ON NON-LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER: IV. THE GENERAL EQUATION $\ddot{y}+kf(y)\dot{y}+g(y)=b\,k\,p(\varphi),\quad \varphi=t+\alpha$

BY

J. E. LITTLEWOOD

in Cambridge

§ 1. We enter now on our complete account of the more general equation

$$\ddot{y}+k\,f(y)\,\dot{y}+g(y)=b\,k\,p(\varphi),\quad \varphi=t+\alpha.$$

The functions f, g, p are fixed, b is non-negative, and k is large and positive. We proceed to state the long list of assumptions about f, g, p. It may help towards easier reading to imagine that f and g are polynomials and p a trigonometrical polynomial: in so far as hypotheses about the smoothness of f, g, p are concerned our arguments are not essentially different from what they would then be, and the reader may trust us to have taken care of the details. He may similarly take on trust details about the constants connected with these functions, and the various appeals to the f, g, p dictionary (§ 3) that occur in the arguments.

p has continuous p'', is periodic with period normalized to 2π, has mean value 0, and is skew-symmetric, i.e. $p(\pi+\varphi)=-p(\varphi)$. Any integral $\int p\,d\varphi$ is periodic; we define $p_1(\varphi)$ be that one for which the mean value is 0. It also is skew-symmetric. It is now an essential assumption that p_1 *attains its upper* (*and consequently also its lower*) *bound once only in a period*. We normalize p to make 1 the upper bound of p_1, to be attained at $\frac{1}{2}\pi$. So $p_1(\frac{1}{2}\pi)=-p_1(-\frac{1}{2}\pi)=1$, $p(\pm\frac{1}{2}\pi)=0$. $p'(-\frac{1}{2}\pi)$ is non-negative; we suppose it a *positive* constant a_2.

$f(y)$ is even, with continuous f''. It has a single pair of zeros, normalized to ± 1; $f'(1)$ is a *positive* constant a_1, and f has a positive lower bound in (say) $y\geq 2$. We define $F(y)=\int_0^y f(y)\,dy$; F is odd. We normalize f to make $F(-1)$ (certainly positive) $\frac{2}{3}$. This will make $\frac{2}{3}$ the critical value of b, as for van der Pol's equation; the behaviour for $b>\frac{2}{3}$ is crude, and we suppose for simplicity that $0<b<2$ as before.

We define H, a constant >1, by $F(H)=F(-1)=\frac{2}{3}$.[1]

The final assumptions are about g: we suppose it odd, with continuous g''. We should in any case suppose that g' has a positive lower bound. In order to avoid certain complications we suppose $g'\geq 1$.[2]

§ 2. Constants L are throughout positive constants depending only on the functions f, g, p, and the constants implied in O's are always of type L.

Before going on we state the essential

LEMMA 1. *Suppose (as always) that* $0\leq b\leq 2$. *Then every trajectory* Γ *ultimately satisfies*

$$|y|\leq L_0^*, \qquad |\dot{y}|\leq L_0^* k$$

where we may suppose $L_0^*>20$, *say.*[3] *If it satisfies these at* $t=t_0$, *then it will satisfy* $|y|\leq L^*$, $|\dot{y}|\leq L^* k$ *for* $t\geq t_0$.

If $b>0$, Γ *(strictly) crosses* $y=0$ *infinitely often.*

This is proved (for still more general f, g, p) elsewhere.[4]

§ 3. In the light of Lemma 1 we define, slightly extending the natural meaning of the adjective, an "eventual" Γ to be one satisfying $|y|\leq L_0^*$, $|\dot{y}|\leq L_0^* k$ at the (arbitrary) origin of time $t=0$. It then satisfies $|y|\leq L^*$, $|\dot{y}|\leq L^* k$ for $t\geq 0$.

We may observe that this ultimate behaviour holds (for a suitable L_0^* and L^*) subject to very general conditions on f, g, p.[4] Once granted this, it is enough for our further purposes that the more stringent conditions we impose on f, g, p should hold in the restricted range $|y|\leq L^*$.

We give for convenience of reference a "dictionary" of f, g, p.

LEMMA 2. $p(\varphi)$[5] *has continuous* p''. *It has period* 2π *and mean value* 0, *and* $p(\varphi+\pi)=-p(\varphi)$. $p_1(\varphi)$, *the integral of* p *with mean value* 0, *has* $p_1(\varphi+\pi)=-p_1(\varphi)$. p_1 *attains its upper bound, which is* 1, *at* $\varphi\equiv\frac{1}{2}\pi$, *and nowhere else, and attains its lower bound* -1 *at* $\varphi\equiv-\frac{1}{2}\pi$ *and nowhere else.*

$$p(\pm\tfrac{1}{2}\pi)=0,\ p'(\pm\tfrac{1}{2}\pi)=\mp a_2,\ a_2>0;\ p_1(\pm\tfrac{1}{2}\pi)=\pm 1. \tag{1}$$

---

[1] In van der Pol's equation this critical height H has the value 2. We choose to normalize the critical b to $\frac{2}{3}$ rather than H to 2.

[2] Since the period is normalized to 2π this is a real restriction on one parameter of the equation and may be unnecessary. We could alternatively assume that $f'\geq 0$ in $1\leq y\leq H$.

[3] Constants with *'s (and with or without suffixes) are permanent (see § 4 below).

[4] See M. L. CARTWRIGHT and J. E. LITTLEWOOD, *Ann. of Math.*, 48 (1948).

[5] φ is the phase, and is of the form $t+\alpha$, since the period is 2π. We work sometimes in φ, sometimes in t. We have of course

$$p'=dp/d\varphi=dp/dt=\dot{p}.$$

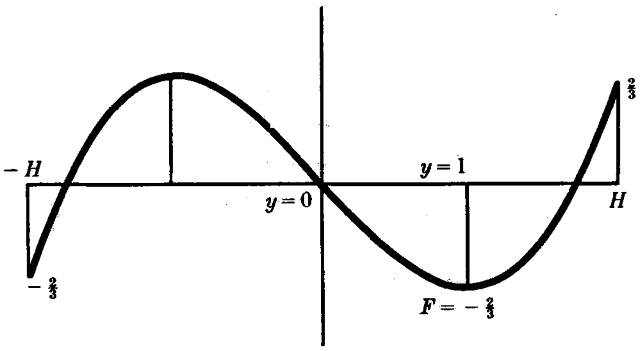

Fig. 1. Graph of $F(y)$.

For $|\psi|\leq\pi$ *we have*

$$|p(-\tfrac{1}{2}\pi+\psi)|\leq L|\psi|, \tag{2}$$

$$1+p_1(-\tfrac{1}{2}\pi+\psi)=\tfrac{1}{2}a_2\psi^2+O(\psi^3), \tag{3}$$

$$L\psi^2\leq 1+p_1(-\tfrac{1}{2}\pi+\psi)\leq L\psi^2, \tag{4}$$

with corresponding results for $p_1(\frac{1}{2}\pi+\psi)=-p_1(-\frac{1}{2}\pi+\psi)$.

f *is even, with continuous* f''. g *is odd, with continuous* g''.

$$f(\pm 1)=0,\qquad F(\mp 1)=\pm\tfrac{2}{3},\qquad f'(\pm 1)=\pm a_1,\quad a_1>0. \tag{5}$$

$$F(H)=F(-1)=\tfrac{2}{3},\quad H>1. \tag{6}$$

$$y=1+\eta;\qquad f(y)=a_1\eta+O(\eta^2)\quad (|y|\leq L^*);\qquad f(y)/\eta\gtrless L\quad (0\leq y\leq L^*). \tag{7}$$

$$\left.\begin{aligned}&F(y)-F(1)=\tfrac{1}{2}a_1\eta^2+O(\eta^3)\quad (|y|\leq L^*);\\ &F(y)-F(1)\geq 0\quad (y\geq -H);\\ &L\eta^2\leq F(y)-F(1)\leq L\eta^2\quad (-\tfrac{1}{2}(1+H)\leq y\leq L^*);\end{aligned}\right\} \tag{8}$$

$$g'' \text{ is continuous, and } 1\leq g'\leq L^*, \text{ for } |y|\leq L^*. \tag{9}$$

(1) is agreed, and (2), (3) follow from (1) and the continuity of p''. The second inequality in (4) is a trivial corollary of (3); the first, however, depends on the fact that the lower bound $p_1(-\frac{1}{2}\pi)$ is attained at $\varphi\equiv-\frac{1}{2}\pi$ only.

The results about f and F are either agreed, or simple consequences of (5) and the continuity of f''. (9) is agreed.

§ 4. Notation for upper bounds. We use L (as we said above) for positive constants depending only on the functions f, g, p; and we use $A(x, y, \ldots)$ for positive constants depending only on these functions and the $x, y, \ldots$ In the rather rare cases when A is used as an index [as $\log^A k$, or k^{-A}] it means a positive absolute constant. We use D for constants $A(\delta)$ depending on a δ whose rôle is similar to that of the δ in the Introduction[1] (§ 12). This δ is to be thought of as "small": it has in the end to be less than some definite L; we suppose always, and tacitly, that δ satisfies any inequality $\delta < L$ called for by the run of the argument. Each L, D, $A(\ )$, as it occurs will in general depend on previously occurring ones; the chain, e.g., of D's *could* be made one of explicitly defined constants.

Many L's and D's do not need identification. Where they need identification throughout a single argument we use *temporary* suffixes, restarting the suffixes again at 1 on the next occasion. We sometimes use dashes in the same way (when suffixes are too thick on the ground). Where constants need *permanent* identification we use stars (as well as suffixes): thus D_1^* (§ 20) is always the same D. Suffixes to things other than constants L, D are used in many distinct senses; we hope that these are sufficiently disparate not to be confused; our notational problems are very difficult.

The upper bounds implied in O's are always L's.

We have to employ Lemmas with undetermined non-negative or positive constants d, d'; these are blank cheques, constants chosen in different ways on different occasions; when chosen, they may be 0, or L, or D, but are always one of these. The assertions of the Lemmas, which involve such things as $A(d, d')$, $k_0(d, d')$, consequently involve D's at worst, when they actually come to be applied. (Indeterminate constants other than d, d' are sometimes used, but only temporarily and with *ad hoc* explanations.)

The constant b requires some discussion. It is always (as explained in § 1) subject to $0 \le b \le 2$, and for some results no restriction other than this is necessary. But both $b = 0$ and $b = \frac{2}{3}$ are generally critical, and bounds of various things depend on the nearness of b to 0 or $\frac{2}{3}$. Behaviour when $b = 0$ has considerable interest of its own, and our first intention was to introduce a second "δ", δ', and a hypothesis $b \ge \delta'$ in the relevant contexts. By leaving the orders of δ and δ' independent we should arrive at results which were at least pointers to the case of small b (the real answers probably require such b to be a function of k). The complications of having more than one δ, however, proved almost prohibitive, and we adopted the simplification of making all δ the same. It turned out in the end, however, that even the

[1] See Paper III *Acta Math.* Vol. 97 (1957). This paper will be referred to in future as the Introduction. Both papers are based on joint work with M. L. Cartwright.

assumption $b \geq \delta \, (\delta' = \delta)$ led to a very great increase of complication; and our final hypothesis (where the critical values 0, $\frac{2}{3}$ are relevant) is $b \in B$, where B is the range

$$\tfrac{1}{100} \leq b \leq \tfrac{2}{3} - \tfrac{1}{100}. \tag{1}$$

We regret this masking of behaviour for small b, but it seems the lesser evil.

When $b \in B$ an $A(b)$ or $A(b, \delta)$, if continuous in b, as it always is in practice, lies between two L's or D's respectively.

The dependence of $k_0 = k_0(x, y, \ldots)$ on constants (cf. Introduction §§ 5, 9) requires only a short explanation. $k_0(x, y, \ldots)$ is always an $A(x, y, \ldots)$ and depends only on f, g, p and the $x, y, \ldots$. Where we have *Lemmas* containing (undetermined) d's the k_0 naturally depends on thes d's. The k_0's of *Theorems* generally depend on δ, but never on undetermined parameters (which Theorems never contain).

A k_0 in a Lemma or Theorem is "sufficiently large". It has to be continually re-chosen as the argument proceeds. Suppose, for example, we have proved $X < D_1 k^{-\frac{1}{2}}$ where $k > k_0$. We then have, e.g., $X < k^{-\frac{1}{3}}$ for $k > k_0'$, where $k_0' = \max(k_0, D_1^6)$, and say "$X < D_1 k^{-\frac{1}{2}} < k^{-\frac{1}{3}}$ by re-choice of k_0". It would be intolerable to mention all the re-choices, and, once having directed attention strongly to the point, we shall more and more frequently suppose tacitly that any necessary rechoice is being made.

§ 5. We now seriously begin our long and intricate story, which, after the literary excursions of the Introduction, we shall not try to lighten. What we have aimed at is to make things as easy as may be for a reader who omits the *proofs* of the Lemmas (or merely skims them for the general idea) and concentrates on their statements (and of course the connecting explanations). We have taken pains to make the chain of statements as lucid and efficient as we can. Each Lemma of the chain, further, has almost always a self-contained proof; clumsinesses that happen inside these proofs do not carry over outside. Part of the plan is to collect all needed results of a similar kind into one Lemma at a time, and some of the Lemmas are long "dictionaries".

§ 6. LEMMA 3. (i) *Let $0 \leq b \leq 2$, and let d be a non-negative and d' a positive constant. Then there is a $k_0(d, d')$ such that when $k \geq k_0$, the following properties hold.*

Suppose that an eventual trajectory has a piece XYZ lying entirely in $y \geq 1 - dk^{-\frac{1}{2}}$; suppose also that (a) XY has time length at least d', (b) YZ contains a point at which $\varphi \equiv -\frac{1}{2}\pi$, (c) YZ has time length at least $k^{-\frac{1}{2}} \log k$. Then for any Q of YZ,

$$|\dot{y}| < A(d, d'), \qquad |\ddot{y}| < A(d, d')\, k^{\frac{1}{2}}, \qquad |\dddot{y}| < A(d, d')\, k; \tag{1}$$

$$\dot{y} f = b\, p(\varphi) + O(A(d, d')\, k^{-\frac{1}{2}}); \tag{2}$$

$$\dot{y} f = b\, p(\varphi) + O(A(d, d')\, k^{-1} |y - 1|^{-1}). \tag{3}$$

In the identity

$$F - F(1) = C + b(1 + p_1(\varphi)) - k^{-1} \int_0^t g\, dt - \dot{y}\, k^{-1} \tag{4}$$

we may substitute $\dot{y} = O(A(d, d'))$ *in the stretch* YZ.

(ii) *Let* $0 \le b \le 2$, *and suppose that* d *is a positive constant, and that* $k \ge k_0$, *where* k_0 *is a certain* $k_0(d)$. *Then at a* Q *that has been preceded by a piece of an eventual trajectory lying in* $y \ge 1 + d$, *and lasting a time* $k^{-1} \log^2 k$ *at least, we have*

$$|\dot{y}| < A(d), \qquad |\ddot{y}| < A(d), \qquad |\dddot{y}| < A(d);$$

with various consequences, e.g. (2) *is valid with error term improved to* $O(A(d)\, k^{-1})$, *or*

$$|\dot{y} f - b\, p(\varphi)| < A_1(d)\, k^{-1}. \tag{5}$$

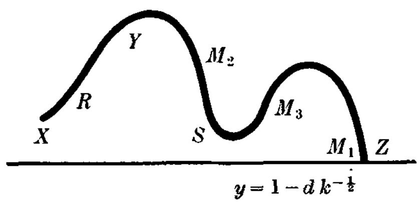

Fig. 2. M_1, M_2, M_3 correspond to cases (i), (ii), (iii) respectively.

We abbreviate constants $A(d, d')$ to Λ.

We begin by proving the result $|\dot{y}| < \Lambda$ in (i). On the trajectory we have always $|y| \le L_1$. In any piece of the trajectory of time interval d' $|\dot{y}|$ cannot *everywhere* exceed $2L_1/d'$. Hence (see fig. 2) there is an R of XY with $|\dot{y}_R| < L/d'$. Let $|\dot{y}|$ attain its maximum v for RZ at M. We may in what follows suppose that v is greater than any particular Λ that arises, since otherwise we have what we want. (In particular we systematically reject alternatives $v < \Lambda$ as they present themselves.) We suppose always $v > 1$.

We may suppose M not at R ($M = R$ would give what we want). By the hypothesis about YZ, RZ contains a point, S within 2π on one side or the other of M, for which $\varphi_S \equiv -\frac{1}{2}\pi$. Let $\varphi_M \equiv -\frac{1}{2}\pi + \psi$, where $|\psi| \le 2\pi$.

§ 7. We may suppose, by prolonging YZ to the next intersection[1] with the line $y=1-dk^{-\frac{1}{2}}$, that Z lies *on* this line (the hypotheses being satisfied *a fortiori* in the new case). We have now to distinguish three cases:

(i) M identical with Z (when $\dot{y}_M$ is negative by the geometry and $\dot{y}_M=-v$);

(ii) M is not Z, $\dot{y}_M=-v$;

(iii) M is not Z, $\dot{y}_M=+v$.

In cases (i) and (ii) $\dot{y}_M=-v$. From the $\dot{y}$-identity, writing g_1 for $\int g\,dt$, we have

$$\dot{y}_S-\dot{y}_M=-k\,(F(y_S)-F(1))+k\,(F(y_M)-F(1))+b\,k\,(p_1(\varphi_S)-p_1(\varphi_M))-[g_1]_M^S.$$

The left-hand side is $\dot{y}_S+v\geq 0$. On the right the first term is non-positive by Lemma 2 (8); the second is not greater than $L\,k\,\eta_M^2$ by Lemma 2 (8); the fourth is less than L since $|t_S-t_M|\leq 2\pi$. We have, therefore,

$$0\leq L\,k\,\eta_M^2+b\,k\,(p_1(\varphi_S)-p_1(\varphi_M))+L.$$

Now
$$-p_1(\varphi_S)+p_1(\varphi_M)=1+p_1(-\tfrac{1}{2}\pi+\psi)\geq L\,\psi^2,$$

by Lemma 2 (4). Hence

$$b\,\psi^2\leq L\eta_M^2+L\,k^{-1}\quad \textit{in cases (i) and (ii).} \tag{1}$$

Next, in either of the cases (ii), (iii), M is strictly interior to RZ, and consequently $\ddot{y}_M=0$; whence by substitution in the differential equation

$$\dot{y}_M\,f(y_M)=b\,p\,(\varphi_M)-g\,(y_M)\,k^{-1}\quad \textit{in cases (ii) and (iii).} \tag{2}$$

By (7) and (2) of Lemma 2, and since $\varphi_M=-\frac{1}{2}\pi+\psi$, we have

$$v\,|\eta_M|<b\,L\,|\psi|+L\,k^{-1}\quad \textit{in cases (ii) and (iii).} \tag{3}$$

In case (ii) we have (1) and (3), and so also

$$v^2\,\eta_M^2<b^2\,L^2\,\psi^2+L^2\,k^{-2}<L\,b\,\psi^2+L\,k^{-1}<L\,(L\,\eta_M^2+L\,k^{-1})+L\,k^{-1}<L_1\,\eta_M^2+L\,k^{-1}.$$

Since we may suppose $v^2>2\,L_1$, this gives

$$|\eta_M|<L\,k^{-\frac{1}{2}},$$

and *a fortiori* $|\eta_M|<\Lambda\,k^{-\frac{1}{2}}$. This last inequality, just proved for case (ii), is true also in case (i) (when $\eta_M=\eta_Z=-d\,k^{-\frac{1}{2}}$). In either case we now have $b\,\psi^2<\Lambda\,k^{-1}$.

---

[1] This need not happen *immediately*.

Hence, summing up:

$$\textit{In cases (i) and (ii)} \begin{cases} \dot{\eta}_M = -v, \\ b\psi^2 < \Lambda k^{-1}, \\ |\eta_M| < \Lambda k^{-\frac{1}{2}}. \end{cases} \tag{4}$$

We continue to take cases (i) and (ii) together, and now consider the reversed motion (r.m.) from M; if τ is its time variable we have $t = \varphi_M - \tau$. The $\dot{y}$-identity for this r.m. is

$$-\frac{dy}{d\tau} = -v - k(F(y) - F(1)) + k(F(y_M) - F(1)) +$$
$$+ bk(p_1(-\tfrac{1}{2}\pi + \psi - \tau) - p_1(-\tfrac{1}{2}\pi + \psi)) + \int_0^\tau g\, d\tau,$$

with $y(0) = 1 + \eta_M$, $(dy/d\tau)_0 = v$. We write $\eta = y - 1 = k^{-\frac{1}{2}} z$, $\tau = k^{-\frac{1}{2}} x$, $P = P(z) = k(F(y) - F(1))$, $P_0 = P(z_0)$. Then $P, P_0 \geq 0$, $z_0 = k^{\frac{1}{2}} \eta_M > -\Lambda$. The differential equation becomes

$$\frac{dz}{dx} = v + P - P_0 + bkp(-\tfrac{1}{2}\pi + \psi)\tau + O(k\tau^2) + O(\tau).$$

Now by (4) and Lemma 2 (2), $bkp(-\frac{1}{2}\pi + \psi) = O(k\psi) = O(k^{\frac{1}{2}}\Lambda)$; and when we substitute from this and for τ the last differential equation becomes

$$\frac{dz}{dx} = v + P - P_0 + O(\Lambda x) + O(\Lambda x^2). \tag{5}$$

Let $X = \log^{\frac{1}{2}}(v+2)$; then $X < L \log^{\frac{1}{2}} k$, and (since the r.m. lasts a time $k^{-\frac{1}{2}} \log k$, which corresponds to a range $\gamma^{-1} \log k$ of x, without y reaching L^*) (5) has a solution in $0 \leq x \leq X$ that is bounded by $Lk^{\frac{1}{2}}$.

We show next that either $v <$ a certain Λ, as desired, or else dz/dx, initially positive, remains positive throughout $0 \leq x \leq X$. If dz/dx ever vanishes, let it vanish *first* at $x = \xi \leq X$; then in $(0, \xi)$ $z \geq z_0 > -\Lambda$. Now if $z_0 < 0$, then $P - P_0 \geq -P_0 > -\Lambda$; and if $z_0 \geq 0$, then $P - P_0 = \int_{z_0}^{z} (\textit{positive})\, dz \geq 0$; in either case

$$\frac{dz}{dx} > v - \Lambda - \Lambda\xi - \Lambda\xi^2 > v - \Lambda - \Lambda X^2 > v - \Lambda - \Lambda \log(v+2),$$

which is positive at $x = \xi$ (contrary to hypothesis) unless $v < \Lambda$.

We may suppose, then, that $dz/dx > 0$ and $z \geq z_0$ in $(0, X)$. In this range we have certainly $-1 < y < L^*$, and so, by Lemma 2 (8),

$$Lz^2 \leq P \leq Lz^2.$$

It follows now that
$$P-P_0>\begin{cases} Lz^2-\Lambda & (z_0<0),\\ L(z-z_0)^2 & (z_0\geq 0).\end{cases} \tag{6}$$

For if $z_0<0$, then $z_0=O(\Lambda)$, $P_0<\Lambda$, and $P-P_0>Lz^2-\Lambda$. If $z_0\geq 0$, then

$$P-P_0=k\int_{\eta_M}^{\eta} f\,d\eta \geq k\int_{\eta_M}^{\eta} L\eta\,d\eta,$$

by Lemma 2 (7), and so

$$P-P_0\geq Lk(\eta^2-\eta_M^2)=L(z^2-z_0^2)\geq L(z-z_0)^2,$$

since $z\geq z_0$; and this completes the proof of (6).

From (5) and (6) we have in $(0, X)$ for the case $z_0<0$,

$$\frac{dz}{dx}>v-\Lambda-\Lambda X-\Lambda X^2+Lz^2>v-\Lambda-\Lambda\log(v+2)+Lz^2, \tag{7}$$

and a similar inequality with $L(z-z_0)^2$ in place of the last term for the case $z_0\geq 0$.

Now either v is less than a certain Λ, as desired, or else (7) gives, in, e.g., the case $z_0<0$,

$$\frac{dz}{dx}>\tfrac{1}{2}v+Lz^2,$$

and then
$$\log^{\frac{1}{2}}(v+2)=X=\int_0^X dx\leq\int_{z_0}^{z}\frac{dz}{\frac{1}{2}v+Lz^2}<\left|\int_0^{z_0}\right|+\left|\int_0^{\infty}\right|<2\int_0^{\infty}=Lv^{-\frac{1}{2}},$$

and $v<\Lambda$. In the case $z_0\geq 0$, the alternative to $v<\Lambda$ is

$$\frac{dz}{dx}>\tfrac{1}{2}v+L(z-z_0)^2,$$

and the rest is much as before.

We have now proved $|\dot{y}|<\Lambda$ in cases (i) and (ii).

§ 8. It remains to consider case (iii), in which $y_M=v$: here we have to pay close attention to *signs* (of ψ and η_M).

We recall the identity (2) of § 7 [valid for case (iii)].

$$\dot{y}_M f(y_M)=b\,p(\varphi_M)-g(y_M)\,k^{-1}.$$

This gives, by Lemma 2 (7) (whatever the sign of η_M)

$$b\,p(\varphi_M)<Lv\eta_M+Lk^{-1}, \tag{1}$$

(algebraically, note) and also

$$v\,|\eta_M| < L\,b\,|p(\varphi_M)| + L\,k^{-1}, \tag{2}$$

$$< L\,b\,|\psi| + L\,k^{-1}, \tag{3}$$

by Lemma 2 (2) (since $\varphi_M = -\tfrac{1}{2}\pi + \psi$).

In the $\dot y$-identity between M and S of § 7 we have now for the left-hand side $\dot y_S - \dot y_M$ the lower bound $-2v$ in place of the original 0; the conclusion (1) of § 7 is consequently modified to

$$b\,\psi^2 < L\,\eta_M^2 + L\,v\,k^{-1}. \tag{4}$$

Combining this with (3) (and using $b < L$, $v > 1$) we have

$$v^2\,\eta_M^2 < L_1\,\eta_M^2 + L\,v\,k^{-1},$$

and unless $v^2 \le 2\,L_1$, which we can reject, we have

$$v^2\,\eta_M^2 < L\,v\,k^{-1},\quad |\eta_M| < L\,(v\,k)^{-\frac{1}{2}}. \tag{5}$$

We prove next that either v *is less than a certain* Λ (which we reject), *or else* $\dot y > 0$ *for a time interval* $k^{-\frac{1}{2}}$ *before* M. Suppose the second alternative false; then there is a stationary point Σ, with $\dot y = 0$, at time $\tau \le k^{-\frac{1}{2}}$ before M, and we may suppose it the nearest such point to M. The point Σ is in XM (since XM has time-length at least $d' > k^{-\frac{1}{2}}$); hence $y_\Sigma \ge 1 - d\,k^{-\frac{1}{2}}$, $-d\,k^{-\frac{1}{2}} \le \eta_\Sigma \le \eta_M$, and so $\eta_\Sigma^2 \le \eta_M^2 + d^2\,k^{-1}$. Consequently

$$0 - v = \dot y_\Sigma - \dot y_M = -\,k\,(F(y_\Sigma) - F(1)) + k\,(F(y_M) - F(1)) + b\,k\,(p_1(\varphi_M - \tau) - p_1(\varphi_M)) + [g_1]_\Sigma^M$$

$$\ge -L\,k\,\eta_\Sigma^2 + 0 + b\,k\,(-\tau\,p(\varphi_M) - L\,\tau^2) + 0,$$

$$b\,k\,\tau\,p(\varphi_M) \ge v - L\,(k\,\eta_M^2 + \Lambda) + 0 - b\,k\,L\,k^{-1} + 0 \ge v - L\,v^{-1} - \Lambda,$$

by (5). Unless v is less than a certain Λ, which we reject, this is greater than $\tfrac{1}{2}v$, and then

$$b\,p(\varphi_M) \ge \tfrac{1}{2}\,v\,k^{-1}\,\tau^{-1} \ge \tfrac{1}{2}\,v\,k^{-\frac{1}{2}}. \tag{6}$$

On the other hand (1) and (5) give

$$b\,p(\varphi_M) < L\,v^{\frac{1}{2}}\,k^{-\frac{1}{2}} + L\,k^{-1}$$

which contradicts (6) unless $v < L$, which we can reject. Then $\dot y > 0$ for an interval $k^{-\frac{1}{2}}$ before M, as stated.

By (5) we now have, for $t_M - k^{-\frac{1}{2}} \le t \le t_M$,

$$-\,d\,k^{-\frac{1}{2}} \le \eta \le \eta_M < L\,v^{-\frac{1}{2}}\,k^{-\frac{1}{2}}. \tag{7}$$

The $\dot{y}$-identity between t and t_M gives

$$\begin{aligned}\dot{y}&=v+k(F(y_M)-F(y))+bk(p_1(\varphi)-p_1(\varphi_M))+[g_1]_t^{t_M}\\ &\geq v+k(F(y_M)-F(y))+bk(t-t_M)p(\varphi_M)-Lk(t-t_M)^2+0.\end{aligned}\tag{8}$$

We now distinguish the cases (i) $\eta>0$, (ii) $\eta\leq 0$, and prove in each case that $\dot{y}>\frac{1}{2}v$ at the point in question (of the $k^{-\frac{1}{2}}$ interval), or else $v<\Lambda$.

In (i) $y_M\geq y\geq 1$ and $F(y_M)-F(y)=\int_y^{y_M} f\,dy\geq 0.$

In (ii) $|\eta|\leq dk^{-\frac{1}{2}}$, and so

$$F(y_M)-F(y)=(F(y_M)-F(1))-(F(y)-F(1))\geq 0-L\eta^2,$$

by Lemma 2 (8), $\qquad > -\Lambda k^{-1}.$

This last inequality is therefore true in either case, and then (8) gives

$$\begin{aligned}\dot{y}&\geq v-\Lambda-Lk\cdot k^{-\frac{1}{2}}\,\mathrm{Max}\,(0,bp(\varphi_M))-L+0\\ &\geq v-\Lambda-Lk^{\frac{1}{2}}v|\eta_M|,\end{aligned}$$

by (1), $\qquad \geq v-\Lambda-Lv^{\frac{1}{2}},$

by (5), $\qquad >\frac{1}{2}v,$

unless $v<\Lambda$.

Ignoring the $v<\Lambda$ alternatives, then, we have $\dot{y}>\frac{1}{2}v$ throughout the time interval $k^{-\frac{1}{2}}$ before M. But then at time $t_M-k^{-\frac{1}{2}}$ we have

$$\eta\leq\eta_M-\tfrac{1}{2}vk^{-\frac{1}{2}}\leq Lv^{-\frac{1}{2}}k^{-\frac{1}{2}}-\tfrac{1}{2}vk^{-\frac{1}{2}}.$$

The left-hand side being at least $-dk^{-\frac{1}{2}}$; this leads to $v<\Lambda$, which is therefore established.

We take next the (easier) proof that $|\ddot{y}|<\Lambda k^{\frac{1}{2}}$ on YZ. Let X_1 be the point of XY of time halfway between t_X and t_Y. We have $|\dot{y}|<\Lambda_1$ (say) on X_1Z (by the $\dot{y}$ result). Then we cannot have $|\ddot{y}|>2\Lambda_1/(\frac{1}{2}d')$ on the whole of X_1Y (or $\dot{y}$ would somewhere exceed Λ_1); there is therefore a point R of X_1Y at which $|\ddot{y}_R|\leq\Lambda$. Let the maximum of $|\ddot{y}|$ for RZ occur at M. We may suppose M not at R, which would give what we want. This time we distinguish two cases:

(α) $|\eta_M|\leq dk^{-\frac{1}{2}}$ (this includes the case $M=Z$),

(β) $\eta_M>dk^{-\frac{1}{2}}$.

In case (α) we use the fact that there is an S of RZ, within 2π on one side or the other of M, with $\varphi_S\equiv-\frac{1}{2}\pi$, and then, by the $\dot{y}$-identity,

$$b\,k\,(1+p_1(\varphi_M))=b\,k\,(p_1(\varphi_M)-p_1(\varphi_S))$$
$$=\dot y_M-\dot y_S+k\,(F(y_M)-F(1))-k\,(F(y_S)-F(1))+[g_1]_S^M$$
$$\le\Lambda+\Lambda+L\,k\,\eta_M^2-0+2\,\pi\,L<\Lambda.$$

By Lemma 2 (2) the left-hand side is at least $b\,k\,L\,\psi^2$, where $\varphi_M\equiv-\frac{1}{2}\pi+\psi$, $|\psi|\le 2\pi$; hence $b\,\psi^2<\Lambda\,k^{-1}$, and

$$b\,|p(\varphi_M)|<b\,L\,|\psi|<L\,(b\,\psi^2)^{\frac{1}{2}}<\Lambda\,k^{-\frac{1}{2}}.$$

Since
$$|f(y_M)|<L\,|\eta_M|<\Lambda\,k^{-\frac{1}{2}},$$
we have
$$\ddot y_M=|-k\,f\,\dot y-g+b\,k\,p|_M\le k\,\Lambda\,k^{-\frac{1}{2}}+L+k\,\Lambda\,k^{-\frac{1}{2}}<\Lambda\,k^{\frac{1}{2}},$$
which proves what we want.

In case (β) M is strictly interior to RZ; consequently $\dddot y_M=0$, or

$$0=-k\,\ddot y_M\,f(y_M)-k\,f'(y_M)\,\dot y_M^2-g'(y_M)\,\dot y_M+b\,k\,p'(\varphi_M).$$

Since $|f(y_M)|>L\,|\eta_M|>\Lambda\,k^{-\frac{1}{2}}$, by Lemma 2 (7),

$$|\ddot y_M|\le\Lambda\,k^{\frac{1}{2}}\,|-f'(y_M)\,\dot y_M^2-k^{-1}\,g'(y_M)\,\dot y_M+b\,p'(\varphi_M)|$$
$$<\Lambda\,k^{\frac{1}{2}}\,(L\Lambda+k^{-1}\,L\,\Lambda+L)<\Lambda\,k^{\frac{1}{2}},$$

which completes the proof that $|\ddot y|<\Lambda\,k^{\frac{1}{2}}$ for YZ.

The proof of $|\dddot y|<\Lambda\,k$ is much like that of the $\ddot y$ result, but simpler, since the term $b\,k\,p''$ is crudely $O(k)$. We differentiate once more and use $y^{iv}=0$ in one half of the argument (as for $\ddot y$): it is this that requires us to assume the existence of continuous second derivatives of f, g, p.[1]

We have now established (1) of the Lemma: (2) and (4) are immediate consequences. For (3) we have

$$f\,\ddot y=b\,p'-g'\,\dot y\,k^{-1}-f'\,\dot y^2-\dddot y\,k^{-1}=O(\Lambda),$$

and so $\ddot y=O(\Lambda\,|y-1|^{-1})$, and we have only to substitute this in

$$\dot y\,f-b\,p=-g\,k^{-1}-\ddot y\,k^{-1}.$$

This completes the proof of part (i).

§ 9. In part (ii) let $\tau=\log^2 k/k$, and consider the r.m. from the point concerned as time origin, over the time $0\le t\le\tau$. Let $T=\int_0^t f\,dt$. Since $y\ge 1+d$, we have $f>L\,d$ (Lemma 2 (7)), $T\ge t\,L\,d$. The r.m. is

---

[1] We need the inequality for $\dddot y$: it is not a luxury.

$$\ddot{y}=kf(y)\dot{y}-g+bkp,[1] \tag{1}$$

or

$$\frac{d}{dt}(\dot{y}e^{-kT})=(bkp-g)e^{-kT}. \tag{2}$$

From (1) we have

$$\dddot{y}=kf\ddot{y}+(kf'\dot{y}^2-g'\dot{y}+bk\dot{p}),$$

and so

$$\frac{d}{dt}(\ddot{y}e^{-kT})=(kf'\dot{y}^2-g'\dot{y}+bk\dot{p})e^{-kT}. \tag{3}$$

From (2)

$$\dot{y}e^{-kT}-\dot{y}_0=\int_0^t(bkp-g)e^{-kT}dt=\int_0^t O(k)e^{-kT}dt$$

$$=O(k)\int_0^t e^{-kT}dt=O(k)\int_0^t e^{-ktLd}dt=O(d^{-1}),$$

Hence, either $\dot{y}_0=O(d^{-1})$, or else

$$|\dot{y}e^{-kT}|>\tfrac{1}{2}|\dot{y}_0|>1.$$

The last alternative makes $|\dot{y}|>\exp(Ld\log^2 k)$ at time τ, contrary to $\dot{y}=O(Lk)$. Thus $\dot{y}_0=O(d^{-1})$, as desired.

For $\ddot{y}_0$ we argue similarly from (3), substituting $\dot{y}=O(Ld^{-1})$ on the right-hand side. For $\dddot{y}$ the argument is similar.

This completes the proof of Lemma 3.

§ 10. We take next the key Lemma B of the Introduction, (Lemma 5, below) prefacing it by Lemma C (Lemma 4, below); we restate them for convenience (they are unaltered in form, except for an addition to Lemma B).

LEMMA 4. *Suppose y_1, y_2 are respectively solutions of*

$$\dot{y}=\Phi(y,t)+R_{1,2},$$

where Φ is continuous in (y,t), $R_{1,2}$ continuous and $R_1>R_2$ for $t\geq t_0$.

(i) *If now $y_1(t_0)\geq y_2(t_0)$, then $y_1>y_2$ for $t>t_0$.*

(ii) *The conclusion is true if $R_1>R_2$ for $t>t_0$ only, provided we know independently that $y_1>y_2$ for small positive $t-t_0$.*

For the proof see § 14 of the Introduction.

[1] The argument of p involves $-t$ and a constant, but neither detail affects the reasoning.

LEMMA 5. *Consider the (Riccati) equation, for* $x \geq 0$,

$$\frac{dz}{dx} = z^2 - x^2 + 1 + \alpha - 2\beta x, \quad z(0) = 0,$$

where $\alpha \geq -1$, *and* β *further satisfies* $\beta < 0$ *when* $\alpha = -1$, *so that* z *is positive for small positive* x.

There is a $\beta_0 = \beta_0(\alpha)$ *such that*

(i) *if* $\beta > \beta_0$ [*or* $0 > \beta > \beta_0$ *when* $\alpha = -1$], *then* z *changes sign to negative at an* $x = x_0(\alpha, \beta) > 0$;

(ii) *if* $\beta < \beta_0$, *then* $z \to +\infty$ *at an asymptote* $x = x_0(\alpha, \beta) > 0$;

(iii) *if* $\beta = \beta_0$, *there is a solution in* $(0, \infty)$ *for which* $z \geq 0$ *and*

$$z = x + \beta_0 + F(x, \alpha),$$

where $F(x, \alpha)$ *is continuous in* (x, α)[1] *and* $F = O(1/x)$ *as* $x \to \infty$.

Further $\beta_0(\alpha)$ *and* $\gamma_0(\alpha) = \alpha + \beta_0^2(\alpha)$ *are continuous and (strictly) increasing.* $\beta_0(\alpha)$ *is large with large positive* α.

Finally $\beta_0(\alpha)$ *has the sign of* α (*and* $\beta_0(0) = 0$).

(iv) *If* $\beta = \beta_0(\alpha)$, $0 < l_1 \leq 1 + \alpha \leq l_2$, *then* $dz/dx > A(l_1, l_2) > 0$.

z is positive for small positive x, since $z'(0) > 0$ if $\alpha > -1$, and $z'(0) = 0$, $z''(0) = -2\beta > 0$ if $\alpha = -1$.

Let $z = u + x + \beta$, $\gamma = \alpha + \beta^2$ (and $\gamma_0 = \alpha + \beta_0^2$); the equation becomes

$$\frac{du}{dx} = u^2 + 2(x + \beta)u + \gamma = u(u + 2x + 2\beta) + \gamma, \quad u(0) = -\beta. \tag{1}$$

Let C_z, C_u be the curves $z = z(x)$, $u = u(x)$ (both for $x \geq 0$), determined by the equations and their initial conditions, and let Γ_u be the hyperbola

$$u(u + 2x + 2\beta) + \gamma = 0.$$

The slope of C_u can vanish only at a point of Γ_u.

For given $\alpha \geq -1$ there are 3 mutually exclusive possibilities in respect of β: (A)[2] C_z has a vertical asymptote where $z \to +\infty$; (B) C_z crosses Ox (from positive to negative z) (C) neither (A) nor (B) happens; we say in the respective cases that $\beta \in (A)$, (B), (C) (the classes vary with α).

---

[1] We include this obvious fact because it is explicitly used later.

[2] Initial of "asymptote".

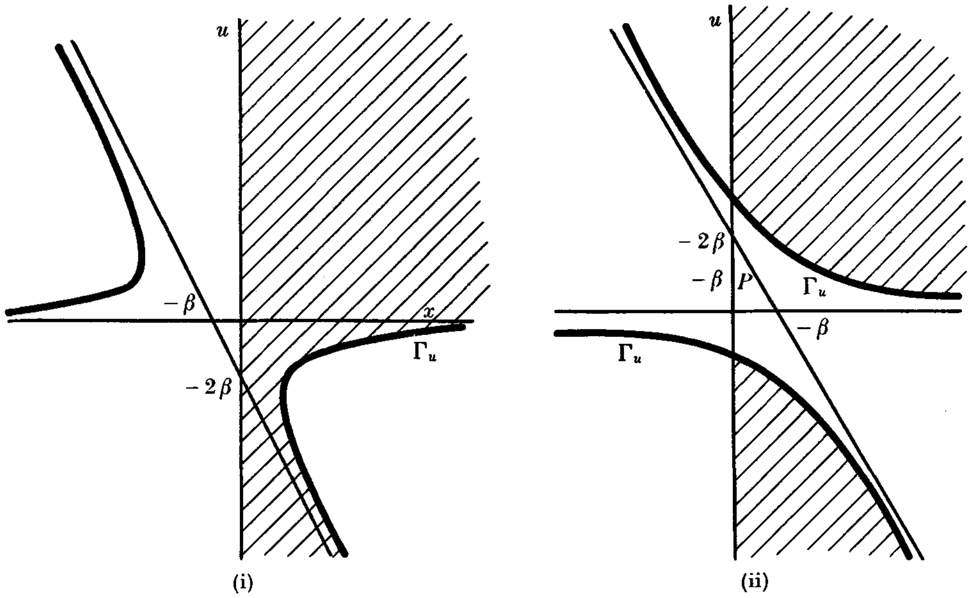

Fig. 3. (i) $\alpha, \beta > 0$. The region $du/dx > 0$ is shaded. (ii) $\alpha, \beta < 0$. The region $du/dx > 0$ is shaded. (*Erratum:* in Fig. 3 (i) $-\beta$ should be placed between -2β and origo.)

