THE CRITICAL LATTICES OF A STAR-SHAPED OCTAGON

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Introduction

1. Let $\mathfrak{R}$ denote the region formed by the boundary and interior of the star-shaped octagon whose sides are segments of the lines $x = \pm l(y \pm 1)$, $y = \pm l(x \pm 1)$, where $l > 1$. Let A, B, C, D denote the vertices

$$(1, 0),\ \left(\frac{l}{l-1}, \frac{l}{l-1}\right),\ (0, 1),\ \left(-\frac{l}{l-1}, \frac{l}{l-1}\right)$$

respectively, and let A', B', C', D' denote their images in the origin O. Let the angle ABC be 2θ, and so l is equal to $\tan(45° + \theta)$. Then for $30° \leqslant \theta < 45°$ Mordell [3] has in effect shown that the determinant of a critical lattice of $\mathfrak{R}$ is

$$\frac{l^2(l^2+4l+5)}{(l^2+2l-1)^2}.$$

He has also shown that there are two critical lattices, which can be regarded as being defined by squares whose vertices and the mid-points of whose sides lie on the boundary of $\mathfrak{R}$; for $30° < \theta < 45°$ these are the only critical lattices, for $\theta = 30°$ there are two further critical lattices.

By similar methods I shall prove that the determinant of a critical lattice of $\mathfrak{R}$ is

$$1 + \frac{1}{2l} \quad \text{if } 22\tfrac{1}{2}° \leqslant \theta \leqslant 30°,$$

$$\frac{2l^2(l+1)(3l+1)}{(3l^2-1)^2} \quad \text{if } \theta_0 \leqslant \theta \leqslant 22\tfrac{1}{2}°,$$

and

$$\frac{l^2(l^2+4l+5)}{(l^2+2l-1)^2} \quad \text{if } 15° \leqslant \theta \leqslant \theta_0,$$

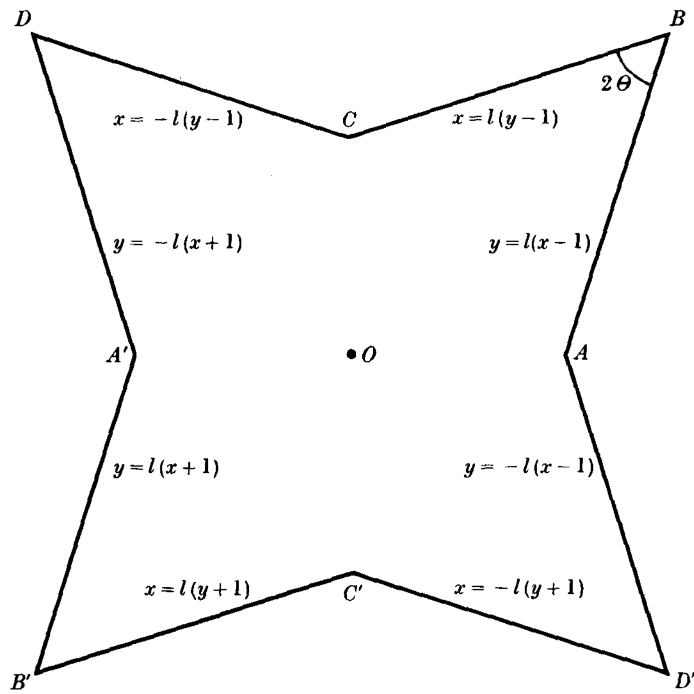

Diagram 1.

where θ_0 is given by

$$3l^6 + 4l^5 - 7l^4 - 24l^3 - 7l^2 + 4l + 3 = 0. \tag{1}$$

The critical lattices will be described later.

The region $\mathfrak{R}$, which consists of two intersecting parallelograms, depends on a parameter l. I shall thus find the determinant of a critical lattice of $\mathfrak{R}$ for a range of values of the parameter. The only other result of this nature is, I believe, that due to Mahler [1], who considered the region formed by the two intersecting ellipses

$$x^2 + y^2 = 1, \quad \lambda x^2 + \frac{1}{\lambda} y^2 = 1,$$

where $\lambda > 1$. By applying his general theory of lattice points in two-dimensional star domains he found the determinant of a critical lattice of this region for the range of values of the parameter λ given by

$$2 \leqslant \lambda + 1/\lambda \leqslant 25.$$

In section 11 I shall give a brief account of the ideas which suggested the above conclusions.

Finally I shall prove that if, and only if, l takes one of the values

$$(1+\sqrt{n^2+1})/n \quad (n=1, 2, \ldots) \quad \text{or} \quad \sqrt{(n+1)/(n-1)} \quad (n=2, 3, \ldots)$$

then the determinant of a critical lattice of $\mathfrak{R}$ is equal to that of one of the two intersecting parallelograms of which $\mathfrak{R}$ is composed (see diagram 8).

The substitution $t=l+l^{-1}$ reduces (1) to the form

$$3t^3+4t^2-16t-32=0,$$

which has precisely one real root t_0. Since $2<t_0<3$ it follows that (1) has two distinct positive roots l_0, l_0^{-1} (where $l_0>1$). It is easily verified that

$$2.00<l_0<2.01,$$

and so $\quad 0.3333<\tan\theta_0=(l_0-1)/(l_0+1)<0.3356,$

whence $\quad 18°25'<\theta_0<18°34'.$

The following table will be useful:

θ	l
30°	$2+\sqrt{3}$
$22\frac{1}{2}$°	$1+\sqrt{2}$
15°	$\sqrt{3}$

Proof of Result for $22\frac{1}{2}° \leqslant \theta \leqslant 30°$

2. *Introduction* (see diagram 2).

THEOREM I. *If* $22\frac{1}{2}° \leqslant \theta \leqslant 30°$, *i.e. if* $1+\sqrt{2} \leqslant l \leqslant 2+\sqrt{3}$, *then the determinant of a critical lattice of* $\mathfrak{R}$ *is*

$$\Delta = 1+1/2l.$$

Moreover the lattice Λ_1 *generated by* $A(1,0)$, $L(\frac{1}{2}, \Delta)$ *and its image* Λ_1' *in the line* $x=y$ *are critical. For* $22\frac{1}{2}° \leqslant \theta < 30°$ Λ_1, Λ_1' *are the only critical lattices; for* $\theta=30°$ *there are two further critical lattices, viz.* Λ_0 *generated by the points* $(\frac{1}{4}(\sqrt{3}-1),\ \frac{1}{4}(3\sqrt{3}-1))$, $(\frac{1}{4}(3\sqrt{3}-1),\ -\frac{1}{4}(\sqrt{3}-1))$, *and its image* Λ_0' *in the y-axis.*

Λ_0, Λ_0' can be regarded as being defined by squares whose vertices and the mid-points of whose sides lie on the boundary of $\mathfrak{R}$; see Mordell [3]. I shall show at the end of section 4 that these lattices are admissible for $\mathfrak{R}$.

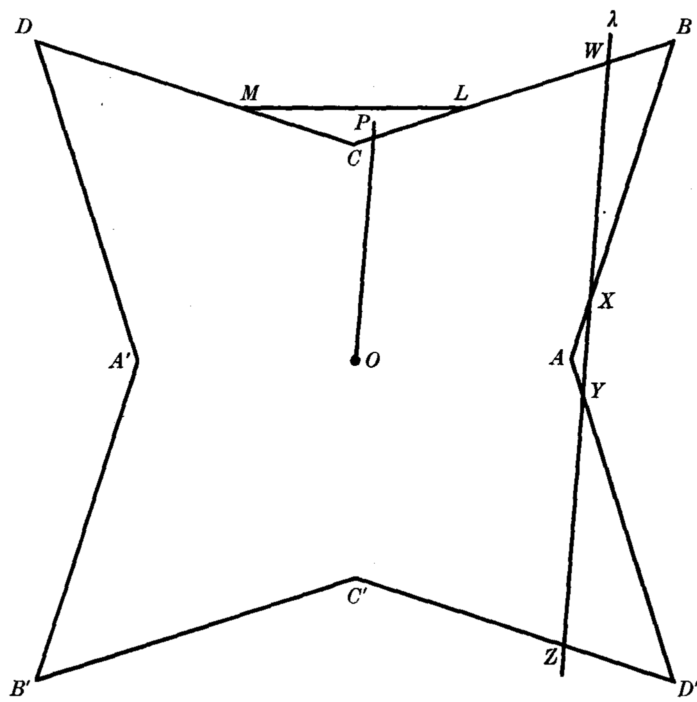

Diagram 2.

LEMMA 1. *Λ_1 is admissible for $\mathfrak{R}$.*

Proof. Since $2\Delta > l/(l-1)$, the lattice line $y = 2\Delta$ contains no point of $\mathfrak{R}$. Further the lattice line $y = \Delta$ meets the sides BC, CD in the points $L(\frac{1}{2}, \Delta)$, $M(-\frac{1}{2}, \Delta)$ respectively, and L, M are points of Λ_1. Finally the point $L + A$, i.e. $(\frac{3}{2}, \Delta)$, lies on or to the right of AB; for the equation of AB is $lx - y - l = 0$, and $\frac{3}{2}l - \Delta - l \geqslant 0$ according as $l \geqslant 1 + \sqrt{2}$. This completes the proof of the lemma.

Let now Λ be any lattice of determinant Δ. I shall prove either that Λ is one of the critical lattices mentioned in the enunciation of theorem I, or that Λ contains a point (other than O) in the interior of $\mathfrak{R}$.

Consider the rectangle of area 4Δ defined by

$$|x| < 1, \quad |y| \leqslant \Delta.$$

Every point of this rectangle is either

(i) an interior point of $\mathfrak{R}$; or

(ii) a point of $\mathfrak{R}_1$, where $\mathfrak{R}_1$ is the region formed by the interior and boundary of the triangle CLM; or

(iii) a point of $\mathfrak{R}_1'$, where $\mathfrak{R}_1'$ is the image of $\mathfrak{R}_1$ in O.

By Minkowski's theorem this rectangle contains a primitive(1) point $P(x_1, y_1)$ of Λ. If P is an interior point of $\mathfrak{R}$ there is no more to prove. If not it can be assumed that P lies in $\mathfrak{R}_1$ and, without loss of generality, that $x_1 \geqslant 0$. I shall prove either that Λ is one of the above-mentioned critical lattices, or that the lattice line

$$\lambda: \; xy_1 - x_1 y = \Delta$$

(which is parallel to OP and at a perpendicular distance Δ/OP from it) contains a point of Λ in the interior of $\mathfrak{R}$.

