EXTREMAL LENGTH AND FUNCTIONAL COMPLETION

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Introduction

The present study arose from an attempt to characterize structurally the completion of certain classes of functions connected with vector analysis and partial differential equations. As examples may be mentioned the class of irrotational vector fields or of solenoidal vector fields, the class of Beppo Levi functions (characterized by a finite Dirichlet integral), or the graph of a system of linear first order partial differential operators with constant coefficients. The completion refers to an L^p-metric, $p \geq 1$, and takes place within a given region X in Euclidean n-dim. space R^n. Restricting the attention to sufficiently differentiable functions or vector fields, one may characterize the classes in question by certain classical relations involving integration over arbitrary smooth curves or surfaces. (Thus the irrotational vector fields in R^3 are characterized by the vanishing of the circulation along closed curves homolog zero, the solenoidal fields by the vanishing of the flow through closed surfaces homolog zero, etc.) These restricted classes are, however, incomplete in the L^p-metric, and the question arises how to describe the structure of the functions (or vector fields) in the completed classes. It is known that these completions may be viewed as "weak extensions" (cf., e.g., Friedrichs [12, 13], Weyl [36], Hörmander [20]). Thus a vector field $f \in L^p(X)$ is irrotational (in the generalized sense) if, and only if, $\int_X (f \cdot \operatorname{rot} v)\, dx = 0$ for every smooth field v which vanishes outside some compact subset of the given region X. The same idea is fundamental in the theory of distributions due to Schwartz [34].

In order to obtain a new insight as to the structure of such completions, one may return to the integral relation valid for sufficiently smooth functions or fields from the class in question. When passing to more general functions or fields, such as those from an L^p-class, one may consider the extended class which arises when one requires that the integral relation shall remain valid for *all* curves or surfaces in question. This idea has been used by several authors, e.g., Bôcher [5], Evans [11]. It turns out, however, that such extensions are usually incomplete just like the original classes. The problem arises, therefore, *to which extent* the integral relation subsists within the L^p-completion of the class in question. The answer may be expressed in terms of a concept which will be called an *exceptional system* of curves or surfaces. This concept is independent of the particular class of functions or vector fields; it depends solely on the exponent p of the L^p-class. *A system* E *of curves (or surfaces) in* R^n *is called exceptional of order* p *if there exists a Baire function* $f \in L^p(R^n)$, $f \geq 0$, *such that the line integral (or surface integral) of* f *over every curve (or surface) from* E *equals* $+\infty$.(1) In terms of this concept, the completion within L^p is characterized, in each of the above cases, by the validity of the appropriate integral relation for "*almost every*" curve (or surface), i.e., for every curve (or surface) which does not belong to some exceptional system of order p. (Cf. Theorem 10, Chapter III, for the case of irrotational vector fields.)

This notion of exceptional systems of curves or surfaces in R^n is directly connected with the concept of *extremal length* introduced by A. Beurling in the early thirties, though not published until 1951 in the article by Ahlfors and Beurling [1] containing

(1) Cf. the well-known fact that a point-set $E \subset R^n$ is of Lebesgue measure 0 if, and only if, there exists a Baire function $f \in L^p(R^n)$, $f \geq 0$, such that $f(x) = +\infty$ for every $x \in E$.

applications to the theory of analytic functions of a complex variable. Later contributions to the theory of extremal length were made by Jenkins [21], Hersch [18, 19], and others. If $\mathbf{E}$ denotes a system of plane curves C with line elements ds, the extremal length of $\mathbf{E}$ was defined in [1] by the following expression

$$\lambda(\mathbf{E}) = \sup_{\varrho} L_\varrho(\mathbf{E})^2/A_\varrho,$$

where
$$L_\varrho(\mathbf{E}) = \inf_{C \in \mathbf{E}} \int_C \varrho\, ds; \qquad A_\varrho = \int_{R^2} \varrho^2\, dx \quad (dx = dx_1\, dx_2),$$

and the weight function $\varrho = \varrho(x) = \varrho(x_1, x_2)$ ranges over all non-negative Baire(1) functions such that $L_\varrho(\mathbf{E})$ and A_ϱ are not simultaneously 0 or $+\infty$.

For our purpose it is preferable to operate with the *module* $\mathsf{M} = 1/\lambda$ rather than the extremal length λ itself. Moreover, we shall replace the exponent 2 by an arbitrary exponent p, $1 \le p < \infty$, and pass from plane curves to k-dimensional surfaces in R^n, $1 \le k \le n-1$. This generalized module may perhaps be called the module of order p. If $\mathbf{E}$ now denotes an arbitrary system of k-dim. surfaces in R^n, its module $\mathsf{M}(\mathbf{E}) = 1/\lambda_p(\mathbf{E})$ of order p may be defined equally well by

$$\mathsf{M}_p(\mathbf{E}) = \inf_{f \wedge \mathbf{E}} \int f^p\, dx \quad (dx = dx_1 \ldots dx_n),$$

where the symbol $f \wedge \mathbf{E}$ (f is associated with the system $\mathbf{E}$) means that f is a non-negative Baire function, defined in R^n, such that

(1) It was not required in [1] that ϱ should be a Baire function; it was merely assumed that the integrals $\int_{R^2} \varrho^2\, dx$ and $\int_C \varrho\, ds$ should be defined, the latter for every $C \in \mathbf{E}$. The present restriction to Baire functions causes, however, no change in the value of the extremal length λ because the Lebesgue measurable weight-function $\varrho \ge 0$ may be replaced by a Baire function $\bar{\varrho} \ge \varrho$ which equals ϱ almost everywhere. (We might even restrict the attention to *lower semi-continuous* functions since there corresponds to any function $\varrho \ge 0$, $\varrho \in L^p(R^n)$, and any $\varepsilon > 0$ a lower semi-continuous function $\varrho_\varepsilon \ge \varrho$ such that $\int_{R^n} \varrho_\varepsilon^p\, dx < \int_{R^n} \varrho^p\, dx + \varepsilon$.)

On the other hand, we may equally well admit quite general functions $\varrho \ge 0$ provided $\int_{R^2} \varrho^2\, dx$ is replaced by the corresponding *upper Lebesgue integral* (and $\int_C \varrho\, ds$ by the corresponding upper or lower Lebesgue integral, or even the lower Darboux integral). Hersch [18] uses, however, the *upper Darboux integral* $\overline{\int_{R^2}} \varrho^2\, dx$ (and the lower Darboux integral $\underline{\int_C} \varrho\, ds$), but this leads to an extremal length λ^* which in general is smaller than the above extremal length λ. The module $\mathsf{M}^* = 1/\lambda^*$ in the sense of Hersch is related to the above module $\mathsf{M} = 1/\lambda$ like the exterior Jordan measure to the exterior Lebesgue measure. In particular, M^* is not countably sub-additive (cf. Theorem 1 (b), § 1).

$$\int_S f\, d\sigma \geq 1 \quad \text{for every surface } S \in \mathbf{E}.$$

(A similar definition for the case $0 < p < 1$ would lead to the value $\mathsf{M}_p(\mathbf{E}) = 0$ for every system $\mathbf{E}$ of surfaces (or curves).)

In terms of this straightforward generalization of the concept of extremal length, it is easily verified (Theorem 2) that *a system* $\mathbf{E}$ *of curves or surfaces in* R^n *is exceptional of order* p *if, and only if, the module of order* p *of* $\mathbf{E}$ *equals zero:* $\mathsf{M}_p(\mathbf{E}) = 0$.

In Chapter I some elementary properties of M_p are studied under more general circumstances (systems of measures μ instead of systems of surfaces). As a set-function, M_p has some resemblance with an exterior measure (cf. Theorem 1 and the remark following it). Theorem 3 contains, in particular, a generalization of the well-known fact that mean-convergence implies convergence almost everywhere of a suitable subsequence. This result indicates the role of exceptional systems by the above instances of functional completion.

Chapter II deals with k-dimensional Lipschitz surfaces in R^n, $1 \leq k \leq n-1$. The principal problem treated in this chapter is the characterization of those subsets of R^n for which the system of all k-dimensional surfaces intersecting the set, is exceptional of order p. For $p = 2$, it is found that these sets are identical with the sets whose exterior capacity of order $2k$ equals 0. There are substitute results for $p \neq 2$ (Theorems 6, 7, and 8). The proofs of these results depend on the theory of generalized potentials of functions from the class L^p and on the theory of singular integrals (Hilbert transforms) in R^n. The last section of the chapter is devoted to the study of certain simple systems of curves or hypersurfaces for which M_2 and λ_2 may be expressed in terms of the capacity of a condensor, or a thermal conductivity, cf. Theorem 9. Methodically, these results are related to the method of prescribed level surfaces, devised by Pólya and Szegö [31] with the purpose of estimating capacities etc. The fact that critical points may occur causes, however, some technical complications.

Chapter III describes the role of exceptional systems of curves in connection with (generalized) irrotational vector fields and Beppo Levi functions. A vector field $f \in L^p(X)$ is irrotational in a region $X \subset R^n$ if, and only if, the circulation of f vanishes along almost every closed curve homolog zero in X (Theorem 10). Next, a Beppo Levi function (of order p) in X is defined as a primitive u of a differential form

$$f(x) \cdot dx = \sum_{i=1}^{n} f_i(x)\, dx_i,$$

where each $f_i \in L^p(X)$, in the sense that, along almost every curve $C \subset X$,

$$u(b) - u(a) = \int_a^b f(x) \cdot dx,$$

a and b being arbitrary points of C. The field $f = (f_1, \ldots, f_n)$ is necessarily irrotational (Theorem 11). The class of Beppo Levi functions of order p in X is denoted by $BL^p(X)$. It follows at once from the definition of Beppo Levi functions that the equations $\partial u / \partial x_i = f_i(x)$ subsist almost everywhere in X (Theorem 12), so that a Beppo Levi function may be defined equivalently as a function which is absolutely continuous along almost every curve and whose gradient belongs to $L^p(X)$. In consequence of Theorem 7, a function $u \in BL^2(X)$ is determined only quasi-everywhere, i.e., except in some set of exterior capacity 0. There is a substitute result for $p \neq 2$, $p > 1$, (Theorem 13). Finally it is shown that, for $p > 1$, the class $BL^p(X)$ is the perfect pseudo-functional completion in the sense of Aronszajn [2] of the class of smooth functions whose gradients belong to $L^p(X)$. In particular, the class $BL^2(X)$ is identical with the class of "fonctions (BL) précisées" in the sense of Deny and Lions [10], which in turn is identical with a class of functions considered by Aronszajn and Smith [3].

Other applications of the concept of exceptional systems of surfaces or curves, in particular to systems of linear partial differential operators, will be described in a subsequent article.

CHAPTER I

The Module of a System of Measures

1. The module of order p

We consider measures in a fixed abstract set X. (By a measure in X is meant a countably additive, σ-finite set-function with non-negative values (the value $+\infty$ being admitted), defined on a σ-field of subsets of X.) The completion of a measure μ is denoted by $\bar{\mu}$. The domain of $\bar{\mu}$ consists of all sets $E \subset X$ such that $A \subset E \subset B$ for suitable A and B from the domain of μ with $\mu(B - A) = 0$; then $\bar{\mu}(E) = \mu(A) = \mu(B)$.

One such measure in X will be kept fixed throughout the present chapter. This basic measure will be denoted by m and its domain of definition by $\mathfrak{M}$. It is assumed that $X \in \mathfrak{M}$. (By the applications described in the following chapters X will be Euclidean n-dimensional space R^n, $\mathfrak{M}$ will be the system of Borel subsets of R^n, m the n-dimensional Borel measure and hence $\bar{m}$ the n-dimensional Lebesgue measure.)

We shall now consider other measures, or rather systems (=sets) of other measures, in relation to this fixed measure m. We denote by $\mathbf{M}$ the system of all measures μ in X whose domains contain the domain $\mathfrak{M}$ of m. With an arbitrary system $\mathbf{E}$ of measures $\mu \in \mathbf{M}$ we associate the class of all non-negative m-measurable functions f defined in X and subjected to the condition

$$\int_X f \, d\mu \geq 1 \quad \text{for every } \mu \in \mathbf{E}.$$

We write $f \wedge \mathbf{E}$ to signify that f is associated with the system $\mathbf{E}$ in this manner. The module M_p is now defined as follows:

$$\mathsf{M}_p(\mathbf{E}) = \inf_{f \wedge \mathbf{E}} \int_X f^p \, dm \quad (0 < p < \infty),$$

interpreted as $+\infty$ if no functions are associated with $\mathbf{E}$.(1) As a partial motivation for this definition it may be mentioned that the measure $m(E)$ of an arbitrary set $E \in \mathfrak{M}$ equals the minimum of $\int_X f(x)^p \, dm(x)$ when f ranges over all non-negative m-measurable functions such that $f(x) \geq 1$ everywhere in E. A minimizing function f is the characteristic function χ_E for E. This analogy expresses, by the way, an actual connection between the measure m and the module M_p in the special case where the system $\mathbf{E}$ consists of "Dirac measures". (With any $x \in X$ is associated the Dirac measure χ_x defined by $\chi_x(A) = \chi_A(x) = 1$ or 0 according as A does or does not contain x.) If $\mathbf{E}$ denotes a system of such measures χ_x, obtained by taking for x the points of some given set $E \in \mathfrak{M}$, then it follows easily that $\mathsf{M}_p(\mathbf{E}) = m(E)$. Returning to general systems of measures, we shall establish a few simple properties of M_p.

THEOREM 1. *The module M_p is monotone and countably sub-additive:*

(a) $$\mathsf{M}_p(\mathbf{E}) \leq \mathsf{M}_p(\mathbf{E}') \qquad \text{if } \mathbf{E} \subset \mathbf{E}'.$$

(b) $$\mathsf{M}_p(\mathbf{E}) \leq \sum_i \mathsf{M}_p(\mathbf{E}_i) \quad \text{if } \mathbf{E} = \bigcup_i \mathbf{E}_i.$$

Proof. The monotony of M_p follows at once from the fact that $f \wedge \mathbf{E}'$ implies $f \wedge \mathbf{E}$ if $\mathbf{E} \subset \mathbf{E}'$. The subadditivity may be proved as follows. If $f(x) = \sup_i f_i(x)$, where each f_i is a non-negative m-measurable function defined in X, then f is likewise such a function, and

$$\int_X f^p \, dm \leq \sum_i \int_X f_i^p \, dm.$$

(1) The only case in which no functions are associated with $\mathbf{E}$ in the manner explained above is the case where $\mathbf{E}$ contains the measure $\mu \equiv 0$.

To see this, we define, for an arbitrary index n,

$$g_n(x) = \max\{f_1(x), \ldots, f_n(x)\}; \qquad X_i = \{x \in X : f_i(x) = g_n(x)\}.$$

Then g_n is m-measurable, $X_i \in \mathfrak{M}$, and $X = \bigcup_{i=1}^{n} X_i$. Hence,

$$\int_X g_n^p\,dm \le \sum_{i=1}^{n} \int_{X_i} g_n^p\,dm = \sum_{i=1}^{n} \int_{X_i} f_i^p\,dm \le \sum_{i=1}^{\infty} \int_X f_i^p\,dm.$$

The desired inequality now follows for $n \to \infty$ since $g_n(x) \to f(x)$ monotonically, and hence $\int_X g_n^p\,dm \to \int_X f^p\,dm$. Next, let f_i be so chosen that $f_i \wedge \mathbf{E}_i$ and

$$\int_X f_i^p\,dm \le \mathsf{M}_p(\mathbf{E}_i) + \varepsilon \cdot 2^{-i}.$$

Then $f \wedge \mathbf{E}$, and

$$\mathsf{M}_p(\mathbf{E}) \le \int_X f^p\,dm \le \sum_{i=1}^{\infty} \int_X f_i^p\,dm \le \sum_{i=1}^{\infty} \mathsf{M}_p(\mathbf{E}_i) + \varepsilon.$$

Remark. If, in particular, the systems $\mathbf{E}_i$ are "separate" in the sense that there exist mutually disjoint sets $S_i \in \mathfrak{M}$ such that $\mu(X - S_i) = 0$ when $\mu \in \mathbf{E}_i$, then the sign of equality holds in Theorem 1, (b). In fact, if $f \wedge \mathbf{E}$, and hence $f \wedge$ each $\mathbf{E}_i$, and if we define functions f_i by $f_i(x) = f(x)$ or $= 0$ according as $x \in S_i$ or $x \notin S_i$, then $f_i \wedge \mathbf{E}_i$.