In the first place we have by continuity:

The classes (A), (B) are open, and vary continuously with α, (2)

and from Lemma 4 we have:

If $\beta \in (A)$ so does $\beta' < \beta$; if $\beta \in (B)$ so does $\beta' > \beta$. (3)

Thus (A) and (B), unless one of them is null, are infinite open intervals, separated by the complementary (C), which is either a closed interval or a single point.

We aim first at proving the following results:

$(a)_+$ If $\alpha > 0$, a small positive $\beta \in (A)$, and every large positive $\beta \in (B)$.

$(a)_-$ If $-1 \leq \alpha < 0$, then $\beta = -|\alpha|^{\frac{1}{3}} \in (A)$, and any small negative $\beta \in (B)$.

(b) For $\alpha \neq 0$, if $\beta \in (C)$, then $z \geq 0$,

$$z = x + \beta + F(x, \alpha), \quad F = O(1/x),$$

and (C) contains exactly one β.

$(c)_+$ This β is large with large positive α.

Suppose these results established. From $(a)_+$, $(a)_-$, $(b)_+$, $(c)_+$ and (2) (continuity) it follows that a unique continuous $\beta_0(\alpha)$ exists for all α (including $\alpha = 0$), that $\beta_0(\alpha) \in (C)$, that $\beta_0(\alpha)$ has the sign of α, and that $\beta_0(\alpha)$ is large with large positive α. Further (i), (ii), (iii) of the Lemma hold. It remains only to prove (iv) and that $\beta_0(\alpha)$ and $\alpha+\beta_0^2(\alpha)$ are monotonic increasing, and these we postpone, going on now to prove $(a)_+$ to $(c)_+$. We can divide (b) into the two cases $(b)_+$ and $(b)_-$ corresponding to $\alpha>0$ and $\alpha<0$.

Begin with the results involving $\alpha>0$, namely $(a)_+$, $(b)_+$, $(c)_+$.

In $(a)_+$ and $(b)_+$ we may suppose $\alpha>0$, $\beta>0$. So $\gamma>0$, Γ_u is as in fig. 3 (i) and does not cut Ou (the equation with $x=0$ has no real roots).

C_u[1] starts at $P(0, -\beta)$ with positive slope α. If C_u cuts Ox before cutting Γ_u, the slope cannot vanish thereafter, we have $du/dx>u^2+\gamma$, and C_u, and so also C_z, has a vertical asymptote at an $x=x_0(\alpha,\beta)$; β belongs to Class (A).

If C_u cuts Γ_u before cutting Ox, the slope of C_u becomes negative and remains negative thereafter; u is negative at the crossing, $u=-\lambda$ say, and subsequently decreases, so that $|u|\geq\lambda$. Next, we must have $u<-\frac{3}{2}(x+\beta)$ for *some* large x, since the contrary inequality for all large x, combined with (1), would imply

$$u+2x+2\beta\geq\tfrac{1}{2}(x+\beta)$$

and

$$\frac{du}{dx}<-\tfrac{1}{2}|u|(x+\beta)+\gamma\leq-\tfrac{1}{2}\lambda(x+\beta)+\gamma,$$

so that u would go to $-\infty$ at least as fast as $-\frac{1}{4}\lambda x^2$, a contradiction. Since $u<-\frac{3}{2}(x+\beta)$ implies $z<-\frac{1}{2}(x+\beta)$, this last is true for some large x; consequently C_z must cross Ox at a positive $x=x_0(\alpha,\beta)$, and β belongs to class (B).

To sum up: β belongs to class (A) if C_u cuts Ox before Γ_u, and to class (B) if it cuts Γ_u before Ox.

For a small positive β, C_u, having slope $\gamma>\alpha$ at P, clearly cuts Ox first; and we have the first part of $(a)_+$.

Suppose next that β is large and positive. Then dz/dx is large and negative for some positive x. For suppose not, then C_z does not go to $-\infty$ for finite x. Further, by Lemma 4, C_z is below the curve

$$\frac{d\zeta}{dx}=\zeta^2+1+\alpha,\quad \zeta(0)=0,$$

which has an asymptote at $x=\frac{1}{2}\pi(1+\alpha)^{-\frac{1}{2}}=c$, say, and satisfies $\zeta<c'$, say, in $0\leq x\leq\frac{1}{2}c$. In the range $\frac{1}{4}c\leq x\leq\frac{1}{2}c$ we have, on the one hand $z<c'$, and on the other, by hypothesis, $dz/dx>-K$, where K is independent of β, and so $z>-\frac{1}{2}cK$. Hence at $x=\frac{1}{4}c$,

[1] C_u is an auxiliary curve for proving facts about C_z.

$$\frac{dz}{dx}<z^2+1+\alpha-2\beta\cdot\tfrac{1}{4}c<(c'+\tfrac{1}{2}cK)^2+1+\alpha-\tfrac{1}{2}\beta c,$$

and as this is large and negative we have a contradiction.

So dz/dx and *a fortiori* du/dx, is large and negative for some positive x. But for such x C_u must have already crossed Γ_u (since du/dx is positive until Γ_u is crossed). C_u cannot have first crossed Ox, since it would then continue to move upwards. This establishes the second half of $(a)_+$.

When β is of class (C), z remains positive. Further, C_u cuts neither Ox nor Γ_u, and consequently approaches Ox between Ox and the asymptotic branch of Γ_u; henc $u=O(1/x)$, and $z=x+\beta+O(1/x)$. If this happened for two distinct β's, β_1 and $\beta_2>\beta_1$, we should have $z_2=z_1+(\beta_2-\beta_1)+O(1/x)>z_1$ for large x, whereas $z_2<z_1$ by Lemma 4. We have accordingly proved $(b)_+$.

That $\beta_0(\alpha)$ is large with large positive α is evident; if α is large and β is not, the large initial slope of C_u will take it across Ox, and β will not be of class (C). This is $(c)_+$, and we have proved all the $\alpha>0$ results.

§ 11. We now take up the $\alpha<0$ results, namely $(a)_-$, $(b)_-$. In $(a)_-$ we have $\alpha<0$, $\beta<0$.

When $\gamma=0$, or $\beta=-|\alpha|^{\frac{1}{2}}$, the u equation is

$$\frac{du}{dx}=u^2+2(x+\beta)u,\quad u(0)=|\beta|.$$

This is soluble in finite terms, and the solution has an upward asymptote: this proves $(a)_-$.

For small negative β we have $\gamma<0$, and Γ_u is as in fig. 3 (ii). Since du/dx vanishes on and only on Γ_u, C_u certainly cannot cross the lower branch of Γ_u. Hence if C_u crosses Ox (as it clearly does for a small negative β), u takes a negative value $-\lambda$ and thereafter decreases further. If we now had $u\geq-\frac{3}{2}(x+\beta)$ for all large x, we should have

$$u+2(x+\beta)\geq\tfrac{1}{2}(x+\beta),\qquad \frac{du}{dx}\leq-\lambda\tfrac{1}{2}(x+\beta)+\gamma$$

for large x, and u would go to $-\infty$ like $-\frac{1}{4}\lambda x^2$ at least, thereby crossing the lower branch of Γ_u, which is impossible. Hence for some large x $u<-\frac{3}{2}(x+\beta)$, and so $z<-\frac{1}{2}(x+\beta)$, C_z crosses Ox, and β belongs to class (B). $(a)_-$ is now proved.

For a β of class (C) C_u must go to ∞ between Ox and the upper branch of Γ_u, since if it crosses the Γ_u, u would subsequently increase; for large x we should have

$du/dx > u^2 + xu$ with a vertical asymptote. Hence $u = O(1/x)$, and the rest of the proof of $(b)_-$ is the same as for $(b)_+$.

It remains to prove $\beta_0(\alpha)$ and $\gamma_0(\alpha)$ (strictly) increasing, and finally (iv).

Consider C_1, C_2, the C_z for $(\alpha_1, \beta_0(\alpha_1))$, $(\alpha_2, \beta_0(\alpha_2))$ respectively, and let $\zeta = z_1 - z_2$. We have

$$\zeta' = P\zeta - 2bx + a, \qquad \zeta(0) = 0, \qquad P = z_1 + z_2,$$

where $a = \alpha_1 - \alpha_2$, $b = \beta_0(\alpha_1) - \beta_0(\alpha_2)$, and this gives

$$\zeta = e^{P_1}\int_0^x (-2bx + a)\, e^{-P_1}\, dx, \quad P_1 = \int_0^x P\, dx.$$

As $x \to \infty$, $P = 2x + O(1)$, $P_1 = x^2 + O(x)$.

If now $\alpha_1 > \alpha_2$ and so $a > 0$, ζ will tend to ∞ like e^{P_1} unless $b > 0$. Since $\zeta = O(1)$, $b > 0$ and $\beta_0(\alpha)$ is increasing.

Next, again with $\alpha_1 > \alpha_2$, let $\eta = z_1(x) - z_2(x + \beta_1 - \beta_2)$ where we write $\beta_{1,2} = \beta_0(\alpha_{1,2})$, $\gamma_{1,2} = \gamma_0(\alpha_{1,2})$. We find

$$\eta' = P\eta + (\gamma_1 - \gamma_2), \qquad P = z_1(x) + z_2(x + \beta_1 - \beta_2),$$

$$\eta = e^{P_1}\Big(\eta(0) + (\gamma_1 - \gamma_2)\int_0^x e^{-P_1}\, dx\Big).$$

We have again $P = 2x + O(1)$, $P_1 = x^2 + O(x)$. Also $\eta(0) < 0$ and we shall have $\eta \to -\infty$, which is false, unless $\gamma_1 - \gamma_2 > 0$; $\gamma_0(\alpha)$ is incresing.

In (iv) C_z cannot cross the hyperbola $z^2 - x^2 - 2\beta_0 x + 1 + \alpha = 0$, since its dz/dx would thereafter be negative, and $dz/dx > 0$ for $x \geq 0$; also $dz/dx \to 1$ as $x \to \infty$. dz/dx has a positive minimum $A(\alpha)$, continuous in α, and the desired results follows.

§ 12. "Linkage of v, ω, V at U for a settled trajectory".[1] This, in full detail, and for general f, g, p, is our next task.

There are in point of fact two distinct sets of circumstances in which we need to establish "linkage" at U on $y = 1$ between v and ω (and V, which is a combination of v and ω). One, discussed at length in the Introduction, is the case of arrival at U after a "long descent" to $y = 1$, with possible dips. Here we establish not only the linkage, but (from Lemma 3) also an upper bound for $|\omega|$ (one of order $k^{-\frac{1}{3}}$). The other becomes important only much later. In this, on the one hand nothing is assumed about the previous history of the trajectory earlier than a time $k^{-\frac{1}{3}}\log k$ before U; on the other hand we are *given* that ω is of order $k^{-\frac{1}{3}}$. We give a separate

---

[1] Cp. Introduction § 11, 12.

Lemma for each case; when we come to *proofs*, however, it is natural to establish first the restriction on ω in the first case, after which everything reduces to proving the second case (where the ω-restriction is a hypothesis).

In dealing with linkage, we naturally transform our variables v, ω (as in the Introduction) to parameters α, β appropriate for the application of Lemma 5. The statement of the two Lemmas is further complicated by (i) the necessity of working with undetermined d, d'; (ii) the need for a specific error-term in the "linkage".

We set out first some permanent notation. For a trajectory (in the first instance otherwise unrestricted) arriving at U on $y=1$ from above, let

$$\varphi_U=-\tfrac{1}{2}\pi-\omega,\ |\omega|\leq\pi;\qquad -\dot{y}_U=v;\qquad V=v+bk(1+p_1(-\tfrac{1}{2}\pi-\omega));\tag{1}$$

and let the change of variables to α, β be defined by

$$\left.\begin{aligned}\beta&=2^{-\frac{1}{2}}a_1^{\frac{1}{2}}a_2^{-\frac{3}{2}}b^{\frac{1}{2}}k^{\frac{1}{2}}(-p(-\tfrac{1}{2}\pi-\omega)),\\ 1+\alpha&=v/v^*,\qquad V^*=v^*=a_1^{-\frac{1}{2}}a_2^{\frac{1}{2}}b^{\frac{1}{2}}.[1]\end{aligned}\right\}\tag{2}$$

$\beta_0(\alpha)$ is the function of Lemma 5.

We have now

LEMMA 6. *Let $\frac{1}{100}\leq b\leq 2$, and let d be a non-negative, and d' a positive constant. Suppose that an eventual trajectory Γ satisfies the two following sets of conditions (A) and (B):*

(A) it ends with a piece WU (U on $y=1$) lying in $y\geq 1$ and of time-length at least $k^{-\frac{1}{2}}\log k$.

(B) WU is preceded by a piece XYW; the whole of XW (and so of XU) is in $y\geq 1-dk^{-\frac{1}{2}}$; XY has time-length at least d'; and YU contains a point at which $\varphi\equiv-\frac{1}{2}\pi$.

If now further $k\geq k_0(d,d')$, then we have upper bounds (for v, ω, V, α, β) as follows, in which Λ is an abbreviation for $A(d,d')$:

(a) $|\omega|<\Lambda k^{-\frac{1}{2}}$,

(b) $0\leq v\leq V\leq\Lambda$;

(c) $|\beta|<\Lambda$

(d) $0\leq 1+\alpha<\Lambda$.

[1] We *define* V^* by $V^*=v^*$.

And we have linkage given (in terms of α, β) by

(e) $$\beta = \beta_0(\alpha) + O(k^{-\frac{1}{2}} \log^A k).$$[1]

We note for convenience the asymptotic relations[2] *(for k large, ω small)*

(f) $$\left.\begin{aligned} &\beta \sim 2^{-\frac{1}{2}} a_1^{\frac{1}{2}} a_2^{\frac{1}{2}} b^{\frac{1}{2}} k^{\frac{1}{2}} \omega, \qquad V - v \sim \tfrac{1}{2} a_2 b k \omega^2, \\ &V = V^* (1 + \alpha + \beta_0^2(\alpha)) + O(\Lambda k^{-\frac{1}{2}} \log^A k). \end{aligned}\right\}$$

Finally we have (for reference)

(g) $$V^* = v^*, \; L < V^* < L.$$

LEMMA 7. *The conclusions of Lemma 6, with d absent from k_0 and Λ's, are valid (in form*[3]*) when (B) is replaced by*

(B') $$|\omega| \leq d' k^{-\frac{1}{2}}.$$

We prove first (a) of Lemma 6. In Lemma 6 Lemma 3 (4) is valid for YU, so that for points YU

$$F(y) - F(1) = C + b(1 + p_1(\varphi)) + O(\Lambda k^{-1}). \tag{3}$$

Now YU contains a point S where $\varphi \equiv -\frac{1}{2}\pi$, and so $1 + p(\varphi) = 0$; also $F(y_S) - F(1) \geq 0$; hence $C > -\Lambda k^{-1}$, and taking $y = 1$, $\varphi = \varphi_U$ in (3), we have

$$b(1 + p_1(\varphi_U)) < -C + O(\Lambda k^{-1}) < O(\Lambda k^{-1}),$$

and so from Lemma 2 (4)[4]

$$\omega^2 = O(\Lambda k^{-1}),$$

as desired.

§ 13. Everything now reduces to proving Lemma 7; for in Lemma 6 we have proved (a), i.e. condition (B') is fulfilled with $A(d, d')$ for d', and this leads to the same final results. Our arguments are now based on (B') and the fact that the r.m. from U does not go outside $1 \leq y \leq L^*$ within a time $k^{-\frac{1}{2}} \log k$.

In the $\dot{y}$-identity for the direct motion (d.m.) from U, viz.

$$\dot{y} = -v - k(F(y) - F(1)) + bk(p_1(\varphi) - p_1(\varphi_U)) - \int_{t_u}^{t} g\, dt,$$

[1] We do not aim at best possible powers of $\log k$ in the error term, the more so that we can absorb a factor Λ by changing the A.

[2] These are straightforward calculations from (a), ..., (e), and the properties of the functions p, p_1.

[3] d' has a new meaning in Lemma 7, and d does not occur.

[4] The special assumption about p_1 is involved.

we write $t=\tau_U-\tau$ to obtain the r.m. with time variable zero at U. This gives

$$\frac{dy}{d\tau}=v+k(F(y)-F(1))-bk(p_1(-\tfrac{1}{2}\pi-\omega-\tau)-p_1(-\tfrac{1}{2}\pi-\omega))+O(\tau)$$

$$=v+k(F(y)-F(1))+bkp(-\tfrac{1}{2}\pi-\omega)\tau-\tfrac{1}{2}bkp'(-\tfrac{1}{2}\pi-\omega)\tau^2+O(k\tau^3)+O(\tau),$$

with $y(0)=1$, or $\eta(0)=0$. In this we write

$$\eta=y-1=ck^{-\frac{1}{3}}z,\qquad \tau=\gamma k^{-\frac{1}{3}}x,$$

where c, γ are given in terms of the fundamental constants by

$$\tfrac{1}{2}c\gamma a_1=1,\ \tfrac{1}{2}\gamma^3c^{-1}ba_2=1,$$

and then write

$$\alpha+1=\gamma c^{-1}v,\qquad \beta=\tfrac{1}{2}bk^{\frac{1}{3}}(-p(-\tfrac{1}{2}\pi-\omega))\gamma^2c^{-1},$$

which yield the values of § 12 for α, β.

The result of the substitutions is

$$\frac{dz}{dx}=1+\alpha+\psi(z)-\frac{p'(-\tfrac{1}{2}\pi-\omega)}{p'(-\tfrac{1}{2}\pi)}x^2-2\beta x+O(k^{-\frac{1}{3}}x^3)+O(k^{-\frac{1}{3}}x),$$

where $\psi(z)=\psi(z,k)=\gamma c^{-1}k(F(y)-F(1))$. Since $\omega=O(\Lambda k^{-\frac{1}{3}})$, the coefficient of x^2 is $-1+O(\Lambda k^{-\frac{1}{3}})$. Thus the r.m. from U, in (z, x) form, is

$$\frac{dz}{dx}=1+\alpha+\psi(z)-x^2-2\beta x+x\varepsilon(x),\quad z(0)=0, \tag{1}$$

where, over the range $0\le x\le\gamma^{-1}\log k$,

$$\varepsilon(x)=O(\Lambda k^{-\frac{1}{3}})(1+x+x^2), \tag{2}$$

and

$$\beta=O(k^{\frac{1}{3}}\omega)=O(\Lambda). \tag{3}$$

The solution z is finite and non-negative in the range, y satisfies $1\le y\le L^*$, and so, by Lemma 2 (8), ψ satisfies

$$\psi=z^2+O(k^{-\frac{1}{3}}z^3), \tag{4}$$

and

$$L_1z^2\ge\psi(z)\ge L_2z^2. \tag{5}$$

From this state of things [and for suitable $k_0(d, d')$] we have to deduce the results of the Lemma.

We begin by proving

$$1+\alpha<\Lambda \tag{6}$$

(which is (d) of the Lemma).

For $x \leq 1$ (and suitable $k_0(d, d')$) we have

$$|-x^2 - 2\beta x + x\varepsilon(x)| < 1 + \Lambda + \Lambda k^{-\frac{1}{2}} < \Lambda_1.$$

Suppose now that $1 + \alpha \geq \Lambda_1 + l$; then from (1)

$$\frac{dz}{dx} \geq l + \psi(z) > l + L_2 z^2;$$

by Lemma 4 z is above the solution of $dz/dx = L_2 z^2 + l$, which has an asymptote at $x = \frac{1}{2}\pi (l L_2)^{-\frac{1}{2}}$. This number is less than 1 if l is a suitable chosen L, and we have then a contradiction with "$z < \infty \quad (0 \leq x \leq 1)$". Hence $1 + \alpha \geq \Lambda_1 + l$ implies $l < L$, and this proves (6).

For $0 \leq x \leq \gamma^{-1} \log k$ (and suitable k_0) we have

$$|1 + \alpha - x^2 - 2\beta x + x\varepsilon(x)| < \Lambda + \gamma^{-2} \log^2 k + \Lambda \log k + \Lambda k^{-\frac{1}{2}} \log^3 k < 2\gamma^{-2} \log^2 k.$$

Hence
$$\frac{dz}{dx} = \psi + 2\vartheta\gamma^{-2} \log^2 k \quad (0 \leq x \leq \gamma^{-2} \log k), \tag{7}$$
where $|\vartheta| \leq 1$.

We prove next that in the shorter range $0 \leq x \leq \frac{1}{2}\gamma^{-2} \log k$

$$\frac{dz}{dx} < \log^3 k. \tag{8}$$

For suppose not, so that $dz/dx = \log^3 k$ for the first time at an $x = \xi$ satisfying $0 \leq \xi \leq \frac{1}{2}\gamma^{-1} \log k$. Consider now the range from ξ to ξ_1, where ξ_1 is either $\gamma^{-1} \log k$, or else that $x > \xi$ at which first $dz/dx = 0$, whichever is least. In (ξ, ξ_1) z is non-decreasing and so

$$\frac{dz}{dx} \geq \psi - 2\gamma^{-2} \log^2 k \geq L_2 z^2 - 2\gamma^{-2} \log^2 k \geq L_2 z^2(\xi) - 2\gamma^{-2} \log^2 k = (L_2/L_1)(L_1 z^2(\xi) - L \log^2 k) > 0,$$

since
$$L_1 z^2(\xi) \geq \psi(\xi) = \left(\frac{dz}{dx} - 2\vartheta\gamma^{-2} \log^2 k\right)_{x=\xi} = \log^3 k - 2\vartheta_\xi \gamma^{-2} \log^2 k > \tfrac{1}{2} \log^3 k. \tag{9}$$

Hence the alternative $dz/dx = 0$ does not happen first, so that $\xi_1 = \gamma^{-2} \log k$. Thus in $(\xi, \gamma^{-1} \log k)$

$$z \geq z(\xi) > L \log^{3/2} k,$$

$$\frac{dz}{dx} > L_2 z^2 - 2\gamma^{-2} \log^2 k \geq \tfrac{1}{2} L_2 z^2,$$

$$\tfrac{1}{2} \log k \leq \log k - \xi = \int_{\xi}^{\log k} dx < \int_{z(\xi)}^{\infty} \frac{dz}{\frac{1}{2} L_2 z^2} < \frac{L}{z(\xi)} < 1$$

by (9). This being false, we have established (8).

For the range $0\leq x\leq\frac{1}{2}\gamma^{-1}\log k$ we now have $|z|\leq\frac{1}{4}\gamma^{-1}\log^4 k$, by (8); also $|\varepsilon_1(x)|<\Lambda k^{-\frac{1}{2}}\log^2 k$, by (2). From these and (1), (4) the z, x equation now becomes

$$\frac{dz}{dx}=1+\alpha+z^2-x^2-2\beta x+x\varepsilon_1(x),\quad z(0)=0,$$

$$|\varepsilon_1(x)|<\Lambda k^{-\frac{1}{2}}\log^{12}k<k^{-\frac{1}{2}}\log^A k.$$

Let $\theta=\beta-\beta_0(\alpha)$, let $\zeta=\zeta(x,\alpha)$ be the solution in $0\leq x\leq\frac{1}{2}\gamma^{-1}\log k$ of

$$\frac{d\zeta}{dx}=1+\alpha+\zeta^2-x^2-2\beta_0(\alpha)x,\quad \zeta(0)=0,$$

and let $u=z-\zeta$. We shall prove that $|\theta|\leq 2k^{-\frac{1}{2}}\log^{A_1}k$, thereby establishing the remaining result (e) of the Lemma. Suppose that, on the contrary, $|\theta|>2k^{-\frac{1}{2}}\log^{A_1}k$, and suppose first that θ is positive. Then $2\theta-\varepsilon_1(x)\geq\theta$. Now u satisfies

$$\frac{du}{dx}=u(\zeta+z)-(2\theta-\varepsilon_1(x))x,\quad u(0)=0,$$

and by Lemma 4 $u\leq w$, where

$$\frac{dw}{dx}=w(\zeta+z)-\theta x,\quad w(0)=0,$$

and so
$$w=-\theta\exp\Big(\int_0^x(\zeta+z)\,dx\Big)\int_0^x x\exp\Big(-\int_0^x(\zeta+z)\,dx\Big)\,dx. \tag{10}$$

Now, by Lemma 5 (iii), $|\zeta-x|=|\beta_0(\alpha)+F(x,\alpha)|<\Lambda$, since $-1\leq\alpha<\Lambda$, and by (8) we have $0\leq z\leq x\log^3 k$. Hence

$$\begin{aligned}\int_0^x x\exp\Big(-\int_0^x(\zeta+z)\,dx\Big)\,dx&\geq\int_0^x x\exp\Big(-\int_0^x(x+\Lambda+x\log^3 k)\,dx\Big)\,dx\\&=\int_0^x x\exp\left(-\Lambda x-\tfrac{1}{2}(1+\log^3 k)x^2\right)dx\\&\geq\Lambda\log^{-3}k\end{aligned}\tag{11}$$

for $x=1$ and therefore for $x\geq 1$. So for $x\geq 1$ we have from (10)

$$|w|=-w\geq\theta\exp\Big(\int_0^x\zeta\,dx\Big)\cdot\Lambda\log^{-3}k,$$

$$|w|\geq(2k^{-\frac{1}{2}}\log^{A_1}k)\cdot\exp\left(\tfrac{1}{2}x^2-\Lambda x\right)\cdot\Lambda\log^{-3}k\quad(1\leq x\leq\tfrac{1}{2}\gamma^{-1}\log k). \tag{12}$$

On the other hand,

$$|w| = -w \le -u = \zeta - z \le |\zeta| + |z| < x + \Lambda + x \log^3 k. \tag{13}$$

(12) and (13) are incompatible (for a suitable k_0) when $x = \frac{1}{2}\gamma^{-1} \log k$, and the assumed inequality for θ is false.

In the case of negative θ, assuming $\theta < -2k^{-\frac{1}{2}} \log^{A_1} k$, we have w non-negative, $u \ge w$, and so $w \le z - \zeta \le z + |\zeta|$, and the rest of the argument is the same.

This completes the proof of Lemma 7 (and Lemma 6).

§ 14. LEMMA 8. ("*Dip or shoot-through at a U*".) *Let* $\frac{1}{100} \le b \le 2$. *Let the piece* WU *of* Γ *satisfy the conditions* (A), (B) *of Lemma* 6. *Abbreviate constants* $A(d, d', \delta)$ *to* Δ.[1]

(i) *Suppose* $V \ge V^* + \delta$; *then for* $k \ge k_0(d, d', \delta)$ *the d.m. from* U_1 *shoots through and reaches*[2] $y = -\frac{1}{2}(1 + H)$ *in time at most* $\Delta k^{-\frac{1}{2}}$. *Up to this moment we have* $-\dot{y} \ge V^* > L$, *and*

$$-\dot{y} = v + k(F(y) - F(1)) + O(\Delta);$$

and finally the velocity of arrival at $y = -\frac{1}{2}(1 + H)$ *satisfies* $-\dot{y} > Lk$.

(ii) *Suppose* $V \le V^* - \delta$; *then for* $k \ge k_0(d, d', \delta)$, (a) *the d.m. from* U *makes a dip of depth* $\Delta k^{-\frac{1}{2}}$ *at most below* $y = 1$, *emerging at time* $\Delta k^{-\frac{1}{2}}$ *at most later. It then* (b) *pursues approximately the curve* C_1, *the branch of*

$$F(y) - F(1) = b(1 + p_1(\varphi))$$

lying in $y \ge 1$, *and* (c) *if* Γ *has been above* $y = 1 - dk^{-\frac{1}{2}}$ *for a time* 3π *before* u *it arrives at* $y = 1$ *again at a time approximately* 2π *later.*

In either case Γ satisfies, up to its arrival at U, the hypotheses, and therefore the conclusions, of Lemma 6.

The d.m. from U, taking $t = 0$ at U, is

$$\begin{aligned} -\dot{y} &= v + k(F(y) - F(1)) - bk(p_1(\varphi_U + t) - p_1(\varphi_U)) + g_1 \\ &= \{v + k(F(y) - F(1))\} - bk\{p(-\tfrac{1}{2}\pi - \omega)t + \tfrac{1}{2}p'(-\tfrac{1}{2}\pi - \omega)t^2\} + O(kt^3) + O(t). \quad (1) \\ &= \{v + k(F(y) - F(1))\} + O(k\omega t) + O(kt^2) + O(kt^3) + O(t). \quad (2) \end{aligned}$$

---

[1] In applications Δ become D's. The blank cheques d, d' are still involved, via the hypotheses (A), (B).

[2] What we do (while we are about it), is to follow the shoot-through up to a point a distance L *below* $y = -1$: this is a more convenient place than $y = -1$ for the next starting point.

Let c, γ, α, β be the numbers and $\psi(z)$ the function of § 13, and write $y = 1 - c k^{-\frac{1}{2}} \zeta$, $t = \gamma k^{-\frac{1}{2}} x$; (1) then gives (with an $\varepsilon(x)$ different from that of § 13)

$$\left.\begin{aligned} \frac{d\zeta}{dx} = 1 + \alpha + \psi(-\zeta) - x^2 + 2\beta x + x\varepsilon(x), \quad \zeta(0) = 0, \\ \varepsilon(x) = O(\Delta k^{-\frac{1}{2}})(1 + x + x^2). \end{aligned}\right\} \tag{3}$$

(that is, formally, (1) of § 13 with $-\zeta$ for z and $-\beta$ for β).

Case (i). $V \geq V^* + \delta$. By Lemma 6 (f) we have $\alpha + \beta_0^2(\alpha) > L\delta$, and so, by Lemma 5, $\alpha > A(\delta)$ and $\beta_0(\alpha) > A_1(\delta)$. Since

$$|\beta - \beta_0(\alpha)| < \Delta k^{-\frac{1}{2}} \log^A k < \tfrac{1}{2} A_1(\delta),$$

by Lemma 6 (e), we have $\beta > A(\delta)$.

Consider now (3) for the range of x after 0 to the value for which (for the first time) $\zeta = 0$, or $|\zeta| = k^{\frac{1}{10}}$, or $x = k^{\frac{1}{20}}$, whichever happens first. In this range $2\beta + \varepsilon(x) > 0$ and $\psi(-\zeta) = \zeta^2 + O(k^{-\frac{1}{2}+\frac{3}{10}})$, and so

$$\frac{d\zeta}{dx} > 1 + \tfrac{1}{2}\alpha + \zeta^2 - x^2.$$

By Lemma 4 $\zeta \geq w$, where

$$\frac{dw}{dx} = 1 + \tfrac{1}{2}\alpha + w^2 - x^2, \quad w(0) = 0.$$

By Lemma 5 $w \geq 0$ and w has an asymptote to $+\infty$ at $x = x_0(\alpha) < A_2(\delta)$. Hence two of the alternatives fail, and ζ reaches the value $+k^{\frac{1}{10}}$ before $x = A_2(\delta)$ at most, which corresponds to $t = \Delta k^{-\frac{1}{2}}$ at most, and then $-\dot{y} = (c/\gamma)\, d\zeta/dx > L\zeta^2 - \Delta > L k^{\frac{1}{5}}$. Further $\zeta \geq x$, since ζ is not less, by Lemma 4, than the solution of

$$\frac{du}{dx} = 1 + u^2 - x^2, \quad u(0) = 0,$$

which is $u = x$; *hence* $d\zeta/dx > 1 + \frac{1}{2}\alpha > 1$ *throughout, equivalent to* $-\dot{y} > V^*$.

Return now to (1). We have $-\dot{y} > V^*$ up to a time $t_1 < \Delta k^{-\frac{1}{2}}$, and at $t = t_1$, $y - 1 = -c k^{\frac{1}{10} - \frac{1}{2}}$. Consider the range from $t = t_1$ until either $-\dot{y} = V^*$, or $y = -\frac{1}{2}(1 + H)$, or $t - t_1 = k^{-\frac{1}{2}}$, whichever happens first. In this range (2) gives

$$(-\dot{y}) - \{v + k(F(y) - F(1))\} = O(\Delta), \tag{4}$$

since $\omega = O(\Delta k^{-\frac{1}{2}})$. In particular

$$-\dot{y} > k(F(y) - F(1)) - \Delta_1 > L_2 k (1-y)^2 - \Delta_1$$

by Lemma 2 (8). Further $-\dot{y} > V^* > 0$ and $k(1-y)^2 \geq k(1-y)^2_{t=t_1} = Lk^{\frac{1}{5}}$, and so

$$-\dot{y} > \tfrac{1}{2} L_2 k (1-y)^2. \qquad (5)$$

Now this motion, *if uninterrupted*, makes y go to $-\infty$ in time $O(k^{-\frac{1}{2}-\frac{1}{10}})$ with $-\dot{y} > V^*$ throughout. We infer that of the three alternatives it is $y = -\frac{1}{2}(1+H)$ that happens first, and in time at most $(\Delta+1)k^{-\frac{1}{2}}$ after U, and then, by (5), $-\dot{y} > Lk$. This completes the proof of case (i).

§ 15. *Case* (ii). $V \leq V^* - \delta$. Much of this is parallel to case (i). By Lemma 6 (f) we have $\alpha < -A(\delta)$, so that, by Lemma 5, $\beta_0(\alpha) < -A_1(\delta)$; also $|\beta - \beta_0(\alpha)| < \frac{1}{2} A_1(\delta)$ and so $\beta < -\frac{1}{2} A_1(\delta)$. Consider the ζ, x equation [(3) of § 14] for the range of x after 0 to the value for which (for the first time) $\zeta = 0$, or $|\zeta| = k^{\frac{1}{10}}$, or $x = k^{\frac{1}{20}}$, whichever happens first. In this range $2\beta + \varepsilon(x) < 0$, and $\psi(-\zeta) = \zeta^2 + O(k^{-\frac{1}{2}+\frac{3}{10}})$, and so

$$\frac{d\zeta}{dx} < 1 + \tfrac{1}{2}\alpha + \zeta^2 - x^2.$$

By Lemma 4 $\zeta \leq w$, where

$$\frac{dw}{dx} = 1 + \tfrac{1}{2}\alpha + w^2 - x^2, \quad w(0) = 0.$$

By Lemma 5 (since $\alpha < 0$) w, initially positive, becomes negative at $x = A(\alpha)$ and is bounded by an $A(\alpha)$ before this point. We infer that obvious alternatives fail, and that the d.m. from U' makes a dip, as described in (ii).

§ 16. Let the dip emerge at U', with $y'_{U'} = v' \geq 0$.[1] We take $t = 0$ at U', and we have now to discuss the d.m. from U', for which

$$\dot{y} = v' - k(F(y) - F(1)) + bk(p_1(\varphi_{U'} + t) - p_1(\varphi_{U'})) - g_1.$$

Now for $t \leq k^{-\frac{2}{5}}$

$$bk(p_1(\varphi_{U'} + t) - p_1(\varphi_{U'})) = bktp(-\tfrac{1}{2}\pi + \theta),$$

where $\theta = -\omega + (\varphi_{U'} - \varphi_U) + \vartheta t$, which is (a) small, and (b) greater than $-\omega$, which is positive with $-\beta$ (Lemma 6). Since $p'(-\frac{1}{2}\pi)$ is positive,

$$bkp(-\tfrac{1}{2}\pi + \theta) \geq bkp(-\tfrac{1}{2}\pi - \omega) = Lb^{\frac{2}{3}}|\beta|k^{\frac{1}{3}} > \Delta k^{\frac{1}{3}}.$$

[1] The dashes in U', v' are temporary notation only, inside the proofs, and while we are dealing with dips.

Hence for $t \leq k^{-\frac{2}{5}}$ the d.m. has

$$\dot{y} \geq -k(F(y)-F(1))+(\Delta k^{\frac{1}{2}}-L)t=\Phi(y,t).$$

Since the first term in Φ is $-k(L(y-1)^2+O(y-1)^3)$, the locus $\Phi=0$ has a branch starting at $t=0$, $y=1$, and lying above $y=1$ for $0<t\leq k^{-\frac{2}{5}}$, and $\Phi \geq 0$ for points between this and $y=1$. Since $\dot{y} \geq 0$ for small positive t it clearly follows that $\dot{y} \geq 0$ and $y \geq 1$ so long as $t \leq k^{-\frac{2}{5}}$. During this time, and afterwards until y next descends to $y=1$, Γ satisfies the conditions of Lemma 3, with $d=\Delta$, and therefore satisfies the two relations

$$\dot{y}f=bp(\varphi)+O(\Delta k^{-\frac{1}{2}}), \tag{1}$$

$$F-F(1)=C+b(1+p_1(\varphi))+O(\Delta k^{-1}). \tag{2}$$

But from time $k^{-\frac{2}{5}}$ to a suitable[1] L_1 we have $-\frac{1}{2}\pi+k^{-\frac{2}{5}} \leq \varphi \leq L_1+|\omega|$, and $bp(\varphi)>bLk^{-\frac{2}{5}}$. The right side of (1) is then positive (for suitable k_0); so y remains a time at least L in $y \geq 1$.

By taking $t=0$, $y=1$ in (2) we see that C satisfies $C=O(b\omega)+O(\Delta k^{-\frac{1}{2}})$. It follows that, to error $O(k^{-A})$, Γ pursues up to $t=2\pi$ the branch C_1 of

$$F-F(1)=b(1+p_1(\varphi))$$

in $y \geq 1$, as (b) of Lemma 8 (ii) asserts, and the error is of the form $O(k^{-A})$.

§ 17. Finally we have, in (c), to deal with the point mentioned in (ii) of § 13 of the Introduction, and prove that Γ does reach $y=1$ near the end of C_1, and in fact by time $t_{U'}+2\pi$ at latest. Suppose this false, and (with the notation of Lemma 6) let $y_1=y(t)$, $y_2=y(t+2\pi)$, $w=y_2-y_1$, and consider the range $\mathcal{R}$, $\tau \leq t \leq t_{U'}$, where $\tau=t_{U'}-k^{-\frac{1}{2}}\log k$. We are to show that $y_2=1$ for some t of $\mathcal{R}$. Suppose on the contrary that $y_2>1$ in $\mathcal{R}$.

Now the $\dot{y}$-identities for $y_{1,2}$ are of the forms

$$\dot{y}=-kF(y)+R_{1,2} \tag{1}$$

where, since $\int_t^{t+2\pi} p\,dt=0$, we have $R_2-R_1=-\int_t^{t+2\pi} g\,dt$,

and so
$$R_2-R_1<-L, \tag{2}$$

since $y_1, y_2>1-dk^{-\frac{1}{2}}$ in $\mathcal{R}$. It follows, by Lemma 4, that $w<0$, and so $y_2<1$, at $t=t_{U'}$, provided $w<0$ at *some* point of $\mathcal{R}$. If the desired result is false, then, we must

---

[1] e.g. an L_1 such that $-\frac{1}{2}\pi+L_1$ is halfway between $-\frac{1}{2}\pi$ and the next zero of $p(\varphi)$.

have both $y_2 > 1$ and $y_2 - y_1 = w > 0$ throughout $\mathcal{R}$: we proceed to show that these hypotheses lead to a contradiction.[1]

Let $\mathcal{R}'$ be the middle third of $\mathcal{R}$. In $\mathcal{R}'$ (and indeed in $\mathcal{R}$) the results of Lemma 3 (i), and in particular its (2), are valid for y_1.[2] So we have in $\mathcal{R}'$

$$-\dot{y}_1 f(y_1) = -b\,p(\varphi) + O(\Delta k^{-\frac{1}{2}});$$

and in this we have

$$\varphi - (-\tfrac{1}{2}\pi) < -\tfrac{1}{3} k^{-\frac{1}{2}} \log k + O(\Delta k^{-\frac{1}{2}}) < -L k^{-\frac{1}{2}} \log k,$$

and so $-b\,p(\varphi) > L k^{-\frac{1}{2}} \log k$. Since $|\dot{y}_1| < \Delta$, by (1) of Lemma 3 (i), and $y_1 > 1$, this gives

$$f(y_1) > \Delta k^{-\frac{1}{2}} \log k. \tag{2}$$

Now, by (1), $$\frac{dw}{dt} = -kX + R_2 - R_1 < -kX - L,$$

where $$X = F(y_2) - F(y_1) = \int_{y_1}^{y_2} f\,dy > w f(y_1) > \Delta w k^{-\frac{1}{2}} \log k,$$

by (2). So in $\mathcal{R}'$, or $t_1 \le t \le t_2$, say,

$$\frac{dw}{dt} < -qw - L, \qquad q = \Delta k^{\frac{1}{2}} \log k,$$

$$\frac{d}{dt}(w e^{qt}) < -L e^{qt},$$

$$w < w(t_1) e^{-q(t-t_1)} - L e^{-qt} \int_{t_1}^{t} e^{qt}\,dt$$

$$< L e^{-q(t-t_1)} - L q^{-1} (1 - e^{-q(t-t_1)}).$$

This is negative for, say, $t = t_1 + k^{-\frac{1}{4}}$, and gives the desired contradiction.

This completes the proof of Lemma 8.

§ 18. We consider now a series of successive dips, $U_1 U_1'$, $U_2 U_2'$, ..., of which the first, $U_1 U_1'$, is subject to the hypotheses of Lemma 8 (ii). If the depth of the dip $U_1 U_1'$ is $d_1 k^{-\frac{1}{2}}$ we have $d_1 < \Delta$. Then Γ, taken up to U_2, satisfies the conditions of Lemma 6 with Max (d, d_1) for new d and the same d' as before. Hence, with obvious notation, v_2, ω_2, V_2 satisfy the bounding and linkage relations (a) to (g) of Lemma 6, with the new d, d'.

---

[1] The remainder of the argument is different from the one in the Introduction, because we are making weaker assumptions.

[2] The hypotheses of Lemma 8, of Lemma 6 and of Lemma 3 (i), are effectively the same.

Further, U_1U_2 is the curve C_1, to error $O(k^{-A})$. Now the $\dot{y}$-identity between U_1 and U_2 can be written in the form

$$V_2-V_1=\int_{U_1}^{U_2} g\,dt,$$

and we have in consequence

$$V_2-V_1=M+O(k^{-A}),\tag{1}$$

where $M=\int_{-\pi}^{\pi} g(Y)\,dt$, and $y=Y(t)$ is the equation of the curve C_1. M is an $A(b)$ (depending only on b and the fixed functions) and lies between two L's.

