [3] studied several properties of isosymmetric operators.

The aim of this paper is to initiate the study of complex isosymmetric and (m, n, C)-isosymmetric operators which are classes of operators that contains complex symmetric operators. We give some properties of these classes of operators.

2. Complex isosymmetric operators

We define complex isosymmetric operators as follows:

Definition 2.1. Let C be a conjugation on $\mathcal{H}$, and let $T \in \mathcal{B}(\mathcal{H})$. We define

$$\Delta(T; C) := T^{*2}CTC - T^*CT^2C - T^* + CTC,$$

and T is said to be *complex isosymmetric with a conjugation* C if

$$\Delta(T; C) = T^{*2}CTC - T^*CT^2C - T^* + CTC = 0.$$

From the definition of complex isosymmetric operators, it is easy to see that if T is complex symmetric with a conjugation C, then T is complex isosymmetric with a conjugation C.

The authors in [1] studied (m, C)-isometric operators. Let $m \in \mathbb{N}$, and let C is a conjugation on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be an (m, C)-*isometric operator* if

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{*m-k}CT^{m-k}C = 0.$$

It is easy to see that if $T^*CTC = I$ (i.e., T is $(1, C)$-isometry), then T is complex isosymmetric with a conjugation C.

Example 2.2. Let $\mathcal{H} = \ell^2(\mathbb{N})$, and let $C : \mathcal{H} \to \mathcal{H}$ be the canonical conjugation given by

$$C(\sum_{n=1}^{\infty} x_n e_n) = \sum_{n=1}^{\infty} \overline{x_n} e_n,$$

where $\{e_n\}$ is the orthonormal basis of $\mathcal{H}$ with $Ce_n = e_n$ and $\{x_n\}$ is a sequence in $\mathbb{C}$ with $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. Let S be the unilateral shift on ℓ^2. Since $CSC = S$, we have $S^*CSC = I$. Hence it is easy to see that S is complex isosymmetric with a conjugation C.

Example 2.3. Let C be a conjugation on $\mathbb{C}^2$ given by $C(x,y) = (\overline{y}, \overline{x})$ for $x, y \in \mathbb{C}$, and let $T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ for some nonzeros $a, b, c \in \mathbb{C}$. Then T is complex isosymmetric with a conjugation C if and only if $ac = 1$ or $a = c$. Indeed, since

$$T^* - CTC = \begin{pmatrix} \overline{a} - \overline{c} & 0 \\ 0 & \overline{c} - \overline{a} \end{pmatrix},$$

it follows that

$$T^*(T^* - CTC)CTC - (T^* - CTC) = 0 \Leftrightarrow (\overline{ac} - 1)(\overline{a} - \overline{c}) = 0.$$

Hence T is complex isosymmetric with a conjugation C if and only if $ac = 1$ or $a = c$. In particular, if $a = c$, then T is complex symmetric with a conjugation C. If $ac = 1$ and $a \neq c$, then T is not $(1, C)$-isometry. For instance, if $R = \begin{pmatrix} 2 & b \\ 0 & \frac{1}{2} \end{pmatrix}$, for some nonzero $b \in \mathbb{C}$, then R is complex isosymmetric with a conjugation C which is not $(1, C)$-isometry.

Theorem 2.4. *Let $T \in \mathcal{B}(\mathcal{H})$, and let C be a conjugation on $\mathcal{H}$. Then the following statements hold:*

(i) *T is complex isosymmetric with a conjugation C if and only if $(T^*CTC - I)CTC$ is complex symmetric with a conjugation C;*

(ii) *If T is invertible, then T is complex isosymmetric with a conjugation C if and only if T^{-1} is complex isosymmetric with a conjugation C.*

Proof. (i) Suppose that T is complex isosymmetric with a conjugation C. Then

$$\begin{aligned} & T^{*2}CTC - T^*CT^2C - T^* + CTC = 0 \\ \Longleftrightarrow\ & T^{*2}CTC - T^* = T^*CT^2C - CTC \\ \Longleftrightarrow\ & T^*(T^*CTC - I) = (T^*CTC - I)CTC. \end{aligned}$$

By the final equation, it holds

$$\begin{aligned} (T^*CTC - I)CTC &= C(CT^*CT^2 - T)C \\ &= C(T^{*2}CTC - T^*)^*C \\ &= C(T^*(T^*CTC - I))^*C \\ &= C((T^*CTC - I)CTC)^*C. \end{aligned}$$

Therefore, $(T^*CTC - I)CTC$ is complex symmetric. The converse implication is clear.

