UNITARY REPRESENTATIONS OF GROUP EXTENSIONS. I

BY

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Introduction

Let $\mathfrak{G}$ be a separable locally compact group. Continuing the convention adopted in our papers [11] and [12] we shall abbreviate the term "continuous unitary representation of $\mathfrak{G}$" to "representation of $\mathfrak{G}$". If $\mathfrak{K}$ is a proper closed subgroup of $\mathfrak{G}$ whose representations are in a suitable sense "all known" one may pose the following two questions. (a) Which representations of $\mathfrak{K}$ are the restrictions to it of irreducible representations of $\mathfrak{G}$? (b) Given such a representation of $\mathfrak{K}$ how can one construct all irreducible representations of $\mathfrak{G}$ of which it is the restriction? When $\mathfrak{K}$ is the identity subgroup question (a) has a trivial answer (apart from questions of dimension) and question (b) is essentially the same as that of determining all irreducible representations of $\mathfrak{G}$? However, for other choices of $\mathfrak{K}$, questions (a) and (b) can furnish a useful breakdown of the problem of determining all irreducible representations of $\mathfrak{G}$ into two more accessible components. It is the primary purpose of this paper to discuss questions (a) and (b) and their application to the determination of the representations of $\mathfrak{G}$ in the special case in which $\mathfrak{K}$ is normal. Actually we shall find it more convenient to deal with the slight variation in which we identify representations of $\mathfrak{K}$ which are quasi equivalent in the sense defined on page 195 of [12].

For the special case in which $\mathfrak{K}$ is not only normal but commutative and in which $\mathfrak{G}$ is a semi direct product of $\mathfrak{K}$ and $\mathfrak{G}/\mathfrak{K}$ this program has been carried out in outline in our paper

(1) In large part the material in this paper has been described in outline in each of the following: (a) Two lectures given in Paris in October 1954 under the auspices of the "Colloque Henri Poincaré". (b) A series of ten lectures on group representations given under the auspices of the Princeton University physics department and supported by the Eugene Higgins fund. (c) A course in group representations given during the 1955 summer quarter at the University of Chicago. (d) A paper presented by title at the 1955 summer meeting of the American Mathematical Society (Abstract 61-6-726 t). Mimeographed lecture notes of the University of Chicago course have been issued by the University of Chicago mathematics department and it is possible that the Centre Nationale des Recherches Scientifiques will publish a volume containing the texts of the lectures presented at the "Colloque Henri Poincaré".

[10]. (See also [11] for a clarification and reformulation of parts of [10]). When the automorphisms of the character group $\hat{\mathfrak{K}}$ of $\mathfrak{K}$ defined by the inner automorphisms of $\mathfrak{G}$ had orbits in $\hat{\mathfrak{K}}$ which were "sufficiently smooth" question (a) was given a complete answer and question (b) was reduced to the problem of finding the irreducible representations of certain subgroups of $\mathfrak{G}/\mathfrak{K}$. It is natural to expect the same kind of reduction of question (b) even when the commutativity and semi direct product hypotheses are dropped. As we shall see, however, the abandonment of either hypothesis leads to a situation in which we may have to study not the ordinary representations of subgroups of $\mathfrak{G}/\mathfrak{K}$ but certain "projective" representations; that is homomorphisms of these subgroups into the quotient groups of the unitary group by the subgroup of constant operators.

At first sight this last circumstance would seem to be a serious obstacle in the way of using our program inductively to determine the representations of complicated groups in several stages. However, it turns out to be possible to carry out the whole discussion from the beginning for projective representations themselves, and when this is done it is still only projective representations which appear in answering question (b). We shall thus concern ourselves throughout with projective representations; ordinary representations, of course, being included as a special case.

The abandonment of the hypothesis that $\mathfrak{K}$ is commutative leads to another difficulty in that the rather complete duality theory for locally compact abelian groups is no longer available. However, combining the von Neumann theory of direct integrals with a theory of "Borel structure" in the set $\hat{\mathfrak{K}}$ of equivalence classes of irreducible representations of $\mathfrak{K}$ we obtain a partial substitute. This substitute, worked out in [13] expressly for the needs of the present article, yields a decomposition theory for representations fully as complete as in the abelian case whenever the group $\mathfrak{K}$ has only type I representations and a "sufficiently regular" Borel structure in $\hat{\mathfrak{K}}$.

Using the material in [13] and working from the beginning with projective representations we obtain a generalization of the results of [10] in which the semi direct product hypothesis is dropped altogether and $\mathfrak{K}$ is allowed to be any closed normal subgroup of $\mathfrak{G}$ to which the decomposition theory of [13] applies. As in [10] we get a complete theory only when certain "orbits" in $\hat{\mathfrak{K}}$ are "sufficiently smooth". We hope to study the situation for non smooth orbits in a later article.

We begin the paper with four sections on the general theory of projective representations. Section one contains the basic definitions. Section two contains an extension to the infinite case of a classical device which enables one to deduce theorems about projective representations from corresponding ones about ordinary representations. In sections three and four certain known results about direct integral decompositions and about induced representations are generalized so as to apply to projective representations. Sections five and six contain a detailed account of the material on systems of imprimitivity sketched in [10]; somewhat generalized to fit the needs of the present paper and somewhat modified in other respects. In sections seven and eight we apply the results of the earlier sections to the study of our main problem. Our principal result is theorem 8.4. The final section nine contains applications and examples. Here, amongst other things, we show how a problem arising in quantum field theory can be formulated as the problem of finding certain projective representations of a certain discrete group, we find quite explicitly the irreducible representations of the solvable group of all 3×3 unimodular real matrices with zeros above the main diagonal, and we prove a theorem that can be used to show that many solvable groups have only type I representations.

As far as purely algebraic aspects of our problem are concerned a large part of what we do is contained in a well known paper of Clifford [3] dealing with finite dimensional representations of discrete groups and in earlier work to which he refers. Infinite dimensional projective representations of topological groups have been considered by Wigner in [17] and more recently and systematically by Bargmann in [1]. Bargmann is chiefly concerned with the problem of finding all possible multipliers (see section 1 for definition) for a given group. Since we consider this problem only briefly in section nine, and then for a different class of groups, there is very little overlap between this paper and Bargmann's. Wigner's paper studies the possible projective representations of the inhomogeneous Lorentz group. Given his determination of the possible multipliers for this group his results are deducible from our theory just as his results on the ordinary representations of this group were deduced from the theory in [10]. Mention should also be made of a very recent paper of Takenouchi [15] which discusses briefly a special situation falling under our general theory.

1. Elementary facts about projective representations

Let $\mathfrak{G}$ be a separable locally compact group. By a *projective representation* L of $\mathfrak{G}$ we shall mean a mapping $x \to L_x$ of $\mathfrak{G}$ into the group of all unitary transformations of some separable Hilbert space $\mathfrak{H}(L)$ onto itself such that (a) $L_e = I$ where e is the identity of $\mathfrak{G}$ and I is the identity operator, (b) For all x and y in $\mathfrak{G}$, L_{xy} is a constant multiple $\sigma(x,y)$ of $L_x L_y$, and (c) For each ϕ and ψ of $\mathfrak{H}(L)$ the function $x \to (L_x(\phi), \psi)$ is a Borel function on $\mathfrak{G}$. The function σ: $x,y \to \sigma(x,y)$ is uniquely determined by L and will be called the *multiplier* of L. By a σ *representation* of $\mathfrak{G}$ we shall mean a projective representation whose multiplier is σ. It is easy to see that the multiplier σ of the projective representation L has the following properties: (a) $\sigma(e) = \sigma(e,x) = \sigma(x,e) = 1$ and $|\sigma(x,y)| = 1$ for all x and y

in $\mathfrak{G}$. (b) $\sigma(xy,z)\sigma(x,y) = \sigma(x,yz)\sigma(y,z)$ for all x,y and z in $\mathfrak{G}$. (c) σ is a Borel function on $\mathfrak{G} \times \mathfrak{G}$. We call any function from $\mathfrak{G} \times \mathfrak{G}$ to the complex numbers which has these three properties a *multiplier* for $\mathfrak{G}$. As we shall see later every multiplier for $\mathfrak{G}$ is the multiplier of some projective representation L of $\mathfrak{G}$.

For a fixed choice of σ one can develop a theory of σ representations which in most respects is completely analogous to the theory of ordinary representations—that is to the theory of σ representations with $\sigma(x,y) \equiv 1$. If L and M are σ representations we say that they are *equivalent* if there exists a unitary transformation U from $\mathfrak{H}(L)$ onto $\mathfrak{H}(M)$ such that $UL_xU^{-1} = M_x$ for all $x \in \mathfrak{G}$. If $\mathfrak{H}_1$ is a closed subspace of $\mathfrak{H}(L)$ such that $L_x(\mathfrak{H}_1) \subseteq \mathfrak{H}_1$ for all $x \in \mathfrak{G}$ then the restriction of each L_x to $\mathfrak{H}_1$ defines a new σ representation $L^{\mathfrak{H}_1}$ of $\mathfrak{G}$ such that $\mathfrak{H}(L^{\mathfrak{H}_1}) = \mathfrak{H}_1$ and which we may refer to as a sub σ representation of L. It is easy to prove that the orthogonal complement of $\mathfrak{H}_1$ also defines a sub σ representation and it is clear that in an obvious sense L is the "direct sum" of these two subrepresentations. When there are no proper sub σ representations of L we say that L is irreducible; otherwise that it is reducible. If L and M are σ representations of $\mathfrak{G}$ then we denote by $\mathbf{R}(L,M)$ the set of all intertwining operators for L and M where by an *intertwining operator* we mean a bounded linear operator T from $\mathfrak{H}(L)$ to $\mathfrak{H}(M)$ such $TL_x = M_xT$ for all $x \in \mathfrak{G}$. We call L a *factor* σ representation (or a *primary* σ representation) if the center $\mathbf{CR}(L,L)$ of $\mathbf{R}(L,L)$ contains only multiples of the identity; that is if $\mathbf{R}(L,L)$ is a factor in the sense of von Neumann and Murray. The general theory largely reducing the study of general representations to that of irreducible representations and factor representations extends without essential change to σ representations.

When it comes to the formation of Kronecker products of projective representations some of the parallelism with ordinary representations disappears. The Kronecker product of two σ representations is not a σ representation in general but a σ^2 representation. More generally let L be a σ_1 representation of $\mathfrak{G}_1$ and let M be a σ_2 representation of $\mathfrak{G}_2$. Then $x,y \to L_x \times M_y^*$ will be a $\sigma_1 \times \sigma_2$ representation of $\mathfrak{G}_1 \times \mathfrak{G}_2$ where $\sigma_1 \times \sigma_2((x_1,y_1),(x_2,y_2)) = \sigma_1(x_1,x_2)\sigma_2(y_1,y_2)$. We call this representation the *outer Kronecker product* $L \times M$ of L and M and we call $\sigma_1 \times \sigma_2$ the *outer product* of the two multipliers σ_1 and σ_2. When $\mathfrak{G}_1 = \mathfrak{G}_2 = \mathfrak{G}$ then restriction of $L \times M$ to the diagonal $\tilde{\mathfrak{G}}$ consisting of all x,y with $x = y$ defines a projective representation of $\mathfrak{G}$ whose multiplier is simply the product of the multipliers σ_1 and σ_2. We call this $\sigma_1\sigma_2$ representation of $\mathfrak{G}$ the *Kronecker product* $L \otimes M$ of L and M.

Similarly if L is a σ representation of $\mathfrak{G}$ then $x \to (L_x^*)^{-1}$ (where A^* denotes the adjoint of A as an operator in the dual $\overline{\mathfrak{H}(L)}$ of $\mathfrak{H}(L)$; the canonical anti linear mapping of $\overline{\mathfrak{H}(L)}$ on $\mathfrak{H}(L)$ being ignored) is a $1/\sigma$ representation of $\mathfrak{G}$ which we call the adjoint $\bar{L}$ of L. We note that $L \otimes \bar{L}$ is always an ordinary representation of $\mathfrak{G}$.

We remark that when σ_1 and σ_2 are distinct multipliers for $\mathfrak{G}$ the theory of the σ_1 representations of $\mathfrak{G}$ can be as different from the theory of the σ_2 representations of $\mathfrak{G}$ as the ordinary representation theories of two different groups. As we shall see, for example, we can choose $\mathfrak{G}$ and σ so that while $\mathfrak{G}$ is commutative it has factor σ representations which are not of type I. We can also choose $\mathfrak{G}$ and σ so that while $\mathfrak{G}$ is commutative it has (to within equivalence) just one irreducible σ representation and that one is infinite dimensional.

On the other hand there is a simple relation which may exist between pairs of multipliers and which implies a complete parallelism between the corresponding σ representation theories. Let ϱ be a Borel function from $\mathfrak{G}$ to the complex numbers of modulus one such that $\varrho(e) = 1$. On setting $\sigma_\varrho(x,y) = \varrho(xy)/\varrho(x)\varrho(y)$ we verify at once that σ_ϱ is a multiplier for $\mathfrak{G}$. Let σ_1 and σ_2 be any two multipliers for $\mathfrak{G}$ such that $\sigma_2 = \sigma_\varrho \sigma_1$. Then if L is a σ_1 representation of $\mathfrak{G}$ we compute easily that L', where $L'_x = \varrho(x) L_x$, is a σ_2 representation. Moreover it is not difficult to see that $L \to L'$ is a one-to-one correspondence between the σ_1 representations of $\mathfrak{G}$ and the σ_2 representations of $\mathfrak{G}$ which preserves equivalence, irreducibility etc. in such a manner that once the theory of the σ_1 representations of $\mathfrak{G}$ has been worked out that for the σ_2 representations follows at once. Accordingly when $\sigma_2 = \sigma_\varrho \sigma_1$ for some Borel function ϱ such that $\varrho(e) = 1$ we shall say that σ_1 and σ_2 are *similar* multipliers. Multipliers of the form σ_ϱ we shall call *trivial* multipliers. It is obvious that the multipliers for $\mathfrak{G}$ form an Abelian group under multiplication and that the trivial multipliers form a subgroup. Let us denote these two groups by the symbols $\mathcal{M}'_{\mathfrak{G}}$ and $\mathcal{J}_{\mathfrak{G}}$. The group $\mathcal{M}'_{\mathfrak{G}} \mid \mathcal{J}_{\mathfrak{G}}$ whose elements are the similarity classes of multipliers for $\mathfrak{G}$ we shall call the multiplier group of $\mathfrak{G}$ and denote by $\mathcal{M}_{\mathfrak{G}}$.

2. A relationship between ordinary and projective representations

In many cases generalizing a theorem about ordinary representations to a corresponding theorem about σ representations presents no difficulties at all; the most obvious minor modifications in the ordinary proof leading at once to a proof for σ representations. However this is not always so and even when it is it may be a tedious task to make sure. Fortunately there is a simple device which often enables one to pass almost directly from the theorem for trivial σ to the theorem for general σ. This device is as follows. If σ is any multiplier for $\mathfrak{G}$ we define a new group $\mathfrak{G}^\sigma$ whose elements are pairs (λ, x) with λ a complex number of modulus one and $x \in \mathfrak{G}$ and in which two pairs are multiplied according to the rule: $(\lambda,x)(\mu,y) = (\lambda\mu/\sigma(x,y), xy)$. There is no difficulty in verifying that $\mathfrak{G}^\sigma$ thus defined is indeed a group with identity $(1, e)$ and with $(\sigma(x,x^{-1})/\lambda,\ x^{-1})$ as the inverse of λ, x. The obvious topology, namely the direct product of the complex number topology with that in $\mathfrak{G}$, will not do for our purposes since with it $\mathfrak{G}^\sigma$ is not in general a topological group. Making use however of Theorem 7.1 of [13] we can introduce a suitable topology. Let $\mathcal{K}$ denote the compact group of all complex numbers of modulus one. $\mathcal{K}$ and $\mathfrak{G}$ then, as separable locally compact groups, have natural Borel structures which are "standard" in the sense described in [13]. The direct product of these defines a standard Borel structure in $\mathfrak{G}^\sigma$ with respect to which $x, y \rightarrow xy^{-1}$ is readily seen to be a Borel function. Thus $\mathfrak{G}^\sigma$ is a standard Borel group in the sense of section 5 of [13]. Moreover it is trivial to verify that the direct product of Haar measure in $\mathcal{K}$ with a right invariant Haar measure in $\mathfrak{G}$ is a right invariant measure in $\mathfrak{G}^\sigma$. Thus Theorem 7.1 of [13] applies and tells us that $\mathfrak{G}^\sigma$ admits a unique locally compact topology under which it is a separable locally compact group whose associated Borel structure is that just described. We suppose $\mathfrak{G}^\sigma$ equipped with this topology. Now for each σ representation L of $\mathfrak{G}$ let $L^0_{\lambda, x} = \lambda L_x$ and designate by L^0 the mapping $\lambda, x \rightarrow L^0_{\lambda, x}$. We have then

THEOREM 2.1. *For each σ representation L of $\mathfrak{G}$ the mapping L^0 is an ordinary representation of $\mathfrak{G}^\sigma$. Moreover the correspondence $L \rightarrow L^0$ is one-to-one and has for its range the set of all ordinary representations of $\mathfrak{G}^\sigma$ which reduce on $\mathcal{K}$ to a multiple of the one dimensional representation $\lambda, e \rightarrow \lambda$.*

Proof. The proof is straightforward and may be left to the reader. In subsequent sections we shall give a number of examples of the use of the correspondence described in Theorem 2.1 in deducing theorems about σ representations from theorems about ordinary representations.

Lest the reader suppose that this correspondence might be used to eliminate the consideration of projective representations altogether we hasten to point out that in the main problem of this paper its application leads around a circle. In fact $\mathfrak{G}^\sigma$ is itself a group extension of $\mathfrak{G}$ by the group $\mathcal{K}$ of complex numbers of modulus one. Thus while the problem of finding the σ representations of $\mathfrak{G}$ can be reduced to that of finding certain ordinary representations of $\mathfrak{G}^\sigma$ the latter problem leads back to that of finding the σ representations of $\mathfrak{G}^\sigma/\mathcal{K} = \mathfrak{G}$.

As a further application of Theorem 2.1 we establish a connection between projective representations (as defined in section 1) and the continuous homomorphisms of $\mathfrak{G}$ into the "projective group". Let $\mathcal{U}(\mathfrak{H})$ denote the group of all unitary transformations of the Hilbert space onto itself and let $\mathcal{U}^e(\mathfrak{H})$ denote its quotient group modulo the normal subgroup $\mathcal{K}$ of scalar multiples of the identity I. Then for each ϕ and ψ in $\mathfrak{H}$, $|(U(\phi), \psi)|$ depends only upon the $\mathcal{K}$ coset to which U belongs and hence defines a function on $\mathcal{U}^e(\mathfrak{H})$. Let us denote this function by $f_{\phi, \psi}$. We shall say that the homomorphism $x \rightarrow M_x$ from $\mathfrak{G}$ into $\mathcal{U}^e(\mathfrak{H})$

is continuous if $f_{\phi,\psi}(M_x)$ is a continuous function of x for all ϕ and ψ in $\mathfrak{H}$. This definition is easily seen to be equivalent to that given by Bargmann in [1].

THEOREM 2.2. *Let σ be a multiplier for the separable locally compact group $\mathfrak{G}$ and let h be the canonical mapping of $\mathcal{U}(\mathfrak{H}(L))$ on $\mathcal{U}^{\varrho}(\mathfrak{H}(L))$. Then $x \to h(L_x)$ is a continuous homomorphism of $\mathfrak{G}$ into $\mathcal{U}^{\varrho}(\mathfrak{H}(L))$. Conversely every continuous homomorphism of $\mathfrak{G}$ into $\mathcal{U}^{\varrho}(\mathfrak{H})$ is of the form $x \to h(L_x)$ for some σ representation L of $\mathfrak{G}$.*

Proof. To prove the first statement we need only show that $|(L_x(\phi),\psi)|$ is continuous in x for all ϕ and ψ in $\mathfrak{H}(L)$. But $\lambda, x \to L^0_{\lambda,x}$ is an ordinary representation $\mathfrak{G}^\sigma$. Hence $(L^0_{\lambda,x}(\phi),\psi)$ is continuous on $\mathfrak{G}^\sigma$. Hence $|\lambda(L_x(\phi),\psi)| = |(L_x(\phi),\psi)|$ is continuous on $\mathfrak{G}^\sigma$. Hence $|(L_x(\phi),\psi)|$ is continuous on $\mathfrak{G}$. To prove the converse let $\mathfrak{H}$ be a Hilbert space and let $x \to M_x$ be a homomorphism of $\mathfrak{G}$ into $\mathcal{U}^{\varrho}(\mathfrak{H})$ such that $f_{\phi,\psi}(M_x)$ is continuous in x for all ϕ and ψ in $\mathfrak{H}$. We must choose a representative L_x of each M_x in such a manner that $(L_x(\phi),\psi)$ is a Borel function of x for all ϕ and ψ in $\mathfrak{H}$. Let $\phi_1, \phi_2, \ldots$ be an orthonormal basis for $\mathfrak{H}$. For each $x \in \mathfrak{G}$ the representatives V_x of M_x differ from one another by factors e^{it}. Hence for each $j = 1, 2, \ldots$ $(V_x(\phi),\psi)$ is zero for every representative or for none. Let j be the least integer such that $(V_x(\phi_1),\phi_j) \neq 0$ for any representative V_x and choose as L_x the (obviously unique) representative such that $(L_x(\phi_1),\phi_j) > 0$. Now let θ,ψ,ϕ be any three elements in $\mathfrak{H}$ and let O be the set of all x in $\mathfrak{G}$ for which $(L_x(\phi),\theta) \neq 0$. We show next that $(L_x(\psi),\theta)/(L_x(\phi),\theta)$ is continuous on O. Let $\varrho(x) = |(L_x(\phi),\theta)|/(L_x(\phi),\theta)$. Then $\varrho(x)(L_x(\phi),\theta) = |(L_x(\phi),\theta)|$ and is continuous in x. Hence we need only show that $(N_x(\psi),\theta)$ is continuous where $N_x = \varrho(x)L_x$. Let $(N_x(\psi),\theta) = u(x) + iv(x)$ where u and v are real valued function and let $(N_x(\phi),\theta) = w(x)$. Then w is real and continuous and different from zero in O. Moreover $|(N_x(\phi+\psi),\theta)|$ is continuous in x and so is $|(N_x(\phi + i\psi),\theta|$. Hence $(w(x) + u(x))^2 + v(x)^2$ and $(w(x) - v(x))^2 + u(x)^2$ are continuous. Hence $w(x)^2 + v(x)^2 + u(x)^2 + 2w(x)u(x)$ and $w(x)^2 + v(x)^2 + u(x^2) - 2w(x)v(x)$ are continuous. Also $u(x)^2 + v(x)^2$ is continuous, so $w(x)^2 + 2w(x)u(x)$ and $w(x)^2 - 2w(x)v(x)$ are continuous. Hence $2u(x) + 1$ and $1-2v(x)$ are continuous on the set O where $w(x) \neq 0$. Hence $(N_x(\psi),\theta)$ is continuous on O as was to be proved. It follows at once that $(L_x(\phi_1),\theta)$ is continuous on the set O_j where $(L_x(\phi_1),\phi_j) \neq 0$ and hence that $(L_x(\phi_1),\theta)$ is a Borel function of x for all θ. Now let ψ be any element of $\mathfrak{H}$ with $\|\psi\| = 1$. Then ψ is part of a basis and by the argument just given there exists an element L'_x in each M_x so that $(L'_x(\psi),\theta)$ is a Borel function of x for all θ. Let $\theta_1,\theta_2,\ldots$ be a countable dense subset of $\mathfrak{H}$ and let S_j be the set of all x for which $(L'_x(\psi),\theta_j)(L_x(\phi_1),\theta_j) \neq 0$. Then $\mathfrak{G} = \bigcup_{j=1}^{\infty} S_j$. Let $L'_x = \varrho(x)L_x$. Then on S_j $\varrho(x)$ is the quotient of $(L'_x(\psi),\theta_j)$ by the product of $(L_x(\psi),\theta_j)/(L_x(\phi_1),\theta_j)$ with $(L_x(\phi_1),\theta_j)$ and hence is a Borel function of x there. Hence ϱ is a Borel function of x. Hence $(L_x(\psi),\theta)$ is a Borel function of x for all ψ and θ and the proof is complete.