There are now three alternatives concerning V_2; (i) $V_2\geq V^*+\delta$, (ii) $V_2\leq V^*-\delta$, (iii) $V^*-\delta<V_2<V^*+\delta$; the first two correspond to (i) and (ii) of Lemma 8, the third we describe as the "gap" case. In (i) there is a shoot-through as in Lemma 8 (1); in (ii) there is another dip U_2U_2', of depth $d_2k^{-\frac{1}{2}}<\Delta k^{-\frac{1}{2}}$, followed by $U_2'U_3$, approximately a period length of the curve C_1, as in Lemma 8 (ii). And so on. If k_0 is successively rechosen in the obvious way we shall arrive at a final $U_n=U$, with $n\leq 1+[(V^*+\delta)/(M-Lk^{-A})]$, and either with a $V(=V_n)$ in the gap $V^*\pm\delta$, or else a $V\geq V^*+\delta$ and a shoot-through. We have $n<L$, and the final k_0 is a $k_0(L,d,d',\delta)$, where d,d' are the parameters conditioning Γ at the start. Finally, Lemma 6 is valid (with new d,d' of type Δ) for the stretch ending in U, and Lemma 3 (i) is similarly valid and up to U.

§ 19. We continue these additions to Lemma 8 by pursuing the shoot-through of § 14 a stage further. We assume the (minimum) hypotheses of Lemma 8 (i) [namely those of Lemma 6, together with $V\geq V^*+\delta$]. As we have seen, $\varphi_U=-\frac{1}{2}\pi-\omega$, where $\omega=O(\Delta k^{-\frac{1}{2}})$,[1] and Γ arrives, at K', say,[2] on $y=-\frac{1}{2}(1+H)$, with $\dot{y}<-Lk$, and in time $\tau=O(\Delta k^{-\frac{1}{2}})$ after U. We proceed to show that $\dot{y}$ vanishes, within time $\Delta k^{-\frac{1}{2}}$ after U, at an inverted vertex $\mathfrak{Z}'$ for which

$$|y_{\mathfrak{Z}'}+H|<\Delta_1k^{-1},\qquad |\varphi_{\mathfrak{Z}'}+\tfrac{1}{2}\pi|<\Delta_2k^{-\frac{1}{2}}.\tag{1}$$

Consider Γ beyond K' up to the time when first $\dot{y}=0$. The differential equation can be written

$$\frac{d}{dt}(\dot{y}\,e^{kf_1})=e^{kf_1}(bkp-g),$$

where $f_1=\int f\,dt$. If we take $t=0$ at K' this gives

[1] We retain the notation Δ for $A(d,d',\delta)$.

[2] The dash attached to K, and $\mathfrak{Z}$ below, is there to conform with what is later systematic notation.

$$\dot y - \dot y_0 e^{-kf_1} = e^{-kf_1} b k \int_0^t p e^{kf_1} dt - e^{-kf_1} \int_0^t g e^{kf_1} dt$$

$$= b k p(t) \int_0^t e^{-kE} d\tau - \int_0^t g(\tau) e^{-kE} d\tau - b k \int_0^t (p(t) - p(\tau)) e^{-kE} d\tau, \tag{2}$$

where $E = \int_\tau^t f(y) d\tau$. Since $y < -1 - L$ throughout we have $f \gtrless L$ and $e^{-kE} \gtrless e^{-Lk(t-\tau)}$; also the second and third terms on the right side of (2) are

$$O\left(\int_0^t e^{-Lk(t-\tau)} d\tau\right) + O(k) \int_0^t (t-\tau) e^{-Lk(t-\tau)} d\tau = O\left(\frac{1}{k}\right).$$

Also $\dot y_0 = O(k)$. Hence (2) gives

$$\dot y > -Lk e^{-Lkt} + b p(-\tfrac{1}{2}\pi + \psi)(1 - e^{-Lkt}) - L k^{-1} \tag{3}$$

where $\psi = \varphi + \frac{1}{2}\pi = -\omega + (t - t_u)$. Then $p(-\frac{1}{2}\pi + \psi) = a_2 \psi + O(\psi^2)$ (for small ψ), and the right side of (3) is certainly positive if (i) $\psi \geq k^{-\frac{1}{3}}$ and (ii) $t > k^{-\frac{1}{3}}$. Since $t_{K'} - t_U < \Delta k^{-\frac{1}{3}}$, $\dot y$ accordingly vanishes at a time after U at most $\Delta k^{-\frac{1}{3}}$, so that $|\varphi_{Z'} + \frac{1}{2}\pi| < \Delta_2 k^{-\frac{1}{3}}$, the second half of (1).

Further, by the $\dot y$-identity between U and Z',

$$0 = \dot y_{Z'} = b k (1 + p_1(\varphi_{Z'})) - V - k (F(y_{Z'}) - F(1)) - g_1$$
$$= O(k(\Delta k^{-\frac{1}{3}})^2) + O(1) - k(F(y_{Z'}) - F(1))$$
$$= O(\Delta) - k(F(y_{Z'}) - F(-H)),$$

so that $F(y_{Z'}) - F(-H) = O(\Delta k^{-1})$, and $|y_{Z'} + H| < \Delta k^{-1}$, as desired.

Incidentally we have at any point of UZ'

$$\dot y = -kF(y) + \tfrac{2}{3}k + O(\Delta) = -k(F(y) - F(1)) + O(\Delta). \tag{4}$$

For convenience of reference we add to this summing up the result (of Lemma 8)

$$|\dot y| > V^* > L \text{ over } UK'. \tag{5}$$

§ 20. We are supposing always that $\frac{1}{100} \leq b \leq \frac{2}{3} - \frac{1}{100}$.

When we take $d = 0$ and $d' = 1$, the numbers $\Delta_{1,2}$ of (1) of § 19 become definite D's, $A_1(\delta)$, $A_2(\delta)$, which we may suppose to increase as δ decreases. *We now define* (note the change from δ to $\frac{1}{2}\delta$) $D_0^* = \mathrm{Max}(A_1(\frac{1}{2}\delta), A_2(\frac{1}{2}\delta))$. [In the special case of van der Pol's equation D_0^*, with δ for $\frac{1}{2}\delta$, is the D_0 of the Introduction (§ 10).] We are now in a position to prove the key-Lemma 9, and the further results in Lemmas 10, 11.[1]

We denote by (S) the set of initial conditions

$$(S)\qquad |\dot{y}_0|\leq D_0^*k^{-1},\ |y_0-H|\leq D_0^*k^{-1},\ |\varphi_0-\tfrac{1}{2}\pi|\leq D_0^*k^{-\frac{1}{2}},$$

and we denote also by S the class ("stream") of Γ's satisfying (S). There is a certain $D_1^*\geq D_0^*$ that we define later (§ 24). We denote by (S_1), S_1, etc., the conditions and "stream" obtained by replacing D_0^* by D_1^*. S_1 contains S.

In considering behaviour connected with the boundaries $y=\pm 1$ of the region Σ, or $|y|\leq 1$, we have so far had only to consider one of them at a time, and have standardized to $y=1$. Any such behaviour happens also in "inverted" form: we systematically use dashes to denote the inverted form of the undashed thing. Thus, to a U on $y=1$, with $\varphi_u\equiv-\frac{1}{2}\pi-\omega$, $V=-\dot{y}(U)+bk(1+p_1(-\frac{1}{2}\pi-\omega))\geq V^*-\delta$ there corresponds a U' on $y=-1$, $\varphi_{U'}\equiv\frac{1}{2}\pi-\omega'$, $V'=\dot{y}(U')+bk(1-p_1(\frac{1}{2}\pi-\omega'))\geq V^*-\delta$, and so generally. We shall state results for one form only at a time, taking the opposite form as understood; but sometimes the uninverted, sometimes the inverted, happens to be the more convenient. This use of dashes among others should never lead to confusion: our practice will be that if a dash *can* mean an inversion it does (and if it cannot it does $k_0(\delta)$ not).

§ 21. LEMMA 9. *Provided $k\geq k_0(\delta)$, a Γ of S_1 (a fortiori one of S) does not reach a certain $y=1+L$, a fortiori does not reach $y=1$, before time Lk at least. After possible dips it will arrive at $y=1$ at U, where either* (i) *V is in the gap $V^*\pm\delta$, or else* (ii) *$V\geq V^*+\delta$. In case* (ii), *and, more generally, when $V\geq V^*+\frac{1}{2}\delta$, there is a shoot-through $U\mathfrak{Z}'$ ending at an inverted vertex $\mathfrak{Z}'$ satisfying the inverted forms of (S),*[2] *namely*

$$(S')\qquad |\dot{y}_0'|\,(=0)\leq D_0^*k^{-1},\qquad |y_0'+H|\leq D_0^*k^{-1},\qquad |\varphi_0'+\tfrac{1}{2}\pi|\leq D_0^*k^{-\frac{1}{2}},$$

so that Γ belongs to an S', and repeats the behaviour just described until, if ever, it "arrives at a gap" (arrives at $y=\pm 1$ with V or V' in $V^*\pm\delta$).[3]

LEMMA 10. (*Linkage at U*). *For the Γ of Lemma 9 we define* (repeating some earlier definitions for convenience of reference)

---

[1] These involve 3 parameters y_0, $\dot{y}_0$, φ_0, so that the Γ are multiply represented.

[2] *Not* (S_1), of course.

[3] The argument will show that whatever D, D' say, is chosen in place of D_1^*, the results of the Lemma are valid provided $k\geq k_0(L,\delta,D')$: strictly speaking we should employ a blank cheque d, ultimately chosen to be D_1^*, but this seems hardly necessary. (To introduce D_1^* before Lemma 9 would waste space.)

$$\varphi_U = -\tfrac{1}{2}\pi - \omega, \quad |\omega| \leq \pi; \qquad -\dot{y}_U = v; \qquad V = v + b\,k\,(1 + p_1(-\tfrac{1}{2}\pi - \omega)) \quad (\geq v); \tag{1}$$

$$V^* = v^* = a_1^{-\frac{1}{3}}\, a_2^{\frac{2}{3}}\, b^{\frac{1}{3}}; \tag{2}$$

$$1 + \alpha = v/V^*, \qquad \beta = -2^{-\frac{1}{3}}\, a_1^{\frac{1}{3}}\, a_2^{-\frac{2}{3}}\, b^{\frac{1}{3}}\, k^{\frac{1}{3}}\, p\,(-\tfrac{1}{2}\pi - \omega). \tag{3}$$

Then we have

$$V^* - \delta \leq V < L; \qquad |\omega| < L k^{-\frac{1}{3}}; \qquad L < 1 + \alpha < L; \qquad |\beta| < L; \tag{4}$$

and the linkage

$$\beta = \beta_0(\alpha) - O(k^{-\frac{1}{3}} \log^A k). \tag{5}$$

Lemma 11. *A Γ of Lemma 9 satisfies, from its start up to U,*

$$|\dot{y}| < D, \quad |\ddot{y}| < D k^{\frac{1}{3}}; \tag{1}$$

$$\dot{y}\, f(y) = b\,p(\varphi) + O(D k^{-\frac{1}{3}}). \tag{2}$$

Over UK' [K' is on $y = -\frac{1}{2}(1+H)$] *we have*

$$-\dot{y} > V^* > L. \tag{3}$$

Over $U\mathfrak{Z}'$ we have

$$\dot{y} = -k\,(F(y) - F(1)) + O(D) = -k\,F(y) - \tfrac{2}{3}k + O(D). \tag{4}$$

Over an interval of time-length 1, say, ending at U we have $-\dot{y} > L$. More generally, for an arrival (possibly earlier than U) at $y = 1$ with $v > L_1$ we have $-\dot{y} > L_2(L_1)$ over the unit interval ending with the arrival, where L_2 depends on L_1.[1] (5)

§ 22. *Proof of Lemma* 9. In the notation of Lemma 3 we take X at the start, and Y at time $t = \log^2 k/k$, or when y first reaches $\frac{1}{2}(1+H)$, whichever happens first. Over XY, writing $f_1 = \int_0^t f\,dt$, we have

$$\frac{d}{dt}(-\dot{y}\, e^{kf_1}) = u\, e^{kf_1}, \quad u = -b\,k\,p + g. \tag{1}$$

We have
$$\varphi = \tfrac{1}{2}\pi + O(D k^{-\frac{1}{3}}) + O(\log^2 k/k) = \tfrac{1}{2}\pi + O(D k^{-\frac{1}{3}}),$$
$$p(\varphi) = O(D k^{\frac{2}{3}}), \qquad u = O(D k^{\frac{2}{3}}).$$

(1) gives
$$-\dot{y} = -\dot{y}_0\, e^{-kf_1} + \int_0^t u(t')\, e^{-k(f_1(t) - f_1(t'))}\, dt'.$$

The exponential in the integral is $< e^{-Lk(t-t')}$; hence

$$|\dot{y}| < D k^{-1} \cdot 1 + D k^{\frac{2}{3}} \int_0^t e^{-Lk(t-t')}\, dt' < D k^{-\frac{1}{3}}. \tag{2}$$

[1] Strictly speaking the L_1 should be a blank cheque: there is actually only one application, in § 42, when L_1 is a particular L.

This gives $\qquad y>y_0-Lt>H-Dk^{-1}-Lk^{-1}\log^2 k>\tfrac{1}{2}(1+H),$

so that it is $t=\log^2 k/k$ that happens first, and XY has length $\log^2 k/k$.

We note for later use that over XY we have both (2) and, since $p=O(Dk^{-\frac{1}{2}})$,

$$\ddot{y}=O(k\dot{y})+O(kp)+O(1)=O(Dk^{\frac{1}{2}}). \tag{3}$$

Next we have, for the whole of Γ,

$$F-F(1)=C+b(1+p_1(\varphi))-k^{-1}\int_0^t g\,dt-\dot{y}k^{-1},$$

in which, on substituting $t=0$,

$$C=F(y_0)-F(1)-b(1+p_1(\varphi_0))+\dot{y}_0k^{-1}.$$

Now
$$F(H)=-F(1)=\tfrac{2}{3},$$

$$F(y_0)=F(H)+O(y_0-H)=\tfrac{2}{3}+O(Dk^{-1}),$$

$$1+p_1(\varphi_0)=2-(p_1(\tfrac{1}{2}\pi)-p_1(\varphi_0))=2+O((\varphi_0-\tfrac{1}{2}\pi))^2=2+O(Dk^{-1})),$$

and so $C=\tfrac{4}{3}-2b+O(Dk^{-1})$, and the equation is

$$F-F(1)=(\tfrac{4}{3}-2b)+b(1+p_1(\varphi))-k^{-1}\int_0^t g\,dt-\dot{y}k^{-1}+O(Dk^{-1}). \tag{4}$$

There is a constant l, an L, such that $F(1+l)-F(1)=\tfrac{1}{200}$. Consider the stretch of Γ from X to the first arrival at $y=1+l$. Lemma 3 (ii) is valid, with $d=l$, and we have $|\dot{y}|<L$. Substituting this in (4) and taking t to be the time of arrival at $y=1+l$, we have

$$\begin{aligned}k^{-1}\int_0^t g\,dt&>(\tfrac{4}{3}-2b)+b(1+p_1(\varphi))-Lk^{-1}-Dk^{-1}-(F(1+l)-F(1))\\&>\tfrac{2}{100}+0-Dk^{-1}-\tfrac{1}{200}>\tfrac{1}{200},\end{aligned}$$

so that $t>Lk$. *A fortiori* the stretch from Y to the first arrival U_1 has time-length at least $Lk\,(>1)$, and contains points with $\varphi\equiv-\tfrac{1}{2}\pi$. It follows in the first place that Lemma 3 (i), with u_1 for Z and a new Y is valid for XYu_1, whence $|\dot{y}|<L$ and in particular $v_1<L$. By § 18 we have the arrival at U described in Lemma 9, with D's for Δ's, and at U we have $V<V_1+(n-1)M+Lk^{-A}<L$ (since $n, M<L$), and so $\alpha<L$.

Next, for the case $V>V^*+\delta$, we recall the results of § 19 about descent to $\mathfrak{Z}'$: the constants $\Delta_{1,2}$ have now $d, d'=0, 1$, and become $A_1(\delta)$, $A_2(\delta)$. For the extended case $V>V^*+\tfrac{1}{2}\delta$ (the last part of Lemma 9 we have to consider) we have the desired result, about $D_0^*=\text{Max}\,(A_1(\tfrac{1}{2}\delta),\ A_2(\tfrac{1}{2}\delta))$.

Proof of Lemma 10. We have proved (5), and V, $\alpha < L$, and from (5) $\alpha < L$ implies $\beta < L$. This implies, *since* ω *is small*,[1] $|\omega| < Lk^{-\frac{1}{2}}$ by the definition (3) of and β the properties of $p(\varphi)$. There remains of Lemma 10 only $1+\alpha > L$. Now $\alpha + \beta_0^2(\alpha) = V/V^* - 1 + O(k^{-\frac{1}{2}}\log^A k) > -L\delta$, and $\alpha + \beta_0^2(\alpha)$ decreases from 0 as α decreases from 0, so that $\alpha < -L$ is possible only for a *small* L, and $1+\alpha > L$ as desired.

Proof of Lemma 11. We recall the last sentence of § 18, in whose consequences, since the original d, d' are 0 and 1, Δ's become D's. We have accordingly (from Lemma 3 (i)) (1) and (2) of Lemma 11, except for a time $\log^2 k/k$ at the beginning. For this stretch the first is included in (2) and (3) above, and the second follows from (3). The results (4) and (3) of the Lemma are respectively (4) of § 19, in which Δ becomes D, and (5) of § 19.

There remains only (5) of Lemma 11. In the first place it is enough to prove this for time $k^{-\frac{1}{2}}\log\log k$ before U. For then we have in the remaining time stretch, $\eta = y - 1 > Lk^{-\frac{1}{2}}\log\log k$; Lemma 3 (i) (3) is valid [with D for $A(d, d')$], and so

$$\begin{aligned}\dot{y} &= b\,p(\varphi)/f + O(Dk^{-1}\eta^{-2})\\ &= b\,p(\varphi)/f + O(D(\log\log k)^{-2})\\ &> L\,p(\varphi)/(t_U - t) - D(\log\log k)^{-2}, \quad \text{since } f < L\eta < L(t_U - t),\\ &> L, \text{ since } -(\varphi + \tfrac{1}{2}\pi) > \omega + (t_U - t) > \tfrac{1}{2}(t_U - t).\end{aligned}$$

For the stretch $k^{-\frac{1}{2}}\log\log k$ before U we have for the r.m., in the notation of § 12, over a stretch that becomes $x \leq L\log\log k$,

$$\frac{dz}{dx} = 1 + \alpha + \psi(z) - \frac{p'(-\frac{1}{2}\pi - \omega)}{p'(-\frac{1}{2}\pi)}x^2 - 2\beta x + O(k^{-\frac{1}{2}}x^3) + O(k^{-\frac{1}{2}}x), \quad z(0) = 0.$$

The coefficient of x^2 is $-1 + O(Dk^{-\frac{1}{2}})$, and $\beta = \beta_0(\alpha) + O(k^{-\frac{1}{2}}\log^A k)$. Also $\psi = z^2 + O(k^{-\frac{1}{2}}z^3)$; and we know that

$$\frac{dz}{dx} \leq L|\dot{y}| < L, \quad \text{so that } |z| \leq Lx.$$

It follows that (over $x \leq L\log\log k$)

$$\frac{dz}{dx} = 1 + \alpha + z^2 - x^2 - 2\beta_0 x + O(k^{-A}).$$

[1] Because U is near the curve C_1. Lemma 6 has $|\omega| < \Lambda k^{-\frac{1}{2}}$, but we cannot use this since the Λ would become a D (because the dips have depths $O(Dk^{-\frac{1}{2}})$). The proof in Lemma 6 is also different (and there is a similar one in the Introduction).

Let ζ be the solution of

$$\frac{d\zeta}{dx}=1+\alpha+\zeta^2-x^2-2\beta_0 x, \quad \zeta(0)=0,$$

and $z=\zeta+u$, so that

$$\frac{du}{dx}=u(2\zeta+u)+O(k^{-A}), \quad u(0)=0. \tag{1}$$

We have $u\geq w$, where

$$\frac{dw}{dx}=2\zeta w+O(k^{-A}), \quad w(0)=0,$$

so that

$$w>-Lk^{-A}e^{2\zeta_1}\int_0^x e^{-2\zeta_1}dx, \tag{2}$$

We have $\zeta=x+O(1)$, $\zeta\geq 0$, so that

$$-w<Lk^{-A}e^{Ax^2}<Lk^{-A}e^{L(\log\log k)^2}<k^{-A}.$$

It follows from (1) that

$$\frac{du}{dx}>2u\zeta-Lk^{-A}>2w\zeta-Lk^{-A}>-Lk^{-A}$$

Finally Lemma 5 (iv) gives $d\zeta/dx>A(l_1, l_2)=L$, where l_2 is any upper bound l of $1+\alpha$, so that

$$\frac{dz}{dx}>L-Lk^{-A},$$

and $|\dot{y}|\geq L\dfrac{dz}{dx}>L$, as desired.

§ 23. We now introduce some permanent notation for important points (or time-points) connected with a trajectory Γ, which we suppose to start in some S_1,[1] and not to meet a gap, or, more generally, to have $V, V'\geq V^*+\frac{1}{2}\delta$ at any U or U' concerned, in the range under consideration.

Z's are time-points where $\varphi\equiv\frac{1}{2}\pi$, N's points[2] where $\varphi\equiv\frac{1}{2}\pi$ (a Z' is accordingly an N and N' a Z, but the *letter*, Z or N, corresponds to the aspect we are emphasizing). In a "long descent" Z's correspond (very) approximately to "vertices" or maxima (and Z''s in an "ascent" to "inverted vertices"). $\mathfrak{Z}$ is a *real* vertex after a

[1] To be defined in § 24 following.

[2] Initials of zenith and nadir.

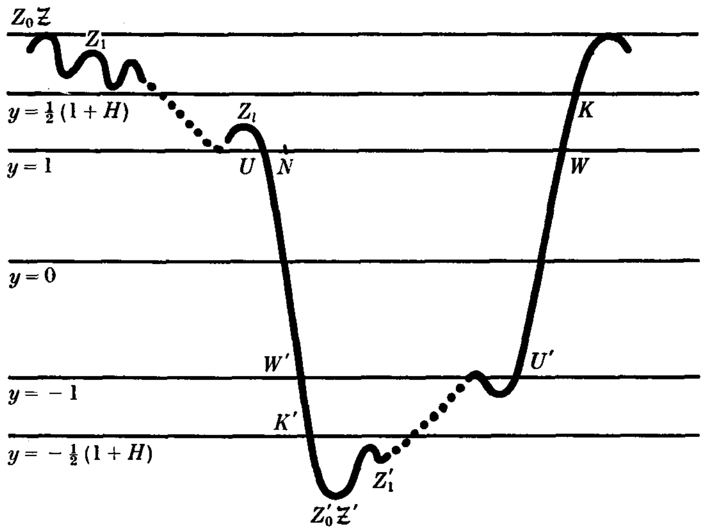

Fig. 4.

shoot-through.[1] Z_0 is the first Z after U', Z_l is the last Z before U. N's (in a descent) give approximate minima, but this is unimportant. The last few waves of a descent are near $y=1$, and may meet it, near the points N concerned: an intersection U_n, and in particular U, is "near" the N (a distance $O(Dk^{-\frac{1}{2}})$ away). It is convenient to be able to say that U_n, or U, is "at" the$N$ concerned. When two Γ have their U's near the same N, we say they have U's "together at N", or simply "together". All this about Z's and N's happens also, with dashes, in "inverted" form.

An intersection of a shoot-through from a U with $y=-1$ we call W' and that with $y=-\frac{1}{2}(1+H)$ we call K'.[2] This allocation of dashes to W' and K' corresponds to treating U to U' as a "half-cycle" (a "dashed" one). Another important kind of half-cycle, however, is Z_1 to Z_1'. After Lemma 9, a Γ that is "gap-free", or one that always has $V, V' > V^* + \frac{1}{2}\delta$ at U, U', enters a new S_1 (actually an S) after each shoot-through, and executes successive half-cycles U to U', U' to (the next) $U, \ldots$; or again Z_1 to Z_1', to $Z_1, \ldots$.

We understand by $\mathfrak{Z}$, U, W, K both points of Γ, and also time-points (abscissae). We use all the letters freely as names of *times*, and speak, e.g., of "the time-range $Z_1 - 1$ to $Z_1 + 1$".

The y-range $|y| \le 1$ we call Σ.

---

[1] Exceptionally a "start" in an S_1 "at" Z_0 may (also may not) have a $\mathfrak{Z}$; this is never important.

[2] This notation has occurred by anticipation in § 19.

§ 24. Let $\lambda=[V^*/M]+1$ (the integer λ is a special $A(L,b)$). Consider now a Γ of S from its start "at" Z_0 to $Z_{2\lambda}$. By Lemma 3 (ii) we have $\ddot{y}=O(D)$, and consequently

$$\dot{y}f(y)=bp-(\ddot{y}+g)/k=bp+O(Dk^{-1});$$

and in particular, since $p(Z)=0$,

$$|\dot{y}(Z_m)|<D_1k^{-1}\quad(1\le m\le 2\lambda).$$

In the $\dot{y}$-identity

$$F(y)=F(y_0)+b(p_1-p_1(\varphi_0))-k^{-1}\int_0^t g\,dt-(\dot{y}-\dot{y}_0)/k,$$

we have $$p_1(\varphi_0)=p_1(\tfrac{1}{2}\pi+O(Dk^{-\frac{1}{2}}))=1+O(Dk^{-1}),$$

and $$F(y_0)=F(H+O(Dk^{-1}))=F(H)+O(Dk^{-1}).$$

At Z_m we have $p_1=1$; so

$$|F(y_{Z_m})-F(H)|<Dk^{-1},$$

and so $$|y_{Z_m}-H|<D_2k^{-1}\quad(1\le m\le 2\lambda).$$

Let now $D^*=\operatorname{Max}(D_0^*,D_1,D_2,L_3^*)$, where L_3^* is a certain L defined in § 67,[1] and define the stream S^* by

$$(S^*)\qquad |y_0-H|\le D^*k^{-1},\qquad |\dot{y}_0|\le D^*k^{-1},\qquad |\varphi_0-\tfrac{1}{2}\pi|\le D^*k^{-\frac{1}{2}}.$$

We write $\Gamma(2m\pi)$ for Γ translated a time $2m\pi$ forward, with the obvious meaning for $S(2m\pi)$ etc. Then we see that S^* *includes* S, *the streams* $S(2m\pi)$, $m=1,2,\ldots\lambda$ *from their starts on, and* $S(-2m\pi)$, $m=1,2,\ldots,\lambda$ *from* Z_0 *on.*

We now repeat the ("expansion") process above, but starting with S^* in place of S and D^* in place of D_0^*, and taking this time $\lambda=2$. A Γ of S^* satisfies, for $m=1,2$, $|\dot{y}_{z_m}|<D_3k^{-1}$, $|y_{z_m}-H|<D_4k^{-1}$. We now take the D_1^* of § 20 to be $\operatorname{Max}(D^*,D_3,D_4)$, and define S_1 (as in Lemma 9) by

$$(S_1)\qquad |y_0-H|\le D_1^*k^{-1},\qquad |\dot{y}_0|\le D_1^*k^{-1},\qquad |\varphi_0-\tfrac{1}{2}\pi|\le D_1^*k^{-\frac{1}{2}}.$$

S_1 will now contain S^* and $S^*(-2\pi)$ from Z_0 on.[2]

To sum up, S, S^*, S_1 are continuous simply-connected convex[3] streams (in respect

[1] This is needed for the topological argument occurring much later. The definition of L_3^* in § 67 starts from first principles, and can be read now, but it is too long to be incorporated here.

[2] Also $S^*(2\pi)$ from its start, but we do not need this.

[3] These are important properties (even though we do not need their full force). We cannot define, e.g., S^* as $\sum_{-\lambda}^{\lambda}S(2m\pi)$, since this sum need not be connected.

of their initial conditions). S_1 contains S^*, and $S^*(-2\pi)$ from Z_0 on. S^* contains S and $S(\pm 2m\pi)$ for $m=1,2,\ldots,$ from their starts or from Z_0 on. S^* is the most important stream, but for some of its properties we need to call on its slight enlargement S_1; the practical upshot is that we shall be concerned with S_1 alone for some considerable time.

After Lemma 9, a Γ of S_1 that has $V \ge V^* + \frac{1}{2}\delta$ at its first U, shoots through, and then becomes a member of the S at the ensuing Z_0, so that we have only an S to deal with after the first shoot-through. (None the less we have to make a prolonged study of S_1.)

§ 25. We proceed to develop the properties of a Γ connected with $\tau = k\int_0^t f(y)\,dt$, for ultimate use in "$T$-form" (see the Introduction, § 18). These are concerned as much with "reversed" trajectories, or motions (r.m.), as direct ones (d.m.), and we distinguish the cases throughout. We begin with an important and rather delicate result about a shoot-through.

LEMMA 12. *Suppose a d.m. Γ of S_1, so far gap-free, has a U' with $V' \ge V^* + \frac{1}{2}\delta$, and so shoots through upwards, and suppose the time range $t_1 \le t \le t_2$ is in $U'K$. Then*

$$e^{-k\int_{t_1}^{t_2} f\,dt} = \varrho\,\frac{\dot y(t_2)}{\dot y(t_1)}$$

where $D < \varrho < D$. The corresponding ("reversed") result for the r.m. has $\varrho\,\dot y(t_1)/\dot y(t_2)$ on the right-hand side.

By Lemma 8, writing $y=-1+\eta$, $G(\eta)=\frac{2}{3}-F(y)$, we have $\dot y > V^*$, $v' > V^* > L$, and

$$v' + kG - D_1 < \dot y < v' + kG + D_1. \tag{1}$$

Taking $t_1 = 0$, $t_2 = t$ we have

$$-k\int_0^t f\,dt = \int_{(y_0)}^{(y)} \frac{k\,dG(y)}{\dot y} \le \int \frac{k\,dG}{\operatorname{Max}(V^*, v' + kG - D_1)}. \tag{2}$$

$G(\eta)$ increases from $\eta=0$ to 2, after which $G > L$. Let η^* be the η (near $y=-1$) for which $kG(\eta^*) = D_1$, so that $D < k^{\frac{1}{2}}\eta^* < D$ (we need these inequalities later in § 29). If y_0 and y are both less than $-1+\eta^*$, then $\dot y_0$ and $\dot y$ both lie between V^* and D, and $\dot y/\dot y_0$ lies between two D's. Also, by (2),

$$0\leq -k\int_0^t f\,dt\leq\int_0^{\eta^*}\frac{k\,dG}{V^*}=\frac{D_1}{V^*}<D,$$

$\exp\,(-k\int f\,dt)$ lies between D's, and so $\exp\,(-k\int f\,dt)/(\dot{y}/\dot{y}_0)$ does, as desired.

Suppose next that $y_0\leq -1+\eta^*<y$. Then

$$-k\int_0^t f\,dt\leq\int_0^{\eta^*}\frac{k\,dG}{V^*}+\int_{\eta^*}^{\eta}\frac{k\,dG}{v'+kG-D_1}$$

$$=\frac{D_1}{V^*}+\log\frac{v'+kG-D_1}{v'+D_1-D_1}$$

$$<D+\log(v'+kG-D_1)<\log(D\dot{y}).$$

On the other hand

$$-k\int_0^t f\,dt\geq\int_{\eta^*}^{\eta}\frac{k\,dG}{v'+kG+D_1}=\log\frac{v'+kG+D_1}{v'+D_1+D_1}$$

$$>\log\{D(v'+kG+D_1)\}\geq\log(D\dot{y}).$$

Hence $\exp\left(-k\int_0^t f\,dt\right)/\dot{y}$ lies between two D's, and combining this with the particular value $t=0$, $\dot{y}=\dot{y}_0$, we have $\exp\left(-k\int_0^t f\,dt\right)/(\dot{y}/\dot{y}_0)$ between two D's as desired.

If both y_0, $y>-1+\eta^*$ we have, on the one hand (since $v'>V^*$),

$$-k\int_0^t f\,dt<\int_{\eta_0}^{\eta}\frac{k\,dG}{v'+kG-D_1}=\log\frac{v'+kG-D_1}{v'+kG(\eta_0)-D_1}$$

$$<\log\frac{\dot{y}}{v'+kG(\eta_0)-D_1},\tag{3}$$

and on the other

$$-k\int_0^t f\,dt>\int_{\eta_0}^{\eta}\frac{k\,dG}{v'+kG+D_1}=\log\frac{v'+kG+D_1}{v'+kG(\eta_0)+D_1}$$

$$>\log\frac{\dot{y}}{v'+kG(\eta_0)+D_1}.\tag{4}$$

Finally $\{v'+kG(\eta_0)-D_1\}/\{v'+kG(\eta_0)+D_1\}$ increases, with $G(\eta_0)$, as η_0 increases from η^* to 2, after which it is $1+O(Dk^{-1})$: at η^* it has the value $v'/(v'+2D_1)>D$. So

$$v' + kG(\eta_0) - D_1 > D(v' + kG(\eta_0) + D_1) > D\dot{y}_0, \tag{5}$$

whence also
$$v' + kG(\eta_0) + D_1 < D(v' + kG(\eta_0) - D_1) < D\dot{y}_0. \tag{6}$$

It follows from (5), (6), (3) and (4) that $\exp(-k\int f\,dt)/(\dot{y}/\dot{y}_0)$ lies between two D's. This completes the proof of the Lemma.

§ 26. For a Γ, whether d.m. or r.m., we define $\tau = \tau(t) = k\int_0^t f\,dt$. The origin $t=0$ is here arbitrary; this is partly a convenient abbreviation, and all results generalize to a range (t_0, t) with $\tau - \tau(t_0)$ for τ. Note that a given stretch PQ of a Γ has $t=0$, $t=t$ at P, Q for a d.m., and at Q, P for the r.m., but the associated τ is the same in each case.[1] We state results (as always) for one kind of half-cycle only.

LEMMA 13. *Suppose throughout that $t \geq 0$, and that the range $(0,t)$ belongs to a Γ starting in some S_1 and thereafter having $V, V' \geq V^* + \frac{1}{2}\delta$ at all the U, U' concerned. Then*

(a) $e^{-\tau} < Dk$ *(d.m. or r.m.),*

(b) $\int_0^t e^{-\tau}\,dt < Dk^{\frac{1}{2}}$, $\quad e^{-\tau}\int_0^t e^{\tau}\,dt < Dk^{\frac{1}{2}}$ *(each d.m. or r.m.),*

(c) $e^{-\tau} < e^{-Lk}$ *if* $t \geq 1$ *(d.m. or r.m.).*

(d) *If the range $(0,t)$ is in (d.m.) WU[2] (a fortiori if it is in a long descent Z_1U),*

$$\tau > -D, \quad \int_0^t e^{-\tau}\,dt < Dk^{-\frac{1}{2}}, \quad e^{-\tau}\int_0^t e^{\tau}\,dt < Dk^{-\frac{1}{2}} \quad \textit{(each for d.m. or r.m.).}$$

(e) *If $(0,t)$ is in the (d.m.) range $(W, W + k^{\frac{1}{2}})$,*

$$\int_0^t e^{-\tau}\,dt < Dk^{-1}, \quad e^{-\tau}\int_0^t e^{\tau}\,dt < Dk^{-1} \quad \textit{(each for d.m. or r.m.).}$$

(f) *If $(0,t)$ is in an r.m. WU',*

$$\int_0^t e^{\tau}\,dt < Dk^{-\frac{1}{2}} \quad \textit{(r.m.).}$$

---

[1] Note however that e.g. the two $\int e^{-\tau}\,dt$ are *not* the same; thus, $\int_0^t e^{-\tau}\,dt$ for r.m. becomes $e^{-\tau}\int_0^t e^{\tau}\,dt$ for d.m. over the same range.

[2] See § 23 for the definition of W. (i) Where a pair of letters (here W, U or Z_1, U) occur together like this it is naturally understood that the second is the *first* of its kind after the first. (ii) The r.m. corresponding e.g. to the d.m. WU is called UW, but a d.m. UW would be a different "piece" of Γ (namely $UW'\mathfrak{Z}' \ldots Z_1'U'W$); we have accordingly to note in the text that, strictly speaking, WU is a d.m. There is, however, an obvious convention that where nothing is said a d.m. is in question.

Also
$$\int_W^{U'} e^{\tau-\tau_W}\,dt < Dk^{-1} \quad (r.m.).$$ [1]

(g) *For* $t \geq k^{-1}$ *we have the lower bounds*

$$\int_0^t e^{-\tau}\,dt > Lk^{-1}, \quad e^{-\tau}\int_0^t e^{\tau}\,dt > Lk^{-1} \quad (\textit{each for d.m. or r.m.}).$$

Proof of (a) *and* (c). We begin with (a) in the special case $t \leq 1$. Then Γ can enter Σ at most once, and the worst case is when Γ is in Σ throughout $(0, t)$, since τ is increasing when y is outside Σ. *This* case is covered by Lemma 12, since

$$L < |\dot{y}_0|, \quad |\dot{y}| < Lk.$$

Consider next (c).[2] This is the *same* result for d.m. and r.m., and we operate with a d.m. Let G be any time-interval of length 1 (of the kind the Lemma is concerned with). Now in the first place (α), a shoot-through, $U'\mathfrak{Z}$ say, has y increasing and lasts only time $O(Dk^{-\frac{1}{2}})$; also $\mathfrak{Z}Z_2$, say, is above $y=1+L$. We show next that (β), if L_1 is a small enough L, a G contained in Z_1U[3] is below $y=1+L_1$ during one time-interval at most, of length small with L_1. Now (i), over G, Γ is, to error $o(1)$, part of a curve $F(y)=C+b(1+p_1(\varphi))$ which does not go below $y=1$ (§ 16); moreover since $F(y)$ increases with y in $y \geq 1$, and because of the special hypothesis about p_1 (§ 1 and Lemma 2), the line $y=1+L_1$, for suitably small L_1 (and in a time-interval limited to 1) cuts this curve at most twice, and then at points a distance apart small with L_1. Next, (ii), by Lemma 3 (i) (2), at any crossing of $y=1+L_1$ by Γ we have both $|\dot{y}|=|bp(\varphi)/f+O(Dk^{-\frac{1}{2}})|>L$ and $|\ddot{y}|<D$. It follows that the first and last crossings must be the only ones, since any other must be within $o(1)$ of either the first or the last (Γ and the curve differing by $o(1)$), and this is incompatible with (ii). This establishes (β).

A little consideration of (α) and (β) shows that for *any* G Γ is in $|y| \leq 1+L_1$ for a single interval I at most, and that $G-I$ has time-measure $>L$ (it consists in general of two intervals). Now $\dot{\tau} \geq 0$ except in I, and the increment $\Delta\tau$ over I satisfies $e^{-\Delta\tau} < Dk$, by the special case of (a). In $G-I$ we have $\dot{\tau} > Lk$ and the increment of τ over $G-I$ is at least $Lk|G-I| > Lk$. This establishes (c).

It is easy to see that as a result of (c) we may suppose in all the results to be proved that $t \leq 1$. For example, if $n < t \leq n+1$, we have

[1] This is the special case where $t=0$ is W. $\int_0 e^{\tau}\,dt$ is very sensitive to the position of $t=0$ (and we must avoid a fallacious "a fortiori").

[2] The natural order of the proofs is different from that of the results.

[3] $\mathfrak{Z}Z_2$ and Z_1U overlap by a length >1.

$$\int_0^t e^{-\tau} dt = \sum_{m=0}^{n-1} e^{-\tau(m)} \int_m^{m+1} e^{-(\tau-\tau(m))} dt + e^{-\tau(n)} \int_n^t e^{-(\tau-\tau(n))} dt$$

$$\leq \sum_0^{n-1} e^{-mLk} I_m + e^{-nLk} I_n,$$

where the I's are of the form $\int_0^t e^{-\tau} dt$ with $t \leq 1$, and an upper bound for the I's carries over to the left-hand side with an extra factor L only. Since this argument applies both to d.m. and r.m., and since $\int e^{-\tau} dt$ for d.m. becomes $e^{-\tau} \int e^{\tau} dt$ for r.m. and *vice versa*, we have disposed also of the latter form. In other results the reduction to $t \leq 1$ is trivial (or irrelevant).

The result (a) has been proved already for $t \leq 1$ and obviously extends, by (c), to the general case. We have, then, proved (a) and (c), and may suppose in what follows always that $t \leq 1$.

§ 27. *Proof of* (b). We now provisionally assume (d), postponing its proof, and consider (b). In (b) it is enough to prove the first part (for both d.m. and r.m.), because of the interchange of $\int e^{-\tau} dt$ and $e^{-\tau} \int e^{\tau} dt$ between d.m. and r.m. (similar cases will recur). Next, we can reduce the proof of this first part to the special cases when the relevant stretch of d.m. lies respectively in (i) $U' - 1$ to U', (ii) $U'W$, (iii) WU.[1] For, assuming the special cases, and remembering that we need consider only one kind of half-cycle, suppose $(0, t)$ overlaps some of (i) to (iii); suppose, e.g., it overlaps all three. We have then, for the d.m. case, writing $\tau_{U'}$ for $\tau(U')$, etc.,

$$\int_0^t e^{-\tau} dt = \int_0^{U'} e^{-\tau} dt + e^{-\tau_{U'}} \int_{U'}^{W} e^{-(\tau-\tau_{U'})} dt + e^{-\tau_{U'}} \cdot e^{-(\tau_W - \tau_{U'})} \int_W^t e^{-(\tau-\tau_W)} dt.$$

The first two integrals on the right belong to the special cases (with t-origins at the lower limits) and are (by hypothesis) $< Dk^{\frac{1}{2}}$; and the third, by (d), is $< Dk^{-\frac{1}{2}}$. Also $e^{-\tau_{U'}} \leq 1$ (since $\dot{\tau} \geq 0$ in (i)), and $e^{-(\tau_W - \tau_{U'})} < Dk$, by (a). So

$$\int_0^t e^{-\tau} dt < Dk^{\frac{1}{2}} + 1 \cdot Dk^{\frac{1}{2}} + 1 \cdot Dk \cdot Dk^{-\frac{1}{2}} < Dk^{\frac{1}{2}},$$

as desired.