Hence $$\int_{S_i} f^p\,dm = \int_X f_i^p\,dm \ge \mathsf{M}_p(\mathbf{E}_i),$$

and consequently $$\int_X f^p\,dm \ge \sum_i \int_{S_i} f^p\,dm \ge \sum_i \mathsf{M}_p(\mathbf{E}_i),$$

which implies that $\mathsf{M}_p(\mathbf{E}) \ge \sum_i \mathsf{M}_p(\mathbf{E}_i)$.

To the above elementary properties of the module M_p (or the generalization of extremal length $\lambda_p = 1/\mathsf{M}_p$) one may add Lemmas 1, 2, and 3 in Ahlfors and Beurling [1], p. 115, all of which may be extended to the present case of an arbitrary order p and systems of measures instead of families of curves. We shall say that a system $\mathbf{E}$ of measures $\mu \in \mathbf{M}$ is *minorized* by a system $\mathbf{E}'$ of such measures if there corresponds to any $\mu \in \mathbf{E}$ a measure $\mu' \in \mathbf{E}'$ such that $\mu \ge \mu'$ (that is, $\mu(A) \ge \mu'(A)$ for every point-set $A \in \mathfrak{M}$).

The three lemmas may now be generalized as follows:

(c) *If* $\mathbf{E}$ *is minorized by* $\mathbf{E}'$, *then* $\lambda_p(\mathbf{E}) \geq \lambda_p(\mathbf{E}')$.

(d) *If* $p>1$, *if the systems* $\mathbf{E}_1, \mathbf{E}_2, \ldots$ *are separate, and if a system* $\mathbf{E}$ *is minorized by each* $\mathbf{E}_i$, *then*

$$\lambda_p(\mathbf{E})^{\frac{1}{p-1}} \geq \sum_i \lambda_p(\mathbf{E}_i)^{\frac{1}{p-1}}.$$

(e) *If the systems* $\mathbf{E}_1, \mathbf{E}_2, \ldots$ *are separate, and if each* $\mathbf{E}_i$ *is minorized by a system* $\mathbf{E}$, *then*

$$\lambda_p(\mathbf{E})^{-1} \geq \sum_i \lambda_p(\mathbf{E}_i)^{-1}; \quad \text{i.e.,} \quad \mathsf{M}_p(\mathbf{E}) \geq \sum_i \mathsf{M}_p(\mathbf{E}_i).$$

Statement (e) contains the above remark as a special case and is proved exactly like it. In particular, the sign of equality holds if $\mathbf{E} = \bigcup_i \mathbf{E}_i$, where the $\mathbf{E}_i$ are separate (but otherwise arbitrary) systems. Statement (c) is easily verified. By the proof of (d), it is convenient to express the definition of the "extremal length" λ_p in the following form:

$$\lambda_p(\mathbf{E}) = \sup_f L_f(\mathbf{E})^p; \qquad f \geq 0, \quad f \in L^p(m); \int f^p\, dm = 1,$$

where $L_f(\mathbf{E}) = \inf_{\mu \in \mathbf{E}} \int f\, d\mu$. If $\lambda_p(\mathbf{E}_i) = 0$ for some i, the corresponding term may be neglected. If $\lambda_p(\mathbf{E}_i) = +\infty$ for some i, it follows from (c) that $\lambda_p(\mathbf{E}) = +\infty$. Thus we may assume that $0 < \lambda_p(\mathbf{E}_i) < +\infty$ for every i, and also that $0 < \lambda_p(\mathbf{E}) < +\infty$. To any given number $\varepsilon_i > 0$ corresponds a function $f_i \geq 0$, $f_i \in L^p(m)$, such that

$$\int f_i^p\, dm = 1, \quad \text{and} \quad L_{f_i}(\mathbf{E}_i) > \lambda_p(\mathbf{E}_i)^{1/p} - \varepsilon_i.$$

Choosing disjoint sets S_i so that $\mu(X - S_i) = 0$ when $\mu \in \mathbf{E}_i$, we may assume, moreover, that $f_i = 0$ in $X - S_i$. Define $f(x) = \sum_i t_i f_i(x)$, where $t_i \geq 0$, $\sum_i t_i^p = 1$. It follows that

$$\int f^p\, dm = \sum_i t_i^p \int f_i^p\, dm = \sum_i t_i^p = 1.$$

Hence,

$$\lambda_p(\mathbf{E}) \geq L_f(\mathbf{E})^p.$$

Let $\mu \in \mathbf{E}$. By assumption, there are measures $\mu_i \in \mathbf{E}_i$ such that $\mu \geq \mu_i$. Consequently,

$$\int f\, d\mu = \sum_i t_i \int f_i\, d\mu \geq \sum_i t_i \int f_i\, d\mu_i \geq \sum_i t_i L_{f_i}(\mathbf{E}_i) \geq \sum_i t_i \lambda_p(\mathbf{E}_i)^{1/p} - \sum_i t_i \varepsilon_i.$$

It follows that

$$L_f(\mathbf{E}) \geq \sum_i t_i \lambda_p(\mathbf{E}_i)^{1/p} - \sum_i t_i \varepsilon_i,$$

and hence

$$\lambda_p(\mathbf{E}) \geq \left(\sum_i t_i \lambda_p(\mathbf{E}_i)^{1/p}\right)^p.$$

In Hölder's inequality

$$\left(\sum_i t_i \lambda_p(\mathbf{E}_i)^{1/p}\right)^p \leq \sum_i t_i^p \cdot \left(\sum_i \lambda_p(\mathbf{E}_i)^{\frac{1}{p-1}}\right)^{p-1}$$

the sign of equality holds if, and only if, the numbers t_i^p are proportional to the numbers $\lambda_p(\mathbf{E}_i)^{1/(p-1)}$. This optimal choice of the multipliers t_i leads to the desired inequality.

The sign of equality holds in (d) if, in particular, $\mathbf{E} = \sum_i \mathbf{E}_i$, where the $\mathbf{E}_i$ are separate (but otherwise arbitrary) systems. In fact,

$$L_f(\mathbf{E}) \leq \sum_i L_f(\mathbf{E}_i)$$

for arbitrary $f \geq 0$, $f \in L^p(m)$, since $\mu_i \in \mathbf{E}_i$ implies $\sum_i \mu_i \in \mathbf{E}$. Defining $t_i = \left\{\int_{S_i} f^p\, dm\right\}^{1/p}$, and $f_i(x) = t_i^{-1} f(x)$ or $=0$ according as x belongs or does not belong to S_i, we have $f \geq \sum_i t_i f_i$, and

$$\int f^p\, dm \geq \sum_i t_i^p. \tag{1}$$

On the other hand, $L_f(\mathbf{E}_i) = t_i L_{f_i}(\mathbf{E}_i)$, and consequently

$$L_f(\mathbf{E}) \leq \sum_i L_f(\mathbf{E}_i) = \sum_i t_i L_{f_i}(\mathbf{E}_i) \leq \sum_i t_i \lambda_p(\mathbf{E}_i)^{1/p}.$$

Applying Hölder's inequality as above, we obtain

$$L_f(\mathbf{E})^p \leq \sum_i t_i^p \cdot \left(\sum_i \lambda_p(\mathbf{E}_i)^{\frac{1}{p-1}}\right)^{p-1}. \tag{2}$$

Combining (1) and (2), we arrive at the desired inequality:

$$\lambda_p(\mathbf{E}) \leq \left(\sum_i \lambda_p(\mathbf{E}_i)^{\frac{1}{p-1}}\right)^{p-1}.$$

2. Exceptional systems of measures

A system $\mathbf{E} \subset \mathbf{M}$ will be called *exceptional of order* p (abbreviated: p-exc) if $\mathsf{M}_p(\mathbf{E}) = 0$.

The well-known fact concerning point-sets $E \subset X$ that $\bar{m}(E) = 0$ if, and only if, there exists a function $f \in L^p(m)$, $f \geq 0$, such that $f(x) = +\infty$ for every $x \in E$ (the value of $p > 0$ being irrelevant), may be generalized as follows:

THEOREM 2. *A system* $\mathbf{E} \subset \mathbf{M}$ *is p-exc if, and only if, there exists a function* $f \in L^p(m)$, $f \geq 0$, *such that*

$$\int_X f\, d\mu = +\infty \quad \textit{for every } \mu \in \mathbf{E}.$$

Proof. If f has these properties, then $n^{-1}f \wedge \mathbf{E}$ for every $n=1, 2, \ldots$; and $\int (n^{-1}f)^p\, dm = n^{-p}\int f^p\, dm \to 0$ as $n \to \infty$; hence $\mathsf{M}_p(\mathbf{E}) = 0$. Conversely, let $\mathsf{M}_p(\mathbf{E}) = 0$ and choose a sequence of functions $f_n \wedge \mathbf{E}$ so that $\int f_n^p\, dm < 4^{-n}$. Writing $f(x) = \{\sum_n 2^n f_n(x)^p\}^{1/p}$, we infer that $\int f^p\, dm = \sum_n 2^n \int f_n^p\, dm < \infty$; on the other hand, $\int f\, d\mu \geq \int 2^{n/p} f_n\, d\mu \geq 2^{n/p}$ for every $\mu \in \mathbf{E}$ and every $n = 1, 2, \ldots$, and hence $\int f\, d\mu = +\infty$.

A proposition concerning measures μ which belong to some specified system $\mathbf{E} \subset \mathbf{M}$, is said to hold for *almost every* $\mu \in \mathbf{E}$, *of order* p, (abbreviated: p-a.e. $\mu \in \mathbf{E}$) if the system of all measures $\mu \in \mathbf{E}$ for which the proposition fails to hold is exceptional of order p. This amounts to the existence of a function $f \in L^p(m)$, $f \geq 0$, such that the proposition holds for every $\mu \in \mathbf{E}$ for which $\int_X f\, d\mu < \infty$.

THEOREM 3. (a) *Any subsystem of a p-exc system is p-exc.*

(b) *The union of a finite or denumerable family of p-exc systems is p-exc.*

(c) *If $p > q$, then every p-exc system of finite measures is q-exc.*

(d) *If $E \subset X$ and $\bar{m}(E) = 0$, then $\bar{\mu}(E) = 0$ for p-a.e. $\mu \in \mathbf{M}$.*

(e) *If $f \in L^p(\bar{m})$, then $f \in L^1(\bar{\mu})$ for p-a.e. $\mu \in \mathbf{M}$.*

(f) *If a sequence of functions $f_i \in L^p(\bar{m})$ converges in the mean of order p with respect to $\bar{m}$ to some function f, i.e.,*

$$\lim_{i\to\infty} \int_X |f_i - f|^p\, d\bar{m} = 0,$$

then there is a subsequence of indices i_ν tending to ∞ with the property that, for p-a.e. $\mu \in \mathbf{M}$, f_{i_ν} converges to f in the mean of order 1 with respect to $\bar{\mu}$:

$$\lim_{\nu\to\infty} \int_X |f_{i_\nu} - f|\, d\bar{\mu} = 0 \quad \textit{for p-a.e. } \mu \in \mathbf{M}.$$

Statements (d), (e), (f) *remain valid if $\bar{m}$ and $\bar{\mu}$ are replaced by m and μ, respectively.*

Proof. Statements (a) and (b) are contained in Theorem 1. To prove (c), let $\mathbf{E}$ denote a p-exc system of finite measures $\mu \in \mathbf{M}$, and let $f \in L^p(m)$, $f \geq 0$, be so chosen that $\int f\, d\mu = +\infty$ for every $\mu \in \mathbf{E}$. Now, $f^{p/q} \in L^q(m)$, and an application of Hölder's inequality shows that $\int f^{p/q}\, d\mu = +\infty$ when $\mu \in \mathbf{E}$ since $\mu(X) < \infty$ and $p/q > 1$. In fact,

$$+\infty = \int f\, d\mu \leq \left(\int f^{p/q}\, d\mu\right)^{q/p} \mu(X)^{1-q/p}.$$

As to statements (d), (e), (f), we begin by proving the corresponding statements in which $\bar{m}$ and $\bar{\mu}$ are replaced by m and μ, respectively. The statement corresponding to (e) is then contained in Theorem 2, while (d) may be proved as follows. Let $E \in \mathfrak{M}$, $m(E)=0$, and $f(x)=+\infty$ for $x \in E$, $f(x)=0$ for $x \notin E$. Then f belongs to $L^p(m)$ and

$$\int f\,d\mu = (+\infty)\cdot\mu(E) = +\infty \quad \text{for every } \mu \text{ such that } \mu(E)>0.$$

As to (f), we choose an increasing sequence of integers i_ν so that

$$\int_X |f_{i_\nu}(x)-f(x)|^p\,dm(x) < 2^{-p\nu-\nu},$$

and write $g_\nu(x)=|f_{i_\nu}(x)-f(x)|$. Introducing the systems

$$\mathbf{A}_\nu = \{\mu \in \mathbf{M} : \int g_\nu\,d\mu > 2^{-\nu}\}, \quad \mathbf{B}_n = \bigcup_{\nu>n} \mathbf{A}_\nu, \quad \text{and } \mathbf{E} = \bigcap_n \mathbf{B}_n,$$

we have $2^\nu g_\nu \wedge \mathbf{A}_\nu$ and hence

$$\mathsf{M}_p(\mathbf{A}_\nu) \le \int (2^\nu g_\nu)^p\,dm = 2^{p\nu}\int g_\nu^p\,dm < 2^{-\nu}.$$

This implies, in view of Theorem 1, that

$$\mathsf{M}_p(\mathbf{E}) \le \mathsf{M}_p(\mathbf{B}_n) \le \sum_{\nu>n} \mathsf{M}_p(\mathbf{A}_\nu) < 2^{-n}.$$

Consequently, $\mathsf{M}_p(\mathbf{E})=0$. To every $\mu \in \mathbf{M}-\mathbf{E}$ corresponds an index n such that $\mu \notin \mathbf{B}_n$, i.e., $\int |f_{i_\nu}-f|\,d\mu = \int g_\nu\,d\mu \le 2^{-\nu}$ for every $\nu>n$. Hence $\lim_{\nu\to\infty}\int |f_{i_\nu}-f|\,d\mu=0$. It remains to reduce the original statements (d), (e), (f) to the above corresponding statements in which $\bar{m}$ and $\bar{\mu}$ were replaced by m and μ, respectively. As to (d), let $E \subset X$ and assume that $\bar{m}(E)=0$. There exists a set $E^* \in \mathfrak{M}$ such that $m(E^*)=0$ and $E^* \supset E$. The system of all measures μ such that $\bar{\mu}(E)>0$ is, therefore, contained in the p-exc system of all measures μ such that $\mu(E^*)>0$. As to (e), the function f may be replaced by an equivalent m-measurable function f^*. Applying (d) to the set $E=\{x : f(x) \neq f^*(x)\}$, we infer that $\bar{\mu}(E)=0$ for p-a.e. μ; in particular,

$$\int |f|\,d\bar{\mu} = \int |f^*|\,d\bar{\mu} = \int |f^*|\,d\mu \quad (<\infty) \quad \text{for } p\text{-a.e. } \mu.$$

Statement (f) may be treated in a similar manner, and the proof is complete.

Simple examples show that the infimum in the definition of $\mathsf{M}_p(\mathbf{E})$ is not necessarily attained by any function $f \wedge \mathbf{E}$. However, the following theorem subsists for any order $p > 1$ and any system $\mathbf{E}$ of measures $\mu \not\equiv 0$, $\mu \in \mathbf{M}$: *There exists a function* $f \geq 0$ *such that* $\int_X f^p \, dm = \mathsf{M}_p(\mathbf{E})$ *and* $\int_X f \, d\mu \geq 1$ *for p-a.e.* $\mu \in \mathbf{E}$. (The former property of f obviously only depends on the m-equivalence class of f, and so does the latter by virtue of Theorem 3 (d).) The existence of f is clear if $\mathsf{M}_p(\mathbf{E}) = +\infty$; and if $\mathsf{M}_p(\mathbf{E}) < +\infty$, it is a consequence of the well-known facts that the Banach space $L^p(m)$ is uniformly convex when $p > 1$, and that any convex, closed, and non-empty subset of a uniformly convex Banach space contains a unique vector with minimal norm (cf., e.g., Nagy [28], p. 7). For any system $\mathbf{E}$ of measures $\mu \in \mathbf{M}$, $\mu \not\equiv 0$, the set of all functions $f \in L^p(m)$, $f \geq 0$, such that $\int f \, d\mu \geq 1$ for p-a.e. $\mu \in \mathbf{E}$, is convex and non-empty, and it is closed in $L^p(m)$ by virtue of Theorem 3 (f). From the uniqueness of the minimal vector follows that the minimal function f is *uniquely determined* up to a set of measure $m = 0$. Simple examples show that the restriction $p > 1$ is essential for the existence as well as for the uniqueness of f.