3. Decomposition theory for σ representations

The global multiplicity theory for ordinary representations described in section one of [12] extends almost word for word to σ representations. In fact inspection shows that this theory is almost completely independent of the object being represented and can be formulated without difficulty for quite general representations of quite general objects. At any rate the lemmas and theorems formulated on pages 194 through 198 of [12] are all true for σ representation; the proofs being as given for ordinary representations. It follows in particular that the study of type I σ representations may be reduced to the study of multiplicity free σ representations. The correspondence between multiplicity free representations and measure classes in the dual object developed in sections nine and ten of [13] also has a complete analogue for σ representations. However the proofs in the ordinary case do not apply in quite so immediate a fashion. We devote the rest of this section to the supplementary considerations needed to show that the results are also valid for σ representations.

Let $\mathfrak{G}$ be a separable locally compact group and let σ be a multiplier for $\mathfrak{G}$. We denote the set of all equivalence classes of σ-representations of $\mathfrak{G}$ by $\mathfrak{G}^{r,\sigma}$ and the set of all equivalence classes of irreducible σ-representations of $\mathfrak{G}$ by $\hat{\mathfrak{G}}^{\sigma}$. We call $\hat{\mathfrak{G}}^{\sigma}$ the *σ-dual* of $\mathfrak{G}$. We introduce a Borel structure [13, section 1] in $\hat{\mathfrak{G}}^{\sigma}$ just as we did for ordinary representations in [13] by regarding $\hat{\mathfrak{G}}^{\sigma}$ as a subspace of $\mathfrak{G}^{r,\sigma}$ and $\mathfrak{G}^{r,\sigma}$ as a quotient space of $\mathfrak{G}^{c,\sigma}$. Here $\mathfrak{G}^{c,\sigma}$ is the space of all "concrete" σ representations of $\mathfrak{G}$; concrete σ representation and the Borel structure in $\mathfrak{G}^{c,\sigma}$ being defined by obvious analogy with the corresponding ordinary concepts. The Borel structure in $\mathfrak{G}^{c}$ was shown in [13] to be standard by mapping it onto $\mathfrak{A}^{c}_{\mathfrak{G}}$ where $\mathfrak{A}_{\mathfrak{G}}$ is the group algebra of $\mathfrak{G}$ and then applying a corresponding theorem for representations of Banach algebras. Presumably a corresponding proof would work for $\mathfrak{G}^{c,\sigma}$. Rather than define and discuss σ group algebras however we apply Theorem 2.1 of this paper. This theorem gives a one-to-one mapping of $\mathfrak{G}^{c,\sigma}$ onto a certain subset of $\mathfrak{G}_1^c$ where $\mathfrak{G}_1 = \mathfrak{G}^{\sigma}$ and this mapping is obviously a Borel isomorphism. Moreover the range of this mapping is obviously a Borel set. Since $\mathfrak{G}_1^c$ is standard it follows that $\mathfrak{G}^{c,\sigma}$ is also standard. Hence the first statement of the corollary to theorem 9.1 of [13] is true when $\mathfrak{G}^{c}$ and $\hat{\mathfrak{G}}$ are replaced by $\mathfrak{G}^{c,\sigma}$ and $\hat{\mathfrak{G}}^{\sigma}$ respectively; that is Theorems 8.1 through 8.6 of [13] remain true when $\mathfrak{A}^{c}$ and $\hat{\mathfrak{A}}$ are replaced by $\mathfrak{G}^{c,\sigma}$ and $\hat{\mathfrak{G}}^{\sigma}$ respectively.

We turn now to section ten. The discussion preceding Theorem 10.1 applies equally well to σ representations except for the verification that the integrated representation $x \to M_x$ is indeed a σ representation. A different argument is needed to show that $(M_x(\phi),\psi)$ is always a Borel function since the $(L_x(\phi),\psi)$ are not known to be continuous in x. However one can apply Theorem 2.1 of the present paper to the $L^0_{\lambda,x}$ and deduce that $\lambda, x \rightarrow \lambda M_x$ is a representation whence in follows that $x \rightarrow M_x$ is a σ representation. Similar use of Theorem 2.1 allows us to deduce the truth of Theorems 10.1, 10.2 and 10.3 for σ representations from the fact that they are true for ordinary representations. The mappings $C \rightarrow \mathfrak{L}(C)$ and $L \rightarrow \mathfrak{C}(L)$ defined in the two paragraphs preceding Theorem 10.4 may be defined in the same manner for σ representations and it is obvious that $\mathfrak{L}(C^0) = \mathfrak{L}(C)^0$ and $\mathfrak{C}(L^0) = \mathfrak{C}(L)^0$ where $C \rightarrow C^0$ is the mapping of measure classes in $\hat{\mathfrak{G}}^\sigma$ into measure classes in $\hat{\mathfrak{G}}_1$ induced by the canonical map of $\hat{\mathfrak{G}}^\sigma$ into $\hat{\mathfrak{G}}_1$. (Here $\mathfrak{G}_1 = \mathfrak{G}^\sigma$ of course.) Mautner's theorem for σ representation is of course an immediate consequence of Theorem 2.1 of the present paper and Mautner's theorem for ordinary representations. Theorems 10.4 through 10.7 may now be generalized to σ representations using the obvious facts that ^{0} commutes with $\mathfrak{L}$ and $\mathfrak{C}$ and that T intertwines L and M if and only if it intertwines L^0 and M^0. The proof of Theorem 10.8 applies without change to σ representations.

The σ dual of a group may have of course quite different properties from the ordinary dual. In particular the ordinary dual may be smooth and of type I without this being true for the σ dual for all σ.

4. Induced σ representations

Let σ be a multiplier for the separable locally compact group $\mathfrak{G}$ and let $\mathcal{G}$ be a closed subgroup of $\mathfrak{G}$. Then the restriction of σ to $\mathcal{G}$ is a mulitplier for $\mathcal{G}$ and we may speak of the σ representations of $\mathcal{G}$ as well as of $\mathfrak{G}$. In particular the restriction to $\mathcal{G}$ of a σ representation of $\mathfrak{G}$ is a σ representation of $\mathcal{G}$. In [10], [11] and [12] we have discussed a process for going from ordinary representations L of $\mathcal{G}$ to certain ordinary representations U^L of $\mathfrak{G}$ which we called induced representations. We show now that this process can be generalized so as to work for σ representations as well. The definition can be given most rapidly by making use of Theorem 2.1. Let θ denote the identity mapping of $\mathcal{G}^\sigma$ into $\mathfrak{G}^\sigma$. The range of θ is the inverse image of the closed subgroup $\mathcal{G}$ under the canonical homomorphism of $\mathfrak{G}^\sigma$ on $\mathfrak{G}$. Hence this range is a closed subgroup of $\mathfrak{G}^\sigma$ and is accordingly locally compact. Since θ is obviously both an algebraic isomorphism and a Borel isomorphism it follows from the argument in the last few lines of the proof of Theorem 7.1 of [13] that it is a homeomorphism as well. Now let L be an arbitrary σ representation of $\mathcal{G}$. Then L^0 is an ordinary representation of $\mathcal{G}^\sigma$ which may be regarded as an ordinary representation of the closed subgroup $\theta(\mathcal{G}^\sigma)$ of $\mathfrak{G}^\sigma$. We form U^{L^0} as described in [11] and note that it follows from Theorem 12.1 of [11] and Theorem 2.1 of the present paper that U^{L^0} is of the form V^0 for some uniquely determined σ representation V of $\mathfrak{G}$. Actually U^{L^0} is only defined up to an equivalence. However $L \rightarrow L^0$ preserves equivalences so V is well defined up to an equivalence. We call V the σ representation of $\mathfrak{G}$ induced by the σ representation L of $\mathcal{G}$ and denote it by U^L.

We show next that an equivalent definition of U^L may be given which is analogous to the definition given for ordinary representations in [11] and reduces to it when $\sigma = 1$. Let μ be a quasi invariant measure in $\mathfrak{G}/\mathcal{G}$ and let us denote by ${}^{\mu}\mathfrak{H}^L$ the set of all functions f from $\mathfrak{G}$ to $\mathfrak{H}(L)$ such that:

(a) $(f(x),\phi)$ is a Borel function of x for all $\phi \in \mathfrak{H}(L)$,

(b) $f(\xi x) = \sigma(\xi,x)L_\xi(f(x))$ for all $\xi \in \mathcal{G}$ and all $x \in \mathfrak{G}$,

(c) $\int (f(x),f(x))\,d\mu(z) < \infty$.

The meaning of the integral in (c) is to be found in the fact that the integrand is constant on the right $\mathcal{G}$ cosets and hence defines a function on $\mathfrak{G}/\mathcal{G}$. For each $f \in {}^{\mu}\mathfrak{H}^L$ set $\|f\| = \sqrt{\int (f(x),f(x))\,d\mu(z)}$. Now let ϱ be the Borel function on $\mathfrak{G}$ which serves to define the Radon Nikodym derivatives of the translates of μ as described in section one of [11]. For each $f \in {}^{\mu}\mathfrak{H}^L$ and each $y \in \mathfrak{G}$ let $V_y(f) = g$ where $g(x) = \sqrt{\varrho(xy)/\varrho(x)}\,f(xy)/\sigma(x,y)$.

THEOREM 4.1. *${}^{\mu}\mathfrak{H}^L$ is a vector space with respect to the obvious definitions of addition and scalar multiplication. It becomes a Hilbert space under $\|\;\|$ when functions equal almost everywhere are identified. For each $y \in \mathfrak{G}$ and each $f \in {}^{\mu}\mathfrak{H}^L$, $V_y(f)$ is also in ${}^{\mu}\mathfrak{H}^L$ and $f \to V_y(f)$ defines a unitary operator V'_y in the Hilbert space associated with ${}^{\mu}\mathfrak{H}^L$. $y \to V'_y$ is a σ representation of $\mathfrak{G}$ which is equivalent to the induced representation U^L defined above.*

Proof. For each member f of ${}^{\mu}\mathfrak{H}^L$ let f^0 be the function from $\mathfrak{G}^\sigma$ to $\mathfrak{H}(L)$ such that $f^0(\lambda,x) = \lambda f(x)$. It is routine to verify that $f \to f^0$ is a one-to-one linear mapping of the set of all functions from $\mathfrak{G}$ to $\mathfrak{H}(L)$ which satisfy (a) and (b) of the definition of ${}^{\mu}\mathfrak{H}^L$ onto the set of all function from $\mathfrak{G}^\sigma$ to $\mathfrak{H}(L)$ which satisfy (a) and (b′) where (b′) is (b) with $\sigma = 1$, $\mathcal{G} = \theta(\mathcal{G}^\sigma)$, $L = L^0$ and $\mathfrak{G} = \mathfrak{G}^\sigma$. Now let k denote the natural one-to-one mapping of $\mathfrak{G}/\mathcal{G}$ on $\mathfrak{G}^\sigma/\mathcal{G}^0$ where $\mathcal{G}^0 = \theta(\mathcal{G})$. It is easy to see that k is a Borel isomorphism. We define a quasi invariant measure μ' in $\mathfrak{G}^\sigma/\mathcal{G}^0$ by letting $\mu'(E) = \mu(k^{-1}(E))$ and we verify without difficulty that $\int (f^0(\lambda,x),f^0(\mu,x))\,d\mu' = \int (f(x),f(x))\,d\mu$. Thus $f \to f^0$ is one-to-one and onto from ${}^{\mu}\mathfrak{H}^L$ to ${}^{\mu'}\mathfrak{H}^{L^0}$. Since we know from [11] that ${}^{\mu'}\mathfrak{H}^{L^0}$ is a Hilbert space it follows that ${}^{\mu}\mathfrak{H}^L$ is also and that $f \to f^0$ is unitary from one onto the other. Now a straightforward calculation shows that $f \to f^0$ takes the correspondence $f \to V_y(f)$ over into the correspondence $f^0 \to (1/\eta)\,U^{L^0}_{\eta,y}(f^0)$ for all η. All statements of the theorem now follow from this and known facts about ordinary representations in a straightforward and obvious manner.

We devote the rest of this section to the deduction of theorems about induced σ

representations from the corresponding theorems about ordinary induced representations by making use of the equivalence between U^{L^0} and $(U^L)^0$. In some cases a certain reformulation of the theorems is necessary.

THEOREM. 4.2. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be closed subgroups of the separable locally compact group $\mathfrak{G}$ such that $\mathcal{G}_1 \subset \mathcal{G}_2$. Let σ be a multiplier for $\mathfrak{G}$ and let L be a σ representation of $\mathcal{G}_1$. Let M be the σ representation of $\mathcal{G}_2$ induced by L. Then U^L and U^M are equivalent σ representations of $\mathfrak{G}$.*

Proof. Form $\mathfrak{G}^\sigma$, $\mathcal{G}_1^\sigma$, and $\mathcal{G}_2^\sigma$ and identify the latter two groups with their canonical images in $\mathfrak{G}^\sigma$ so that we have $\mathcal{G}_1^\sigma \subset \mathcal{G}_2^\sigma \subset \mathfrak{G}^\sigma$. Let N be the representation of $\mathcal{G}_2$ induced by L^0. It follows from Theorem 4.1 of [11] that U^{L^0} and U^N are equivalent and it follows from Theorem 4.1 of the present paper that N and M^0 are equivalent. Thus U^N and U^{M^0} are equivalent. Hence U^{M^0} and U^{L^0} are equivalent. Since (Theorem 4.1 of this paper) $L \rightarrow L^0$ commutes with $L \rightarrow U^L$ we conclude that $(U^M)^0$ and $(U^L)^0$ are equivalent and hence that U^M and U^L are equivalent.

Let σ be any multiplier for $\mathfrak{G}$ and let E denote the identity subgroup of $\mathfrak{G}$. Since σ reduces to one on E the one dimensional identity representation I is a σ representation of E and hence induces a σ representation of $\mathfrak{G}$. We call this σ representation of $\mathfrak{G}$ the *regular σ representation* since it reduces to the ordinary regular representation when $\sigma = 1$. In particular we see that there exist σ representations for every multiplier σ. It is an immediate corollary of Theorem 4.2 that U^L is the regular σ representation of $\mathfrak{G}$ whenever L is the regular σ representation of $\mathcal{G}$.

THEOREM 4.3. *Let σ be a multiplier for the separable locally compact group $\mathfrak{G}$ and let L be a σ representation of the closed subgroup $\mathcal{G}$. Then the $(1/\sigma) = \bar{\sigma}$ representations of $\mathfrak{G}$, $U^{\bar{L}}$ and $\overline{U^L}$, are equivalent.*

Proof. Observe first that $\lambda, x \rightarrow \bar{\lambda}, x$ sets up an isomorphism between $\mathfrak{G}^\sigma$ and $\mathfrak{G}^{\bar{\sigma}}$ which preserves Borel sets and hence is a homeomorphism as well. Moreover this correspondence is easily seen to take the representation $\overline{L^0}$ of $\mathfrak{G}^\sigma$ into the representation $(\bar{L})^0$ of $\mathfrak{G}^{\bar{\sigma}}$, L being a σ representation of $\mathcal{G}$. Arguments similar to those used in proving Theorem 3.2 now enable us to deduce the truth of the present theorem from the corresponding theorem about ordinary representation—Theorem 5.1 of [11].

THEOREM 4.4. *Let σ_1 and σ_2 be multipliers for the separable locally compact groups $\mathfrak{G}_1$ and $\mathfrak{G}_2$. Let L and M be σ_1 and σ_2 representations respectively of the closed subgroups $\mathcal{G}_1$ and $\mathcal{G}_2$ of $\mathfrak{G}_1$ and $\mathfrak{G}_2$. Then the $\sigma_1 \times \sigma_2$ representations $U^{L \times M}$ and $U^L \times U^M$ of $\mathfrak{G}_1 \times \mathfrak{G}_2$ are equivalent.*

Proof. Observe first that the mapping $(\lambda,x),(\mu,y)\to\lambda\mu,x,y$ is a homomorphism of $\mathfrak{G}_1^{\sigma_1}\times\mathfrak{G}_2^{\sigma_2}$ on $(\mathfrak{G}_1\times\mathfrak{G}_2)^{\sigma_1\times\sigma_2}$ whose kernel $\mathfrak{G}_0$ is the set of all $(\lambda,e),(\mu,e)$ with $\mu=1/\lambda$. (Here e denotes the identity element of the appropriate group.) One can then verify without difficulty that $U^0\times V^0$ reduces to the identity on $\mathfrak{G}_0$ and via the canonical mapping of $(\mathfrak{G}_1^{\sigma_1}\times\mathfrak{G}_2^{\sigma_2})/\mathfrak{G}_0$ on $(\mathfrak{G}_1\times\mathfrak{G}_2)^{\sigma_1\times\sigma_2}$ goes over into $(U\times V)^0$. This remark having been made it is not difficult to deduce the truth of the theorem from that of Theorem 5.2 of (11) along the lines indicated in the proof of Theorem 4.2 above. In this deduction one has need of the following lemma whose proof is quite straightforward and may also be left to the reader.

LEMMA 4.1. *Let $\mathcal{G}$ be a closed subgroup of the separable locally compact group $\mathfrak{G}$ and let $\mathcal{H}$ be a closed normal subgroup of $\mathfrak{G}$ with $\mathcal{H}\subseteq\mathcal{G}$. Let L be a representation of $\mathcal{G}$ which is the identity on $\mathcal{H}$ and let L' be the corresponding representation of $\mathcal{G}/\mathcal{H}$. Then U^L is the identity on $\mathcal{H}$ and the corresponding representation of $\mathfrak{G}/\mathcal{H}$ is equivalent to $U^{L'}$.*

COROLLARY (of Theorem 4.4). *If L is a σ_1 representation of $\mathfrak{G}_1$ and we form U^L, regarding σ_1 as the restriction to $\mathfrak{G}_1\times e$ of $\sigma_1\times\sigma_2$, then U^L is equivalent to the Kronecker product of L with the σ_2 regular representation of $\mathfrak{G}_2$.*

We consider next the question of generalizing the first main theorem (Theorem 12.1) of (11). There is a difficulty in that the transform of a σ representation L by an inner automorphism need not be a σ representation. Indeed if $M_x=L_{sxs^{-1}}$ then M is a σ' representation where $\sigma'(x,y)=\sigma(sxs^{-1},sys^{-1})$ and σ^1 will not in general be equal to σ. This difficulty is easily overcome by making use of the fact that σ and σ' are similar multipliers.

LEMMA 4.2. *Let σ be a multiplier for the group $\mathfrak{G}$, let s be an element of $\mathfrak{G}$ and let $\sigma'(x,y)=\sigma(sxs^{-1},sys^{-1})$. Then the multipliers σ and σ' are similar. Indeed $\sigma'(x,y)/\sigma(x,y)=g_s(xy)/(g_s(x)g_s(y))$ where $g_s(x)=(\sigma(sx,s^{-1})\sigma(s,x))/\sigma(s^{-1},s)$.*

Proof. Using repeatedly the fundamental identity defining a multiplier we have the following string of equalities:

$$
\begin{aligned}
&\sigma(sxs^{-1},sys^{-1})\\
&\quad=(\sigma(sx,ys^{-1})\sigma(s^{-1},sys^{-1}))/\sigma(sx,s^{-1})\\
&\quad=(\sigma(sxy,s^{-1})\sigma(sx,y)\sigma(s^{-1},sys^{-1}))/(\sigma(y,s^{-1})\sigma(sx,s^{-1}))\\
&\quad=(\sigma(sxy,s^{-1})\sigma(s^{-1},sys^{-1})\sigma(s,xy)\sigma(x,y))/(\sigma(y,s^{-1})\sigma(sx,s^{-1})\sigma(s,x))\\
&\quad=[\sigma(x,y)][(\sigma(sxy,s^{-1})\sigma(s,xy))/(\sigma(sx,s^{-1})\sigma(s,x)]\cdot\\
&\qquad\qquad\cdot[(\sigma(s^{-1},s)\sigma(e,ys^{-1}))/(\sigma(s,ys^{-1})\sigma(y,s^{-1}))]\\
&\quad=[\sigma(x,y)][(\sigma(sxy,s^{-1})\sigma(s,xy))/(\sigma(sx,s^{-1})\sigma(s,x))]\cdot\\
&\qquad\qquad\cdot[(\sigma(s^{-1},s)\sigma(e,ys^{-1})\sigma(y,s^{-1}))/(\sigma(sy,s^{-1})\sigma(y,s^{-1})\sigma(s,y))].
\end{aligned}
$$

Since $\sigma(e,z)=1$ for all z we get on regrouping and dividing by $\sigma(x,y)$

$$\sigma(sxs^{-1}, sys^{-1})/\sigma(x,y)$$
$$=[(\sigma(sxy,s^{-1})\sigma(s,xy))/\sigma(s^{-1},s)][\sigma(s^{-1},s)/(\sigma(sx,s^{-1})\sigma(s,x))][\sigma(s^{-1},s)/(\sigma(sy,s^{-1})\sigma(s,y))]$$

and this completes the proof.

COROLLARY. *Let $\mathcal{G}$ be a closed subgroup of the separable locally compact group $\mathfrak{G}$, let σ be a multiplier for $\mathfrak{G}$ and let L be a σ representation of $\mathcal{G}$. Then for all $s\in\mathfrak{G}$ the mapping $x\to(\sigma(s^{-1},s)/\sigma(sx,s^{-1})\sigma(s,x))L_{sxs^{-1}}$ is a σ representation of the subgroup $s^{-1}\mathcal{G}s$.*

We shall denote the σ representation defined in the corollary by L^s. We leave it to the reader to verify that $(L^s)^t=L^{st}$ for all s and t in $\mathfrak{G}$.