(ii) Suppose that T is complex isosymmetric with a conjugation C. Since

$$\begin{aligned} & T^{*2}CTC - T^*CT^2C - T^* + CTC \\ =\ & C(CT^{*2}CT - CT^*CT^2 - CT^*C + T)C, \end{aligned}$$

it follows that T is complex isosymmetric with a conjugation C if and only if CTC is complex isosymmetric with a conjugation C. Assume that T^{-1} is complex isosymmetric with a conjugation C. Since $CT^{-1}C$ is complex isosymmetric and

$$(T^{-1})^{*2}CT^{-1}C - (T^{-1})^*C(T^{-1})^2C - (T^{-1})^* + CT^{-1}C = 0,$$

it follows that

$$\begin{aligned} 0 &= T^{*2}\Big((T^{-1})^{*2}CT^{-1}C - (T^{-1})^{*}C(T^{-1})^{2}C - (T^{-1})^{*} + CT^{-1}C\Big)CT^{2}C \\ &= CTC - T^{*} - T^{*}CT^{2}C + T^{*2}CTC. \end{aligned}$$

Hence T is complex isosymmetric with a conjugation C. The converse implication is similar. □

Let us recall that the Hardy–Hilbert space, denoted by H^2, consists of all analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the unit disc $\mathbb{D}$ such that $\|f\|_2 := (\sum_{n=0}^{\infty} |a_n|^2)^{\frac{1}{2}} < \infty$.

Example 2.5. Let C be a conjugation defined by $Cf(z) = \overline{f(\overline{z})}$, and let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of H^2. If we put $\mathcal{C} = C \oplus C$, then $\mathcal{C}$ is clearly a conjugation on $H^2 \oplus H^2$. Assume that

$$T = \begin{pmatrix} S & e_0 \otimes e_0 \\ 0 & I \end{pmatrix} \in \mathcal{L}(H^2 \oplus H^2),$$

where S is the unilateral shift on H^2. Then

$$\mathcal{C}T\mathcal{C} = \begin{pmatrix} CSC & C(e_0 \otimes e_0)C \\ 0 & I \end{pmatrix} = T$$

and

$$T^{*}\mathcal{C}T\mathcal{C} - I = \begin{pmatrix} 0 & 0 \\ 0 & e_0 \otimes e_0 \end{pmatrix}.$$

Therefore, we have

$$T^{*}(T^{*}\mathcal{C}T\mathcal{C} - I) = (T^{*}\mathcal{C}T\mathcal{C} - I)\mathcal{C}T\mathcal{C} = \begin{pmatrix} 0 & 0 \\ 0 & e_0 \otimes e_0 \end{pmatrix}$$

and it is complex symmetric with a conjugation $\mathcal{C}$. Hence T is complex isosymmetric with a conjugation $\mathcal{C}$ from Theorem 2.4 (i). However, T is neither $(1, \mathcal{C})$-isometry nor complex symmetric with a conjugation $\mathcal{C}$.

Now we study some properties of $\Delta(T; C)$.

Theorem 2.6. *Let $T \in \mathcal{B}(\mathcal{H})$, and let C be a conjugation on $\mathcal{H}$. Then $\Delta(T; C)$ is skew complex symmetric with a conjugation C.*

Proof. If

$$\Delta(T; C) = T^{*2}CTC - T^{*}CT^{2}C - T^{*} + CTC,$$

then

$$\begin{aligned} C(\Delta(T; C))^{*}C &= C(CT^{*}CT^{2} - CT^{*2}CT - T + CT^{*}C)C \\ &= T^{*}CT^{2}C - T^{*2}CTC - CTC + T^{*} \\ &= -\Delta(T; C). \end{aligned}$$

Hence $\Delta(T; C)$ is skew complex symmetric with a conjugation C. □

For an operator $T \in \mathcal{B}(\mathcal{H})$, the spectrum and the approximate point spectrum are denoted by $\sigma(T)$ and $\sigma_{ap}(T)$, respectively.