Chapter II

The Module of a System of Surfaces

Notations. By R^n we denote the Euclidean n-dimensional space with a fixed Cartesian coordinate system. The origin is denoted by 0, and a point x is identified with the vector from 0 to x. The coordinates of a point x will be denoted by x_ν, $\nu = 1, 2, \ldots, n$, and we write $x = (x_1, x_2, \ldots, x_n)$ and $|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}$. The closure of a set $E \subset R^n$ is denoted by $\bar{E}$. The open ball $\{y \in R^n : |y - x| < r\}$ is denoted by $B_r(x)$, and we write $B_r'(x) = \{y \in R^n : |y - x| \geq r\}$. If $x = 0$, we may use the notations B_r and B_r'. The unit sphere in R^n is denoted by $\Omega = \Omega_n = \{x \in R^n : |x| = 1\}$. The usual surface measure on Ω is denoted by ω, and we write $\omega_n = \omega(\Omega_n)$ for the total surface measure of Ω_n, the value of which is

$$\omega_n = 2\pi^{n/2}/\Gamma(n/2).$$

The system of Borel subsets of a given Borel set $X \subset R^n$ is denoted by $\mathfrak{B}(X)$, in particular by $\mathfrak{B}$ if $X = R^n$. The n-dimensional Borel measure is denoted by m_n and the n-dimensional Lebesgue measure by $\bar{m}_n$. The Lebesgue classes L^p refer to $\bar{m}_n$, and we write $L^p(X)$ if the functions are defined only in a subset $X \subset R^n$. Likewise, mean convergence refers to $\bar{m}_n$ unless otherwise stated. We write $\|f\|_p = \left(\int_X |f(x)|^p \, dx\right)^{1/p}$.

By $C^h(X)$ is denoted the class of all real-valued continuous functions in X (open) having continuous partial derivatives of order h everywhere in X. In the case $h=0$ (continuous functions), we may write simply $C(X)$. If, in addition, each function is required to vanish outside some compact subset of X, we obtain the subclasses $C_0^h(X)$, in particular $C_0(X)$ for $h=0$. The functions in these subclasses will be understood as defined in the entire space R^n, vanishing outside X. The symbol

$$\frac{\partial^h}{\partial x_{\alpha_1}\partial x_{\alpha_2}\ldots\partial x_{\alpha_h}}$$

for a partial differentiation of order h will be written shortly as D_α, where

$$\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_h).$$

The order h of the differentiation will be denoted by $|\alpha|$. When speaking about "all derivatives $D_\alpha u$ of orders $|\alpha|\le k$" of a function u, we include the derivative of order 0, the function u itself.

1. Lipschitz image and Lipschitz surface

A subset $X\subset R^n$ is called a *Lipschitz image* of a subset $T\subset R^k$ if there exists a Lipschitz transformation of T onto X. A *Lipschitz transformation* of T onto X is a one-to-one mapping of T onto X such that

$$c^{-1}|t'-t''|\le|x'-x''|\le c|t'-t''| \tag{1}$$

whenever $x',x''\in X$ correspond to $t',t''\in T$. Here c denotes a suitable constant. In the sequel it will be assumed that T is a non-void open subset of R^k, the dimension k being kept fixed, $1\le k_{(\le)}n$. Hence T and X are countable unions of compact sets. In view of a theorem of Rademacher [32], each of the Lipschitz functions x_i has almost everywhere in T a total differential with respect to $t=(t_1,\ldots,t_k)$. Thus, for a.e. fixed $t\in T$ and every $i=1,\cdots,n$,

$$x_i'-x_i=\sum_{j=1}^{k}a_{ij}(t_j'-t_j)+o(|t'-t|)\quad\text{as } t'\to t, \tag{2}$$

where $x_i=x_i(t)$, $x_i'=x_i(t')$, $t'\in T$, and $a_{ij}=\partial x_i/\partial t_j$ evaluated at the point t. This implies, in particular, the existence of a "tangent plane" Π at x, given parametrically by the differential mapping of R^k onto Π obtained by neglecting the remainder term in (2) and allowing t' to take arbitrary values $t'\in R^k$. In fact, it will be shown presently that the rank of the matrix $\{a_{ij}\}$ equals k. With any symbol $\alpha=(\alpha_1,\ldots,\alpha_k)$

where each α_j, $j=1, \ldots, k$, is one of the numbers $1, 2, \ldots, n$, we associate the Jacobian minor

$$q_\alpha = \frac{\partial (x_{\alpha_1}, \ldots, x_{\alpha_k})}{\partial (t_1, \ldots, t_k)} = \text{det. } \{a_{\alpha_i, j}\} \tag{3}$$

evaluated at the point t. These minors form the components of an antisymmetric tensor of rank k. The quantity

$$q = \left(\frac{1}{k!} \sum_{|\alpha|=k} q_\alpha^2 \right)^{\frac{1}{2}} \tag{4}$$

represents the ratio of k-dimensional area by the above differential mapping of R^k onto the tangent plane Π. The functions q_α and q are measurable. We proceed to prove the inequalities

$$c^{-k} \le q \le c^k. \tag{5}$$

(In particular, $q \neq 0$, so that the rank of $\{a_{ij}\}$ is indeed k.) Writing $\tau = t' - t$, $\xi = x' - x$, we consider the linear mapping of R^k into R^n given by the equations

$$\xi_i = \sum_{j=1}^{k} a_{ij} \tau_j. \tag{6}$$

It follows easily from (1) and (2) that, by the mapping (6), the linear ratio $|\xi|/|\tau|$ in any direction is between c^{-1} and c. Now, there exist k orthogonal unit vectors $\xi^{(1)}, \ldots, \xi^{(k)}$ whose images $\eta^{(1)}, \ldots, \eta^{(k)}$ by the mapping (6) are likewise mutually orthogonal. Since $c^{-1} \le |\eta^{(j)}| \le c$, it follows that the ratio of k-dim. area $q(t) = |\eta^{(1)}| \ldots |\eta^{(k)}|$ is between c^{-k} and c^k.

A subset $E \subset X$ is a Borel set if, and only if, the corresponding subset $F \subset T$ is a Borel set (in R^k). Using these notations, we define

$$\mu_X(E) = \int_F q(t)\, dt. \tag{7}$$

From the results of Rademacher [32] concerning transformations of integrals it follows in a well-known manner that $\mu_X(E)$ is independent of the parametric representation of the Lipschitz image X. (In particular, μ_X equals the restriction of m_n to X if $k=n$.) Since q is bounded, we have $\mu_X(E) < \infty$ for every bounded set $E \in \mathfrak{B}(X)$. It follows that μ_X is a (σ-finite) measure on X. We call it the *surface measure* on X. By integrations with respect to μ_X (or $\bar{\mu}_X$), we shall write $d\sigma$ instead of $d\mu_X$ (or $d\bar{\mu}_X$). The following lemma will be used in the next section.

LEMMA 1. *Given a Lipschitz image $X \subset R^n$ of an open set $T \subset R^k$, and a point $x^* \in X$. There exists a constant K, depending only on X and x^*, such that the inequality*

$$|u(x^*)| \leq K \int_X \sum_{|\alpha| \leq k} |D_\alpha u| \, d\sigma$$

holds for any function $u \in C^k(R^n)$.

Proof: Denote by t^* the point of T which corresponds to x^* by a parametric representation $t \to x(t)$ of the Lipschitz image X. Since T is open, there exists a closed ball $A \subset T$ with the centre t^*. Denoting by a the radius of A and by c the constant introduced in (1), we consider the closed ball $B \subset R^n$ with the centre x^* and the radius a/c. The inverse image $F = \{t \in T : x(t) \in B\}$ is then contained in A in view of (1). Now, choose a function $\varphi \in C^k(R^n)$ so that $\varphi(x^*) = 1$ and $\varphi(x) = 0$ outside B, and keep φ fixed. Corresponding to an arbitrary function $u \in C^k(R^n)$ we write $f(x) = \varphi(x) \cdot u(x)$. Then $f \in C^k(R^n)$, and $f(x) = 0$ outside B. Hence any derivative $D_\alpha f$ of order $|\alpha| \leq k$ is continuous in R^n and vanishes outside B. This implies that the composite function $D_\alpha f(x(t))$ is continuous in T and vanishes outside A. For $|\alpha| = k$, write

$$p_\alpha(t) = \frac{\partial x_{\alpha_1}}{\partial t_1} \frac{\partial x_{\alpha_2}}{\partial t_2} \cdots \frac{\partial x_{\alpha_k}}{\partial t_k},$$

the argument of $\partial x_{\alpha_j} / \partial t_j$ being

$$t^{(j)} = (t_1, \ldots, t_j, t^*_{j+1}, \ldots, t^*_k).$$

With this notation, it is easily verified that

$$f(x^*) = \int_Q \sum_{|\alpha| = k} D_\alpha f(x(t)) \, p_\alpha(t) \, dt, \tag{8}$$

where
$$Q = \{t \in A : t_j \leq t^*_j \text{ for every } j = 1, \ldots, k\}.$$

In fact, integrating first with respect to t_k and writing

$$\tau = (t_1, \ldots, t_{k-1}), \ \gamma = (\alpha_1, \ldots, \alpha_{k-1}),$$

we obtain, since $t^{(1)}, \ldots, t^{(k-1)}$ are independent of t_k,

$$\int^{t^*_k} \sum_{|\alpha| = k} D_\alpha f(x(t)) \, p_\alpha(t) \, dt_k = \sum_\gamma \left\{ \int^{t^*_k} \sum_{\alpha_k = 1}^{n} \frac{\partial D_\gamma f}{\partial x_{\alpha_k}} \frac{\partial x_{\alpha_k}}{\partial t_k} \, dt_k \right\} p_\gamma(\tau)$$
$$= \sum_\gamma D_\gamma f(x(t^{(k-1)})) \, p_\gamma(\tau).$$

Here
$$p_\gamma(\tau)=\frac{\partial x_{\alpha_1}}{\partial t_1}\cdots\frac{\partial x_{\alpha_{k-1}}}{\partial t_{k-1}},$$

the argument of $\partial x_{\alpha_j}/\partial t_j$ being $t^{(j)}$ as above, $j=1, \ldots, k-1$. In a similar way one may integrate successively with respect to $t_{k-1}, \ldots, t_1$, and the formula (8) results. Since $|\partial x_i/\partial t_j|\leq c$, we have $|p_\alpha(t)|\leq c^k$. Moreover, $dt=d\sigma/q(t)\leq c^k\, d\sigma$, and hence, from (8),
$$|f(x^*)|\leq c^{2k}\int_X \sum_{|\alpha|=k}|D_\alpha f(x)|\, d\sigma(x).$$

Clearly,
$$D_\alpha f=D_\alpha(\varphi\cdot u)=\sum_{|\beta|\leq k}\varphi_{\alpha,\beta}D_\beta u,$$

where the coefficients $\varphi_{\alpha,\beta}=\varphi_{\alpha,\beta}(x)$ are certain functions of x derived from the function φ. Hence
$$|u(x^*)|=|f(x^*)|\leq c^{2k}\int_X\sum_{|\beta|\leq k}\Big(\sum_{|\alpha|=k}|\varphi_{\alpha,\beta}|\Big)|D_\beta u|\, d\sigma$$
$$\leq K\int_X\sum_{|\beta|\leq k}|D_\beta u|\, d\sigma,$$

where
$$K=c^{2k}\max_{|\beta|\leq k}\ \max_{x\in B}\sum_{|\alpha|=k}|\varphi_{\alpha,\beta}(x)|$$

is a finite constant. This completes the proof of Lemma 1.

Now, let $1\leq k\leq n-1$. A non-void subset $S\subset R^n$ will be called a k-dimensional *Lipschitz surface* (or manifold) in R^n if there corresponds to every point $x^*\in S$ an open set $U\subset R^n$ such that $x^*\in U$, and $S\cap U$ is a Lipschitz image of some open set $T\subset R^k$.(1) From Lindelöf's covering theorem follows that S is a Borel subset of R^n; in fact, S is a countable union of compact sets. Moreover, it is easily verified that there exists one and only one measure μ_S defined on $\mathfrak{B}(S)$ which agrees with the surface measure μ_X on every Lipschitz image $X=S\cap U$ of the above type. This measure μ_S is called the *surface measure* on S. By integrations we shall write $d\sigma$ instead of $d\mu_X$ (or $d\bar{\mu}_X$).

2. Exceptional systems of surfaces

In order to apply the notions and results of Chapter I to systems (=sets) of surfaces, in particular curves, in R^n, we take $X=R^n$, $\mathfrak{M}=\mathfrak{B}$, $m=m_n$ and hence $\bar{m}=\bar{m}_n$.

(1) According to this definition, a connected 1-dimensional Lipschitz surface means a simple continuous curve, either "closed" and rectifiable, or "open" and locally rectifiable. The restriction to simple curves is, however, not necessary; the results of the present paper are likewise valid for systems of parametrically given continuous curves (with or without multiple points), provided they are locally rectifiable in the sense that every arc corresponding to a compact sub-interval, is rectifiable. Moreover, no point of a curve should correspond to an interval of parameter values.

We denote by $\mathfrak{S}^k$ the system of all k-dimensional Lipschitz surfaces in R^n. For any system $\mathbf{E} \subset \mathfrak{S}^k$ we define its module $\mathsf{M}_p(\mathbf{E})$ as the module M_p of the system of measures μ_S, $S \in \mathbf{E}$, these measures being extended in such a way to $\mathfrak{B} = \mathfrak{B}(R^n)$ that $\mu_S(R^n - S) = 0$. Thus, for any p such that $0 < p < \infty$,

$$\mathsf{M}_p(\mathbf{E}) = \inf_{f \wedge \mathbf{E}} \int_{R^n} f(x)^p \, dx,$$

where $f \wedge \mathbf{E}$ means that f is a non-negative Baire function such that

$$\int_S f \, d\sigma \geq 1 \quad \text{for every } S \in \mathbf{E}.$$

The results of Chapter I may be carried over. In particular (Theorem 2), a system $\mathbf{E}$ of k-dimensional surfaces is exceptional of order p, i.e. $\mathsf{M}_p(\mathbf{E}) = 0$, if and only if there exists a Baire function $f \geq 0$ such that $f \in L^p$ and yet $\int_S f \, d\sigma = +\infty$ for every $S \in \mathbf{E}$. (Instead of a Baire function, one might equally well consider a lower semi-continuous function; cf. the note on p. 173.)

For systems of surfaces, the only values of p which are of interest are the values $p \geq 1$. *If* $0 < p < 1$, *any system of surfaces is p-exc.* For the sake of simplicity, we restrict ourselves, by the proof, to systems of regular C^1-manifolds rather than general Lipschitz surfaces. In view of Theorem 3 (b), it suffices to prove that, when $0 < p < 1$, the system of all k-dim. surfaces which intersect the cube $Q_a = \{x \in R^n : |x_\nu| < a \text{ for } \nu = 1, \ldots, n\}$ is p-exc for any a. Choose a Baire function $\varphi(t) \geq 0$ in the interval $I_a = \{t : -a < t < a\}$ so that $\varphi \in L^p$, and yet $\int_I \varphi(t) \, dt = +\infty$ for every interval $I \subset I_a$. Define

$$f(x) = \sum_{\nu=1}^{n} \varphi(x_\nu) \quad \text{for } x \in Q_a; \qquad f(x) = 0 \quad \text{for } x \notin Q_a.$$

Then $f \in L^p$, $f \geq 0$, and $\int_S f \, d\sigma = +\infty$ for every k-dimensional regular C^1-manifold S which intersects Q_a. In fact, let $x^* \in S \cap Q_a$, and let $t \to x(t)$ be a C^1-homeomorphism of an open set $T \subset R^k$ onto an open neighbourhood X of x^* in S, with the property that the rank of the matrix $\{\partial x_i / \partial t_j\}$ equals k. If t^* denotes the point which is mapped into x^*, there exist numbers $\alpha_1, \ldots, \alpha_k$ such that

$$\frac{\partial(x_{\alpha_1}, \ldots, x_{\alpha_k})}{\partial(t_1, \ldots, t_k)} \neq 0 \quad \text{at } t^*.$$

Since this Jacobian q_α is continuous, the mapping $t \to x' = (x_{\alpha_1}, \ldots, x_{\alpha_k})$ is a C^1-homeomorphism of some open neighbourhood N of t^* onto some open set $N' \subset R^k$. Hence

$$\int_S f \, d\sigma \geq \int_X f \, d\sigma \geq \int_X \varphi(x_{\alpha_1}) \, d\sigma \geq \int_T \varphi(x_{\alpha_1}) \, | q_\alpha(t) | \, dt$$
$$\geq \int_N \varphi(x_{\alpha_1}) \, | q_\alpha(t) | \, dt = \int_{N'} \varphi(x_{\alpha_1}) \, dx' = +\infty.$$

Throughout the rest of the present paper, only values $p \geq 1$ will be considered.