THEOREM 4.5. *Let $\mathfrak{G}$ be a separable locally compact group and let $\mathcal{G}_1$ and $\mathcal{G}_2$ be closed subgroups of $\mathfrak{G}$ which are regularly related* (1) *in the sense of* ([11], p. 127). *Let σ be a multiplier for $\mathfrak{G}$ and let L be a σ representation of $\mathcal{G}_1$. For each $x\in\mathfrak{G}$ consider the subgroup $\mathcal{G}_2\cap(x^{-1}\mathcal{G}_1x)$ of $\mathfrak{G}$ and the restriction to this subgroup of the representation L^x of $x^{-1}\mathcal{G}_1x$. Let ${}^xV$ denote the σ representation of $\mathcal{G}_2$ induced by this restriction. Then to within equivalence ${}^xV$ depends only upon the double coset $\mathcal{G}_1x\mathcal{G}_2=d(x)$ to which x belongs so that we may write ${}^dV={}^xV$ where $d=d(x)$. Moreover U^L restricted to $\mathcal{G}_2$ is a direct integral over the set $\mathbf{D}$ of $\mathcal{G}_1:\mathcal{G}_2$ double cosets, with respect to any admissible measure in $\mathbf{D}$, of the representations ${}^dV$.*

Proof. We form $\mathfrak{G}^\sigma$, $\mathcal{G}_1^\sigma$, and $\mathcal{G}_2^\sigma$ and as above identify $\mathcal{G}_1^\sigma$ and $\mathcal{G}_2^\sigma$ with the corresponding subgroups of $\mathfrak{G}^\sigma$. We compute easily that the canonical homomorphism of $\mathfrak{G}^\sigma$ on $\mathfrak{G}$ sets up a one-to-one correspondence between the $\mathcal{G}_1^\sigma:\mathcal{G}_2^\sigma$ double cosets on the one hand and the $\mathcal{G}_1:\mathcal{G}_2$ double cosets on the other. It follows that $\mathcal{G}_1^\sigma$ and $\mathcal{G}_2^\sigma$ are regularly related so that we may apply Theorem 12.1 of [11] to the restriction to $\mathcal{G}_2$ of the representation U^{L^0} of $\mathfrak{G}^\sigma$. Now let x be any element of $\mathfrak{G}$ and consider the component of this restriction associated with the double coset $\mathcal{G}_1^\sigma(1,x)\mathcal{G}_2^\sigma$. By Theorem 12.1 it is the representation of $\mathcal{G}_2$ induced by the representation $\lambda,\eta\to L^0_{(1,x)(\lambda,\eta)(1,x)^{-1}}$ of the subgroup $\mathcal{G}_2^\sigma\cap((1,x)^{-1}\mathcal{G}_1^\sigma(1,x))$. But

$$L^0_{(1,x)(\lambda,\eta)(1,x)^{-1}}=(\lambda\sigma(x,x^{-1})/(\sigma(x,\eta)\sigma(x\eta,x^{-1}))L_{x\eta x^{-1}}=\lambda L^x_\eta$$

and $(1,x)^{-1}\mathcal{G}_1^\sigma(1,x)=(x^{-1}\mathcal{G}_1x)^\sigma$. Thus the component in question is the representation of $\mathcal{G}_2^\sigma$ induced by the representation $(L^x)^0$ of the subgroup $(\mathcal{G}_2\cap x^{-1}\mathcal{G}_1x)^\sigma$. But $U^{(L^x)^0}$ is just

(1) Making use of the notions of [13] this theorem and the next may be given a somewhat neater formulation, as follows. If we regard $\mathfrak{G}$ as a Borel group and $\mathbf{D}$ as a quotient space of $\mathfrak{G}$ the unique invariant measure class in $\mathfrak{G}$ defines a measure class $\mathcal{C}$ in $\mathbf{D}$. Saying that $\mathcal{G}_1$ and $\mathcal{G}_2$ are regularly related is then the same thing as saying that $\mathcal{C}$ is a countably separated measure class. Moreover by Theorem 6.2 of [13] saying that $\mathcal{C}$ is countably separated is the same as saying that $\mathcal{C}$ is standard. Finally the direct integrals which appear in the conclusion of these theorems may be described simply as the integral of the V^d over $\mathbf{D}$ with respect to the standard measure class $\mathcal{C}$.

$(U^{L^x})^0$. Moreover as we have already observed passing from M to M^0 commutes with the taking of direct integrals. The truth of the present theorem follows easily from these remarks and Theorem 12.1 of [11].

We deduce the final theorem of this section from Theorem 4.5 just as we deduced Theorem 12.2 from Theorem 12.1 in [11]. We leave details to the reader.

THEOREM 4.6. *Let $\mathfrak{G}$, $\mathcal{G}_1$, $\mathcal{G}_2$ be as in Theorem 4.5 and let σ and τ be multipliers for $\mathfrak{G}$. Let L and M be σ and τ representations of $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively. For each x and y in $\mathfrak{G}$ consider the restrictions of L^x and M^y to $(x^{-1}\mathcal{G}_1 x) \cap (y^{-1}\mathcal{G}_2 y)$. Let ${}^{x,y}V$ denote the representation of $\mathfrak{G}$ induced by the Kronecker product of these two projective representations of $(x^{-1}\mathcal{G}_1 x) \cap (y^{-1}\mathcal{G}_2 y)$. Then (to within equivalence) ${}^{x,y}V$ depends only upon the double coset $\mathcal{G}_1 xy^{-1}\mathcal{G}_2$. $d(xy^{-1})$ to which xy^{-1} belongs so that we may write ${}^dV = {}^{x,y}V$ where $d = d(xy^{-1})$. Moreover $U^L \otimes U^M$ is equivalent to the direct integral of the ${}^dV$ with respect to any admissible measure in the set $\mathbf{D}$ of $\mathcal{G}_1 : \mathcal{G}_2$ double cosets.*

5. Systems of imprimitivity

The notion of "system of imprimitivity" for infinite dimensional group representations introduced in [10] makes sense as it stands for σ representations. In this section and the next we shall study this notion for σ representations in considerably more detail than was done in [10] for ordinary representations. In particular we shall give in section six an independent proof of the σ generalization of the main theorem of [10]. One could presumably deduce this generalization from the special case proved in [10] by applying Theorem 2.1. However in view of the central importance of the theorem for this paper and the fact that [10] contains only the outline of a proof the former alternative seems more desirable. We begin with a brief account of projection valued measures.

Let S be a metrically standard Borel space [13]. By a *projection valued mesaure* on S we shall mean a mapping P, $E \to P_E$, of the Borel subsets of S into the projections on some separable Hilbert space $\mathfrak{H}(P)$ such that $P_{E \cap F} = P_E P_F = P_F P_E$, $P_S = I$, $P_0 = 0$ and $P_E = \sum_{j=1}^{\infty} P_{E_j}$ whenever $E = \bigcup_{j=1}^{\infty} E_j$ and the E_j are disjoint. We say that P and Q are *equivalent* if there exists a unitary map U of $\mathfrak{H}(P)$ on $\mathfrak{H}(Q)$ such that $UP_E U^{-1} = Q_E$ for all E. If P^1, P^2, ... are projection valued measures we define their *direct sum* $P = P^1 \oplus P^2 \oplus \ldots$ to be the projection valued measure such that $\mathfrak{H}(P) = \mathfrak{H}(P^1) \oplus \mathfrak{H}(P^2) \oplus \ldots$ and $P_E(\phi_1, \phi_2, \ldots) = P^1_E(\phi_1), P^2_E(\phi_2), \ldots$ As is well known [14] there exists for each P an element ϕ of $\mathfrak{H}(P)$ such that $P_E(\phi) = 0$ if and only if $P_E = 0$. Thus the Borel measure $E \to (P_E(\phi), \phi)$ has as null sets exactly the sets E for which $P_E = 0$. Thus every P has associated with it a unique measure class (section 6 of [13]) which we shall call *the measure class of* P and denote by C^P. The well known analysis of projection valued measures on the real line which yields the unitary equivalence theory of self adjoint operators (see Halmos [7]) applies without essential change to projection valued measures on S. The results may be described as follows. The algebra $\mathbf{R}(P)$ of all bounded linear operators T such that $TP_E = P_E T$ for all E is commutative if and only if there exists an element ϕ in $\mathfrak{H}(P)$ such that the $P_E(\phi)$ have $\mathfrak{H}(P)$ as their closed linear span. Such a P is said to be *uniformly one dimensional.* If μ is any finite Borel measure in S and P_E^μ is defined as the bounded linear operator $f \to \psi_E f$ where $f \in \mathbf{L}^2(\mathrm{S},\mu)$ and ψ_E is the characteristic function of E then P^μ is a uniformly one dimensional projection valued measure whose associated measure class is that containing μ. P^μ and P^ν are equivalent if and only if μ and ν lie in the same measure class and every uniformly one dimensional P is equivalent to some P^μ. Thus $\mu \to P^\mu$ sets up a one-to-one correspondence between measure classes in S and equivalence classes of uniformly one dimensional projection valued measures on S. We say that P and Q are *disjoint* if their associated measure classes are disjoint ([13] section 10) and that P is uniformly k dimensional $k=\infty, 1, 2, \ldots$ if P is the direct sum of k replicas of some uniformly one dimensional Q. Every projection valued measure is uniquely of the form $P^{n_1} \oplus P^{n_2} \oplus \ldots$ where $n_1, n_2, \ldots$ is a subsequence of $\infty, 1, 2, \ldots$, each P^{n_j} is uniformly n_j dimensional and the P^{n_j} are mutually disjoint. As remarked in section 3 the multiplicity theory described in section 1 of [12] is really very general. In particular it applies to projection valued measures and may be used to obtain an alternative derivation of the reduction to the uniformly one dimensional case just described.

Let L be a σ representation of the separable locally compact group $\mathfrak{G}$. By a *system of imprimitivity for* L we shall mean the pair consisting of a projection valued measure P with $\mathfrak{H}(P) = \mathfrak{H}(L)$ and an anti homomorphism h of $\mathfrak{G}$ into the group of all Borel automorphisms of the domain S of P such that (a) If $[x]y$ denotes the action of $h(y)$ on x then $y,x \to [x]\,y$ is a Borel function, and (b) $L_y P_E L_y^{-1} = P_{[E]y^{-1}}$ for all $y \in \mathfrak{G}$ and all Borel sets $E \subseteq \mathrm{S}$. We shall call S the *base* of the system of imprimitivity. The measure class of P we shall refer to as the *measure class of the system.* We note that if $P_E = 0$ then $P_{[E]y} = L_{y^{-1}} P_E L_{y^{-1}}^{-1} = 0$ for all $y \in \mathfrak{G}$. Thus the measure class of the system is invariant under the action of $\mathfrak{G}$ on S. We shall say that the system of imprimitivity P, h is *ergodic* if no measure class in S which is invariant under $\mathfrak{G}$ is strictly "absolutely continuous" with respect to the measure class of P in the sense of being associated with a properly larger family of null sets.

THEOREM 5.1. *The system of imprimitivity P, h fails to be ergodic if and only if there exists a P_{E_0} different from 0 and I such that $P_{E_0} L_y = L_x P_{E_0}$ for all $y \in \mathfrak{G}$.*

Proof. If P fails to be ergodic let μ be a member of the measure class of P and let ν be a member of an invariant measure class having more null sets than μ. Let ϱ be a Borel function which is a Radon Nikodym derivative of ν with respect to μ and let E_0 be the set on which ϱ is 0. Since the ν null sets are invariant, E_0 and $E_0 y$ differ by a μ null set for all y. Hence $P_{E_0} = P_{[E_0]y}$ for all y. Hence P_{E_0} commutes with all L_y. P_{E_0} is obviously not 0 or I. Conversely if E_0 exists then $P_{E_0} = P_{[E_0]y}$ for all y and $E \to \mu(E \cap E_0)$ defines an invariant measure class with a properly larger family of null sets.

THEOREM 5.2. *Let P, h be an ergodic system of imprimitivity for the σ representation L. Then P is uniformly k dimensional for some $k = \infty, 1, 2, \ldots$.*

Proof. Let $P = P^{n_1} \oplus P^{n_2} \oplus \ldots$ be the canonical decomposition of P where P^{n_j} is uniformly n_j dimensional. Let Q_j be the projection on the subspace corresponding to the summand P^{n_j}. It follows from the theory of the decomposition (loc. cit.) that the Q_j depend only upon the range of P. Since P and its unitary transform by each L_y have the same range it follows that each L_y commutes with each Q_j. Hence each pair P^{n_j}, h is a system of imprimitivity for a subrepresentation of L. Hence the measure class associated with each P^{n_j} is invariant under $\mathfrak{G}$. Since these measure classes are obviously absolutely continuous with respect to the measure class of P it follows from the ergodicity hypothesis that they are identical with the measure class of P. Since they are at the same time mutually disjoint we have a contradiction unless there is only one term. This completes the proof.

Now let $\mathfrak{G}$ be a separable locally compact group and let h be an anti homomorphism of $\mathfrak{G}$ into the group of Borel automorphisms of S such that $x, y \to [x]y = h(y)(x)$ is a Borel function. Let C be any measure class in S invariant under $\mathfrak{G}$ and let $k = \infty, 1, 2, \ldots$. As we have seen there is to within equivalence just one uniformly k dimensional projection valued measure on S whose measure class is C. Call it P. We devote the balance of this section to a partial analysis of the family of all possible σ representations of $\mathfrak{G}$ having P, h as a system of imprimitivity. We begin by exhibiting a certain canonical ordinary representation with this property. Choose a finite member μ of C and realize P as a projection valued measure with $\mathfrak{H}(P)$ the set of all square summable functions with respect to μ from S to some fixed k dimensional Hilbert space $\mathfrak{H}_k$ and P_E multiplication by the characteristic function of E. For each $y \in \mathfrak{G}$ let ϱ_y be a μ measurable function on S which is a Radon Nikodym derivative of the measure $E \to \mu([E]y)$ with respect to μ and let $\varrho(y,x) = \varrho_y(x)$. For each $f \in \mathfrak{H}(P) = \mathfrak{L}^2(\mathrm{S}, \mu, \mathfrak{H}_k)$ and each $y \in \mathfrak{G}$ let $W_y(f) = g$ where $g(x) = f([x]y)\varrho(y,x)$.

THEOREM 5.3. *For each $y \in \mathfrak{G}$, W_y is a unitary operator. Moreover $y \to W_y$ is a representation of $\mathfrak{G}$ having P as a system of imprimitivity.*

Proof. It follows easily from the definition of ϱ_y that for all y_1 and y_2 in $\mathfrak{G}$ we have $\varrho(y_1 y_2, x) = \varrho(y_1, x)\varrho(y_2, [x]y_1)$ for μ almost all x in S. Moreover making use of this almost everywhere identity there is no difficulty in verifying not only that each W_y is unitary but also that $W_{y_1} W_{y_2} = W_{y_1 y_2}$ for all y_1 and y_2 in $\mathfrak{G}$. Thus to show that W is a representation we need only show that $(W_y(f), g)$ is μ measurable as a function of y whenever f and g are members of $\mathfrak{H}(P)$. This will follow from the Fubini theorem once we know that the arbitrary choices in the ϱ_y may be made so that ϱ is measurable on $\mathfrak{G} \times \mathrm{S}$. That these choices may be so made we prove by applying Lemma 3.1 of [9]. If E is any measurable subset of S then $\int \varrho(y,x)\psi_E(x)\,d\mu(x) = \mu((E)y)$ where ψ_E is the characteristic function of E. Thus Lemma 3.1 of [9] will apply once we know that for each E, $\mu([E]y)$ is measurable in y. There is also a boundedness restriction in the statement of the lemma but examination of the proof shows that only the existence of the integrals is actually used. To prove the measurability of $\mu([E]y)$ choose a Borel set F differing from E by a μ null set. Since $\mu([E]y) = \mu([F]y)$ for all y it will suffice to prove that the latter function is measurable. Let T be the mapping $y, x \to y, [x]y$. T is then a one-to-one map of $\mathfrak{G} \times \mathrm{S}$ on $\mathfrak{G} \times \mathrm{S}$ and T^{-1} takes y, x into $y, [x]y^{-1}$. Thus T and T^{-1} are both Borel functions so T is a Borel automorphism. Thus $T(\mathfrak{G} \times F)$ is a Borel set and hence so is $T(\mathfrak{G} \times F) \cap (y \times \mathrm{S}) = y \times [F]y$. Let ν be a finite measure in $\mathfrak{G}$ having the same null sets as Haar measure. Applying the Fubini theorem to the characteristic function of $T(\mathfrak{G} \times F)$ and the measure $\nu \times \mu$ we see that $\mu([F]y)$ is measurable in y and hence that W is a representation. To show that P is a system of imprimitivity for W one needs only compute $(W_y P_E W_y^{-1}(f))(x)$ and apply the almost everywhere identity involving ϱ cited above.

It is easy to see that (to within equivalence) W is uniquely determined by h, k and the measure class of P and indeed that this is true for the pair P, W. We shall call it the *permutation representation of* $\mathfrak{G}$ *defined by* h, k and C^P.

Now let $\mathcal{U}_P$ denote the group of all unitary operators in $\mathfrak{H}(P)$ which commute with all P_E. It follows from the identity $W_y P_E W_y^{-1} = P_{[E]y^{-1}}$ that $W_y \mathcal{U}_P W_y^{-1} = \mathcal{U}_P$. Thus each $y \in \mathfrak{G}$ defines an automorphism $V \to y(V) = W_y V W_y^{-1}$ of $\mathcal{U}_P$ and the mapping of $\mathfrak{G}$ into the group of automorphisms of $\mathcal{U}_P$ so defined is a homomorphism.

THEOREM 5.4. *Let σ be any multiplier for $\mathfrak{G}$ and let Q, $y \to Q_y$ be any function from $\mathfrak{G}$ to $\mathcal{U}_P$ which satisfies the following three conditions*:

(a) $Q_{y_1 y_2} = \sigma(y_1, y_2) Q_{y_1} y_1(Q_{y_2})$ *for all y_1 and y_2 in $\mathfrak{G}$.*

(b) $Q_e = I$.

(c) $(Q_y(f), g)$ *is a Borel function of y for all f and g in $\mathfrak{H}(P)$.*

Then $y \to Q_y W_y$ is a σ representation of $\mathfrak{G}$ having P, h as a system of imprimitivity. Conversely if L is any σ representation of $\mathfrak{G}$ having P, h as a system of imprimitivity then there exists a unique function $Q, y \to Q_y$, from $\mathfrak{G}$ to $\mathcal{U}_P$ satisfying (a), (b) *and* (c) *and such that $L_y = Q_y W_y$ for all y.*

Proof. Let $L_y = Q_y W_y$, where each $Q_y \in \mathcal{U}_P$. Then $L_{y_1 y_2} = \sigma(y_1, y_2) L_{y_1} L_{y_2}$ if and only if $Q_{y_1 y_2} = \sigma(y_1, y_2) Q_{y_1} W_{y_1} Q_{y_2} W_{y_1}^{-1} = \sigma(y_1, y_2) Q_{y_1} y_1(Q_{y_2})$; that is if and only if (a) is satisfied. Moreover since $L_e = Q_e W_e$ it follows that $L_e = I$ if and only if (b) is satisfied. Finally since

$$(L_y(f), g) = (Q_y W_y(f), g) = (W_y(f), Q_y^*(g)) = \sum_{n=1}^{\infty} (W_y(f), f_n)(Q_y(f_n), g)$$

and

$$(Q_y(f), g) = (L_y W_y^*(f), g) = (W_y^*(f), L_y^*(g)) = \sum_{n=1}^{\infty} (f, W_y(f_n))(L_y(f_n), g)$$

where $\{f_n\}$ is a complete orthonormal set for $\mathfrak{H}(P)$, it follows that $y \to L_y$ is a σ representation if and only if $y \to Q_y$ satisfies conditions (a), (b) and (c). Since

$$L_y P_E L_y^{-1} = Q_y W_y P_E W_y^{-1} Q_y^{-1} = Q_y P_{[E]y^{-1}} Q_y^{-1} = P_{[E]y^{-1}}$$

it follows that P, h is always a system of imprimitivity for L. Finally if L is any representation of $\mathfrak{G}$ with P, h as a system of imprimitivity then

$$W_y^{-1} L_y P_E L_y^{-1} W_y = W_y^{-1} P_{[E]y^{-1}} W_y = P_E.$$

Thus $Q_y = W_y^{-1} L_y \in \mathcal{U}_P$. This completes the proof of the theorem.

THEOREM 5.5. *Let Q and Q' satisfy* (a), (b) *and* (c) *of Theorem 5.4 and let L and L' be the corresponding σ representations of $\mathfrak{G}$. Then there exists a unitary transformation of $\mathfrak{H}(P)$ onto $\mathfrak{H}(P)$ which carries each L_y into L'_y and each P_E into itself if and only if there exists $V \in \mathcal{U}_P$ such that for all y $Q'_y = V Q_y (y(V))^{-1}$. Moreover there exists a proper closed subspace of $\mathfrak{H}(P)$ which is invariant under all P_E and all L_y if and only if there exists a member V of $\mathcal{U}_P$ not a multiple of the identity such that $Q_y y(V) = V Q_y$ for all $y \in \mathfrak{G}$.*

Proof. The second statement follows from the first on taking $Q = Q'$. To prove the first we note that the unitary operators taking each P_E into itself are just the members of $\mathcal{U}_P$ and that $V Q_y W_y V^{-1} = Q'_y W_y$ if and only if $Q'_y = V Q_y W_y V^{-1} W_y^{-1} = V Q_y y(V)^{-1}$.

We observe next that the members of $\mathcal{U}_P$ may be described in terms of functions from S to the group of all unitary maps of $\mathfrak{H}_k$ on $\mathfrak{H}_k$. Indeed let A, $x \to A(x)$ be any such function which is a Borel function in the sense that $((A(x)(\phi), \psi))$ is a complex valued Borel function for all ϕ and ψ in $\mathfrak{H}_k$. Then $f \to g$ where $g(x) = A(x)(f(x))$ is a unitary operator on $\mathfrak{H}(P)$ which clearly belongs to $\mathcal{U}_P$. We denote this operator by $A^{\sim}$. It is easy to verify that $A_1^{\sim} = A_2^{\sim}$

if and only if $A_1(x) = A_2(x)$ for almost all x and that $W_y A^{\sim} W_y^{-1} = B^{\sim}$ where $B(x) = A([x]y)$. Moreover it follows from the discussion given in section 4 of [9] that every member of $\mathcal{U}_P$ is of the form $A^{\sim}$. In particular every Q_y is of the form $A^{\sim}$. Thus each of our functions Q can be replaced by a function R on $\mathfrak{G} \times S$ with values amongst the unitary operators mapping $\mathfrak{H}_k$ on $\mathfrak{H}_k$ and we deduce the following variant of Theorem 5.4.