The general idea of the proof of this last statement is as follows. Let λ meet the lines CB, BA, AD', $D'C'$ in the points W, X, Y, Z respectively; for future reference the coordinates of W, X, Y, Z (insofar as they are well-defined points of intersection) are given in the table below:

W	$\left(\dfrac{l\Delta + lx_1}{ly_1 - x_1}, \quad \dfrac{\Delta + ly_1}{ly_1 - x_1}\right)$	CB
X	$\left(\dfrac{\Delta - lx_1}{y_1 - lx_1}, \quad \dfrac{l\Delta - ly_1}{y_1 - lx_1}\right)$	BA
Y	$\left(\dfrac{\Delta + lx_1}{y_1 + lx_1}, \quad -\dfrac{l\Delta - ly_1}{y_1 + lx_1}\right)$	AD'
Z	$\left(\dfrac{l\Delta - lx_1}{ly_1 + x_1}, \quad -\dfrac{\Delta + ly_1}{ly_1 + x_1}\right)$	$D'C'$

It will be shown in Lemma 3 that, unless P is one of at most three points, its co-ordinates satisfy either the inequality (3) or the inequality (4). If (3) is satisfied then W, X, Y, Z lie on the sides(2) CB, BA, AD', $D'C'$ respectively and $XY < OP$ while (by Lemma 2) $WZ > 2OP$. If (4) is satisfied then Y, Z lie on the sides AD', $D'C'$ respectively and $YZ > OP$. Since λ contains points of Λ equally spaced at a distance OP apart there is, in either case, a point of Λ in the interior of $\mathfrak{R}$. The exceptional points mentioned above may lead to the critical lattices.

I shall prove Lemmas 2 and 3 in section 3 and the theorem itself in section 4.

(1) A point P of a lattice is *primitive* if (i) it is distinct from O, and (ii) the lattice line OP contains no lattice point lying between O and P.

(2) I make a distinction between the infinite straight *line* CB and the *segment* (or *side*) CB. The latter consists of those points of the line CB which lie between, or coincide with one of, the points C, B.

3. Lemma 2. *Let* $P(x_1, y_1)$ *be any point of* $\mathfrak{R}_1$, *then*

$$l\Delta + \tfrac{1}{4}l^2 > l^2(y_1 - \tfrac{1}{2})^2 - x_1^2. \tag{2}$$

(This result shows that $WZ > 2OP$.)

Proof. The equation

$$l\Delta + \tfrac{1}{4}l^2 = (ly - \tfrac{1}{2}l - x)(ly - \tfrac{1}{2}l + x) \tag{2'}$$

is that of a hyperbola with asymptotes $ly \pm x = \frac{1}{2}l$, i.e. lines parallel to CD, CB and intersecting in $(0, \frac{1}{2})$. The lowest point of the upper branch of this hyperbola is given by

$$x = 0, \quad y = \tfrac{1}{2} + \sqrt{\Delta/l + \tfrac{1}{4}}.$$

Since

$$\tfrac{1}{2} + \sqrt{\Delta/l + \tfrac{1}{4}} > \Delta,$$

$\mathfrak{R}_1$ lies in the open region bounded by the upper branch of the hyperbola (2′) and its asymptotes, and so (2) holds for every point $P(x_1, y_1)$ of $\mathfrak{R}_1$.

Lemma 3. *Let* $P(x_1, y_1)$ *be any point of* $\mathfrak{R}_1$ *for which* $x_1 \geqslant 0$. *Then, provided* P *is not one of the points* $(0, 1)$, $(\frac{1}{2}, \Delta)$ *or (if* $\theta = 30°$) $(\frac{1}{4}(\sqrt{3} - 1), \frac{1}{4}(3\sqrt{3} - 1))$, *either*

$$2l\Delta < y_1^2 - l^2x_1^2 + 2ly_1 \tag{3}$$

or

$$l^2 - (l^2 - 1)\Delta > (ly_1 + x_1 - l)(y_1 + lx_1 - l). \tag{4}$$

(The inequalities (3), (4) show that $XY < OP$, $YZ > OP$ respectively.)

Proof. The equation

$$\mathcal{H}_1:\ 2l\Delta + l^2 = (y + l)^2 - l^2x^2 = (y + l - lx)(y + l + lx) \tag{3'}$$

is that of a hyperbola ($\mathcal{H}_1$) with asymptotes AB, $A'D$. $\mathcal{H}_1$ intersects BC in the points C $(0, 1)$, $R\{2l(l+1)/(l^4 - 1),\ (l^4 + 2l + 1)/(l^4 - 1)\}$, and intersects LM in the points $S(\Delta/l, \Delta)$, $T(-\Delta/l, \Delta)$. S lies to the left of or coincides with L according as $\Delta/l <$ or $= \frac{1}{2}$, i.e. according as $l >$ or $= 1 + \sqrt{2}$; R lies to the left of or coincides with L according as $2l(l+1)/(l^4 - 1) <$ or $= \frac{1}{2}$, i.e. according as $l >$ or $= 1 + \sqrt{2}$. For every point P lying inside the upper branch of $\mathcal{H}_1$ the inequality (3) is satisfied.

Now consider the equation

$$\mathcal{H}_2 : l^2 - (l^2 - 1)\Delta = (ly + x - l)(y + lx - l). \tag{4'}$$

Since $l^2 - (l^2 - 1)\Delta <$ or $= 0$ according as $l >$ or $= 1 + \sqrt{2}$ it follows that (4′) is, for $l > 1 + \sqrt{2}$, the equation of a hyperbola with asymptotes CD, AD' and, for $l = 1 + \sqrt{2}$, that of the two straight lines CD, AD'. This "hyperbola" ($\mathcal{H}_2$) intersects LM in the

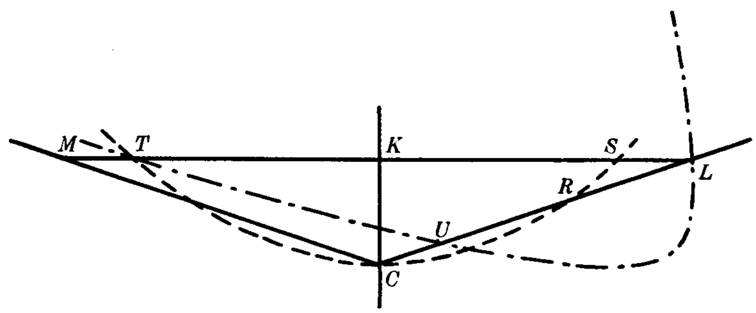

Diagram 3.
The hyperbola $\mathcal{H}_1$ - - - - -
The hyperbola $\mathcal{H}_2$ - · - · - · - · -

points $L(\frac{1}{2}, \Delta)$, $T(-\Delta/l, \Delta)$, and BC in the points L,

$$U\{l^2-2l-1)/2(l^2+1),\ (2l^3+l^2-1)/2l(l^2+1)\}.$$

U lies to the right of or coincides with C according as $(l^2-2l-1)/2(l^2+1) >$ or $=0$, i.e. according as $l >$ or $= 1+\sqrt{2}$; also U lies to the left of or coincides with R according as $(l^2-2l-1)/2(l^2+1) <$ or $= 2l(l+1)/(l^4-1)$, i.e. according as $l <$ or $= 2+\sqrt{3}$. For every point P lying inside the "upper branch" of $\mathcal{H}_2$ the inequality (4) is satisfied.

For $l=2+\sqrt{3}$ the points R, U both become

$$(\tfrac{1}{4}(\sqrt{3}-1),\ \tfrac{1}{4}(3\sqrt{3}-1)).$$

Now let K be the mid-point of LM (and so K lies inside the upper branch of $\mathcal{H}_1$). Then, for $22\frac{1}{2}° < \theta < 30°$, it follows from what has already been proved that

(i) every point of the quadrilateral $CRSK$, except for the vertices C, R, S, lies inside the upper branch of $\mathcal{H}_1$; and

(ii) every point of the triangle LRS, except for the vertex L, lies inside the upper branch of $\mathcal{H}_2$ (since R, S lie inside the upper branch of $\mathcal{H}_2$).

Therefore every point of the triangle CLK, except for the vertices C, L, lies inside either the upper branch of $\mathcal{H}_1$ or the upper branch of $\mathcal{H}_2$.

This, with slight modifications in the wording of the last paragraph when $\theta = 22\frac{1}{2}°$ or when $\theta = 30°$, completes the proof of the lemma.

4. *Proof of Theorem I.*

Let $P(x_1, y_1)$ be any point of $\mathfrak{R}_1$ for which $x_1 \geqslant 0$. Then $ly_1 - x_1 \geqslant l > 0$ (since P does not lie below CB) and $y_1 + lx_1 > 0$, $ly_1 + x_1 > 0$ (since $x_1 \geqslant 0$, $y_1 > 0$); further, if the inequality (3) is satisfied,

$$y_1 - l x_1 > \frac{2l(\Delta - y_1)}{y_1 + l x_1} \geqslant 0$$

(since $y_1 \leqslant \Delta$). It follows (see section 2) that W, Y, Z and, if (3) is satisfied, X are well-defined points of intersection.

Suppose firstly that the inequality (3) *is satisfied.* Then

$$l^2 x_1^2 < y_1^2 + 2l y_1 - 2l\Delta,$$

and so $$l x_1 - l y_1 + \Delta(l-1) < \sqrt{y_1^2 + 2l y_1 - 2l\Delta} - l y_1 + \Delta(l-1). \tag{5}$$

Now $$\{l y_1 - \Delta(l-1)\}^2 - (y_1^2 + 2l y_1 - 2l\Delta) = (y_1 - \Delta)\{(l^2-1)y_1 - \Delta(l-1)^2 - 2l\}. \tag{6}$$

Since $\Delta(l-1)/(l+1) + 2l/(l^2-1) > \Delta \geqslant y_1$, each factor on the right of (6) is $\leqslant 0$; therefore

$$\{l y_1 - \Delta(l-1)\}^2 \geqslant y_1^2 + 2l y_1 - 2l\Delta.$$

Also, since $y_1 \geqslant 1$,

$$l y_1 - \Delta(l-1) \geqslant l - \Delta(l-1) = (l+1)/2l > 0$$

and so $$l y_1 - \Delta(l-1) \geqslant \sqrt{y_1^2 + 2l y_1 - 2l\Delta},$$

whence, by (5), $$l x_1 - l y_1 + \Delta(l-1) < 0. \tag{7}$$

I now show that W, X, Y, Z lie on the sides CB, BA, AD', $D'C'$ respectively. For from the inequalities at the beginning of this paragraph, together with (7) and $0 \leqslant x_1 \leqslant \frac{1}{2}$, $1 \leqslant y_1 \leqslant \Delta$, it follows that

$$0 < \frac{l(\Delta - x_1)}{l y_1 + x_1} \leqslant \frac{l(\Delta + x_1)}{l y_1 - x_1} < \frac{l}{l-1}$$

and $$1 \leqslant \frac{\Delta + l x_1}{y_1 + l x_1} \leqslant \frac{\Delta - l x_1}{y_1 - l x_1} < \frac{l}{l-1}$$

(the last part of each of these inequalities following from (7)). The first inequality proves the assertion for W and Z, the second for X and Y.