Corollary 2.7. *Let $T \in \mathcal{B}(\mathcal{H})$, and let C be a conjugation on $\mathcal{H}$. Then*

$$\sigma(\Delta(T;C)) = \sigma_{ap}(\Delta(T;C)) \cup \big(-\sigma_{ap}(\Delta(T;C))\big).$$

Proof. It is known from [4, Page 222] that for an arbitrary $T \in \mathcal{B}(\mathcal{H})$, $\sigma(T) = \sigma_{ap}(T) \cup \sigma_{ap}(T^*)^*$. Since $\Delta(T;C)$ is skew complex symmetric, it follows from [2, Lemma 2.5] that $\sigma_{ap}(\Delta(T;C)) = -\sigma_{ap}(\Delta(T;C)^*)^*$. Hence

$$\sigma(\Delta(T;C)) = \sigma_{ap}(\Delta(T;C)) \cup \big(-\sigma_{ap}(\Delta(T;C))\big).$$

□

Definition 2.8. For $T \in \mathcal{B}(\mathcal{H})$ and a conjugation C on $\mathcal{H}$, let

$$\alpha(T;C) := T^* - CTC$$

and

$$\beta(T;C) := T^*CTC - I.$$

Then the following lemma is clear. So the proof is omitted.

Lemma 2.9. *Let $T \in \mathcal{B}(\mathcal{H})$, and let C be a conjugation on $\mathcal{H}$. Then the following statements are equivalent:*
(i) *T is complex isosymmetric with a conjugation C;*
(ii) *$T^*\alpha(T;C)CTC = \alpha(T;C)$;*
(iii) *$T^*\beta(T;C) = \beta(T;C)CTC$.*

Theorem 2.10. *Let C be a conjugation on $\mathcal{H}$, and let $T = \begin{pmatrix} N & E \\ 0 & X \end{pmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$, and let $\mathcal{C} = C \oplus C$. Then the following statements hold:*
(i) *Suppose that N is a $(1,C)$-isometric operator and that $N^*CE = CEX$ and that $E = NEX$ hold. Then T is complex isosymmetric with a conjugation $\mathcal{C}$ if and only if X is complex isosymmetric with a conjugation C;*
(ii) *Suppose that N is complex symmetric with a conjugation C and that $EX = NE$ holds. Then T is complex isosymmetric with a conjugation $\mathcal{C}$ if and only if X is complex isosymmetric with a conjugation C and $E = NEX$ holds.*

Proof. It is clear that $\mathcal{C} = C \oplus C$ is a conjugation on $\mathcal{H} \oplus \mathcal{H}$. Since $T = \begin{pmatrix} N & E \\ 0 & X \end{pmatrix}$, it holds $\mathcal{C}T\mathcal{C} = \begin{pmatrix} CNC & CEC \\ 0 & CXC \end{pmatrix}$, and so

$$\beta(T;\mathcal{C}) = \begin{pmatrix} \beta(N;C) & N^*CEC \\ E^*CNC & E^*CEC + \beta(X;C) \end{pmatrix}.$$

Therefore we obtain

$$\begin{aligned} &\beta(T;\mathcal{C})\mathcal{C}T\mathcal{C} \\ &= \begin{pmatrix} \beta(N;C)CNC & \beta(N;C)CEC + N^*CEXC \\ E^*CN^2C & E^*CNEC + E^*CEXC + \beta(X;C)CXC \end{pmatrix} \end{aligned} \tag{2.1}$$

and

$$T^*\beta(T;\mathcal{C}) = \begin{pmatrix} N^*\beta(N;C) & N^{*2}CEC \\ E^*\beta(N;C)+X^*E^*CNC & E^*N^*CEC+X^*E^*CEC+X^*\beta(X;C) \end{pmatrix}. \tag{2.2}$$