THEOREM 5.6. *Let σ be a multiplier for $\mathfrak{G}$ and let R, y, $x \to R(y,x)$ be a function from $\mathfrak{G} \times S$ to the group of all unitary maps of $\mathfrak{H}_k$ on $\mathfrak{H}_k$. Suppose that R satisfies the following three conditions*:

(a) *For each y_1 and y_2 in $\mathfrak{G}$ we have $R(y_1 y_2, x) = \sigma(y_1, y_2) R(y_1, x) R(y_2, [x] y_1)$ for μ almost all x in S.*

(b) *$R(e,x)$ in the identity for almost all x in S.*

(c) *For all ϕ and ψ in $\mathfrak{H}_k$, $(R(y,x)(\phi),\psi)$ is measurable as a function on $\mathfrak{G} \times S$, and for each $y \in \mathfrak{G}$ is measurable as a function on S.*

Then if we set $R_y^{-}(x) = R(y,x)$ the function $y \to (R_y^{-})^{\sim}$ satisfies (a), (b) *and* (c) *of Theorem 5.4 and hence $y \to (R_y^{-})^{\sim} W_y$ is a σ representations of $\mathfrak{G}$ having P, h as a system of imprimitivity. Conversely if L is any σ representation of $\mathfrak{G}$ having P, h as a system of imprimitivity then there exists R satisfying* (a), (b) *and* (c) *above such that $L_y = (R_y^{-})^{\sim} W_y$ for all $y \in \mathfrak{G}$. R_1 and R_2 lead to the same L if and only if they are almost everywhere equal.*

Proof. Except for certain measure theoretic points which we shall now discuss the statements of the theorem are obvious consequences of the preceding remarks and Theorem 5.4. To show that $(R_y^{-})^{\sim}$ satisfies (c) of Theorem 5.4 we must show that $\int (R(y,x)(f(x)), g(x)) d\mu(x)$ is a Borel function of y for each f and g in $\mathfrak{H}(P)$. It is enough to consider the case in which $f(x) = a(x)\phi$ and $g(x) = b(x)\psi$ where a and b are complex valued functions and ϕ and ψ are members of $\mathfrak{H}_k$ since $\mathfrak{H}(P)$ has a basis consisting of such functions. In this case however the expression reduces to $\int a(x)\overline{b(x)}(R(y,x)(\phi),\psi) d\mu(x)$ which is measurable in y by the Fubini theorem. But then the argument of Theorem 5.4 shows that $y \to (R_y^{-}) W_y = L_y$ is measurable in y and has the algebraic properties of a σ representation. Define L^0 as in section 2. Then $(L^0_{\lambda,y}(f),g)$ is measurable on $\mathfrak{G}^{\sigma}$ for all f and g in $\mathfrak{H}(P)$. Since L^0 has the algebraic properties of an ordinary representation it is an ordinary representation. Hence L is a σ representation of $\mathfrak{G}$. Hence $((R_y^{-})^{\sim}(f),g)$ is a Borel function as was to be proved. In proving the converse the measure theoretic point to be established is that the arbitrary choices involved in putting the Q_y in the form $(R_y^{-})^{\sim}$ can be made in such a manner that (c) of the present theorem holds. Let $\phi_1, \phi_2, \ldots$ be an orthonormal basis for $\mathfrak{H}_k$. It will of course suffice to choose the R_y^{-} so that (c) holds when ϕ and ψ are chosen from amongst the ϕ_i.

First of all let us put the Q_y in the form $(R_y^-)^\sim$ in an arbitrary manner. Let $r_{ij}(y,x) = (R(y,x)(\phi_i),\phi_j)$. Let E be any μ measurable subset of S and let ψ_E be its characteristic function. Then $\psi_E\phi_i$ and $\psi_E\phi_j$ are members of $\mathfrak{H}(P)$ so that $(Q_y(\psi_E\phi_i),\psi_E\phi_j)$ is a Borel function of y. But this last expression is equal to $\int \psi_E(x)(R(y,x)(\phi_i),\phi_j)d\mu(x) = \int r_{ij}(y,x)\psi_E(x)d\mu(x)$. It follows from Lemma 3.1 of [9] that for each i and j there is a measurable function r'_{ij} on $\mathfrak{G} \times$ S such that for each $y \in \mathfrak{G}$ we have $r_{ij}(y,x) = r'_{ij}(y,x)$ for almost all x. The matrix $\| r'_{ij}(y,x) \|$ will then be that of a unitary operator $R'(y,x)$ for almost all pairs y,x. Let $R''(y,x) = R'(y,x)$ whenever $R'(y,x)$ exists as a unitary operator and let $R''(y,x)$ be the identity operator otherwise. Then for all y in $R''(y,x) = R(y,x)$ for μ almost all x and R'' satisfies (c) of the present theorem. This completes the proof.

6. Transitive systems of imprimitivity

Let P, h be a system of imprimitivity for the σ representation L of the separable locally compact group $\mathfrak{G}$. We shall say that the system is *transitive* if the range of h is a transitive group of transformations of S onto S; that is if given x_1 and x_2 in S there exists $y \in \mathfrak{G}$ such that $[x_1]y = x_2$. Let P, h and P', h' be systems of imprimitivity for the same σ representation L. We shall say that P, h and P', h' are *strongly equivalent* if there exists a Borel isomorphism ϕ of the base S of P onto the base S$'$ of P' such that $P'_{\phi[E]} = P_E$ for all E and $h'(y) = \phi h(y)\phi^{-1}$ for all $y \in \mathfrak{G}$. If there exists a Borel subset S_0 of S such that S_0 is invariant under the action of $\mathfrak{G}$ and $P_{S_0} = I$ we shall call the system of imprimitivity obtained by restricting P and the $h(y)$ to S_0 a *trivial contraction* of P, h. Each P has an obvious unique extension to the Borel field generated by its domain and the subsets of its P null sets. If P' is a contraction of this extension to some Borel field which includes the domain of P and is such that P', h is a system of imprimitivity we shall call P', h a *partial completion* of P, h. We shall say that the systems P, h and P', h' are *equivalent* if a partial completion of a trivial contraction of one is strongly equivalent to a partial completion of a trivial contraction of the other. It is clear that ergodicity and dimensionality are preserved under passage to an equivalent system.

THEOREM 6.1. *Let P, h be a transitive system of imprimitivity for a σ representation L of the separable locally compact group $\mathfrak{G}$. Let* S *be the base of P and let x_0 be a point of* S. *Let $\mathcal{G}$ be the subgroup of $\mathfrak{G}$ consisting of all y such that $[x_0]\,y = x_0$. Then $\mathcal{G}$ is closed and the function $y \to [x_0]\,y$ maps the coset space $\mathfrak{G}/\mathcal{G}$ onto* S *in such a manner as to set up a strong equivalence between a partial completion of P, h and a system of imprimitivity P', h' where the base of P' is $\mathfrak{G}/\mathcal{G}$ and h' defines the canonical action of $\mathfrak{G}$ on $\mathfrak{G}/\mathcal{G}$.*

Proof. It follows from the definition of system of imprimitivity that $y \to [x_0]y$ is a Borel mapping. Hence if $\mathfrak{G}/\mathcal{G}$ is equipped with the quotient Borel structure then $y \to [x_0]y$ defines a one-to-one Borel map θ of $\mathfrak{G}/\mathcal{G}$ onto S. Let α be a finite member of the unique invariant measure class in $\mathfrak{G}$ and let α' be the image of α in $\mathfrak{G}/\mathcal{G}$. Since S is metrically standard there exists a Borel subset E_0 of S such that E_0 as a subspace is standard, $\mathrm{S} - E_0$ is a P null set and $\alpha'((\mathfrak{G}/\mathcal{G}) - \theta^{-1}(E_0)) = 0$. Since E_0 is countably separated so is $\theta^{-1}(E_0)$. Since $\theta^{-1}(E_0)$ is a Borel image of a Borel subset of the standard Borel space $\mathfrak{G}$, $\theta^{-1}(E_0)$ is analytic and hence metrically standard by Theorem 4.2 of [13]. Hence the hypotheses of Theorem 7.2 of [13] are satisfied and we may conclude that $\mathcal{G}$ is closed and that $\mathfrak{G}/\mathcal{G}$ is a standard Borel space. It follows now from Theorem 3.2 of [13] that θ restricted to $\theta^{-1}(E_0)$ is a Borel isomorphism. Hence the transforms by θ of the Borel subsets of $\mathfrak{G}/\mathcal{G}$ differ from Borel subsets of S by null sets. The remaining statements of the theorem now follow trivially.

THEOREM 6.2. *Every transitive system of imprimitivity is ergodic.*

Proof. By Theorem 6.1 we need only consider the case in which the base of the system is a coset space with respect to a closed subgroup $\mathcal{G}$ of $\mathfrak{G}$ and on which $\mathfrak{G}$ acts canonically. If the system were not ergodic there would exist two distinct non trivial invariant measure classes in $\mathfrak{G}/\mathcal{G}$. This is impossible by Theorem 1.1 of [11].

While the converse of Theorem 6.2 is very far from being true there in general is an important special case in which it is true.

THEOREM 6.3 *Let P, h be an ergodic system of imprimitivity with base* S. *Let* $\mathrm{S}^{\sim}$ *be the space of all "orbits" of* S *under* $\mathfrak{G}$ *where a subset of* S *is an orbit if it is the set of all* $[x_0]y$ *for some fixed* x_0 *in* S. *If the quotient Borel structure in* $\mathrm{S}^{\sim}$ *is metrically countably separated then* P, h *is equivalent to a transitive system.*

Proof. It will clearly suffice to show that $P_{E'} = 0$ where E' is a Borel set which is the complement of some orbit. Since $\mathrm{S}^{\sim}$ is metrically countably separated P, h is equivalent to a system whose orbit space is countably separated. Hence we may suppose that $\mathrm{S}^{\sim}$ is countably separated and hence that every orbit is a Borel set. Now if $\mathbf{O}$ is an orbit such that $P_{\mathbf{O}} \neq 0$ then $P_{\mathrm{S}-\mathbf{O}} = 0$ since otherwise we would be able to deduce an immediate contradiction from the assumed ergodicity. Thus we need only show that there exists at least one orbit which is not a P null set. Let μ be a finite measure in the measure class of P and suppose that $\mu(\mathbf{O}) = 0$ for every orbit $\mathbf{O}$. Let $E_1, E_2, \dots$ be the inverse images in S of a countable separating family for $\mathrm{S}^{\sim}$. Then each orbit is an intersection of the members of a subsequence of the E_j. Because of the ergodicity hypothesis each $\mu(E_j)$ is either zero or $\mu(\mathrm{S})$ and any intersection of E_j's with measures equal to $\mu(\mathrm{S})$ has itself this measure. Hence every orbit is contained in an E_j with $\mu(E_j) = 0$. Hence S is covered by a countable family of sets of measure 0. Thus $\mu(\mathrm{S}) = 0$ and this contradiction completes the proof.

We devote the rest of this section to a detailed study of the situation analyzed in Theorems 5.4 to 5.6 in the special case in which S is the space $\mathfrak{G}/\mathcal{G}$ of right cosets defined by some closed subgroup $\mathcal{G}$ of $\mathfrak{G}$ and $\mathfrak{G}$ acts canonically on $\mathfrak{G}/\mathcal{G}$. Given any function g on $\mathfrak{G}/\mathcal{G}$ there exists a unique function g_0 on $\mathfrak{G}$ such that $g_0(\xi x) = g_0(x)$ and $g(c(x)) = g_0(x)$ for all $\xi \in \mathcal{G}$ where $c(x)$ denotes the right $\mathcal{G}$ coset to which x belongs. In particular we may replace the function R of Theorem 5.6 by a function R_0 defined on $\mathfrak{G} \times \mathfrak{G}$. Conditions (a), (b), and (c) of that theorem may evidently be expressed as follows in terms of R_0 (a'). For each y_1 and y_2 in $\mathfrak{G}$ we have $R_0(y_1 y_2, x) = \sigma(y_1, y_2) R_0(y_1, x) R_0(y_2, x y_1)$ for almost all x in $\mathfrak{G}$. (b') $R_0(e,x)$ is the identity for almost all x in $\mathfrak{G}$. (c') For all ϕ and ψ in $\mathfrak{H}_k$ $(R_0(y,x)(\phi),\psi)$ is measurable on $\mathfrak{G} \times \mathfrak{G}$ and for each y is measurable on $\mathfrak{G}$. In what follows it will be convenient to adopt the convention that an operator valued function $q \to A(q)$ is measurable, Borel, continuous etc. if this is so in the usual sense for the complex valued functions $(A(q)(\phi),\psi)$ for all ϕ and ψ in the relevant Hilbert space.

We show next that the identity in (a') may in a certain sense be "solved".

LEMMA 6.1. *Let B be any measurable function from $\mathfrak{G}$ to the unitary operators of $\mathfrak{H}_k$ and let σ be a multiplier for $\mathfrak{G}$. Let $R_B(y,x) = (B^{-1}(x) B(xy))/\sigma(x,y)$. Then R_B satisfies* (a'), (b'), *and* (c') *above. Conversely if R satisfies* (a'), (b') *and* (c') *there exists a Borel function B such that $R(y,x) = R_B(y,x)$ for almost all pairs y,x. $R_{B_1}(y,x) = R_{B_2}(y,x)$ for almost all pairs if and only if there exists a unitary operator C such that $B_2(x) = C B_1(x)$ for almost all x.*

Proof. The proofs of the first and third statements are immediate and may be left to the reader. To prove the second let S be a Borel function such that $S(y,x) = R(y,x)$ for almost all pairs y,x. Then $S(y_1, y_2, x) = \sigma(y_1,y_2) S(y_1,x) S(y_2, x y_1)$ for almost all triples y_1, y_2, x. Hence there exists $x_0 \in \mathfrak{G}$ such that $S(y_1 y_2, x_0) = \sigma(y_1,y_2) S(y_1,x_0) S(y_2, x_0 y_1)$ for almost all pairs y_1, y_2 in $\mathfrak{G} \times \mathfrak{G}$. But whenever this last equation holds we have

$$S(y_2, x_0 y_1) = (S^{-1}(y_1,x_0) S(y_1 y_2, x_0))/\sigma(y_1,y_2)$$

so that $$S(y_2,y_3) = (1/\sigma(x_0^{-1} y_3, y_2)) S^{-1}(x_0^{-1} y_3, x_0)\, S(x_0^{-1} y_3 y_2, x_0)$$

where $y_3 = x_0 y_1$. But

$$\sigma(x_0^{-1} y_3, y_2) = \sigma(x_0^{-1}, y_3 y_2) \sigma(y_3, y_2)/\sigma(x_0^{-1}, y_3).$$

Thus for almost all pairs y_2, y_3 we have

$$S(y_2,y_3) = (S(x_0^{-1} y_3, x_2)/\sigma(x_0^{-1}, y_3))^{-1} (S(x_0^{-1} y_3 y_2, x_0)/\sigma(x_0^{-1}, y_3 y_2)) (1/\sigma(y_3, y_2)).$$

Hence we need only take $B(y) = S(x_0^{-1}y, x_0)/\sigma(x_0^{-1}, y)$ in order to complete the proof of the lemma.

LEMMA 6.2. *Let R satisfy* (a'), (b') *and* (c') *above and in addition let $R(y,\xi x) = R(y,x)$ for all ξ in the closed subgroup $\mathcal{G}$ of $\mathfrak{G}$ and all x and y in $\mathfrak{G}$. Then the B of Lemma* 6.1 *may be chosen so that $B(\xi x) = \sigma(\xi,x)L_\xi B(x)$ for all $\xi \in \mathcal{G}$ and all $x \in \mathfrak{G}$ where $\xi \to L_\xi$ is a σ representation of $\mathcal{G}$ which to within equivalence is uniquely determined by R.*

Proof. First let B_1 be any unitary operator valued Borel function such that $R_{B_1}(y,x) = R(y,x)$ for almost all pairs y,x. Then for each $\xi \in \mathcal{G}$ we have $R_{B_1}(y,\xi x) = R_{B_1}(y,x)$ for almost all pairs y,x. Thus

$$(B_1^{-1}(\xi x)\, B_1(\xi x y))/\sigma(\xi x, y) = (B_1^{-1}(x)\, B_1(xy))/\sigma(x,y)$$

or equivalently

$$B_1(\xi x y)\, B_1^{-1}(xy)/\sigma(\xi, xy) = (B_1^{-1}(x)\, B_1(\xi x))/\sigma(\xi, x)$$

again for almost all pairs y,x. In other words for each $\xi \in \mathcal{G}$ the unitary operator $(B_1(\xi x)\cdot B_1^{-1}(x))/\sigma(\xi,x)$ is almost everywhere equal to an operator L_ξ which is independent of x. We verify at once that $L_e = I$ and that $L_{\xi_1\xi_2} = \sigma(\xi_1,\xi_2)L_{\xi_1}L_{\xi_2}$ for all ξ_1 and ξ_2 in $\mathcal{G}$. To show that L is a σ representation we need only show that it is a Borel function of ξ. But let α be any finite member of the unique invariant measure class in $\mathfrak{G}$. Then if ϕ and ψ are in $\mathfrak{H}_k$, $\int (B_1(\xi x)(B_1^{-1}(x)(\phi), \psi)/\sigma(\xi,x))\,d\alpha(x)$ is a Borel function and is equal to $\alpha(\mathfrak{G})(L_\xi(\phi),\psi)$. The next thing we show is that B_1 can be changed on a set of measure zero so that the identity in question holds for all pairs $\xi, x \in \mathcal{G} \times \mathfrak{G}$. We do this in two stages first constructing B_2 in which the almost everywhere restriction applies only to x and not to ξ. Note first that since both sides are Borel functions we may pass from the given almost everywhere condition to the conclusion that for almost all x we have $B_1(\xi x) = \sigma(\xi,x)L_\xi B_1(x)$ for almost all ξ. Let β be a member of the unique invariant measure class in $\mathcal{G}$ such that $\beta(\mathcal{G}) = 1$. For each ϕ and ψ in $\mathfrak{H}_k$ let $A(\phi,\psi,x) = (1/\sigma(\xi,x))(L_\xi^{-1}B_1(\xi x)(\phi),\psi)\,d\beta(\xi)$. Then for each ϕ and ψ, $A(\phi,\psi,x)$ is a Borel function of x and for almost all x in $\mathfrak{G}$ is equal to $(B_1(x)(\phi),\psi)$. Moreover for all x we have $\|A(\phi,\psi,x)\| \leqslant \|\phi\|\,\|\psi\|$ so that there exists a unique bounded operator $B_2(x)$ such that $A(\phi,\psi,x) = (B_2(x)(\phi),\psi)$. $x \to B_2(x)$ is clearly a Borel function of x and furthermore

$$\begin{aligned}(B_2(\xi_0 x)(\phi),\psi) &= A(\phi,\psi,\xi_0 x)\\ &= \int (1/\sigma(\xi,\xi_0 x))(L_\xi^{-1}B_1(\xi\xi_0 x)(\phi),\psi)\,d\beta(\xi)\\ &= \int (1/\sigma(\xi\xi_0^{-1},\xi_0 x))(L_{\xi\xi_0^{-1}}^{-1}B_1(\xi x)(\phi),\psi)\,d\beta'(\xi)\end{aligned}$$

where β' is another measure in $\mathcal{G}$ having the properties assumed for β. Thus

$$(B_2(\xi_0 x)(\phi),\psi)$$
$$= \int (\sigma(\xi,\xi_0^{-1})/(\sigma(\xi,x)\sigma(\xi_0^{-1},\xi_0 x)\sigma(\xi,\ \xi_0^{-1}))(L_{\xi_0^{-1}}^{-1} L_\xi^{-1} B_1(\xi x)(\phi),\psi)\,d\beta(\xi)$$
$$= \int (\sigma(\xi_0,\xi_0^{-1})\sigma(\xi_0,x))/(\sigma(\xi_0^{-1},\xi_0)\sigma(\xi,x)(L_\xi^{-1} B_1(\xi x)(\phi),L_{\xi_0}^*(\psi))\,d\beta(\xi)$$
$$= \sigma(\xi_0,x)\int (1/\sigma(\xi,x))(L_\xi^{-1} B_1(\xi x)(\phi),L_{\xi_0}^*(\psi)\,d\beta(\xi).$$

Let N_1 be a Borel subset of $\mathfrak{G}$ which is of measure zero and outside of which $B_1(\xi x) = \sigma(\xi,x)L_\xi B_1(x)$ for almost all $\xi \in \mathcal{G}$. Then for $x \in \mathfrak{G} - N_1$ we have

$$(B_2(\xi_0 x)(\phi),\psi) = \sigma(\xi_0,x)(B_1(x)(\phi),L_{\xi_0}^*(\psi))$$

for all $\xi_0 \in \mathcal{G}$ and all ϕ and ψ in $\mathfrak{H}_k$. Thus for all $x \in \mathfrak{G} - N_1$ we have $B_2(\xi x) = \sigma(\xi x)B_1(x)$ for all $\xi \in \mathcal{G}$ and in particular (taking $\xi = e$) we have $B_2(x) = B_1(x)$. Hence $B_2(\xi\gamma) = \sigma(\xi,x)\cdot L_\xi B_2(x)$ for all $x \in \mathfrak{G} - N_1$ and all $\xi \in \mathcal{G}$. Now let N be the Borel set outside of which $B(x)$ is unitary and $B_2(\xi x) = \sigma(\xi,x)L_\xi B_2(x)$ for all $\xi \in \mathcal{G}$. Then $N \subseteq N_1$ and hence is a null set. Moreover if $x \in \mathfrak{G} - N$ and $\xi_0 \in \mathcal{G}$ then

$$\begin{aligned} B_2(\xi\xi_0 x) &= \sigma(\xi\xi_0,x)L_\xi L_{\xi_0}\sigma(\xi\xi_0)B_2(x) \\ &= \sigma(\xi,\xi_0 x)\sigma(\xi_0,x)L_\xi L_{\xi_0}B_2(x) \\ &= \sigma(\xi,\xi_0 x)L_\xi B_2(\xi_0 x) \quad \text{for all } \xi \in \mathcal{G}. \end{aligned}$$

Thus N is a union of right $\mathcal{G}$ cosets. Suppose that there exists a Borel function B_3 such that $B_3(\xi x) = \sigma(\xi,x)L_\xi B_3(x)$ for all $\xi \in \mathcal{G}$ and all $x \in \mathcal{G}$. Then if we set $B(x) = B_3(x)$ for all $x \in N$ and $B(x) = B_2(x)$ for all $x \in \mathfrak{G} - N$ we will have a B with the desired properties. We may construct such a B_3 as follows. By Lemma 1.1 of [11] there exists a Borel set S in $\mathfrak{G}$ which intersects each right $\mathcal{G}$ coset in just one point. Each $x \in \mathfrak{G}$ is uniquely representable in the form ξs where $\xi \in \mathcal{G}$ and $s \in S$. We set $B_3(x) = \sigma(\xi,s)L_\xi$ and verify at once that B_3 satisfies the required identity. If c is the canonical mapping of $\mathfrak{G}$ on $\mathfrak{G}/\mathcal{G}$ and c_1 is the restriction of c to S then c_1^{-1} is a Borel function. Moreover $c_1^{-1}(c(\xi s)) = s$. Thus ξ and s are Borel functions of x. It follows that B_3 is a Borel function of x. We complete the proof of the lemma by showing that the equivalence class of the representation L is uniquely determined by R. If $R_{B_1}(y,x) = R_{B_2}(y,x)$ for almost all pairs y,x then by Lemma 6.1 there exists a unitary operator C such that $B_1(x) = CB_2(x)$ for almost all x. Let $B_1(\xi x) = \sigma(\xi,x)L_\xi B_1(x)$ and $B_2(\xi x) = \sigma(\xi,x)M_\xi B_2(x)$ for all ξ,x in $\mathcal{G} \times \mathfrak{G}$. Then for all $\xi \in \mathcal{G}$

$$L_\xi = (B_1(\xi x)B_1^{-1}(x))/\sigma(\xi,x) = (CB_2(\xi x)B_2^{-1}(x)C^{-1})/\sigma(\xi,x) = CM_\xi C^{-1}$$

for almost all x. Thus L and M are equivalent as was to be proved.