Also $WZ > 2OP$ and $XY < OP$. To prove the first of these it suffices to show that

$$\frac{\Delta + l y_1}{l y_1 - x_1} + \frac{\Delta + l y_1}{l y_1 + x_1} > 2 y_1;$$

this is so since $$l(\Delta + l y_1) > l^2 y_1^2 - x_1^2$$

by (2). To prove the second of these it suffices to show that

$$\frac{l(\Delta - y_1)}{y_1 - l x_1} + \frac{l(\Delta - y_1)}{y_1 + l x_1} < y_1;$$

this is so since $$2l(\Delta - y_1) < y_1^2 - l^2 x_1^2$$
by (3).

I have now shown that, if (3) is satisfied, there is a point (other than O) of Λ in the interior of $\mathfrak{R}$.

Suppose secondly that the inequality (4) *is satisfied.* Since

$$l x_1 + l y_1 - (l-1)\Delta \geqslant l - (l-1)\Delta > 0,$$

it follows that
$$1 \leqslant \frac{\Delta + l x_1}{y_1 + l x_1} < \frac{l}{l-1}$$

and
$$0 < \frac{l(\Delta - x_1)}{l y_1 + x_1} < \frac{l}{l-1}.$$

Therefore Y, Z lie on the sides AD', $D'C'$ respectively.

Further $YZ > OP$. To prove this it suffices to show that

$$-\frac{l(\Delta - y_1)}{y_1 + l x_1} + \frac{\Delta + l y_1}{l y_1 + x_1} > y_1;$$

this is so since, by (4),

$$\Delta(1 - l^2) + l(y_1 + l x_1) + l(l y_1 + x_1) > (y_1 + l x_1)(l y_1 + x_1).$$

I have now shown that, if (4) is satisfied, there is a point (other than O) of Λ in the interior of $\mathfrak{R}$.

Suppose lastly that P is one of the two (or, if $\theta = 30^\circ$, three) exceptional points mentioned in the enunciation of Lemma 3. Then either Λ is one of the critical lattices mentioned in the enunciation of Theorem I, or there is a point (other than O) of Λ in the interior of $\mathfrak{R}$.

Thus if $\theta = 30^\circ$ and P is the point $(\frac{1}{4}(\sqrt{3}-1),\ \frac{1}{4}(3\sqrt{3}-1))$ then, as before, W, X, Y, Z lie on the sides CB, BA, AD', $D'C'$ respectively but now $XY = YZ = OP$ (since the inequalities (3) and (4) become equalities). By substituting the known numerical values of l, Δ, x_1, y_1 it is easily verified that $WX < OP$, and that the lattice line $x y_1 - x_1 y = 2\Delta$ has no point in common with $\mathfrak{R}$. It follows that (i) the lattice Λ_0 generated by $P\{\frac{1}{4}(\sqrt{3}-1),\ \frac{1}{4}(3\sqrt{3}-1)\}$, $Y\{\frac{1}{4}(3\sqrt{3}-1),\ -\frac{1}{4}(\sqrt{3}-1)\}$ is admissible for $\mathfrak{R}$; (ii) either there is a point (other than O) of Λ in the interior of $\mathfrak{R}$, or Λ is the critical lattice Λ_0.

Similarly, if P is the point $(0, 1)$ then $W(\Delta, 1 + l^{-1}\Delta)$, $X(\Delta, \frac{1}{2})$, $Y(\Delta, -\frac{1}{2})$, $Z(\Delta, -1 - l^{-1}\Delta)$ lie on the sides CB, BA, AD', $D'C'$ respectively and $XY = 1 = OP$. It is easily verified that now $\frac{1}{2} < WX$, $YZ \leqslant 1$ with equality only if $\theta = 22\frac{1}{2}^\circ$. Therefore either there is a point (other than O) of Λ in the interior of $\mathfrak{R}$, or Λ is the critical lattice Λ_1'.

Finally if P is the point $(\frac{1}{2}, \Delta)$ then Y is the point $(1, 0)$ and Z is the point

$$\left\{\frac{l(\Delta-\frac{1}{2})}{l\Delta+\frac{1}{2}},\ -\frac{\Delta(1+l)}{l\Delta+\frac{1}{2}}\right\}=(\tfrac{1}{2},\ -\Delta).$$

Y, Z lie on AD', $D'C'$ respectively and $YZ=OP$. In this case either there is a point (other than O) of Λ in the interior of $\mathfrak{R}$, or Λ is the critical lattice Λ_1.

This completes the proof of Theorem I.

Proof of Result for $\theta_0 \leqslant \theta \leqslant 22\frac{1}{2}^\circ$

5. *Introduction* (see diagram 4).

THEOREM II. *If* $\theta_0 \leqslant \theta \leqslant 22\frac{1}{2}^\circ$, *i.e. if* $l_0 \leqslant l \leqslant 1+\sqrt{2}$, *then the determinant of a critical lattice of* $\mathfrak{R}$ *is*

$$\Delta=\frac{2l^2(l+1)(3l+1)}{(3l^2-1)^2}.$$

Moreover the lattice Λ_2 *generated by*

$$S\left\{\frac{l(l+1)}{3l^2-1},\ \frac{l(3l+1)}{3l^2-1}\right\},\quad T\left\{-\frac{l(l+1)}{3l^2-1},\ \frac{l(3l+1)}{3l^2-1}\right\},$$

and its image Λ_2' *in the line* $x=y$ *are critical. For* $\theta_0 < \theta \leqslant 22\frac{1}{2}^\circ$ Λ_2, Λ_2' *are the only critical lattices; for* $\theta=\theta_0$ *there are two further critical lattices* Λ_3, Λ_3'.

Λ_3 can be regarded as being defined by a square whose vertices and the mid-points of whose sides lie on the boundary of $\mathfrak{R}$; see the enunciation of Theorem III for the co-ordinates of a pair of points generating it. Λ_3' is the image of Λ_3 in the y-axis.

The lattice Λ_2 can be regarded as being defined by the line parallel to the x-axis which has equal intercepts made on it by the sides AB, BC, CD, DA'; in fact, since

$$1<\frac{l(3l+1)}{3l^2-1}<\frac{l}{l-1},$$

the line $y=l(3l+1)/(3l^2-1)$ meets the sides AB, BC, CD, DA' in the points

$$R\left\{\frac{3l(l+1)}{3l^2-1},\ \frac{l(3l+1)}{3l^2-1}\right\},\ S,\ T,\quad U\left\{-\frac{3l(l+1)}{3l^2-1},\ \frac{l(3l+1)}{3l^2-1}\right\}$$

respectively and $RS=ST=TU=2l(l+1)/(3l^2-1)$.

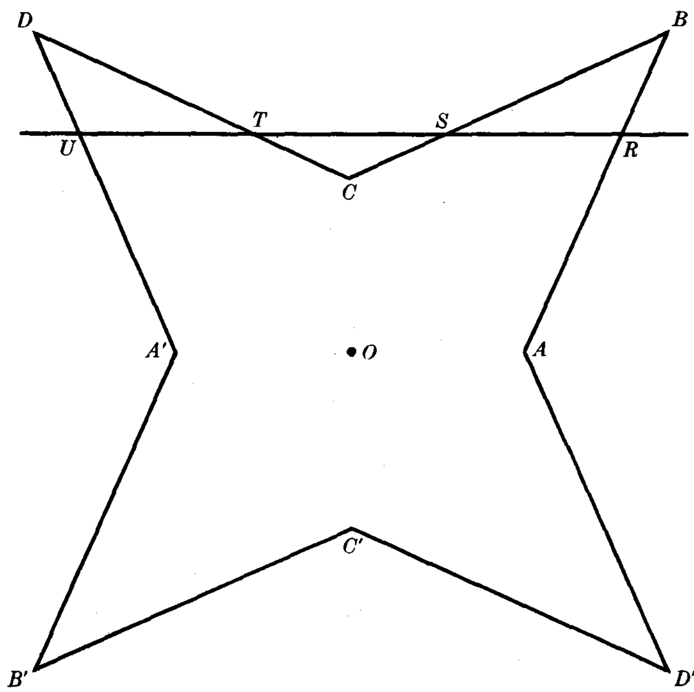

Diagram 4.

LEMMA 4. *Λ_2 is admissible for $\mathfrak{R}$.*

Proof. Firstly the lattice line $y = 2l(3l+1)/(3l^2-1)$ contains no point of $\mathfrak{R}$; for

$$\frac{2l(3l+1)}{3l^2-1} > \frac{l}{l-1}$$

if $l > (2+\sqrt{7})/3$, which is the case here since $l \geqslant l_0 > 2$.

Secondly the point $\{2l(l+1)/(3l^2-1),\ 0\}$ lies outside $\mathfrak{R}$ for $\theta_0 \leqslant \theta < 22\frac{1}{2}°$ and lies on the boundary of $\mathfrak{R}$ (coinciding with A) for $\theta = 22\frac{1}{2}°$.

This completes the proof of the lemma.

Let now Λ be any lattice of determinant Δ. I shall prove either that Λ is one of the critical lattices mentioned in the enunciation of Theorem II, or that there is a point (other then O) of Λ in the interior of $\mathfrak{R}$.

Consider the rectangle of area 4Δ defined by

$$|x| \leqslant \frac{2l(l+1)}{3l^2-1}, \quad |y| < \frac{l(3l+1)}{3l^2-1}.$$

By Minkowski's theorem this rectangle contains a primitive point $P(x_1, y_1)$ of Λ. If

P is an interior point of $\mathfrak{R}$ there is no more to prove. Otherwise it can, without loss of generality, be assumed that P belongs either

(i) to the region $\mathfrak{R}_1$ formed by the interior of the triangle CST together with the two sides CS, CT but excluding the end-points S, T; or

(ii) to the region $\mathfrak{R}_2'$ formed by the interior and boundary of the triangle (which reduces to a point when $\theta = 22\frac{1}{2}°$) whose sides are the lines AB, AD' and $x = 2l(l+1)/(3l^2-1)$.

Let $\mathfrak{R}_1'$, $\mathfrak{R}_2$ denote the images of $\mathfrak{R}_1$, $\mathfrak{R}_2'$ respectively in the line $x = y$. Since

$$\frac{2l(l+1)}{3l^2-1} < \frac{l(3l+1)}{3l^2-1}$$

it follows that $\mathfrak{R}_2'$ lies in $\mathfrak{R}_1'$. Therefore P lies either in $\mathfrak{R}_1$ or in $\mathfrak{R}_1'$ and so, without essential loss of generality, it can be assumed to lie in $\mathfrak{R}_1$.

As before I consider the lattice line

$$\lambda : x y_1 - x_1 y = \Delta.$$

Its intersections W, X, Y, Z (1) with the lines CB, BA, AD', $D'C'$ respectively are well-defined. For if $P(x_1, y_1)$ is any point of $\mathfrak{R}_1$, then $ly_1 \pm x_1 \geqslant l > 0$; further

$$y_1 - lx_1 > \frac{l(3l+1)}{3l^2-1} - \frac{l^2(l+1)}{3l^2-1},$$

since the right-hand side is the value of $y - lx$ at the point S (where $y - lx$ obviously takes a lower value than at any point of $\mathfrak{R}_1$), and so

$$y_1 - lx_1 > \frac{-l(l^2-2l-1)}{3l^2-1} \geqslant 0;$$

similarly $y_1 + lx_1 > 0$.