THEOREM 4. *By a Lipschitz transformation of an open set $X \subset R^n$ onto an open set $Y \subset R^n$, any p-exc system of k-dim. Lipschitz surfaces contained in X is transformed into a p-exc system of k-dim. Lipschitz surfaces in Y.*

Proof. Denote by c the constant introduced in the preceding section, now associated with the Lipschitz transformation $x \to y = \varphi(x)$ of X onto Y. It was mentioned that the Jacobian

$$J = \frac{\partial (y_1, \ldots, y_n)}{\partial (x_1, \ldots, x_n)}$$

exists a.e. in X; its absolute value $|J|$ represents the volume ratio dy/dx, and it was shown that $c^{-n} \leq |J| \leq c^n$ a.e. in X. If g denotes a non-negative Baire function in Y and if $g \in L^p(Y)$, then the corresponding function f defined in X by $f(x) = g(\varphi(x))$ is a Baire function in $L^p(X)$. Moreover,

$$c^{-n} \int_X f(x)^p \, dx \leq \int_Y g(y)^p \, dy \leq c^n \int_X f(x)^p \, dx.$$

It is easily shown that φ maps any k-dim. Lipschitz surface $S_x \subset X$ onto a surface $S_y \subset Y$ of the same kind. The ratio of k-dim. surface area $\varrho(x) = d\sigma_y / d\sigma_x$ is subjected a.e. on S_x to the inequalities $c^{-k} \leq \varrho(x) \leq c^k$. This may be shown in a manner similar to the procedure by the proof of (5), §1. Hence,

$$c^{-k} \int_{S_x} f \, d\sigma_x \leq \int_{S_y} g \, d\sigma_y \leq c^k \int_{S_x} f \, d\sigma_x.$$

If $\mathbf{E}_x$ denotes a system of k-dim. Lipschitz surfaces in X and $\mathbf{E}_y$ the corresponding system in Y, then it follows easily that

$$c^{-kp-n} \mathsf{M}_p(\mathbf{E}_x) \leq \mathsf{M}_p(\mathbf{E}_y) \leq c^{kp+n} \mathsf{M}_p(\mathbf{E}_x).$$

This implies, in particular, the statement of the theorem.

A *Lipschitz family of k-dimensional surfaces* will be defined in the following way. Let T denote a non-void open set in R^n with points $t=(t_1, \ldots, t_n)$. Writing $t'=(t_{k+1}, \ldots, t_n)$, we consider the orthogonal projection of T onto the $(n-k)$-dimensional t'-plane. This projection T' is an open set in the t'-plane. For every $t' \in T'$, the set of points $t \in T$ whose projection is t', is a k-dim. plane Lipschitz surface $T_{t'}$. By a Lipschitz transformation $t \to x = \varphi(t)$ of T onto some (open) set $X \subset R^n$, the family $\mathbf{E}$ of these plane surfaces $T_{t'}$ is transformed into a family $\mathbf{F}$ of Lipschitz surfaces $S_{t'} = \varphi(T_{t'})$. Such a family will be called a Lipschitz family of k-dim. surfaces in R^n. Restricting the parameter point $t'=(t_{k+1}, \ldots, t_n)$ to some set $E' \subset T'$, we obtain a system of k-dim. Lipschitz surfaces $S_{t'}$. Let $p \geq 1$. *In order that this restricted system be p-exc, it is sufficient and, provided* $p=1$ *or* $m_n(X) < \infty$, *necessary that* $\bar{m}_{n-k}(E')=0$. By virtue of Theorem 4 we may, in fact, just as well consider the system of plane surfaces $T_{t'}$, $t' \in E'$, instead of the surfaces $S_{t'}$. If $\bar{m}_{n-k}(E')=0$, choose a Borel set A' so that $E' \subset A' \subset T'$ and $m_{n-k}(A')=0$. Write $f(t)=+\infty$ if $t' \in A'$ and $f(t)=0$ otherwise. Then $f \in L^p$ for any p, and $\int_{T_{t'}} f \, d\sigma = +\infty$ for any $t' \in E'$. Conversely, let $p=1$, or let $m_n(X) < \infty$ and hence $m_n(T) < \infty$; and assume that the system of plane surfaces $T_{t'}$, $t' \in E'$, is p-exc. If f has the properties stated in Theorem 2, then so has the function g defined by $g(t)=f(t)$ for $t \in T$, $g(t)=0$ otherwise. It follows that $g \in L^1(R^n)$. In view of Fubini's theorem, g is integrable over $T_{t'}$ for a.e. $t' \in T'$.

A Lipschitz family of surfaces is a very special system of surfaces. In the following sections more interesting systems of surfaces will be considered. It will appear that the notion: a p-exc system of k-dim. Lipschitz surfaces, depends effectively on the value of $p \geq 1$.

3. The system of all surfaces intersecting a given point set

We denote by $\mathbf{S}^k(E)$ the system of all k-dimensional Lipschitz surfaces(1) which intersect a given non-void set $E \subset R^n$. According to a previous remark, $\mathbf{S}^k(E)$ is p-exc for any $p<1$, irrespective of the choice of E. For $p \geq 1$, the problem whether $\mathbf{S}^k(E)$ is p-exc depends, besides on E, largely on the number kp, as shown by Theorem 8. We begin by showing that $\mathbf{S}^k(E)$ is *not* p-exc when $kp > n$. This is implied by the following theorem.

(1) The results of the present section (Theorems 5–8) would remain valid even if only very regular k-dim. surfaces were considered (e.g. connected analytic manifolds). In fact, by the proofs of the necessity parts (of Theorems 5 and 6) only k-dim. circular disks are considered. For a comment in the opposite direction concerning the case $k=1$, see the note on p. 186.

THEOREM 5. *The system of all k-dimensional Lipschitz surfaces which pass through a given point of R^n, is exceptional of order p if, and only if, $kp \leq n$.*

Proof. Let the given point be the origin 0, and write $|x| = r$. If $kp < n$, define $f(x) = r^{-k}$ for $r < 1$, $f(x) = 0$ for $r \geq 1$. It follows that $f \in L^p$ since $\int_0^1 r^{-kp} r^{n-1} dr < \infty$ when $kp < n$. If a k-dim. Lipschitz surface S passes through 0, then so does a Lipschitz image $X \subset S$ of some open set $T \subset R^k$. Using the notations introduced in § 1, we may assume that $x = 0$ corresponds to $t = 0$. Then $|x| \leq c|t|$, and hence, in view of (5), § 1,

$$\int_X f\, d\sigma = \int_T f(x(t))\, q(t)\, dt \geq c^{-2k} \int_T |t|^{-k} dt = +\infty.$$

Next, if $kp = n$, take $f(x) = r^{-k} (\log (2/r))^{-\alpha}$ for $r < 1$, and $f(x) = 0$ for $r \geq 1$. If $k/n < \alpha \leq 1$, then $f \in L^p$, and $\int_S f\, d\sigma = +\infty$ for any Lipschitz surface S passing through 0. Finally, in the case $kp > n$, we may exhibit a system of plane k-dim. surfaces through 0 which is *not* p-exc. We shall use the following integral formula for the mean value of the integrals of a Baire function $f(x)$ over all k-dim. planes L passing through the origin 0 in R^n:

$$\int_{\mathbf{L}^k} \left(\int_L f(x)\, d\sigma(x) \right) d\mu(L) = \frac{\omega_k}{\omega_n} \int_{R^n} |x|^{k-n} f(x)\, dx. \tag{1}$$

Here $\mathbf{L}^k$ denotes the system of all k-dim. *linear subspaces* (= "planes" through 0) in R^n, and μ is a certain measure defined on a σ-field of subsystems of $\mathbf{L}^k$ and invariant under orthogonal transformations of the space R^n (with 0 as a fixed point). The formula holds in the sense that, if one side of (1) is defined, then so is the other, and they are equal. This is the case, in particular, if $f \geq 0$. The measure μ is well known in an explicit form from integral geometry (cf. Herglotz [17] and Blaschke [4]), and a proof of (1) may be based upon this explicit expression. An alternative procedure is employed in a note [15], based on the theory of invariant integration in compact topological groups. Returning to the case $kp > n$ of Theorem 5, we consider for any plane $L \in \mathbf{L}^k$ the intersection $S = L \cap B_1$ with the unit ball B_1 in R^n. The system of all these special k-dim. Lipschitz surfaces S is not exceptional of order p when $kp > n$. In fact, if g is a non-negative Baire function in L^p, we may apply (1) to the function f defined by $f(x) = g(x)$ for $x \in B_1$, $f(x) = 0$ otherwise:

$$\int_{\mathbf{L}^k} \left(\int_{L \cap B_1} g(x)\, d\sigma(x) \right) d\mu(L) = \frac{\omega_k}{\omega_n} \int_{B_1} |x|^{k-n} g(x)\, dx. \tag{2}$$

The integral on the right is finite in view of Hölder's inequality since $g \in L^p(B_1)$, and $|x|^{k-n} \in L^{p/(p-1)}(B_1)$ when $kp > n$; in fact,

$$\int_0^1 r^{(k-n)\frac{p}{p-1}} r^{n-1} dr = \int_0^1 r^{\frac{kp-n}{p-1}-1} dr < \infty.$$

It follows that $\displaystyle\int_{L \cap B_1} g(x)\, d\sigma(x) < \infty$ for μ-a.e. $L \in \mathbf{L}^k$.

For an arbitrary (non-void) set $E \subset R^n$, the question whether or not $\mathrm{S}^k(E)$ is p-exc is connected with potential theory, as shown by the following theorem.

THEOREM 6. *Let $p \geq 1$ and $kp \leq n$. In order that the system $\mathrm{S}^k(E)$ of all k-dimensional Lipschitz surfaces which intersect a given set $E \subset R^n$ be exceptional of order p, it is necessary and, when $p > 1$, sufficient that there exist a function $f \in L^p(R^n)$, $f \geq 0$, whose potential of order k*

$$U_k^f(x) = \int_{R^n} |x-y|^{k-n} f(y)\, dy$$

equals $+\infty$ for every $x \in E$, without being identically infinite. For $p = 1$, the stated condition is sufficient under the additional assumption that $f \in Z$, i.e.,

$$\int_{R^n} f(x) \log^+ f(x)\, dx < \infty.$$

Remark. Let $f \geq 0$ be *locally integrable* in R^n (i.e., integrable over bounded sets). In order that the potential U_α^f of order α, $0 < \alpha < n$, of f be *not identically infinite*, it is necessary and sufficient that

$$\int_{R^n} (1 + |x|)^{\alpha-n} f(x)\, dx < \infty,$$

or, equivalently, that $\displaystyle\int_{B_a'} |y|^{\alpha-n} f(x-y)\, dy < \infty$ for some, and hence for any, pair (a, x) where $0 < a < \infty$ and $x \in R^n$. When one of these equivalent conditions is fulfilled, U_α^f is locally integrable, in particular $U_\alpha^f(x) < \infty$ a.e. in R^n. An application of Hölder's inequality shows that the above condition is fulfilled if $f \in L^p$ provided $p \geq 1$ and $\alpha p < n$. Hence, in the formulation of Theorem 6, the words "without being identically infinite" could be dispensed with, except in the case $kp = n$.

Proof of Theorem 6. The proof of the *necessity* of the stated condition is again based on the integral formula (2). For an arbitrary fixed $x \in R^n$, write $g(y) = f(x-y)$, and apply (2):

$$\int_{\mathbf{L}^k} \left(\int_{L \cap B_1} f(x-y)\, d\sigma(y) \right) d\mu(L) = \frac{\omega_k}{\omega_n} \int_{B_1} |y|^{k-n} f(x-y)\, dy.$$

Now, if $\mathbf{S}^k(E)$ is p-exc, there exists a Baire function $f \in L^p$, $f \geq 0$, such that

$$\int_S f\, d\sigma = +\infty$$

for every $S \in \mathbf{S}^k(E)$, in particular for every k-dim. "circle" $S = L \cap B_1$, $L \in \mathbf{L}^k$, and hence

$$\int_{L \cap B_1} f(x-y)\, d\sigma(y) = +\infty \quad \text{when } L \in \mathbf{L}^k, \quad x \in E.$$

Combining these two formulae, we infer that $\int_{B_1} |y|^{k-n} f(x-y)\, dy = +\infty$ for every $x \in E$. If $kp < n$, the function f has the properties stated in the theorem. In fact, $f \in L^p$, and $U_k^f(x) = +\infty$ when $x \in E$ since

$$U_k^f(x) \geq \int_{B_1} |y|^{k-n} f(x-y)\, dy;$$

and, finally, $U_k^f \not\equiv \infty$ as it was mentioned at the end of the above remark. If $kp = n$, the function f may be replaced by

$$f_1(x) = (1 + |x|)^{-\alpha} f(x) \quad (\alpha > 0).$$

Again, $f_1 \in L^p$, and $U_k^{f_1}(x) = +\infty$ for every $x \in E$ since $f_1(x-y) \geq (2 + |x|)^{-\alpha} f(x-y)$ when $y \in B_1$; and, finally, $U_k^{f_1} \not\equiv \infty$ since

$$\int_{R^n} (1 + |x|)^{k-n} f_1(x)\, dx = \int_{R^n} (1 + |x|)^{k-n-\alpha} f(x)\, dx < \infty$$

in view of Hölder's inequality.

The proof of the *sufficiency* is based on the theory of singular integrals (Hilbert transforms) in R^n as developed by Calderon and Zygmund [6]. We begin with a simple lemma concerning the convolution $\varphi * f$ of two functions. By *local mean convergence* is understood mean convergence over every bounded subset; in particular the functions in question must be locally integrable.

LEMMA 2. *Let* $p \geq 1$. (a) *If* $\varphi \in L^1$ *and* $f \in L^p$, *then* $\varphi * f \in L^p$, *and*

$$\| \varphi * f \|_p \leq \| \varphi \|_1 \, \| f \|_p.$$

(b) *If* $\varphi_\varepsilon \to \varphi$ *in the mean of order* 1, *and if* $f \in L^p$, *then* $\varphi_\varepsilon * f \to \varphi * f$ *in the mean of order* p.

(c) *If* $\varphi_\varepsilon \to \varphi$ *locally in the mean of order* 1, *and if* $f \in L^p$ *and* f *vanishes outside some bounded set, then* $\varphi_\varepsilon * f \to \varphi * f$ *locally in the mean of order* p.

Proof. It follows from Hölder's inequality that

$$|(f*\varphi)(x)|^p \le \left(\int |f(x-y)|\cdot|\varphi(y)|^{1/p}|\varphi(y)|^{1-1/p}\,dy\right)^p$$
$$\le \int |f(x-y)|^p|\varphi(y)|\,dy\cdot\left(\int|\varphi(y)|\,dy\right)^{p-1}.$$

Hence
$$\int |f*\varphi|^p\,dx \le \iint |f(x-y)|^p|\varphi(y)|\,dx\,dy\cdot\left(\int|\varphi(y)|\,dy\right)^{p-1}$$
$$= \int |f(x)|^p\,dx\left(\int|\varphi(y)|\,dy\right)^p.$$

This proves statement (a). Statements (b) and (c) are easily derived.