In the r.m. case, when the order is $OWU't$, we have

$$\int_0^t e^{-\tau} dt = \int_0^W e^{-\tau} dt + e^{-\tau_W} \int_W^{U'} e^{-(\tau-\tau_W)} dt + e^{-\tau_W} \cdot e^{-(\tau_{U'} - \tau_W)} \int_U^t e^{-(\tau-\tau_{U'})} dt.$$

[1] The restriction $t \leq 1$ ensures that the stretch can have at most one (connected) piece in Σ.

By hypothesis the first two integrals are $O(Dk^{\frac{1}{3}})$, and the third is $O(Dk^{-\frac{1}{3}})$ by (d). Also $e^{-\tau_W}\leq 1$, and $e^{-(\tau_{U'}-\tau_W)}<Dk$ by (a). It follows that $\int_0^t e^{-\tau}dt<Dk^{\frac{1}{3}}$ as desired.

Take now the special cases. (i) and (iii) are covered by (d), and it remains to prove $\int_0^t e^{-\tau}dt<Dk^{\frac{1}{3}}$ for a range in $U'W$, and for both d.m. and r.m. Now by Lemma 12 $e^{-\tau}<D|\dot{y}/\dot{y}_0|$, $D|\dot{y}_0/\dot{y}|$ in the two cases. In either case $e^{-\tau}<Dk/L$, and (b) follows since the range of integration $\leq U'W<Dk^{-\frac{1}{3}}$.

§ 28. *Proof of* (d). We give next the postponed proof of (d), with $t\leq 1$. We have $\dot{\tau}\geq 0$ except in dips, which last a time $O(Dk^{-\frac{1}{3}})$ and have depth $O(Dk^{-\frac{1}{3}})$, so that in a dip $\dot{\tau}=kf>-Lk|y-1|>-Dk^{\frac{1}{3}}$. It follows (since a unit time interval can overlap at most one dip) that $\tau>-D$.

The third result is equivalent to the second (with the two parts reversed) and we take this (of course for both d.m. and r.m.). Next, it is enough to prove (both d.m. and r.m.) for the special cases when $(0,t)$ lies respectively in (i) WK, (ii) KZ_1, (iii) Z_1Z_l, (iv) Z_lU. [See Fig. 4, § 23.] This follows by the argument used for (b), here simpler because the factors $e^{-\tau_K}$, $e^{-\tau_{Z_l}-\tau_K}$, etc., (for either d.m. or r.m.) are less than $e^{-\Delta\tau}<e^D=D$ (by the first part). We have, in fact, $\int_0^t e^{-\tau}dt<D$ times (a sum of integrals belonging to special cases). We take now the special cases, in each of which we mostly consider the d.m. and r.m. together, and are doing so unless the contrary is indicated.

In case (i) $e^{-\tau}\leq 1$ and $t\leq|K-W|<Lk^{-1}$, so $\int_0^t e^{-\tau}dt<Lk^{-1}$.

In case (ii) $\dot{\tau}=kf>Lk$ and $\int_0^t e^{-\tau}dt<\int_0^\infty e^{-Lkt}dt<Lk^{-1}$.

In case (iv) we have, writing $y=1+\eta$, $\dot{\eta}>L$ for $|t-U|\leq 1$ (Lemma 11 (5)), and so, over Z_lU, $\eta\geq L|t-U|$, $\dot{\tau}\geq Lk|t-U|$. For a d.m., writing $t=uk^{-\frac{1}{2}}$, $U=u_0k^{-\frac{1}{2}}$, we have then

$$\tau\geq Lk\int_0^t|t-U|\,dt\geq L\int_0^u|u-u_0|\,du,$$

and so
$$\int_0^t e^{-\tau}dt<\int_0^\infty \exp\left\{-L\int_0^u|u-u_0|\,du\right\}k^{-\frac{1}{2}}du<Lk^{-\frac{1}{2}}$$

(the last inequality being independent of the value of u_0); this is the desired result with the stronger L for D.

For an r.m., and so $U \leq 0 \leq t$, we have $\dot{\tau} \geq Lkt$, $\tau \geq Lkt^2$, and

$$\int_0^t e^{-\tau}\,dt < \int_0^\infty e^{-Lkt^2}\,dt < Dk^{-\frac{1}{2}},$$

as desired.

Take finally case (iii) in which once more the argument applies at once to d.m. and r.m. Let N be the time nearest $t=0$ at which $\varphi \equiv -\frac{1}{2}\pi$; this is in $Z_1 Z_l$.[1] Over $Z_1 Z_l$ we have $|\dot{y}| < D$, and so, over the range including both $(0, t)$ and N (of length at most $\pi + 1$) we have

$$F(y) - F(1) = b(1 + p_1(\varphi)) + C + O(Dk^{-1}). \tag{1}$$

At N we have $1 + p_1 = 0$; also F has a minimum at $y = 1$ and $F(y) - F(1) \geq 0$; hence $C > -Dk^{-1}$. Since $F(y) - F(1) \leq L\eta^2$, (1) now gives $\eta^2 > L(1 + p_1) - Dk^{-1}$, or

$$\eta^2 > L|t - N|^2 - Dk^{-1}, \tag{2}$$

by Lemma 2 (4) $[\varphi + \frac{1}{2}\pi = t - N]$. But also $\eta > -Dk^{-\frac{1}{2}}$, and this combined with (2) is easily verified to give

$$\eta > D|t - N| - Dk^{-\frac{1}{2}}. \tag{3}$$

Now (for the η's concerned) $f \geq L_1\eta$ when $\eta \geq 0$, $f \geq L_2\eta$ when $\eta < 0$. So

$$\begin{aligned}\tau = k\int_0^t f\,dt &\geq L_1 k \int_{(\eta \geq 0)}^t \eta\,dt + L_2 k \int_{(\eta < 0)}^t \eta\,dt\\ &\geq L_1 k \int_0^t \eta\,dt + L_2 k \int_{(\eta<0)}^t (-Dk^{-\frac{1}{2}})\,dt\\ &> Dk\int_0^t |t - N|\,dt - D,\end{aligned}$$

by (3) and because the range in which $\eta < 0$ is $O(Dk^{-\frac{1}{2}})$. Writing now, as above, $t = uk^{-\frac{1}{2}}$, $N = u_1 k^{-\frac{1}{2}}$, we have

$$\tau > D\int_0^u |u - u_1|\,du - D,$$

$$\int_0^t e^{-\tau}\,dt < \int_0^\infty \exp\left(-D\int_0^u |u - u_1|\,du - D\right) k^{-\frac{1}{2}}\,du < Dk^{-\frac{1}{2}},$$

as desired.

[1] In the extreme case of an r.m. with $t=0$ at Z_1 we take, of the two equidistant N's, the one in $Z_1 Z_l$.

We have now disposed of all the special cases, and so have completed the proof of (d).

§ 29. *Proof of* (e), (f) *and* (g). In (e) the two parts are equivalent, and we take the first. In this we have $y>1+L$, and so $\dot{\tau}>Lk$, except in the range $(W, W+k^{-1})$, in which $\dot{\tau}\geq 0$. It follows (for d.m. and r.m.) that

$$\int_0^t e^{-\tau}\,dt<\int_0^\infty e^{-Lkt}\,dt+\int_0^{k^{-1}} 1\cdot dt<Lk^{-1}.$$

In (f) we have, by Lemma 12 (reversed),

$$e^{-\tau}>D\,|\dot{y}_0/\dot{y}\,|,$$

$$\int_0^t e^{\tau}\,dt\leq \int_0^{U'} D\left|\frac{\dot{y}}{\dot{y}_0}\right|dt=D\frac{\eta_0}{|\dot{y}_0|},$$

where $\eta_0=y_0+1$. If $\eta_0\leq 2\eta^*$ defined in the proof of Lemma 12, this is $O(D\eta^*/L)<Dk^{-\frac{1}{3}}$; and if $\eta_0>2\eta^*$ it is

$$O(D\eta_0/\{k(G(\eta_0)-G(\tfrac{1}{2}\eta_0))\})=O(D\eta_0/(Lk\eta_0^2))$$
$$<Dk^{-1}\eta^{*-1}<Dk^{-\frac{1}{3}}.$$

We have therefore proved the first part of (f). For the second

$$\int_W^{U'} e^{\tau}\,dt<\int_W^{U'}\frac{D|\dot{y}|\,dt}{|\dot{y}_W|}=\frac{D}{|\dot{y}_W|}<Dk^{-1},$$

as desired.

In (g) the two parts are equivalent, and in the first

$$\tau<k\int_0^{k^{-1}} L\,dt=L\quad \text{for } t\leq k^{-1},$$

and

$$\int_0^t e^{-\tau}\,dt\geq \int_0^{k^{-1}} e^{-L}\,dt>Lk^{-1}.$$

§ 30. Lemma 14. *Let* Γ *satisfy the hypotheses of Lemma* 13, *and suppose further that* $0\leq t\leq 3\pi$. *Then, for d.m. or r.m.*

$$J=\int_0^t e^{-\tau(\eta)}\,d\eta\int_0^\eta e^{\tau(\xi)}\,d\xi=\iint\limits_{0\leq\xi\leq\eta\leq t} e^{-\tau_\eta+\tau_\xi}\,d\xi\,d\eta<Dk^{-\frac{1}{3}}.$$

LEMMA 15. *There is an* L_2^* *with the following properties. Let* Γ *satisfy the hypotheses of Lemma* 13, *and suppose further that* $(0, t)$ *is contained in* (W, U_1) *and that* $t \le L_2^* k$. *Then*[1] *for the d.m. (only)*

$$L_1^* J = L_1^* \iint_{0 \le \xi \le \eta \le t} e^{-\tau_\eta + \tau_\xi}\, d\xi\, d\eta < \tfrac{1}{2}.$$

We include here, for later convenience, a result with a special time-origin, and an integrand of opposite type, namely $\exp(\tau_\eta - \tau_\xi)$.

LEMMA 16. *Let* Γ *satisfy the hypothesis of Lemma* 13, *and let* $t = 0$ *be the special point* U'. *If further* $t \le 3\pi$, *then, for the d.m.*,

$$\iint_{0 \le \xi \le \eta \le t} e^{\tau_\eta - \tau_\xi}\, d\xi\, d\eta \le D \int_0^t e^{\tau}\, dt.$$

Proof of Lemma 14 *for r.m.* We take first the cases in which $t = 0$ is outside Σ, and the r.m. Γ over $(0, t)$ enters or crosses Σ downwards (for the standard case WU').

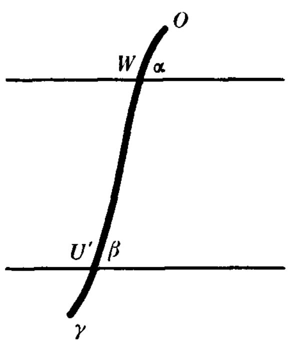

Fig. 5.

We begin with the most complicated case in which Σ is crossed, and write α, β, γ for W, U', t. Then $J \le u_1 + u_2 + u_3 + u_4$, where the ranges of ξ and η in the four u's are respectively restricted by (i) $0 \le \xi \le \eta \le \alpha$; (ii) $\alpha \le \eta \le \beta$, $\xi \le \beta$; (iii) $\beta \le \eta \le \gamma$, $\xi \le \beta$; (iv) $\beta \le \xi \le \eta \le \gamma$. (There is some overlapping, and in (ii) we drop the restriction $\xi \le \eta$.) We have (changing some names of variables of integration[2]), references (a), (b), ..., being to the various parts of Lemma 13,

$$u_1 = \int_0^\alpha dt \left(e^{-\tau} \int_0^t e^{\tau_\xi}\, d\xi\right) = \int_0^\alpha dt\, (O(D k^{-\frac{1}{2}})) \le D k^{-\frac{1}{2}}, \tag{1}$$

by (d).

[1] L_1^* is the upper bound of g' in $|y| \le L^*$ [§ 1].

[2] This cannot confuse, since the changes are made only in identities.

Next, $$u_2=\Bigl(\int_\alpha^\beta e^{-\tau}\,dt\Bigr)\Bigl(\int_0^\beta e^{\tau}\,dt\Bigr)=\Bigl(\int_\alpha^\beta e^{-(\tau-\tau_\alpha)}\,dt\Bigr)\Bigl(\int_0^\beta e^{\tau-\tau_\alpha}\,dt\Bigr), \tag{2}$$

and $$u_3=\Bigl(\int_\beta^\gamma e^{-\tau}\,dt\Bigr)\Bigl(\int_0^\beta e^{\tau}\,dt\Bigr)=\Bigl(\int_\beta^\gamma e^{-(\tau-\tau_\beta)}\,dt\Bigr)\cdot e^{(\tau_\alpha-\tau_\beta)}\cdot\Bigl(\int_0^\beta e^{\tau-\tau_\alpha}\,dt\Bigr). \tag{3}$$

The last factors on the right-hand sides of (2) and (3) are each

$$\int_0^\beta e^{\tau-\tau_\alpha}\,dt=e^{-\tau_\alpha}\int_0^\alpha e^{\tau}\,d\tau+\int_\alpha^\beta e^{\tau-\tau_\alpha}\,dt=O(Dk^{-1})+O(Dk^{-1})=O(Dk^{-1}),$$

by (e) and (f) respectively. The first factor on the right in (2) is $O(Dk^{\frac{1}{2}})$, by (b). The first factor on the right in (3) is $O(Dk^{-\frac{1}{2}})$, by (d) [applied to the (opposite kind of) half-cycle whose d.m. ends at U'], and the second factor is $O(Dk)$, by (a). Thus

$$u_2=O(Dk^{-1})\,O(Dk^{\frac{1}{2}})=O(Dk^{-\frac{1}{2}}),$$
$$u_3=O(Dk^{-1})\,O(Dk^{-\frac{1}{2}})\,O(Dk)=O(Dk^{-\frac{1}{2}}).$$

Finally $$u_4=\int_\beta^\gamma dt\Bigl(e^{-\tau}\int_\beta^t e^{\tau_\xi}\,d\xi\Bigr)=\int_\beta^\gamma dt\cdot O(Dk^{-\frac{1}{2}})=O(Dk^{-\frac{1}{2}}),$$

by (d) [again for the half-cycle ending at U']. This completes the proof for the case considered.

The cases when O is above $y=1$ and $t=\gamma$ is in or above Σ are effectively particular cases of the foregoing one. u_3 und u_4 disappear, and the treatment of u_1 is as before. If there is a u_2 it becomes

$$u_2=\Bigl(\int_\alpha^\gamma e^{-(\tau-\tau_\alpha)}\,dt\Bigr)\Bigl(e^{-\tau_\alpha}\int_0^\alpha e^{\tau}\,dt+\int_\alpha^\gamma e^{\tau-\tau_\alpha}\,dt\Bigr)\cdot$$

The first factor is $O(Dk^{\frac{1}{2}})$ by (b) as before; the first term of the second factor is $O(Dk^{-1})$, as before; and the second one is $O(Dk^{-1})$ by the second part of (f).

Take next the case when $t=0$ is in Σ. When Γ crosses $y=-1$, or when $\gamma>\beta$, we have (with overlapping)

$$J\le u_1+u_2+u_3,$$

where the variables in $u_{1,2,3}$ are subject respectively to (i) $0\le\xi\le\beta,\ 0\le\eta\le\beta$; (ii) $0\le\xi\le\beta,\ \beta\le\eta\le\gamma$; (iii) $\beta\le\xi\le\eta\le\gamma$. We have by Lemma 12, reversed for r.m., $\exp(-\tau_\eta+\tau_\xi)<D\dot{y}(\xi)/\dot{y}(\eta)$. So

$$u_1\le D\left(\int_0^\beta \dot{y}(\xi)\,d\xi\right)\left(\int_0^\beta\frac{dt}{\dot{y}}\right)\le D\cdot 2\cdot(\beta/L)<Dk^{-\frac{1}{2}}$$

(since $\beta<Dk^{-\frac{1}{2}}$).

Next
$$u_2 = \Big(\int_0^\beta e^{\tau_\xi - \tau_\beta}\, d\xi\Big)\Big(\int_\beta^\gamma e^{-(\tau_\eta - \tau_\beta)}\, d\eta\Big);$$

the first factor is $O\left(D\int_0^\beta \left|\frac{\dot y(\xi)}{\dot y(\beta)}\right| d\xi\right) < D$, since $|\dot y_\beta| > L$; the second is $O(D k^{-\frac{1}{2}})$, by (d); and so $u_2 < D k^{-\frac{1}{2}}$.

Finally
$$u_3 \leq \int_\beta^\gamma d\eta \int_\beta^\eta e^{-(\tau_\eta - \tau_\xi)}\, d\xi \leq \int_\beta^\gamma d\eta\, D k^{-\frac{1}{2}} < D k^{-\frac{1}{2}}$$

by (d). So $J < D k^{-\frac{1}{2}}$, as desired.

When $\gamma < \beta$ we have $J \leq \iint\limits_{0 \leq \xi \leq \eta \leq \beta}$, which is u_1 of the previous case and accordingly $O(D k^{-\frac{1}{2}})$.

There remains to be considered only the case when Γ has no point in common with Σ and then we have

$$I \leq \int_0^{3\pi} d\eta \Big(e^{-\tau_\eta}\int_0^\eta e^{\tau_\xi}\, d\xi\Big) \leq \int_0^{3\pi} d\eta\, D k^{-\frac{1}{2}} = D k^{-\frac{1}{2}},$$

by (d). This completes the proof of Lemma 14 for r.m.

§ 31. *Proof of Lemma* 14 *for d.m.* We begin with the most complicated case when Σ is crossed (upwards), and write α, β, γ for U', W, t. We have

$$\iint\limits_{0 \leq \xi \leq \eta \leq t} e^{-\tau_\eta + \tau_\xi}\, d\xi\, d\eta \leq u_1 + u_2 + u_3 + u_4 + u_5,$$

where in the five u's we have respectively (i) $0 \leq \xi \leq \eta \leq \alpha$; (ii) $0 \leq \xi \leq \alpha$, $\alpha \leq \eta \leq \beta$; (iii) $\alpha \leq \xi \leq \beta$, $\alpha \leq \eta \leq \beta$; (iv) $0 \leq \xi \leq \beta$, $\beta \leq \eta \leq \gamma$; (v) $\beta \leq \xi \leq \eta \leq \gamma$.

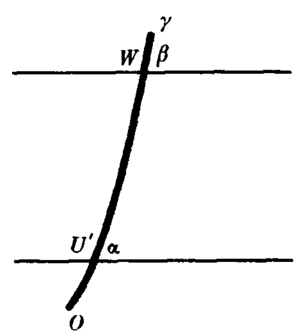

Fig. 6.

We have

$$u_1 \le \int_0^{\alpha} d\eta \int_0^{\eta} e^{-\tau_\eta+\tau_\xi}\, d\tau < \int_0^{3\pi} d\eta\, D k^{-\frac{1}{2}} = D k^{-\frac{1}{2}},$$

by (d).

Next

$$u_2 = \left(\int_0^{\alpha} e^{\tau-\tau_\alpha}\, dt\right)\left(\int_{\alpha}^{\beta} e^{-(\tau-\tau_\alpha)}\, dt\right)\cdot$$

The first factor is $O(Dk^{-\frac{1}{2}})$, by (d). The second, by Lemma 12, is

$$O\left(D\int_{\alpha}^{\beta} \frac{\dot{y}}{\dot{y}_\alpha}\, dt\right) < D\cdot\frac{2}{L} = D, \text{ since } \dot{y}_\alpha > L.$$

So $u_2 < Dk^{-\frac{1}{2}}$.

In u_3 we have $\exp(-\tau_\eta+\tau_\xi) < D\dot{y}(\eta)/\dot{y}(\xi)$, and so

$$u_3 < D\cdot 2\int_{\alpha}^{\beta} \frac{dt}{\dot{y}} < D(\beta-\alpha) < Dk^{-\frac{1}{2}},$$

since the time over $\alpha\beta$ is $O(Dk^{-\frac{1}{2}})$.

Next,

$$u_4 = \left(\int_0^{\beta} e^{(\tau-\tau_\beta)}\, dt\right)\left(\int_{\beta}^{\gamma} e^{-(\tau-\tau_\beta)}\, dt\right)\cdot$$

The first factor is $O(Dk^{\frac{1}{2}})$, by (b); the second is $O(Dk^{-1})$ since $y > 1+L$ for $t-\beta > k^{-1}$ and so

$$\int_{\beta}^{\gamma} \le \int_{\beta}^{\beta+k^{-1}} 1.\ dt + \int_{\beta+k^{-1}}^{\infty} e^{-Lkt}\, dt.$$

So

$$u_4 < Dk^{-\frac{1}{2}}.$$

Finally

$$u_5 = \int_{\beta}^{\gamma} d\eta \int_{\beta}^{\eta} e^{-\tau_\eta+\tau_\xi}\, d\xi < \int_{\beta}^{\gamma} d\eta\, Dk^{-\frac{1}{2}} < Dk^{-\frac{1}{2}}$$

by (d).

Take next the case when 0 is in Σ; we may suppose (*a fortiori*) that $\gamma > \beta$. We have $J \le u_1+u_2+u_3$, when in $u_{1,2,3}$ we have respectively (i) $0\le\xi\le\beta$, $0\le\eta\le\beta$; (ii) $0\le\xi\le\beta$, $\beta\le\eta\le\gamma$; (iii) $\beta\le\xi\le\eta\le\gamma$.

We have $u_3 < Dk^{-\frac{1}{2}}$ as for u_5. Also (practically as for u_3 of the last case)

$$u_1 < D\int_0^{\beta}\int_0^{\beta} \frac{\dot{y}(\eta)}{\dot{y}(\xi)}\, d\xi\, d\eta < D\int_0^{\beta} \frac{dt}{\dot{y}} < D\beta < Dk^{-\frac{1}{2}},$$

since $\dot{y} > L$.

Finally
$$u_2=\left(\int_0^{\beta} e^{\tau}\,dt\right)\left(\int_{\beta}^{\gamma} e^{-\tau}\,dt\right),$$

in which the second factor is $O(Dk^{-\frac{1}{2}})$ by (d), and $e^{\tau}\leq 1$ in the first, so that $u_2=O(Dk^{-\frac{1}{2}})$.

There remains the case when Γ in $(0, t)$ has no point in common with Σ, and then
$$J=\int_0^t d\eta\left(e^{-\tau_\eta}\int_0^{\eta} e^{\tau_\xi}\,d\xi\right)$$

in which the bracket is $O(Dk^{-\frac{1}{2}})$ by (d). This completes the proof of Lemma 14 for a d.m.

§ 32. *Proof of Lemma* 15. If $t=0$ in WZ_1 and the L_2^* (of the condition $t\leq L_2^* k$) is chosen small enough, we have $y>1+L$ except in $(W, W+k^{-1})$, during which $\tau_\eta-\tau_\xi\geq 0$. So
$$\int_0^{\eta} e^{-\tau_\eta+\tau_\xi}\,d\xi\leq\int_0^{k^{-1}} 1\cdot d\xi+\int_{k^{-1}}^{\infty} e^{-Lk\xi}\,d\xi=O(k^{-1}),$$

and $L_1^*\,J<\frac{1}{2}$ provided L_2^* is small enough. We may therefore suppose $(0, t)$ is in $Z_1 U$. Let $N_1, N_2, \ldots, N_\nu$ be the N's (points with $\varphi\equiv-\frac{1}{2}\pi$) contained in $(0, t)$, so that $0\leq N_1\leq\cdots\leq N_\nu\leq U$.[1] The equation of Γ between N_n and N_{n+1} is
$$F(y)-F(1)=b(1+p_1(\varphi))+C_n+O(Dk^{-1}), \qquad (1)$$

where
$$-(C_n-C_{n-1})=\frac{1}{k}\int_{N_{n-1}}^{N_n} g(y)\,dt>Lk^{-1}. \qquad (2)$$

At N_n we have $F(y)-F(1)\geq 0$, $1+p_1(\varphi)=0$, so that $C_n>-Dk^{-1}$. By adjusting the error term in (1) we may therefore suppose $C_n\geq C_\nu\geq 0$. For $n\leq\nu$ let $J_n^{(1)}$, $J_n^{(2)}$ be
$$\iint e^{-\tau_\eta+\tau_\xi}\,d\xi\,d\eta,$$

with integrations over the respective ranges
$$N_n-\tfrac{3}{2}\pi\leq\xi\leq\eta\leq N_n,\quad N_n\leq\xi\leq\eta\leq N_n+\tfrac{3}{2}\pi$$

(these overlap neighbours with $n-1$ and $n+1$ respectively). Now
$$\exp\,(-\tau_\eta+\tau_\xi)<e^{-Lk}\ \text{if}\ \eta-\xi\geq\tfrac{1}{2}\pi;\ \text{also}\ \iint e^{-\tau_\eta+\tau_\xi}\,d\xi\,d\eta=O(Dk^{-\frac{1}{2}})$$

[1] The N near U is *after* Z_1U, and in the extreme case when $(0, t)$ extends to U, N_ν is approximately 2π before U.

for a range of η of length $\leq\frac{3}{2}\pi$ (and $\xi\leq\eta$), by Lemma 14; in particular $J_n^{(1)}, J_n^{(2)}=O(Dk^{-\frac{1}{2}})$ (uniformly in n). By rejecting appropriate areas of integration with $\eta-\xi>\frac{1}{2}\pi$ (contributing $O(k^2e^{-Lk})$ to J) we are left with something less than the sums of the $J_n^{(1)}$ and $J_n^{(2)}$;

$$J\leq\sum_{n=1}^{\nu}(J_n^{(1)}+J_n^{(2)}+O(Dk^{-\frac{1}{2}})).^1 \tag{3}$$

We proceed to evaluate $J_n^{(1)}, J_n^{(2)}$: they are effectively alike, and it is enough to consider $J_n^{(2)}$. It follows from (2) and $C_\nu\geq 0$ that $C_n\geq(\nu-n)Lk^{-1}$. For the range $(N_n-\frac{3}{2}\pi,\ N_n+\frac{3}{2}\pi)$ we have[2] for $y=1+\eta$

$$\begin{aligned} L\eta^2 &\geq F(y)-F(1)=b(1+p_1(\varphi))+L(\nu-n)k^{-1}-Dk^{-1}\\ &>L(t-N_n)^2+L(\nu-n)k^{-1}-Dk^{-1}. \end{aligned}\tag{4}$$

Also
$$\eta>-Dk^{-\frac{1}{2}}. \tag{5}$$

It follows from (4) and (5) [cf. § 28 (2)] that

$$\eta>L\,|t-N_n|+L(\nu-n)^{\frac{1}{2}}k^{-\frac{1}{2}}-Dk^{-\frac{1}{2}},$$

and consequently

$$\eta>L\,|t-N_n|+L(\nu-n)^{\frac{1}{2}}k^{-\frac{1}{2}}>0 \qquad (n\leq\nu-D). \tag{6}$$

We proceed to show that

$$J_n^{(2)}<Lk^{-1}\log\frac{Lk}{\nu-n} \qquad (n\leq\nu-D_1). \tag{7}$$

From this (and the corresponding upper bound for $J_n^{(1)}$) the result of Lemma 15 will follow. For we may replace $\sum_1^{\nu}$ in (3) by $\sum_1^{\nu-D_1}$, and then, after (7), and supposing $t\leq 2\pi\Lambda k$ and so $\nu\leq\Lambda k$ (where Λ will be chosen presently), we have for $J^{(2)}=\sum_n J^{(2)}$,

$$\begin{aligned} J^{(2)} &< Lk^{-1}\sum_1^{\nu-1}\log\frac{Lk}{\nu-n}+Dk^{-\frac{1}{2}}<L_1k^{-1}\sum_1^{\Lambda k}\log\frac{L_2k}{n}+Dk^{-\frac{1}{2}}\\ &< L_1k^{-1}\int_0^{\Lambda k}\log\frac{L_2k}{x}\,dx+Dk^{-\frac{1}{2}}\\ &< L_1k^{-1}\{\Lambda k\log(L_2k)-\Lambda k\log(\Lambda k)+\Lambda k\}+Dk^{-\frac{1}{2}}\\ &< L_1\Lambda(\log(L_2/\Lambda)+1)+Dk^{-\frac{1}{2}}\\ &< \tfrac{1}{6}/L_1^*, \end{aligned}$$

[1] Incidentally, we have got rid of the odd pieces $(0, N_1)$, (N_ν, t).

[2] We have momentarily two meanings for η, but they are easily distinguishable.

provided Λ is chosen to be a sufficiently small L. If then L_2^* is chosen sufficiently small we shall have, for $t \le L_2^* k$,

$$L_1^* J \le L_1^* (J^{(1)} + J^{(2)} + D k^{-\frac{1}{2}}) < \tfrac{1}{6} + \tfrac{1}{6} + D k^{-\frac{1}{2}} < \tfrac{1}{2},$$

as desired.

Consider then (7), and in $J_n^{(2)}$ write $\xi = N_n + k^{-\frac{1}{2}} u$, $\eta = N_n + k^{-\frac{1}{2}} v$, so that $0 \le u \le v \le \frac{3}{2}\pi k^{\frac{1}{2}}$. We have from (6), and $f \ge L(y-1)$

$$\tau_\eta - \tau_\xi \ge L k \int_\xi^\eta (t - N_n)\, dt + L k^{\frac{1}{2}} (\nu - n)^{\frac{1}{2}} (\eta - \xi)$$

$$= L \int_u^v s\, ds + L(\nu - n)^{\frac{1}{2}} (v - u),$$

$$J_n^{(2)} \le L k^{-1} \iint_{0 \le u \le v \le {}^3/_2 \pi k^{\frac{1}{2}}} e^{-E}\, du\, dv,$$

where $$E = L(v^2 - u^2) + L(\nu - n)^{\frac{1}{2}} (v - u) \ge L(v - u)(v + (\nu - n)^{\frac{1}{2}}).$$

Performing the u-integration first we have

$$J_n^{(2)} \le \frac{L}{k} \int_0^{{}^3/_2 \pi k^{\frac{1}{2}}} \frac{1 - e^{-Lv(v + (N-n)^{\frac{1}{2}})}}{v + (\nu - n)^{\frac{1}{2}}}\, dv < \frac{L}{k} \int_0^{{}^3/_2 \pi k^{\frac{1}{2}}} \frac{dv}{v + (\nu - n)^{\frac{1}{2}}}$$

$$= \frac{L}{k} \log\left(\frac{(3\pi/2) k^{\frac{1}{2}}}{(\nu - n)^{\frac{1}{2}}} + 1\right) < \frac{L}{k} \log \frac{Lk}{\nu - n},$$

the desired result (7).

§ 33. *Proof of Lemma* 16. It is enough to prove $\int_0^t e^{-\tau}\, dt < D$, or again the worst case $\int_0^{3\pi} = \int_0^K + \int_K^{3\pi} < D$. In $\int_0^K$ we have $e^{-\tau} < D\dot{y}/\dot{y}(0) < D\dot{y}$ by Lemma 12, and $\int_0^K \le D \int_0^K \dot{y}\, dt = DK = D$.

Also $$\int_K^{3\pi} = e^{-\tau_K} \int_K^{3\pi} e^{-(\tau - \tau_K)} dt < Dk \int_K^{3\pi} e^{-Lk(t-K)} dt < D,$$

by (a), and since $\tau > Lk$ in $(K, 3\pi)$.

§ 34. We now take up the question of the behaviour of two neighbouring $\Gamma_{1,\,2}$, more particularly of their "convergence". We suppose always in what follows that $\Gamma_{1,\,2}$ have started in some S_1 and have been gap-free *before*[1] the moment under

[1] If the moment is one of an arrival at $y = \pm 1$ we are *not* assuming anything about this.

consideration, with their $U_{1,2}$ and $U'_{1,2}$ "together" at the relevant N and N'. After Lemma 9 $\Gamma_{1,2}$ belong to the S_1 (and indeed the S) at the Z_0 following each common shoot-through. and we may suppose without loss of generality that Z_0, and its neighbouring Z_1, are the *first* such points.

We set out some permanent notation.

If X is a number associated with a Γ (e.g. $y(t)$, V, ω) we denote $X(\Gamma_2)-X(\Gamma_1)$ by ΔX. We write $w=y_2-y_1=\Delta y$,

$$u=\frac{\Delta F}{\Delta y},\quad \gamma=\frac{\Delta g}{\Delta y}\quad (\gamma \text{ satisfies } 1\le\gamma\le L_1^*).$$

We consider sometimes d.m., sometimes r.m., from an "arbitrary" origin $t=0$. For *either* d.m. or r.m. we define[1]

$$T=k\int_0^t u\,dt, \text{ so that } \dot{T}=ku=k\frac{\Delta F}{\Delta y};$$

and
$$w_1=\int_0^t \Delta g\,dt=\int_0^t \gamma w\,dt.$$

Since γ lies between L's w_1 can be thought of as a modified integral of w; it is exactly this when $g=y$. The suffix is used to suggest such an integration.

We define also, each for either d.m. or r.m.,

$$w_0=w(0),\qquad \dot{w}_0=\dot{w}(0),\qquad \dot{T}_0=\dot{T}(0).$$

For d.m. we define $c_0=\dot{w}_0+\dot{T}_0 w_0$, for r.m. $\bar{c}_0=\dot{w}_0-\dot{T}_0 w_0$, and for general t $c(t)=\dot{w}+\dot{T}w$, $\bar{c}(t)=\dot{w}-\dot{T}w$. $c(t)$ satisfies the identity

$$c(t_2)-c(t_1)=-\int_{t_1}^{t_2}\Delta g\,dt=-w_1(t_2)+w_1(t_1), \tag{1}$$

as is easily verified.

For the d.m. and r.m. with the same origin $t=0$ the w_0 and $\dot{T}_0$ are the same, the $\dot{w}_0$ equal and opposite; and $\bar{c}_0=-c_0$ (generally $\bar{c}(t)=-c(-t)$). As for τ in §§ 25–34, for a given stretch of time taken in opposite direction as $(0,t)$, the two T are the same (but e.g. the $\int_0^t e^{-T}dt$ are not the same).

[1] This runs parallel to our use of τ in §§ 25–34.

The w differential equation for d.m. has the two equivalent forms

$$(W)\qquad \begin{cases} \dot{w} = -\dot{T}w + c_0 - w_1, \\ \dfrac{d}{dt}(e^T w) = e^T (c_0 - w_1) \end{cases}$$

and that for an r.m. the forms[1]

$$(\overline{W})\qquad \begin{cases} \dot{w} = \dot{T}w + \bar{c}_0 - w_1, \\ \dfrac{d}{dt}(e^{-T} w) = e^{-T} (\bar{c}_0 - w_1). \end{cases}$$

§ 35. We shall often have to use certain developments from (W), $(\overline{W})$; we label them for reference, with "barred" labels for r.m. results.

For d.m. $$w = w_0 e^{-T} + e^{-T} \int_0^t e^T (\bar{c}_0 - w_1)\, dt \qquad (W_1)$$

We generally normalize (by renumbering $\Gamma_{1,2}$ if necessary) to w being initially positive; i.e. $w_0 > 0$, or, if $w_0 = 0$, then $\dot{w}_0 > 0$.

With normalized w we have, up to the next[2] intersection (if any)

$$\left.\begin{aligned} & w \le w_0 e^{-T} + c_0 \varphi(t) \quad \text{(up to intersection)}, \\ & \varphi = e^{-T} \int_0^t e^T\, dt. \end{aligned}\right\} \qquad (W_2)$$

Substituting this in the $w_1 = \int \gamma w\, dt$ in (W_1), in which $1 \le \gamma \le L_1^*$, we have, for normalized w,

$$\left.\begin{aligned} & w e^T \ge w_0 - w_0 \psi + c_0 \left(\int_0^t e^T dt - \vartheta \chi \right) \quad \text{(to intersection)}, \\ & \psi = L_1^* \iint\limits_{0 \le \xi \le \eta \le t} e^{T_\eta - T_\xi}\, d\xi\, d\eta, \qquad \chi = L_1^* \iiint\limits_{0 \le \xi \le \eta \le \zeta \le t} e^{T_\xi - T_\eta + T_\zeta}\, d\xi\, d\eta\, d\zeta, \end{aligned}\right\} \qquad (W_3)$$

and we use ϑ always for numbers satisfying $0 \le \vartheta \le 1$.[3]

If we drop w_0 from the right-hand side of the inequality we obtain the two further inequalities

---

[1] These are of course valid for the $w = y_2 - y_1$ of any pair $\Gamma_{1,2}$ whatever.

[2] That is, the next after $t = 0$ if $w_0 = 0$, $\dot{w}_0 > 0$.

[3] The ϑ is needed in the inequality because c_0 may be negative.

$$\left.\begin{aligned} w &\geq -w_0 e^{-T}\psi + c_0\varphi\Big(1-\vartheta\chi \Big/ \int_0^t e^T dt\Big) \quad \text{(to intersection)},\\ w &\geq c_0\varphi\Big(1-\vartheta\chi \Big/ \int_0^t e^T dt - (w_0/c_0)\psi \Big/ \int_0^t e^T dt\Big) \end{aligned}\right\} \qquad (W_4)$$

of which the second (though true generally) is specially appropriate for use when $c_0 > 0$.

For r.m.

$$w = w_0 e^T + e^T \int_0^t e^{-T}(\bar{c}_0 - w_1)\,dt. \qquad (\overline{W}_1)$$

For normalized w we have, up to the next intersection (if any)

$$w e^{-T} \leq w_0 + \bar{c}_0 \int_0^t e^{-T} dt \quad \text{(to intersection)}, \qquad (\overline{W}_2)$$

and substituting as in the d.m. case we obtain[1]

$$\left.\begin{aligned} & w e^{-T} \geq w_0(1-\bar{\psi}) + \bar{c}_0\Big(\int_0^t e^{-T} dt - \vartheta\bar{\chi}\Big) \quad \text{(to intersection)},\\ & \bar{\psi} = L_1^* \iint_{0\leq\xi\leq\eta\leq t} e^{-T_\eta + T_\xi}\, d\xi\, d\eta, \qquad \bar{\chi} = L_1^* \iiint_{0\leq\xi\leq\eta\leq\zeta\leq t} e^{-T_\xi + T_\eta - T_\zeta}\, d\xi\, d\eta\, d\zeta. \end{aligned}\right\} \qquad (\overline{W}_3)$$

The special value $t=1$ is important here, and we define the constant μ_0 (associated with the origin $t=0$) by

$$\mu_0 = \int_0^1 e^{-T} dt \quad \text{(r.m.)}.$$

If there is no intersection of the r.m. up to $t=1$ we can (using $c_0 = -\bar{c}_0$) translate $(\overline{W}_{2,3})$ (with $t=1$) into the following d.m. result:

given there is no intersection of the d.m. for time 1 *before* $t=0$, *and that* $w_0 > 0$, *we have*[2]

$$\left.\begin{aligned} & \mu_0 c_0 \leq w_0 \\ & \mu_0 c_0 (1 - \vartheta\bar{\chi}(1)/\mu_0) > w_0(1-\bar{\psi}(1)) - \zeta, \end{aligned}\right\} \qquad (W_5)$$

where ζ, as always and constantly in what follows, denotes a number of the form $O(e^{-Dk})$. The ζ in the second inequality comes from $w_0 e^{-T(-1)}$.

The discussion that follows is long and intricate. The main work is to prove

---

[1] Our treatments of d.m. and r.m. do not run quite parallel, which is why $(\overline{W}_3)$ and $(W_{3,4})$ do not.

[2] μ_0 is here of course, $\int_0^1 e^{-T} dt$ for the r.m. (or $\int_0^1 e^{-T(-t')} dt'$ for the d.m.).

that either $\Gamma_{1,2}$ do not intersect, or else they differ by $O(\zeta)$ after a certain point. We concentrate first on the cases of non-intersection.

We start (for the present) at a Z_1, which we may suppose to be the first one after the start in S_1 at Z_0 and we study the behaviour of w over the half-cycle $Z_1 Z_1'$.

We shall use a mesh of trajectories Γ of S_1, "intermediate" to $\Gamma_{1,2}$ and with consecutive ordinates at Z_1 differing by at most k^{-a}, where a is either 10 or 11.

§ 36. In the Lemmas that follow we take for granted the hypothesis that $\Gamma_{1,2}$ are Γ's of the S_1 at Z_0, and that (in accordance with our convention that letters denote points the *first* of their kind). Z_1 is a period later than Z_0.

LEMMA 17. *Let Ω be an arbitrary time-origin in (Z_1-1, Z_1+1). Then $w_0 = w(\Omega) = O(Dk^{-1})$. Also, normalizing to $w_0 \geq 0$, we have a linkage of $c(\Omega)$, $w(\Omega)\,[c_0, w_0]$:*

$$\mu_0 c_0 - \zeta < w_0 < 2\mu_0 c_0 + \zeta;$$

and in this $\mu_0 \lesssim L k^{-1}$, so that

$$L k^{-1} c_0 - \zeta < w_0 < L k^{-1} c_0 + \zeta.$$

In particular all this is true for $w(Z_1)$, $c(Z_1)$, and further $\dot{w}(Z_1) = O(Dk^{-1})$.

COROLLARY 1. *There is a ζ' such that if $w(Z_1) > \zeta'$ then $c(Z_1) > L k w(Z_1)\ (>0)$.*

COROLLARY 2. *If $w(\Omega) = 0$, then $\dot{w}(\Omega) = O(\zeta)$; in particular this is true for $\Omega = Z_1$.*

We have for a Γ of the S_1 at Z_0, "starting" at t_0, y_0, $\dot{y}_0$, say,[1]

$$F = F(y_0) + b(p_1(\varphi) - p_1(\varphi_0)) - \frac{1}{k}\left(\int_{t_0}^{t} g\,dt + \dot{y} - \dot{y}_0\right). \tag{1}$$

As in § 22, $p_1(\varphi_0) = 1 + O(Dk^{-1})$. $F(y_0) = F(H) + O(Dk^{-1})$. Let G be the range (Z_1-3, Z_1+3) (this includes both the d.m. and the r.m. from Ω to time $t=2$). Lemma 3 (ii) (1) is valid with $\dot{y} = O(1)$, $\ddot{y} = O(1)$ in G. Hence we have in G, for $\Gamma_{1,2}$, $\Delta F = O(Dk^{-1})$ and so $w = O(Dk^{-1})$; also

$$\dot{y} = b\,p/f - (\ddot{y} + g)/(kf)$$

for $y_{1,2}$, and so $\dot{w}(Z_1) = O(Dk^{-1})$, since $p(Z_1) = 0$.

[1] We should naturally take $\mathfrak{Z}$ were it not that Z_0 may be the first Z and Γ have no $\mathfrak{Z}$. Cf. § 23.