Moreover W, X, Y, Z are interior points of the sides CB, BA, AD', $D'C'$ respectively. For, if $P(x_1, y_1)$ is any point of $\mathfrak{R}_1$, it follows that

$$y_1 - x_1 > \frac{l(3l+1)}{3l^2-1} - \frac{l(l+1)}{3l^2-1},$$

since the right-hand side is the value of $y - x$ at the point S (where $y - x$ obviously takes a lower value than at any point of $\mathfrak{R}_1$), and so

$$y_1 - x_1 > \frac{2l^2}{3l^2-1} = \frac{\Delta(3l^2-1)}{(l+1)(3l+1)} \geqslant \frac{l-1}{l}\Delta;$$

---

(1) The co-ordinates of W, X, Y, Z were given in Section 2.

similarly
$$y_1 + x_1 > \frac{l-1}{l}\,\Delta.$$

Since also
$$|x_1| < \frac{l(l+1)}{3l^2-1} < \Delta, \quad |y_1| < \frac{l(3l+1)}{3l^2-1} \leqslant \Delta,$$

it follows that
$$0 < \frac{l(\Delta + x_1)}{l y_1 - x_1}, \quad \frac{l(\Delta - x_1)}{l y_1 + x_1} < \frac{l}{l-1}$$

(and so W, Z are interior points of the sides CB, $D'C'$), and
$$1 < \frac{\Delta - l x_1}{y_1 - l x_1}, \quad \frac{\Delta + l x_1}{y_1 + l x_1} < \frac{l}{l-1}$$

(and so X, Y are interior points of the sides BA, AD').

The general idea of the proof of Theorem II is now as follows. (1) I shall prove in Lemmas 5 and 6 that, except for the point $E\{0, 2l(l+1)/(3l^2-1)\}$, the region $\mathfrak{R}_2$ lies

(i) between the upper branch of the hyperbola
$$\mathcal{H}_1:\ \tfrac{8}{3} l \Delta + \tfrac{1}{9} l^2 = (l y - x - \tfrac{1}{3} l)(l y + x - \tfrac{1}{3} l)$$
and its asymptotes;

(ii) inside the upper branch of the hyperbola
$$\mathcal{H}_2:\ l(\Delta - y) = y^2 - l^2 x^2.$$

These results imply that if P lies in $\mathfrak{R}_2$ and does not coincide with E then $WZ > 3OP$ and $XY < 2OP$. If P coincides with E then $WX = XY = YZ = OP$. This proves Theorem II when P lies in $\mathfrak{R}_2$.

I next consider the case when P lies in $\mathfrak{R}_1$ but not in $\mathfrak{R}_2$. There is no loss of generality in assuming $x_1 \geqslant 0$. The hyperbola
$$\mathcal{H}_3:\ l^2 - \Delta(l^2-1) = (l y + x - l)(y + l x - l)$$
passes through E and has the lines CD, AD' as asymptotes. The upper branch of $\mathcal{H}_3$ cuts the side CB in points I, J which lie respectively below and above the line $y = 2l(l+1)/(3l^2-1)$ (see diagram 5). The co-ordinates of I, J are given by
$$l^2 - \Delta(l^2-1) = (l y + x - l)(y + l x - l), \quad x = l(y-1).$$

If y_0 denotes the ordinate of J, then y_0 is the greater root of
$$2(l^2+1)y^2 - 2(2l^2+l+1)y + 2l^2 + l + \Delta(l - l^{-1}) = 0.$$

(1) To avoid special cases I shall, for the remainder of this section, assume that $\theta_0 < \theta < 22\tfrac{1}{2}°$.

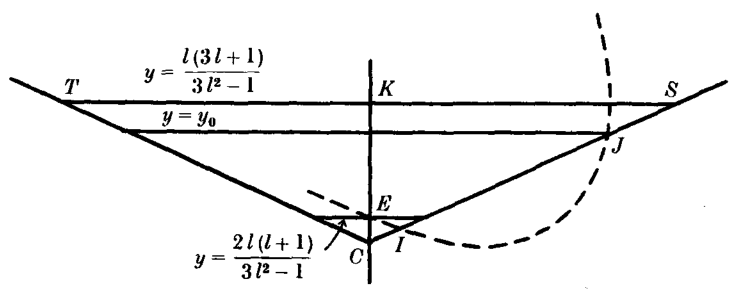

Diagram 5. The hyperbola $\mathcal{H}_3$ – – – – – –.

$\mathfrak{R}_1: ly + x - l \geq 0,\ ly - x - l \geq 0,\ y < \frac{l(3l+1)}{3l^2-1}.$

$\mathfrak{R}_2: ly + x - l \geq 0,\ ly - x - l \geq 0,\ y \leqslant \frac{2l(l+1)}{3l^2-1}.$

$\mathfrak{R}_3: ly + x - l \geq 0,\ ly - x - l \geq 0,\ y < y_0.$

In particular, if P lies in $\mathfrak{R}_3$ (that part of $\mathfrak{R}_1$ lying below the line $y = y_0$) but not in $\mathfrak{R}_2$ then P lies inside the upper branch of $\mathcal{H}_3$. I shall show that if P lies inside the upper branch of $\mathcal{H}_3$ then $YZ > OP$, and so Theorem II is proved in this case also.

Finally if P lies in that part of $\mathfrak{R}_1$ not already covered, i.e. if P lies neither in $\mathfrak{R}_2$ nor inside the upper branch of $\mathcal{H}_3$, then (still assuming $x_1 \geqslant 0$) I shall show that P lies inside the upper branch of the hyperbola

$$\mathcal{H}_4:\ l(y_0 - \Delta) = (x + ly - l)(x - y_0).$$

This result implies that the side $D'C'$ and the line $x = y_0$ now make an intercept on λ of length greater than OP. It follows that there is a point (other than O) of Λ either in the interior of $\mathfrak{R}$ or in the image of $\mathfrak{R}_3$ in the line $x = y$. The proof of Theorem II is then completed by appealing to the proofs covering the first two cases.

6. Lemma 5. *Let $P(x_1, y_1)$ be any point of $\mathfrak{R}_2$; then*

$$\tfrac{2}{3} l\Delta + \tfrac{1}{9} l^2 \geqslant (ly_1 - x_1 - \tfrac{1}{3} l)(ly_1 + x_1 - \tfrac{1}{3} l), \tag{8}$$

with equality if and only if P and E coincide.

(This result shows that $WZ > 3OP$.)

Proof. The equation

$$\mathcal{H}_1:\ \tfrac{2}{3} l\Delta + \tfrac{1}{9} l^2 = (ly - x - \tfrac{1}{3} l)(ly + x - \tfrac{1}{3} l) \tag{8'}$$

is that of a hyperbola ($\mathcal{H}_1$) with asymptotes $ly \pm x - \frac{1}{3} l = 0$, i.e. lines parallel to CD, CB intersecting in the point $(0, \frac{1}{3})$. The lowest point of the upper branch of $\mathcal{H}_1$ is

E, and E lies on the upper boundary of $\mathfrak{R}_2$. The lemma follows since (8) (with inequality) is satisfied at all points lying between the upper branch of $\mathcal{H}_1$ and its asymptotes.

Lemma 6. *Let* $P(x_1, y_1)$ *be any point of* $\mathfrak{R}_2$; *then*

$$l(\Delta - y_1) < y_1^2 - l^2 x_1^2. \tag{9}$$

(This result shows that $XY < 2OP$.)

Proof. The equation

$$\mathcal{H}_2: \; l\Delta + \tfrac{1}{4}l^2 = (y - lx + \tfrac{1}{2}l)(y + lx + \tfrac{1}{2}l) \tag{9'}$$

is that of a hyperbola ($\mathcal{H}_2$) with asymptotes $y \pm lx + \frac{1}{2}l = 0$. The lowest point of the upper branch of $\mathcal{H}_2$ is given by

$$x = 0, \quad y = -\tfrac{1}{2}l + \sqrt{l\Delta + \tfrac{1}{4}l^2}.$$

The inequality (9) is satisfied at all points lying inside the upper branch of $\mathcal{H}_2$. To show that all points of $\mathfrak{R}_2$ lie inside the upper branch of $\mathcal{H}_2$ it suffices to show that

(i) C lies inside it; and

(ii) the upper branch of $\mathcal{H}_2$ intersects the line BC in a point lying above the line $y = 2l(l+1)/(3l^2 - 1)$.

To show (i) I have to show that

$$-\tfrac{1}{2}l + \sqrt{l\Delta + \tfrac{1}{4}l^2} < 1,$$

i.e. that $l\Delta < l + 1$; this is the case since

$$\begin{aligned} l + 1 - l\Delta &= (l+1)(3l^4 - 2l^3 - 6l^2 + 1)/(3l^2 - 1)^2 \\ &= (l+1)\{l^3(l-2) + 2l^2(l^2 - 4) + 2l^2 + 1\}/(3l^2 - 1)^2 \end{aligned}$$

and $l \geqslant l_0 > 2$. To show (ii) I have to show that the greater root of

$$(l^4 - 1)y^2 - (2l^4 + l)y + l\Delta + l^4 = 0 \tag{10}$$

is greater than $2l(l+1)/(3l^2 - 1)$. Since the lowest point of the upper branch of $\mathcal{H}_2$ lies below the line BC while, for numerically large x, $\mathcal{H}_2$ lies above this line, it follows that the quadratic (10) has real roots. Since the arithmetic mean of the roots of (10) is $(2l^4 + l)/2(l^4 - 1)$ it therefore suffices to show that

$$\frac{2l^4 + l}{2(l^4 - 1)} > \frac{2l(l+1)}{3l^2 - 1},$$

i.e. that

$$2l^5 - 4l^4 - 2l^3 + 3l^2 + 4l + 3 > 0.$$

This is so since

$$2l^5-4l^4-2l^3+3l^2+4l+3=2l^4(l-2)-(2l+1)(l^2-2l-1)+2$$

and $1+\sqrt{2}\geqslant l\geqslant l_0>2$.

Lemma 7. *Let $P(x_1, y_1)$ be any point of $\mathfrak{R}_1$ which does not lie in $\mathfrak{R}_2$ and for which $x_1\geqslant 0$. Let y_0 be the greater root of*

$$2(l^2+1)y^2-2(2l^2+l+1)y+2l^2+l+\Delta(l-l^{-1})=0. \tag{11}$$

Then for $\theta_0\leqslant\theta<22\tfrac{1}{2}°$

either $$l^2-\Delta(l^2-1)>(ly_1+x_1-l)(y_1+lx_1-l) \tag{12}$$

or $$l(y_0-\Delta)>(x_1+ly_1-l)(x_1-y_0) \tag{13}$$

except when $\theta=\theta_0$ and $P=(ly_0-l,\ y_0)=J$ in which case (12) *and* (13) *both become equalities; further the inequality* (12) *holds when $\theta=22\tfrac{1}{2}°$.*

(The inequality (12) shows that $YZ>OP$; (13) shows that the side $D'C'$ and the line $x=y_0$ make an intercept on λ of length greater than OP.)