Consider now the kernel $\varphi(x)=|x|^{k-n}$ and, as an approximation, the function

$$\varphi^{(\varepsilon)}(x) = (|x|^2+\varepsilon^2)^{\frac{k-n}{2}}; \quad x \in R^n,\ \varepsilon \text{ real}. \tag{3}$$

The function $\varphi^{(\varepsilon)}(x)$ has continuous partial derivatives of all orders with respect to the variables $x_1, \ldots, x_n, \varepsilon$ except at the point $(x, \varepsilon) = (0, 0)$. By an arbitrary derivation D_α with respect to $x_1, \ldots, x_n$ one obtains

$$D_\alpha \varphi^{(\varepsilon)}(x) = H_\alpha(x,\varepsilon)\,(|x|^2+\varepsilon^2)^{\frac{k-n}{2}-|\alpha|}, \tag{4}$$

where $H_\alpha(x, \varepsilon)$ is a homogeneous polynomial in $x_1, \ldots, x_n, \varepsilon$ of degree $|\alpha|$. Hence there is a number C_1 independent of (x, ε) such that

$$|H_\alpha(x,\varepsilon)| \le C_1\,(|x|^2+\varepsilon^2)^{\frac{|\alpha|}{2}}, \tag{5}$$

and
$$|D_\alpha\varphi^{(\varepsilon)}(x)| \le C_1\,(|x|^2+\varepsilon^2)^{\frac{k-|\alpha|-n}{2}} \le C_1|x|^{k-|\alpha|-n}. \tag{6}$$

The continuity of $D_\alpha\varphi^{(\varepsilon)}$ implies, in particular, that

$$D_\alpha\varphi^{(\varepsilon)}(x) \to D_\alpha\varphi(x) \quad \text{as } \varepsilon\to 0, \tag{7}$$

pointwise for $x \neq 0$. If $|\alpha| < k$, the same limit relation (7) holds in the sense of local mean convergence of order 1. This follows easily from the theorem of Lebesgue on the interchange of integration and passage to a limit when an integrable majorant exists. In fact, the majorant $C_1|x|^{k-|\alpha|-n}$ obtained in (6) is locally integrable when $|\alpha| < k$.

Now, let $f \ge 0$, $f \in L^p$, where $p \ge 1$, and assume that $f(x) = 0$ outside some bounded set. Consider the convolutions

$$u(x)=\int_{R^n}\varphi(x-y)\,f(y)\,dy=U_k^f(x); \qquad u^{(\varepsilon)}(x)=\int_{R^n}\varphi^{(\varepsilon)}(x-y)\,f(y)\,dy. \tag{8}$$

For $\varepsilon \neq 0$, say $\varepsilon>0$, $u^{(\varepsilon)}\in C^\infty(R^n)$, and $D_\alpha u^{(\varepsilon)}=(D_\alpha\varphi^{(\varepsilon)})*f$ for any α. Moreover

$$\lim_{\varepsilon\to 0} u^{(\varepsilon)}(x)=u(x) \tag{9}$$

pointwise everywhere in R^n since the integration and the monotone limit process may be interchanged. In view of the above remarks and Lemma 2(c),

$$D_\alpha u^{(\varepsilon)}\to(D_\alpha\varphi)*f \tag{10}$$

locally in the mean of order p, as $\varepsilon\to 0$, provided $|\alpha|<k$.

In the case $|\alpha|=k$, the above crude method cannot be applied since $D_\alpha\varphi$ fails to be locally integrable. Nevertheless, it may be proved that, even in this case, $D_\alpha u^{(\varepsilon)}$ converges (even "globally") in the mean of order p, as $\varepsilon\to 0$, provided $p>1$. The corresponding statement for $p=1$ does not hold in general. However, under the additional assumption $f\in Z$, $D_\alpha u^{(\varepsilon)}$ does converge in the mean of order 1 over every set of finite measure, in particular over every bounded set, i.e., locally. These statements are formulated as a separate lemma (Lemma 3) given below.

The sufficiency of the condition stated in Theorem 6 may now be proved as follows. Since any subset of R^n may be covered by a countable family of bounded sets, we may assume, in view of Theorem 3 (b), that the given set E is bounded. Now, let $f_1\in L^p$, $f_1\geq 0$, $U_k^{f_1}=+\infty$ in E, and $U_k^{f_1}\not\equiv\infty$. (If $p=1$, it is assumed, in addition, that $f_1\in Z$.) Choose a radius a so that $E\subset B_a$ and put $f(x)=f_1(x)$ for $x\in B_a$, $f(x)=0$ for $x\in B_a'$. Then f has, likewise, the properties stated in the sufficiency part of Theorem 6. In particular,(1)

$$u=U_k^f=\varphi*f=+\infty \quad \text{everywhere in } E. \tag{11}$$

With this function f we form, besides the potential u, the "approximate potentials" $u^{(\varepsilon)}=\varphi^{(\varepsilon)}*f$ as in (8). According to the above discussion, any derivative $D_\alpha u^{(\varepsilon)}$ of order $|\alpha|\leq k$ converges locally in the mean of order p as $\varepsilon\to 0$. Writing $g_\alpha^{(\varepsilon)}(x)=D_\alpha u^{(\varepsilon)}(x)$ for $x\in B_a$, $g_\alpha^{(\varepsilon)}(x)=0$ for $x\in B_a'$, we infer that the functions $g_\alpha^{(\varepsilon)}$ converge in the mean

(1) It suffices to prove that $U_k^{f_1-f}(x)<\infty$ when $|x|<a$. Clearly, $B_{a-|x|}\subset B_a(x)$ and $B'_{a-|x|}\supset B'_a(x)$. Hence

$$U_k^{f_1-f}(x)=\int_{B'_a(x)}|y|^{k-n}f_1(x-y)\,dy\leq\int_{B'_{a-|x|}}|y|^{k-n}f_1(x-y)\,dy<\infty$$

because $U_k^{f_1}\not\equiv\infty$. (Cf. the remark following the formulation of Theorem 6.)

of order p over R^n as $\varepsilon \to 0$. In view of Theorem 3, (b) and (f), there is a sequence $\{\varepsilon_\nu\}$ tending to 0 such that each derivative $D_\alpha u^{(\varepsilon_\nu)}$, $0 \le |\alpha| \le k$, converges in the mean of order 1 over $S \cap B_a$ for p-a.e. k-dim. Lipschitz surface S. We conclude that the system $\mathfrak{S}^k(E)$ is p-exc. because the derivatives $D_\alpha u^{(\varepsilon_\nu)}$ do *not* all converge in the mean of order 1 over $S \cap B_a$ if $S \in \mathfrak{S}^k(E)$, that is, if S intersects E. In fact, let $x^* \in S \cap E$ and choose a k-dim. Lipschitz image X (of an open set) so that $x^* \in X \subset S \cap B_a$. (Recall that $E \subset B_a$.) If each of the sequences $\{D_\alpha u^{(\varepsilon_\nu)}\}$ were mean-convergent over $S \cap B_a$, and hence over X, it would follow from Lemma 1, § 1, that the numerical sequence $\{u^{(\varepsilon_\nu)}(x^*)\}$ would be bounded, which is impossible since $x^* \in E$ and hence, in view of (9) and (11),

$$\lim_{\nu \to \infty} u^{(\varepsilon_\nu)}(x^*) = u(x^*) = +\infty.$$

Except for Lemma 3, the proof of Theorem 6 is now completed.

LEMMA 3. *Let k be a positive integer. Denote by*

$$K^{(\varepsilon)} = D_\alpha \varphi^{(\varepsilon)} = \frac{\partial^k \varphi^{(\varepsilon)}}{\partial x_{\alpha_1} \ldots \partial x_{\alpha_k}}$$

an arbitrary derivative of order $|\alpha| = k$ of the function

$$\varphi^{(\varepsilon)}(x) = (|x|^2 + \varepsilon^2)^{\frac{k-n}{2}}, \quad x \in R^n.$$

Consider, for $\varepsilon > 0$, the convolution integral

$$g^{(\varepsilon)}(x) = \int_{R^n} K^{(\varepsilon)}(x-y) f(y)\, dy.$$

(a) *If $f \in L^p$, $1 < p < \infty$, then $g^{(\varepsilon)}$ converges in the mean of order p over R^n as $\varepsilon \to 0$.*

(b) *If $f \in Z$, i.e., f is measurable and $\int_{R^n} |f(x)| \log^+ |f(x)|\, dx < \infty$; and if moreover $f(x) = 0$ outside some bounded set, then $g^{(\varepsilon)}$ converges in the mean of order 1 over every subset of R^n of finite measure as $\varepsilon \to 0$.*

This lemma may be derived from the theory of singular integrals in R^n as developed by Calderon and Zygmund [6]. The kernel

$$K(x) = D_\alpha \varphi(x) = \frac{H_\alpha(x, 0)}{|x|^{n+k}} \tag{12}$$

fulfills the requirements listed on p. 89 of [6]. The "smoothness" condition is satisfied with $\omega(t) = t$ since $|x|^n K(x)$ has continuous partial derivatives for $x \neq 0$. The decisive condition

$$(M=)\int_{\Omega} K(x)\,d\omega(x)=\int_{\Omega} H_\alpha(x,0)\,d\omega(x)=0 \tag{13}$$

is likewise satisfied by the kernel (12). The following simple consideration may easily be turned into a formal proof of this fact. Denote by G the compact group of all orthogonal substitutions in R^n (with 0 as a fixed point). In a new coordinate system in R^n, the transition to which is given by an orthogonal substitution $g \in G$, the differential operator D_α is transformed into some linear homogeneous differential operator $P_g(D)$ of the same order $|\alpha|=k$. (Here $P_g(D)$ is obtained from a polynomial $P_g(\zeta)=P_g(\zeta_1,\ldots,\zeta_n)$ by the substitutions $\zeta_\nu=\partial/\partial x_\nu$, $\nu=1,\ldots,n$.) Since $\varphi(x)=|x|^{k-n}$ depends on $|x|$ only, i.e., $\varphi(gx)=\varphi(x)$, it follows that

$$\int_{\Omega} P_g(D)\,\varphi(x)\,d\omega(x)=M \quad \text{for every } g\in G. \tag{14}$$

Now consider the differential operator $P(D)$ whose corresponding polynomial $P(\zeta)$ is the mean value $\int_G P_g(\zeta)\,dg$, where dg refers to the Haar measure on G. It follows from (14) that

$$\int_{\Omega} P(D)\,\varphi(x)\,d\omega(x)=M. \tag{15}$$

Being invariant under G, the polynomial $P(\zeta)$ depends only on $|\zeta|=(\zeta_1^2+\cdots+\zeta_n^2)^{\frac{1}{2}}$:

$$P(\zeta_1,\ldots,\zeta_n)=P(|\zeta|,\ 0,\ldots,0).$$

As $P(\zeta)$ is homogeneous of degree $|\alpha|=k$, we infer that $P(\zeta)\equiv 0$ if k is odd, whereas $P(\zeta)$ equals a constant times $(\zeta_1^2+\cdots+\zeta_n^2)^{k/2}$ if k is even. In view of (15), the desired result $M=0$ is now obvious for odd k, and for even k it follows from the well-known identity

$$\Delta^m(|x|^{2m-n})=0 \quad \text{in } R^n-\{0\},$$

applied to $m=k/2$.

For any $\varepsilon>0$ write[1]

$$\tilde{f}_\varepsilon(x)=\int_{B_\varepsilon'(x)} K(x-y)\,f(y)\,dy. \tag{16}$$

In view of [6], Theorem 7, p. 108, $\tilde{f}_\varepsilon$ converges in the mean of order p as $\varepsilon\to 0$, provided f satisfies the assumptions stated in our Lemma 3. (In the case (b), the mean convergence applies to sets of finite measure only.) It may be noted that,

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[1] In CALDERON and ZYGMUND [6] our $\tilde{f}_\varepsilon$ is denoted by $\tilde{f}_\lambda$ with $\lambda=1/\varepsilon$. Moreover, our $B_\varepsilon(x)$ and $B_\varepsilon'(x)$ are called $\Gamma_\varepsilon(x)$ and $\Gamma_\varepsilon'(x)$, respectively.

according to Theorems 1 and 2, p. 116 and p. 118 in [6], the limit $\tilde{f}$ of $\tilde{f}_\varepsilon$ likewise exists pointwise a.e. in R^n, and thus $\tilde{f}(x)$ equals the Hilbert transform of f given a.e. by the singular integral

$$\tilde{f}(x) = \int_{R^n} K(x-y) f(y)\, dy,$$

interpreted as the Cauchy principal value.

To complete the proof of Lemma 3, it remains to be shown that $g^{(\varepsilon)} - \tilde{f}_\varepsilon$ converges in the mean of order p over R^n when $f \in L^p$, $1 \leq p < \infty$. This may be done in the manner described for the case $k=1$, $n=2$ on p. 125 of [6]:

$$g^{(\varepsilon)}(x) - \tilde{f}_\varepsilon(x) = \int_{R^n} K^{(\varepsilon)}(x-y) f(y)\, dy - \int_{B_\varepsilon'(x)} K(x-y) f(y)\, dy = \frac{1}{\varepsilon^n} \int_{R^n} N\left(\frac{x-y}{\varepsilon}\right) f(y)\, dy,$$

where
$$N(x) = \begin{cases} K^{(1)}(x) - K(x) & \text{for } |x| \geq 1, \\ K^{(1)}(x) & \text{for } |x| < 1. \end{cases}$$

If $|x| = r \geq 1$, it follows from (4) and (12) that

$$\begin{aligned}(1+r^2)^{\frac{n+k}{2}} N(x) &= H_\alpha(x, 1) - \left(1 + \frac{1}{r^2}\right)^{\frac{n+k}{2}} H_\alpha(x, 0) \\ &= H_\alpha(x, 1) - H_\alpha(x, 0) - \left(\left(1 + \frac{1}{r^2}\right)^{\frac{n+k}{2}} - 1\right) H_\alpha(x, 0).\end{aligned} \tag{17}$$

Now, $\varphi^{(-\varepsilon)}(x) = \varphi^{(\varepsilon)}(x)$, and hence $H_\alpha(x, -\varepsilon) = H_\alpha(x, \varepsilon)$, so that only even powers of ε occur in the homogeneous polynomial $H_\alpha(x, \varepsilon)$. Thus the degree of the polynomial $H_\alpha(x, 1) - H_\alpha(x, 0)$ is $\leq |\alpha| - 2 = k - 2$, and there is a constant C_2 such that

$$|H_\alpha(x, 1) - H_\alpha(x, 0)| \leq C_2 r^{k-2} \quad \text{for } r \geq 1.$$

Moreover,
$$\left(1 + \frac{1}{r^2}\right)^{\frac{n+k}{2}} - 1 \leq C_3 r^{-2} \quad \text{for } r \geq 1,$$

C_3 denoting a suitable constant. Finally, it follows from (5) that $|H_\alpha(x, 0)| \leq C_1 r^k$. In view of these three inequalities, we infer from (17) that

$$r^{n+k} |N(x)| \leq (1+r^2)^{\frac{n+k}{2}} |N(x)| \leq (C_2 + C_1 C_3) r^{k-2},$$

i.e.,
$$|N(x)| \leq C_4 r^{-2-n} \quad \text{for } r \geq 1. \tag{18}$$

On the other hand, it follows from (4) and (5) that

$$|N(x)| = |H_\alpha(x,1)| \cdot (1+r^2)^{-\frac{n+k}{2}} \leq C_1 (1+r^2)^{-\frac{n}{2}} \quad \text{for } r<1. \tag{19}$$

Defining $C_5 = \max(C_4, 2^{-n/2} C_1)$, and

$$\psi(r) = \begin{cases} C_5 \left(\frac{2}{1+r^2}\right)^{\frac{n}{2}} & \text{for } r<1 \\ C_5\, r^{-2-n} & \text{for } r \geq 1, \end{cases}$$

we obtain from (18) and (19) the inequality

$$|N(x)| \leq \psi(|x|) \quad \text{for } x \in R^n.$$

Since $\psi(r)$ is a decreasing function of r, and $\int\limits_{R^n} \psi(|x|)\,dx < \infty$, it follows from Lemma 2, p. 113 in [6], that

$$g^{(\varepsilon)}(x) - \bar{f}_\varepsilon(x) = \frac{1}{\varepsilon^n} \int\limits_{R^n} N\left(\frac{x-y}{\varepsilon}\right) f(y)\,dy \to A \cdot f(x) \tag{20}$$

in the mean of order p over R^n as $\varepsilon \to 0$. Here $A = \int\limits_{R^n} N(y)\,dy$ is a constant (depending on n, k, and α).(1) It may be noted that (20) holds likewise in the sense of pointwise convergence a.e. in R^n. This follows from Lemma 1, p. 111 in [6].

This completes the proof of Lemma 3 and hence of Theorem 6. In order to obtain a further characterization of the sets E such that $S^k(E)$ is p-exc, one may apply the following theorem from potential theory.