We prove next that for origin Ω:

$$\bar{\varphi}(t)<Lk^{-1},\ \ \bar{\chi}(t)\leq Lk^{-1}\int_0^t e^{-T}dt \quad \textit{for the r.m., and } t\leq 1. \tag{2}$$

In G we have $y_{1,2}>1+L$, $\dot{T}>Lk$, $T\geq Lkt$ (d.m. or r.m.). For r.m.

$$e^{-T_\eta+T_\xi}\leq \exp\Big(-\int_\xi^\eta Lk\,dt\Big)=\exp(-Lk(\eta-\xi)),$$

and so

$$\bar{\varphi}\leq L_1^*\int_0^1 d\eta\int_0^\eta e^{-Lk(\eta-\xi)}d\xi<Lk^{-1},$$

and consequently also

$$\bar{\chi}\leq L_1^*\Big(\int_0^t e^{-T_\xi}d\xi\Big)\Big(\iint_{0\leq\eta\leq\zeta\leq t} e^{-T_\zeta+T_\eta}d\eta\,d\zeta\Big)=\bar{\varphi}\int_0^t e^{-T}dt\leq Lk^{-1}\int_0^t e^{-T}dt$$

so that (2) is proved. Since $Lk<\dot{T}<Lk$, $\mu(\Omega)=\int_0^1 e^{-T}dt$ with origin at $t=\Omega$ (r.m.) lies between two $\int^1 e^{-Lkt}dt=Lk^{-1}$,

$$Lk^{-1}<\mu(\Omega)<Lk^{-1}, \tag{3}$$

as stated in the Lemma.

Consider now the r.m. from Ω to $t=1$, or the first intersection of $\Gamma_{1,2}$ (if any), whichever happens first. We distinguish two cases: (i) an intersections happens first, (ii) no intersection before $t=1$.

Case (i). Consider the r.m. from the intersection as new origin for a (further) time 1 or to the next intersection, whichever happens first. We have $\dot{w}_0<0$, and $w\leq 0$ in the range. $(\overline{W}_3)$, written with $-w$ for w to normalize, is valid, with $w_0=0$, $\bar{c}_0=-\dot{w}_0=|\dot{w}_0|$. Since the results (2) are valid we infer that

$$-we^{-T}\geq \tfrac{1}{2}|\dot{w}_0|\int_0^t e^{-T}dt,\ \text{ or }\ we^{-T}\leq -\tfrac{1}{2}\dot{w}_0\int_0^t e^{-T}dt. \tag{4}$$

This shows, first that there is no (second) intersection, and so, secondly, that (4) is valid at $t=1$, when it gives $\dot{w}_0=O(\zeta)$. Incidentally this establishes Corollary 2.

Consider now the d.m. from the intersection of $\Gamma_{1,2}$ up to Ω. There is no intersection before Ω, and (W_2) is valid with $w_0=0$, $c_0=|\dot{w}_0|$, so that

$$0\leq w\leq|\dot{w}_0|\varphi<\zeta,\quad\text{and}\quad w_1=O\Big(\int_0^1|w|\,dt\Big)=O(\zeta).$$

In particular $w(\Omega) = O(\zeta)$, and also, taking $t = t_\Omega$ in $\dot{w} = -\dot{T}w + \dot{w}_0 - w_1$, we have $\dot{w}(\Omega) = O(\zeta)$, and so $c(\Omega) = O(\zeta)$. To sum up, in case (i) (of intersection) $w(\Omega)$, $c(\Omega) = O(\zeta)$.

Case (ii). In this case $(\overline{W}_{2,3})$ and (2) are valid over $0 \le t \le 1$ of the r.m. with $w_0 = w(\Omega)$, $\bar{c}_0 = -c(\Omega)$; in particular they are valid at $t = 1$. Since the left side of $(\overline{W}_2)$ is positive we have

$$w(\Omega) - \mu(\Omega)\, c(\Omega) > 0. \tag{5}$$

Substituting in $(\overline{W}_3)$, with $t = 1$, from the inequalities (2) for $\bar{\varphi}$, $\bar{\chi}$, and observing that the left side is $O(\zeta)$, we have

$$\zeta > \tfrac{1}{2}\, w(\Omega) - \mu(\Omega)\,(1 - \tfrac{1}{2}\vartheta)\, c(\Omega). \tag{6}$$

(5) and (6) give the c, w linkages at Ω of the Lemma. These are proved for case (ii), but are valid also (trivially) in case (i), when $w(\Omega)$, and $c(\Omega)$ are $O(\zeta)$. The remaining results of the Lemma follow in virtue of (3), and Corollary 1 is a trivial consequence of the Lemma.

§ 37. We must now introduce a mesh. With $a = 10$ or 11 (always) we can divide the interval between the ordinates of $\Gamma_{1,2}$ at Z_1 (which is $O(Dk^{-1})$ into at most $N < Dk^{a-1}$ equal intervals of common length $\varepsilon \le k^{-a}$ [$N = 1$ if $w(Z_1) \le k^{-a}$]. By continuity, a Γ of the S_1 of §§ 19 and 24 at Z_0 can be found to pass through each point of division.[1] We will denote a consecutive pair of these by $\Gamma_{3,4}$, and take over the w, T, c, μ etc., notation with the understanding that they refer to the pair $\Gamma_{3,4}$.

The Lemmas immediately following are restricted to the half-cycle $Z_1 Z_1'$, but not always to the range $Z_1 U$ of Lemma 3 (d) (which is only part, though the worst part, of $Z_1 Z_1'$). They are all about an arbitrary pair $\Gamma_{3,4}$ of the k^{-a} mesh;[1] they will mention $\Gamma_{3,4}$ explicitly as a safeguard.

§ 38 We give a short name to a hypothesis often made about $\Gamma_{3,4}$: namely that *starting at Z_1 as consecutives of a k^{-a} mesh* $(0 < w(Z_1) \le k^{-a})$, *they have their $U_{3,4}$ together and have* $V_{3,4} \ge V^* + \delta$. We call this hypothesis (H). We sometimes say also that $\Gamma_{1,2}$ satisfy (H), with the obvious meaning. We denote the earlier and later of $U_{3,4}$ by U_-, U_+ (we do not as yet know even that $U_{3,4}$ will be together), and by Γ_- and Γ_+ the corresponding Γ of $\Gamma_{3,4}$.

---

[1] Their assertions are to be true for *both* values of a.

Suppose that $(0, t)$ is in (Z_1, U_-), or again that it is in (Z_1, Z_1') and $\Gamma_{3,4}$ satisfy (H); suppose further that in $(0, t)$ we have $|w|<k^{-5}$. Then for T (formed from $\Gamma_{3,4}$) we have $\dot{T}=\dot{\tau}+O(k\cdot k^{-5})$, where τ is formed from Γ_3, and

$$T=\tau+O(k^{-3}),\ e^{\pm T}=e^{\pm\tau}(1+O(k^{-3})),$$

since $t<Lk$. Further, in the (H) case with t in (U_-, Z_1'), we have $\dot{y}(\Gamma_+)>L$ over a range of time length L before U_+, and since $|w(U_-)|<k^{-5}$ it is easy to see that $|U_3-U_4|<Lk^{-5}$. Similarly and more crudely we have $|W_3'-W_4'|<Lk^{-5}$; points like U, W' are "displaced" by $O(k^{-5})$ as between $\Gamma_{3,4}$. It follows that, subject to the hypotheses just mentioned, *we can take over Lemmas* 13 *to* 16, *about a* τ, *"in T-form", i.e. with our present* T *in place of the* τ, *with the understanding that the factor* $\frac{1}{2}$ *on the right-hand side in Lemma* 15 *is replaced by* $\frac{2}{3}$ (to cover a factor $1+O(k^{-3})$).[1] We shall be constantly using this principle and will refer to it shortly by saying "Lemma so and so in T-form". References (a), (b), ... will be to parts of Lemma 13.

§ 39. LEMMA 18. *Suppose that* t_0 *is in* (Z_1, U_-), *and that in* (Z_1, t_0) $\Gamma_{3,4}$ *do not intersect and* $0<w<k^{-5}$. *Then*

(i) $w(t_0)$, $c(t_0)$ *satisfy the linkage relations*

$$\mu(t_0)\,c(t_0)\le w(t_0), \qquad \mu(t_0)\,c(t_0)>\tfrac{1}{2}w(t_0)-\zeta,$$

or

$$w(t_0)<2\mu(t_0)\,c(t_0)+\zeta.$$

(ii) $\mu(t_0)$ *satisfies*[2] $Lk^{-1}<\mu(t_0)<Dk^{-\frac{1}{2}}$, *and* $\mu(U_-)\lesssim Lk^{-\frac{1}{2}}$.

We have further in (Z_1, t_0)

$$w<Dk^{\frac{1}{2}}\,w(Z_1)+\zeta.$$

The range $Z_1\le t_0\le Z_1+1$ is covered by Lemma 17, and we may suppose $t_0>Z_1+1$. Consider the r.m. from t_0 as origin over $t\le 1$; this lies in (Z_1, t_0) (with no intersection and $w<k^{-5}$).

By (W_5) with $t=1$ (and w_0 for $w(t_0)$, etc.)

$$\mu_0 c_0\le w_0, \tag{1}$$

$$\mu_0 c_0(1-\vartheta\bar{\chi}(1)/\mu_0)>w_0(1-\bar{\psi}(1))-\zeta. \tag{2}$$

[1] Note (i) that the τ-lemmas involve a hypothesis $V\ge V^*+\frac{1}{2}\delta$ where there is a U, duly fulfilled under our present hypotheses; (ii) since the τ-lemmas sometimes mention U, W', etc., we have to cover the effect of their "displacement".

[2] For unrestricted t we have $Lk^{-1}<\mu(t_0)<Dk^{\frac{1}{2}}$.

By Lemma 14 in T-form we have

$$\bar{\psi}(1) = L_1^* \iint\limits_{0\le\xi\le\eta\le 1} e^{-T_\eta + T_\xi}\, d\xi\, d\eta < D k^{-\frac{1}{2}},$$

$$\bar{\chi}(1) = L_1^* \iiint\limits_{0\le\xi\le\eta\le\zeta\le 1} e^{-T_\zeta + T_\eta - T_\xi}\, d\xi\, d\eta\, d\zeta \le L \Big(\int_0^1 e^{-T_\xi} d\xi\Big) \Big(\iint\limits_{0\le\eta\le\zeta\le t} e^{-T_\zeta + T_\eta}\, d\eta\, d\zeta\Big)$$

$$< L\mu_0 \cdot L\bar{\psi}(1) < D k^{-\frac{1}{2}} \mu_0.$$

(1) and (2) now give the linkages of the Lemma.

The inequalities for $\mu(t_0) = \mu_0$ are cases of (g), (d) of Lemma 13, in T-form. For the special point $t_0 = U_-$ we have $\mu(U_-) > L \int_0^1 e^{-\tau} dt$ (r.m.). In the r.m. from U_- $|\dot{y}_3|$ lies between two L's, $\dot{\tau}$ between two $Lk|y-1|$'s or two Lkt's, $e^{-\tau}$ between two $\exp(-Lkt^2)$'s, and so μ between two $Lk^{-\frac{1}{2}}$'s as desired.

For the last part we observe that in (Z_1, t_0) $c(t)$ is decreasing, by (1) of § 34,[1] so that $c(t_0) < c(Z_1)$. So

$$w(t_0) < 2\mu(t_0)\, c(Z_1) + \zeta < D k^{-\frac{1}{2}} c(Z_1) + \zeta,$$

and by Lemma 17 $c(Z_1) < Lkw(Z_1) + \zeta$, so that $w(t_0) < Dk^{\frac{1}{2}} w(Z_1) + \zeta$, which is equivalent to the desired result.

§ 40. LEMMA 19. *Suppose that* $(0, t_0)$ *is in* (Z_1, U_-), *that* $t_0 \le L_2^* k$, *and that there is no intersection of* $\Gamma_{3\,4}$, *and* $0 < w < k^{-5}$, *in* (Z_1, t_0). *Then in* $(0, t_0)$

(i) $w \le w_0 e^{-T} + c_0 \varphi(t)$,

(ii) $w \ge c_0 \varphi(t)(1 - \frac{2}{3}\vartheta - \vartheta' D k^{-\frac{1}{2}} w_0/c_0)$

(iii) $w > \frac{1}{4} c_0 \varphi(t) - \zeta$,

where (we recall)

$$\varphi = e^{-T} \int_0^t e^T dt.$$

The first inequality is a case of (W_2).

By the second inequality in (W_4),

$$w \ge c_0 \varphi \Big(1 - \vartheta\chi \Big/ \int_0^t e^T dt - \frac{w_0}{c_0}\psi \Big/ \int_0^t e^T dt\Big). \tag{1}$$

[1] It is decreasing in any range in which $w > 0$.

Now

$$\psi\le L\iint\limits_{0\le\xi\le\eta\le t} e^{T_\eta-T_\xi}\,d\xi\,d\eta\le L\Bigl(\int_0^t e^{T_\eta}\,d\eta\Bigr)\Bigl(\int_0^t e^{-T_\xi}\,d\xi\Bigr)\le Dk^{-\frac{1}{3}}\int_0^t e^{T}\,dt, \tag{2}$$

$$\chi\le L_1^*\iiint\limits_{0\le\xi\le\eta\le\zeta} e^{T_\xi-T_\eta+T_\zeta}\,d\xi\,d\eta\,d\zeta\le L_1^*\Bigl(\int_0^t e^{T_\zeta}\,d\zeta\Bigr)\Bigl(\iint\limits_{0\le\xi\le\eta\le t} e^{-T_\eta+T_\xi}\,d\xi\,d\eta\Bigr) \tag{3}$$

in which the first bracket is $\int_0^t e^T\,dt$. Substituting in (1) from (2) and (3) we have

$$w\ge c_0\,\varphi\,\Bigl(1-\vartheta L_1^*\iint\limits_{0\le\xi\le\eta\le t} e^{-T_\eta+T_\xi}\,d\xi\,d\eta-\vartheta' Dk^{-\frac{1}{3}}\,w_0/c_0\Bigr)\cdot \tag{4}$$

By Lemma 15 in T-form the factor of ϑ is $<\frac{2}{3}$, and (ii) follows.

Rewrite (ii) as

$$w\ge(1-\tfrac{2}{3}\vartheta)\,c_0\,\varphi-\vartheta' Dk^{-\frac{1}{3}}\,w_0\,\varphi. \tag{5}$$

By Lemma 18 $w_0<2\mu_0 c_0+\zeta$ and $c_0>-\zeta$. If $c_0>0$ the first of these substituted in (5) gives (iii); if $c_0\le 0$ the right side of (iii) is negative and the inequality trivial since $w_0>0$.

The following special result involves rather similar reasoning, and in order not to interrupt a later argument we include it here.

LEMMA 20. *Suppose that there is no intersection of* $\Gamma_{3,4}$, *and* $w<k^{-5}$, *in* $(Z_1,\ U_-+t_0)$, *where* $0\le t_0\le 3\pi$. *Suppose further that* $U_{3,4}$ *are together, with* $V_{3,4}\ge V^*+\delta$ *(so that* U_- *is* U_3). *Then for* $0\le t\le t_0$ *we have [with* U_3 *as effective origin]*

$$w(U_3+t)\ge c(U_3)\,\varphi_{U_3}(t)\,(1-\vartheta Dk^{-\frac{1}{3}}-\vartheta' Dw(U_3)/c(U_3)).$$

By (W_4) we have (1) as before, but the rest is different. By (3) and Lemma 14 in T-form the factor of ϑ is $O(Dk^{-\frac{1}{3}})$. Also

$$\psi\Big/\int_0^t e^T\,dt\le L\iint\limits_{0\le\xi\le\eta\le t} e^{T_\eta-T_\xi}\,d\xi\,d\eta\Big/\int_0^t e^T\,dt<D$$

by Lemma 16 in T-form (and inverted). This gives the desired result.

§ 41. We divide the range (Z_1, U_-) at $t_1=Z_1,\ t_2,\ \ldots,\ t_{\nu+1}=U_-$ in the following way. Each t_n-t_{n-1} lies between $L_1 k$ and $L_2^* k$; where L_1 is suitably small, and for $n\le\nu$ t_ν is at a Z ($t_{\nu+1}=U_-$ is exceptional). It is evidently possible to make such a division, and we have $\nu<L$. We write w_n, c_n, μ_n etc. for $w(t_n)$, $c(t_n)$, $\mu(t_n)$, etc., and $\varphi_n(t)$ for $\varphi(t)=$ "$e^{-T}\int_0^t e^T\,dt$" formed from the d.m. with origin t_n. We have

$$\varphi_{n-1}(t_n) = \int_{t_n-1}^{t_n} e^{-(T(t_n)-T)}\,dt + \int_{t_{n-1}}^{t_n-1},$$

in which the first term is μ_n, and (since T increases by at least Lk when t increases by 1) the second is $\vartheta\zeta$; thus

$$\varphi_{n-1}(t_n) = \mu_n + \vartheta\zeta. \tag{1}$$

Further, since $\dot{T}$ lies between two Lk's in the range of integration concerned in μ_n, we have

$$\mu_n \lesssim L k^{-1} \quad (n \leq \nu). \tag{2}$$

We have now

LEMMA 21. *There is a* ζ_1 *(independent of* n*) such that if* $\Gamma_{3,4}$ *have* $0 < w(Z_1) \leq k^{-10}$ *and* $w(Z_1) > \zeta_1$, *then for each* n *of* $1 \leq n \leq \nu + 1$:

(a) *there is no intersection before* t_n;

(b) $0 < w < k^{-5}$ *up to* t_n.

Further, for $1 \leq n \leq \nu$.

(c) $\frac{3}{4}\mu_n c_n \leq w_n, \qquad \mu_n c_n \geq \frac{1}{5} w_n > 0, \qquad w_n \geq L w_{n-1} \quad (n > 1);$

(d) $w_n \geq L w(Z_1).$

We have stated the Lemma in a form suited to an inductive proof. We suppose throughout that $w(Z_1) > \zeta_1$, successively rechoosing ζ_1 (smaller) as the run of the argument requires it.

For $n = 1$ $(t_1 = Z_1)$ (c) is true, by Lemma 17, provided ζ_1 is suitably chosen (the last part does not arise), and (a), (b), (d) are trivial. Suppose now that (a) to (d) are true up to $n-1$, and consider them for n. Consider the d.m. from t_{n-1} till the first intersection, if any, or $w = k^{-5}$, or $t = t_n$, whichever happens first. After (a), (b) for $n-1$, Lemma 19 (ii) is valid in the range, with t_{n-1} for $t = 0$. In this, by (c) for $n-1$, $c_{n-1} > 0$ and $w_{n-1}/c_{n-1} \leq 5\mu_{n-1} < Lk^{-1}$. Hence (in the range)

$$w \geq c_{n-1}\varphi_{n-1}\left(1 - \tfrac{2}{3}\vartheta - \vartheta' D k^{-\frac{3}{2}}\right) \geq \tfrac{1}{4} c_{n-1}\varphi_{n-1} > 0. \tag{3}$$

This shows incidentally that an intersection is not the first event.

Next, over *any* range from Z_1 in which there is no intersection we have, by (W_2),

$$w \leq w(Z_1)\, e^{-(T - T_{Z_1})} + c(Z_1)\, \varphi_{Z_1}(t).$$

This is valid in particular over our present range, where it gives $w<k^{-5}$, since $c(Z_1)\varphi_{Z_1}<Lkw(Z_1)\cdot Dk^{-\frac{1}{2}}<Dk^{-10+\frac{1}{2}}$ by Lemma 18, and Lemma 13 (d) in T-form. So $w=k^{-5}$ is not the first event. Hence we arrive at t_n with no intersection since Z_1, and $w<k^{-5}$ throughout. (3) is now valid at t_n and gives

$$w_n \geq \tfrac{1}{4}\mu_n c_{n-1}>0. \tag{4}$$

Next, Lemma 18 is valid with t_n for t_0, so that

$$\mu_n c_n \leq w_n, \tag{5}$$

$$\mu_n c_n > \tfrac{1}{2} w_n - \zeta'. \tag{6}$$

For n restricted by $1<n\leq\nu$ we have $\mu_n<L\mu_{n-1}$ (by (2)). Hence from (4) $w_n \geq L\mu_{n-1}c_{n-1}$, and so, by (c) for $n-1$.

$$w_n \geq L w_{n-1} \quad (n\leq\nu). \tag{7}$$

By iteration this further gives (since $\nu<L$)

$$w_n \geq L_1 w(Z_1) \quad (n\leq\nu). \tag{8}$$

It follows from (6) and (8) that if ζ_1, is increased so that $\zeta_1>6\zeta'/L_1$ we have (when $w(Z_1)>\zeta_1$) $\mu_n c_n \geq \frac{1}{3}w_n$.[1] This, together with (5) and (7) gives (c) for n (when $n\leq\nu$). (d) is true for $n\,(n\leq\nu)$ by (8).

We have now completed the induction from $n-1$ to n [(a) and (b) for $n\leq\nu+1$, (c) and (d) for $n\leq\nu$], and proved Lemma 21.

In particular $\Gamma_{3,4}$ do not intersect before U_-.

We record some further consequences of what we have proved.

LEMMA 21. COROLLARY. *With the hypotheses of Lemma 21 we have*

(e) $$w(U_-)>Lk^{\frac{1}{2}}w(Z_1).$$

(f) $$w<Lk^{\frac{1}{2}}w(Z_1)+\zeta \text{ in } (Z_1+1,\ U_-), \text{ in particular at } U_-.$$

(g) $$Lk^{\frac{1}{2}}w(U_-)>c(U_-)>Lkw(Z_1)-\zeta>Lk^{\frac{1}{2}}w(U_-)-\zeta.$$

(h) $$w(U_-),\ \dot{w}(U_-),\ c(U_-)=O(k^{-7}).$$

We have from (4), (c) for ν, and (2)

$$w(U_-)=w_{\nu+1}>L\mu_{\nu+1}c_\nu>L\mu_{\nu+1}\mu_\nu^{-1}w_\nu>Lk\mu_{\nu+1}w(Z_1),$$

[1] Since this ζ_1 is independent of n it is clear that $\mu_n c_n \geq \frac{1}{3}w_n$ will be true for the earlier values of n also.

in which $\mu_{\nu+1}=\mu(U_-)$, and lies between two $Lk^{-\frac{1}{2}}$, by Lemma 18 (ii). Hence we have (e).

Next, (W_2) gives in (Z_1+1, U_-)

$$\begin{aligned} w &\leq w(Z_1)\,e^{-(T-T_{Z_1})}+c(Z_1)\,\varphi_{Z_1}(t) \\ &< \zeta + c(Z_1)\,(\mu(t)+\vartheta\zeta) < \zeta + Lk\,w(Z_1)\,\mu(t) \\ &< Dk^{\frac{1}{2}}\,w(Z_1)+\zeta, \end{aligned}$$

which is (f).

By (6) with $n=\nu+1$,

$$c(U_-) > L\mu^{-1}(U_-)\,w(U_-)-\zeta > Lk^{\frac{1}{2}}\cdot Lk^{\frac{1}{2}}\,w(Z_1)-\zeta,$$

giving the second part of (g), and the third follows by (f) (for U_-). The first part is a case of (5).

The results of (h) follow from (f), (g), and

$$|\dot{w}(U_-)| \leq |c(U_-)| + Lk\,w(U_-).$$

§ 42. LEMMA 22. *There is a ζ_2 with the following properties. Suppose that $\Gamma_{1,2}$ have their $w(\Gamma_{1,2}, Z_1) > \zeta_2$, and have their $U_{1,2}$ together. Then the intermediates of the k^{-a} mesh*[1] *have their U's in the interval (U_1, U_2), and do not intersect before U_2, so that $U_{3,4}$ have $U_3 < U_4$, and no intersection before U_3.*

Further, the V's decrease as the U's increase $(V_1 > V_3 > V_4 > V_2)$.

We have also

(a) *$w < Lk^{\frac{1}{2}}\,w(Z_1)$ in (Z_1+1, U_3).*

(b) *The ratio of any two of $kw(Z_1)$, $k^{\frac{1}{2}}w(U_3)$, $c(U_3)$ lies between two L's.*

(c) *$c(U_3)$, $w(U_3)$, $\dot{w}(U_3)=O(k^{-7})$.*

(d)[2] *For the original $\Gamma_{1,2}$ we have $w < Lk^{\frac{1}{2}}\,w(Z_1)$ in (Z_1+1, U_1).*

The final (d) follows from (a), and addition over the non-intersection mesh.

We suppose $w(\Gamma_{1,2}, Z_1) > Dk^{11-1}\,\zeta'$, where D is chosen so that the inequality makes $w(\Gamma_{3,4}, Z_1) > \zeta'$, and we rechoose ζ' as we proceed so that successive requirements are fulfilled. In the first place we choose ζ' so that the $\Gamma_{3,4}$ do not intersect before U_-.

---

[1] A k^{-11} mesh is a case of a k^{-10} one (Lemma 21 has k^{-10}).

[2] (d) is not used until § 68.

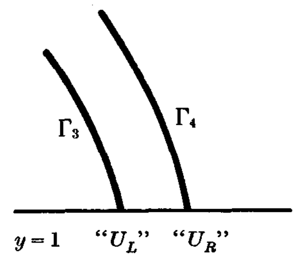

Fig. 7.

We need momentarily to deal again with possible arrivals of a Γ at $y=1$ (downward and) earlier than its U. We call such a point of arrival "U", and the associated $-\dot{y}+bk(1+p_1)$, "V". Suppose now till further notice that $\Gamma_{3,4}$ arrive at "U_L", "U_R", as in the figure, we sometimes abbreviate the "U"'s to L, R. In any case $\Gamma_{3,4}$ do not intersect before "U_L", and it is Γ_3 that arrives at "U_L", Γ_4 at "U_R". With obvious notation we have now the identities

$$\text{"}V_L\text{"}=\text{"}V_R\text{"}+\gamma,\ \gamma=c(R)-\int_L^R g(y_L)\,dt=c(L)-\int_L^R g(y_4)\,dt. \tag{1}$$

For this we have

$$[-\dot{y}_3+bk(1+p_1)]_L^R=k[F(y_3)-F(1)]_L^R+\int_L^R g(y_3)\,dt,$$

$$[-\dot{y}_4+\dot{w}+bk(1+p_1)]_R-\text{"}V_L\text{"}=-k[\Delta F]_R-0+\int_L^R g(y_3)\,dt,$$

$$-\gamma=\text{"}V_R\text{"}-\text{"}V_L\text{"}=[-\dot{w}-k\,\Delta F]_R+\int_L^R g(y_3)\,dt,$$

whence the first form for γ. The second follows by (1) of § 34.

We suppose now that *one* of "U_L", "U_R" is U_-. Then by the first form for γ when "U_R" is U_-, and by the second when "U_L" is, we have

$$\gamma>c(U_-)-L(\text{"}U_R\text{"}-\text{"}U_L\text{"}). \tag{2}$$

We show next that (in either of these cases)

$$|\dot{y}_4|>L \text{ in the range } (\text{"}U_L\text{"}, \text{"}U_R\text{"}). \tag{3}$$

If U_R is U_- we have "V_R" $\geq V^*-\delta$, and, in the notation of Lemma 6, $1+\alpha>L$ at "U_R"; then the desired result follows from Lemma 11 (5). If "U_L" is U_-, then $-\dot{y}_4(L)=v_3-\dot{w}(U_-)>L-k^{-7}>L$. Also $s=y_4(L)-1=w(L)<k^{-5}$. The retardation $|\ddot{y}_4|$ is less than Lk^2, and cannot reduce the velocity by more than Lk^{-3} in the y-interval s. Hence $|\dot{y}_4|>L$ in ("U_L", "U_R"), as desired.

It follows now from (3) that $|``U_L" - ``U_R"| < Lw(L) < Lk^{\frac{1}{2}} w(Z_1) + \zeta$ (by (f) of § 41), and so, from (2), that

$$\gamma > c(U_-) - Lk^{\frac{1}{2}} w(Z_1) - \zeta > (L_1 k - L_2 k^{\frac{1}{2}}) w(Z_1) - \zeta,$$

by (g) of § 41. For suitable ζ_2 we have accordingly $\gamma > 0$, and so

$$``V_L" > ``V_R". \tag{4}$$

Let Γ_5 be the Γ next before Γ_2. To establish the Lemma it is enough to prove that $U_5 < U_2$, that $U_{5,2}$ are together, and that $V_5 > V_2$. For we can start again with Γ_5 as a new Γ_2, and repeat the process. If U_- is the earlier of $U_{5,2}$ there are two cases: (i) U_2 is U_-; (ii) U_5 is U_-. In (i) there is no intersection of $\Gamma_{5,2}$ before U_2, and Γ_5 has a "U_5" "together" with U_2 and "U_5" $< U_2$. By (4) "V_5" $> V_2 \geq V^* - \delta$, so that "U_5" is U_5, and $U_{5,2}$ are together with $V_5 > V_2$. This completest he proof for case (i).

In case (ii) we have $-\dot{y}_2(U_5) = v_5 - \dot{w}(U_5) > L - k^{-A} > L$, and Γ_2 must reach $y = 1$ (near U_5), since a retardation Lk^2 cannot destroy velocity L in space $w(U_5) < k^{-5}$. So Γ_2 has a "U_2" near U_5. We have what we want [after (4)] provided "U_2" is U_2. But if "U_2" is not U_2, then Γ_1, which certainly meets $y = 1$ at a "U_1" near and earlier than U_5 (because of non-intersection) has "U_1" not U_1 (since $U_{1,2}$ are together). Since, by (4), "V_1" $> V_5 \geq V^* - \delta$, this is false, and the proof of case (ii) is completed.

The remaining results of the Lemma are immediate consequences of (e) to (h) of § 41, when the ζ' of $w(Z_1) > \zeta'$ is suitably rechosen.

§ 43. We are now in a position to follow $\Gamma_{3,4}$ beyond U_3.

LEMMA 23. *If ζ' is suitably rechosen, then, provided $w(Z_1) > \zeta'$, and $\Gamma_{1,2}$ satisfy (H), the $\Gamma_{3,4}$ have $w < k^{-5}$ and no intersection in (U_3, Z_1') [so none in (Z_1, Z_1')]. Also*

$$w(Z_1') \lessgtr Lw(Z_1), \quad c(Z_1') \lessgtr Lkw(Z_1).$$

Lemma 22 is valid, and we continue to rechoose ζ'. Consider the d.m. from U_3 until the first intersection, or $w = k^{-5}$, or $t = Z_1'$, whichever happens first. Over this range Lemma 20 is valid. It follows from Lemma 22 (b) that $c(U_3) > 0$ and also that the bracket in Lemma 20 exceeds $\frac{1}{2}$, so that

$$w > \tfrac{1}{2} c(U_3) \varphi_{U_3}(t) > 0, \tag{1}$$

and in particular an intersection is not the first event.

By (W_2)
$$w \leq w(Z_1)\,e^{-(T-T_{Z_1})}+c(Z_1)\,\varphi_{Z_1}(t)<k^{-5}, \tag{2}$$

since $\exp\{-(T-T_{Z_1})\}<Dk$, and $\varphi_{Z_1}(t)<Dk^{\frac{1}{2}}$ by (a) and (b) of Lemma 13. Hence $t=Z_1'$ is the first event, and there is no intersection in (Z_1, Z_1').

(1) is valid at Z_1', and gives, after (b) of Lemma 22,

$$w(Z_1')>Lk\,w(Z_1)\,\varphi_{U_3}(Z_1'),$$

in which (since $Z_1'-U_3>1$) $\varphi_{U_3}>\mu(Z_1')>Lk^{-1}$. Hence $w(Z_1')>Lw(Z_1)$.

By (2) with $t=Z_1'$, by the method of § 41 (1), by (2) of § 41, and by (c) of Lemma 13,

$$w(Z_1')<\zeta+c(Z_1)(\mu(Z_1')+\zeta)<\zeta+Lk^{-1}c(Z_1)<\zeta+Lw(Z_1),$$

by Lemma 17. For suitable ζ' this gives $w(Z_1')<Lw(Z_1)$.

Finally, by Lemma 17

$$c(Z_1')\lesssim Lk\,w(Z_1')\pm\zeta\lesssim Lk\,w(Z_1)\pm\zeta,$$

and for suitable ζ' this gives $c(Z_1')\lesssim Lkw(Z_1)$. This completes the proof of Lemma 23.

§ 44. We can now state the following key result about non-intersection.

LEMMA 24. *There is a ζ_1^* with the following properties. Let $\Gamma_{1,2}$ belong to S_1 at Z_0. Then:*

(i) *if $w(\Gamma_{1,2}, Z_1)>\zeta_1^*$, $\Gamma_{1,2}$ do not intersect in (Z_1, U_1). Also the ratios of any two of $kw(Z_1)$, $k^{\frac{1}{2}}w(U_1)$, $c(U_1)$ lie between two L's.*

(ii) *if, further, $\Gamma_{1,2}$ satisfy (H), of § 38, then there is no intersection before U_2'. Also the ratio of any two of $kw(Z_1)$, $kw(Z_1')$, $c(Z_1)$, $c(Z_1')$, $k^{\frac{1}{2}}w(U_1)$, $c(U_1)$, $k^{\frac{1}{2}}w(U_2')$, $c(U_2')$, lies between two L's.*

We employ a k^{-11} mesh at Z_1: for suitably chosen ζ_1^* this makes $w(Z_1)>\zeta'$, and for suitable ζ' the results of Lemmas 22, 23 are valid. Hence there is no intersection of the $\Gamma_{3,4}$ up to U_3 and U_4' in cases (i) and (ii) respectively. Also in the second case $w(Z_1')$ lies between two $Lw(Z_1)$. So on the one hand $w(Z_1')>\zeta'$ for a suitably rechosen ζ_1^*, and on the other the Γ at Z_1' constitute a k^{-10} mesh.[1] Lemma 22 (inverted) is accordingly valid over (Z_1', U_-'); $\Gamma_{3,4}$ do not intersect before U_4'; $w(Z_1')\lesssim Lw(Z_1)$, and $c(Z_1')\lesssim Lkw(Z_1)$. Further, in virtue of Lemma 22 (b), we have $Lkw(Z_1)>c(U_3)>Lk^{\frac{1}{2}}w(U_3)>Lkw(Z_1)$ in the first case, and $Lkw(Z_1)>$

[1] The device of two meshes is avoided in the Introduction, but the apparently simple line taken there does not fit in with the present lay-out.

$Lkw(Z_1') > c(U_4') > Lk^{\frac{1}{2}}w(U_4') > Lkw(Z_1') > Lkw(Z_1)$ in the second. Finally, the non-intersection and the various inequalities are additive, and extend from the pairs $\Gamma_{3,4}$ to the original $\Gamma_{1,2}$. This completes the proof of Lemma 24.

§ 45. We now take up the (easier) question about intersection. We suppose that $\Gamma_{1,2}$ belong to the S_1 at Z_0, but there is no mesh in what follows, and w, c, etc., refer always to $\Gamma_{1,2}$.

We restart the dashes to ζ's (the ζ' used above have served their turn). We have now (U_- being the earlier of $U_{1,2}$, U_-' the earlier of $U_{1,2}'$).

LEMMA 25. *Let $\Gamma_{1,2}$ belong to S_1. Given a ζ', then if $|w(Z_1)| < \zeta'$ we have $|w|$, $|\dot{w}|$, $|c| < \zeta(\zeta')$ over (Z_1, U_-). This is true also over (Z_1, U_-') provided $\Gamma_{1,2}$ satisfy (H).*

In applications the ζ' becomes a definite ζ, and the ζ denoted by $\zeta(\zeta')$ to show its dependence on ζ' also becomes one.[1]

With a hypothesis $|w(Z_1)| < \zeta'$, and normalization to $w(Z_1) \geq 0$, we have $0 \leq w(Z_1) < k^{-11}$ *a fortiori*, and we may take over for $\Gamma_{1,2}$ various results of Lemmas about $\Gamma_{3,4}$. We abbreviate (within the present proof) $\zeta(\zeta')$ to ζ.

By Lemma 17, with Z_1 for Ω, we have $c(Z_1) = O(\zeta)$. Consider the d.m. from Z_1, normalizing to w being initially positive. Up to the first intersection, or till $w = k^{-5}$,[2] or till $t = Z_1'$, (W_2) gives

$$0 \leq w \leq w(Z_1) + c(Z_1)\varphi(t) < \zeta,$$

and so also $\dot{w} = O(\zeta)$ in virtue of

$$\dot{w} = -\dot{T}w + c(Z_1) + O\left(\int |w|\,dt\right). \tag{1}$$

Since $w = k^{-5}$ is clearly not the first event we have w, $\dot{w} = O(\zeta)$ up to the first intersection or Z_1'.

If there is an intersection before U_-, consider the d.m. from this intersection till the next intersection, or $w = k^{-5}$, or $t = Z_1'$. $(W_{2,3})$ are valid with $w_0 = 0$, $\dot{w}_0 = O(\zeta)$, and become (normalizing since w is negative)

$$0 \leq -w < |\dot{w}_0|\,\varphi(t) < \zeta < k^{-5} \tag{2}$$

$$-w \geq |\dot{w}_0|\,e^{-T}\left(\int_0^t e^T\,dt - \vartheta\chi\right). \tag{3}$$

By § 40 (3) and Lemma 15 in T-form (see § 38) the factor of $|\dot{w}_0|$ in (3) exceeds

---

[1] It seems unnecessary and wasteful to call on a blank cheque notation here.

[2] Without $w \leq k^{-5}$ we have no information about φ, in spite of the drastic hypothesis.

$\frac{1}{3}\varphi(t)$ provided $t\le U_-$ and $t\le L_2^* k$, and it follows that a second intersection before U_- cannot occur within time $L_2^* k$ after the first. Up to the new intersection, if any, (2) gives $w=O(\zeta)$ and so, by (1), $\dot{w}=O(\zeta)$, and $c(t)=\dot{w}+\dot{T}w=O(\zeta)$. The argument can evidently be repeated, and we arrive in at most L steps at U_- with at most L intersections, and w, $\dot{w}=O(\zeta)$ throughout.

For the second part, consider the d.m. from U_- as origin, and let $w_0=w(U_-)$ be, say, positive. Then until the next intersection, or $w=k^{-5}$, or $t=Z_1'$, we have (whatever the sign of c_0) by (W_2)

$$w\le w_0 e^{-T}+c_0\varphi=O(\zeta),$$

since w, $\dot{w}$, and so c_0, are $O(\zeta)$ at U_-. There is nothing more to prove unless there is an intersection before Z_1'. If there is one, consider the d.m. from it as origin. We have $\dot{w}_0=O(\zeta)$. Until the next intersection, or $|w|=k^{-5}$, or $t=Z_1'$, we have by $(W_{2,3})$, normalizing,

$$0\le -w\le|\dot{w}_0|\,\varphi(t)<\zeta<k^{-5}, \tag{4}$$

$$-w\ge|\dot{w}_0|\,e^{-T}\Big(\int_0^t e^T\,dt-\vartheta\chi\Big). \tag{5}$$

Since the range is $<3\pi$ (and $|w|<k^{-5}$) we have $\vartheta\chi\le\frac{2}{3}\int_0^t e^T\,dt$ by §40 (3) and Lemma 14 in T-form, so that an intersection is not the first event. Hence there is no further intersection up to Z_1', and further (4) gives $w=O(\zeta)$, and (1) gives $\dot{w}=O(\zeta)$ also.

We now have $w(Z_1')=O(\zeta)$, and can apply the first part to the range (Z_1', U_-'). This completes the proof of the Lemma.

§ 46. We record for reference a number of identities and near identities, not all of them new. In them $\Gamma_{1,2}$ belong to S_1 at Z_0, they have their $U_{1,2}$ together, and we consider the range to U_-', the earlier of $U_{1,2}'$ (about which we make no assumptions).

$$\left.\begin{aligned} c(t)&=\dot{w}+\dot{T}w=\dot{w}+k\,\Delta F;\\ c(t_2)-c(t_1)&=-\int_{t_1}^{t_2}\Delta g\,dt=-[w_1]_{t_1}^{t_2}; \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} &c(U_1)=-\Delta V+\int_{U_1}^{U_2} g(y_2)\,dt,\ \ c(U_{1,2})=-\Delta V+O(\Delta\omega);\\ &c(U_2')=\Delta V'-\int_{U_2'}^{U_1'} g(y_1)\,dt\ \text{ and }\ c(U_{1,2}')=\Delta V'+O(\Delta\omega'),\\ &\text{provided } U_{1,2}' \text{ are together.} \end{aligned}\right\} \tag{2}$$

$$\left.\begin{array}{l}\text{If } -\Delta\omega>0 \text{ [equivalent to } U_1<U_2], \text{ then } w(U_1)\lesssim L(-\Delta\omega).\\ \text{If } U'_{1,2} \text{ are together and } \Delta\omega'>0, \text{ then } w(U'_2)\lesssim L(\Delta\omega').\end{array}\right\} \tag{3}$$

(1) is old, and the identities for $c(U_1)$, $c(U'_2)$ are proved on the lines of (1) of § 42. The inequalities in (2) follow from the identities and the second part of (1). In, say, the second part of (3) we have $(U'_2<U'_1)$

$$w(U'_2)=-1-y_1(U'_2)=-\int_{U'_1}^{U'_2}\dot{y}_1\,dt=\int_{U'_2}^{U'_1}\dot{y}_1\,dt$$

in which $L<\dot{y}_1<L$ (by Lemma 11 (5)), and the range has (positive) length $\Delta\omega'$.

We have now

LEMMA 26. *Suppose $\Gamma_{1,2}$ of S_1, with $w(Z_1)>0$, satisfy (H). Then*

$$w>Lk^{-1}\cdot(-\Delta V)-\zeta$$

over a time Lk beyond Z'_1+1.[1]

In the first place we may suppose $w(Z_1)>$ any relevant ζ'. For if $w(Z_1)\leq\zeta'$ $w, \dot{w}, c$ are $O(\zeta)$ in (Z_1, Z'_1) by Lemma 25. By (3) $\Delta\omega=O(\zeta)$, and by (2) $\Delta V=O(\zeta)$ and the inequality of the Lemma reduces to $O(\zeta)>O(\zeta)-\zeta$ and is true trivially.

In particular we choose ζ' so that $w(Z_1)>\zeta_1^*$ when, by Lemma 24, $\Gamma_{1,2}$ do not intersect before $U'_2(=U'_-)$, and further so that Lemmas 22, 23 are valid.