Proof. Consider the equation

$$\mathcal{H}_3:\ l^2-\Delta(l^2-1)=(ly+x-l)(y+lx-l). \tag{12'}$$

Now $$l^2-\Delta(l^2-1)=l^2(l^2-2l-1)(3l^2-2l-3)/(3l^2-1)^2$$

and so is $<$ or $=0$ according as $l<$ or $=1+\sqrt{2}$ $(3l^2-2l-3>0$ since $l\geqslant l_0>2>(1+\sqrt{10})/3)$. Therefore (12′) is, for $\theta_0\leqslant\theta<22\tfrac{1}{2}°$, the equation of a hyperbola with asymptotes CD, AD' and, for $\theta=22\tfrac{1}{2}°$, that of the two straight lines CD, AD'. This "hyperbola" ($\mathcal{H}_3$) passes through E and meets the y-axis in the further point

$$\{0,\quad (l^2-1)(3l+1)/(3l^2-1)\},$$

which lies above ST since

$$\frac{(l^2-1)(3l+1)}{3l^2-1}>\frac{l(3l+1)}{3l^2-1}.$$

Further $\mathcal{H}_3$ meets the line BC in the points I, J given by

$$2(l^2+1)y^2-2(2l^2+l+1)y+2l^2+l+\Delta(l-l^{-1})=0,$$

i.e. by (11). The roots of (11) are real since the substitution $y=2l(l+1)/(3l^2-1)$ in the left-hand side gives $l(l^2-2l-1)(2l^3-l^2-4l-3)/(3l^2-1)^2$, and this is $\leqslant 0$ since

$$2l^3-l^2-4l-3=(l-2)(2l^2+3l+2)+1>0$$

for $l \geqslant l_0 > 2$; this shows further that, for $\theta_0 \leqslant \theta < 22\frac{1}{2}°$, I, J lie one on each side of the line $y = 2l(l+1)/(3l^2-1)$.

If $\theta = 22\frac{1}{2}°$ $\mathcal{H}_3$ reduces to a pair of straight lines and the region $\mathfrak{R}_2$ consists of the single point C (for E coincides with C when $\theta = 22\frac{1}{2}°$). The points of intersection of $\mathcal{H}_3$ and the line BC are now, as is easily verified, C and S. Since (12) is satisfied at all points lying inside the "upper branch" of $\mathcal{H}_3$ it follows that the lemma is proved when $\theta = 22\frac{1}{2}°$.

Suppose now that $\theta_0 \leqslant \theta < 22\frac{1}{2}°$. The lines AD', BC meet in the point

$$\left\{\frac{l(l-1)}{l^2+1},\ \frac{l(l+1)}{l^2+1}\right\}$$

and so
$$y_0 < \frac{l(l+1)}{l^2+1} < \frac{l(3l+1)}{3l^2-1} < \Delta.$$

The equation
$$\mathcal{H}_4:\ l(y_0-\Delta) = (x+ly-l)(x-y_0) \tag{13'}$$

is therefore that of a hyperbola ($\mathcal{H}_4$) with asymptotes CD, $x = y_0$, and the inequality (13) is satisfied at all points lying inside the upper branch of $\mathcal{H}_4$. In particular (13) is satisfied if P is the point $K\{0,\ l(3l+1)/(3l^2-1)\}$ (i.e. the mid-point of ST) or the point S, for in either case it is equivalent to $y_0 > 2l(l+1)/(3l^2-1)$, which is true since J (the upper of the two points I, J) lies above the line $y = 2l(l+1)/(3l^2-1)$; it follows that K and S lie inside the upper branch of $\mathcal{H}_4$ (it is clear that they do not lie inside the lower branch).

The upper branch of $\mathcal{H}_4$ meets the line BC in points given by

$$2ly^2 - 2(2l+y_0)y + (2l+y_0+\Delta) = 0. \tag{14}$$

The reality of the roots of (14) is implied by (15). The point $J(ly_0-l,\ y_0)$ will lie between these two points or coincide with one of them provided that

$$2(l-1)y_0^2 - (4l-1)y_0 + (2l+\Delta) \leqslant 0, \tag{15}$$

i.e. provided that
$$y_0 \geqslant \frac{l^2+3l+\Delta(l^2+2l-1)/l}{l^2+4l+1}$$

(since y_0 satisfies (11)). Since

$$y_0 = \frac{(2l^2+l+1)+\sqrt{(2l^2+l+1)^2-2(l^2+1)\{2l^2+l+\Delta(l-l^{-1})\}}}{2(l^2+1)}$$
$$= \frac{(2l^2+l+1)+\sqrt{(l+1)(2l^2-l+1)-2(l^4-1)\Delta/l}}{2(l^2+1)},$$

it follows that (15) is satisfied provided that

$$\sqrt{(l+1)(2l^2-l+1)-2(l^4-1)\Delta/l}$$
$$\geqslant \frac{-15l^7-5l^6+63l^5+53l^4+11l^3-7l^2-3l-1}{(3l^2-1)^2(l^2+4l+1)}. \tag{16}$$

This inequality will be satisfied if

$$27l^{14}+72l^{13}+69l^{12}-104l^{11}-345l^{10}-736l^9-887l^8$$
$$-848l^7-559l^6-248l^5-17l^4+56l^3+45l^2+16l+3\geqslant 0. \tag{17}$$

Since the left-hand side of (17) is the product of

$$3l^6+4l^5-7l^4-24l^3-7l^2+4l+3$$

and
$$9l^8+12l^7+28l^6+28l^5+30l^4+20l^3+12l^2+4l+1$$

it follows that (17) is true, with equality only if $\theta=\theta_0$ (when the first factor vanishes). Since

$$-15l^7-5l^6+63l^5+53l^4+11l^3-7l^2-3l-1$$
$$=-5(l-1)(3l^6+4l^5-7l^4-24l^3-7l^2+4l+3)$$
$$+8(l^5-4l^4+12l^3+6l^2-l-2)$$

and
$$l^5-4l^4+12l^3+6l^2-l-2$$
$$=(l-2)\{l^3(l-2)+8l^2+22l+43\}+84$$

is positive when $l=l_0$, it follows that the right-hand side of (16) is positive when $l=l_0$ and so there is equality in (15) if and only if $\theta=\theta_0$.

Now if $\theta_0<\theta<22\frac{1}{2}°$ it follows from what has already been proved that

(i) every point of the triangle SKJ lies inside the upper branch of $\mathcal{H}_4$ (since S, K and J lie inside it);

(ii) every point of the triangles JKE, EJI, except for the vertices E, I, J, lies inside the upper branch of $\mathcal{H}_3$ (since E, I, J lie on and K lies inside the upper branch of $\mathcal{H}_3$).

Therefore the region defined by

$$0\leqslant x\leqslant l(y-1),\quad \frac{2l(l+1)}{3l^2-1}<y<\frac{l(3l+1)}{3l^2-1}$$

(i.e. that part of $\mathfrak{R}_1$ which does not lie in $\mathfrak{R}_2$ and for which $x\geqslant 0$) lies either inside the upper branch of $\mathcal{H}_3$ or inside the upper branch of $\mathcal{H}_4$. This still holds if $\theta=\theta_0$ except that the point J now lies on both $\mathcal{H}_3$ and $\mathcal{H}_4$.

7. *Proof of Theorem II.*

Suppose firstly that P lies in $\mathfrak{R}_2$. If P is distinct from E then $WZ > 3\,OP$ and $XY < 2\,OP$. To prove the first of these it suffices to show that

$$\frac{\Delta + l y_1}{l y_1 - x_1} + \frac{\Delta + l y_1}{l y_1 + x_1} > 3\, y_1;$$

this is so since

$$2l(\Delta + l y_1) > 3(l y_1 - x_1)(l y_1 + x_1)$$

by (8). The proof that $XY < 2\,OP$ follows in the same way from (9). Thus there is a point (other than O) of Λ in the interior of $\mathfrak{R}$. If P coincides with E then W, X, Y, Z are the images of R, S, T, U respectively in the line $x = y$ and

$$WX = XY = YZ = OP.$$

It follows either that there is a point (other than O) of Λ in the interior of $\mathfrak{R}$ or that Λ is the critical lattice Λ_2'.

Assume henceforth that P is any point of $\mathfrak{R}_1$ which does not lie in $\mathfrak{R}_2$ and for which $x_1 \geqslant 0$.

Suppose secondly that (12) *is satisfied.* It follows that $YZ > OP$ and so there is a point (other than O) of Λ in the interior of $\mathfrak{R}$. This completes the proof if $\theta = 22\tfrac{1}{2}°$. It also completes the proof if $y_1 < y_0$ (i.e. if P lies in $\mathfrak{R}_3$) or if $x_1 = 0$; for it follows from the last paragraph of section 6 that (12) is satisfied at every point (other than the vertices E, I, J) of the quadrilateral $KEIJ$.

Suppose thirdly that (13) *is satisfied and that $x_1 > 0$.* Then the point $Z + P$ lies either in the interior of $\mathfrak{R}$ or in the image of $\mathfrak{R}_3$ in the line $x = y$. For the abscissa of $Z + P$ is

$$\frac{l(\Delta - x_1)}{l y_1 + x_1} + x_1,$$

which is $< y_0$ by (13), and the ordinate of $Z + P$ is $< -1 + l(3l + 1)/(3l^2 - 1) < 1$ (this inequality is necessary to ensure that the point $Z + P$ does not lie above $\mathfrak{R}$). It follows that there is a point (other than O) of Λ either in the interior of $\mathfrak{R}$ or in the image of $\mathfrak{R}_3$ in the line $x = y$. If the second alternative holds, the proof of the theorem is completed by appealing to the proofs covering the first two cases.

Suppose lastly that $\theta = \theta_0$ and P coincides with J. In this case $YZ = OP$ and either there is a point (other than O) of Λ in the interior of $\mathfrak{R}$ or Λ is the critical lattice Λ_3' defined in Theorem III. To verify this second statement it is sufficient to show that the angle POY is now a right-angle and so P and Y are now the mid-points of two sides of a square (one of whose vertices is Z) whose vertices and the mid-points of whose sides lie on the boundary of $\mathfrak{R}$. That the angle POY is now right-angle follows from the fact that (12) and (13) are now equalities; the geometrical significance of these equalities is that Y lies on the line $x = y_0$ as well as on the side AD' and so has co-ordinates $(y_0, \ -l y_0 + l)$. This gives the desired result since now $P = J(l y_0 - l, \ y_0)$.

Proof of Result for $15^\circ \leqslant \theta \leqslant \theta_0$

8. *Introduction* (see Diagram 6).