THEOREM A. *Let $p \geq 1$ and $0 < \alpha p \leq n$. In order that there exist a function $f \geq 0$, $f \in L^p(R^n)$, such that $U^f_\alpha(x) = +\infty$ for every $x \in E$, but $U^f_\alpha \not\equiv \infty$, it is*

(a) *necessary that* $\begin{cases} \operatorname{cap}_{\alpha p} E = 0, \text{ provided } 1 \leq p \leq 2. \\ \operatorname{cap}_{\alpha p - \varepsilon} E = 0 \text{ for every } \varepsilon > 0, \text{ provided } p > 2; \end{cases}$

(b) *sufficient that* $\begin{cases} \operatorname{cap}_{\alpha p} E = 0, \text{ provided } p \geq 2. \\ \operatorname{cap}_{\alpha p + \varepsilon} E = 0 \text{ for some } \varepsilon > 0, \text{ provided } 1 \leq p < 2. \end{cases}$

Remark. When $\alpha p < n$, the requirement that $U^f_\alpha \not\equiv \infty$ is always fulfilled for $f \in L^p$, as pointed out in the remark following Theorem 6. The capacity of order α in R^n refers to the kernel $|x|^{\alpha-n}$ when $0 < \alpha < n$, and to the kernel $\log(1/|x|)$ when $\alpha = n$. By capacity is meant exterior capacity. Note that the conditions (a) and (b) are identical in the case $p = 2$.

(1) It follows from (13) that $A = \lim\limits_{a\to\infty} \int\limits_{B_a} K^{(1)}(x)\,dx$.

Part (a) was established for the case $n=1$, $\alpha p<n$, by Littlewood and du Plessis [26], who also showed that the ε cannot be dispensed with when $p>2$. Using their method, J. Deny [9] treated part (a) for general n in the case $\alpha=1$, $1\leq p\leq 2$, $\alpha p<n$. Part (b) is related to a result of H. Cartan [8], Théorème 3 bis, p. 96. A complete treatment of both parts (a) and (b) of Theorem A will appear in [16].

Combining Theorem 6 and Theorem A, we obtain

THEOREM 7. *Let $p\geq 1$ and $kp\leq n$. In order that the system $\mathfrak{S}^k(E)$ of all k-dimensional Lipschitz surfaces which intersect a given set $E\subset R^n$ be exceptional of order p, it is*

(a) *necessary that* $\begin{cases} \operatorname{cap}_{kp} E=0, \text{ provided } 1\leq p\leq 2, \\ \operatorname{cap}_{kp-\varepsilon} E=0 \text{ for every } \varepsilon>0, \text{ provided } p>2; \end{cases}$

(b) *sufficient that* $\begin{cases} \operatorname{cap}_{kp} E=0, \text{ provided } p\geq 2, \\ \operatorname{cap}_{kp+\varepsilon} E=0 \text{ for some } \varepsilon>0, \text{ provided } 1\leq p<2. \end{cases}$

Thus, in the case $p=2$, the condition $\operatorname{cap}_{2k} E=0$ is both necessary and sufficient.

In view of Theorem 7, it is convenient to make use of the concept of capacitary dimension as introduced by Pólya and Szegö [30]. Corresponding to an arbitrary set $E\subset R^n$, there is exactly one real number α, $0\leq\alpha\leq n$, such that

$$\text{(a)}\quad \operatorname{cap}_{n-\alpha-\varepsilon} E=0 \quad \text{for every } \varepsilon>0, \text{ and}$$

$$\text{(b)}\quad \operatorname{cap}_{n-\alpha+\varepsilon} E>0 \quad \text{for every } \varepsilon>0.$$

(When $\alpha=0$, only part (a) applies, and when $\alpha=n$, only part (b) applies.) This number α is called the capacitary dimension of E. In view of certain relations between capacities and Hausdorff measures established for closed sets by Myrberg [27] and Frostman [14], and for arbitrary sets by Kametani [22, 23] and Carleson [7], the capacitary dimension is identical with the Hausdorff dimension. We denote it by "dim" and write co-dim $E=n-\dim E$. From Theorem 7 together with Theorem 5 is thus obtained the following weaker result, in which it is merely assumed that $p\geq 1$.

THEOREM 8. *If $kp<$ co-dim E, then $\mathfrak{S}^k(E)$ is exceptional of order p. If $kp>$ co-dim E, then $\mathfrak{S}^k(E)$ is not exceptional of order p unless E, and hence $\mathfrak{S}^k(E)$, is void.*

4. The case $p=2$

The extremal length as introduced by Beurling refers to the most important case in which the order p equals 2. This is also the only case in which Theorem 7 gives a condition at the same time necessary and sufficient in order that $\mathfrak{S}^k(E)$ be

p-exc, the condition being: $\mathrm{cap}_{2k} E = 0$. For systems of curves ($k = 1$), the capacity in question is then of order 2, and this is the classical harmonic capacity: logarithmic capacity in the plane, Newton capacity in R^3, and capacity with respect to the kernel $|x|^{2-n}$ for general $n \geq 3$. As pointed out in [1], the extremal length λ_2 for a system of plane curves is a conformal invariant.

It is known that there is in the case $p = 2$ an even closer connection between extremal length and capacity than expressed in Theorem 7. Since no complete proof seems to have been published(¹), we shall give a detailed treatment of a typical case.

THEOREM 9. *Let K be an arbitrary compact subset of R^n, $n \geq 3$, and G the unbounded component of $R^n - K$. Denote by $\mathbf{C} = \mathbf{C}(\infty, K)$ the system of all curves $C \subset G$ connecting the point at infinity of R^n with K, and by $\mathbf{H} = \mathbf{H}(\infty, K)$ the system of all closed hypersurfaces $H \subset G$ which separate the point at infinity from K. Then*

$$\mathsf{M}_2(\mathbf{C}) = \frac{1}{\mathsf{M}_2(\mathbf{H})} = a_n \,\mathrm{cap}_2 K.$$

Here $a^n = (n-2)\,\omega_n$ is the elementary flux in R^n, i.e. the constant in Poisson's formula and in Gauss' theorem; in particular $a_3 = 4\pi$. A *curve* C is said to connect the point (or sphere) at infinity with K if the curve has a parametric representation $t \to x(t)$, $a < t < b$, such that $|x(t)| \to \infty$ as $t \to a$ and $x(t) \to K$ as $t \to b$ (in the sense that the distance between $x(t)$ and K approaches 0 when $t \to b$). The curves considered in the theorem should be locally rectifiable, but the theorem would subsist even if only analytic curves were admitted. By a *closed hypersurface* is meant a compact $(n-1)$-dimensional Lipschitz surface, not necessarily connected. The theorem would remain valid if only analytic manifolds were considered. If $H_1, \ldots, H_N$ denote the connected components of a closed hypersurface H, and if $I_1, \ldots, I_N$ are the interior (bounded) regions determined by $H_1, \ldots, H_N$, then H is said to separate the point at infinity from K if $I_1, \ldots, I_N$ are mutually disjoint and $K \subset \bigcup_{k=1}^{N} I_k$.

By the proof we shall use certain well-known results from potential theory (see e.g.

(¹) The proof of Theorem 9 and related theorems is simple in the special case where there are no critical points for the vector field grad u defined presently, and where, in addition, K is sufficiently regular (so that the classical Dirichlet problem may be solved in the unbounded component G of $R^n - K$). HERSCH [19] treats the case where K is homeomorphic to a solid sphere in R^3. Contrary to the situation in the corresponding two-dimensional problem, critical points can, however, occur even in this simple case, e.g. if K is obtained by removing a thin "slice" from a solid torus; cf. a similar example due to J. J. GERGEN, Amer. J. Math., 52 (1930), 198–200. For $n \geq 3$, there are no critical points if K is convex (WALSH [35]).

Frostman [14]), or, equivalently, results concerning the Dirichlet problem. Corresponding to an arbitrary compact set $K \subset R^n$ of positive capacity (of order 2) there is a (unique) measure $\mu \geq 0$, supported by K (in fact, by the boundary of the unbounded component G of $R^n - K$), with the following properties:

(a) The potential $u = U_2^\mu$ of order 2 of μ is ≤ 1 everywhere in R^n.

(b) $u = 1$ everywhere in $R^n - G$ except at the irregular points (if any) of the boundary of G.

This measure μ is called the equilibrium distribution on K. Its total mass $\mu(K)$ equals $\operatorname{cap}_2 K$. The equilibrium potential $u = U_2^\mu$ is superharmonic and positive everywhere in R^n, bounded away from 0 in any bounded part of R^n, and harmonic and < 1 in G. Moreover, u is regular at infinity; in particular $\lim_{|x| \to \infty} |x|^{n-2} u(x) = \operatorname{cap}_2 K$. Finally,(1)

$$\int_G |\operatorname{grad} u|^2 \, dx = \int_H \frac{\partial u}{\partial e} \, d\sigma = a_n \, \mu(K) = a_n \operatorname{cap}_2 K \tag{1}$$

when $H \in \mathbf{H}(\infty, K)$, and e and $d\sigma$ denote the inward unit normal and the surface element on H. It is convenient to define $\operatorname{grad} u = 0$ in $R^n - G$.

Corresponding to an arbitrary compact set $K \subset R^n$ and a number $\varepsilon > 0$ there exists a compact set K^* without irregular points, such that $K^* \supset K$ and $0 < \operatorname{cap}_2 K^* < \operatorname{cap}_2 K + \varepsilon$. One may, for example, choose K^* as the set of all points within a suitably small distance from K.

Proof of Theorem 9. 1°. $\mathsf{M}_2(\mathbf{H}) \leq (a_n \operatorname{cap}_2 K)^{-1}$. We may assume that $\operatorname{cap}_2 K > 0$. If $u = U_2^\mu$ denotes the equilibrium potential associated with K, then

$$|\operatorname{grad} u| / (a_n \operatorname{cap}_2 K) \wedge \mathbf{H}$$

since, for every surface $H \in \mathbf{H}$,

$$\int_H |\operatorname{grad} u| \, d\sigma \geq \int_H \frac{\partial u}{\partial e} \, d\sigma = a_n \operatorname{cap}_2 K.$$

Hence $$\mathsf{M}_2(\mathbf{H}) \leq (a_n \operatorname{cap}_2 K)^{-2} \int |\operatorname{grad} u|^2 \, dx = (a_n \operatorname{cap}_2 K)^{-1}.$$

2°. $\mathsf{M}_2(\mathbf{C}) \leq a_n \operatorname{cap}_2 K$. Choose a compact set K^* without irregular points so that $K^* \supset K$ and $0 < \operatorname{cap}_2 K^* < \operatorname{cap}_2 K + \varepsilon$. Denote by u^* the equilibrium potential associated with K^*. Then $|\operatorname{grad} u^*| \wedge \mathbf{C}$ since, for every curve $C \in \mathbf{C}$,

(1) The validity of the Gauss-Green integral formula for a region bounded by a closed Lipschitz hypersurface follows from the results of SCHAUDER [33], Chapter III, by which only "one-sided" Lipschitz conditions are assumed. (Cf. Theorem XIX, p. 47, for the case of a connected boundary.)

$$\int_C |\operatorname{grad} u^*|\, ds \geq \int_C \operatorname{grad} u^* \cdot dx = \int_C du^* = 1.\text{(1)}$$

Hence
$$\mathsf{M}_2(\mathbf{C}) \leq \int |\operatorname{grad} u^*|^2\, dx = a_n \operatorname{cap}_2 K^*,$$

from which the desired inequality follows when $\varepsilon \to 0$.

3°. $\mathsf{M}_2(\mathbf{C}) \geq a_n \operatorname{cap}_2 K$. We may assume that $\operatorname{cap}_2 K > 0$. Let u denote the equilibrium potential associated with K, and η the (positive) minimum of the lower semi-continuous function u on K. A point $x \in G$ is called *critical* if $\operatorname{grad} u$ vanishes at x. The set of all critical points will be denoted by X, and the set of all critical values $\gamma = u(x)$, $x \in X$, will be called Γ. Since u is regular at infinity, the set X is bounded, and the critical values are therefore bounded away from 0. We denote by E_α the equipotential set $\{x \in R^n : u(x) = \alpha\}$.

LEMMA 4. *For every $\alpha \notin \Gamma$, $0 < \alpha < \eta$, the equipotential set E_α is a compact analytic $(n-1)$-dimensional manifold (not necessarily connected) separating the point at infinity from K. In particular, $E_\alpha \in \mathbf{H}$.*

Proof. Clearly E_α is a bounded subset of G since $u \to 0$ as $|x| \to \infty$, and $u \geq \eta$ in $R^n - G$. From the lower semi-continuity of u follows that the set $\{x \in R^n : u(x) \leq \alpha\}$ is closed, and it is a subset of G, in which u is continuous. Hence E_α is closed. From a classical theorem on implicit functions follows that E_α is an analytic manifold because $\operatorname{grad} u \neq 0$ on E_α. Denote by H_k an arbitrary connected component of E_α and by I_k and J_k the corresponding interior and exterior regions. The boundary of each of these regions is H_k. From the superharmonicity of u follows that $u > \alpha$ everywhere in I_k; the alternative $u \equiv \alpha$ in I_k would imply $u \equiv \alpha$ in G since G contains points of I_α. Any two of the interior regions are disjoint. In fact, if $x^0 \in I_j \cap I_k$, then $I_j \subset I_k$ since otherwise the region I_j would contain some boundary point $\xi \in H_k$ of I_k, and this is impossible because $u = \alpha$ on H_k whereas $u > \alpha$ in I_j. Similarly $I_k \subset I_j$, and hence $I_j = I_k$. It remains to be shown that every point of K (or, more generally, of $\{x \in R^n : u(x) > \alpha\}$) belongs to some I_k. Denoting as above by e the unit normal on H_k pointing into I_k, we have $\partial u / \partial e = \operatorname{grad} u \cdot e > 0$. (Clearly, $\partial u/\partial e \geq 0$ since $u > \alpha$ in I_k, and the sign of equality cannot occur since $|\partial u/\partial e| = |\operatorname{grad} u| \neq 0$ on E_α because $\alpha \notin \Gamma$.) It follows that $u < \alpha$ in J_k everywhere sufficiently close to H_k. Now, let $u(x^0) > \alpha$, and consider a half-line L connecting x^0 with the point at infinity. Since

(1) Throughout the rest of the article, we reserve the dot "$\cdot$" to scalar multiplication of vectors. In this manner, the "dx" in line integrals will not be misunderstood as a volume element.

$u(x) \to 0$ as $|x| \to \infty$, the closed set $\{x \in L : u(x) \leq \alpha\}$ is not void, and hence it contains a point ξ closest to x^0. Since $u(\xi) \leq \alpha < \eta$, ξ must belong to G, where u is continuous. Thus $u(\xi) = \alpha$, and $\xi \in H_k$ for some k. The segment on L between x^0 and ξ, the latter point being excluded, must belong to I_k or to J_k since $u > \alpha$ on the segment. But the segment cannot belong to J_k since it was shown above that $u < \alpha$ in J_k everywhere sufficiently close to H_k. Thus it has been proved that the interior regions I_k constitute the totality of connected components of the open set $\{x \in R^n : u(x) > \alpha\}$. (Hence $\{x \in R^n : u(x) < \alpha\} = \bigcap_k J_k$.)

If we interpret the vector field $v = \operatorname{grad} u$ as a stationary velocity field in G, the orthogonal trajectories of the sets E_α are the *lines of flow*. For any given point $\xi \in G$, the initial value problem

$$\frac{dx}{dt} = v(x); \quad x = \xi \text{ for } t = 0, \tag{2}$$

has a unique solution $x = x(\xi, t) = T_t \xi$ in G. This solution may be continued within G in some maximal open interval $\tau^-(\xi) < t < \tau^+(\xi)$, and $x(\xi, t)$ depends analytically on (ξ, t). If $\xi \in X$, then $x(\xi, t) = \xi$ for $-\infty < t < +\infty$. For fixed $\xi \in G - X$, the equation $x = x(\xi, t)$ determines an analytic curve F_ξ, called a line of flow, as t ranges over the interval $\tau^-(\xi) < t < \tau^+(\xi)$. Such a line of flow never passes through a critical point because of the uniqueness of the solution of (2). The potential u increases with t along a line of flow since

$$\frac{du}{dt} = \operatorname{grad} u \cdot \frac{dx}{dt} = |v|^2 > 0. \tag{3}$$

In particular, a line of flow has no multiple points. Moreover, the limits

$$u^+(\xi) = \lim_{t \to \tau^+(\xi)} u(x(\xi, t)); \qquad u^-(\xi) = \lim_{t \to \tau^-(\xi)} u(x(\xi, t))$$

exist.