Next, it is enough to take the case of consecutives of the k^{-11} mesh, for by Lemma 22 these satisfy $V_{3,4}\geq V^*+\delta$, and the desired inequality is additive (whether or not the mesh has intersections, though in fact it has not). For suitable ζ' we now have, by Lemma 24, non-intersection, and $w<k^{-5}$ up to U'_4; also

$$w(Z'_1)\lesssim Lw(Z_1). \tag{4}$$

For suitable ζ' we have, by Lemma 22 (b), replacing the two $w(Z_1)$ by $w(Z'_1)$ in virtue of (4),

$$Lkw(Z'_1)<Lk^{\frac{1}{2}}w(U_3)<c(U_3)<Lk^{\frac{1}{2}}w(U_3)<Lkw(Z'_1). \tag{5}$$

Next, by (2) and (3)

$$c(U_3)=-\Delta V+O(w(U_3))=-\Delta V+O(k^{-\frac{1}{2}}c(U_3)),$$

so that $c(U_3)\lesssim L(-\Delta V)$, and by (5),

$$w(Z'_1)\lesssim Lk^{-1}(-\Delta V). \tag{6}$$

Next, by Lemma 19 (ii) (inverted), with $t=0$ at Z'_1 and $t_0=L_2^*k$, we have for $0\leq t\leq t_0$

$$w\geq c(Z'_1)\,(1-\tfrac{2}{3}\vartheta-\vartheta' Dk^{-\frac{1}{2}}w(Z'_1)/c(Z'_1))\,\varphi(t). \tag{7}$$

[1] By taking more, but unnecessary, trouble, we could replace Z'_1+1 by Z_0+L.

By Lemma 23 $$c(Z_1')>Lkw(Z_1') \tag{8}$$

(for a suitable ζ'). The large bracket in (7) is then $>L$, and from (7) and (8) we have

$$w>Lkw(Z_1')\,\varphi(t). \tag{9}$$

For $t\geq 1$ $\varphi(t)\geq\mu(t)>Lk^{-1}$, and so, by (6) and (9),

$$w>Lk^{-1}(-\Delta V)$$

over a range from $Z_1'+1$ of length Lk, as desired.

§ 47. LEMMA 27. *If $\Gamma_{1,2}$ of S_1 have $w(Z_1)>0$, then $c(t)>-\zeta$ in (Z_1, U_-); if they further satisfy (H), then $c(t)>-\zeta$ in (Z_1, U_-').*

COROLLARY. *If $\Gamma_{1,2}$ have "U_1", "U_2" together, with "U_1" to the left, then "V_1" $>$ "V_2" $-O(k^{-A})$.*

If $w(Z_1)>\zeta'$ for a suitable ζ' Lemma 24 (i) and (ii) give $c(U_1)>0$, $c(U_2')>0$ in the respective cases. Further $\Gamma_{1,2}$ do not intersect in (Z_1, U_1), (Z_1, U_2') respectively, and $c(t)$ is decreasing in the respective ranges by § 46 (1). So the desired results are true when $w(Z_1)>\zeta'$. When $w(Z_1)\leq\zeta'$ we have $c=O(\zeta)$ in the respective ranges, by Lemma 25. This completes the proof.

The corollary follows from the result of the lemma, the identities (2) of § 46 which (being identities) are naturally true also for "U_1", "U_2", and the fact that $|$"U_1" $-$ "U_2"$|=O(Dk^{-\frac{1}{3}})$.

§ 48. We now introduce "pseudo-V's". We have again to consider downward arrivals at $y=1$ other than U's, and again use the "U", "V" notation of § 42, with "inverse" notation for $y=-1$. The points N, primarily at points where $\varphi\equiv-\frac{1}{2}\pi$, now become specially important, and we use N also for the trajectory point on $y=1$, and also for its *neighbourhood*, along with the "at" notation. We say a Γ is gap-free *before* N if it has not met a gap at an *earlier* N or N' (N' corresponds to $y=-1$ and $\varphi\equiv\frac{1}{2}\pi$); and gap-free before and at N if in addition Γ does not meet a gap "at" N.

We define

$$\mathfrak{V}(t)=\mathfrak{V}(\Gamma, t)=-\dot{y}-k(F(y)-F(1))+bk(1+p_1). \tag{1}$$

This satisfies $$\mathfrak{V}(t_2)-\mathfrak{V}(t_1)=\int_{t_1}^{t_2} g\,dt, \tag{2}$$

which is a variant of the $\dot{y}$-identity. There is a corresponding inversion

$$\mathfrak{V}'(t) = \dot{y} + k(F(y) - F(-1)) + bk(1 - p_1), \tag{3}$$

but $\mathfrak{V}$, $-\mathfrak{V}'$ are essentially the same function, since

$$\mathfrak{V}(t) + \mathfrak{V}'(t) = k(F(1) - F(-1) + 2b) = -(\tfrac{4}{3} - 2b)k. \tag{4}$$

If Γ has a "U" at N we have

$$\mathfrak{V}(\text{“}U\text{”}) = -\dot{y}(\text{“}U\text{”}) + bk(1 + p_1(\text{“}U\text{”})) = \text{“}V\text{”},$$

and
$$\mathfrak{V}(N) - \mathfrak{V}(\text{“}U\text{”}) = \int_{\text{“}U\text{”}}^{N} g\,dt = O(N - \text{“}U\text{”}) = O(Dk^{-\frac{1}{2}}).$$

Thus to error $O(Dk^{-\frac{1}{2}}) = O(k^{-A})$, $\mathfrak{V}(N) = $ "V", and in particular $\mathfrak{V}(N) = V$ if "U" is U. If there is a U at N and $0 < r < L$ we have (see § 18)

$$\mathfrak{V}(N - 2r\pi) = V - \int_{N-2r\pi}^{N} g\,dt = V - rM + O(k^{-A}), \tag{5}$$

and this is true whether or not there is a "U" at $N - 2r\pi$. Our pseudo-"V"'s are the $\mathfrak{V}(N)$ (their inverses $\mathfrak{V}'(N')$): they agree to error $O(k^{-A})$ with V's and "V"'s where there are U's or "U"'s but further exist where either there is no "U", or where we do not know in the first instance that there is one. Their full connexions with actual V's and "V"'s are set out in the following Lemma, which is incidentally vital. In this V_0 stands for $V^* \beta_0^2(-1)$ (in the notation of Lemmas 5 and 6). After Lemma 6 the "V" of a tangential "U" (with $v = 0$) is $V_0 + O(k^{-A})$, and V_0 is effectively the minimum value of a "V".

LEMMA 28. *Let Γ belong to S^**[1] *at Z_0, and let its U be at N_0. Let $N \le N_0$. Then:*

(a) *If there is no "U" at N, then $\mathfrak{V}(N) < V_0 + O(k^{-A})$. In particular, if $\mathfrak{V}(N) > V_0 + \frac{1}{4}\delta$ there is a "U" at N. If there is a "U" at N, then $\mathfrak{V}(N) > V_0 + O(k^{-A})$.*

(b) *If $\mathfrak{V}(N) > V^* - \frac{3}{4}\delta$ there is a U at N.*

(c) *If Γ has a U at N then $V^* - \delta \le V < V^* + M - \frac{3}{4}\delta$.*

(a) *First part.* Since $\mathfrak{V}(N)$ increases by $M + O(k^{-A})$ for increase 2π of N up to the U we may suppose (*a fortiori*) that there is a "U" at $N + 2\pi$ (and none at N). Now Γ and $\Gamma(-2\pi)$, which is Γ moved backwards a distance 2π, i.e. the trajectory $y(t + 2\pi)$, belong to S_1.[2] The difference of their y's at Z_1 is $y(Z_1) - y(Z_2)$ for the

[1] Not S_1.

[2] This is the sole *raison d'être* of S_1. It might seem that (a), the key to Lemma 28, is "obvious" and susceptible of some other easy proof. Actually it is rather deep, and a proof more from first principles would set up a "linkage" like that of β, α in Lemma 6 at a "U", but, since there is no "U", generalized to a neighbourhood of N. Our view is that we have chosen the lesser evil.

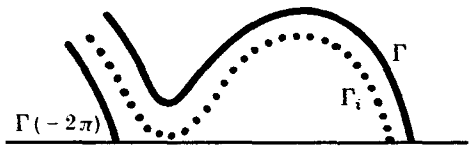

Fig. 8.

trajectory Γ, and we have

$$k(F(y_{Z_1})-F(y_{Z_2}))=\int_{Z_1}^{Z_2} g\,dt-\dot{y}_{Z_1}+\dot{y}_{Z_2}>L.$$

This difference is accordingly $>Lk^{-1}$. By Lemma 24 Γ and $\Gamma(-2\pi)$ do not intersect before their U_-, which is $U-2\pi>$ "U" $-2\pi=N$, and they form a "tube". By continuity an intermediate Γ_i exists. touching $y=1$ near N. By Lemma 27 Cor. its "V_i" at $N+2\pi$ exceeds that of Γ by $O(Dk^{-\frac{1}{2}})$ (Lemma 6), and so

$$\mathfrak{V}(\Gamma_i,\ N+2\pi)>\mathfrak{V}(\Gamma,\ N+2\pi)+O(k^{-A}).$$

We now have, on the one hand

$$\mathfrak{V}(\Gamma_i,\ N)>\mathfrak{V}(\Gamma,\ N)+O(k^{-A}),$$

and on the other, since Γ_i has $v=0$ at N,

$$\mathfrak{V}(\Gamma_i,\ N)<\text{“}V_i\text{”}(N)+O(k^{-A})<V_0+O(k^{-A}),$$

and the first part of (a) follows. The second is trivial and the third old.

(b) Since $V^*>V_0+L$ the hypothesis implies a "U" at N. But then

$$\text{“}V\text{”}=\mathfrak{V}(N)+O(k^{-A})>V^*-\delta,$$

and the "U" is a U.

(c) We must have $\mathfrak{V}(N-2\pi)\leq V^*-\frac{7}{8}\delta$; for if there is no "$U$" at $N-2\pi$ we have $\mathfrak{V}(N-2\pi)<V_0+O(k^{-A})<V^*-\frac{7}{8}\delta$, by (a); and if there is a "U" its "V" is $<V^*-\delta$, and $\mathfrak{V}(N-2\pi)<$ "V" $+O(k^{-A})<V^*-\frac{7}{8}\delta$. Hence

$$\mathfrak{V}(N)<\mathfrak{V}(N-2\pi)+M+O(k^{-A})<V^*+M-\tfrac{3}{4}\delta,$$

as desired.

§ 49. LEMMA 29. *Given an N, the S^* at a suitable Z_0 (depending on N) contains a continuous stream of Γ, each with a "U" at N, and with $\mathfrak{V}(N)$ ranging from $V^*-3\delta$ to $V^*+M-4\delta$. This includes a substream with "V" ranging from $V^*-2\delta$ to $V^*+M-5\delta$.*[1]

[1] The "U" and "V" are of course U and V when $V\geq V^*-\delta$. The bounds could be made wider but we state what is actually used.

Take any Γ of S^*; call it Γ_0, and let its U be at N_0, so that

$$V^* - \delta + o(1) < \mathfrak{V}(\Gamma_0, N_0) < V^* + M - \tfrac{3}{4}\delta + o(1).$$

Then $$\mathfrak{V}(\Gamma_0(4\pi), N_0) < V^* + M - \tfrac{3}{4}\delta + o(1) - 2M + o(1) < V^* - 3\delta,$$

also $\Gamma_0(4\pi)$ belongs to S^*.

Now we can "interpolate" between two Γ's, by taking the segment in the r.p. (representative point, see § 1 of the Introduction) space between the extreme ones, and obtain a stream with all the intermediate values[1] of $\mathfrak{V}(N_0)$. Thus we can find a Γ_1 "between" Γ_0 and $\Gamma_0(4\pi)$ with $\mathfrak{V}(\Gamma_1, N_0) = V^* - 3\delta$. This has its U not earlier than N_0, since otherwise there would be a "first" intermediate Γ from $\Gamma_0(4\pi)$ with $\mathfrak{V}(N_0) \leq V^* - 3\delta$, but U earlier than N_0; on the other hand, by the continuity of $\mathfrak{V}$ this "first" Γ would have $\mathfrak{V}(N_0) \geq V^* - \delta + M + o(1) > V^* - 3\delta$. It now follows from Lemma 28 that Γ_1 has its U at $N_0 + 2\pi$, and $\Gamma_1(-2\pi)$ (also belonging to S^*) has its $\mathfrak{V}(N_0) = V^* - 3\delta + M + o(1)$, and U not earlier than (in fact at) N_0. We can now interpolate between Γ_1 and $\Gamma_1(-2\pi)$ in a similar manner, with $\mathfrak{V}(N_0)$ ranging from $V^* - 3\delta$ to $V^* + M - 4\delta$. And the argument from continuity of $\mathfrak{V}(N_0)$ shows again that the Γ concerned have their U not earlier than N_0. This being so, they have each a "U" at N_0, by Lemma 28, and so "V" $= \mathfrak{V} + o(1)$. Since "V" varies continuously in the stream we have the result about the range of "V".

The Lemma is thus true for N_0, and we have only to "translate" S^* by $N - N_0$.

§ 50. LEMMA 30. *Suppose $\Gamma_{1,2}$ belong to S^* at Z_0, and have $U_{1,2}$ together at N, with $V_{1,2} \geq V^* + \delta$. Let U'_- be at N', and let $\mathfrak{V}_{1,2} = \mathfrak{V}(\Gamma_{1,2}, N)$, $\mathfrak{V}'_{1,2} = \mathfrak{V}(\Gamma_{1,2}, N')$, $\Delta\mathfrak{V} = \mathfrak{V}_2 - \mathfrak{V}_1$. Then* (i) *we have the following results.*

(a) $$L\,|\Delta\mathfrak{V}| - k^{-\frac{1}{3}} < |\Delta\mathfrak{V}'| < \alpha\,|\Delta\mathfrak{V}| + k^{-\frac{1}{3}},$$

where α is an L satisfying $0 < \alpha < 1$.

(b) *Either* $$|\Delta\mathfrak{V}|,\ k\,|w(Z_1)|,\ |\Delta\mathfrak{V}'| < k^{-\frac{1}{3}},$$

or else $$\operatorname{sgn}\Delta\mathfrak{V}' = -\operatorname{sgn}\Delta\mathfrak{V} = \operatorname{sgn} w(Z_1).$$

(c) *If $kw(Z_1) > k^{-\frac{1}{3}}$, or if $-\Delta\mathfrak{V} > k^{-\frac{1}{3}}$, or if $\Delta\mathfrak{V}' > k^{-\frac{1}{3}}$, then U'_- is U'_2.*

(ii) *If further $U'_{1,2}$ are together, we have the following results:*

(d) $$L\,|\Delta V| - \zeta < |\Delta V'| < \alpha\,|\Delta V| + \zeta.$$

(e) *Either* $$|\Delta V|,\ k\,|w(Z_1)|,\ |\Delta V'| < \zeta_2^*,$$

or else $$L\,|\Delta V| < |\Delta V'| < \alpha\,|\Delta V| \quad \textit{and} \quad \operatorname{sgn}\Delta V' = -\operatorname{sgn}\Delta V = \operatorname{sgn} w(Z_1).$$

---

[1] The whole segment may have Γ's and $\mathfrak{V}$'s outside the extremes; we then take the appropriate subsegment.

In what follows we use α generally for an L satisfying $0<\alpha<1$, rechoosing it (larger) as the argument proceeds.

We normalize to $w(Z_1)>0$. We have

$$-\Delta \mathfrak{V}=c(N), \qquad \Delta \mathfrak{V}'=c(N'), \tag{1}$$

$$\Delta \mathfrak{V}'+\Delta \mathfrak{V}=-\int_N^{N'} \Delta g\, dt. \tag{2}$$

If $w(Z_1)\leq \zeta_1^*$, we have $w, \dot{w}=O(\zeta)$ in (N, U'_-), $\Delta \mathfrak{V}$ and $\Delta \mathfrak{V}'$ are $O(\zeta)$, and so are ΔV, $\Delta V'$ in part (ii). The various parts of the Lemma are all true trivially or vacuously.[1]

We suppose then, in both (i) and (ii), that $w(Z_1)>\zeta'\geq\zeta_1^*$, rechoosing ζ' as we proceed. Then in the first place $\Gamma_{1,2}$ do not intersect before U'_-. We have then, from (2), (3) of § 46, and Lemma 24,

$$c(U_1)\lesssim Lkw(Z_1), \qquad c(U'_-)\lesssim Lkw(Z_1), \tag{3}$$

$$-\Delta V=c(U_1)+O(w(U_1))=c(U_1)(1+O(Dk^{-\frac{1}{2}}))\gtrsim Lkw(Z_1)>0. \tag{4}$$

In (ii) we have in addition $U'_-=U'_2$,

$$c(U'_2)\lesssim Lkw(Z_1) \tag{5}$$

$$\Delta V'=c(U'_2)(1+O(Dk^{-\frac{1}{2}}))\gtrsim Lkw(Z_1)>0. \tag{6}$$

In both (i) and (ii) we have $c(N)-c(U_1)=O(Dk^{-\frac{1}{2}})$, $c(N')-c(U'_-)=O(Dk^{-\frac{1}{2}})$, so that, by (1) and (3)

$$\left.\begin{aligned} -\Delta \mathfrak{V}&\gtrless c(U_1)\mp Dk^{-\frac{1}{2}}\gtrless Lkw(Z_1)\mp Dk^{-\frac{1}{2}}\\ \Delta \mathfrak{V}'&\gtrless c(U'_-)\mp Dk^{-\frac{1}{2}}\gtrless Lwk(Z_1)\mp Dk^{-\frac{1}{2}}.\end{aligned}\right\} \tag{7}$$

Also, as we saw in § 48,

$$-\Delta V=-\Delta \mathfrak{V}+O(Dk^{-\frac{1}{2}}), \tag{8}$$

and similarly, in (ii),

$$\Delta V'=\Delta \mathfrak{V}'+O(Dk^{-\frac{1}{2}}). \tag{9}$$

Next we have in (i), by (2),

$$\Delta \mathfrak{V}'=(-\Delta \mathfrak{V})-\int_N^{N'} \Delta g\, dt.$$

---

[1] Provided $\zeta_2^*\geq k\zeta_1^*$, which we suppose.

In the integral we have $\Delta g > Lw > 0$ everywhere, and $w > Lk^{-1}(-\Delta V) - \zeta$ over a part range of length Lk, by Lemma 26. So

$$\Delta \mathfrak{V}' < (-\Delta \mathfrak{V}) - Lk(Lk^{-1}(-\Delta V) - \zeta) < (1-L)(-\Delta \mathfrak{V}) + Dk^{-\frac{1}{2}}, \tag{10}$$

by (8).

Also (7), (3), (4), and (8) give

$$\Delta \mathfrak{V}' > Lkw(Z_1) - Dk^{-\frac{1}{2}} > Lc(U_1) - Dk^{-\frac{1}{2}} > L(-\Delta V) - Dk^{-\frac{1}{2}} > L(-\Delta \mathfrak{V}) - Dk^{-\frac{1}{2}}. \tag{11}$$

We now have (a) from (10) and (11), and (b) from (7).

In (c) we have, after (7), $\Delta \vartheta' > Lk^{-\frac{1}{2}}$ in all three alternatives. Suppose, if possible, that U'_- is not U'_2 but U'_1. By the non-intersection of $\Gamma_{1,2}$ there is a "U'_2" at N', and $\mathfrak{V}'_2(N') <$ "V'_2" $+ O(Dk^{-\frac{1}{2}})$. On the other hand

$$\mathfrak{V}'_2(N') > \mathfrak{V}'_1(N') + Lk^{-\frac{1}{2}} > V'_1 + O(Dk^{-\frac{1}{2}}) + Lk^{-\frac{1}{2}}.$$

Combination of these gives "V'_2" $> V'_1 \geq V^* - \delta$, and "U'_2" is U'_2 after all. Thus (c), and so the whole of part (i), is established.

Consider now part (ii) (in which *all* the numbered results hold). We have

$$c(U'_2) - c(U_1) = -\int_{U_1}^{U'_2} \Delta g\, dt < -L \int_{U_1}^{U'_2} w\, dt < -L((-\Delta V) - \zeta), \tag{12}$$

since $w > 0$ in (U_1, U'_2), and by Lemma 26 $w > Lk^{-1}((-\Delta V) - \zeta)$ over a range of length Lk. It follows from (4), (6), and (12) that

$$\Delta V' < (1 + O(k^{-A}))\{(-\Delta V) - (1 + O(k^{-A}))L(-\Delta V)\} + \zeta$$

which gives ($-\Delta V$ and $\Delta V'$ being positive, by (4) and (5))

$$\Delta V' < (1-L)(-\Delta V) + \zeta, \tag{13}$$

and so the second inequality of (d). The first one follows from (4) and (6).[1]

In (e), if one of $(-\Delta V) > \zeta'$, $kw(Z_1) > \zeta'$, $\Delta V' > \zeta'$ holds, then all three hold with $L_1 \zeta'$ in place of ζ', and with a suitable ζ' and $kw(Z_1) > L_1 \zeta'$ (3) to (13) are valid, and $-\Delta V$, $\Delta V'$ are positive. With a fresh choice of ζ' and a diminished α (d) holds without the ζ's. Since, finally, we have (sufficiently) normalized to $w(Z_1) > 0$, we have now established (e).

This completes the proof of the Lemma.

---

[1] Without the ζ because we are (sufficiently) supposing $w(Z_1) > \zeta'$.

§ 51. We now need the result that, roughly, for fixed V the ensuing V' varies smoothly with b. The exact result deals in $\mathfrak{V}$ and is as follows: we postpone the proof, which is rather long.

LEMMA 31. *There is a function $\theta(b)$, continuous in B, or $\frac{1}{100}\le b\le\frac{2}{3}-\frac{1}{100}$, depending only on the functions f, g, p (and so independent of k and δ, and of $\mathfrak{V}(N)$ below), and satisfying $L<\theta\le 2$, with the following properties. Suppose that a Γ of S^* has its U at N, with $V\ge V^*+\delta$, and that the ensuing U' is at N'. Let δb be a small negative increment of b, satisfying $0\le-\delta b<L_1k^{-1}$, and leaving $b+\delta b$ in B, where L_1 is an L to be chosen later. With the new b let a Γ of S^* have the same $\mathfrak{V}(N)$ as before.*[1] *Then with the usual incremental notation we have*

$$\delta\mathfrak{V}'(N'-2\pi)=\theta(b)\,k\,\delta b+O(k^{-A}).\tag{1}$$

N' does not jump to the left (in the change from b to $b+\delta b$) if $(-k\,\delta b)>\delta$ (i.e. if $-\delta b$ is not too small). In this case (1) is valid with N' in place of $N'-2\pi$.

We have for V^*, M, qua functions of b,

$$\delta V^*,\ \delta M=O(k^{-A}).$$

It is further true that

$$\delta\theta=O(k^{-A})\quad \textit{for}\quad \tfrac{1}{100}+Lk^{-1}\le b\le\tfrac{2}{3}-\tfrac{1}{100}.^{2}$$

§ 52. For each b of B now let $P=V^*+\delta^{\frac{1}{2}}$, $Q=V^*+M-\delta^{\frac{1}{2}}$. Let Γ_P be a Γ of S^* with its U at N and with $\mathfrak{V}(\Gamma_P,N)=P$, and let the ensuing U' be at N'. Let Γ_Q be a Γ of S^* with $\mathfrak{V}(\Gamma_Q,N)=Q$. Such trajectories Γ_P, Γ_Q exist by Lemma 29. With the notation of Lemma 30 and $\Gamma_1=\Gamma_Q$, $\Gamma_2=\Gamma_P$, we have $-\Delta\mathfrak{V}=Q-P>L$, and so, by (c) of the Lemma, Γ_Q has its U' not earlier than N_0'.[3] We abbreviate $\mathfrak{V}'(\Gamma_{P,Q},N_0')$ to P', Q'. With these understandings we now have

LEMMA 32. *The interval B consists of a set B_1 of intervals i_1, a set B_2 of intervals i_2, and a third "excluded" set E of intervals of total length $O(\delta^{\frac{1}{2}})$. For a b of B_1 P', Q' both lie in the range $(P+\delta^{\frac{1}{2}},\ Q-\delta^{\frac{1}{2}})$; for a b of B_2 P' is in $(P+\delta^{\frac{1}{2}},\ Q-\delta^{\frac{1}{2}})$, Q' below $V^*-2\delta^{\frac{1}{2}}$. The state of things is described graphically in figs. 9 and 10, and it*

[1] (1) The initial conditions of S^* do not involve b, (2) there is a small "sheaf" of possible Γ; owing to "play", and our error terms have to cover this; (3) it is a logical possibility (when Γ is very extreme in S^*) that there may not exist a second Γ with the new b. In this case the Lemma is true vacuously: the point does not arise in applications.

[2] This could be extended to the whole of B if need be.

[3] An abbreviation for "U' is not at an N_1' earlier than N'".

is further true that in both cases, PP', QQ' "cross and shrink"; P' is above Q', and the lengths[1] *PQ, $P'Q'$ satisfy*

$$L \cdot PQ < P'Q' < \alpha \cdot PQ,$$

where α is an L satisfying $0 < \alpha < 1$.

Let b_0 be the right-hand end of an i_1 or an i_2, and let N'_0 be the N' for $b = b_0$. Then for b of the i_1 or i_2, N' is not earlier than N'_0. Further a Γ of S^* with $\mathfrak{V}(N) = R$ where $P \leq R \leq Q$, has its U' not earlier than N' $(\geq N'_0)$, and we can define[2] *$R' = \mathfrak{V}'(\Gamma_R, N')$. Also if R, R_1 are two R's of (P, Q), RR' and $R_1 R'_1$ cross and shrink; more precisely*

$$L \cdot RR_1 - k^{-\frac{1}{3}} < R'R'_1 < \alpha \cdot RR_1 + k^{-\frac{1}{3}}$$

(in particular either of R, R_1 may be either of P and Q).

The length $P'Q'$ is $l(b) + O(\delta^{\frac{1}{2}})$, where $l(b) < \alpha M$. $l(b)$ is independent of k, δ. The length of i_1 is $k^{-1}((M - l)/\theta + O(\delta^{\frac{1}{3}}))$, that of i_2 is $k^{-1}(l/\theta + O(\delta^{\frac{1}{3}}))$ where in each case the b of M, l, θ is taken at the right-hand end of the i. Both lengths lie between two Lk^{-1}.[3]

We begin with some preliminary observations. We abbreviate k^{-1} to ε, and we shall be working to errors $O(\delta^{\frac{1}{3}})$. Our $i_{1,2}$ are going to have lengths of order ε, and we are to ignore a total length $O(\delta^{\frac{1}{3}})$ in b, absorbable in E. We therefore start from a b_0 which is inside B by an amount $\delta^{\frac{1}{2}}$ at each end. We then consider a *decrease* of b from b_0 of amount $L\varepsilon$. Inside this stretch, which we will call I, we can ignore stretches of length $O(\varepsilon\delta^{\frac{1}{3}})$ (absorbable in E). In particular we may decrease an "inconvenient" initial b_0 by any convenient amount $L\varepsilon\delta^{\frac{1}{3}}$.

Next, the variations of M, V^*, P, Q, over an I are $O(k^{-A})$ (Lemma 31), which is very small compared with $\delta^{\frac{1}{3}}$; the upshot of this is that we can effectively suppose $M = M(b_0)$, etc., over I; the applications of this principle will be made more or less tacitly to avoid further complicating a rather tangled story.

The position N' of the Lemma is determined, for each b, by Γ_P.[4] If N'_0 is the N' for $b = b_0$, $\mathfrak{V}'(\Gamma_P, b_0, N'_0)$ lies between $V^* - 2\delta$ and $V^* + M + 2\delta$. Lemma 31 tells

---

[1] Taken signless throughout, as are RP, $R'Q'$, etc. below.

[2] (1) Recall that N' is determined by Γ_P (for each b), (2) the definition of R' is consistent with that of P' and Q'.

[3] The complications, and in particular the different powers of δ involved, arise because (i) we wish the excluded intervals to be a small proportion of B, (ii) as a result of this, the behaviour when b is near an end of an i is rather extreme.

[4] We suppose throughout that a single representative Γ_P or Γ_Q is selected, for each b, from the two (small) sheaves of possible ones.

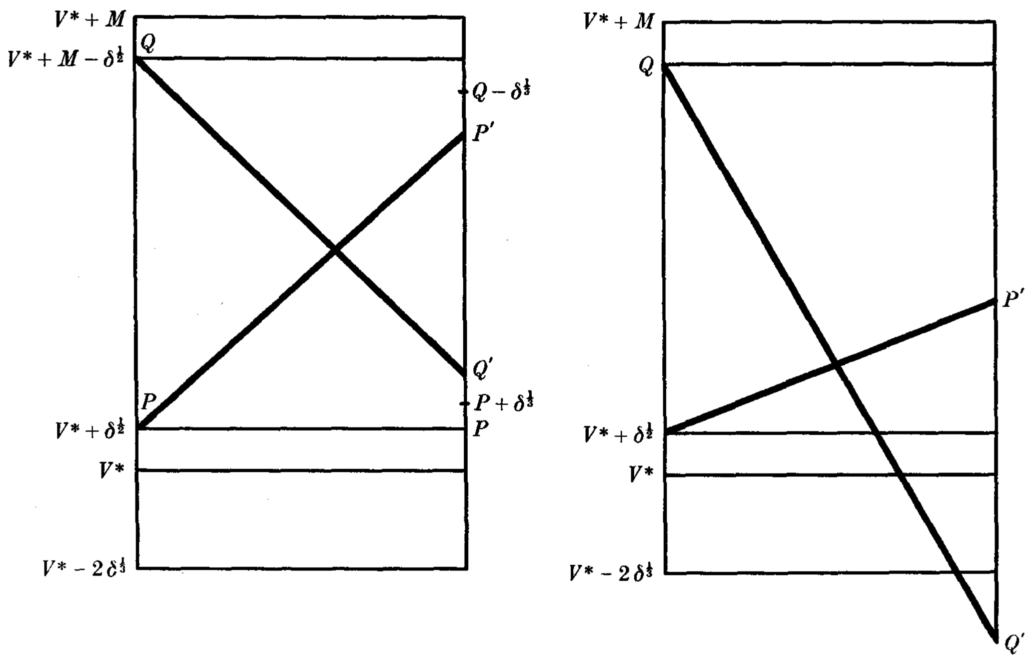

Fig. 9. (i) b belonging to B_1; (ii) b belonging to B_2.

us that a decrease $\varepsilon\delta^{\frac{1}{3}}$ of b from b_0, (i) leaves N' not earlier than N_0' from then and during subsequent decrease,[1] (ii) decreases $\mathfrak{V}'(\Gamma_P, N')$ by at least $L\delta^{\frac{1}{3}}$.[2] If now, for b_0, $\mathfrak{V}'(\Gamma_P, N_0')$ is above Q, then a suitable decrease $L\varepsilon\delta^{\frac{1}{3}}$ of b will bring it below $Q-\delta^{\frac{1}{3}}$ (but within $O(\delta^{\frac{1}{3}})$). If, on the other hand, P', for b_0, is below $P+\delta^{\frac{1}{3}}$, a suitable decrease $L\varepsilon\delta^{\frac{1}{3}}$ will bring $\mathfrak{V}'(\Gamma_P, N_0')$ below $V^*-2\delta^{\frac{1}{3}}$ (but within $O(\delta^{\frac{1}{3}})$). In this case, by Lemma 28, and (i), the N' of Γ_P is now at $N_0'+2\pi$; and if we then shift to the new b_0, and take the diagram with the appropriate new $N'\,(=N_0'+2\pi)$, we shall have P' (slightly) below $Q-\delta^{\frac{1}{3}}$. *We may accordingly suppose* (*by absorption in* E) *that for* $b=b_0$ P' *lies between* $P+\delta^{\frac{1}{3}}$ *and* $Q-\delta^{\frac{1}{3}}$.[3]

For this (new) b_0 let R be a value in $P\le R\le Q$, and let Γ_R be a Γ of S^* with $\mathfrak{V}(N)=R$. We shall prove now that

(a) Γ_R *has its* U' *not earlier than* N_0'.

---

[1] If we were dealing in *increasing* b this could fail, and our discussion would be even more awkward than it is.

[2] The $\delta\mathfrak{V}'(N')$ form of Lemma 31 is available.

[3] The completion of the diagram, and division into two cases, depends on, and awaits, the discussion of Q'.

Taking this momentarily for granted we may then define (consistently) $R' = \mathfrak{V}'(\Gamma_R, N'_0)$. Then we have further, for any two R, R_1 of (P, Q)

$$\text{(b)} \qquad L \cdot RR_1 - k^{-\frac{1}{3}} < R'R'_1 < \alpha \cdot RR_1 + k^{\frac{1}{3}}.$$

(b) follows from (a) and Lemma 30, and we turn to the proof of (a). With the notation of Lemma 30, and $\Gamma_1 = \Gamma_R$, $\Gamma_2 = \Gamma_P$, let U'_- be at N'; we have to show that $N' \geq N'_0$. There are two cases: (α) $R - P < k^{-\frac{1}{3}}$; (β) $R - P \geq k^{-\frac{1}{3}}$.

Case (α). By Lemma 30 we have

$$\mathfrak{V}'(\Gamma_R, N') - \mathfrak{V}'(\Gamma_P, N') = O(k^{-\frac{1}{3}}). \tag{1}$$

If $N' < N'_0$, then on the one hand

$$\mathfrak{V}'(\Gamma_P, N') = \mathfrak{V}'(\Gamma_P, N'_0) - M + O(Dk^{-\frac{1}{2}}) \leq (Q - \delta^{\frac{1}{3}}) - M + O(Dk^{-\frac{1}{2}}) < V^* - \delta^{\frac{1}{3}}, \tag{2}$$

and on the other U'_R is at N', so that

$$\mathfrak{V}'(\Gamma_R, N') > V^* - \delta + O(Dk^{-\frac{1}{2}}). \tag{3}$$

Since the combination (1), (2), (3) is impossible, (a) is true in case (α).

Case (β). By Lemma 30 (c) U'_- is U'_P and $N \geq N'_0$ as desired.

This is all for a single b_0 for which P' satisfies $P_0 + \delta^{\frac{1}{3}} < P' < Q - \delta^{\frac{1}{3}}$. We now let b decrease through I. Abbreviate[1] $\mathfrak{V}'(\Gamma_{P,Q}, N'_0)$ to $\mathfrak{V}'_{P,Q}$. By Lemma 31 $\mathfrak{V}'_P$, $\mathfrak{V}'_Q$ descend with the same constant velocity $\theta(b_0)$ (with respect to kb), to error $O(k^{-A})$, and their difference remains constant, to this error. This constant we denote by $l(b_0)$; $l(b)$ lies between two L's.

Further, as we have seen, N' will not jump to the left of N'_0, and, after (a), Γ_R has its U' not earlier than $N' (\geq N'_0)$; this is true in particular of Γ_Q.

Let us suppose, momentarily, that $\mathfrak{V}'_P$ starts below and within $O(\delta^{\frac{1}{3}})$ of the value Q, then over a stretch of length $\varepsilon((M - l)/\theta(b_0) + O(\delta^{\frac{1}{3}}))$ from b_0 $\mathfrak{V}'_P$ and $\mathfrak{V}'_Q$ lie between $P + \delta^{\frac{1}{3}}$ and $Q - \delta^{\frac{1}{3}}$. Since N' has not jumped to the *left* it follows from Lemma 28 that N' (determined by Γ_P) is still N'_0, and $\mathfrak{V}'_P$ and $\mathfrak{V}'_Q$ can be identified with P', Q'. We have accordingly P', Q' lying in the range (P, Q); we are in case (i), and the stretch of b is an i_1.

As b continues to decrease there ensues a stretch of length $O(\varepsilon\delta^{\frac{1}{3}})$, which we consign to E, beyond which $\mathfrak{V}'_Q$ is below $V^* - 2\delta^{\frac{1}{3}}$. Next comes a stretch of length $\varepsilon(l/\theta + O(\delta^{\frac{1}{3}}))$ during which $\mathfrak{V}'_P$ remains above the value $P + \delta^{\frac{1}{3}}$, ending by being only

[1] We do not have $P' = \mathfrak{V}'_P$ unless N', determined by Γ_P, is still N'_0 for the new b.

$O(\delta^{1/3})$ above. As before, $N'=N_0'$, and ϑ_P', ϑ_Q' are identified with P', Q'. P', Q' are accordingly disposed as in case (ii), and the stretch of b is an i_2.

The ensues a stretch of length $O(\varepsilon\delta^{1/3})$, consigned to E, after which ϑ_P' is just below $V^*-2\delta^{1/2}$, and ϑ_Q' is an amount $l+O(\delta^{1/3})>L$ lower still. We make a fresh start from here, with new b_0. It is by now clear that the new N_0' is the old one $+2\pi$. The diagram, based on the new N_0' will be case (i); moreover we are in the situation momentarily taken above as starting point (with new b_0, N'). We can now repeat the processes described, and a little reflection will convince the reader that we have established all the results set out in the Lemma.[1]

§ 53. There are further developments for case (ii). Consider a point R between P and Q, and a Γ_R of S^* with $\vartheta(N)=R$, and let us define[2] $R'=\vartheta'(\Gamma_R, N')$. For Γ_R we have, after Lemma 32, that U_R' is not earlier than N', that RR', QQ' cross and shrink, with error $O(k^{-A})$, as do RR', PP'. We recall that in case (ii) $Q'<V^*-2\delta^{1/2}$, $P'>P=V^*+\delta^{1/2}$. Since a continuous stream of Γ_R exists for $P\le R\le Q$, it follows by continuity that there is an $R_0=V_*$ such that $R_0'=V^*$. There is a (small) range of possible V_*; we suppose that a unique representative is selected for each b (of an i_2).

Next, V_* lies between $P+\delta^{1/3}\,(=V^*+\delta^{1/2}+\delta^{1/3})$ and $Q-\delta^{1/3}\,(=V^*+M-\delta^{1/2}-\delta^{1/3})$. For if, e.g., $R_0=V_*<P+\delta^{1/3}$ we should have $R_0P<\delta^{1/3}$, $R_0'P'<\alpha\cdot R_0P+k^{-1/3}<(\alpha\delta^{1/3}+k^{-1/3})$, and so

$$V^*=R_0'>P'-(\alpha\,\delta^{1/3}+k^{-1/3})>V^*+\delta^{1/3}-(\alpha\,\delta^{1/3}+k^{-1/3})>V^*,$$

a contradiction. The other case is similar.

When b decreases through i_2, V_* decreases, to error $O(\delta^{1/3})$, from $Q+O(\delta^{1/3})$ to $P+O(\delta^{1/3})$. (The speed is actually approximately constant, though we shall not prove this.)

We are now in a position to prove

L EMMA 33. *Suppose $b\in B_1$. Then if R belongs to the range $\mathcal{R}$, or $V^*+\delta^{1/2}\le R\le V^*+M-\delta^{1/2}$, Γ_R[3] has its successive $U, U', \ldots$ at $N, N+\tfrac{1}{2}p,\ N+2(\tfrac{1}{2}p), \ldots$ where $p=2(2n-1)\pi=2(N'-N)$.*

---

[1] The suffix 0 in N_0' is scaffolding for the proof, and disappears from some of the statements of the Lemma.

For the statements about R', note that the b_0 of (a) and (b) may be any b of an i_1 or i_2.

[2] Consistently with the definitions of P', Q'.

[3] Recall that there is a small sheaf of Γ_R for a given R; the various results are true whichever members of the sheaf or sheaves are taken.

If R, R_1 belong to $\mathcal{R}$, then Γ_R, Γ_{R_1} quasi-converge; that is to say $y(\Gamma_{R_1}) - y(\Gamma_R) = O(\zeta)$ and $\dot{y}(\Gamma_{R_1}) - \dot{y}(\Gamma_R) = O(\zeta)$ for large t.

For $b \in B_2$ there are two cases:

(α) *If R belongs to $\mathcal{R}_\alpha$, or $V^* + \delta^{\frac{1}{2}} \leq R \leq V_* - \delta^{\frac{2}{5}}$, then Γ_R has its successive U, U', ... at N, $N + \frac{1}{2}p$, $N + 2(\frac{1}{2}p)$, ...*

If R, R_1 belong to $\mathcal{R}_\alpha$, then Γ_R, Γ_{R_1} quasi-converge.

(β) *If R belongs to $\mathcal{R}_\beta$, or $V_* + \delta^{\frac{2}{5}} \leq R \leq V^* + M - \delta^{\frac{1}{2}}$, then Γ_R has its successive, U, U', ... at N, $N + \frac{1}{2}p_1$, $N + 2(\frac{1}{2}p_1)$, ... where $p_1 = p + 4\pi = 2(2n+1)\pi$.*

If R, R_1 belong to $\mathcal{R}_\beta$ then Γ_R, Γ_{R_1} quasi-converge. $\mathcal{R}_\alpha$ is a proportion $1 - L\,\delta^{\frac{1}{15}}$ at least of (V^*, V_*), which has length at least $\delta^{\frac{1}{3}}$. $\mathcal{R}_\beta$ is the same proportion at least of $(V_*, V^* + M)$, which has length at least $\delta^{\frac{1}{3}}$.

The last clause is a consequence of what was proved above.

It will be enough to take the more difficult $b \in B_2$, for which the argument is easily adapted to $b \in B_1$.