Theorem III. If $15^\circ \leqslant \theta \leqslant \theta_0$, *i.e. if* $\sqrt{3} \leqslant l \leqslant l_0$, *then the determinant of a critical lattice of* $\mathfrak{R}$ *is*

$$\Delta = \frac{l^2(l^2+4l+5)}{(l^2+2l-1)^2}.$$

Moreover the lattice Λ_3 *generated by*

$$E\left\{\frac{l(l+2)}{l^2+2l-1}, \ \frac{l}{l^2+2l-1}\right\}, \quad F\left\{\frac{l(l+1)}{l^2+2l-1}, \ \frac{l(l+3)}{l^2+2l-1}\right\},$$

and its image Λ_3' *in the line* $x = y$ *are critical. For* $15^\circ \leqslant \theta < \theta_0$ Λ_3, Λ_3' *are the only critical lattices; for* $\theta = \theta_0$ *there are two further critical lattices* Λ_2, Λ_2'.

See the enunciation of Theorem II for the co-ordinates of a pair of points generating Λ_2; Λ_2' is the image of Λ_2 in the line $x = y$.

The lattices Λ_3, Λ_3' can be regarded as being defined by squares whose vertices and the mid-points of whose sides lie on the boundary of $\mathfrak{R}$, and whose centres are at O. There are two, and only two, such squares, viz. $FHF'H'$ and $LNL'N'$, the nomenclature of the vertices being fixed by taking E as the mid-point of FH' and L, N' as the images of F, H respectively in the line $x = y$. Let E, G, E', G' denote the mid-points of $H'F$, FH, HF' $F'H'$ respectively and let K, M, K', M' denote the mid-points of $N'L$, LN, NL', $L'N'$ respectively.

Lemma 8. Λ_3 *is admissible for* $\mathfrak{R}$.

Proof. Λ_3 consists of the points $\alpha E + \beta G$ where α, β are integers. The required result follows after proving that

(i) if $\max(|\alpha|, |\beta|) \geqslant 3$ then the point $\alpha E + \beta G$ does not lie in $\mathfrak{R}$; and

(ii) if $\max(|\alpha|, |\beta|) = 2$ then the point $\alpha E + \beta G$ lies either outside $\mathfrak{R}$ or on the boundary of $\mathfrak{R}$.

The points for which $\max(|\alpha|, |\beta|) = 1$ all lie on the boundary of $\mathfrak{R}$.

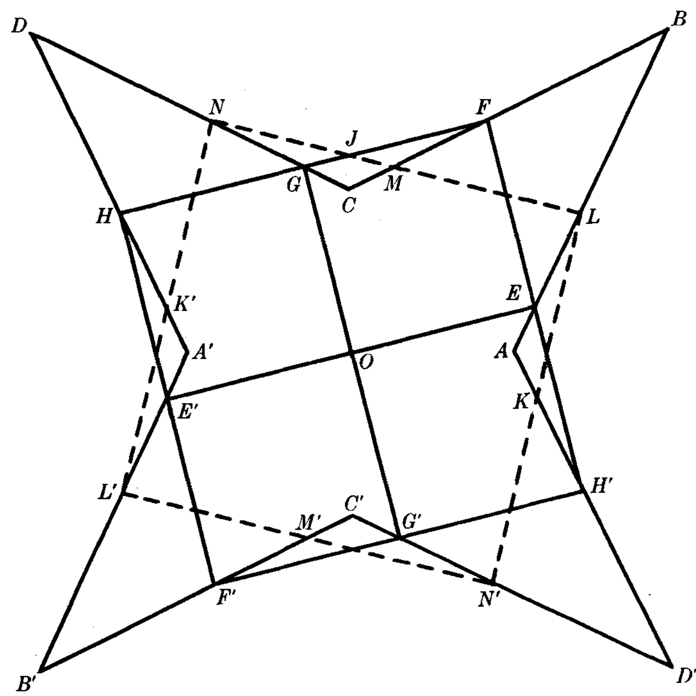

Diagram 6.

To show (i) it suffices to show that B lies to the left of the lattice line containing those lattice points for which $\alpha = 3$. This line has equation

$$(l+2)\,x + y = 3\,\frac{l\,(l^2+4\,l+5)}{l^2+2\,l-1},$$

and so it suffices to show that

$$3\,\frac{l\,(l^2+4\,l+5)}{l^2+2\,l-1} - (l+2)\,\frac{l}{l-1} - \frac{l}{l-1} > 0;$$

the left-hand side of this inequality is equal to $2\,l\,(l^3+2\,l^2-l-6)/(l-1)\,(l^2+2\,l-1)$

and $$l^3+2\,l^2-l-6 = l\,(l^2-1) + 2\,(l^2-3)$$

is positive for $l \geqslant \sqrt{3}$.

To show (ii) it suffices, by symmetry, to show that the points

$$\alpha E + \beta G \quad (\alpha = 2;\ \beta = -1, 0, 1, 2)$$

lie either outside $\mathfrak{R}$ or on the boundary of $\mathfrak{R}$. The points E and $E+G$ (i.e. F) lie on the boundary of $\mathfrak{R}$ and so the points $2\,E$ and $2\,E+2\,G$ lie ouside $\mathfrak{R}$. The abscissa of the point $2E-G$ or $3E-F$ is $l(2l+5)/(l^2+2l-1)$, which is greater than $l/(l-1)$; therefore $2E-G$ lies to the right of BD' and so outside $\mathfrak{R}$. There remains the point $2E+G$ or $E+F$

$$\left\{\frac{l(2l+3)}{l^2+2l-1},\ \frac{l(l+4)}{l^2+2l-1}\right\}.$$

This lies on or to the right of the side AB according as $l=$ or $>\sqrt{3}$.

This completes the proof of the lemma.

Let now Λ be any lattice of determinant Δ. I shall prove either that Λ is one of the critical lattices mentioned in the enunciation of Theorem III, or that Λ contains a point (other than O) in the interior of $\mathfrak{R}$. Consider the parallelogram of area 4Δ defined by

$$|y+lx|<l, \quad |(l+2)y+x|\leqslant\frac{l(l^2+4l+5)}{l^2+2l-1};$$

the boundary of this parallelogram consists of the lines AD', $A'D$ and LN, $L'N'$. By Minkowski's theorem this parallelogram contains a primitive point Q_1 of Λ. If Q_1 is an interior point of $\mathfrak{R}$ there is no more to prove. Otherwise it can, without loss of generality, be taken as lying inside or on the boundary of the triangle CMN. In a similar way it is possible to show that, if Λ is admissible for $\mathfrak{R}$, there exist primitive points Q_2, Q_3, Q_4 of Λ inside or on the boundaries of the triangles CFG, AKL, $AH'E$ respectively.

LEMMA 9. *If Λ is admissible for $\mathfrak{R}$ and is distinct from Λ_3, Λ_3', then the points Q_1, Q_2, Q_3, Q_4 are not all distinct.*

Proof. Suppose the points Q_1, Q_2, Q_3, Q_4 are all distinct. I show first that O, Q_1, Q_2 are not collinear. For if they are, the co-ordinates (x', y') of the lattice point Q_1-Q_2 satisfy the inequalities

$$|x'|\leqslant\frac{l(l+1)}{l^2+2l-1} \quad \text{(i.e. the abscissa of } F\text{)}$$

and

$$|y'|\leqslant\frac{l(l+3)}{l^2+2l-1}-1 \quad \text{(i.e. the ordinate of } F-C\text{)}.$$

Therefore

$$|x'|<1, \quad |y'|<1$$

and so the point Q_1-Q_2 is a point (other than O) of Λ in the interior of $\mathfrak{R}$; this contradicts the hypothesis on Λ. Similarly O, Q_3, Q_4 are not collinear.

I next prove that Q_1 cannot lie in the region $\mathfrak{R}_1$ formed by the interior and boundary of the quadrilateral $GCMJ$, where J is the point of intersection of LN,

FH. For if it did, the triangle OFG of area $\frac{1}{2}\Delta$ would contain two points Q_1, Q_2 of Λ where O, Q_1, Q_2 are not collinear; this is only possible if $Q_1 = G$, $Q_2 = F$ and so $\Lambda = \Lambda_3$. (1)

Similarly Q_2 cannot lie in $\mathfrak{R}_1$, and Q_3, Q_4 cannot lie in the image $\mathfrak{R}_1'$ of $\mathfrak{R}_1$ in the line $x = y$.

Let Q_1', Q_2' denote the images of Q_1, Q_2 in O. Then I show that Q_3, Q_4 lie strictly between the lattice lines $Q_1 Q_2$ and $Q_1' Q_2'$. For Q_1 lies on or below JN and to the left of the y-axis (because Q_1 lies inside or on the triangle CMN but not in $\mathfrak{R}_1$) while Q_2 lies above the line JM and to the right of the y-axis (because Q_2 lies inside or on the triangle CFG but not in $\mathfrak{R}_1$); therefore the lattice line $Q_1 Q_2$ passes above L. Similarly the lattice line $Q_1' Q_2'$ passes below H'.

Now the area of the triangle OQ_1Q_2 is $\frac{1}{2}\Delta$; for on the one hand its area does not exceed that of the triangle OFN, which is

$$\frac{l^2(l^2+4l+3)}{(l^2+2l-1)^2} < \Delta;$$

on the other hand its area is a positive integral multiple of $\frac{1}{2}\Delta$. Therefore Q_1, Q_2 generate the lattice Λ and all points of Λ lying strictly between the lattice lines $Q_1 Q_2$ and $Q_1' Q_2'$ lie on the line through O parallel to $Q_1 Q_2$, i.e. O, Q_3, Q_4 are collinear.

This contradiction completes the proof of the lemma.

Since Q_1, Q_2, Q_3, Q_4 are not all distinct it can, without loss of generality, be assumed that $Q_1 = Q_2 = P(x_1, y_1)$, say, and so there is a point $P(x_1, y_1)$ of Λ in $\mathfrak{R}_1$. As before I consider the lattice line

$$\lambda:\ xy_1 - x_1 y = \Delta.$$

Its intersections W, X, Y, Z (2) with the lines CB, BA, AD', $D'C'$ respectively are well-defined. For if $P(x_1, y_1)$ is any point of $\mathfrak{R}_1$ then

(i) $$l y_1 \pm x_1 \geqslant l > 0;$$

(ii) $$y_1 \pm l x_1 \geqslant \frac{l(l+2)-l^2}{l^2+2l-1},$$

since the right-hand side is the value of, say, $y - lx$ at the point

$$M\left\{\frac{l}{l^2+2l-1},\ \frac{l(l+2)}{l^2+2l-1}\right\},$$

and so $y_1 \pm l x_1 > 0$.

(1) I use here the *Lemma*. Let OAB be a triangle having one of its vertices at the origin and area $\frac{1}{2}\Delta$. Then if two points P, Q of a lattice Λ of determinant Δ lie inside or on the boundary of this triangle and if O, P, Q are not collinear it follows that either $P = A$, $Q = B$ or $P = B$, $Q = A$. Further P, Q generate Λ.

(2) The co-ordinates of W, X, Y, Z were given in Section 2.