By Lemma 2.9 and equations (2.1) and (2.2), T is complex isosymmetric with a conjugation $\mathcal{C}$ if and only if

$$\begin{cases} \beta(N;C)CNC = N^*\beta(N;C), \\ \beta(N;C)CEC + N^*CEXC = N^{*2}CEC, \\ E^*CN^2C = E^*\beta(N;C) + X^*E^*CNC, \\ E^*CNEC + E^*CEXC + \beta(X;C)CXC = E^*N^*CEC + X^*E^*CEC + X^*\beta(X;C). \end{cases} \tag{2.3}$$

(i) Assume that N is $(1,C)$-isometry. Then $\beta(N;C)=0$, and so (2.3) becomes

$$\begin{cases} N^*CEXC = N^{*2}CEC, \\ E^*CN^2C = X^*E^*CNC, \\ E^*CNEC + E^*CEXC + \beta(X;C)CXC = E^*N^*CEC + X^*E^*CEC + X^*\beta(X;C). \end{cases} \tag{2.4}$$

Since $N^*CE = CEX$ and $E = NEX$ hold, it follow from (2.4) that

$$\beta(X;C)CXC = X^*\beta(X;C).$$

For the last equation, if $N^*CE = CEX$ and $E = NEX$, then

$$\begin{aligned} E^*CNEC + E^*CEXC &= X^*E^*N^*CNCCEC + E^*(N^*CE)C \\ &= X^*E^*CEC + E^*N^*CEC. \end{aligned}$$

The first and second equations clearly hold. Hence X is complex isosymmetric with a conjugation C. The converse implication holds by similar arguments.

(ii) Assume that T is complex symmetric with a conjugation C and that X is complex isosymmetric with a conjugation C. Since $EX = NE$ and $N^* = CNC$, it follows that $X^*E^* = E^*N^* = E^*CNC$, and so $X^*E^*C = E^*CN$ and $N^*CE = CEX$ hold. Hence $E^*CNEC + E^*CEXC = X^*E^*CEC + E^*N^*CEC$ holds. Therefore (2.3) becomes

$$\begin{cases} (CN^2C - I)CEC + CNEXC = CN^2EC, \\ E^*N^{*2} = E^*(N^{*2} - I) + X^*E^*N^*. \end{cases}$$

This gives that

$$\begin{cases} CEC = CNEXC, \\ E^* = X^*E^*N^*, \end{cases}$$

which is equivalent to $E = NEX$. The converse implications hold by similar arguments. □

Corollary 2.11. *Let* $T = \begin{pmatrix} V & E \\ 0 & X \end{pmatrix}$ *on* $\mathcal{H}\oplus\mathcal{H}$ *such that* V *is* $(1,C)$*-isometry. If* $V^*CEC = 0$ *and* $X^*(E^*CEC + X^*CXC - I) = (E^*CEC + X^*CXC - I)CXC$*, then* T *is complex isosymmetric with a conjugation* $\mathcal{C} = C \oplus C$.

Proof. Let $A = (E^*CEC + X^*CXC - I)$. Then

$$T^*(T^*\mathcal{C}T\mathcal{C} - I) = (T^*\mathcal{C}T\mathcal{C} - I)\mathcal{C}T\mathcal{C} \Leftrightarrow X^*A = ACXC.$$

Since $X^*A = ACXC$, it follows that T is complex isosymmetric with a conjugation $\mathcal{C}$. □

Theorem 2.12. *Let C be a conjugation on $\mathcal{H}$, and let $T \in \mathcal{B}(\mathcal{H})$. Suppose that $\mathcal{M} = \ker(T^*CTC - I)$ is invariant for T and C. Then T has the following block operator:*

$$T = \begin{pmatrix} V & E \\ 0 & X \end{pmatrix} \quad \text{on } \mathcal{M} \oplus \mathcal{M}^\perp$$

such that V is a $(1, C_1)$-isometric with a conjugation C_1 on $\mathcal{M}$ and $E^*C_1VC_1 = 0$ on $\mathcal{M}$, where $C_1 = C_{|\mathcal{M}}$ and $C_2 = C_{|\mathcal{M}^\perp}$.