Now let μ be a fixed member of the unique invariant measure class in $\mathrm{S} = \mathfrak{G}/\mathcal{G}$ and let P and W be as in section 5. Let h define the canonical action of $\mathfrak{G}$ on $\mathfrak{G}/\mathcal{G}$. Let B be any

Borel function from $\mathfrak{G}$ to the unitary operators in $\mathfrak{H}_k$ such that $B(\xi x) = \sigma(\xi,x)L_\xi B(x)$ for all $\xi, x \in \mathcal{G} \times \mathfrak{G}$ where L is some σ representation of $\mathcal{G}$. Clearly L is uniquely determined by B. Accordingly we shall write L^B for L. Functions B having the above described properties so that L^B is defined we shall call $\sigma - \mathcal{G}$ functions. Let B any $\sigma - \mathcal{G}$ function and form R_B as indicated in Lemma 6.1. Let R'_B be the function on $\mathfrak{G} \times \mathbf{S}$ such that $R'_B(y,c(x)) = R_B(y,x)$ for all y and x in $\mathfrak{G} \times \mathfrak{G}$. R'_B then satisfies (a), (b) and (c) of Theorem 5.6 so that the function $y \to (R'_{B,\bar{y}})^{\sim}$ satisfies (a), (b) and (c) of Theorem 5.4 and hence defines a representation of $\mathfrak{G}$ having P as a system of imprimitivity. This representation of $\mathfrak{G}$ depends only upon B and will be denoted by V^B. Now let B_1 and B_2 be any two $\sigma - \mathcal{G}$ functions and let T be any member of the vector space $\mathbf{R}(L^{B_1}, L^{B_2})$ of all intertwining operators for L^{B_1} and L^{B_2}. Then for each $x \in \mathfrak{G}$ and each $\xi \in \mathcal{G}$ we have

$$\begin{aligned} B_2^{-1}(\xi x)\, T\, B_1(\xi x) &= B_2^{-1}(\xi x)\, T L_\xi^{B_1} B_1(x)\sigma(\xi,x) \\ &= \sigma(\xi,x)\, B_2^{-1}(\xi x) L_\xi^{B_2}\, T\, B_1(x) \\ &= B_2^{-1}(x)\, T\, B_1(x). \end{aligned}$$

Thus setting $C_T(c(x)) = B_2^{-1}(x)\, T_1\, B_1(x)$ defines C_T unambiguously as an operator valued function on $\mathfrak{G}/\mathcal{G}$. We let $T^{\sim}$ be the bounded linear transformation of $\mathfrak{H}(P)$ into itself such that $T^{\sim}(f)(t) = C_T(t)f(t)$ for all $t \in \mathbf{S}$. We may now state:

THEOREM 6.4. *Every σ representation of $\mathfrak{G}$ having P, h as a system of imprimitivity is of the form V_B for some $\sigma - \mathcal{G}$ function B. Moreover if B_1 and B_2 are two $\sigma - \mathcal{G}$ functions then $T \to T^{\sim}$ is an isomorphism of the vector space $\mathbf{R}(L^{B_1}, L^{B_2})$ onto the vector space of all members of $\mathbf{R}(V^{B_1}, V^{B_2})$ which commute with all P_E. If $B_1 = B_2$ then $T \to T^{\sim}$ is a ring isomorphism as well.*

Proof. The first statement is an immediate consequence of Theorems 5.4 and 5.6 and Lemmas 6.1 and 6.2. That $T \to T^{\sim}$ is a vector space isomorphism in general and a ring isomorphism when $B_1 = B_2$ is evident. Thus to complete the proof of the theorem we have only to establish that the range of $T \to \tilde{T}$ is as asserted. Now it follows from the discussion on page 320 of [9] that the bounded linear transformations S of $\mathfrak{H}(P)$ into itself which commute with all P_E are just those such that $S(f)(t) = C(t)f(t)$ for all $f \in \mathfrak{H}(P)$ where C is a bounded Borel function from $\mathbf{S}$ to the bounded linear transformations of $\mathfrak{H}_k$ onto $\mathfrak{H}_k$. Thus what remains to be shown is that C defines a member of $\mathbf{R}(V^{B_1}, V^{B_2})$ if and only if C is of the form C_T for some $T \in \mathbf{R}(L^{B_1}, L^{B_2})$. Now an obvious calculation shows that S is in $\mathbf{R}(V^{B_1}, V^{B_2})$ if and only if for each $y \in \mathfrak{G}$ we have $C(t)R'_{B_1}(y,t) = R'_{B_2}(y,t)C([t]y)$ for almost all t in $\mathbf{S}$. Now let $C_0(x) = C(c(x))$ where c is the canonical mapping of $\mathfrak{G}$ on $\mathfrak{G}/\mathcal{G}$. Our condition then becomes: $C_0(x)R_{B_1}(y,x) = R_{B_2}(y,x)C_0(xy)$ for almost all x in $\mathfrak{G}$ or $C_0(x)B_1^{-1}(x)B_1(xy) = B_2^{-1}(x)\; B_2(xy)C_0(xy)$ for almost all x in $\mathfrak{G}$ or $B_2(x)C_0(x)B_1^{-1}(x) = B_2(xy)C_0(xy)B_1^{-1}(xy)$ for almost all x in $\mathfrak{G}$. Applying the Fubini theorem we say that the last form of our condition is equivalent to the assertion that $B_2(x)C_0(x)B_1^{-1}(x)$ is almost everywhere equal to some constant operator T; in other words that there exists a bounded operator T such that $C_0(x)=B_2^{-1}(x)\cdot TB_1(x)$ for almost all x. Since C_0 is constant on the right $\mathcal{G}$ cosets it follows from the condition that for almost all x we have $B_2^{-1}(\xi x)TB_1(\xi x) = B_2^{-1}(x)TB_1(x)$; that is $B_2^{-1}(x)(L_\xi^{B_2})^{-1}TL_\xi^{B_1}B_1(x) = B_2^{-1}(x)TB_1(x)$ and hence that $TL_\xi^{B_1}=L_\xi^{B_2}T$ for all $\xi\in\mathcal{G}$. The truth of the theorem now follows at once.

COROLLARY 1. *There exists a unitary operator which sets up an equivalence between the pair P, V^{B_1} and the pair P, V^{B_2} if and only if L^{B_1} and L^{B_2} are equivalent.*

COROLLARY 2. *L^{B_1} is irreducible if and only if no proper closed subspace of $\mathfrak{H}(P)$ is invariant under all V^{B_1} and all P_E.*

In the course of the proof of Lemma 6.2 it was shown that every σ representation L of $\mathcal{G}$ is of the form L^B for some σ-$\mathcal{G}$ function B. It follows moreover from Corollary 1 to Theorem 6.4 that the equivalence class of the pair P, V^B does not depend upon the particular B chosen and in addition does not change when L is replaced by an equivalent σ representation of $\mathcal{G}$. Thus the problem of finding the most general σ representation of $\mathfrak{G}$ having P, h as a system of imprimitivity is equivalent (equivalent σ representations being identified) to the problem of finding the most general σ representation of the subgroup $\mathcal{G}$. In particular we have a natural correspondence which assigns to each equivalence class of σ representations of $\mathcal{G}$ a well defined equivalence class of σ representations of $\mathfrak{G}$. It is natural to compare this correspondence with the correspondence $L\to U^L$ set up in section 4 and not surprising to find them identical.

THEOREM 6.5. *Let $\mathfrak{G}$, P etc. be as in Theorem 6.4 and the immediately preceding discussion. Let B be any σ-$\mathcal{G}$ function. Then the pair P, V^B is equivalent to the pair P, U^{L^B} where U^{L^B} denotes as usual the representation of $\mathfrak{G}$ induced by the σ representation L^B of $\mathcal{G}$.*

Proof. Let f be any Borel function from S to $\mathfrak{H}_k$ and let f^0 be the Borel function $f\circ c$ where c is the canonical map of $\mathfrak{G}$ on $\mathrm{S}=\mathfrak{G}/\mathcal{G}$. Let f^{0B} be the function $x\to B(x)(f^0(x))$. Then f^{0B} is a Borel function and

$$f^{0B}(\xi x) = B(\xi x)f^0(\xi x) = \sigma(\xi,x)L_\xi^B(B(x)(f^0(x))) = \sigma(\xi,x)L_\xi^B(f^{0B}(x))$$

for all ξ, $x\in\mathcal{G}\times\mathfrak{G}$. Conversely it follows at once on multiplication by $B^{-1}(x)$ that every Borel function g from $\mathfrak{G}$ to $\mathfrak{H}_K$ such that $g(\xi x)=\sigma(\xi,x)L_\xi^B(g(x))$ for all ξ, $x\in\mathcal{G}\times\mathfrak{G}$ is uniquely of the form f^{0B}. Now

$$(f^{0B}(x),f^{0B}(x)) = (B(x)(f^0(x)),B(x)(f^0(x)) = (f^0(x),f^0(x))$$

since each $B(x)$ is unitary. It follows at once that f is in $\mathbf{L}^2(\mathbf{S},\mu,\mathfrak{H}_k)$ if and only if f^{0B} is in ${}^{\mu}\mathfrak{H}^L$ and that $f \to f^{0B}$ sets up a unitary map V_0 of the first Hilbert space on the second. An obvious calculation which we leave to the reader shows that $V_0 P_E = P_E V_0$ for all E and that $V_0^{-1} U_x^{L^B} V_0 = V_x^B$ for all $x \in \mathfrak{G}$. This completes the proof of the theorem.

It is easy to see directly that an induced representation U^L always has associated with it a canonical system of imprimitivity based on $\mathfrak{G}/\mathcal{G}$. It is P, h where P_E is the operation of multiplying by the characteristic function of E and h is the canonical action of $\mathfrak{G}$ on $\mathfrak{G}/\mathcal{G}$. Clearly the canonical unitary transformation setting up an equivalence between ${}^{\mu_1}U^L$ and ${}^{\mu_2}U^L$ where μ_1 and μ_2 are different quasi invariant measures in $\mathfrak{G}/\mathcal{G}$ also sets up an equivalence between the corresponding P's. Thus the pair P, U^L is defined to within equivalence by L. We may now formulate the main theorem of this section. Its proof is an immediate consequence of the foregoing considerations and will be left to the reader.

THEOREM 6.6. *Let $\mathfrak{G}$ be a separable locally compact group, let $\mathcal{G}$ be a closed subgroup of $\mathfrak{G}$ and let σ be a multiplier for $\mathfrak{G}$. Let V be any σ representation of $\mathfrak{G}$ and let P' be any projection valued measure based on $\mathfrak{G}/\mathcal{G}$ such that P', h is a system of imprimitivity for V. Then there exists a σ representation L of $\mathcal{G}$ such that the pair P' V is equivalent to the pair P, U^L where P, h is the canonical system of imprimitivity for U^L based on $\mathfrak{G}/\mathcal{G}$. If L_1 and L_2 are two σ representations of $\mathcal{G}$ and P_1, h and P_2, h are the corresponding canonical systems of imprimitivity then the pairs P_1, U^{L_1} and P_2, U^{L_2} are equivalent if and only if L_1 and L_2 are equivalent σ representations of $\mathcal{G}$. Finally the commuting algebra of L is isomorphic to the intersection of the commuting algebras of P and U^L.*

7. The restriction of a σ representation to a normal subgroup

Let $\mathfrak{G}$ be a separable locally compact group and let $\mathcal{K}$ be a closed normal subgroup of $\mathfrak{G}$. Let σ be a multiplier for $\mathfrak{G}$. For each $s \in \mathfrak{G}$ the mapping $L \to L^s$ defined in the corollary to Lemma 4.2 takes σ representations of $\mathcal{K}$ into σ representations of $\mathcal{K}$. We shall call a σ representation L of $\mathcal{K}$ such that L and L^s are equivalent an *invariant* σ representation (with respect to the given imbedding of $\mathcal{K}$ in $\mathfrak{G}$). If L is an irreducible σ representation of $\mathcal{K}$ we shall denote the set of all L^s (or rather the set of their equivalence classes) by $\mathbf{O}_L$ and call it the *orbit* of L. Of course every member of $\mathbf{O}_L$ is irreducible and given irreducible σ representations L and M we have either $\mathbf{O}_L = \mathbf{O}_M$ or $\mathbf{O}_L \cap \mathbf{O}_M = 0$. When $\mathcal{K}$ is compact every σ representation of $\mathcal{K}$ is a direct sum of irreducible σ representations and is determined to within equivalence by the multiplicities with which the irreducible σ representations occur. It is easy to see in this case that the invariant representations of $\mathcal{K}$ are just those for which these multiplicities are constant on the orbits. It is natural to call an invariant σ representation *transitive* (we are still assuming $\mathfrak{K}$ compact) if all multiplicities are zero outside of a single orbit. Every invariant σ representation is then uniquely a direct sum of transitive σ representations no two of which have any irreducible sub σ representation in common. Guided by these considerations in the compact case we make the following definition in the general case. The invariant σ representation L of the normal subgroup $\mathfrak{K}$ is *ergodic* if it cannot be written as the direct sum of two invariant σ representations which are disjoint in the sense of section 1 of [12]. As explained in section 3 of the present paper the discussion of section 1 of [12] applies without change to σ representations. Following a suggestion of M. Krasner however we shall make one change in terminology and refer to factor representations as primary representations.

THEOREM 7.1. *Let M be a σ representation of the separable locally compact group $\mathfrak{G}$ and let $M^{(\mathfrak{K})}$ denote the restriction of M to the closed normal subgroup $\mathfrak{K}$ of $\mathfrak{G}$. Then $M^{(\mathfrak{K})}$ is invariant. If in addition M is primary then $M^{(\mathfrak{K})}$ is ergodic.*

Proof. For all $\xi, s \in \mathfrak{K} \times \mathfrak{G}$ we have

$$\begin{aligned}(M^{(\mathfrak{K})})^s_\xi &= (\sigma(s^{-1},s)/\sigma(s\xi,s^{-1})\sigma(s,\xi))\, M^{(\mathfrak{K})}_{s\xi s^{-1}}\\ &= (\sigma(s^{-1},s)/(\sigma(s\xi,s^{-1})\sigma(s,\xi))\, M_{s\xi s^{-1}}\\ &= (\sigma(s^{-1},s)/\sigma(s,\xi))\, M_{s\xi} M_{s^{-1}}\\ &= \sigma(s^{-1},s)\, M_s M_\xi M_{s^{-1}}\\ &= M_s M_\xi M_s^{-1} = M_s M^{(\mathfrak{K})}_\xi M_s^{-1}.\end{aligned}$$

Thus M_s sets up an equivalence between $M^{(\mathfrak{K})}$ and $(M^{(\mathfrak{K})})^s$. Now let E be any projection in $\mathbf{R}(M^{(\mathfrak{K})}, M^{(\mathfrak{K})})$ such that ${}^E M^{(\mathfrak{K})}$ and ${}^{1-E}M^{(\mathfrak{K})}$ are disjoint and invariant. If we can show that $E \in \mathbf{R}(M,M)$ then any intertwining operator for ${}^E M$ and ${}^{1-E}M$ will be such, a fortiori, for ${}^E M^{(\mathfrak{K})}$ and ${}^{1-E}M^{(\mathfrak{K})}$ and hence will be zero. Thus it will follow that ${}^E M$ and ${}^{1-E}M$ are disjoint and this contradiction to the primariness of M will prove the second statement of the theorem. Thus it will suffice to show that $M_s E = E M_s$ for all $s \in \mathfrak{G}$. But since ${}^E M^{(\mathfrak{K})}$ and ${}^{1-E}M^{(\mathfrak{K})}$ are invariant there exists for each s a unitary operator U_s in $\mathfrak{H}(M) = \mathfrak{H}(M^{(\mathfrak{K})})$ which commutes with E and is such that $U_s (M^{(\mathfrak{K})})^s_\xi U_s^{-1} = M^{(\mathfrak{K})}_\xi$ for all $\xi \in \mathfrak{K}$. Thus

$$U_s M_s M^{(\mathfrak{K})}_\xi M_s^{-1} U_s^{-1} = U_s (M^{(\mathfrak{K})})^s_\xi U_s^{-1} = M^{(\mathfrak{K})}_\xi$$

for all $\xi, s \in \mathfrak{K} \times \mathfrak{G}$. Thus $U_s M_s \in \mathbf{R}(M^{(\mathfrak{K})}, M^{(\mathfrak{K})})$ for all $s \in \mathfrak{G}$. Thus for all s $U_s M_s$ commutes with E and since U_s does so does M_s. This completes the proof.

THEOREM 7.2. *Let $\mathfrak{K}$ be as in the preceding theorem and let L be an invariant σ representation of $\mathfrak{K}$. Then the type* I, *type* II *and type* III *components of L are all invariant and if L is of type* I *then for all $j = \infty, 1, 2, \ldots$ the uniformly of multiplicity j component of L is invariant.*

Proof. First let L be invariant and of type X where $X = \mathrm{I}$, II or III. Then $L^s_\xi = (\sigma(s^{-1},s)/\sigma(s\xi,s^{-1})\sigma(s,\xi))L_{s\xi s^{-1}}$. Thus $\mathbf{R}(L^s,L^s) = \mathbf{R}(L,L)$. Hence L^s is also of type X. Let $L = L^{\mathrm{I}} \oplus L^{\mathrm{II}} \oplus L^{\mathrm{III}}$ where L^X is of type X and not every term need be present. Then $L^s = L^{\mathrm{I}s} \oplus L^{\mathrm{II}s} \oplus L^{\mathrm{III}s}$. But since L and L^s are equivalent and the decomposition is unique it follows that $(L^X)^s$ and L^X are equivalent as was to be proved. The proof of the second statement is analogous.

COROLLARY 1. *If L is ergodic and invariant then L is either of type* I, *type* II *or type* III *and if L is of type* I *it is uniformly of multiplicity k for some $k = \infty$,* 1, 2, ...

COROLLARY 2. *If M is a primary representation of $\mathfrak{G}$ then $M^{(\mathfrak{N})}$ is of type* I, *type* II *or type* III *and if it is of type* I *it is uniformly of multiplicity k for some $k = \infty$,* 1, 2, ...

At this point we need to consider the decomposibility of the representations of the normal subgroup $\mathfrak{N}$ into irreducible parts. To this end we apply the theory developed in section 10 of [13] and adapted to σ representations in section 3 of the present paper. We recall that the set $\hat{\mathfrak{N}}^\sigma$ of all equivalence classes of irreducible σ representations of $\mathfrak{N}$ equipped with a certain natural Borel structure is called the σ dual of $\mathfrak{N}$. It turns out that if $\hat{\mathfrak{N}}^\sigma$ is as much as metrically countably separated as a Borel space then it is actually metrically standard. We say then that $\hat{\mathfrak{N}}^\sigma$ is metrically smooth. If every σ representation of $\mathfrak{N}$ is of type I we say that $\hat{\mathfrak{N}}^\sigma$ is of type I. We have mappings $L \to \mathcal{C}(L)$ and $C \to \mathcal{L}^\sigma(C)$ where L is an equivalence class of multiplicity free σ representations of $\mathfrak{N}$, $\mathcal{C}(L)$ is a measure class in $\hat{\mathfrak{N}}$, C is a standard measure class in $\hat{\mathfrak{N}}^\sigma$ and $\mathcal{L}^\sigma(C)$ is an equivalence class of σ representations of $\mathfrak{N}$. When $\mathfrak{N}$ is metrically smooth and of type I then the mappings $\mathcal{C}$ and $\mathcal{L}^\sigma$ invert one another and set up a one-to-one correspondence between the measure classes in $\hat{\mathfrak{N}}^\sigma$ and the equivalence classes of multiplicity free σ representations of $\mathfrak{N}$.

THEOREM 7.3. *Let $\mathfrak{N}$ be a closed normal subgroup of the separable locally compact group $\mathfrak{G}$. Then for each multiplier σ the mapping $L, s \to L^s$ is a Borel function from $\hat{\mathfrak{N}}^\sigma \times \mathfrak{G}$ to $\hat{\mathfrak{N}}^\sigma$.*

Proof. Let R denote the set of all irreducible σ representations of $\mathfrak{N}$ in some fixed Hilbert space $\mathfrak{H}_0$ where we do not identify equivalent σ representations. In view of the definition of the Borel structure in $\hat{\mathfrak{N}}^\sigma$ it will suffice to show that $L, s \to (L^s_x(\phi),\psi)$ is a Borel function on $R \times \mathfrak{G}$ for each fixed x in $\mathfrak{G}$ and each ϕ and ψ in $\mathfrak{H}_0$. But

$$(L^s_x(\phi),\psi) = (\sigma(s^{-1},s)/\sigma(sx,s^{-1})\sigma(s,x))(L_{sxs^{-1}})\phi),\psi).$$

Moreover for each x, sxs^{-1} is a continuous function of s. Thus we need only show that $(L_s(\phi),\psi)$ is a Borel function on $R \times \mathfrak{G}$. But $(L_s(\phi),\psi) = (1/\lambda)(L^0_{\lambda,s}(\phi),\psi)$ where L^0 is the ordinary representation of the group extension $\mathfrak{G}^0$ defined in section 2. Moreover since $(L^0_{\lambda,s}(\phi),\psi)$

is continuous as a function of $\mathfrak{G}^0$ for each fixed L, ϕ and ψ it follows from the proof of Lemma 9.2 of [11] that for each fixed ϕ and ψ $(L_\lambda^0(\phi),\psi)$ is a Borel function on $R \times \mathfrak{G}$. Hence $(L_s \cdot (\phi),\psi)$ is a Borel function on $\hat{\mathfrak{K}}^\sigma \times \mathfrak{G}$ and the truth of the theorem follows at once.

It follows in particular from Theorem 7.3 that $L \to L^s$ is a Borel automorphism of $\hat{\mathfrak{K}}^\sigma$ for each $s \in \mathfrak{G}$. Let us denote this automorphism of $\mathfrak{K}$ by $h(s)$. Then $s \to h(s)$ is an anti homomorphism of $\mathfrak{G}$ into the group of Borel automorphisms of $\hat{\mathfrak{K}}^\sigma$. We shall refer to it as defining the *canonical action of* $\mathfrak{G}$ *on* $\hat{\mathfrak{K}}^\sigma$. According to Theorem 7.3 it satisfies condition (a) in the discussion preceding Theorem 5.1 and thus is capable of being the second term in a system of imprimitivity based on $\hat{\mathfrak{K}}^\sigma$ whenever the latter is a metrically standard Borel space. The definition of ergodicity for a system of imprimitivity given in that same discussion actually involved only the measure class of the projection valued measure and not the representation. Hence we may speak unambiguously of ergodic invariant measure classes in $\hat{\mathfrak{K}}^\sigma$ (with respect to the canonical action of $\mathfrak{G}$ on $\hat{\mathfrak{K}}^\sigma$).