Moreover, W, X, Y, Z are interior points of the sides CB, BA, AD', $D'C'$ respectively. For, if P is any point of $\mathfrak{R}_1$, $y_1 \pm x_1 \geqslant l(l+1)/(l^2+2l-1)$, since the right-hand side is the value of, say, $y-x$ at the point M; it follows that $y_1 \pm x_1 > \Delta(l-1)/l$. Also $|x_1| \leqslant l/(l^2+2l-1)$ (i.e. the abscissa of M) and $|y_1| \leqslant l(l^2+4l+5)/(l+2)(l^2+2l-1)$ (i.e. the ordinate of J) and so $|x_1| < \Delta$, $|y_1| < \Delta$. Therefore

$$0 < \frac{l(\Delta + x_1)}{ly_1 - x_1}, \quad \frac{l(\Delta - x_1)}{ly_1 + x_1} < \frac{l}{l-1}$$

(and so W, Z are interior points of the sides CB, $D'C'$),

and
$$1 < \frac{\Delta - lx_1}{y_1 - lx_1}, \quad \frac{\Delta + lx_1}{y_1 + lx_1} < \frac{l}{l-1}$$

(and so X, Y are interior points of the sides BA, AD').

The general idea of the proof of Theorem III is now as follows. I can, without loss of generality, assume that $x_1 \geqslant 0$. It will then be shown in Lemma 11 that, unless P is one of at most two points, its co-ordinates satisfy either the inequality (19) or the inequality (20). If (19) is satisfied then $YZ > OP$. If (20) is satisfied then $WZ > 3OP$ while, by Lemma 10, $XY < 2OP$. In either case it follows that there is a point (other than O) of Λ in the interior or $\mathfrak{R}$. The exceptional points mentioned above may lead to critical lattices.

9. LEMMA 10. *Let $P(x_1, y_1)$ be any point of $\mathfrak{R}_1$; then*

$$XY < 2OP.$$

Proof. I show first that X does not lie to the right of

$$L\left\{\frac{l(l+3)}{l^2+2l-1}, \quad \frac{l(l+1)}{l^2+2l-1}\right\},$$

i.e. that
$$\frac{\Delta - lx_1}{y_1 - lx_1} \leqslant \frac{l(l+3)}{l^2+2l-1}.$$

This inequality is equivalent to

$$(l+3)y_1 - (l+1)x_1 \geqslant \frac{l(l^2+4l+5)}{l^2+2l-1}, \tag{18}$$

which holds for all points P of $\mathfrak{R}_1$ since

(i) the lines $(l+3)y - (l+1)x = \text{constant}$ have gradient $(l+1)/(l+3)$ and

$$(l+1)/(l+3) \geqslant 1/l \qquad \text{(i.e. the gradient of } CB\text{)}$$

for $l \geqslant \sqrt{3}$; and

(ii) the right-hand side of (18) is the value of

$$(l+3)\,y-(l+1)\,x \quad \text{at the point } M.$$

Similarly Y does not lie to the right of H'.

It follows that $XY \leqslant LH' = 2l(l+1)/(l^2+2l-1) < 2 \leqslant 2OP$.

LEMMA 11. *Let $P(x_1, y_1)$ be any point of $\mathfrak{R}_1$ for which $x_1 \geqslant 0$. Then either*

$$l^2-\Delta(l^2-1) > (ly_1+x_1-l)(y_1+lx_1-l) \tag{19}$$

or

$$2l(\Delta+ly_1) > 3(ly_1-x_1)(ly_1+x_1), \tag{20}$$

except when (i) *P is the point*

$$M\left\{\frac{l}{l^2+2l-1},\ \frac{l(l+2)}{l^2+2l-1}\right\},$$

in which case (19) *becomes an equality and* (20) *is false; or* (ii) *$\theta=\theta_0$ and P is the point*

$$\left\{0,\ \frac{2l(l+1)}{3l^2-1}\right\},$$

in which case (19) *and* (20) *both become equalities.*

(The inequalities (19), (20) show that $YZ > OP$, $WZ > 3OP$ respectively.)

Proof. Consider the equation

$$\mathcal{H}_1:\ l^2-\Delta(l^2-1) = (ly+x-l)(y+lx-l). \tag{19'}$$

Now
$$l^2-\Delta(l^2-1) = -2l^2(l^2-3)/(l^2+2l-1)^2$$

and so is $<$ or $=0$ according as $l >$ or $=\sqrt{3}$. Therefore (19′) is, for $15^\circ < \theta \leqslant \theta_0$, the equation of a hyperpola with asymptotes CD, AD' and, for $\theta = 15^\circ$, that of the two straight lines CD, AD'. This "hyperbola" ($\mathcal{H}_1$) passes through M and intersects CM in a further point U given by

$$y=\frac{l^4+3l^3-l-1}{(l^2+1)(l^2+2l-1)}.$$

U lies strictly between C and M unless $\theta = 15^\circ$, when it coincides with C. Further $\mathcal{H}_1$ intersects the positive y-axis in points R, S (R being the lower) given by

$$y^2-(l+1)\,y+\frac{l(l^2-1)(l^2+4l+5)}{(l^2+2l-1)^2}=0, \tag{21}$$

and R, S lie one on each side of J; for when

$$y=\frac{l(l^2+4l+5)}{(l+2)(l^2+2l-1)} \quad \text{(i.e. the ordinate of } J)$$

the left-hand side of (21) is the equal to $-2l(l^2+4l+5)/(l+2)^2(l^2+2l-1)^2<0$ (this verifies incidentally that the roots of (21) are, in fact, real and so positive). The inequality (19) is satisfied at all points P lying inside the "upper branch" of $\mathcal{H}_1$.

Now consider the equation

$$\mathcal{H}_2:\ \tfrac{2}{3}l\Delta+\tfrac{1}{9}l^2=(ly-x-\tfrac{1}{3}l)(ly+x-\tfrac{1}{3}l). \qquad (20')$$

This is the equation of a hyperbola ($\mathcal{H}_2$) with asymptotes $ly\pm x=\frac{1}{3}l$. The inequality (20) is satisfied at all points lying between the upper branch of $\mathcal{H}_2$ and its asymptotes.

$\mathcal{H}_2$ meets the positive y-axis in a point T given by the positive root of

$$y^2-\frac{2}{3}y-\frac{2\Delta}{3l}=0. \qquad (22)$$

T lies between R and S if the positive root of (22) lies between the two roots of (21) (which are both positive). This is so if (1)

$$3l^6+4l^5-7l^4-24l^3-7l^2+4l+3<0,$$

which holds when $15^\circ\leqslant\theta<\theta_0$. If $\theta=\theta_0$ T coincides with one of the two points R, S; it is implicit in what immediately follows that it must coincide with the lower of these two points, namely R.

$\mathcal{H}_2$ intersects the line BC in the point V given by $y=\Delta/2l+3/4$.

Now
$$\frac{l^4+3l^3-l-1}{(l^2+1)(l^2+2l-1)}<\frac{\Delta}{2l}+\frac{3}{4}\leqslant\frac{l(l+2)}{l^2+2l-1};$$

the first inequality follows since it is equivalent to

$$\begin{aligned}0&>l^6+6l^5+3l^4-28l^3-29l^2-2l+1\\&=(l+1)(l^2+4l+1)(l^3+l^2-7l+1),\end{aligned}$$

and this is true since l^3+l^2-7l+1 increases for $l\geqslant\frac{1}{3}(-1+\sqrt{22})$ (which is the case if $l\geqslant\sqrt{3}$) and

$$l_0^3+l_0^2-7l_0+1<(2.01)^3+(2.01)^2-7(2)+1<0;$$

the second inequality follows since it is equivalent to

$$0\leqslant l^4+2l^3-2l^2-6l-3=(l^2-3)(l^2+2l+1).$$

In other words V lies between U and M or coincides with M according as $l>$ or $=\sqrt{3}$.

(1) If each of the two quadratic equations

$$ay^2+by+c=0,\quad a'y^2+b'y+c'=0$$

has distinct real roots, then the roots "interlace" provided that

$$(ac'-a'c)^2-(ab'-a'b)(bc'-b'c)<0.$$

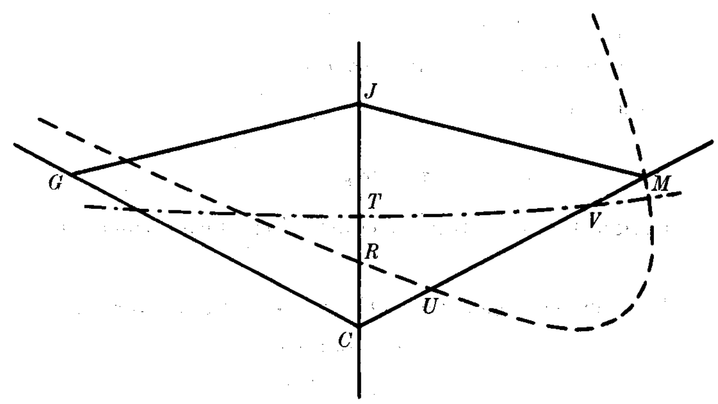

Diagram. 7.

The hyperbola $\mathcal{H}_1$ – – – – –
The hyperbola $\mathcal{H}_2$ – · – · – · – · –

Now suppose that $15^\circ < \theta < \theta_0$ and consider a point moving along the upper branch of $\mathcal{H}_2$; for numerically large values of the abscissa it does not lie inside the upper branch of $\mathcal{H}_1$ but at the points T, V it does. The upper branches of $\mathcal{H}_1$, $\mathcal{H}_2$ intersect in at most two points; for from the position of the asymptotes it follows that the lower branches of $\mathcal{H}_1$, $\mathcal{H}_2$ intersect in at least one point, and further that $\mathcal{H}_1$, $\mathcal{H}_2$ intersect "at infinity". Therefore the upper branches of $\mathcal{H}_1$, $\mathcal{H}_2$ intersect in precisely two points; further one of these points lies to the left of T and the other to the right of V. It follows that all points on the arc TV of $\mathcal{H}_2$ lie inside the upper branch of $\mathcal{H}_1$. Therefore every point lying inside or on the boundary of the triangle CMJ, with the exception of M, lies either inside the upper branch of $\mathcal{H}_1$ or between the upper branch of $\mathcal{H}_2$ and its asymptotes. The argument in this paragraph completes the proof of the lemma if $15^\circ < \theta < \theta_0$. Slight modifications are required if $\theta = 15^\circ$ or if $\theta = \theta_0$; further, if $\theta = \theta_0$, T and R coincide in the point

$$\{0,\ 2l(l+1)/(3l^2-1)\}$$

mentioned in the enunciation of the lemma (this can be verified by using (1)).

10. *Proof of Theorem III.*

If (19) is satisfied then, as before, $YZ > OP$, and if (20) is satisfied then $WZ > 3OP$. Since, by Lemma 10, $XY < 2OP$ it follows that, unless P is one of the exceptional points mentioned in the enunciation of Lemma 11, there is a point (other than O) of Λ in the interier of $\mathfrak{R}$. If P coincides with M then $YZ = OP$ and either there is a point (other than O) of Λ in the interior of $\mathfrak{R}$ or Λ is the critical lattice

Λ_3'. Finally, if $\theta=\theta_0$ and P is the point $\{0, 2l(l+1)/(3l^2-1)\}$ then $WX=XY=YZ=OP$ and either there is a point (other than O) of Λ in the interior of $\mathfrak{R}$ or Λ is the critical lattice Λ_2'.