We begin by proving the two addenda:

(a) *In (α), for all R of $\mathcal{R}_\alpha$, U'_R is at N', and $\mathcal{V}'(\Gamma_R, N')$ and V'_R lie in a range $(V^* + L\,\delta^{\frac{2}{5}},\ V_* - L\,\delta^{\frac{1}{3}})$, and so in $\mathcal{R}_\alpha$ diminished by $L\,\delta^{\frac{2}{5}}$ at each end.*

(b) *In (β), for all R of $\mathcal{R}_\beta$, U'_R is at $N' + 2\pi$, and $\mathcal{V}'(\Gamma_R, N' + 2\pi)$ and V'_R lie in the range $(V_* + L\,\delta^{\frac{1}{3}},\ V^* + M - L\,\delta^{\frac{2}{5}})$, and so in $\mathcal{R}_\beta$ diminished by $L\,\delta^{\frac{2}{5}}$ at each end.*

It will be enough to take the slightly more complicated (b). The range $\mathcal{R}_\beta$ is (T, Q), where $T = V_* + \delta^{\frac{2}{5}}$. By Lemma 32 [$R R_1$ and $R' R'_1$ cross and shrink] T' lies between two values $V^* - L\,\delta^{\frac{2}{5}}$, and Q' lies between $T' - \alpha \cdot QT$ and $T' - L \cdot QT$. Since $QT = Q V_* - \delta^{\frac{2}{5}} > \delta^{\frac{1}{3}} - \delta^{\frac{2}{5}} > L\,\delta^{\frac{1}{3}}$, it follows by easy calculations that $Q' + M$ and $T' + M$ lie in $(V_* + L\,\delta^{\frac{1}{2}},\ V^* + M - L\,\delta^{\frac{2}{5}})$, and for an R of TQ $R' + M$ lies in this interval, to error $k^{-\frac{1}{3}}$, so that (with new L)

$$V_* + L\,\delta^{\frac{1}{3}} < R' + M < V^* + M - L\,\delta^{\frac{2}{5}}. \tag{1}$$

Since by Lemma 32 U'_R is not earlier than N', it follows from Lemma 28 that U'_R is at $N' + 2\pi$. Then $\mathcal{V}'(\Gamma_R, N' + 2\pi)$ and V'_R are $R' + M + O(D k^{-\frac{1}{2}})$, and they satisfy (1) with new L. This proves (b).

After (b) we use, for case (β), a diagram based on $N' + 2\pi$ for the right-hand ordinate, or $N + p_1$. [In case (α) we use $N' = N + p$.] Since R' is in the diminished

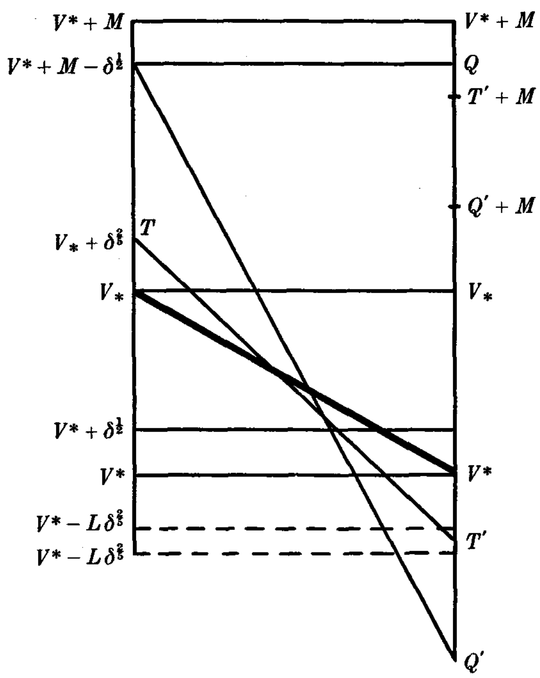

Fig. 10.

$\mathcal{R}_\beta$, the U_R next after U'_R is at $N'+2p_1$, the next U'_R is at $N'+3p_1$, and so on; also $(R')'$, which we call R'' is $\mathfrak{V}(\Gamma_R, N+2p_1)$, and so on for R''', ... The V'_R, V''_R, ... corresponding to the U'_R, U''_R, ... differ from R', R'', ... by $O(Dk^{-\frac{1}{2}})$, and lie inside a diminished $\mathcal{R}_\beta$. Starting now from a pair of Γ's, Γ_R, Γ_{R_1}, we can apply Lemma 30 in its ΔV form. We have

$$|V_{R_1}^{(m+1)}-V_R^{(m+1)}|<\alpha|V_{R_1}^{(m)}-V_R^{(m)}|+\zeta,$$

from which it follows that

$$V_{R_1}^{(m)}-V_R^{(m)}=O(\zeta)\text{ for large }m. \tag{2}$$

It remains to deduce that $y(\Gamma_{R_1})-y(\Gamma_R)=O(\zeta)$ and $\dot{y}(\Gamma_{R_1})-\dot{y}(\Gamma_R)=O(\zeta)$ for large t.[1] Since (2) is true for *all* large m, it is enough, after Lemma 25, to prove, in the notation and context of § 46, etc., that $\Delta V=O(\zeta)$ implies $w(Z_1)=O(\zeta)$. We may suppose (normalizing) that $w(Z_1)>\zeta_1^*$ (else we have what we want). Then by Lemma 24

[1] The conclusion is "obvious", but it happens that it was not convenient in earlier Lemmas to record just the combination required.

$$c(U_1) \lesssim L k^{\frac{1}{2}} w(U_1) \lesssim L k w(Z_1). \tag{3}$$

The quasi-identities (2) and (3) of § 46 combine to give

$$c(U_1) = -\Delta V + O(w(U_1)),$$

and from this and (3)

$$-\Delta V = c(U_1)(1 + O(k^{-\frac{1}{2}})),$$

$$|\Delta V| > L c(U_1) > L k w(Z_1),$$

so that $w(Z_1) < \zeta$, as desired.

§ 54. *Theorem* 1.[1] *For $b \in B_1$, when $R \in \mathcal{R}$, Γ_R converges to a periodic Γ of period $(2n-1)2\pi$. For $b \in B_2$, when $R \in \mathcal{R}_\alpha$, Γ_R converges to a periodic Γ of period $(2n-1)2\pi$; when $R \in \mathcal{R}_\beta$ it converges to a periodic Γ of period $(2n+1)2\pi$.*

Before going on to the proof of this some remarks about the scope of its results may be welcome. The theorem proves at once (after Lemma 29) that for $b \in B_1$ there is a sheaf of Γ's in any S^* which all converge to one of a set[2] of $2n-1$ with period $(2n-1)2\pi$ and that for $b \in B_2$ there is one sheaf converging to one of a set of Γ of period $(2n-1)2\pi$, and another sheaf converging to a set of period $(2n+1)2\pi$. It is fairly clear that, *roughly speaking, and in some sense, most* trajectories behave in one of these ways (for $b \in B_1 + B_2$). To make this statement precise would give a good deal of trouble, and it would be of doubtful value since any particular formulation would probably seem rather arbitrary.[3] We shall be content, therefore, to sketch the general setting.

In the first place it is clear from the work of Lemmas 6 to 9 that we could, if called upon, prove that Γ's satisfying much wider conditions (than starting in an S^*) sooner or later enter an S or S'. Next, the Γ's of an S can be classified, in a fairly natural way, by their $y(Z_1)$'s. The Lemmas connecting $w(Z_1)$ and $c(U_1)$ (which is more or less $-\Delta V$) for a mesh go to show, and could be developed to do so rigorously, that a small proportion of the possible range of V, or its near equivalent $\mathcal{V}(\Gamma, N)$, which range is approximately $(V^*, V^* + M)$, corresponds to a small

[1] For convenience of reference we recall: R and Γ_R are defined in § 52, Γ_R being a Γ of S^* defined in § 24 with $\mathcal{V}(\Gamma, N) = R$. $\mathcal{R}$ is $V^* + \delta^{\frac{1}{2}} \leq R \leq V^* + M - \delta^{\frac{1}{2}}$. $\mathcal{R}_\alpha$ is $V^* + \delta^{\frac{1}{2}} \leq R \leq V_* - \delta^{2/5}$; $\mathcal{R}_\beta$ is $V_* + \delta^{2/5} \leq R \leq V^* + M - \delta^{\frac{1}{2}}$. The very fundamental V_* is discussed in the Introduction, and defined in § 53. The integer $2n-1$ is $2(N'-N)/2\pi$, and N' is defined (for each b) in Lemma 31, § 51.

"Convergence" means that both $y(\Gamma_R) - y(\Gamma)$ and $\dot{y}(\Gamma_R) - \dot{y}(\Gamma)$ tend to 0.

[2] The set consists of displacements by $0, 2\pi, \ldots (2n-1)2\pi$ of a single Γ.

[3] Because so much would depend on how the initial conditions were weighted. The full truth is probably that all trajectories except a nowhere dense set in the phase space behave so.

proportion of the range of the $y(Z_1)$ in an S. Thus a classification in terms of $R=\mathfrak{V}(\Gamma, N)$ inherits the "naturalness" of one by $y(Z_1)$, and this classification by R is that used in the theorem. Finally the ranges $\mathcal{R}_{\alpha,\beta}$ of R are the total possible ones, with small diminutions.

§ 55. *Proof of Theorem* 1. It is enough to prove that if $\Gamma_{1,2}$ quasi-converge and have all their $V_{1,2}$, $V'_{1,2}>V^*+\delta$, then they converge strictly. For suppose this proved, and consider the part, sufficiently representative, of Theorem 1 about $b\in B_2$ and $R\in\mathcal{R}_\beta$. Our assumption implies that any two Γ_R, Γ_{R_1} $(R, R_1\in\mathcal{R}_\beta)$ converge, and in particular, since (by § 53 (b)), R' belongs to $\mathcal{R}_\beta$, it implies that Γ_R and $\Gamma_R(p_1)$ converge. If then y_m is $y(\Gamma_R)$ at $t=mp_1$, we have $y_{m+1}-y_m\to 0$, $\dot{y}_{m+1}-\dot{y}_m\to 0$. If $(Y, \dot{Y})$ is a limit point of the set of points $(y_m, \dot{y}_m)$, the trajectory Γ_0 with $y=Y$, $\dot{y}=\dot{Y}$ at $t=0$ is clearly periodic. Its R belongs to $\mathcal{R}_\beta$ and by the assumption Γ_R converges to it.

We have accordingly to prove that $w\to 0$, $\dot{w}\to 0$, in the notation of §§ 34 *et seq.* We have $w=O(\zeta)$, $\dot{w}=O(\zeta)$, hypothesis (H) of § 38 is always valid, and we may take over Lemmas 13 to 16, about τ, in T-form.

There are two cases: (i) $\Gamma_{1,2}$ have no intersection beyond some point, (ii) they have an infinity of intersections.

Case (i). Suppose that w is ultimately positive. Then, first, w_1, which is an increasing function, must be bounded. Otherwise, we should have, for an arbitrarily large G and $t>t_0(G)$, $c-w_1<-G$, and so [(W_1 of § 34]

$$w=w_0e^{-T}+e^{-T}\int_0^t(c_0-w_1)\,e^T\,dt$$

$$<(w_0+c_0\int_0^{t_0}e^T\,dt)\,e^{-T}-G\,e^{-T}\int_{t_0}^t e^T\,dt.$$

By Lemma 13 (a) the first term is less than a constant independent of G, while for $t>k^{-1}$ the second is less than $-Lk^{-1}G$, so that w is ultimately negative, a contradiction that shows that w_1 is bounded. Since $w>0$ and $\dot{w}$ is bounded, we must have $w\to 0$; and then $\dot{w}\to 0$ since $\ddot{w}$ is bounded.

§ 56. *Case* (ii). Let the intersections be I_n, $n=1, 2, \ldots$, and $w_n=w(I_n)$. In the first place we have for any t in I_nI_{n+1}

$$|w|<D\,|\dot{w}_n|\,k^{\frac{1}{2}}. \tag{1}$$

For, taking $t=0$ at I_n, and supposing $\dot{w}_n$ and w non-negative, say, we have

$$w = e^{-T}\int_0^t (\dot{w}_n - w_1)\, e^T\, dt$$

$$\le \dot{w}_n\, e^{-T}\int_0^t e^T\, dt < D k^{\frac{1}{2}}\, \dot{w}_n,$$

by (b) of Lemma 13.

Consider now the r.m. from I_{n+1}, taken as $t=0$, up to time $t=1$, or till we reach I_n, whichever happens first. If w is, say, non-negative, we have from $(\overline{W}_3)$ [§ 35], with $\bar{c}_0 = \dot{w}_{n+1}$

$$w\, e^{-T} \ge \dot{w}_{n+1}\left(\int_0^t e^{-T}\, dt - L \iiint_{0\le\xi\le\eta\le\zeta\le t} e^{-T_\zeta + T_\eta - T_\xi}\, d\xi\, d\eta\, d\zeta\right). \tag{2}$$

The triple integral $\le \left(\int_0^t e^{-T}\, dt\right)\left(\iint_{0\le\eta\le\zeta\le t} e^{-T_\zeta + T_\eta}\, d\eta\, d\zeta\right),$

and the second factor $< D k^{-\frac{1}{2}}$, by Lemma 14. Hence, from (2),

$$w \ge \tfrac{1}{2}\, \dot{w}_{n+1}\, e^T \int_0^t e^{-T}\, dt,$$

and in particular I_n is not reached before $t=1$. At $t=1$ we have

$$w(1) \ge \tfrac{1}{2}\, \dot{w}_{n+1}\, e^{T(1)} \int_0^1 e^{-T}\, dt \ge \tfrac{1}{2}\, \dot{w}_{n+1} \cdot e^{Lk} \cdot L k^{-1},$$

$$w(1) > e^{Lk}\, \dot{w}_{n+1}.$$

On the other hand this $w(1)$, being a w of $I_n I_{n+1}$, satisfies (1), or $|w(1)| \le D k^{\frac{1}{2}} |\dot{w}_n|$. Hence $|\dot{w}_{n+1}| \le \frac{1}{2} |\dot{w}_n|$.

We have now $\dot{w}_n \to 0$, and so, from (1), $w \to 0$ uniformly. Finally $\dot{w}$ must also $\to 0$ since $\ddot{w}$ is bounded. Thus $\Gamma_{1,2}$ converge, and the proof is completed.

§ 57. We now take up the postponed proof of Lemma 31. It is more convenient to take the inverted form in which $\vartheta'(N')$ is given the same for the two Γ's and we have to prove $\delta\vartheta(N-2\pi) = \theta(b)\, k\, \delta b + O(k^{-A})$. We use Γ_1 for the Γ with b, Γ_2 for that with $b+\delta b$, and $y_{1,2}$ for their y's. We abbreviate k^{-1} to ε.

Since $$\vartheta(N-2\pi) + \vartheta'(N') = (2b - \tfrac{4}{3})\, k + \int_{N'}^{N-2\pi} g\, dt \tag{1}$$

we have $\delta\mathfrak{V}(N-2\pi)=2k\delta b+\int_{N'}^{N-2\pi}\delta g\,dt$. Let N_n be the nadir $(\varphi\equiv-\frac{1}{2}\pi)$ between Z_n and Z_{n+1}. Let N_ν be the last N_n in the long descent such that $y\geq 1+k^{-\alpha}$ for $N_1\leq t\leq N_\nu$, where $\alpha=\frac{2}{7}$ (the index has to be between $\frac{1}{4}$ and $\frac{1}{3}$). We shall find that $\int_{N'}^{N_1}\delta g\,dt$ and $\int_{N_\nu}^{N-2\pi}\delta g\,dt$ are negligible.

We begin by disposing of $\int_{N'}^{N_1}$; this is crude. In the first place, $\mathfrak{Z}$ is a time $O(Dk^{-\frac{1}{3}})$ after N' and $\int_{N'}^{\mathfrak{Z}}g\,dt=O(Dk^{-\frac{1}{3}})$ for each of $\Gamma_{1,2}$. To this error we may start at the later of the two $\mathfrak{Z}$. From this point to N_1 we have (for each Γ) $\dot{y}=O(1)$ [Lemma 11], and so, by the $\dot{y}$-identity.

$$F(y)=F(\mathfrak{Z})+b(p_1(t)-p_1(\mathfrak{Z}))+O(\varepsilon)$$
$$=F(H)+b(p_1(t)-p_1(Z_0))+O(D\varepsilon),$$

since $\mathfrak{Z}-H=O(D\varepsilon)$, $\mathfrak{Z}-Z_0=O(Dk^{-\frac{1}{3}})$. Hence

$$\delta F(y)=O(\delta b)+O(D\varepsilon)=O(D\varepsilon),$$

and so, since $y>1+L$, $\delta y=O(D\varepsilon)$, and so $\delta g=O(D\varepsilon)$. Thus

$$\int_{N'}^{N_1}\delta g\,dt=O(Dk^{-\frac{1}{3}}),$$

and so

$$\delta\mathfrak{V}(N-2\pi)=2k\delta b+\int_{N_1}^{N_\nu}\delta g\,dt+R+O(Dk^{-\frac{1}{3}}),\quad R=\int_{N_\nu}^{N-2\pi}\delta g\,dt. \tag{2}$$

Consider now the range $(N_1\,N_\nu)$. In an intermediate (N_n,N_{n+1}) we have

$$F(y)=b\,p_1+c_n-\varepsilon\Big(\dot{y}+\int_{N_n}^{t}g\,dt\Big)\quad(1\leq n\leq\nu-1;\ N_n\leq t\leq N_{n+1}), \tag{3}$$

where
$$c_{n+1}-c_n=-\varepsilon\int_{(n)}g\,dt, \tag{4}$$

and we abbreviate $\int_{N_n}^{N_{n+1}}$ to $\int_{(n)}$. Also, writing $\eta=y-1$, we have, by Lemma 3 (3)

$$\dot{y}+b\,p/f=O(\varepsilon\eta^{-2}). \tag{5}$$

It follows from (2) that

$$\delta \vartheta (N-2\pi) = 2k\delta b + \delta (kc_1) - \delta (kc_\nu) + R + O(Dk^{-\frac{1}{3}}). \tag{6}$$

We begin with $\delta (kc_1)$. The formula (3) is actually an identity for all t. Taking $n=1$ and $t=U'$ we have

$$F(-1) = c_1 + b - \varepsilon \{v' + bk(1-p_1(U'))\} + \varepsilon \int_{U'}^{N_1} g\,dt$$

$$= c_1 + b - \varepsilon V' + \varepsilon \int_{U'}^{N'} g\,dt + \varepsilon \int_{N'}^{N_1} g\,dt.$$

In this $V' - \vartheta'(N') = O(Dk^{-\frac{1}{3}})$, $\int_{U'}^{N'} = O(Dk^{-\frac{1}{3}})$. Hence

$$\delta (kc_1) = -k\delta b + O(Dk^{-\frac{1}{3}}) + \int_{N'}^{N_1} \delta g\,dt$$

$$= -k\delta b + O(Dk^{-\frac{1}{3}}), \tag{7}$$

since we saw above that the integral last written is $O(Dk^{-\frac{1}{3}})$. From (6) and (7) we have (summing up for convenience of reference)

$$\delta \vartheta (N-2\pi) = k\delta b - \delta (kc_\nu) + R + O(Dk^{-\frac{1}{3}}),\ R = \int_{N_\nu}^{N-2\pi} \delta g\,dt. \tag{8}$$

Thus, apart from the discussion of R, the calculation of $\delta\vartheta$ is reduced to that of $\delta(kc_\nu)$. Our method for this is to operate with δ on (3), (4), (5), with δb satisfying $0 \le -k\delta b \le L_1$.

§ 58. Consider the range, $\mathcal{R}$, say, from N_1 up to N_ν, or up to the first moment when $|\delta y| = \Lambda\varepsilon\eta^{-1}$, whichever happens first. Here $\Lambda \ge 1$ is a number that will ultimately be chosen to be a certain L_2 (itself depending on the "given" L_1 of the Lemma). Among the consequences we shall deduce from the hypothesis that $|\delta y| \le \Lambda\varepsilon\eta^{-1}$ in $\mathcal{R}$ is an inequality which, when the substitution $\Lambda = L_2$ is made, yields $|\delta y| < \frac{1}{2}L_2\varepsilon\eta^{-1}$ (in $\mathcal{R}$). It follows that $|\delta y| = \Lambda\varepsilon\eta^{-1}$ is not the first event, so that $\mathcal{R}$ extends to N_ν, and further that $|\delta y| < \frac{1}{2}L_2\varepsilon\eta^{-1}$ in (N_1, N_ν). We suppose, tacitly as usual, that the upper bound $k_0 = k_0(\delta, \Lambda)$ is rechosen to suit the argument; with the final substitution k_0 becomes a normal $k_0(\delta)$.

We begin by recording some results for later convenience. In the range $\mathcal{R}$ we have, writing η_n for $y(N_n) - 1$,

$$\eta=y-1\geq k^{-\frac{2}{7}};\qquad |\delta\eta|=|\delta y|\leq\Lambda\varepsilon\eta^{-1};\qquad \eta+\vartheta\delta\eta\gtrless L\eta\quad\text{for } 0\leq\vartheta\leq 1. \tag{9}$$

[1]

$$\int_{(n)}\frac{dt}{\eta}=O(\log\eta_n)=O(\log k);\qquad \int_{(n)}\frac{dt}{\eta^3}=O(\eta_n^{-2})=O(k^{\frac{4}{7}});\qquad \int_{(n)}\frac{dt}{\eta^4}=O(\eta_n^{-3})=O(k^{\frac{6}{7}}). \tag{10}$$

The first part of (9) is true by definition; the second by definition; the third by the first and second.[2]

We have in (N_n, N_{n+1})

$$F(y)-F(1)=(F(N_n)-F(1))+b(1+p_1)-\varepsilon(\dot{y}-\dot{y}(N_n))+\int_{N_n}^{t} g\,dt,$$

and so

$$\eta^2>L(F(y)-F(1))>L\eta_n^2+L\ \mathrm{Min}\ \{(t-N_n)^2,\ (t-N_{n+1})^2\}-L\varepsilon,$$

from which the results of (10) follow by straightforward calculation.

§ 59. We now operate on (3), (4), (5) with δ, observing that $1/f(y)=O(\eta^{-1})$, $\eta+\delta\eta>L\eta$, and so $1/f(y+\delta y)=O(\eta^{-1})$, $\delta(1/f)=O(\eta^{-2}\delta\eta)$. Operating on (3) we have

$$f(y)\,\delta y+O((\delta y)^2)=p_1\,\delta b+\delta c_n-\varepsilon\int_{N_n}^{t}\delta g\,dt-\varepsilon\,\delta\dot{y}. \tag{11}$$

Operating on (5), we have

$$\delta\dot{y}=O(\delta b\,\eta^{-1})+O(\delta\eta\cdot\eta^{-2})+O(\varepsilon\eta^{-2})=O(\Lambda\varepsilon\eta^{-3}). \tag{12}$$

Also
$$(\delta\eta)^2=O(\Lambda^2\varepsilon^2\eta^{-2}),$$

and
$$\left|\int_{N_n}^{t}\delta g\,dt\right|<L\int_{(n)}|\delta\eta|\,dt<L\Lambda\varepsilon\int_{(n)}\eta^{-1}\,dt<L\Lambda\varepsilon\log k,$$

by (10). Substituting from these and (12) in (11) (and combining the worst elements of the errors[3]), we have

$$\delta y=\delta b\,(p_1/f)+\delta c_n/f+O(\Lambda^2\varepsilon^2\eta^{-4}\log k). \tag{13}$$

Next we have, operating on (4),

$$\delta(kc_{n+1})-\delta(kc_n)=-\int_{(n)}\delta g\,dt=-\int_{(n)}(g'\,\delta y+O(\delta y)^2)\,dt.$$

---

[1] Numbering of formulae is consecutive throughout the proof of Lemma 31.

[2] And (to mention it for once) by a rechoice of $k_0(L,\delta,\Lambda)$.

[3] So as to have a single error-term.

Substituting for the δy's in this from (13) we have

$$\delta(k c_{n+1}) - \delta(k c_n)(1 - \varepsilon \int_{(n)} \varphi(y)\,dt) + \delta b \int_{(n)} X\,dt = O(\Lambda^2 \varepsilon^2 \log k) \int_{(n)} \eta^4\,dt = O(\Lambda^2 \varepsilon k^{-\frac{1}{7}} \log k) \tag{14}$$

by (10), where

$$\varphi(y) = g'/f, \qquad X = p_1 g'/f = p_1 \varphi. \tag{15}$$

Now $\int_{(n)} \varphi\,dt = O(\int_{(n)} \eta^{-1}\,dt) = O(\log k)$, by (10), so that

$$1 - \varepsilon \int_{(n)} \varphi\,dt = \exp(-\varepsilon \int_{(n)} \varphi\,dt) + O(\varepsilon^2 \log^2 k). \tag{16}$$

We have further, by (4) (summing), and (7),

$$\delta(k c_n) = \delta(k c_1) - \int_{N_1}^{N_n} \delta g\,dt = O(1) + O(\Lambda\varepsilon) \int_{N_1}^{N_n} \eta^{-1}\,dt = O(\Lambda \log k), \tag{17}$$

since $n = O(k)$. Substituting from (16) and (17) in (14), and keeping only the worst error, we have

$$\delta(k c_{n+1}) - \delta(k c_n) \exp(-\varepsilon \int_{(n)} \varphi\,dt) = -\delta b \int_{(n)} X\,dt + O(\Lambda^2 \varepsilon k^{-\frac{1}{7}} \log k),$$

or writing

$$\psi = \psi(t) = \varepsilon \int_{N_1}^{t} \varphi\,dt, \; \psi_n = \psi(N_n), \; u_n = e^{\psi_n} \delta(k c_n), \tag{18}$$

$$u_{n+1} - u_n = -\delta b\, e^{\psi_{n+1}} \int_{(n)} X\,dt + O(\Lambda^2 e^{\psi_{n+1}} \varepsilon k^{-\frac{1}{7}} \log k). \tag{19}$$

In (N_n, N_{n+1}) we have $\psi - \psi_n = O(\varepsilon \log k)$, and so

$$e^{\psi_{n+1}} \int_{(n)} X\,dt - \int_{(n)} e^{\psi} X\,dt = e^{\psi_n} \int_{(n)} O(\varepsilon \log k)\, X\,dt$$
$$= O(\varepsilon \log k)\, e^{\psi_n} \int_{(n)} \eta^{-1}\,dt = O(\varepsilon e^{\psi_n} \log^2 k).$$

Hence, since $\delta b = O(\varepsilon)$, (19) becomes

$$u_{n+1} - u_n = -\delta b \int_{(n)} e^{\psi} X\,dt + O(\Lambda^2 \varepsilon e^{\psi_n} k^{-\frac{1}{7}} \log k).$$

Since ψ is increasing, $n=O(k)$, and $u_1=k\delta c_1=-k\delta b+O(Dk^{-\frac{1}{7}})$, this gives

$$u_n=-k\delta b-\delta b\int_{N_1}^{N_n} e^{\psi} X\,dt+O(\Lambda^2 e^{\psi_n} k^{-\frac{1}{7}}\log k),$$

or by (18),
$$k\delta b-\delta(kc_n)=\Theta_n k\delta b+O(\Lambda^2 k^{-\frac{1}{7}}\log k), \tag{20}$$

where
$$\Theta_n=1+e^{-\psi_n}+\varepsilon e^{-\psi_n}\int_{N_1}^{N_n} e^{\psi} X\,dt. \tag{21}$$

In this ψ is positive and increasing, and (since $|p_1|\leq 1$) $\varepsilon|X|\leq\varepsilon g'/f=\dot{\psi}$, so that $\varepsilon e^{-\psi_n}\int_{N_1}^{N_n} e^{\psi} X\,dt$ lies between $\pm(1-e^{-\psi_n})$, and so

$$2e^{-\psi_n}\leq\Theta_n\leq 2. \tag{22}$$

It follows from this and (20) that

$$\delta(kc_n)=O(1). \tag{23}$$

We now have from (13),

$$|\delta y|<L\varepsilon\eta^{-1}+L\varepsilon\eta^{-1}+L\Lambda^2(\varepsilon^2\eta^{-3}\log k)\eta^{-1},$$

in which
$$\varepsilon^2\eta^{-3}\log k\leq\varepsilon k^{-\frac{1}{7}}\log k,$$

so that
$$|\delta y|<(L'+L''\Lambda^2 k^{-\frac{1}{7}}\log k)\varepsilon\eta^{-1},$$

where L' and L'' depend on L_1. If we take $\Lambda=3L'$ in this we have $|\delta y|<\frac{3}{2}L'\varepsilon\eta^{-1}=\frac{1}{2}\Lambda\varepsilon\eta^{-1}$. Then, as explained above, $\mathcal{R}$ extends to N_ν, and we have, from (8) and (20),

$$\delta\mathfrak{V}(N-2\pi)=\Theta_\nu k\delta b+O(k^{-\frac{1}{7}}\log k)+\int_{N_\nu}^{N-2\pi}\delta g\,dt. \tag{24}$$

§ 60. We now consider $R=\int_{N_\nu}^{N-2\pi}\delta g\,dt$ (returning later to the further calculation of Θ_ν). The curve Γ_1 between $N_{\nu-1}$ and $N_{\nu+1}$ has an equation of the form

$$F(y)=C+b(1+p_1)+O(\varepsilon),$$

and (since $1+p_1=0$ at N's) this must also be of the form[1]

[1] We write $F(N_{\nu-1})$ for $F(y(N_{\nu-1}))$, etc.

$$F(y) = F(N_{\nu-1}) + b(1+p_1) + O(\varepsilon),$$

and we must also have $F(N_\nu) = F(N_{\nu-1}) + O(\varepsilon)$. By the definition of N_ν, η takes a value $\leq k^{-\alpha}$ somewhere in $(N_\nu, N_{\nu+1})$. On the other hand, since $N_{\nu-1}$ is in $\mathcal{R}$, we have at $N_{\nu-1}$,

$$\eta \geq k^{-\alpha}, \ |\delta\eta| < L\varepsilon\eta^{-1} < Lk^{\alpha-1}, \ \eta + \theta\delta\eta > Lk^{-\alpha}.$$

Since $F - F(1) \gtrless L\eta^2$, it follows by straightforward calculation that

$$F(N_{\nu-1}) - 1, \ F(N_\nu) - 1 \gtrless Lk^{-2\alpha} = Lk^{-\frac{4}{7}}, \tag{25}$$

and

$$\delta F(N_\nu) = \delta F(N_{\nu-1}) + O(\varepsilon) = O(\eta_{\nu-1}\delta\eta_{\nu-1}) = O(\varepsilon). \tag{26}$$

Next, we have for the time from N_ν to $N - 2\pi$

$$(N - 2\pi) - N_\nu = O(k^{\frac{3}{7}}). \tag{27}$$

For over an intermediate stretch (N_n, N_{n+1}) we have

$$\int_{N_n}^{N_{n+1}} g\,dt > M + o(1) > L.$$

Hence

$$F(N_n) - F(1) = (F(N_\nu) - F(1)) - \varepsilon\left(\dot{y}(N_n) - \dot{y}(N_\nu) + \int_{N_\nu}^{N_n} g\,dt\right)$$
$$< Lk^{-\frac{4}{7}} - \varepsilon(-L + L(n-\nu)),$$

by (25). The left-side being non-negative, we have $n - \nu < L + Lk^{1-\frac{4}{7}}$, which is equivalent to (27).

§ 61. We now employ another "Λ-argument"; this time the final choice of Λ is a D. We consider the range $\mathcal{R}_1$, from N_ν up to $t = \tau$, where τ is the earlier of $N - 2\pi$ and the N of U_2, N_0 say,[1] or else the first moment when $|\delta y| = \Lambda k^{-\frac{1}{2}}$, whichever happens first: we shall find that $\tau = N - 2\pi$.

We have in $\mathcal{R}_1$.

$$\int_{N_\nu}^{t} \delta g\,dt = O(\Lambda k^{-\frac{1}{2}})(t - N_\nu) = O(\Lambda k^{-\frac{1}{14}}), \tag{28}$$

by (27). Also, operating on

[1] This is temporary notation, and N_0 is not between Z_0 and Z_1.

$$F(y)=F(N_\nu)+b(p_1-p_1(N_\nu))-\varepsilon(\dot{y}-\dot{y}(N_\nu))-\varepsilon\int_{N_\nu}^{t} g\,dt,$$

and using (26), (27), and (28), we have in $\mathcal{R}_1$,

$$\delta F(y)=O(\varepsilon)+O(\varepsilon\Lambda k^{-\frac{1}{14}})=O(\varepsilon). \tag{29}$$

From this we shall deduce that, in $\mathcal{R}_1$,

$$|\delta y|=|\delta\eta|<D_1 k^{-\frac{1}{2}}. \tag{30}$$

We have (in $\mathcal{R}_1$) $\eta,\ \eta+\delta\eta>-Dk^{-\frac{1}{2}}$, and we distinguish four cases:

(i) $\eta,\ \eta+\delta\eta\geq 0$;
(ii) $\eta,\ \eta+\delta\eta\leq 0$;
(iii) $\eta\geq 0,\quad \eta+\delta\eta\leq 0$;
(iv) $\eta\leq 0,\quad \eta+\delta\eta\geq 0$.

In (i) $$\delta F=\int_{\eta}^{\eta+\delta\eta} f(1+u)\,du\geq\int_{\eta}^{\eta+\delta\eta} Lu\,du\geq\int_{0}^{\delta\eta} Lu\,du=L(\delta\eta)^2,$$

and (30) follows from (29).

In (ii) $$|\delta\eta|\leq|\eta+\delta\eta|+|\eta|<Dk^{-\frac{1}{2}}.$$

In (iii) we must have $\eta<Dk^{-\frac{1}{2}}$, because if $\eta>D'k^{-\frac{1}{2}}$ we should have

$$|\delta(F)|=\left|\int_{\eta+\delta\eta}^{\eta} f(1+u)\,du\right|\geq\left|\int_{0}^{\eta} f(1+u)\,du\right|-Dk^{-1}\geq L\eta^2-Dk^{-1}>LL'k^{-1},$$

which contradicts (29) for a suitable choice of D'. With $\eta<Dk^{-\frac{1}{2}}$ we have

$$|\delta\eta|\leq|\eta|+|\eta+\delta\eta|<Lk^{-\frac{1}{2}}+Dk^{-\frac{1}{2}}<Dk^{-\frac{1}{2}}.$$

Finally in (iv) we have $|\eta|<D'k^{-\frac{1}{2}}$. Then either $|\delta\eta|<D''k^{-\frac{1}{2}}$ which gives what we want, or else $\eta+\delta\eta>(D''-D')k^{-\frac{1}{2}}$, and then

$$\delta F\geq\int_{-D\,k^{-\frac{1}{2}}}^{(D''-D')k^{-\frac{1}{2}}} f(1+u)\,du>\int_{0}^{(D''-D')k^{-\frac{1}{2}}} L'u\,du-\int_{0}^{D'k^{-\frac{1}{2}}} L''u\,du.$$

D' is fixed in this, and if we choose D'' suitably (large enough), we have $\delta F>LD''k^{-1}$, which contradicts (29) if, again, D'' is large enough. We have now proved (30).

§ 62. We now take $\Lambda = 2D_1$: then the event $|\delta\eta| = \Lambda k^{-\frac{1}{2}}$ is not the first, and $\mathcal{R}_1$ extends to N_0, and we have (30), or $|\delta y| < D_1 k^{-\frac{1}{2}}$, in $\mathcal{R}_1$. We have now

$$\int_{N_\nu}^{N_0} \delta g\, dt = O(D k^{-\frac{1}{2}})(N_0 - N_\nu) = O(k^{\frac{3}{7}} D k^{-\frac{1}{2}}) = O(D k^{-\frac{1}{14}}),$$

and so from (24), trivially modified,

$$\delta \mathfrak{V}(N_0) = \Theta_\nu\, k\,\delta b + O(k^{-A}).$$

Next, we have $N_0 \geq N - 2\pi$. For otherwise we should have $N_0 \leq N - 4\pi$ and U_2 at N_0, and consequently

$$\mathfrak{V}(\Gamma_2, N_0) = \mathfrak{V}(\Gamma_1, N_0) + \Theta_\nu\, k\,\delta b + o(1) < \mathfrak{V}(\Gamma_1, N_0) + o(1),$$

which *since* Θ_ν *is positive and* δb *negative*,

$$\begin{aligned} &\leq V_1 - 2M + o(1) \\ &\leq (V^* + M + o(1)) - 2M + o(1) \\ &< V^* - L, \end{aligned}$$

and this is incompatible with U_2 being at N_0. We have, accordingly, U_2 at $N - 2\pi$ or later, and

$$\delta \mathfrak{V}(N - 2\pi) = \Theta_\nu\, k\,\delta b + O(k^{-A}), \qquad \Theta_\nu = 1 + e^{-\psi_\nu} + \varepsilon e^{-\psi_n} \int_{N_1}^{N_\nu} X e^{\psi}\, dt \tag{31}$$

§ 63. We turn to the evaluation of Θ_ν. Let $\varrho_n = \eta_n = y(N_n) - 1$.[1]

We shall need the result:

$$\varrho_1 = \lambda + O(D\varepsilon), \tag{32}$$

where λ *(an* L*) is the positive root of*

$$F(1+\lambda) = \tfrac{2}{3} - 2b, \tag{33}$$

and it is convenient to begin with this. We saw in § 57 that within a range L of N_1 we have

$$F(y) = F(H) + b\,(p_1(t) - p_1(Z_0)) + O(D\varepsilon).$$

Substituting $t = N_1$, $p_1(t) = -1$, $p_1(Z_0) = 1$, $F(H) = \frac{2}{3}$ in this we have

[1] We presently need [see (36) below] a new variable ϱ, whose particular case ϱ_n has the value η_n.

$$F(1+\varrho_1)=\tfrac{2}{3}-2b+O(D\varepsilon),$$

and (32), (33) follow.

§ 64. The equation of Γ_1 in the range (N_n, N_{n+1}), where $n<\nu$, is

$$F(y)=F(1+\varrho_n)+b(1+p_1)+O(\varepsilon). \tag{34}$$

Also

$$\begin{aligned} F(1+\varrho_{n+1})-F(1+\varrho_n) &= -\varepsilon\int_{(n)} g\,dt-\varepsilon(\dot{y}(N_{n+1})-\dot{y}(N_n)) \\ &= -\varepsilon\int_{(n)} g\,dt+O(k^{-\frac{10}{7}}), \end{aligned} \tag{35}$$

since by (5), with $t=N_n$, $\dot{y}(N_n)=O(\varepsilon\eta_n^{-2})=O(\varepsilon k^{2\alpha})$, and similarly for $\dot{y}(N_{n+1})$. Let the (periodic) *curve*

$$F(y)=F(1+\varrho)+b(1+p_1), \tag{36}$$

for a parameter $\varrho>0$, be $y=Y(t,\varrho)=Y(t,\varrho;b)$. Y has a minimum $1+\varrho$ where $t\equiv-\frac{1}{2}\pi$. Let

$$\left.\begin{aligned} &\Phi(\varrho)=\int_0^{2\pi}\varphi\{Y(\varrho,t)\}\,dt \quad [\varphi(y)=g'(y)/f(y)], \\ &G(\varrho)=\int_0^{2\pi} g\{Y(\varrho,t)\}\,dt, \\ &\Omega(\varrho)=f(1+\varrho)\,\Phi(\varrho)/G(\varrho), \\ &P(\varrho)=\int_0^{2\pi} p_1(t)\,\varphi\{Y(\varrho,t)\,dt, \\ &J(\varrho)=P(\varrho)\,\Phi^{-1}(\varrho)\,\Omega(\varrho)\exp\Big(\int_\varrho^\lambda\Omega(\varrho)\,d\varrho\Big)=P(\varrho)\,f(1+\varrho)\,G^{-1}(\varrho)\exp\Big(\int_\varrho^\lambda\Omega(\varrho)\,d\varrho\Big). \end{aligned}\right\} \tag{37}$$

The calculations that follow are inevitably rather long, and it will perhaps be clearest if we set out first the "formal" work, with errors ignored.

We have, by (18),

$$\psi_{n+1}-\psi_n=\varepsilon\int_{(n)}\varphi(y)\,dt\sim\varepsilon\int_0^{2\pi}\varphi\{Y(\varrho_n,t)\}\,dt=\varepsilon\,\Phi_n,$$

where we abbreviate $\Phi(\varrho_n)$ to Φ_n, and similarly for G, Ω, P, J; we also write f_n, F_n for $f(1+\varrho_n)$, $F(1+\varrho_n)$.[1] By (35)

[1] ψ, the only other function to take a suffix n, is a function of t, with $\psi_n=\psi(N_n)$. Otherwise, apart from f and F, where the use is "obvious", the suffix is attached only to functions of ϱ with capital letter names (Φ, G, Ω, P, J).

$$(\varrho_{n+1}-\varrho_n) f_n \sim F_{n+1}-F_n \sim -\varepsilon \int_{(n)} g(y)\,dt \sim -\varepsilon \int_0^{2\pi} g\{Y(\varrho_n, t)\}\,dt = -\varepsilon G_n. \qquad (38)$$

Hence
$$\psi_{n+1}-\psi_n \sim -(\varrho_{n+1}-\varrho_n)\Phi_n f_n G_n^{-1} = -\Omega_n(\varrho_{n+1}-\varrho_n)$$

$$\psi_n \sim \int_{\varrho_n}^{\varrho_1} \Omega\,d\varrho \sim \int_{\varrho_n}^{\lambda} \Omega\,d\varrho. \qquad (39)$$

Next [using (15)]

$$\int_{(n)} X e^{\psi}\,dt \sim e^{\psi_n} \int_{(n)} X\,dt \sim e^{\psi_n} \int_0^{2\pi} p_1(t)\,\varphi\{Y(\varrho_n, t)\}\,dt = P_n e^{\psi_n},$$

and so, by (38) and (39),

$$\varepsilon \int_{(n)} X e^{\psi}\,dt \sim -P_n \exp\Big(\int_{\varrho_n}^{\lambda} \Omega\,d\varrho\Big) f_n G_n^{-1}(\varrho_{n+1}-\varrho_n) = -J_n(\varrho_{n+1}-\varrho_n).$$

So
$$\varepsilon \int_{N_1}^{N_\nu} X e^{\psi}\,dt = \sum_1^{\nu-1} \varepsilon \int_{(n)} \sim \int_{\varrho_\nu}^{\varrho_1} J\,d\varrho \sim \int_{\varrho_\nu}^{\lambda} J\,d\varrho,$$

and, collecting, we have

$$\left.\begin{aligned} \Theta_\nu &= 1 + e^{-\psi_\nu} + \varepsilon e^{-\psi_\nu} \int_{N_1}^{N_\nu} X e^{\psi}\,dt \sim 1 + E(\varrho_\nu)\Big(1 + \int_{\varrho_\nu}^{\lambda} J\,d\varrho\Big), \\ E(\varrho) &= \exp\Big(-\int_{\varrho}^{\lambda} \Omega\,d\varrho\Big), \quad \lambda \text{ the root of } F(1+\lambda) = \tfrac{2}{3} - b. \end{aligned}\right\} \qquad (40)$$

Of the functions of ϱ, Φ, G, Ω, P, J, the first three are positive for $\varrho > 0$, and near $\varrho = 0$ it is not difficult to show that:

$$\left.\begin{aligned} \Phi = O(\log \varrho), \qquad G^{\pm 1} = O(1), \qquad \Omega = O(f(1+\varrho)\Phi) = O(\varrho \log \varrho), \\ P = O(\log \varrho), \qquad J = O(\varrho \log \varrho), \qquad E^{\pm 1} = O(1). \end{aligned}\right\} \qquad (41)$$

Thus we can finally, with negligible error, replace ϱ_ν (which is $O(k^{-\alpha})$) by 0, and obtain finally $\Theta_\nu \sim \theta(b)$, where

$$\theta(b) = 1 + E\Big(1 + \int_0^{\lambda} J(\varrho)\,d\varrho\Big), \qquad E = \exp\Big(-\int_0^{\lambda} \Omega(\varrho)\,d\varrho\Big), \qquad (42)$$

where $J(\varrho)$ is defined in (37). The integrals are convergent uniformly in b, and θ is continuous: it depends only on the functions f, g, p, and on b.