THEOREM 7.4. *Let* $\mathfrak{K}$ *be a closed normal subgroup of the separable locally compact group* $\mathfrak{G}$ *and let* σ *be a multiplier for* $\mathfrak{G}$. *Suppose that* $\hat{\mathfrak{K}}^\sigma$ *is metrically smooth and of type* I. *Let* C *be a measure class in* $\hat{\mathfrak{K}}^\sigma$. *Then* $\mathfrak{L}^\sigma(C)$ *is an invariant* σ *representation of* $\mathfrak{K}$ *if and only if* C *is an invariant measure class under the canonical action of* $\mathfrak{G}$ *on* $\mathfrak{K}$. *Moreover if* C *and* $\mathfrak{L}^\sigma(C)$ *are invariant then* $\mathfrak{L}^\sigma(C)$ *is ergodic if and only if* C *is ergodic.*

Proof. It is easy to see that $(\mathfrak{L}^\sigma(C))^s$ is equivalent to $\mathfrak{L}^\sigma(C^s)$ where C^s is the transform of C by s. Hence $\mathfrak{L}^\sigma(C)$ is invariant if and only if $\mathfrak{L}^\sigma(C)$ and $\mathfrak{L}^\sigma(C^s)$ are equivalent for all s: that is if and only if $C = C^s$ for all s. Now let C and $\mathfrak{L}^\sigma(C)$ be invariant. If $\mathfrak{L}^\sigma(C)$ is not ergodic then $\mathfrak{L}^\sigma(C) = L_1 \oplus L_2$ where L_1 and L_2 are disjoint and invariant. Hence $L_1 = \mathfrak{L}^\sigma(C_1)$ and $L_2 = \mathfrak{L}^\sigma(C_2)$ where C_1 and C_2 are disjoint invariant measure classes. Since C_1 and C_2 have properly more null sets than C, C cannot be ergodic. Conversely if C is invariant and not ergodic there exists an invariant measure class C_1 with properly more null sets than C. Let E_0 be the set on which a Borel Radon Nikodym derivative of some member of C_1 with respect to some member of C is different from zero. Then A is a C_1 null set if and only if $A \cap E_0$ is a C null set. Since C_1 is invariant every $(E_0)^s$ differs from E_0 by a C null set. Let μ be a member of C and let C_2 be the measure class of $E \to \mu((\mathfrak{K} - E_0) \cap E)$. Then C_2 is invariant and $\mathfrak{L}^\sigma(C)$ is equivalent to the direct sum of $\mathfrak{L}^\sigma(C_1)$ and $\mathfrak{L}^\sigma(C_2)$. Hence $\mathfrak{L}^\sigma(C)$ is not ergodic.

When $\hat{\mathfrak{K}}^\sigma$ is countable, as for instance when $\mathfrak{K}$ is compact, then every ergodic invariant measure class in $\mathfrak{K}$ is "concentrated" in an orbit of $\hat{\mathfrak{K}}^\sigma$ under $\mathfrak{G}$; that is there exists such an orbit whose complement is a null set with respect to the measure class. This orbit determines the measure class uniquely and the correspondence thus set up between ergodic invariant measure classes and orbits is one-to-one and onto. When $\hat{\mathcal{K}}^\sigma$ is less special there may be ergodic invariant measure classes which are not concentrated in orbits. These considerations lead us to call an ergodic invariant measure class a *quasi orbit.* A quasi orbit which is concentrated in an orbit we call a *transitive* quasi orbit.

THEOREM 7.5. *Let $\mathcal{K}$ and σ be as in Theorem 7.4. Then there exists one and only one quasi orbit concentrated in each orbit of $\hat{\mathcal{K}}^\sigma$ under $\mathfrak{G}$. Moreover for each $L_0 \in \hat{\mathcal{K}}^\sigma$ the set $\mathcal{G}_{L_0}$ of all $s \in \mathfrak{G}$ such that $L_0{}^s = L_0$ is a closed subgroup of $\mathfrak{G}$ containing $\mathcal{K}$.*

Proof. With evident tiny modifications the first part of the proof of Theorem 6.1 may be read as a proof of the closure of $\mathcal{G}_{L_0}$. If $s \in \mathcal{K}$ then $L^s_x = L_s L_x L_s^{-1}$ for all σ representations L of $\mathcal{K}$. Hence L_s and L are equivalent. Hence $\mathcal{K} \subseteq \mathcal{G}_{L_0}$. The one-to-one map θ of the coset space $\mathfrak{G}/\mathcal{G}_{L_0}$ on the orbit $\mathbf{O}_{L_0}$ defined by $s \to L_0^s$ carries the unique invariant measure class in $\mathfrak{G}/\mathcal{G}_{L_0}$ into an invariant measure class in $\mathcal{K}$ which is concentrated in $\mathbf{O}_{L_0}$. It follows from the argument of Theorem 6.2 that this measure class is ergodic. Now let C be any invariant measure class in $\mathbf{O}_{L_0}$. θ^{-1} carries the Borel structure in $\mathbf{O}_{L_0}$ into a Borel structure in $\mathfrak{G}/\mathcal{G}_{L_0}$ which is perhaps weaker than the given one and carries C into a measure class C' defined on these special Borel sets. Since $\mathcal{K}$ is metrically standard there exists a Borel subset N of $\hat{\mathcal{K}}^\sigma$ of measure zero with respect to C such that $\hat{\mathcal{K}}^\sigma - N$ is standard and hence countably separated. Thus by Theorem 5.1 of [13] θ is a Borel isomorphism when restricted to $\theta^{-1}(\hat{\mathcal{K}}^\sigma - N)$ Hence every Borel subset of $\mathfrak{G}/\mathcal{G}_{L_0}$ differs by a null set from a Borel set on which the members of C' are defined. Hence C' may be canonically extended so that its members are defined on all Borel subsets of $\mathfrak{G}/\mathcal{G}_{L_0}$ and this extended C' is still invariant. Since there is one and only one invariant measure class in $\mathfrak{G}/\mathcal{G}_{L_0}$ the measure class C' is uniquely determined and hence so is C.

If $\mathcal{K}$ is as above and M is any σ representation of $\mathcal{K}$ then M is quasi equivalent ([12] section 1) to a multiplicity free representation L of $\mathcal{K}$ whose equivalence class L^Δ is uniquely determined. Let $C = \mathfrak{C}(L^\Delta)$ so that $L = \mathfrak{L}^\sigma(C)$. For each Borel set E in $\hat{\mathcal{K}}^\sigma$ let C_E denote the measure class whose members are the measures $F \to \mu(F \cap E)$ where μ varies over the members of C. $\mathfrak{L}^\sigma(C_E)$ is then the equivalence class of a sub σ representation L^E of L. Let P_E be the unique projection in the center of $\mathbf{R}(M, M)$ such that the sub σ representation defined by P_E is quasi equivalent to the members of $\mathfrak{L}^\sigma(C_E)$. $E \to P_E$ is evidently a projection valued measure. We call it the *projection valued measure belonging to M.*

THEOREM 7.6. *Let M be a σ representation of the separable locally compact group $\mathfrak{G}$ and let $\mathcal{K}$ be a closed normal subgroup of $\mathfrak{G}$ such that $\hat{\mathcal{K}}^\sigma$ is smooth and of type* I. *Let P be the projection valued measure belonging to the restriction $M^{(\mathcal{K})}$ of M to $\mathcal{K}$. Let h denote the canonical* *action of $\mathfrak{G}$ on $\hat{\mathfrak{K}}^\sigma$. Then P, h is a system of imprimitivity for M based on $\hat{\mathfrak{K}}^\sigma$. This system is ergodic whenever M is primary.*

Proof. In view of Theorem 7.3 we have only to show that $M_y P_E M_y^{-1} = P_{[E]y}$, for all Borel sets $E \subseteq \hat{\mathfrak{K}}^\sigma$ and all $y \in \mathfrak{G}$. Now

$$M_y M_\xi^{(\mathfrak{K})} M_y^{-1} = (M_\xi^{(\mathfrak{K})})^y (\sigma(y^{-1}, y)/(\sigma(y\xi, y^{-1})\sigma(y,\xi))) M^{\mathfrak{K}}_{y\xi y}, \quad \text{for all } \xi, y \in \mathfrak{K} \times \mathfrak{G}.$$

Hence for each fixed y the set of all $M_y M_\xi^{(\mathfrak{K})} M_y^{-1}$ on the one hand and the set of all $M_\xi^{(\mathfrak{K})}$ on the other have the same commuting algebra R. Hence $M_y R M_y^{-1} = R$. Since the P_E are just exactly the projections in the center of R it follows that each $M_y P_E M_y^{-1}$ is of the form P_F for some Borel set $F \subseteq \hat{\mathfrak{K}}^\sigma$. Now P_E and P_F define subrepresentations of $M^{(\mathfrak{K})}$ and $M_y \cdot M^{(\mathfrak{K})} M_y^{-1}$ restricted to the range of P_F is evidently equivalent to $M^{(\mathfrak{K})}$ restricted to the range of P_E. Thus $(M^{(\mathfrak{K})})^y$ restricted to the range of P_F is equivalent to $M^{(\mathfrak{K})}$ restricted to the range of P_E. Let C be the measure class of P. Then $\mathfrak{L}^\sigma(C_E)$ is the equivalence class of the multiplicity free representations quasi equivalent to the restriction of $M^{(\mathfrak{K})}$ to the range of P_E and $\mathfrak{L}^\sigma(([C]y)_F)$ is the equivalence class of the multiplicity free representations quasi equivalent to the restriction of $(M^{(\mathfrak{K})})^y$ to the range of P_F. Hence $C_E = ([C]y)_F$. Hence E and $[F]y$ differ by a C null set. Hence $M_y P_E M_y^{-1} = P_F = P_{[E]y}$, as was to be proved. When M is primary then $M^{(\mathfrak{K})}$ is ergodic by Theorem 7.1 and is of the form nL for some multiplicity free σ representation L by Corollary 2 to Theorem 7.2. Since nL is invariant and ergodic so is L. Now it follows from the definition of P that the measure class of P is $\mathfrak{C}(L^\Delta)$. Since $\mathfrak{C}(L^\Delta)$ is ergodic by Theorem 7.4 it follows that P, h is ergodic.

Let M be any primary σ representation of $\mathfrak{G}$. As we have already remarked the measure class C of the projection valued measure belonging to $M^{(\mathfrak{K})}$ is the unique measure class such that $M^{(\mathfrak{K})}$ is a multiple of the members of $\mathfrak{L}^\sigma(C)$. This measure class, as we have just seen, is ergodic and invariant and hence is what we have called a quasi orbit. We shall call it the quasi orbit associated with M.

8. The σ representations associated with transitive quasi orbits

We are now in a position to discuss problems (a) and (b) of the introduction in the special case in which the closed normal subgroup $\mathfrak{K}$ is such that $\hat{\mathfrak{K}}^\sigma$ is metrically smooth and of type I. It follows from the results of the last section that a σ representation of $\mathfrak{K}$ can be the restriction to $\mathfrak{K}$ of a primary representation of $\mathfrak{G}$ only if it is of the form nL where L is multiplicity free ergodic and invariant. $C \to \mathfrak{L}^\sigma(C)$ sets up a one-to-one correspondence between the equivalence classes of such L's and the quasi orbits in $\hat{\mathfrak{K}}^\sigma$. To say that for every multiplicity free ergodic invariant L there exists $n = \infty, 1, 2, \dots$ such that

nL is the restriction to $\mathfrak{K}$ of some primary σ representation of $\mathfrak{G}$ is to say that every quasi orbit is associated with some primary σ representation of $\mathfrak{G}$. To find all primary σ representations of $\mathfrak{G}$ having a given restriction to $\mathfrak{K}$ (different multiples of the same representation being identified) is to find all primary σ representations of $\mathfrak{G}$ associated with a fixed quasi orbit. In this section we shall discuss these questions in the particular case in which the quasi orbit is transitive. The more difficult intransitive case we hope to treat (less completely) in a subsequent article.

THEOREM 8.1. *Let σ be a multiplier for the separable locally compact group $\mathfrak{G}$ and let $\mathfrak{K}$ be a closed normal subgroup of $\mathfrak{G}$ such that $\hat{\mathfrak{K}}^{\sigma}$ is metrically smooth and of type* I. *Let L_0 be any member of $\hat{\mathfrak{K}}^{\sigma}$, let $\mathcal{G}$ denote the closed subgroup of all $s \in \mathfrak{G}$ such that $(L_0)^s = L_0$ and let $\mathbf{O}$ denote the orbit of L_0 in $\hat{\mathfrak{K}}^{\sigma}$. Then for each primary σ representation L of $\mathcal{G}$ such that $L^{(\mathfrak{K})}$ is equivalent to a multiple of L_0 the induced σ representation U^L of $\mathfrak{G}$ is a primary σ representation whose quasi orbit is concentrated in $\mathbf{O}$. Every such primary representation of $\mathfrak{G}$ may be so obtained and L and U^L determine one another to within equivalence. For each $X =$* I, II, *or* III *U^L is of type X if and only if L is of type X and U^L is irreducible if and only if L is irreducible.*

Proof. Let L be a primary σ representation of $\mathcal{G}$ whose restriction $L^{(\mathfrak{K})}$ to $\mathfrak{K}$ is a multiple n of L^0 and apply Theorem 4.5 to $(U^L)^{(\mathfrak{K})}$. The correspondence $\mathcal{G}x\,\mathfrak{K} \rightarrow \mathcal{G}x$ is clearly one-to-one and Borel set preserving between the space of $\mathcal{G}:\mathfrak{K}$ double cosets and the space of right $\mathcal{G}$ cosets. Thus $\mathcal{G}$ and $\mathfrak{K}$ are regularly related. Moreover the correspondence clearly takes the projection valued measure associated with the direct integral decomposition of Theorem 4.5 into the projection valued measure P defined on $\mathfrak{G}/\mathcal{G}$ which is the first term in the canonical system of imprimitivity associated with U^L. By Theorem 4.5 the σ representation associated with the right coset containing x is just the restriction of L^x to $\mathfrak{K}$. But $(L^x)^{(\mathfrak{K})} = (L^{(\mathfrak{K})})^x$. Moreover $L^{(\mathfrak{K})}$ is a multiple of L_0 and L_0^x and L_0^y are inequivalent whenever x and y are in distinct $\mathcal{G}$ cosets. Thus $(U^L)^{(\mathfrak{K})}$ is a multiple of the multiplicity free σ representation whose measure class is the image in $\mathfrak{K}$ of the measure class of P under the one-to-one mapping c of $\mathfrak{G}/\mathcal{G}$ on $\hat{\mathfrak{K}}^{\sigma}$ defined by $x \rightarrow L_0^x$. It follows in particular that the quasi orbit of U^L is concentrated in $\mathbf{O}$ and that c composed with P is the projection valued measure belonging to $(U^{(L)})^{(\mathfrak{K})}$. Since the $P_{c(E)}$ all lie in the center of the commuting algebra of $(U^L)^{(\mathfrak{K})}$ we conclude that every member of the commuting algebra of U^L commutes with all P_E and hence by Theorem 6.6 that U^L and L have isomorphic commuting algebras. In particular U^L is primary and the statements about the type and reducibility of L and U^L all follow. Suppose now that U^{L_1} and U^{L_2} are equivalent where L_1 and L_2 satisfy the conditions laid upon L above. Let V set up the equivalence. Then V also sets up an equivalence between $(U^{L_1})^{(K)}$ and $(U^{L_2})^{(\mathcal{K})}$ and hence sets up an equivalence between the projection valued measures belonging to these two σ representations. Because of the above described connection between these projection valued measures and the first terms P^1 and P^2 of the canonical systems of imprimitivity associated with U^{L_1} and U^{L_2} respectively we conclude that V sets up an equivalence between the pair P^1, U^{L_1} and the pair P^2, U^{L_2}. Thus L_1 and L_2 are equivalent by Theorem 6.6. Finally let M be any primary σ representation of $\mathfrak{G}$ whose quasi orbit is $\mathbf{O}$. By Theorem 6.1 the mapping c defined above sets up a strong equivalence between a partial completion of the system of imprimitivity for $M^{(\mathcal{K})}$ described in Theorem 7.6 and a system of imprimitivity P', h' for $M^{(\mathcal{K})}$ based on $\mathfrak{G}/\mathcal{G}$. By Theorem 6.6 there exists a σ representation L of $\mathcal{G}$ such that the pair P', M is equivalent to the pair P, U^L where P, h' is the canonical system of imprimitivity for U^L. Thus M is of the form U^L. We complete the proof by showing that $L^{(\mathcal{K})}$ is a multiple of L_0. By Theorem 4.5 and the connection already described between $\mathcal{G}$ cosets and $\mathcal{G}:\mathcal{K}$ double cosets $(U^L)^{(\mathcal{K})}$ is a direct integral over $\mathfrak{G}/\mathcal{G}$ of the $(L^{(\mathcal{K})})^x$ and the corresponding projection valued measure is P. Because of the relationship already described between P and the projection valued measure belonging to $M^{(\mathcal{K})}$ it follows that $(L^{(\mathcal{K})})^x$ must be a multiple of L_0^x for almost all x. But this relationship for a single x implies that $L^{(\mathcal{K})}$ is a multiple of L_0 as was to be proved.

THEOREM 8.1 reduces the study of the primary σ representations of $\mathfrak{G}$ associated with a fixed orbit to the study of certain primary σ representations of a certain closed subgroup $\mathcal{G}$ of $\mathfrak{G}$ which includes $\mathcal{K}$. We show now that this latter study may be reduced to the study of the primary ω representations of $\mathcal{G}/\mathcal{K}$ where ω is a certain multiplier for $\mathcal{G}/\mathcal{K}$ which may be non trivial even when σ is trivial. As a first step we show that an irreducible σ representation of $\mathcal{K}$ may always be extended to an irreducible τ representation of the corresponding $\mathcal{G}$ where τ is a multiplier for $\mathcal{G}$ which agrees with σ on $\mathcal{K}$ but not necessarily elsewhere.

THEOREM 8.2. *Let $\mathcal{K}$ be a closed normal subgroup of the separable locally compact group $\mathcal{G}$. Let σ be a multiplier for $\mathcal{G}$. Let L be an irreducible σ representation of $\mathcal{K}$ such that L^x is equivalent to L for all $x \in \mathcal{G}$. Then there exists a multiplier τ for $\mathcal{G}$ and a τ representation M of $\mathcal{G}$ such that $L_\xi = M_\xi$ for all $\xi \in \mathcal{K}$. τ may be chosen so as to be the product with σ of a multiplier of the form $1/(\omega \circ f)$ where f is the canonical homomorphism of $\mathcal{G} \times \mathcal{G}$ on $\mathcal{G}/\mathcal{K} \times \mathcal{G}/\mathcal{K}$ and ω is a multiplier for $\mathcal{G}/\mathcal{K}$. When τ is so chosen ω is uniquely determined by σ and L up to multiplication by a trivial multipier.*

Proof. Since L^x is equivalent to L for all x there exists for each x a unitary operator M_x such that for all $\xi \in \mathcal{K}$, $M_x L_\xi M_x^{-1} = L_\xi^x$. Since L is irreducible each M_x is uniquely determined up to a multiplicative constant. We shall show that these constants may be chosen so that $x \to M_x$ has the properties stated in the theorem. It will be convenient to do this in stages. Let $\mathcal{U}(\mathfrak{H}(L))$ denote the unitary group of the Hilbert space $\mathfrak{H}(L)$. As shown in the proof of Theorem 8.5 of [13] this group is a standard Borel group in the Borel structure it inherits from the weak operator topology. Moreover as shown in the proof of Theorem 10.8 of [13] there exists a Borel subset S of $\mathcal{U}(\mathfrak{H}(L))$ which intersects each one parameter family $\theta \to (\exp(i\theta))V$ once and only once. Let us choose such an S and then define A_x as the unique member of S such that $A_x L_\xi A_x^{-1} = L_\xi^x$ for all $\xi \in \mathcal{K}$. We show first that $x \to (A_x(\phi), \psi)$ is a Borel function of x for all ϕ and ψ in $\mathfrak{H}(L)$: that is that $x \to A_x$ is a Borel function from $\mathcal{G}$ to $\mathcal{U}(\mathfrak{H}(L))$. Let $\mathbf{E}$ denote the set of all pairs $x, V \in \mathcal{G} \times \mathcal{U}(\mathfrak{H}(L))$ such that $VL_\xi V^{-1} = L_\xi^x$ for all $\xi \in \mathcal{K}$. Then for each Borel subset E of $\mathcal{U}(\mathfrak{H}(L))$ the set E' of all $x \in \mathcal{G}$ with $A_x \in E$ is the projection on $\mathcal{G}$ of $\mathbf{E} \cap (\mathcal{G} \times S)$. If we can show that $\mathbf{E}$ is a Borel set it will follow that E' is the image of a standard Borel space by a one-to-one Borel function and hence is a Borel set by Theorem 3.2 of [13]. Hence it will suffice to prove that $\mathbf{E}$ is a Borel subset of $\mathcal{G} \times \mathcal{U}(\mathfrak{H}(L))$. Let $\mathcal{K}^\sigma$ be the auxiliary group introduced in section 2 of this paper and let L^0 be the ordinary representation of $\mathcal{K}^\sigma$ defined by L. Now it is trivial that $VL_\xi V^{-1} = L_\xi^x$ if and only if $V\lambda L_\xi V^{-1} = \lambda L_\xi^x$ for all complex λ with $|\lambda| = 1$; that is if and only if $VL^0_{\lambda,\xi} V^{-1} = (V^x)^0_{\lambda,\xi}$ for all such λ. Let $\lambda_1, \xi_1; \lambda_2, \xi_2; \ldots$ be dense in $\mathcal{K}^\sigma$ and let $\phi_1, \phi_2, \ldots$ be dense in $\mathfrak{H}(L)$. Then x, $V \in \mathbf{E}$ if and only if $(VL^0_{\lambda_j,\xi_j} V^{-1}(\phi_k), \phi_m) = ((L^x)^0_{\lambda_j,\xi_j}(\phi_k), \phi_m)$ for all $j, k, m = 1, 2, \ldots$; that is if and only if $(VL_{\xi_j} V^{-1}(\phi_k), \phi_m) = (L^x_{\xi_j}(\phi_k), \phi_m)$ for all $j, k, m = 1, 2, \ldots$. But for each fixed triple j, k, m the left hand member of the last equation is clearly a Borel function on $\mathcal{U}(\mathfrak{H}(L))$ and the right hand side, which is equal to $(L_{x\xi_j x^{-1}}(\phi_k), \phi_m)$ multiplied by $(\sigma(x^{-1}, x)/(\sigma(x\xi, x^{-1})\sigma(x, \xi))$, is clearly a Borel function of x. The fact that $\mathbf{E}$ is a Borel set follows at once.