Retrospect

11. In this section I give a brief account of the ideas which suggested the above conclusions.

Mordell [3] had shown that for $30^\circ \leqslant \theta < 45^\circ$ there were two critical lattices (Λ_0, Λ_0') which could be regarded as being defined by squares whose vertices and the mid-points of whose sides lay on the boundary of $\mathfrak{R}$ (compare Diagram 6). For $30^\circ < \theta < 45^\circ$ these were the only critical lattices, but at $\theta = 30^\circ$ two further critical lattices (Λ_1, Λ_1') appeared. Λ_1 could be regarded as being defined by A and the line parallel to the x-axis which had an intercept equal OA made on it by the sides CB, CD (see Diagram 2). It seemed reasonable to assume that the lattice Λ_1, so defined would be critical for θ sufficiently near to and less than 30°. Λ_1 was, in fact, admissible for $\theta \geqslant 22\frac{1}{2}^\circ$ (see Lemma 1), and so the result of Theorem I was suggested.

At $\theta = 22\frac{1}{2}^\circ$ the points $L+A$, $M-A$ of Λ_1 (see Diagram 2) came on to the boundary of $\mathfrak{R}$ and Λ_1 could then be regarded as being defined by the line parallel to the x-axis which had equal intercepts made on it by the sides AB, BC, CD, DA'. This suggested a definition of a critical lattice (Λ_2) for θ sufficiently near to and less than $22\frac{1}{2}^\circ$ (see Diagram 4).

Now the lattices defined by squares whose vertices and the mid-points of whose sides lay on the boundary of $\mathfrak{R}$ had determinant

$$\frac{l^2(l^2+4l+5)}{(l^2+2l-1)^2};$$

therefore these lattices could only be admissible for $\mathfrak{R}$ if

$$\frac{l^2(l^2+4l+5)}{(l^2+2l-1)^2} \geqslant \frac{l^2}{l^2-1},$$

since the right-hand side of this inequality was the determinant of a critical lattice of one of the two intersecting parallelograms of which $\mathfrak{R}$ was composed (see Section 12). This inequality was equivalent to $l \geqslant \sqrt{3}$. These lattices (Λ_3, Λ_3') defined by squares were admissible for θ sufficiently near to and greater than 15°, and critical for $\theta = 15^\circ$. This suggested that Λ_3, Λ_3' were, in fact, critical for θ sufficiently near to and greater than 15°.

Finally Λ_3 had determinant

$$\frac{l^2(l^2+4l+5)}{(l^2+2l-1)^2},$$

and Λ_2 had determinant

$$\frac{2l^2(l+1)(3l+1)}{(3l^2-1)^2}$$

and

$$\frac{l^2(l^2+4l+5)}{(l^2+2l-1)^2} < \text{ or } > \frac{2l^2(l+1)(3l+1)}{(3l^2-1)^2}$$

according as

$$3l^6+4l^5-7l^4-24l^3-7l^2+4l+3 < \text{ or } > 0.$$

This suggested the definition of l_0.

The ideas outlined in the previous three paragraphs suggested the results of Theorems II and III.

In an attempt to find the results for values of l less than $\sqrt{3}$ I found those values of l for which the determinant of a critical lattice of $\mathfrak{R}$ was equal to that of one of the two intersecting parallelegrams of which $\mathfrak{R}$ was composed (see Section 12), and found a critical lattice for each of these values of l; it was then possible to make a reasonable conjecture as to a critical lattice for neighbouring values of each of these values of l. However, I could get no general proof of these conjectures.

A Further Result

12. Let L, M denote the points of intersection of $D'A$, DC and of BC, $B'A'$ respectively, and let L', M' denote their images in O (see Diagram 8). Then L is the point $\{l/(l+1),\ l/(l+1)\}$ and the star-shaped octagon is composed of the two intersecting parallelograms $DLD'L'$, $BMB'M'$; each of these parallelograms has area $4l^2/(l^2-1)$.

Suppose a lattice Λ of determinant $\Delta = l^2/(l^2-1)$ is admissible for the star-shaped octagon. It follows that it is admissible, and therefore critical, for each of its component parallelograms and critical for the star-shaped octagon. Since Λ is a critical lattice of each of the component parallelograms, the mid-points of at least one pair of opposite sides of each parallelogram must be lattice points; see Minkowski [2].

Suppose the mid-point $P\{l/l^2-1),\ l^2/(l^2-1)\}$ of BM is a lattice point and either

(i) the mid-point $Q\{-l/(l^2-1),\ l^2/(l^2-1)\}$ of LD is a lattice point; or

(ii) the mid-point $R\{l^2/(l^2-1),\ -l/(l^2-1)\}$ of LD' is a lattice point.

(i) and (ii) cannot both occur; for if they did L would be a lattice point and Λ would not be admissible for the star-shaped octagon.

Suppose firstly that (i) occurs; then, since O, P, Q are not collinear, the area of the triangle OPQ is $\frac{1}{2}n\Delta$, where n is a positive integer. Therefore l satisfies the equation

$$\frac{2l}{l^2-1}=n,$$

and so takes one of the values

$$\frac{1}{n}\{1+\sqrt{n^2+1}\} \qquad (n=1, 2, \ldots). \tag{23}$$

Since $PQ=2l/(l^2-1)$ it follows that $PQ=n$.

Conversely, if l takes one of the values (23), the lattice generated by A and P contains Q (since $QP=nOA$) and has determinant $l^2/(l^2-1)$. This lattice is admissible for the parallelogram $BMB'M'$ since it contains the mid-point of BM and a point (A) on BM'; similarly it is admissible for the parallelogram $DLD'L'$. Therefore the lattice is admissible and critical for the star-shaped octagon. It should be noted that this lattice and its image in the line $x=y$ are not necessarily the only critical lattices; thus, if $n=4$, giving $l=\frac{1}{4}(1+\sqrt{17})$, the lattice generated by $(2, 0)$ and $\{0, \frac{1}{4}(1+\sqrt{17})\}$ is also admissible and critical.

Suppose secondly that (ii) occurs; then, since O, P, R are not collinear, the area of the triangle OPR is $\frac{1}{2}n\Delta$, where n is a positive integer; in fact $n>1$, since the triangle OPR has area

$$\frac{l^2(l^2+1)}{2(l^2-1)^2}>\frac{l^2}{2(l^2-1)}=\tfrac{1}{2}\Delta.$$

It follows that l satisfies the equation

$$\frac{l^2+1}{l^2-1}=n,$$

and so l takes one of the values

$$\sqrt{(n+1)/(n-1)} \qquad (n=2, 3, \ldots). \tag{24}$$

PR intersects AB, $C'D'$ in the points

$$S\left\{\frac{l(l^3-l+2)}{l^4-1}, \ -\frac{l(l^2-2l-1)}{l^4-1}\right\}, \quad T\left\{\frac{l(2l^3-l^2+1)}{l^4-1}, \ -\frac{l^2(l^2+2l-1)}{l^4-1}\right\},$$

and $PS=RT=\dfrac{1}{n}PR$.

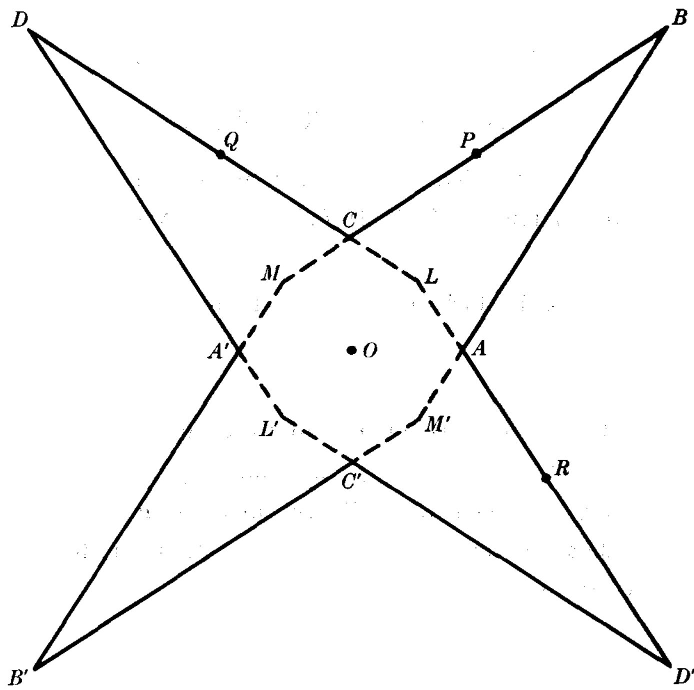

Diagram 8.

Conversely, if l takes one of the values (24), the lattice generated by S and P contains R and T. It is therefore admissible for the parallelograms $BMB'M'$, $DLD'L'$ and so admissible and critical for the star-shaped octagon.

The above results can be summed up in the following theorem.

THEOREM IV. *A necessary and sufficient condition that the determinant of a critical lattice of the star-shaped octagon is equal to the determinant of a critical lattice of one of its component parallelograms is that l takes one of the values*

$$\frac{1}{n}\left(1+\sqrt{n^2+1}\right) \qquad (n=1, 2, \ldots) \tag{23}$$

or

$$\sqrt{(n+1)/(n-1)} \qquad (n=2, 3, \ldots). \tag{24}$$

COROLLARY. *Let Δ denote the determinant of a critical lattice of the star-shaped octagon. Then for $n=2, 3, \ldots$*

$$\frac{1}{n-1}\left(1+\sqrt{n^2-2n+2}\right) \geqslant l \geqslant \sqrt{(n+1)/(n-1)}$$

implies

$$\tfrac{1}{2}\left(1+\sqrt{n^2-2n+2}\right) \leqslant \Delta \leqslant \tfrac{1}{2}(n+1),$$

and $$\sqrt{(n+1)/(n-1)} \geqslant l \geqslant \frac{1}{n}\,(1+\sqrt{n^2+1})$$

implies $$\tfrac{1}{2}\,(n+1) \leqslant \Delta \leqslant \tfrac{1}{2}\,(1+\sqrt{n^2+1}).$$

Proof. If $l_1 > l_2$ the star-shaped octagon corresponding to $l = l_1$ is entirely contained in that corresponding to $l = l_2$. It follows that Δ is non-decreasing with decreasing l.

Further $l^2/(l^2-1) = \tfrac{1}{2}\,(1+\sqrt{n^2+1})$ or $\tfrac{1}{2}\,(n+1)$ according as $l = \frac{1}{n}\,(1+\sqrt{n^2+1})$ or $\sqrt{(n+1)/(n-1)}$.

The above work formed part of my dissertation submitted for the degree of Doctor of Philosophy in the University of Cambridge. I should like to take this opportunity of thanking Professor L. J. Mordell and Mr. R. F. Churchouse for their many helpful and detailed comments, and Dr. C. S. Davis for reading an earlier draft version of sections 8–10.

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