Proof. Since $\mathcal{M}$ is invariant for C, it follows from [5, Proposition 7 (1)] that $\mathcal{M}^\perp$ is invariant for C. Set $C_1 = C_{|\mathcal{M}}$ and $C_2 = C_{|\mathcal{M}^\perp}$. Then C_1 and C_2 are conjugations on $\mathcal{M}$ and $\mathcal{M}^\perp$, respectively, and $C = C_1 \oplus C_2$ holds. Since $\mathcal{M}$ is invariant for T, we have

$$T = \begin{pmatrix} V & E \\ 0 & X \end{pmatrix} \quad \text{on } \mathcal{M} \oplus \mathcal{M}^\perp.$$

Hence it holds

$$T^*CTC - I = \begin{pmatrix} V^*C_1VC_1 - I & V^*C_1EC_2 \\ E^*C_1VC_1 & E^*C_1EC_2 + X^*C_2XC_2 - I \end{pmatrix} \quad \text{on } \mathcal{M} \oplus \mathcal{M}^\perp.$$

If $x \in \mathcal{M}$, then $(T^*CTC - I)(x \oplus 0) = 0$. Hence, we have $V^*C_1VC_1 - I = 0$ and $E^*C_1VC_1 = 0$ on $\mathcal{M}$. Hence V is a $(1, C_1)$-isometric with a conjugation C_1 on $\mathcal{M}$. □

3. (m, n, C)-isosymmetric operators

In this section, we study some properties of (m, n, C)-isosymmetric operators.

Definition 3.1. Let $T \in \mathcal{B}(\mathcal{H})$, and let C be a conjugation on $\mathcal{H}$. Put

$$\begin{cases} \alpha_m(T; C) := \sum_{j=0}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} T^{*m-j}CT^jC, \\ \beta_m(T; C) := \sum_{j=0}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} T^{*m-j}CT^{m-j}C. \end{cases}$$

Then T is said to be an (m, n, C)-*isosymmetric* operator if $\gamma_{m,n}(T; C) = 0$ and

$$\gamma_{m,n}(T; C) := \begin{cases} \sum_{j=0}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} T^{*m-j}\alpha_n(T; C)CT^{m-j}C, \\ \sum_{k=0}^{n} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} T^{*n-k}\beta_m(T; C)CT^kC. \end{cases}.$$

It is easy to see that

$$\gamma_{m+1,n}(T;C) = T^*\gamma_{m,n}(T;C)CTC - \gamma_{m,n}(T;C)$$

and

$$\gamma_{m,n+1}(T;C) = T^*\gamma_{m,n}(T,C) - \gamma_{m,n}(T;C)CTC.$$

Hence if T is (m,n,C)-isosymmetric, then T is (m',n',C)-isosymmetric for all $n' \geq n$ and $m' \geq m$. Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be an (m,C)-*isometric operator* if

$$\sum_{k=0}^{m}(-1)^k \binom{m}{k} T^{*m-k}CT^{m-k}C = 0.$$

From Definition 3.1, it is evident that an (m,C)-isometric operator is (m,n,C)-isosymmetric for any $n \in \mathbb{N}$.

Example 3.2. Let $\mathcal{H} = \ell^2(\mathbb{N})$, and let $C : \mathcal{H} \to \mathcal{H}$ be the canonical conjugation given by

$$C(\sum_{k=1}^{\infty} x_k e_k) = \sum_{k=1}^{\infty} \overline{x_k} e_k,$$

where $\{e_k\}$ is the orthonormal basis of $\mathcal{H}$ with $Ce_k = e_k$ and $\{x_k\}$ is a sequence in $\mathbb{C}$ with $\sum_{k=1}^{\infty} |x_k|^2 < \infty$. Let W be the weighted shift on $\ell^2(\mathbb{N})$ defined by $We_k = \alpha_k e_k$, where $\alpha_k = \sqrt{\frac{k+m}{k+1}}$ for $m > 0$. Then W is (m,n,C)-isosymmetric for any $n \in \mathbb{N}$ (see [1, Example 1.1]).