As the second stage in our definition of M_x we define B_x for each x as follows. If $x \in \mathcal{K}$ we set $B_x = L_x$. If $x \notin \mathcal{K}$ we set $B_x = A_x$. Since $\mathcal{K}$ is a Borel set it follows from the fact that $x \to A_x$ is a Borel function and that $\xi \to L_\xi^x$ is a Borel function that $x \to B_x$ is a Borel function. Moreover since $L_\xi^x = L_x L_\xi L_x^{-1}$ for all x and $\xi \in \mathcal{K}$ we see that $B_x L_\xi B_x^{-1} = L_\xi^x$ for all $\xi, x \in \mathcal{K} \times \mathcal{G}$. Now for all $\xi, x, y \in \mathcal{K} \times \mathcal{G} \times \mathcal{G}$ we have $B_{xy} L_\xi B_{xy}^{-1} = L_\xi^{xy} = (L^x)_\xi^y = (B_x L B_x^{-1})_\xi^y = B_x L_\xi^y B_x^{-1} = B_x B_y L_\xi B_y^{-1} B_x^{-1}$. Thus for each x and y in $\mathcal{G}$ the operator $B_x B_y B_{xy}^{-1}$ commutes with L_ξ for all $\xi \in \mathcal{K}$. Since L is irreducible it follows that there exists a complex number of modulus one, $\tau'(x, y)$ such that $B_{xy} = \tau'(x, y) B_x B_y$. Thus $x \to B_x$ is a τ' representation of $\mathcal{G}$ for the multiplier τ'. Let $\nu(x, y) = \sigma(x, y)/\tau'(x, y)$. Then ν is a multiplier for $\mathcal{G}$ which reduces to the identity on $\mathcal{K}$. However ν need not be of the form $\omega \circ f$. As the third and final stage in the construction of M we show that we may change B so that the corresponding ν is of the desired form. Let I denote the one dimensional identity representation of $\mathcal{K}$. Since

ν is the identity on $\mathcal{H}$ it follows that I is a ν representation. Hence we may form the induced ν representation U^I of $\mathcal{G}$. Call this ν representation W. Now for all $\xi, x \in \mathcal{H} \times \mathcal{G}$ we have $W_{\xi x} = W_\xi W_x \nu(\xi, x)$. Moreover $(I^x)_\xi$ is multiplication by $\nu(x^{-1}, x)/(\nu(x\xi, x^{-1})\nu(x, \xi))$. But

$$L_{x\xi x^{-1}} = B_{x\xi x^{-1}} = \tau'(x\xi, x^{-1}) B_{x\xi} B_{x^{-1}} = \tau'(x\xi, x^{-1})\tau'(x,\xi)/\tau'(x^{-1}, x) B_x L_\xi B_x^{-1}$$

and
$$B_x L_\xi B_x^{-1} = L_x^\xi = \sigma(x^{-1}, x)/(\sigma(x\xi, x^{-1})\sigma(x, \xi) L_{x\xi x^{-1}}.$$

Combining these two equations we deduce at once that $\nu(x^{-1}, x)/(\nu(x\xi, x^{-1})\nu(x,\xi) = 1$ for all ξ, $x \in \mathcal{H} \times \mathcal{G}$. Thus I^x is the one dimensional identity for all $x \in \mathcal{G}$. Hence by Theorem 4.5 W_ξ is the identity for all $\xi \in \mathcal{H}$. Hence $W_{\xi x} = \nu(\xi, x) W_x$ for all ξ, $x \in \mathcal{H} \times \mathcal{G}$. Let C be a regular Borel section ([11] page 103) of $\mathcal{G}$ with respect to $\mathcal{H}$ which meets $\mathcal{H}$ in the identity. For each $x \in \mathcal{G}$ let $c(x)$ denote the unique member of C such that $c(x)x^{-1} \in \mathcal{H}$. Now $\xi, y \to \xi^{-1}y$ is a one-to-one Borel mapping of $\mathcal{H} \times \mathsf{C}$ on $\mathcal{G}$ and hence, since $\mathcal{H} \times \mathsf{C}$ and $\mathcal{G}$ are standard, has a Borel inverse. Since $(c(x)x^{-1})^{-1}c(x) = x$ we see that $x \to c(x)$ is the projection on C of the inverse of this mapping and hence is a Borel function. Let $W'_x = W_{c(y)}$ for all $x \in \mathcal{G}$. Then $W'_x = W_{c(x)x^{-1}x} = \nu(c(x)x^{-1}, x) W_x = g(x) W_x$ where $x \to g(x) = \nu(c(x)x^{-1}, x)$ is a Borel function from $\mathcal{G}$ to the complex numbers of unit modulus and $x \to W'_x$ is constant on the $\mathcal{H}$ cosets in $\mathcal{G}$. Since W is a ν representation of $\mathcal{G}$ it follows that W'_x is a ν' representation of $\mathcal{G}$ where $\nu'(x,y) = \nu(x,y)g(x,y)/g(x)g(y)$. We now define M_x for all $x \in \mathcal{G}$ as $(1/g(x))B_x$. Since B is a σ/ν representation of $\mathcal{G}$ it follows that M is a σ/ν' representation of $\mathcal{G}$. But since W' is constant on the $\mathcal{H}$ cosets of $\mathcal{G}$ and W' is a ν' representation of $\mathcal{G}$ it follows at once that ν' is of the form $\omega \circ f$. That ω is a Borel function follows from Lemma 1.2 of [11]. That $M_\xi = L_\xi$ for all $\xi \in \mathcal{H}$ follows from the fact that W_ξ is the identity. To complete the proof of the theorem we have now only to establish the essential uniqueness of ω. Let N be a $\sigma/\omega' \circ f$ representation of $\mathcal{G}$ which agrees on $\mathcal{H}$ with L. We compute at once that $N_x L_\xi N_x^{-1} = M_x L_\xi \cdot M_x^{-1} = L_\xi^x$ for all ξ, $x \in \mathcal{H} \times \mathcal{G}$ and hence that $N_x = \varrho(x) M_x$ for all x where $\varrho(x)$ is a complex number of unit modulus. Hence $N_{\xi x} = \varrho(\xi x) M_{\xi x}$ for all ξ, $x \in \mathcal{H} \times \mathcal{G}$. Since $(\omega' \circ f)(\xi, x) = (\omega \circ f)(\xi, x) = 1$ we conclude that $\sigma(\xi, x) L_\xi N_x = \varrho(\xi x)\sigma(\xi, x) L_\xi M_x$. Hence $\varrho(\xi x) = \varrho(x)$. Hence ϱ is constant on the $\mathcal{H}$ cosets. Since $(\omega \circ f)(x,y) = (\omega' \circ f)(x,y)(\varrho(xy)/\varrho(x)\varrho(y))$ the desired result follows at once. That ϱ defines a Borel function on $\mathcal{G}/\mathcal{H}$ follows from Lemma 1.2 of [11].

THEOREM 8.3. *Let $\mathcal{H}$, $\mathcal{G}$, σ, L, M, τ, f, and ω be as in the statement of Theorem 8.2. For each ω representation N of $\mathcal{G}/\mathcal{H}$ let N' denote the $\omega \circ f$ representation of $\mathcal{G}$ defined by composing N with the canonical homomorphism of $\mathcal{G}$ on $\mathcal{G}/\mathcal{H}$. Then the mapping $N \to M \otimes N'$ sets up a one-to-one correspondence (equivalent representations being identified) between the set of all primary ω representations of $\mathcal{G}/\mathcal{H}$ and the set of all primary σ representations of $\mathcal{G}$*

which reduce on $\mathcal{K}$ to a multiple of L. For each $X = \mathrm{I}$, II, or III, $M \otimes N'$ is of type X if and only if N is of type X and $M \otimes N'$ is irreducible if and only if N is irreducible.

Proof. Let W be a primary σ representation of $\mathcal{G}$ which reduces on $\mathcal{K}$ to a multiple of L. Replacing W by an equivalent representation if necessary we may suppose that $\mathfrak{H}(W) = \mathfrak{H}(M) \otimes \mathfrak{H}_0$ where $\mathfrak{H}_0$ is a suitable Hilbert space and that $W_\xi = L_\xi \times I_0 = M_\xi \times I_0$ for all $\xi \in \mathcal{K}$. Now for all ξ, $x \in \mathcal{K} \times \mathcal{G}$ we have

$$W_{x\xi x^{-1}} = M_{x\xi x^{-1}} \times I_0 = (M_x M_\xi M_x^{-1} \times I_0)(\tau(x\xi, x^{-1})\tau(x,\xi)/\tau(x^{-1},x))$$

and we have
$$W_{x\xi x^{-1}} = W_x W_\xi W_x^{-1}(\sigma(x\xi,x^{-1})\sigma(x,\xi)/\sigma(x^{-1},x)).$$

We have seen however that I^x regarded as an $\omega \circ f$ representation is equivalent to I for all x. Thus the expressions involving σ and τ must be equal and we may conclude that $W_x W_\xi W_x^{-1} = M_x M_\xi M_x^{-1} \times I_0$. Thus $(M_x \times I_0)(L_\xi \times I_0)(M_x \times I_0)^{-1} = W_x(L_\xi \times I_0) W_x^{-1}$. Thus for each $x \in \mathcal{G}$, $W_x^{-1}(M_x \times I_0)$ commutes with $L_\xi \times I_0$ for all $\xi \in \mathcal{K}$. Since L is irreducible, $W_x^{-1}(M_x \times I_0)$ must be of the form $I \times V_x$ where V_x is a uniquely determined unitary operator in $\mathfrak{H}_0$. (If $T \in \mathfrak{H}_1 \otimes \mathfrak{H}_0$ and hence is an operator from $\overline{\mathfrak{H}_0}$ to $\mathfrak{H}_1$ then $(A \times B)(T) = A\,T\,B^*$. (See [11] section 5 for further details.) Thus $W_x = (M_x \times V_x)$. Now for all x and y in $\mathcal{G}$ $W_{xy} = \sigma(x,y) W_x W_y$. Hence

$$\begin{aligned} M_{xy} \times V_{xy} &= \sigma(x,y)(M_x \times V_x)(M_y \times V_y) = \sigma(x,y)(M_x M_y \times V_x V_y) \\ &= \sigma(x,y)((1/\tau(x,y) M_{xy} \times V_x V_y) = (M_{xy} \times (\sigma(x,y)/\tau(x,y)) V_x V_y). \end{aligned}$$

Therefore
$$M_{xy} \times V_{xy} = M_{xy} \times (\sigma(x,y)/\tau(x,y)) V_x V_y$$

so
$$V_{xy} = (\sigma(x,y)/\tau(x,y)) V_x V_y = (\omega \circ f)(x,y) V_x V_y.$$

The fact that $(V_x(\phi),\psi)$ is a Borel function of x for all ϕ and ψ in $\mathfrak{H}$ follows at once from the corresponding facts about W and M. Thus V is an $\omega \circ f$ representation of $\mathcal{G}$ and $W = M \otimes V$. But V_ξ is the identity for all $\xi \in \mathcal{K}$. Hence (applying Lemma 1.2 of [11] to establish the Borelness of N we see at once that V is of the form N' where N is an ω representation of $\mathcal{G}/\mathcal{K}$. We have thus proved that W is of the form $M \otimes N'$. We shall see below that N must be primary. Now let N_1 and N_2 be any two ω representations of $\mathcal{G}/\mathcal{K}$. Let T be any intertwining operator for $M \otimes N_1'$ and $M \otimes N_2'$. Then T also intertwines the restriction of these σ representations to $\mathcal{K}$; that is $L \otimes I$ and $L \otimes I$. Since L is irreducible T must be of the form $I \times S$. Since $I \times S$ intertwines $M \otimes N_1'$ and $M \otimes N_2'$, S must intertwine N_1' and N_2' and hence must intertwine N_1 and N_2. Conversely if S intertwines N_1 and N_2 then it is obvious that $I \times S$ intertwines $M \otimes N_1'$ and $M \otimes N_2'$. It now follows at once that $M \otimes N_1'$ and $M \otimes N_2'$ are equivalent if and only if N_1 and N_2 are equivalent and that the commuting rings $\mathbf{R}(N,N)$ and $\mathbf{R}(M \otimes N',\ M \otimes N')$ are isomorphic. From these commuting ring isomorphisms it follows at once that N is primary if and only if $M \otimes N'$ is primary that if both are primary they have the same type and that one is irreducible if and only if the other is also. This completes the proof of the theorem.

Combining Theorems 8.1, 8.2 and 8.3 we may state as our final result.

THEOREM 8.4. *Let $\mathfrak{G}$, $\mathfrak{K}$ and σ satisfy the hypotheses of Theorem* 8.1 *and let L_0, $\mathbf{O}$ and $\mathcal{G}$ be defined as in that theorem. Then there exists a multiplier ω for $\mathcal{G}/\mathfrak{K}$ (unique up to multiplication by trivial multipliers) a multiplier τ for $\mathcal{G}$ and a τ representation M of $\mathcal{G}$ such that M restricted to $\mathfrak{K}$ coincides with L_0 and such that $N \to U^{M \otimes N'}$ sets up a one-to-one correspondence (equivalent representations being identified) between the primary ω representations of $\mathcal{G}/\mathfrak{K}$ and the primary σ representations of $\mathfrak{G}$ having $\mathbf{O}$ as orbit. N and $U^{M \otimes N'}$ have the same von Neumann–Murray type and N is irreducible if and only if $U^{M \otimes N'}$ is irreducible.*

9. Applications and examples

Suppose that $\mathfrak{G}$, $\mathfrak{K}$ and σ are as described in the first sentence of Theorem 8.1 and suppose that the only quasi orbits of $\hat{\mathfrak{K}}^\sigma$ under $\mathfrak{G}$ are transitive ones. Then the primary σ representations of $\mathfrak{G}$ described in Theorem 8.4 include *all* primary σ representations of $\mathfrak{G}$ and we reduce the problem of finding the primary representations of $\mathfrak{G}$ to that of finding the orbits of $\hat{\mathfrak{K}}^\sigma$ under $\mathfrak{G}$ and for each such orbit to finding the primary ω representations of a certain subgroup of $\mathfrak{G}/\mathfrak{K}$. Moreover it is easy to find useful sufficient conditions for the absence of non transitive quasi orbits and hence for the possibility of the indicated complete analysis. Let us say that $\mathfrak{K}$ is σ *regularly imbedded* in $\mathfrak{G}$ whenever for each finite Borel measure μ in $\hat{\mathfrak{K}}^\sigma$ the measure $\tilde{\mu}$ in the orbit space $(\hat{\mathfrak{K}}^\sigma)^\sim$ is countably separated. This says slightly less than that $(\hat{\mathfrak{K}}^\sigma)^\sim$ is metrically countably separated since we do not know that every finite Borel measure in $(\hat{\mathfrak{K}}^\sigma)^\sim$ is of the form $\tilde{\mu}$. Now the hypothesis of metric countable separatedness for $\tilde{S}$ made in the statement of Theorem 6.3 is not used in the proof in its full force. All that is actually used is that measures in $\tilde{S}$ of the form $\tilde{\mu}$ are countably separated. Thus we may apply Theorems 6.3 and 7.6 and conclude the truth of

THEOREM 9.1. *If $\mathfrak{G}$, $\mathfrak{K}$ and σ are as described in the first sentence of Theorem* 8.1 *and $\mathfrak{K}$ is σ regularly imbedded in $\mathfrak{G}$ then every quasi orbit of $\hat{\mathfrak{K}}^\sigma$ under $\mathfrak{G}$ is transitive. Hence, in particular, every primary σ representation of $\mathfrak{G}$ is one of those described in Theorem* 8.4.

We have also

THEOREM 9.2. *Let $\mathfrak{G}$, $\mathfrak{K}$ and σ be as described in the first sentence of Theorem* 8.1 *and in addition let $\hat{\mathfrak{K}}^\sigma$ be not only metrically standard but actually standard. Let there exist a Borel set S in $\hat{\mathfrak{K}}^\sigma$ which meets each orbit of $\hat{\mathfrak{K}}^\sigma$ under $\mathfrak{G}$ exactly once. Then $\mathfrak{K}$ is σ regularly imbedded in $\mathfrak{G}$.*

Proof. Apply Theorems 5.2 and 7.3.

It would be interesting to know whether or not it is possible to strengthen Theorem 9.2 along the lines suggested by Theorem 8.6 of [13].

THEOREM 9.3. *Let $\mathfrak{G}$, $\mathcal{K}$ and σ be as described in the first sentence of Theorem 8.1 and let $\mathcal{K}$ be σ regularly imbedded in $\mathfrak{G}$. Suppose that for each $L \in \hat{\mathcal{K}}^{\sigma}$ the subgroup $\mathcal{G}_L$ consisting of all x with $L^x \simeq L$ is such that $(\widehat{\mathcal{G}_L/\mathcal{K}})^{\omega}$ is of type* I *where ω is the essentially unique multiplier in $\mathcal{G}_L/\mathcal{K}$ defined by Theorem* 8.2. *Then $\hat{\mathfrak{G}}^{\sigma}$ is of type* I.

Proof. Apply Theorems 8.4 and 9.1.

THEOREM 9.3 provides an inductive mechanism for establishing the type-I-ness of complicated groups (cf. Dixmier in [4] and [5]). Less directly Theorem 8.4 provides such a mechanism for establishing smoothness and metric smoothness. We are not prepared to formulate a precise theorem but content ourselves with the remark that, generally speaking, when one has an "explicit" enumeration of the irreducible σ representations of a group one can make use of it to show that the group has a smooth σ dual.

For the special case in which $\mathcal{K}$ is commutative, $\sigma \equiv 1$ and $\mathfrak{G}$ is a semi direct product of $\mathcal{K}$ with $\mathfrak{G}/\mathcal{K}$ applications of Theorems 8.4, 9.1, 9.2, and 9.3 to concrete groups have been described in [10] and [11]. In this section we shall discuss some examples to which the results of [10] and [11] do not apply. We begin with a few elementary facts about the existence of non trivial multipliers.

THEOREM 9.4. *Let $\mathcal{K}$ and $\mathcal{H}$ be closed subgroups of the separable locally compact group $\mathfrak{G}$ such that $\mathcal{K}$ is normal, $\mathcal{K} \cap \mathcal{H} = e$ and $\mathcal{K}\mathcal{H} = \mathfrak{G}$. Then every multiplier ν' for $\mathfrak{G}$ is similar to a multiplier ν for $\mathfrak{G}$ which may be uniquely represented in the form:*

(a) $\nu(x_1y_1, x_2y_2) = \sigma(x_1, y_1(x_2))\omega(y_1, y_2)g(x_2, y_1)$ *where x_1 and x_2 are in $\mathcal{K}$, y_1 and y_2 are in $\mathcal{H}$, $y_1(x_2) = y_1x_2y_1^{-1}$,* σ is *a multiplier for $\mathcal{K}$, ω is a multiplier for $\mathcal{H}$, g is a Borel function from $\mathcal{K}\times\mathcal{H}$ to the complex numbers of unit modulus, g is one on $\mathcal{K} \times e$ and σ and g satisfy the two following identities:*

(b) $\sigma(y(x_1), y(x_2)) = \sigma(x_1, x_2)g(x_1x_2, y))/(g(x_1, y)g(x_2, y))$,

(c) $g(x, y_1y_2) = g(y_2(x), y_1)g(x, y_2)$.

Moreover for every choice of σ, ω and g satisfying (b) *and* (c) *the function ν defined by* (a) *is a multiplier for $\mathfrak{G}$.*

Proof. Let ν' be a multiplier for $\mathfrak{G}$ and let V' be any ν' representation. Then $V'_{xy} = \nu'(x, y)V'_xV'_y$ for all x in $\mathcal{K}$ and all y in $\mathcal{H}$. Let $V_{xy} = (1/\nu'(x, y))V'_{xy}$. Then V is a ν representation for a multiplier ν which is similar to ν'. Since $\nu'(x, e) = \nu'(e, y) = 1$ for all $x, y \in \mathcal{K}\times\mathcal{H}$ it follows that $V_{xy} = V'_xV'_y = V_xV_y = A_xB_y$ where A and B denote the restrictions of V to $\mathcal{K}$ and $\mathcal{H}$ respectively. Moreover A is a σ representation of $\mathcal{K}$ and B is an ω representation of $\mathcal{H}$ where σ and ω denote the restrictions of ν to $\mathcal{K}$ and $\mathcal{H}$ respectively. Now

$$V_{x_1y_1x_2y_2} = V_{x_1y_1x_2y_1^{-1}y_1y_2} = V_{x_1y_1(x_2)y_1y_2} = A_{x_1y_1(x_2)}B_{y_1y_2}.$$

Thus $$\nu(x_1y_1,x_2y_2)A_{x_1}B_{y_1}A_{x_2}B_{y_2} = \sigma(x_1,y_1(x_2))\omega(y_1,y_2)A_{x_1}A_{y_1(x_2)}B_{y_1}B_{y_2}$$

and hence $A_{y_1(x_2)}B_{y_1}^{-1}A_{x_2}B_{y_1}$ is identically $\nu(x_1y_1,x_2y_2)/(\sigma(x_1,y_1(x_2))\omega(y_1,y_2))$ times the identity. Hence this last expression depends only on x_2 and y_1. Denoting it by $g(x_2,y_1)$ we obtain (a) and $g(x_2,e)\equiv 1$. Now let σ and ω be arbitrary multipliers for $\mathcal{K}$ and $\mathcal{H}$ respectively and let g be a Borel function from $\mathcal{K}\times\mathcal{H}$ to the complex numbers of modulus one such that $g(x,e)\equiv 1$. Define ν by (a). A straightforward calculation shows that ν is a multiplier for $\mathfrak{G}$ if and only if g and σ satisfy the following identity:

$$\text{(d)}\quad g(x_3,y_1y_2)g(x_2,y_1)\sigma(x_1y_1(x_2),y_1y_2(x_3))\sigma(x_1,y_1(x_2))$$
$$= g(x_2y_2(x_3),y_1)g(x_3,y_2)\sigma(x_1,y_1(x_2)y_1y_2(x_3))\sigma(x_2,y_2(x_3)).$$

Setting $y_2=e$ and using the fact that σ is a multiplier (d) reduces to (b). Moreover using (b) to simplify (d) we get (c). Thus (d) is equivalent to (b) and (c) together and the proof is complete.

We note that when $\sigma\equiv 1$ (b) reduces to the statement that for all y in $\mathcal{H}$, $x\to g(x,y)$ is a homomorphism of $\mathcal{K}$ into the complex numbers of modulus one; that is a member of the group $\hat{\mathcal{K}}$ of all one dimensional unitary representations of $\mathcal{K}$. Thus we have the

COROLLARY. *The multipliers ν for $\mathfrak{G}$ which reduce to one on $\mathcal{K}$ and on $\mathcal{H}$ are just the functions on $\mathfrak{G}\times\mathfrak{G}$ such that $\nu(x_1y_1,x_2y_2)=\chi_{y_1}(x_2)$ where $y\to\chi_y$ is a function from $\mathcal{H}$ to $\mathcal{K}$ such that $\chi_y(x)$ is a Borel function of x and y and $\chi_{y_1y_2}=[\chi_{y_1}]y_2\chi_{y_2}$ for all y_1 and y_2 in $\mathcal{H}$. (Here $[\chi]y(x)=\chi(y(x))$.)*

THEOREM 9.5. *Let $y\to\chi_y$ be as described in the preceding corollary. Then the corresponding multiplier is trivial if and only if there exists a member χ^0 of $\hat{\mathcal{K}}$ such that $\chi_y=[\chi^0]y/\chi^0$ for all $y\in\mathcal{H}$.*

Proof. Let χ^0 be any member of $\hat{\mathcal{K}}$. Let $g(xy)=\chi^0(x)$ for all x,y in $\mathcal{K}\times\mathcal{H}$. Let ν be the trivial multiplier on $\mathfrak{G}$ defined by g. We verify at once that

$$\nu(x_1y_1,x_2y_2)=g(x_1y_1x_2y_2)/g(x_1y_1)g(x_2y_2)=\chi^0(y_1(x_2))/\chi^0(x_2).$$

Conversely let $y\to\chi_y$ define a trivial multiplier. Then there exists a Borel function g from $\mathfrak{G}$ to the complex numbers of modulus one such that

$$g(x_1y_1(x_2)y_1y_2)=g(x_1y_1)g(x_2y_2)\chi_{y_1}(x_2)$$

for all x_1 and x_2 in $\mathcal{K}$ and all y_1 and y_2 in $\mathcal{H}$. If we set $y_1=e$ this becomes $g(x_1x_2y_2)=$

$g(x_1)g(x_2y_2)$. Thus g restricted to $\mathfrak{K}$ defines a member χ^0 of $\mathfrak{K}$ and our original identity becomes

$$\chi^0(x_1)\chi^0(y_1(x_2))g(y_1y_2) = \chi^0(x_1)\chi^0(x_2)g(y_1)g(y_2)\chi_{y_1}(x_2).$$

But since ν reduces to one on $\mathfrak{H}$ it follows that $g(y_1y_2) = g(y_1)g(y_2)$ and we obtain

$$\chi_{y_1}(x_2) = \chi^0(y_1(x_2))/\chi^0(x_2)$$

as was to be proved.