LEMMA 5. *If $\tau^+(\xi) < +\infty$, then $x(\xi, t) \to K$ as $t \to \tau^+(\xi)$.*

Proof. Since u increases along F_ξ, $x(\xi, t)$ belongs to the bounded set

$$\{x \in G : u(x) > u(\xi)\}$$

when $t > 0$, and thus there exists at least one limit point for $x(\xi, t)$ as $t \to \tau^+(\xi)$. The function $|\operatorname{grad} u|$ is bounded in any set $G_0 \subset G$ at positive distance from K, and hence the total arc length described within G_0 by $x(\xi, t)$ for $0 < t < \tau^+(\xi)$ $(< +\infty)$, is finite. A limit point $x^0 \in G$ for $x(\xi, t)$ as $t \to \tau^+(\xi)$ must therefore be the only limit point in R^n. The finiteness of $\tau^+(\xi)$ implies, in view of the uniqueness of solutions of (2), that no critical point can be the limit of $x(\xi, t)$. Neither can a point $x^0 \in G - X$ be a limit point for $x(\xi, t)$ since then $u(x^0) = u^+(\xi)$, and the lines of flow would form, in some neighbourhood of x^0, a regular family of curves orthogonal to $E_{u^+(\xi)}$ and depending on $n-1$ parameters; and thus the line of flow F_ξ could be continued across $E_{u^+(\xi)}$, in contradiction with the definition of $u^+(\xi)$. It follows that, actually, $x(\xi, t) \to K$ as $t \to \tau^+(\xi)$.

After these preparations, the proof of the inequality $\mathsf{M}_2(\mathrm{C}) \geq a_n \operatorname{cap}_2 K$ may be completed as follows. Choose a positive number $\alpha < \eta$ smaller than any critical value of u. According to Lemma 4, the equipotential set E_α is a compact analytic $(n-1)$-dimensional manifold separating the point at infinity from K. (Incidentally, E_α is connected and homeomorphic to the sphere S^{n-1}, but we shall not use this fact.) Since there are no critical points in the exterior region $\{x \in R^n : u(x) < \alpha\}$, it follows by the argument employed in the proof of Lemma 5 (in the case $x^0 \in G - X$) that

$$|x(\xi, t)| \to \infty \quad \text{as} \quad t \to \tau^-(\xi) \tag{4}$$

for every $\xi \in E_\alpha$. (Actually, $\tau^-(\xi) = -\infty$, but this will not be needed.) The transformation $\varphi : (\xi, t) \to x(\xi, t)$ between $E_\alpha \times R^1$ and G, is analytic and one-to-one, and hence its domain $\Delta \subset E_\alpha \times R^1$ and range $D = \varphi(\Delta)$ are open. The set Δ consists of all pairs (ξ, t) with $\xi \in E_\alpha$, $\tau^-(\xi) < t < \tau^+(\xi)$, and D consists of all points $x \in G$ which may be reached in a finite time by lines of flow passing through E_α. In the open set D one may use (ξ, t) as "Gaussian coordinates". The volume element dx at a point $x = x(\xi, t)$ of D is given by

$$dx = (v \cdot e)\, d\sigma\, dt = \frac{\partial u}{\partial e}\, d\sigma\, dt,$$

where the inward unit normal e and the surface element $d\sigma$ at x refer to the surface $T_t E_\alpha$ passing through x. Since $\operatorname{div} v = \Delta u = 0$ in G, it follows from Gauss' theorem (applied to a "tube" formed by lines of flow) that the infinitesimal flux $(v \cdot e)\, d\sigma$ is constant along the tube and hence equal to the corresponding infinitesimal flux through the surface element $d\sigma_\alpha$ on E_α at the point ξ. Here the normal e has the direction given by grad $u = v(\xi)$, and hence(1)

(1) An alternative, more formal proof of (5) depends on the properties of the Jacobian

$$J = \frac{\partial(x_1, \ldots, x_n)}{\partial(t_1, \ldots, t_{n-1}, t)} = \sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial t} = \sum_{i=1}^{n} p_i v_i(x(\xi, t)),$$

where $t' = (t_1, \ldots, t_{n-1})$ is a set of local parameters for the analytic hypersurface E_α, and $p_1, \ldots, p_n$ are certain minors. This Jacobian is independent of t because the transformation $x \to T_t x$ is volume

$$d\,x = |v(\xi)|\, d\,\sigma_\alpha(\xi)\, d\,t. \tag{5}$$

If Δ^+ denotes the set of all pairs (ξ, t) with $\xi \in E_\alpha$ and $0 < t < \tau^+(\xi)$, then $D^+ = \varphi(\Delta^+)$ is contained in the bounded set $I_\alpha = \{x \in R^n : u(x) > u(\xi)\}$. The volume of D^+ is finite and equals

$$m_n(D^+) = \int_{E_\alpha} |v(\xi)|\, \tau^+(\xi)\, d\,\sigma_\alpha(\xi).$$

It follows that $\tau^+(\xi) < \infty$ almost everywhere on E_α. According to Lemma 5 and the limit relation (4), *the line of flow F_ξ connects the point at infinity with K for almost every $\xi \in E_\alpha$.* The system $\mathbf{F}$ of all lines of flow connecting the point at infinity with K, is a sub-system of $\mathbf{C}$. If $f \wedge \mathbf{F}$, in particular if $f \wedge \mathbf{C}$, then

$$\int_{\tau^-(\xi)}^{\tau^+(\xi)} f(x(\xi, t))\, |v(x(\xi, t))|\, d\,t = \int_{F_\xi} f\, d\,s \geq 1 \quad \text{for a.e. } \xi \in E_\alpha$$

since $F_\xi \in \mathbf{F}$ for a.e. $\xi \in E_\alpha$. Applying Schwarz' inequality, we obtain

$$\int_{\tau^-(\xi)}^{\tau^+(\xi)} f(x(\xi, t))^2\, d\,t \int_{\tau^-(\xi)}^{\tau^+(\xi)} |v(x(\xi, t))|^2\, d\,t \geq 1 \quad \text{for a.e. } \xi \in E_\alpha.$$

The second integral equals $u^+(\xi) - u^-(\xi) \leq 1$. A fortiori,

$$\int_{\tau^-(\xi)}^{\tau^+(\xi)} f(x(\xi, t))^2\, d\,t \geq 1 \quad \text{for a.e. } \xi \in E_\alpha.$$

Multiplying by $|v(\xi)|\, d\,\sigma_\alpha(\xi)$ and integrating over E_α, we conclude that, according to (5) and (1),

$$\int_{R^n} f(x)^2\, d\,x \geq \int_D f(x)^2\, d\,x = \int_{E_\alpha} |v(\xi)|\, d\,\sigma_\alpha(\xi) \int_{\tau^-(\xi)}^{\tau^+(\xi)} f(x(\xi, t))^2\, d\,t$$

$$\geq \int_{E_\alpha} |v(\xi)|\, d\,\sigma_\alpha(\xi) = a_n \operatorname{cap}_2 K.$$

Note that it has been proved that $\mathsf{M}_2(\mathbf{C}) = \mathsf{M}_2(\mathbf{F}) = a_n \operatorname{cap}_2 K$, and that the infimum in the definition of M_2 is an actual minimum attained by the function $f = |\operatorname{grad} u|$.

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preserving as a consequence of the equation $\operatorname{div} v = 0$. (Liouville's theorem, see, e.g., KELLOGG [24] p. 35.) And for $t = 0$, the minors p_i serve to define the unit normal e and the surface element $d\,\sigma_\alpha$ on E_α at $\xi = \xi(t_1, \ldots, t_{n-1})$ as follows:

$$e_i\, d\,\sigma_\alpha = p_i\, d\,t',$$

where $d\,t' = d\,t_1 \ldots d\,t_{n-1}$. Hence $J = (e \cdot v)\, d\,\sigma_\alpha / d\,t'$, and

$$d\,x = J\, d\,t'\, d\,t = (e \cdot v)\, d\,\sigma_\alpha\, d\,t = |v(\xi)|\, d\,\sigma_\alpha\, d\,t.$$

4°. $\mathsf{M}_2(\mathbf{H}) \geq (a_n \operatorname{cap}_2 K)^{-1}$. We first assume that $\operatorname{cap}_2 K > 0$ and that K contains no irregular points. The equilibrium potential u then equals 1 everywhere in K, and hence $\eta = 1$. It follows from a result of Kellogg [24], p. 276, that the set Γ of critical values of u is denumerable, and that the critical values may be arranged as an increasing sequence, finite or infinite,

$$(0<)\ \gamma_1 < \gamma_2 < \cdots \quad (<1),$$

where $\lim_i \gamma_i = 1$ if Γ is infinite. In fact, the theorem of Kellogg implies that, when $0 < \alpha < \beta < 1$, all critical points of the "bounded closed region" $\{x \in G : \alpha \leq u(x) \leq \beta\}$ lie on a finite union of equipotential sets E_γ. From Lemma 4 follows that $E_\alpha \in \mathbf{H}$ when $\alpha \notin \Gamma$, $0 < \alpha < 1$.

The critical equipotential sets E_{γ_i} divide the rest of G into open sets $G_0, G_1, \ldots$ containing no critical points. In each of these open sets, say in

$$G_i = \{x \in R^n : \gamma_i < u(x) < \gamma_{i+1}\},$$

we introduce Gaussian coordinates. We shall again use the lines of flow as parameter curves (corresponding to $n-1$ parameters), but the remaining Gaussian coordinate will now be the value of the potential u, $\gamma_i < u < \gamma_{i+1}$, instead of the time t. From (3) we obtain, along any line of flow,

$$d u = |v|^2 d t = |v|\,|d x| = |v|\, d s,$$

where s is the arc length on the line of flow. Hence the volume element is

$$d x = d s\, d\sigma_u = \frac{d u\, d\sigma_u}{|v|}.$$

The system of all equipotential surfaces E_u, $u \notin \Gamma$, will be denoted by $\mathbf{E}$. If $f \wedge \mathbf{E}$, in particular if $f \wedge \mathbf{H}$, then $\int_{E_u} f\, d\sigma_u \geq 1$ for $\gamma_i < u < \gamma_{i+1}$. Moreover, $\int_{E_u} |v|\, d\sigma_u = a_n \operatorname{cap}_2 K$ in view of (1). Inserting these results in Schwarz' inequality

$$\int_{E_u} |v|\, d\sigma_u \int_{E_u} f^2 |v|^{-1} d\sigma_u \geq \Big(\int_{E_u} f\, d\sigma_u\Big)^2,$$

we obtain, after having multiplied by $d u$ and integrated over the interval $\gamma_i < u < \gamma_{i+1}$,

$$\int_{G_i} f^2 d x = \int_{\gamma_i}^{\gamma_{i+1}} d u \int_{E_u} f^2 |v|^{-1} d\sigma_u \geq \frac{\int_{\gamma_i}^{\gamma_{i+1}} d u}{a_n \operatorname{cap}_2 K}.$$

A similar inequality holds for each of the remaining open sets G_i (whether Γ is finite or infinite), and consequently

$$\int_{R^n} f(x)^2\,dx \geq \sum_i \int_{G_i} f(x)^2\,dx \geq \frac{\int_0^1 du}{a_n \operatorname{cap}_2 K} = \frac{1}{a_n \operatorname{cap}_2 K}.$$

Note that it has been proved (under the present restrictions on K) that $\mathsf{M}_2(\mathbf{H}) = \mathsf{M}_2(\mathbf{E}) = (a_n \operatorname{cap}_2 K)^{-1}$, and that the infimum in the definition of M_2 is an actual minimum, attained by the function $|\operatorname{grad} u|/(a_n \operatorname{cap}_2 K)$.

In the general case where K is an arbitrary compact set we choose a compact set K^* without irregular points so that $K^* \supset K$ and $0 < \operatorname{cap}_2 K^* < \operatorname{cap}_2 K + \varepsilon$. Since $\mathbf{H}(\infty, K) \supset \mathbf{H}(\infty, K^*)$, it follows from Theorem 1 (a) that

$$\mathsf{M}_2(\mathbf{H}(\infty, K)) \geq \mathsf{M}_2(\mathbf{H}(\infty, K^*)) = (a_n \operatorname{cap}_2 K^*)^{-1},$$

from which the desired inequality follows for $\varepsilon \to 0$. This completes the proof of Theorem 9.

There are many theorems of the same type as Theorem 9, expressing capacities or conductivities, etc., in terms of the extremal length, or module, of appropriate systems of curves or hypersurfaces. (See, e.g., Hersch [19].) Thus one may replace the point or sphere at infinity in Theorem 9 by a finite external boundary of G and at the same time replace the capacity of K by the capacity of the condensor formed by the external and internal boundary of G. This more general version of Theorem 9 is valid even in the plane (when we define $a_2 = 2\pi$).

CHAPTER III

Applications to Functional Completion

1. Irrotational vector fields

If X is a region (= non-void connected open set) in R^n, we denote by $\mathbf{N}(X)$ the system of all closed curves(1) contained in X and homolog zero in X. A *continuous vector field* $f = f(x) = (f_1, \ldots, f_n)$ is called *irrotational* in X if the circulation

$$\Gamma(C) = \int_C \sum_{i=1}^n f_i\,dx_i = \int_C f \cdot dx$$

(1) As to the type of curves considered, cf. the note on p. 186.

equals 0 along every curve $C \in \mathrm{N}(X)$. According to a classical theorem, a field f whose components belong to $C^1(X)$ is irrotational if, and only if,

$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = 0 \quad \text{everywhere in } X; \quad i, j = 1, \ldots, n.$$

The class of bounded continuous vector fields and the subclass of bounded continuous irrotational vector fields are complete in the topology corresponding to uniform convergence in X. It is, however, often desirable to admit other topologies, in particular the topology corresponding to mean convergence in X. We shall, therefore, consider the class $L^p(X)$ of all vector fields $f = (f_1, \ldots, f_n)$ whose components belong to $L^p(X)$, where p is given, $1 \leq p < +\infty$. The corresponding limit concept is mean convergence of order p for each component. This class of fields $L^p(X)$ is complete in view of the Riesz-Fischer theorem, and the subclass of continuous fields belonging to $L^p(X)$ is dense in $L^p(X)$. (In fact, the class of fields $f \in C_0(X)$ is dense in $L^p(X)$.) The continuous irrotational fields belonging to $L^p(X)$ form a linear subclass $I^p(X)$ of $L^p(X)$. This subclass is not closed in $L^p(X)$. The fields which belong to the closure $\overline{I^p}(X)$ within $L^p(X)$ will now be called *irrotational fields* in $L^p(X)$. Thus a field $f \in L^p(X)$ is irrotational if, and only if, there exists a sequence $\{f^{(\nu)}\}$ of continuous irrotational fields $f^{(\nu)} \in L^p(X)$ converging to f in the sence that $\|f^{(\nu)} - f\|_p \to 0$ as $\nu \to \infty$. No matter how such a field is chosen within its equivalence class, the circulation need not exist (as a Lebesgue integral) along *every* closed curve (even if only very "regular" curves are admitted). For $p \leq n$, this appears from the example on p. 212. Examples of a different nature exist for arbitrary values of p. However, in view of Theorem 3, (b) and (e), each component f_i of an arbitrary field $f \in L^p(X)$ is integrable over p-a.e. curve in X.

THEOREM 10. *In order that a vector field $f \in L^p(X)$ be irrotational, it is necessary and sufficient that the circulation of f vanish along almost every (of order p) closed curve homolog zero in X.*

Proof. As to the necessity of the stated condition, let $f^{(\nu)} \to f$ in $L^p(X)$, and assume that each $f^{(\nu)}$ belongs to $I^p(X)$. According to Theorem 3, (b) and (f), there is a subsequence $\{\nu_q\}$ such that $f^{(\nu_q)} \to f$ in the mean of order 1 on p-a.e. curve in X. Hence, along p-a.e. curve from $\mathrm{N}(X)$, the circulation of f equals the limit of the circulations $(=0)$ of $f^{(\nu_q)}$.