Theorem 3.3. *Let $T \in \mathcal{B}(\mathcal{H})$ and let C be a conjugation on $\mathcal{H}$. Then the following properties hold:*
(i) *If T is invertible, then T is (m,n,C)-isosymmetric if and only if T^{-1} is (m,n,C)-isosymmetric;*
(ii) *If T is (m,n,C)-isosymmetric, then T^k is (m,n,C)-isosymmetric for any $k \in \mathbb{N}$.*

Proof. (i) Let T^{-1} is (m,n,C)-isosymmetric. Then

$$\begin{aligned} 0 &= \sum_{k=0}^{n}(-1)^k \binom{n}{k} (T^{-1})^{*n-k}\beta_m(T^{-1};C)C(T^{-1})^kC \\ &= T^{*m+n}\left(\sum_{k=0}^{n}(-1)^k \binom{n}{k} (T^{-1})^{*n-k}\beta_m(T^{-1};C)C(T^{-1})^kC\right)CT^{m+n}C \\ &= \begin{cases} \gamma_{m,n}(T;C) & \text{if } m+n \text{ is even,} \\ -\gamma_{m,n}(T;C) & \text{if } m+n \text{ is odd.} \end{cases} \end{aligned}$$

Hence T is (m,n,C)-complex isosymmetric. The reverse implication is similar.

(ii) Note that, for any $k \in \mathbb{N}$, the following equation holds:

$$
\begin{aligned}
&(y^k x^k - 1)^m (y^k - x^k)^n \\
&= \Big((yx-1)(y^{k-1}x^{k-1} + y^{k-2}x^{k-2} + \cdots + 1)\Big)^m \Big((y-x)(y^{k-1} + y^{k-2}x + \cdots + x^{k-1})\Big)^n \\
&= \sum_{\ell=0}^{m(k-1)} \sum_{j=0}^{n(k-1)} \lambda_\ell \mu_j y^{m(k-1)-\ell} y^{n(k-1)-j} \big(yx-1\big)^m \big(y-x\big)^n x^j x^{m(k-1)-\ell},
\end{aligned}
$$

where λ_ℓ and μ_j are some constants. From this, we obtain that

$$
\gamma_{m,n}(T^k; C) = \sum_{\ell=0}^{m(k-1)} \sum_{j=0}^{n(k-1)} \lambda_\ell \mu_j T^{*m(k-1)-\ell+n(k-1)-j}\, \gamma_{m,n}(T;C)\, CT^{j+m(k-1)-\ell} C.
$$

Since T is (m,n,C)-isosymmetric; that is, $\gamma_{m,n}(T;C) = 0$, we conclude that T^k is (m,n,C)-isosymmetric for any $k \in \mathbb{N}$. □

Operators T and S are said to be *C-doubly commuting* if $TS = ST$ and $S^*CTC = CTCS^*$. From the equation

$$
\begin{aligned}
&((y_1+y_2)(x_1+x_2)-1)^m((y_1+y_2)-(x_1+x_2))^n \\
&= \sum_{j=0}^{n} \sum_{i+l+h=m} \binom{n}{j} \binom{m}{i,l,h} (y_1+y_2)^i y_2^l (y_1x_1-1)^h (y_1-x_1)^{n-j} (y_2-x_2)^j x_1^l x_2^i,
\end{aligned}
$$

if T and S are C-doubly commuting, then it holds

$$
\begin{aligned}
&\gamma_{m,n}(T+S;C) \\
&= \sum_{j=0}^{n} \sum_{i+l+h=m} \binom{n}{j} \binom{m}{i,l,h} (T^*+S^*)^i S^{*l}\, \gamma_{h,n-j}(T;C)\, \alpha_j(S;C)\, T^l S^i.
\end{aligned} \tag{3.1}
$$

Theorem 3.4. *Let $T \in \mathcal{B}(\mathcal{H})$ be (m,n,C)-isosymmetric, and let N be k-nilpotent. If T and N are C-doubly commuting, then $T+N$ is $(m+2k-2, n+2k-1, C)$-isosymmetric.*