Finally $|p_1| \le 1$, so that, by (37) $|P| \le \Phi$, and

$$E \int_0^\lambda |J|\, d\varrho \le E \int_0^\lambda \Omega \exp\left(\int_\varrho^\lambda \Omega\, d\varrho\right) d\varrho = E(E^{-1} - 1) = 1 - E.$$

Hence by (42) $2E \le \theta(b) \le 2$, and $L \le \theta(b) \le 2$.

§ 65. We now take up the question of errors.[1] We abbreviate, for the range (N_n, N_{n+1}), $Y(\varrho_n, t)$ to $Y^{(n)}$. We recall that for $n \le \nu$

$$\eta \ge k^{-\frac{2}{7}},\ \varrho_n \ge k^{-\frac{2}{7}}. \tag{43}$$

We have, in (N_n, N_{n+1}), always for $n \le \nu$,

$$F(y) = F(1 + \varrho_n) + b(1 + p_1) + O(\varepsilon),$$

$$F(Y^{(n)}) = F(1 + \varrho_n) + b(1 + p_1),$$

and so

$$(y - Y^{(n)})\, f(y + \vartheta(Y^{(n)} - y)) = O(\varepsilon),$$

from which, after (10), we can deduce

$$y - Y^{(n)} = O(\varepsilon \eta^{-1}), \qquad \int_{(n)} |y - Y^{(n)}|\, dt = O(\varepsilon \log k), \tag{44}$$

and

$$\varphi(y) - \varphi(Y^{(n)}) = (y - Y^{(n)})\, \varphi'(y + \vartheta(Y^{(n)} - y)) = O(\varepsilon \eta^{-1} \cdot \eta^{-2}) = O(\varepsilon \eta^{-3}).$$

By (18)

$$\psi_{n+1} - \psi_n = \varepsilon \int_{(n)} \varphi(y)\, dt = \varepsilon \int_{(n)} \varphi(Y^{(n)})\, dt + O(\varepsilon^2) \int_{(n)} \eta^{-3}\, dt$$

$$= \varepsilon \Phi_n + O(k^{-2+\frac{4}{7}}), \quad \text{by (10)}. \tag{45}$$

Next, by (35),

$$F(1 + \varrho_{n+1}) - F(1 + \varrho_n) = -\varepsilon \int_{(n)} g\, dt + O(k^{-\frac{10}{7}})$$

$$= -\varepsilon \int_{(n)} g(Y^{(n)})\, dt + O(\varepsilon) \int_{(n)} |y - Y^{(n)}|\, dt + O(k^{-\frac{10}{7}}) = -\varepsilon G_n + O(k^{-\frac{10}{7}})$$

by (37), (44), and $g' = O(1)$. The left-hand side is

$$(\varrho_{n+1} - \varrho_n)\, f\{1 + \varrho_n + \vartheta(\varrho_{n+1} - \varrho_n)\},$$

[1] We have tried to reduce this to the minimum needed to produce conviction, and omit some minor details of calculations.

and we find successively that

$$(\varrho_{n+1}-\varrho_n)=O(\varepsilon\varrho_n^{-1})=O(k^{-\frac{5}{7}}),\quad \varrho_{n+1}\lesssim L\varrho_n, \tag{46}$$

$$-\varepsilon G_n+O(k^{-\frac{10}{7}})=(\varrho_{n+1}-\varrho_n)f_n(1+O(\varepsilon\varrho_n^{-2})),$$

$$-\varepsilon=(\varrho_{n+1}-\varrho_n)f_n G_n^{-1}+O(k^{-\frac{10}{7}})+O(\varepsilon\varrho_n^{-1}\cdot\varrho_n\cdot\varepsilon\varrho_n^{-2})=(\varrho_{n+1}-\varrho_n)f_n G_n^{-1}+O(k^{-\frac{10}{7}}), \tag{47}$$

which we shall use for transforming sums into integrals. From (45), (46), (47), (37), and since $\Phi_n=O(\log\varrho_n)=O(\log k)$ by (41),

$$\psi_{n+1}-\psi_n=-(\Phi_n/G_n)(-\varepsilon G_n)+O(k^{-\frac{10}{7}})=-\Omega_n(\varrho_{n+1}-\varrho_n)+O(k^{-\frac{10}{7}}\log k)$$

$$=\int_{\varrho_{n+1}}^{\varrho_n}\Omega\,d\varrho+O\big((\varrho_{n+1}-\varrho_n)^2\operatorname*{Max}_{\varrho_{n+1}\le\varrho\le\varrho_n}|\Omega'|\big)+O(k^{-1-\Delta}). \tag{48}$$

We can repeat the argument for (44) with an arbitrary ϱ of $(\varrho_{n+1},\varrho_n)$ in place of ϱ_n, getting

$$y-Y(\varrho,t)=O(\varepsilon\eta^{-1});$$

and this gives also
$$(Y(\varrho,t)-1)^{-1}=O(\eta^{-1})=O(\varrho^{-1}). \tag{49}$$

Further, since $F(Y)=F(1+\varrho)+b(1+p_1)$, we have

$$f(1+\varrho)=f(Y)\frac{\partial Y}{\partial\varrho}, \tag{50}$$

and so, from (49),
$$\frac{\partial Y}{\partial\varrho}=O(\varrho\eta^{-1})=O(1). \tag{51}$$

(49) and (51) enable us to estimate the ϱ-derivatives of the various functions of ϱ. Thus, using (10), we have in (N_n, N_{n+1})

$$\Phi'(\varrho)=\int_0^{2\pi}\varphi'(Y)\frac{\partial Y}{\partial\varrho}dt=O\left(\int_{(n)}\eta^{-2}dt\right)=O(\varrho^{-1})=O(k^{\frac{2}{7}}), \tag{52}$$

$$P'(\varrho)=O(\varrho^{-1})=O(k^{\frac{2}{7}}), \tag{53}$$

similarly
$$G'(\varrho)=\int_0^{2\pi}g'(Y)\frac{\partial Y}{\partial\varrho}dt=O\left(\varrho\int_{(n)}\eta^{-1}dt\right)=O(\varrho\log\varrho). \tag{54}$$

Since $f'(1+\varrho)=O(1)$, $G^{-1}(\varrho)=O(1)$, we have by (41)

$$\Omega'(\varrho)=\frac{d}{d\varrho}(\Phi f(1+\varrho)G^{-1})=O(\varrho\Phi')+O(\Phi)+O(\Phi\varrho^2\log\varrho),$$

$$\Omega'(\varrho)=O(\log\varrho). \tag{55}$$

From $J=Pf(1+\varrho)G^{-1}E^{-1}$, $E'=O(1)$, (53), (54), and (41),

$$J'(\varrho)=O(P'\varrho)+O(P)+O(P\varrho G')+O(J)=O(\log\varrho)=O(\log k). \tag{56}$$

From (55), (46), (48),

$$\psi_{n+1}-\psi_n=\int_{\varrho_{n+1}}^{\varrho_n}\Omega\,d\varrho+O(\varepsilon^2\varrho_n^{-2}\log\varrho_n)+O(k^{-1-A})=\int_{\varrho_{n+1}}^{\varrho_n}\Omega\,d\varrho+O(k^{-1-A}), \tag{57}$$

$$\psi_n=\int_{\varrho_n}^{\lambda}\Omega\,d\varrho+O(k^{-A}). \tag{58}$$

By (45) and (41) we have in (N_n, N_{n+1})

$$\psi-\psi_n\leq\psi_{n+1}-\psi_n=O(\varepsilon\log k), \tag{59}$$

so that
$$X e^{\psi}=p_1(t)\varphi(y)\exp[\psi_n+O(\varepsilon\log k)],$$

and in this

$$\varphi(y)=\varphi(Y^{(n)})+O(\varepsilon\eta^{-1}\operatorname{Max}|\varphi'|)=\varphi(Y^{(n)})+O(\varepsilon\varrho_n^{-3})=\varphi(Y^{(n)})+O(k^{-A}).$$

This leads [ψ is $O(1)$] to

$$\int_{(n)}X e^{\psi}\,dt=e^{\psi_n}\int_0^{2\pi}p_1\varphi(Y^{(n)})\,dt+O(k^{-A})+O\Big(\varepsilon\log k\int_{(n)}|\varphi|\,dt\Big)=e^{\psi_n}P_n+O(k^{-A}).$$

From this, (47), (58), and $P_n=O(\log k)$,

$$\begin{aligned}\varepsilon\int_{(n)}X e^{\psi}\,dt&=(-(\varrho_{n+1}-\varrho_n)f_n G_n^{-1}+O(k^{-1-A}))(e^{\psi_n}P_n+O(k^{-A}))\\&=-(\varrho_{n+1}-\varrho_n)f_n G_n^{-1}P_n\Big(\exp\Big(\int_{\varrho_n}^{\lambda}\Omega\,d\varrho\Big)+O(k^{-A})\Big)+\\&\qquad\qquad+O(k^{-A})|\varrho_{n+1}-\varrho_n|f_n+O(k^{-1-A})\\&=-J_n(\varrho_{n+1}-\varrho_n)+O(k^{-A}|\varrho_{n+1}-\varrho_n|f_n\log k)+O(k^{-1-A})\\&=-J_n(\varrho_{n+1}-\varrho_n)+O(k^{-1-A}),\end{aligned}$$

by (46),

$$= \int_{\varrho_{n+1}}^{\varrho_n} J\,d\varrho + O\left((\varrho_{n+1}-\varrho_n)^2 \operatorname{Max}|J'|\right) + O(k^{-1-A})$$

$$= \int_{\varrho_{n+1}}^{\varrho_n} J\,d\varrho + O(k^{-1-A}) \quad \text{by (46) and (56).}$$

It now follows from (21) and (58) that

$$\Theta_\nu = 1 + E(\varrho_\nu)\left(1 + \int_{\varrho_\nu}^{\lambda} J\,d\varrho\right) + O(k^{-A}).$$

By (41) and $\varrho_\nu = O(k^{-A})$ we can replace ϱ_ν in this by 0, to the same error. This completes the proof of the formula (1) of Lemma 31.

§ 66. If $-k\,\delta b > \delta$ we have (returning to the actual notation in Lemma 31)

$$\mathfrak{V}'(N' - 2\pi,\, b + \delta b) < \mathfrak{V}'(N' - 2\pi,\, b) - L\delta,$$

and this is less than $V^* - \delta - L\delta$, since otherwise $\mathfrak{V}'(N' - 2\pi, b) > V^* - \delta + L\delta$, and (for b) U' would be at $N' - 2\pi$ (or earlier), contrary to hypothesis. This proves the clause of Lemma 31 about N' not jumping to the left, and its consequence, that the formula for $\delta\mathfrak{V}'$ is valid also at N' [$\mathfrak{V}'(\Gamma_{1,2}, N') = \mathfrak{V}'(\Gamma_{1,2}, N' - 2\pi) + M_{1,2}$ to error $O(k^{-A})$, and $M_1 - M_2 = O(k^{-A})$].

It remains to consider δV^*, δM, $\delta\theta$.

That $\delta V^* = O(k^{-1})$ is immediate from the explicit formula (Lemma 10).

M is defined (see § 18) by $M = \int g(Y)\,dt$, where $F(Y) = F(1) + b(1 + p_1)$, and the integration is over a period, say $M = \int_{-\frac{1}{2}\pi}^{\frac{3}{2}\pi}$. Then $M = M_1 + M_2$, where

$$M_1 = \int_{-\frac{1}{2}\pi}^{-\frac{1}{2}\pi + k^{-\frac{1}{3}}} + \int_{\frac{3}{2}\pi - k^{-\frac{1}{3}}}^{\frac{3}{2}\pi}, \qquad M_2 = \int_{\mathfrak{R}},$$

and

$$\mathfrak{R} \text{ is } (-\tfrac{1}{2}\pi + k^{-\frac{1}{3}},\ \tfrac{3}{2}\pi - k^{\frac{1}{3}}).$$

We have $M_1 = O(k^{-\frac{1}{3}})$ for all b, and $\delta M_1 = O(k^{-\frac{1}{3}})$.

In $\mathfrak{R}$ we have $(Y-1)^2 > L(F(Y) - F(1)) > L\tau^2$, where τ is the distance of t from the nearer of $-\frac{1}{2}\pi$, $\frac{3}{2}\pi$, so that $Y - 1 > L\tau > Lk^{-\frac{1}{3}}$, $f(Y) > Lk^{-\frac{1}{3}}$. We now have, for a range of b including the given one,

$$\left|\frac{\partial M_2}{\partial b}\right|=\left|\int_R g'(Y)\frac{\partial Y}{\partial b}dt\right|<L\int_R\left|\frac{\partial Y}{\partial b}\right|dt=L\int_R\frac{1+p_1}{|f(Y)|}dt<Lk^{\frac{1}{2}}.$$

Then $\delta M_2<Lk^{\frac{1}{2}}\delta b<Lk^{-\frac{1}{2}}$, and so finally $|\delta M|<Lk^{-\frac{1}{2}}$.

We prove finally that $\delta\theta=O(k^{-A})$, in the range $\frac{1}{100}\le b\le\frac{2}{3}=\frac{1}{100}-Lk^{-1}$. This includes B_1 and B_2, and could, of course, be extended to the whole of B if necessary.

It is enough to prove the result for $0\le -k\delta b\le\frac{1}{2}L_1$ (since we can make a fresh start at $\frac{1}{2}L_1$ for $\frac{1}{2}L_1$ to L_1). Let $b_0=b,\ b_1=b+\delta b,\ b_2=b+L_1k^{-1}$; let $\theta_0=\theta(b_0)$, $\vartheta_0'=\vartheta'(N'-2\pi,\ b_0)$, and similarly for suffixes 1, 2. Then we have

$$\vartheta_1'-\vartheta_0'=k(b_1-b_0)\,\theta_0+O(k^{-A}),\ \ \vartheta_2'-\vartheta_0'=k(b_2-b_0)\,\theta_0+O(k^{-A}),$$

and so by subtraction $\vartheta_2'-\vartheta_1'=k(b_2-b_1)\,\theta_0+O(k^{-A})$. On the other hand $\vartheta_2'-\vartheta_1'=k(b_2-b_1)\,\theta_1+O(k^{-A})$. Hence $k(b_2-b_1)(\theta_1-\theta_0)=O(k^{-A})$ and finally $\theta_1-\theta_0=O(k^{-A})$ since $k(b_2-b_1)\ge\frac{1}{2}L_1$.

§ 67. We come now to the study of the non-stable motions when $b\in B_2$.[1] The lay-out of §§ 34–56 was designed for the long proof of "convergence", and what we now require sometimes calls for minor variants that would have unduly complicated the account. Where the situation and arguments are reasonably familiar we will abbreviate.

In what remains of the paper we use U for any arrival at $y=1$ from an S^*, gap-free before that point, and similarly for U'. V and V' have the usual associated meanings. If necessary we call the U with $V\ge V^*-\delta$ the "true U".[2]

The locus $y=Y(\varphi)$, or

$$F(-1)-F(Y)=\tfrac{2}{3}-F(Y)=b\,(1-p_1(\varphi)), \tag{1}$$

consists of three disjunct periodic curves $C_{1,2,3}$ as in fig. 11. (Compare fig. 2 of the Introduction.) C_1 and C_2 are each at least distance L from $y=1$.

LEMMA 34. *Suppose that a Γ of an $S^{*\prime}$ has a U' at $y=-1$, near N', or Z_0 of fig. 4, and with $V'\ge V^*-2\delta^{\frac{1}{2}}$, and let $Z_1, Z_2, \ldots$ be (as usual, for given Z_0) 2π, 4π, ... later. Then Γ emerges from $|y|<1$ at latest near Z_1; it crosses $y=1$ with $\dot{y}>Lk$, and then shoots up to near C_1 and begins a long descent.*[3]

---

[1] We suppose from now to the end that $b\in B_2$.

[2] The new U's generally satisfy inequalities like $V\ge V^*-2\delta^{\frac{1}{2}}$, and are "nearly" true U's; or, again, they are true U's for enlarged gaps. It is obvious that we *could* meet all requirements by rechoosing δ to be e.g. the cube of the original one, altering the dependent D's etc. accordingly.

[3] (i) N' and Z_0 are the same time abscissa, under different aspects. (ii) The extremes of behaviour (subject to $\vartheta'\ge V^*-2\delta^{\frac{1}{2}}$) are: on the one hand to enter S^* (Z_0); on the other to approach

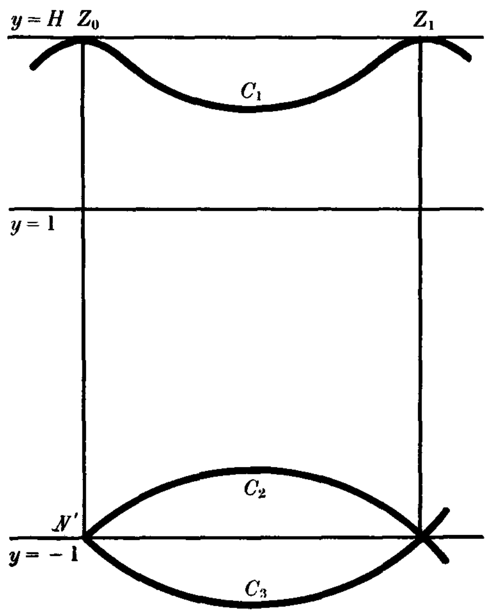

Fig. 11.

COROLLARY. *At* Z_2 *and* Z_3 *we have*

$$|y(Z_{2,3}) - H| < L_3^* k^{-1}, \quad |\dot{y}(Z_{2,3})| < L_3^* k^{-1},$$

when $L_3^* = \text{Max}\,(a, \{(4V^* + 14\pi m)/f(H)\})$, $m = \text{Max}\,|g|$ *for an eventual* Γ, *and* a *is the* A_1 *of Lemma* 3(5) *with* $d =$ *minimum of* $y-1$ *for* C_1. [*Thus* Γ *enters* $S^*(Z_2)$ *and* $S^*(Z_3)$].

We write $\qquad \eta = y + 1, \qquad \Psi(\eta) = F(-1) - F(y) = \frac{2}{3} - F(y).$

Then

$$\dot{y} = V' + k\{(\tfrac{2}{3} - F(y)) - b(1 - p_1)\} + \int_{U'}^{t} (-g)\,dt. \tag{2}$$

This shows that between C_1 and C_2, where the curly bracket is positive, we have $\dot{y}$ positive and of order k except near C_1 and C_2, so that in particular the last clause of the Lemma follows from the "$\dot{y} > Lk$" one.

Let l be a small L. We have two cases: (i) Γ gets above $C_2 + l$ [C_2 translated upwards by l] at t_0 before Z_0; (ii) Γ does not. In (i) we have $\Psi > L$ and $\dot{y} > Lk$

C_1 *near* Z_1 (as the Lemma asserts), but without entering $S^*(Z_1)$; then $S^*(Z_2)$ is the first S^* entered, and Z_3 the first Z *later* than the place of the S^*. This should explain the rôle played in what follows by Z_3. We are approaching a study of the "delta", and spreading out is natural.

from t_0 and until $y=1$ or until Γ returns to C_2+l, which last is clearly impossible. This disposes of case (i).

Case (ii). Γ remains below C_2+l up to Z_1. We will show that

$$-\int_{N'}^{Z_1} g(Y)\,d\varphi > L. \tag{3}$$

It will then follow that if L is chosen small enough,

$$\int_{U'}^{Z_1} (-g)\,dt > L. \tag{4}$$

For (3) we observe that, since $-F$ and g are odd and increasing functions of y in $-1\leq y\leq 1$, we have $-g(Y)=h(F(Y))$, where h is odd and increasing. By (1)

$$-g(Y)=h(c+b\,p_1(\varphi)),\ c=\tfrac{2}{3}-b>0.$$

Since $p_1(\varphi+\pi)=-p_1(\varphi)$, we have

$$-\int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi+2\pi} g(Y)\,d\varphi=\int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi+\pi}\{h(c+b\,p_1(\varphi))+h(c-b\,p_1(\varphi))\}\,d\varphi.$$

The large bracket is positive since $c>0$ and h is increasing, and the value of the integral, depending as it does only on b and the fixed functions, is $>L$, as desired.

We now have (4). Take $t=0$ at Z_1, and consider the range $|t|\leq k^{-\frac{2}{5}}$. In this we have

$$b\,k\,(1-p_1)=\tfrac{1}{2}b\,a_2\,k\,t^2+O(k^{-A}). \tag{5}$$

Hence from (2), (4), and $V'\geq V^*-2\,\delta^{\frac{1}{2}}$, we have

$$\dot{\eta}>V^*+L+k\,(\Psi(\eta)-\tfrac{1}{2}b\,a_2\,t^2). \tag{6}$$

We now distinguish the cases

$$(\alpha)\quad \eta_0=\eta(Z_1)\geq 0,$$
$$(\beta)\quad \eta_0<0.$$

Case (α). By Lemma 4 $[R'_{1,2}]$ we have $\eta\geq u$, where[1]

$$\dot{u}=V^*+L+k\,(\Psi(u)-\tfrac{1}{2}b\,a_2\,t^2),\ u(0)=0;$$

---

[1] With a new L slightly smaller than the old.

and since Ψ is increasing in $-1 \le y \le 1$, we have further $\dot{\eta} > \dot{u}$ so long as $y < 1$. The transformation $u = c k^{-\frac{1}{4}} z$, $t = \gamma k^{-\frac{1}{2}} x$ of § 13 gives

$$\frac{dz}{dx} = 1 + L + k\gamma c^{-1} \Psi(c k^{-\frac{1}{4}} z) - x^2, \; z_0 = 0,$$

and in this $k\gamma c^{-1} \Psi = z^2 + O(k^{-\frac{1}{4}} z^3)$. Hence, up to $t = k^{-\frac{2}{5}}$ or $z = k^{\frac{1}{8}}$, whichever happens first (and this makes $y < 1$) we have

$$\frac{dz}{dx} > 1 + L + z^2 - x^2.$$

By Lemma 5 we have $z = k^{\frac{1}{8}}$ for some $x < L$, or $t < L k^{-\frac{1}{2}} \left(< k^{-\frac{2}{5}}\right)$, and so also $\eta > \dot{u} = L(dz/dx) > L k^{\frac{1}{4}}$ at a time $O\left(k^{-\frac{1}{2}}\right)$ at most after Z_1. As in Lemma 8 this is followed by an increase of $\dot{y}$ to Lk in a short time, so that case (α) is disposed of.

Case (β). We consider the r.m. from Z_1 (as $t = 0$) to $t = k^{-\frac{2}{5}}$. With $\eta = -\zeta$ the r.m. satisfies,

$$\dot{\zeta} > V^* + L + k\Psi(-\zeta) - \tfrac{1}{2} b\, a_2 k t^2, \qquad \zeta_0 = -\eta_0 \ge 0.$$

By Lemma 4 $\zeta > v$, and, since $\Psi(-\zeta)$ is increasing for positive increasing ζ, also $\dot{\zeta} > \dot{v}$, where

$$\dot{v} = V^* + L + k\Psi(-v) - \tfrac{1}{2} b\, a_2 k t^2, \quad v_0 = 0.$$

Transforming and using Lemma 5 as before, we have $\dot{v} > L k^{\frac{1}{4}}$ at a time $O(k^{-\frac{1}{2}})$, and this involves a crude rush to a large y in a short time. Thus (ii) (β) is impossible, and the proof of the Lemma is completed.

Taking now the Corollary we have $y > 1 + d$ in $(Z_2 - 1, Z_3)$, where d is $\frac{1}{2}$ Min $(y - 1)$ for C_1. By Lemma 3 (5)

$$|\dot{y}(Z_{2,3})| \le a k^{-1} \le L_3^* k^{-1},$$

where a is the $A_1(d)$ concerned.

Next, by (2), we have, since $F(-1) = \frac{2}{3} = F(H)$,

$$F(y(Z_{2,3})) = F(H) + k^{-1}(V' - \dot{y}(Z_{2,3})) - k^{-1} \int_{U_1}^{Z_{2,3}} g\, dt,$$

$$|F(y(Z_{2,3})) - F(H)| < L_1 k^{-1}, \qquad L_1 = 2V^* + 7\pi m, \tag{7}$$

where m is max $|g|$ for an eventual trajectory. Since $y(Z_{2,3})$ is near H, we have

$$|F(y(Z_{2,3}))-F(H)|=|(y(Z_{2,3})-H)f\{H+\vartheta(y(Z_{2,3})-H)\}$$
$$\geq|y(Z_{2,3})-H|\tfrac{1}{2}f(H),$$

and the first inequality of the Corollary follows from this and (7).

§ 68. LEMMA 35. (i) *Two* Γ *of* $S^*(Z_0)$ *with* U*'s together, and with the same* V, *where* $V\geq V^*-2\delta^{\frac{1}{2}}$, *differ by* $O(\zeta)$ *at* Z_1. (ii) *If* Γ_1 *of* $S^*(Z_0)$ *has a* U_1 *at* N $V_1\geq V^*-2\delta^{\frac{1}{2}}$, *and if* Γ_2 *starts at* Z_0 *with the same* y_0, *and* $|\dot{y}_0(\Gamma_2)-\dot{y}_0(\Gamma_1)|\leq 2L_3^*k^{-1}$, *then* Γ_2 *has a* U_2 *at* N *with* $V_2-V_1=o(1)$.

(i) Normalizing to $w(Z_1)>0$, suppose that $w(Z_1)>\zeta_1^*$, the number defined in Lemma 24. There is no intersection to U_1, and $-\Delta\omega>0$, and by Lemma 24

$$c(U_1)>Lkw(U_1). \tag{1}$$

We have also (2) and (3) of § 46, giving

$$c(U_1)=-\Delta V+O(\Delta\omega), \tag{2}$$

$$w(U_1)>L|\Delta\omega|.^1 \tag{3}$$

Since $\Delta V=0$ these three are incompatible, and we infer that $w(Z_1)\leq\zeta_1^*$.

(ii) With the w, T, c notation and origin at Z_0 we have $c_0=\dot{w}_0$ and $|\dot{w}_0|\leq 2L_3^*k^{-1}$. We have

$$w=e^{-T}\int_0^t(\dot{w}_0-w_1)e^T\,dt.$$

Until $|w|=|\dot{w}_0|$, or $t=Z_1+1$, whichever happens first, we have

$$|w_1|\leq L\int_0^{2\pi+1}|\dot{w}_0|\,dt<L|\dot{w}_0|,$$

and
$$|w|<L|\dot{w}_0|e^{-T}\int_0^t e^T\,dt.$$

Since $y_{1,2}>1+L$, and so $\dot{T}>Lk$, this gives $|w|<Lk^{-1}|\dot{w}_0|<|\dot{w}_0|$. So $t=Z_1+1$ happens first, and $|w|<Lk^{-1}|\dot{w}_0|<Lk^{-2}$ up to Z_1+1, and in particular at Z_1. In what follows we may suppose $|w(Z_1)|>\zeta_1^*$ and ζ_2 of Lemma 22; in the opposite case easier arguments lead to a final $V_2-V_1=O(\zeta)$. Then by Lemma 22 (*d*) we have, up to U_-, the earlier of the true U's of $\Gamma_{1,2}$

[1] These results, stated for true U's, are true for any U's with $v>L$ (or $V>V_0+L$). The proofs need no real change, but it should be noted that the result "$|\dot{y}|>L$ near U" that is called on is specifically established for a U with $v>L$ in Lemma 11 (5).

$$|w| < L k^{\frac{1}{2}} w(Z_1) < L k^{-\frac{3}{2}}, \tag{4}$$

in particular

$$|w(U_-)| < L k^{-\frac{3}{2}}.$$

And by Lemma 24

$$c(U_-) < L k w(Z_1) < L k^{-1}. \tag{5}$$

There is evidently a good deal to spare, and it is not difficult to deduce (by arguments we have used before) that, whether Γ_2 is above or below Γ_1 near U_1, Γ_2 has a U_2 within Lk^{-1} of U_1, and then (see § 46) that

$$\Delta V = -c(U_1) + O(\Delta\omega) = -c(U_1) + O(w(U_1)) = o(1)$$

as desired.

§ 69. Consider a continuous stream of Γ from an $S^{*\prime}$ arriving at N', or Z_0, on $y=-1$. We shall say that a Γ of the stream "goes through G'" if it has a U' with $V^* - 2\delta^{\frac{1}{2}} \le V' \le V^* + 2\delta^{\frac{1}{2}}$, and that $\Gamma_{1,2}$ go through the $+$ and $-$ ends of G', if $V_1' = V^* + 2\delta^{\frac{1}{2}}$, $V_2' = V^* - 2\delta^{\frac{1}{2}}$. Similarly for a G. For G_* or G'_* we have similar definitions, but with a gap $V_* \pm 2\delta^{\frac{1}{2}}$.

We now recall Lemma 33 and its addenda (a), (b). These (in inverted form) lead to the following consequences.

(i) $\Gamma_{1,2}$, *and more generally*, Γ's *of the stream with* V' *between* $V^* + \delta^{\frac{1}{2}}$ *and* $V^* + 2\delta^{\frac{1}{2}}$, *or between* $V^* - 2\delta^{\frac{1}{2}}$ *and* $V^* - \delta^{\frac{1}{2}}$, *all miss all subsequent gaps.*

(ii) For a G' Γ_1 behaves as follows. It shoots through to a $\mathfrak{Z}$ near N', Z_0, makes a long descent, and has its true U_1 at an N with $N - N' = 2(2n-1)\pi$, and $\mathfrak{V}_1(N)$ lies between $V^* + L\delta^{\frac{2}{5}}$ and $V_* - L\delta^{\frac{1}{3}}$.

By (2) of § 67 with $t = Z_3$ (as $p_1 = 1$, $\dot{y} = o(1)$) we have

$$k(F(Z_3) - F(H)) = V^* + 2\delta^{\frac{1}{2}} - 3C + o(1),$$

where $C = \int_{C_1} g(Y)\,d\varphi$ (the integral being taken over a period of the curve C_1). Hence Γ_1 crosses the ordinate at Z_3 at a point, Q_1 say, with ordinate given by

$$y(\Gamma_1, Z_3) - H = k^{-1}(V^* + 2\delta^{\frac{1}{2}} - 3C)/f(H) + o(k^{-1}). \tag{1}$$

(iii) Γ_2 behaves as follows. It makes a dip, shoots through near Z_1, and enters $S^*(Z_2)$, by Lemma 34 Corollary. By Lemma 33, addendum (b), it has its true U_2 at $N + 4\pi$, with $V_* + L\delta^{\frac{1}{3}} < \mathfrak{V}(N + 4\pi) < V^* + M - L\delta^{\frac{2}{5}}$, and consequently has $V_* - 2M + L\delta^{\frac{1}{3}} < \mathfrak{V}_2(N) < V^* - M - L\delta^{\frac{2}{5}}$.

The calculation for Q_2, the point of Γ_2 at Z_3, gives

$$y(\Gamma_2, Z_3)-H=k^{-1}(V^*-2\delta^{\frac{1}{2}}+M-2C)/f(H)+o(k^{-1}), \tag{2}$$

and incidentally Γ_2 is above Γ_1 from Z_3 to U_2 (since $w(Z_3)>\zeta_1^*$, and there is no intersection).

The relations of the $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are appropriate to G_1, G_* and G_2 being in the delta described in § 29 of the Introduction, and at the right places, of fig. 6 of the Introduction. There is now, by continuity, a sub-stream of the original one through G', a "tube" of Γ's cutting the Z_3 ordinate between Q_1 and Q_2, inclusive, with one (at least) through each point of the segment.[1] No Γ starting inside the tube at Z_3 can cross Γ_1 by more than $O(\zeta)$; consequently *all* Γ *of the tube have their true* U *not earlier than* N. The value of $\mathfrak{V}(N)$ for such a Γ tells us, in the light of Lemma 28, just where the true U is.[2] We observe further that for continuously varying Γ V varies continuously so long as $v=0$ is not involved, and this does not arise in what follows (all V concerned being $\geq V^*-2\delta^{\frac{1}{2}}>V_0+L$).

The final upshot is as follows. There are segments g_1, g_2, g_* (disjunct) of Q_1Q_2 The "sub-tube" through g_1 arrives, after a long descent, at N and "goes through the G_1 there". Similarly for g_2 and $g_*: G_2$ and G_* are at $N+2\pi$. The streams "through" G_1, G_2 are similar in all respects (except for inversion) to the original one "through" G', and the process repeats.[3] The stream "through" G_* consists of "normal" Γ (not near a V-gap); it arrives, after a long ascent, at a new N', $(2n-1)\pi$ beyond $N+2\pi$. If ΔV, $\Delta V'$ represent differences for the extremes of the G_* stream we have [Lemma 30 (e)], $|\Delta V'|>L\Delta V>L\delta^{\frac{2}{5}}$. Since this is large compared with $2\delta^{\frac{1}{2}}$ the G_* stream "surrounds" the G' at N', and there is a sub-stream "through" the G', and the process repeats from there.

We have now, as described in § 30 of the Introduction, a Γ which passes "through" all the G and G' [the G_*, G'_* are intermediaries] of any possible "structures" built from the triple alternatives that occur at successive G or G'. Given such a structure there is, in the first instance, after our discussion above, a stream through the G, G' of any finite piece $-X\leq t\leq X$ of it. The maximum set $S(X)$ of r.p. corresponding to Γ with this property, is bounded, and closed, and shrinks as $X\to\infty$; it must

[1] We "begin" at the "last" (from Γ_1) through Q_1 and "end" at the "first" through Q_2.

[2] For example, if $\mathfrak{V}(N)$ lies between $V^*-M+2\delta$ and $V^*-2\delta$ the true U is at $N+2\pi$, where $V-\mathfrak{V}(N+2\pi)=o(1)$.

[3] The new (e.g.) g_1' on Z_3' will not be an *exact* inversion of g_1.

possess at least one limit-point, which corresponds to a Γ through the structure from $-\infty$ to ∞.

§ 70. It remains finally to show that where the structure is periodic, there is a strictly periodic Γ with the structure. This calls for a topological argument.

We may start the "period" at any G or G'. Suppose, to fix ideas, that G' is followed by G_1. We take the Z_3 associated with the initial G' (6π beyond it); and we take as representative point coordinates of a Γ $\xi = y(Z_3) - H$, $\eta = \dot{y}(Z_3)$.

Consider first a continuous stream[1] from an S' through the first G', and the g_1 at Z_3 explained in § 69. There are Γ's of the stream through the $+$ and $-$ ends of G_1 (and going within ζ of the $+$ and $-$ ends of g_1, and so of G_1), with r.p.'s P_+, P_-, say. Take now the rectangle $\mathfrak{R}$ (see fig. 12) in the (ξ, η) plane bounded by $\xi = \xi(P_\pm) = \xi_\pm$, and $\eta = \pm L_3 k^{-1}$. By (6) of the proof of Lemma 30 ["$w(Z_1) > Lk^{-1}|\Delta V|$"] the Γ of the stream with $V = V^*$ has a $\xi = \xi_0$ such that $(\xi_0 - \xi_+)/(\xi_- - \xi_0)$ lies between L's (it is actually nearly 1), and the ξ of a Γ of the stream for which $V > V^* + \frac{3}{2}\delta^{\frac{1}{2}}$ is at least distant $L(\xi_0 - \xi_+)$ from ξ_-. By Lemma 35 (ii), the Γ *belonging to* $\mathfrak{R}$ on the ordinate through such a ξ have V's differing from the one of the stream by $o(1)$. These Γ's have $V > V^* + \delta^{\frac{1}{2}}$ and miss all subsequent G, G'. Hence we have a vertical dotted line at least $L_1(XY)$ to the right of XZ, to the left of which the Γ's all miss all G, G'; and there is a similar region on the right.

We now abandon the stream we started with, and consider the open set Σ of all points of $\mathfrak{R}$ representing Γ that go strictly through G_1, the G and G' of the period of the structure, and finally through the G_1 at $G_1 + p$, where $p = 2m\pi$ is the period. By what we have just said, it lies between the dotted lines. Since a Γ through an end of a G, G' misses the later ones, the boundary of Σ consists of

(i) a set of intervals, like HR, on XY and ZW, taken open;

(ii) a closed set B_+ of r.p. of Γ that go through the $+$ end of $G_1 + p$, having gone strictly through earlier G, G'; and

(iii) a corresponding set B_-.[2]

The set of Γ corresponding to any path in $\mathfrak{R}$ from P_+ to P_- must have a member through the $+$ end, and one through the $-$ end, of $G_1 + p$. There must therefore be at least one continuum, like HKJ, extending from XY to ZW, consisting of B_+ points, and a similar one for B_-; and there must be a consecutive

[1] This is used for some construction, and later discarded.

[2] B_+ and B_- are not necessarily composed of continuous curves, but this does not affect our argument.

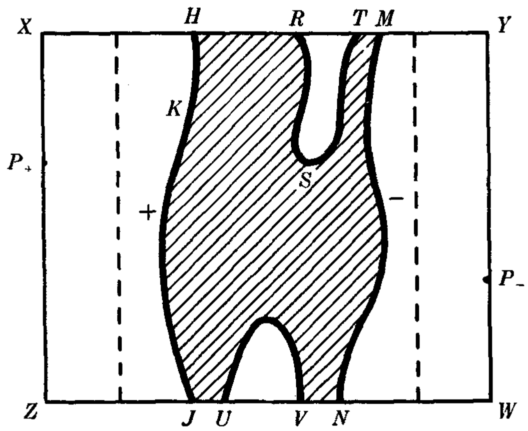

Fig. 12.

pair, with opposite signs, as HKJ, MN, part boundary of a domain Δ contained in $\mathfrak{R}$; Δ is shaded in the figure. The boundary of Δ may further have pieces which are continua of B_+ or B_- points, like RST, UV. We show next that Δ is simply-connected. Suppose that this is not so. Then there is a simple closed curve C in Δ containing points of the frontier of Δ in its interior as well as its exterior domain. Let E be a component of the frontier of Δ in the interior domain of C. It consists entirely of B_+ points or entirely of B_- points, B_+ points, say. Then there is a closed connected set $E + I(E)$ of points not belonging to Δ consisting of E and possibly one or more domains whose frontiers belong to E. It will be sufficient to prove that there is a continuum E_1 of B_+ points such that E_1 meets E and is *not* contained in $E + I(E)$. Points of E_1 and E belong to the frontier of Δ and are connected in the frontier and so E is not a component of the frontier and we have a contradiction.

Now the Γ corresponding to a B_+ point of $E + I(E)$ goes strictly through the + end of $G_1 + p$, i.e. its V at $G_1 + p$ is $V^* + 2\delta$, having gone strictly through the earlier G, G'. The values of t at $G_1 + p$ for all the B_+ points of $E + I(E)$ have a least upper bound τ. If τ is small enough and $\tau \leq t \leq \tau + \tau$ there will be a Γ with r.p. inside C going strictly through the earlier gaps and arriving at $G_1 + p$ at time t with $V = V^* + 2\delta$ (reverse the Γ with the appropriate t, $\dot{y}$ derived from t, V). The r.p. is a B_+ point, and by (iii) is exterior to $E + I(E)$; the range $(\tau, \tau + \tau)$ provides an enlargement of $E + I(E)$ by a continuum E_1 containing exterior B_+ points, as desired. Γ is accordingly simple-connected.

The transformation T from P, (ξ, η), to P', (ξ', η'), where $\xi' = y(Z_3 + p) - H$, $\eta' = \dot{y}(Z_3 + p)$, is topological in $\bar{\Delta}$. We shall show that there is a fixed point of T in Δ, which then corresponds to the desired periodic Γ. Suppose there is no fixed point of T in Δ. Then a continuous vector, or arrow, $P \to P'$, exists for all points P of $\bar{\Delta}$. Now the disposition of the arrows at boundary points of Δ is a follows. If P is a B_+ point, TP (considered as a point of $\mathfrak{R}$ at Z_3) *corresponds to a* Γ' *through the* $+$ *end of* (*the first*) G_1; *further since* Γ *is in* $S^*(Z_2 + p)$ [Lemma 34], *it has arrived at* G_1 *from an* S^*. *By Lemma* 35 (i) *its r.p. is distant* $O(\zeta)$ *from* P_+. The arrow from such a P points nearly at P_+. Similarly for B_- points. A boundary point on XY corresponds to a Γ through all the G, G'; hence its $|\dot{y}(Z_3 + p)| < L_3^* k^{-1} = \eta_0$, by Lemma 34. TP has accordingly $|\eta| < \eta_0$, and the arrow from such a P has a downward component. Similarly one from a boundary point on ZW has an upward one. It follows from these facts, and the continuity of the arrow in $\bar{\Delta}$, *alone*, that when P describes a simple closed contour whose maximum distance from the boundary of Δ is small, the arrow rotates either through $+2\pi$ or -2π (which it is depends on the disposition of the signs on the two continua joining XY, ZW, and the sense of description). This is incompatible with there being no fixed point in Δ.

ERRATA

CORRECTIONS TO THE PAPER: "*On non-linear differential equations of the second order. III. The equation* $\ddot{y} - k(1 - y^2)\dot{y} + y = b\mu k \cos(\mu t + \alpha)$ *for large* k, *and its generalizations*" BY J. E. LITTLEWOOD:

Page 277, line 11 *Read* $O(A(d)k^{-1})$ *for* $O(A(d, d')k^{-1})$

286, line 16 *should read*

$$V' + V = -\left(\tfrac{4}{3} - 2b\right)k - \int_U^{U'} y\,dt, \qquad (1)$$

299, Fig. 5 $(V^* + M)'$ *should be higher.*