We shall be chiefly interested in the case in which $y(x) \equiv x$; that is in the case in which $\mathfrak{G}$ is the ordinary direct product of $\mathfrak{K}$ and $\mathfrak{H}$. Our results yield

THEOREM 9.6. *Let $\mathfrak{G}$ be the direct product of the separable locally compact groups $\mathfrak{K}$ and $\mathfrak{H}$. Then (up to similarity) the multipliers ν for $\mathfrak{G}$ are just the functions on $\mathfrak{G} \times \mathfrak{G}$ such that $\nu(x_1y_1, x_2y_2) = \sigma(x_1,x_2)\omega(y_1,y_2)\chi_{y_1}(x_2)$ where σ is an arbitrary multiplier for $\mathfrak{K}$, ω is an arbitrary multiplier for $\mathfrak{H}$ and $y \to \chi_y$ is a homomorphism of $\mathfrak{H}$ into $\mathfrak{K}$ such that $\chi_y(x)$ is a Borel function on $\mathfrak{K} \times \mathfrak{H}$. The multiplier defined by σ, ω and $y \to \chi_y$ is trivial if and only if σ and ω are trivial and $\chi_y(x) \equiv 1$.*

Example 1. Let $\mathfrak{G} = \mathfrak{K} \times \hat{\mathfrak{K}}$ where $\mathfrak{K}$ is a separable locally compact abelian group and $\hat{\mathfrak{K}}$ is its dual. Let $\nu(x_1,y_1;x_2,y_2) = y_1(x_2)$ for all x_1 and x_2 in $\mathfrak{K}$ and all y_1 and y_2 in $\mathfrak{K}$. By Theorem 9.6 ν is a non trivial multiplier for the abelian group $\mathfrak{G}$. Let us determine the ν representations of $\mathfrak{G}$ by applying Theorem 8.4. Since ν reduces to one on $\mathfrak{K}$, $\hat{\mathfrak{K}}^\nu$ coincides with $\hat{\mathfrak{K}}$ and is certainly smooth and of type I. Let ${}^{y_0}L$ denote the irreducible ν representation of $\mathfrak{K}$ defined by $y_0 \in \hat{\mathfrak{K}}$. Since the inner automorphisms of $\mathfrak{G}$ are all trivial $({}^{y_0}L)_x^{x_1y_1}$ is simply ${}^{y_0}L_x$ multiplied by $\nu(x_1^{-1},y_1^{-1};x_1,y_1)/\nu(x_1xy_1;x_1^{-1},y_1^{-1})\nu(x_1,y_1;x,e)y_1^{-1}(x_1)/y_1(x_1^{-1})\cdot y_1(x) = y_1^{-1}(x)$; that is ${}^{y_0y_1^{-1}}L_x$. Thus there is just one orbit—the whole of $\hat{\mathfrak{K}}$. Moreover the subgroup leaving any one of its elements fixed is just $\mathfrak{K}$. Since $\mathfrak{K}/\mathfrak{K}$ is the identity there is just one irreducible ν representation of $\mathfrak{G}$ associated with this orbit. In other words $\mathfrak{G}$ has to within equivalence *just one irreducible ν representation.* It is infinite dimensional and is the ν representation induced by the identity representation of $\mathfrak{K}$.

It is interesting to compare example 1 with Theorem 1 of [9] which presents exactly the same result from a rather different point of view.

Example 1 points up three ways in which the theory of projective representations differs sharply from the theory of ordinary representations. An abelian group can have infinite dimensional irreducible projective representations and for a given multiplier ν can have a unique irreducible ν representation. The ν representations of a direct product need not be related in a simple way to the ν representations of the factors even when the factors have type I ν duals. Of course if ν is a direct product of multipliers for the factor groups; that is if the $y \to \chi_y$ of Theorem 9.6 is identically one then it is not difficult to prove an analogue of the corollary to Theorem 1.8 of [12].

Example 2. Let $\mathcal{K}$ be as in example 1 but let $\hat{\mathcal{K}}$ be replaced by $\mathcal{H}$ where $\mathcal{H}$ is a countable, discretely topologized, dense subgroup of $\hat{\mathcal{K}}$. Let ν now denote the restriction to $\mathcal{K} \times \mathcal{H}$ of the ν of example 1. Just as before we have $({}^{y_0}L)_x^{x_1, y_1} = {}^{y_0 y_1^{-1}}L_x$ but now y_1 can no longer be an arbitrary element of $\hat{\mathcal{K}}$ but is restricted to lie in $\mathcal{H}$. Thus there are many orbits—one for each $\mathcal{H}$ coset in $\hat{\mathcal{K}}$. Just as before there is exactly one irreducible ν representation of $\mathfrak{G}$ for each of these orbits; namely, the one induced by any member of the orbit. However in this case the irreducible ν representations described in Theorem 8.4 do *not* exhaust the irreducible ν representations of $\mathfrak{G}$. In addition to the orbits in $\hat{\mathcal{K}}$ there is at least one proper quasi orbit. Haar measure in $\hat{\mathcal{K}}$ is ergodic under the group of translations by members of $\mathcal{H}$ and hence defines an ergodic invariant measure class not concentrated in any orbit. The existence of this quasi orbit can be used to show that $\mathfrak{G}$ has primary ν representations which are not of type I in addition to many irreducible ν representations other than those described in Theorem 8.4. Thus a commutative $\mathfrak{G}$ can have non type I projective representations. We shall defer details to our projected paper on the intransitive case.

When $\mathcal{K}$ is a vector group, Theorem 1 of [9] reduces to the theorem of Stone and von Neumann on the uniqueness of sets of operators satisfying the Heisenberg commutation relations. Hence Example 1 above contains this theorem. We shall now indicate a similar connection between Example 2 and the problem of finding all sets of operators satisfying the "anti commutation relations" of quantum field theory. The problem is that of finding all sequences $A_1, A_2, \ldots$ of bounded operators on a Hilbert space $\mathfrak{H}$ such that $A_j A_k + A_k A_j = 0$ and $A_j A_k^* + A_k^* A_j = \delta_{jk}$ for all $j, k = 1, 2, \ldots$. Following H. Weyl ([16], page 252) we let $A_j = \frac{1}{2}(P_{2j-1} + iP_{2j})$ where $i^2 = -1$ and the P_j are self adjoint. We note that the anti commutation relations expressed in terms of the P_j's are $P_j{}^2 = 1$ and $P_j P_k + P_k P_j = 0$ for all $j \neq k$. For operators T with $T^2 = 1$, unitariness is equivalent to self adjointness. Thus our problem is that of finding all sequences $P_1, P_2, \ldots$ of unitary operators such that $P_j^2 = 1$ and $P_j P_k + P_k P_j = 0$ for $j \neq k$. Given such a sequence let $S_1 = P_1$ and let $S_j = iP_{j-1} P_j$ for $j = 2, 3, \ldots$. Then the S_j form a sequence of unitary operators such that $S_j S_{j+1} = -S_{j+1} S_j$ and $S_j S_{j+k} = S_{j+k} S_j$ for all $j = 1, 2, \ldots$ and all $k = 2, 3, \ldots$. Moreover $P_j = (-i)^{j-1} \cdot (S_1 S_2 \ldots S_j)$ for $j = 2, 3, \ldots$. Conversely if $S_1, S_2, \ldots$ is a sequence of unitary operators satisfying the conditions just enunciated and we let $P_j = (-i)^{j-1}(S_1 S_2 \ldots S_j)$ for $j = 2, 3, \ldots$ and $P_1 = S_1$ then $P_j^2 = 1$ and $P_j P_k + P_k P_j = 0$ for $j \neq k$. In other words our problem is also equivalent to that of finding all sequences $S_1, S_2, \ldots$ of unitary operators such that $S_j^2 = 1$, $S_{j+1} S_j = -S_j S_{j+1}$ and $S_j S_{j+k} = S_{j+k} S_j$ for all $j = 1, 2, \ldots$ and $k = 2, 3, \ldots$. Let $\mathcal{K}$ be the direct product of countably many groups of order two and let x_{2j+1} denote the generator of the jth group. For each $j = 1, 2, \ldots$ let y_{2j} denote the unique element of $\mathcal{K}$ such that $y_{2j}(x_{2k+1}) = 1$ whenever $|2j - 2k + 1| \neq 1$ and $y_{2j}(x_{2k+1}) = -1$ whenever $|2j - 2k + 1| = 1$.

Let $\mathcal{H}$ denote the subgroup of $\mathcal{K}$ generated by the y_{2j}. Let ν be the multiplier on $\mathcal{K} \times \mathcal{H}$ described above. A ν representation L of $\mathcal{K} \times \mathcal{H}$ is uniquely determined by its values on the x_{2j+1} and the y_{2j}. Let $S_1, S_2, \ldots$ be a sequence of unitary operators. It is easy to see that there exists a ν representation L of $\mathcal{K} \times \mathcal{H}$ such that $L_{x_{2j+1}} = S_{2j+1}$ and $L_{y_{2j}} = S_{2j}$ for all $j = 1, 2, \ldots$ if and only if the S_j form a solution of the second reformulation of our problem. Thus the problem of finding all sets of operators satisfying the "anti commutation relations" is equivalent to the problem of finding all ν representations of the discrete group $\mathcal{K} \times \mathcal{H}$.

Since this representation problem is a special case of that considered in example 2 it involves the consideration of non transitive quasi orbits and hence is only partly solved by the theory of the preceding sections. A partial solution going beyond that provided by our theory has been sketched in a recent note of Gårding and Wightman [6]. We hope that our projected investigation of the non transitive case will yield general results whose application to the case at hand will include the results of [6]. We remark that example 2 was suggested to us by our study of [6].

Example 3. Let $\mathfrak{G}$ be the group of all 3×3 unimodular real matrices which are zero above the main diagonal. Let $\mathcal{K}$ be the normal subgroup of $\mathfrak{G}$ consisting of all members which are one on the main diagonal. Let $\mathcal{D}$ be the subgroup of $\mathfrak{G}$ consisting of all members which are zero off the main diagonal. We shall determine the (ordinary) irreducible representations of $\mathcal{K}$ and then those of $\mathfrak{G}$. $\mathfrak{G}$ is clearly a semi direct product of $\mathcal{K}$ and $\mathcal{D}$. Let $\langle a, b, c\rangle$ denote the matrix $\begin{pmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ a & c & 0 \end{pmatrix}$ and let (λ, μ, ν) denote the matrix $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}$. Then every member of $\mathfrak{G}$ is uniquely of the form $\langle a, b, c\rangle\, (\lambda, \mu, \nu)$ where a, b, c, λ, μ, and ν are real numbers such that $\lambda\mu\nu = 1$. We compute that

$$\langle a_1, b_1, c_1\rangle \langle a_2, b_2, c_2\rangle = \langle a_1 + a_2 + c_1 b_2,\, b_1 + b_2,\, c_1 + c_2\rangle,$$

that

$$(\lambda_1, \mu_1\, \nu_1)(\lambda_2, \mu_2, \nu_2) = (\lambda_1\lambda_2, \mu_1\mu_2, \nu_1\nu_2)$$

and that

$$(\lambda, \mu, \nu)\langle a, b, c\rangle (\lambda, \mu, \nu)^{-1} = \left\langle \frac{\nu}{\lambda} a, \frac{\mu}{\lambda} b, \frac{\nu}{\mu} c\right\rangle.$$

To determine the irreducible representations of $\mathcal{K}$ we note that the center $\mathcal{Z}$ of $\mathcal{K}$ is the set of all $\langle a, 0, 0\rangle$ and take this normal subgroup of $\mathcal{K}$ as the $\mathcal{K}$ of Theorem 8.4. The quotient group $\mathcal{H}/\mathcal{Z}$ acts trivially on $\mathcal{Z}$ and hence on $\hat{\mathcal{Z}}$. Thus the orbits in $\hat{\mathcal{Z}}$ are the points of $\hat{\mathcal{Z}}$ and the groups holding the points of $\hat{\mathcal{Z}}$ fixed all coincide with $\mathfrak{G}$. Now the points of $\hat{\mathcal{Z}}$ are in one-to-one correspondence with the real numbers in such a fashion that the real number r corresponds to the representation $\langle a, 0, 0\rangle \to \exp\,(iar)$. Let $f_r(a, b, c) = \exp(ir(a - bc))$. We compute that $f_r(\langle a_1, b_1, c_1\rangle \langle a_2, b_2, c_2\rangle) = \exp(-irb_1c_2) f_r(\langle a_1, b_1, c_1\rangle) f_r(\langle a_2, b_2, c_2\rangle)$. Thus

f_r defines a one dimensional projective representation of $\mathcal{K}$, with multiplier $\exp(-irb_1c_2)$, which reduces on $\mathcal{Z}$ to the representation defined by r. Hence the most general irreducible representation of $\mathcal{K}$ is obtained by choosing an r and an irreducible projective representation V of $\mathcal{K}/\mathcal{Z}$ whose multiplier is $\exp(irb_1c_2)$ and then forming $f_r V$. When $r = 0$ we get simply the one dimensional representations of $\mathcal{K}$ defined by the one dimensional representations of the two dimensional vector group $\mathcal{K}/\mathcal{Z}$; that is the representations $W^{s,t}$ where $W^{s,t}_{\langle a,b,c\rangle}$ is multiplication by $\exp(isb + itc)$. When $r \neq 0$ the multiplier $\exp(irb_1c_2)$ of $\mathcal{K}/\mathcal{Z}$ is of the form discussed under Example 1. Thus there is a unique irreducible projective representation V^r of $\mathcal{K}/\mathcal{Z}$ with the multiplier in question and hence a unique irreducible representation $W^r = f_r V^r$ of $\mathcal{K}$ associated with the orbit of $\langle a,0,0\rangle \to \exp(iar)$. In all then, we have one two parameter family $s,t \to W^{s,t}$ of one dimensional representations of $\mathcal{K}$ and one disconnected one parameter family $r \to W^r$ $(r \neq 0)$ of infinite dimensional irreducible representations of $\mathcal{K}$. It follows easily that $\mathcal{K}$ has a smooth type I dual.

To determine the irreducible representations of $\mathfrak{G}$ we shall first determine the orbits in $\mathcal{K}$ under the action of $\mathcal{D}$. $W^{s,t}_{(\lambda,\mu,\nu)\langle a,b,c,\rangle \lambda,\mu,\nu)^{-1}}$ is multiplication by $\exp\left(i\left(\frac{s\mu}{\lambda}b + \frac{t\nu c}{\mu}\right)\right)$ and hence is equal to $W^{s\mu/\lambda,\, t\nu/\mu}_{\langle a,b,c\rangle}$. Thus the one dimensional representations fall into four orbits as follows. $\mathbf{0}_1$ contains all $W^{s,t}$ with $s \neq 0$ and $t \neq 0$. $\mathbf{0}_2$ contains all $W^{s,0}$ with $s \neq 0$, $\mathbf{0}_3$ contains all $W^{0,t}$ with $t \neq 0$, $\mathbf{0}_4$ contains $W^{0,0}$ only. Now $W^r_{(\lambda,\mu,\nu)\langle a,0,0\rangle(\lambda,\mu,\nu)^{-1}} = W^r_{\langle \nu a/\lambda,0,0\rangle}$ which is multiplication by $\exp(i(\nu/\lambda)ar)$. Hence $W^r_{(\lambda,\mu,\nu)\langle a,b,c,\rangle(\lambda,\mu,\nu)^{-1}} = W^{\nu r/\lambda}_{\langle a,b,c\rangle}$. Thus all infinite dimensional irreducible representations of $\mathcal{K}$ lie in a single orbit $\mathbf{0}_\infty$. The subgroup of $\mathcal{D}$ leaving $W^{1,1}$ invariant is the set of all (λ,μ,ν) with $\lambda = \mu = \nu$ and $\lambda^3 = 1$; that is it consists of the identity alone. There is then a single irreducible representation associated with $\mathbf{0}_1$. It is infinite dimensional and is the representation of $\mathfrak{G}$ induced by any one dimensional representation $W^{s,t}$ of $\mathcal{K}$ with $s \neq 0$ and $t \neq 0$. The subgroup $\mathcal{D}_2$ of $\mathcal{D}$ leaving $W^{1,0}$ invariant is the set of all (λ,μ,ν) with $\mu = \lambda$ and $\lambda\mu\nu = 1$; that is, the set of all $(\lambda,\lambda,1/\lambda^2)$ and is isomorphic to the multiplicative group of all non zero real numbers. The possible extensions of $W^{s,t}$ to $\mathcal{K}\mathcal{D}_2$ are in an obvious one-to-one correspondence with the one dimensional representations of $\mathcal{D}_2$ and hence with the non zero real numbers. The irreducible representations of $\mathfrak{G}$ associated with $\mathbf{0}_2$ are the irreducible representations induced by these extensions. They form a family of infinite dimensional representations parameterized by the non zero real numbers. The irreducible representations of $\mathfrak{G}$ associated with $\mathbf{0}_3$ are constructed analogously; the only difference being that $\mathcal{D}_2$ is replaced by the group $\mathcal{D}^3$ of all $(1/\lambda^2,\lambda,\lambda)$. Again we get a family of infinite dimensional representations parameterized by the non zero real numbers. The irreducible representations of $\mathfrak{G}$ associated with $\mathbf{0}_4$ are in an obvious one-to-one correspondence with the irreducible representations of the abelian group $\mathcal{D}$. They are all one dimensional and may be parameterized by the pairs of non zero real numbers. The subgroup $\mathcal{D}_\infty$ of $\mathcal{D}$ leaving W^1 invariant is the set of all $(\lambda, 1/\lambda^2, \lambda)$. It is not hard to show that this group has no non trivial multipliers and hence that W^1 may be extended to be a representation of $\mathcal{K}\mathcal{D}_\infty$. The possible extensions correspond to the irreducible representations of $\mathcal{D}_\infty$. As with $\mathbf{O}_2$ and $\mathbf{O}_3$ we get a family of infinite dimensional irreducible representations parameterized by the non zero real numbers. This time however the inducing representations are themselves infinite dimensional. In all we have one isolated infinite dimensional representation, three families of infinite dimensional representations, each parameterized by the non zero real numbers and one family of one dimensional representations parameterized by the pairs of non zero real numbers.

Example 4. Let $\mathcal{H}$ be a finite group and let $\mathcal{A}$ be group of automorphisms of $\mathcal{H}$. Let $\mathcal{K}$ be the group of all functions f from the integers to $\mathcal{H}$. Equipped with the direct product topology $\mathcal{K}$ becomes a compact group. For each integer n_0 and each $\alpha \in \mathcal{A}$ let α, n_0 denote the automorphism $f \rightarrow f'$ where $f'(n) = \alpha(f(n + n_0))$. The set of all of these automorphisms is a group $\mathcal{C}$ isomorphic to the direct product of $\mathcal{A}$ with the additive group of all integers. Let $\mathfrak{G}$ denote that semi direct product of the compact group $\mathcal{K}$ with the discrete group $\mathcal{C}$ in which the homomorphism from $\mathcal{C}$ into the group of automorphisms of $\mathcal{K}$ is the natural one. If n_1 and n_2 are integers with $n_1 < n_2$ then the set of all f in $\mathcal{K}$ with $f(n) = e$ for $n_1 \leqslant n \leqslant n_2$ is a normal subgroup whose quotient is naturally isomorphic to the direct product of a finite number of replicas of $\mathcal{H}$. The representations of this quotient group define representations of $\mathcal{K}$ and it follows easily from the theory of compact groups that every irreducible representation of $\mathcal{K}$ may be so obtained (with varying n_1 and n_2 of course). Thus the irreducible representations of $\mathcal{K}$ may be put in a natural one-to-one correspondence with those functions M, $n \rightarrow M^n$ from the integers to the irreducible representations of $\mathcal{H}$ such that M^n is the one dimensional identity for all but finitely many values of n. It is clear that the orbits of $\mathcal{K}$ under $\mathcal{C}$ are all infinite except for the one containing the one dimensional identity representation. Thus all irreducible representations of $\mathfrak{G}$, except those trivially derived from representations of $\mathfrak{G}/\mathcal{K} \simeq \mathcal{C}$, are infinite dimensional. The subgroup of $\mathcal{C}$ taking the representation defined by $n \rightarrow M^n$ into one equivalent to itself is the set of all α, 0 such that α takes M^n into a representation equivalent to itself for all n. Hence to determine the irreducible representations of $\mathfrak{G}$ it suffices to determine the irreducible representations of the finite group $\mathcal{H}$, study the way in which the automorphisms in $\mathcal{A}$ act on subsets of these representations and determine the σ representations of certain subgroups of the finite group $\mathcal{A}$ for certain values of σ. If we take $\mathcal{H}$ to be $\mathcal{A}_6$, the alternating group on six elements, and $\mathcal{A}$ to be the group of all automorphisms of $\mathcal{H}$ it is easy to see that all of these things may be done quite explicitly and that non trivial σ's arise. We shall content ourselves here with an indication of the proof that non trivial σ's arise. Let S_6 denote the symmetric group on six elements so that $\mathcal{A}_6$ is a normal subgroup of index two of S_6. As shown on page 209 of [2] the automorphisms of $\mathcal{A}_6$ induced by the inner automorphisms of S_6 form a normal subgroup $\mathcal{A}_0$ of index two in the group $\mathcal{A}$ of all automorphisms. Now it follows from the character table on page 266 of [8] that amongst the irreducible representations of S_6 there are (to within equivalence) just two of dimension ten and it is easy to see that these remain irreducible and become equivalent when restricted to $\mathcal{A}_6$. Let W denote the representation of $\mathfrak{K}$ defined by the function $n \to M^n$ where M^0 is the ten dimensional irreducible representation just described and for $j \neq 0$, M^j is the one dimensional identity representation. The subgroup of $\mathfrak{A}$ taking W into a representation equivalent to itself is the group of all α, 0 and α is restricted to the subgroup of $\mathcal{A}$ taking M^0 into something equivalent to itself. Moreover inspection of the character table of $\mathcal{A}_6$ (easily derived from that of S_6) shows that this later subgroup is the whole of $\mathcal{A}$. The representations of $\mathfrak{G}$ associated with the orbit of W are thus in a natural one-to-one correspondence with the σ representations of $\mathcal{A}$ for some σ. If this σ were trivial there would exist an ordinary representation of $\mathfrak{K}\mathcal{A}$ extending W. Hence there would exist an ordinary representation of $\mathcal{A}_6\mathcal{A}$ extending M^0. Let L be any such representation of $\mathcal{A}_6\mathcal{A}$. By the character table for S_6 and the definition of M^0 we know that L has exactly two inequivalent extensions to the normal subgroup of $\mathcal{A}$ defined by the inner automorphism of S_6. Moreover from the proof given in [2] that there exist automorphisms of $\mathcal{A}_6$ other than those in $\mathcal{A}_6$ it is easy to see that these automorphisms interchange the two extensions of L to $\mathcal{A}_6$. It follows at once from the general theory that there can be no extension of L to $\mathcal{A}$ and hence no extension of W to $\mathfrak{K}\mathcal{A}$. Thus σ cannot be trivial. Thus when $\mathfrak{K}$ is non commutative non trivial multipliers can occur even if $\mathfrak{G}$ is a semi direct product of $\mathfrak{K}$ and $\mathfrak{G}/\mathfrak{K}$.

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