Proof. Since N is k-nilpotent and

$$
\alpha_j(N;C) = \sum_{\mu=0}^{j} (-1)^j \binom{j}{\mu} N^{*j-\mu} C N^\mu C,
$$

we have $\alpha_j(N;C) = 0$ if $j \geq 2k$. From equation (3.1), it holds

$$
\begin{aligned}
&\gamma_{m+2k-2,n+2k-1}(T+N;C) \\
&= \sum_{j=0}^{n+2k-1} \sum_{i+l+h=m+2k-2} \binom{n+2k-1}{j} \binom{m+2k-2}{i,l,h} \\
&\qquad (T^*+N^*)^i N^{*l}\, \gamma_{h,n+2k-1-j}(T;C)\, \alpha_j(N;C) T^l N^i.
\end{aligned}
$$

(1) If $j \geq 2k$ or $i \geq k$ or $l \geq k$, then $\alpha_j(N;C) = 0$ or $N^i = 0$ or $N^{*l} = 0$, respectively.
(2) If $j \leq 2k-1$, $i \leq k-1$, and $l \leq k-1$, then $h = m+2k-2-i-l \geq m$ and $n+2k-1-j \geq n+2k-1-(2k-1) = n$; that is, $\gamma_{h,n+2k-1-j}(T;C) = 0$.

By (1) and (2), we have $\gamma_{m+2k-2,n+2k-1}(T+N;C)=0$. Therefore $T+N$ is $(m+2k-2,n+2k-1,C)$-isosymmetric. □

Corollary 3.5. *Let $S,R\in\mathcal{B}(\mathcal{H})$, and let C be a conjugation on $\mathcal{H}$. Assume that S and R are C-doubly commuting. If S is (m,n,C)-isosymmetric, then the operator $\begin{pmatrix} S & R \\ 0 & S\end{pmatrix}$ on $\mathcal{H}\oplus\mathcal{H}$ is $(m+2,n+3,\mathcal{C})$-isosymmetric, where $\mathcal{C}=C\oplus C$.*

Proof. Put $T=\begin{pmatrix} S & 0 \\ 0 & S\end{pmatrix}$ and $N=\begin{pmatrix} 0 & R \\ 0 & 0\end{pmatrix}$. Then it is clear that $\mathcal{C}$ is a conjugation on $\mathcal{H}\oplus\mathcal{H}$, T is $(m,n,\mathcal{C})$-isosymmetric, and N is 2-nilpotent. Since S and R are C-doubly commuting, it follows that $TN=NT$ and $N^*\mathcal{C}T\mathcal{C}=\mathcal{C}T\mathcal{C}N^*$. Thus T and N are $\mathcal{C}$-doubly commuting. Hence $T+N=\begin{pmatrix} S & R \\ 0 & S\end{pmatrix}$ is $(m+2,n+3,\mathcal{C})$-isosymmetric from Theorem 3.4. □

Note that the equation

$$\begin{aligned}&(y_1y_2x_1x_2-1)^m(y_1y_2-x_1x_2)^n\\ =&\sum_{k=0}^{m}\sum_{j=0}^{n}\binom{m}{k}\binom{n}{j}y_1^{j+k}(y_1x_1-1)^{m-k}(y_1-x_1)^{n-j}(y_2x_2-1)^k(y_2-x_2)^jx_1^kx_2^{n-j}.\end{aligned}$$

From this, if T and S are C-doubly commuting, then it holds

$$\gamma_{m,n}(TS;C)=\sum_{k=0}^{m}\sum_{j=0}^{n}\left(\begin{array}{c} m \\ k\end{array}\right)\left(\begin{array}{c} n \\ j\end{array}\right)T^{*j+k}\gamma_{m-k,n-j}(T;C)\gamma_{k,j}(S;C)T^kS^{n-j}. \tag{3.2}$$

Theorem 3.6. *Let $T\in\mathcal{B}(\mathcal{H})$ be (m,n,C)-isosymmetric, and let $S\in\mathcal{B}(\mathcal{H})$ be an (m',C)-isometric operator and n'-complex symmetric with a conjugation C. If T and S are C-doubly commuting, then TS is $(m+m'-1,n+n'-1,C)$-isosymmetric.*

Proof. From equation (3.2), it holds

$$\begin{aligned}&\gamma_{m+m'-1,n+n'-1}(TS;C)\\ =&\sum_{k=0}^{m+m'-1}\sum_{j=0}^{n+n'-1}\binom{n+n'-1}{j}\binom{m+m'-1}{k}\\ &T^{*j+k}\gamma_{m+m'-1-k,n+n'-1-j}(T;C)\gamma_{k,j}(S;C)T^kS^{n+n'-1-j}.\end{aligned}$$

(1) If $k\geq m'$ or $j\geq n'$, then $\gamma_{k,j}(S;C)=0$.
(2) If $k\leq m'-1$ and $j\leq n'-1$, then $m+m'-1-k\geq m$ and $n+n'-1-j\geq n$; that is, $\gamma_{m+m'-1-k,n+n'-1-j}(T;C)=0$.
By (1) and (2), we have $\gamma_{m+m'-1,n+n'-1}(TS;C)=0$. Hence it completes the proof. □

Corollary 3.7. *Let $T\in\mathcal{B}(\mathcal{H})$, and let C be a conjugation on $\mathcal{H}$ such that $T^*CTC=CTCT^*$. Then the following properties hold:*
(i) *If T is (m,C)-isometric, then T^2 is $(2m-1,1,C)$-isosymmetric.*

(ii) *If T is n-complex symmetric with a conjugation C, then T^2 is $(1, 2n-1, C)$-isosymmetric.*

Proof. Since $T^*CTC = CTCT^*$, the proofs of (i) and (ii) follow from Theorem 3.6. □

For a complex Hilbert space $\mathcal{H}$, let $\mathcal{H} \otimes \mathcal{H}$ denote the completion of the algebraic tensor product of $\mathcal{H}$ and $\mathcal{H}$ endowed a reasonable uniform cross-norm. For operators $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$, $T \otimes S \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ denote the *tensor product* operator defined by T and S. Note that $T \otimes S = (T \otimes I)(I \otimes S) = (I \otimes S)(T \otimes I)$. It is clear that if C and D are conjugations on $\mathcal{H}$, then $C \otimes D$ is a conjugation on $\mathcal{H} \otimes \mathcal{H}$.

Theorem 3.8. *Let $T \in \mathcal{B}(\mathcal{H})$ be (m, n, C)-isosymmetric, and let $S \in \mathcal{B}(\mathcal{H})$ be an (m', D)-isometric operator and n'-complex symmetric with a conjugation D. Then $T \otimes S$ is $(m + m' - 1, n + n' - 1, C \otimes D)$-isosymmetric.*

Proof. $C \otimes D$ is a conjugation on $\mathcal{H} \otimes \mathcal{H}$, and it is clear that if T is (m, n, C)-isosymmetric, then $T \otimes I$ is $(m, n, C \otimes D)$-isosymmetric and if S is (m', D)-isometric and n'-complex symmetric with a conjugation D, then $I \otimes S$ is $(m', C \otimes D)$-isometric and n'-complex symmetric with a conjugation $C \otimes D$. Since $T \otimes I$ and $I \otimes S$ are $C \otimes D$-doubly commuting, it follows from Theorem 3.6 that $T \otimes S$ is $(m + m' - 1, n + n' - 1, C \otimes D)$-isosymmetric. □

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[1]Department of Mathematics, Kanagawa University, Hiratsuka 259-1293, Japan.
E-mail address: chiyom01@kanagawa-u.ac.jp

[2]Department of Mathematics and Statistics, Sejong University, Seoul 143-747, Korea.
E-mail address: jieunlee7@sejong.ac.kr;jieun7@ewhain.net

[3]Department of Mathematics, Cochin university of Science and Technology, Kochi, India.
E-mail address: prasadvalapil@gmail.com

[4]Department of Mathematics, Tohoku Medical and Pharmaceutical University, Sendai 981-8558, Japan.
E-mail address: tanahasi@tohoku-mpu.ac.jp