The sufficiency part will be proved by approximating an arbitrary field f satisfying the stated condition by fields $g^{(\varepsilon)} \in I^p(X)$ such that $\|g^{(\varepsilon)} - f\|_p \to 0$ as $\varepsilon \to 0$. Each of the fields $g^{(\varepsilon)}$ will be defined as a mean value of fields $g(x, t)$, where the parameter t ranges over the ball $B_\varepsilon \subset R^n$. And each of these fields $g(x, t)$ is derived from the given field f by a deformation $x \to y = \varphi_t(x)$ of the underlying region X. We begin by defining such a deformation for every vector $t \in R^n$ of length $|t| < 1$. It is not difficult to construct a function $\varrho = \varrho(x) \in C^1(X)$ with the properties

$$\text{(a) } 0 < \varrho < 1; \qquad \text{(b) } |\operatorname{grad} \varrho| < \tfrac{1}{2}; \qquad \text{(c) } \varrho(x) < \tfrac{1}{2}\delta(x),$$

where $\delta(x)$ is the distance between x and the boundary of X. (If $X = R^n$, put $\delta(x) = +\infty$.) For any vector $t \in B_1$, the deformation

$$x \to y = \varphi_t(x) = x + \varrho(x)\, t$$

is a one-to-one mapping (continuously differentiable) of X into X. (By the method of successive approximations it may be easily shown that φ_t maps X *onto* X; but we shall make no use of this fact.) In view of property (b),

$$|\varrho(x') - \varrho(x'')| \leq \tfrac{1}{2}|x' - x''|,$$

and hence

$$\tfrac{1}{2}|x' - x''| \leq |y' - y''| \leq 2|x' - x''|,$$

so that φ_t is a Lipschitz transformation, and $|\partial y / \partial x_j| \leq 2$, $j = 1, \ldots, n$. The volume ratio is

$$J = \det\{\partial y_i / \partial x_j\} = 1 + t \cdot \operatorname{grad} \varrho;$$

in particular, $\tfrac{1}{2} \leq J \leq 2$. If $f \in L^p(X)$, and $f^*(x) = f(\varphi_t(x))$, then f^* likewise belongs to $L^p(X)$, and

$$\|f^*\|_p \leq 2^{1/p}\|f\|_p \leq 2\|f\|_p. \tag{1}$$

If $C \in \mathbf{N}(X)$, then $C_t = \varphi_t(C)$ likewise belongs to $\mathbf{N}(X)$.

Let $\mathbf{E}$ denote a p-exc system of curves in X. Corresponding to an arbitrary closed curve $C \subset X$ we consider the set T of vectors $t \in B_1$ for which the deformed curve $C_t = \varphi_t(C)$ belongs to $\mathbf{E}$. We show that $m_n(T) = 0$. In view of Theorem 2, there is a Baire function $h \in L^p(X)$, $h \geq 0$, such that the integral of h over any curve from $\mathbf{E}$ equals $+\infty$. For an arbitrary vector $t \in B_1$, we have

$$\int_{C_t} h(y)\, ds_y = \int_C h(x + \varrho(x)\, t) \frac{ds_y}{ds_x}\, ds_x \leq 2 \int_C h(x + \varrho(x)\, t)\, ds_x.$$

Now,

$$\int_{B_1} dt \int_C h(x + \varrho(x)\, t)\, ds_x = \int_C ds_x \int_{B_1} h(x + \varrho(x)\, t)\, dt < \infty.$$

since the inner integral on the right is a continuous function of $x \in X$. (In fact,

$$\int_{B_1} h(x+\varrho(x)t)\,dt = \frac{1}{\varrho(x)^n} \int_{B_{\varrho(x)}(x)} h(z)\,dz,$$

and the integral of h over $B_{\varrho(\tau)}(x)$ is continuous since $h \in L^p(X)$ is locally integrable in X, and $B_{\varrho(x)}(x)$ varies continuously with $x \in X$.) Consequently, $\int_{C_t} h(y)\,ds_y < \infty$ for a.e. $t \in B_1$.

The given field $f \in L^p(X)$ is integrable over C, and the circulation of f vanishes along C, for p-a.e. $C \in \mathrm{N}(X)$, i.e., for every curve $C \in \mathrm{N}(X)$ which does *not* belong to some p-exc system E of curves in X. For any $t \in B_1$, consider the deformed field $g(\ast, t)$ defined in X by

$$g_j(x,t) = \sum_{i=1}^{n} f_i^*(x)\frac{\partial y_i}{\partial x_j} = f_j^*(x) + t\cdot f^*(x)\frac{\partial \varrho(x)}{\partial x_j}, \tag{2}$$

where $f^*(x) = f(\varphi_t(x)) = f(x+\varrho(x)t)$. For an arbitrary curve $C \in \mathrm{N}(X)$, the field f is integrable over $C_t = \varphi_t(C)$, and the circulation of f vanishes along C_t, for a.e. $t \in B_1$:

$$\int_{C_t} f(y)\cdot dy = 0 \quad \text{for a.e. } t \in B_1.$$

It follows from the inequalities

$$\left|\sum_i f_i(y)\frac{\partial y_i}{\partial x_j}\right| \le |f(y)| \left|\frac{\partial y}{\partial x_j}\right| \le 2|f(y)|; \qquad ds_x \le 2\,ds_y$$

that the deformed field $g(\ast, t)$ is integrable over C provided f is integrable over C_t; and then

$$\int_C g(x,t)\cdot dx = \int_C \sum_{i,j} f_i^*(x)\frac{\partial y_i}{\partial x_j}\,dx_j = \int_{C_t} f(y)\cdot dy.$$

Consequently,
$$\int_C g(x,t)\cdot dx = 0 \quad \text{for a.e. } t \in B_1. \tag{3}$$

For any ε, $0 < \varepsilon < 1$, define a field $g^{(\varepsilon)}$ in X by

$$g_j^{(\varepsilon)}(x) = \frac{1}{m_n(B_\varepsilon)} \int_{B_\varepsilon} g_j(x,t)\,dt = \frac{1}{m_n(B_\varepsilon)} \int_{B_\varepsilon} \left(f_j^*(x) + t\cdot f^*(x)\frac{\partial \varrho}{\partial x_j}\right) dt.$$

As above, it is easily verified that $g^{(\varepsilon)}$ is a continuous field in X. Moreover, it follows from (3) by application of Fubini's theorem that

$$\int_C g^{(\varepsilon)}(x)\cdot dx = \frac{1}{m_n(B_\varepsilon)}\int_{B_\varepsilon} dt \int_C g(x,t)\cdot dx = 0$$

for every curve $C \in \mathrm{N}(X)$. Thus $g^{(\varepsilon)}$ is a continuous irrotational field. A standard argument shows that $g^{(\varepsilon)} \in L^p(X)$ and $\|g^{(\varepsilon)} - f\|_p \to 0$ as $\varepsilon \to 0$. In fact, it follows from Hölder's inequality and Fubini's theorem that, for every $j = 1, \ldots, n$,

$$\begin{aligned}\int_X |g_j^{(\varepsilon)}(x) - f_j(x)|^p\, dx &= \int_X \left|\frac{1}{m_n(B_\varepsilon)}\int_{B_\varepsilon} (g_j(x,t) - f_j(x))\, dt\right|^p dx \\ &\leq \int_X dx \frac{1}{m_n(B_\varepsilon)}\int_{B_\varepsilon} |g_j(x,t) - f_j(x)|^p\, dt \\ &= \frac{1}{m_n(B_\varepsilon)}\int_{B_\varepsilon} dt \int_X |g_j(x,t) - f_j(x)|^p\, dx. \qquad (4)\end{aligned}$$

According to (2) and property (b) of ϱ, we have

$$|g_j(x,t) - f_j(x)| \leq |f_j^*(x) - f_j(x)| + \tfrac{1}{2}|t|\,|f^*(x)|. \qquad (5)$$

Since the class $C_0(X)$ is dense in $L^p(X)$, there corresponds to any given number $\eta > 0$ a field $\bar{f} \in C_0(X)$ for which

$$\|f - \bar{f}\|_p < \eta. \qquad (6)$$

From (1) we obtain, writing $\bar{f}^*(x) = \bar{f}(\varphi_t(x))$,

$$\|f^*\|_p \leq 2\|f\|_p; \qquad \|f^* - \bar{f}^*\|_p \leq 2\|f - \bar{f}\|_p < 2\eta. \qquad (7)$$

Since the continuous field $\bar{f}$ vanishes outside some compact set $E \subset X$, $\bar{f}$ is uniformly continuous, and hence there is a number $\theta = \theta(\eta) < 1$ such that $|\bar{f}(y) - \bar{f}(x)|^p\, m_n(E) < \eta^p$ provided $|y - x| \leq \theta$. In view of property a) of ϱ, this condition is fulfilled when $y = \varphi_t(x) = x + \varrho(x)\,t$, and $|t| \leq \theta$. Hence,

$$\|\bar{f}^* - \bar{f}\|_p \leq \eta \quad \text{when } |t| \leq \theta. \qquad (8)$$

Applying Minkowski's inequality to (5), we obtain in view of (6), (7), and (8)

$$\begin{aligned}\|g_j(\ast, t) - f_j\|_p &\leq \|f_j^* - \bar{f}_j^*\|_p + \|\bar{f}_j^* - \bar{f}_j\|_p + \|\bar{f}_j - f_j\|_p + \tfrac{1}{2}|t|\,\|f^*\|_p \\ &< 2\eta + \eta + \eta + \tfrac{1}{2}|t|\,(2\|f\|_p) \\ &= 4\eta + |t|\,\|f\|_p,\end{aligned}$$

provided $|t| \leq \theta$. Consequently,

$$\|g_j(\ast, t) - f_j\|_p < 5\eta \quad \text{when } t < \tau(\eta),$$

where $\tau(\eta) = \min(\theta(\eta), \eta/\|f\|_p)$. Inserting this on the right of (4), we conclude that $\|g_j^{(\varepsilon)} - f_j\|_p \leq 5\eta$ when $\varepsilon < \tau(\eta)$. This completes the proof of Theorem 10.

Remark. It is not necessary to verify the condition of Theorem 10 for general closed curves from $\mathrm{N}(X)$. It suffices to consider boundaries of rectangles $R \subset X$ whose sides are parallel to two of the coordinate axes. Take, for simplicity, the case $n = 2$. Such a rectangle is given by its vertices (a_1, a_2), (a_1, b_2), (b_1, b_2), and (b_1, a_2). It can be proved that a field $f \in L^p(X)$ is irrotational if (and only if) the circulation of f vanishes along the boundary of almost every such rectangle R, i.e., for almost every choice of the four numbers a_1, a_2, b_1, and b_2.

Example. Let $p < n$, and choose α so that $0 < \alpha < (n-p)/p$. Write $|x| = r$ and $u = r^{-\alpha}$. The vector field f defined (a.e.) in the unit ball B_1 by

$$f = \operatorname{grad} u = -\alpha \frac{x}{r^{2+\alpha}}, \quad x = (x_1, \ldots, x_n) \neq 0,$$

belongs to $L^p(B_1)$, but the circulation is not defined along any curve passing through 0 since $u(x) \to \infty$ as $x \to 0$. (No matter how f is changed on a set of measure 0, the circulation will be undefined on "most of" these curves, e.g. on a.e. straight segment through 0.) Nevertheless, the field is irrotational in view of Theorem 5 and Theorem 10 since the circulation of f is 0 along any closed curve not passing through 0. If $p = n$, choose α so that $0 < \alpha < 1 - n^{-1}$, and write $u = (\log(2/r))^\alpha$. The field $f = \operatorname{grad} u$ will then have the same properties as the above field. For $p > n$, the circulation of an arbitrary field in $L^p(X)$ is, according to Theorem 3 (e) and Theorem 5, always defined along "most of" the curves passing through an arbitrary given point of X.

A set $E \subset R^n$ will be called *exceptional* of order p if there exists a function $h \in L^p(R^n)$, $h \geq 0$, such that $U_1^h = +\infty$ everywhere in E, but $U_1^h \not\equiv \infty$. The class of all exceptional sets of order p in R^n is denoted by $\mathfrak{E}^p$. In view of the remark following Theorem 6, the condition $U_1^h \not\equiv \infty$ is always satisfied when $p < n$ (and $h \in L^p(R^n)$). When $p > n$, the class $\mathfrak{E}^p$ contains only the void set. Obviously, any subset of a set from $\mathfrak{E}^p$ belongs to $\mathfrak{E}^p$, and it is easily verified that the union of any sequence of sets from $\mathfrak{E}^p$ belongs to $\mathfrak{E}^p$. An exceptional set has Lebesgue measure 0. From Theorem A follows, in particular, that $E \in \mathfrak{E}^p$ implies $\operatorname{cap}_p E = 0$ provided $1 \leq p \leq 2$, whereas the converse is true when $2 \leq p \leq n$. *A set E is exceptional of order 2 if, and only if,* $\operatorname{cap}_2 E = 0$. (The capacity of order 2 is the classical harmonic capacity: logarithmic capacity in the plane, Newton capacity in R^3, etc.) For the phrase "except in some set $E \in \mathfrak{E}^p$", we shall write briefly: (exc $\mathfrak{E}^p$). From Theorem 10 and the sufficiency part of Theorem 6 we obtain the following corollary: *Consider a vector field* $f \in L^p(X)$, $p > 1$. *If there is a set* $E \in \mathfrak{E}^p$ *such that the circulation of* f *exists and equals* 0 *along every closed curve* $C \in \mathrm{N}(X)$ *which does not intersect* E, *then* f *is irrotational.* This corollary does not admit a direct conversion. However, the following statement is a simple consequence of Theorem 10 and the necessity part of Theorem 6. Let $E \subset X$ and $E \notin \mathfrak{E}^p$, $p \geq 1$. Corresponding to an arbitrary irrotational vector field $f \in L^p(X)$ there exist closed curves intersecting E along which the circulation of f is defined and equal to 0.

2. Beppo Levi functions

In classical vector analysis a function u is called a primitive of a differential form $\sum_i f_i(x)\,dx_i$ (where $f_1, \ldots, f_n$ are continuous functions in a region $X \subset R^n$) if $du = \sum_i f_i\,dx_i$, i.e., $\operatorname{grad} u = f$; or, equivalently, if

$$u(b) - u(a) = \int_a^b \sum_{i=1}^n f_i\,dx_i$$

whenever a and b are points of X, and the integration refers to an arbitrary curve in X leading from a to b. (If X is multiply connected, the various "homology classes" of curves leading from a to b give, in general, different values of the line integral, and u must then be allowed to become multivalued.) In order that there exist, corresponding to a given differential form of the above type, a primitive in X, it is necessary and sufficient that the field $f = (f_1, \ldots, f_n)$ be irrotational in X. A primitive is determined uniquely in X up to an arbitrary additive constant.

We shall now consider the corresponding problem concerning fields

$$f = (f_1, \ldots, f_n) \in L^p(X).$$

For simplicity, we shall assume that the given region X is *simply connected.* A single-valued function u is called a *primitive* of a differential form $\sum_i f_i(x)\,dx_i$ (where $f_1, \ldots, f_n \in L^p(X)$) if, along p-a.e. curve $C \subset X$,

$$u(b) - u(a) = \int_a^b \sum_{i=1}^n f_i\,dx_i, \tag{1}$$

a and b being arbitrary points of C.

THEOREM 11. *In order that there exist, corresponding to a given differential form* $\sum_i f_i\,dx_i$, $f_i \in L^p(X)$, *a primitive in* X, *it is necessary and sufficient that the field* $f = (f_1, \ldots, f_n)$ *be irrotational in* X.

Proof. If u is a primitive of the given differential form, it follows, by application of (1) to closed curves and coinciding points $a=b$, that the field f is irrotational. Conversely, assume that $f \in L^p(X)$ is irrotational in X. Denote by h a non-negative Baire function in $L^p(R^n)$ with the properties that $U_1^h \not\equiv \infty$, and that the circulation of f exists and equals 0 along every closed curve $C \subset X$ such that $\int_C h\,ds < \infty$.[1] The set $E = \{x \in X : U_1^h(x) = +\infty\}$ is then exceptional of order p. If $x \in X - E$, it follows from the integral formula (1), p. 190, (or by a direct argument involving polar coordinates and Fubini's theorem) that $\int_L h\,ds < \infty$ for almost every straight line L through x. This implies that *any two points of* $X - E$ *may be connected by a polygonal line* $L \subset X$ *such that* $\int_L h\,ds < \infty$. (Since X is connected and open, it suffices to verify this in the case where the line segment $\overline{ab}$ determined by the two points a and b belongs to X. Consider the hyperplane H orthogonal to $\overline{ab}$ and passing through the midpoint c. Since $\overline{ab}$ has a positive distance from the boundary of X, the point c has a neighbourhood V in H such that, for every $v \in V$, the segments $\overline{av}$ and $\overline{vb}$ belong to X. Since $a \in X - E$, $\int_{\overline{av}} h\,ds < \infty$ for a.e. $v \in V$ (where "a.e." refers to m_{n-1} in H), and similarly $\int_{\overline{vb}} h\,ds < \infty$ for a.e. $v \in V$. It follows that h is integrable over the polygonal line $\overline{avb}$ for a.e. $v \in V$.) Now, choose arbitrarily a point $x^0 \in X - E$ and keep it fixed. For any $x \in X - E$ define

$$u(x) = \int_{x^0}^{x} f \cdot dx,$$

where the line integral refers to an arbitrary curve $C \subset X$ leading from x^0 to x and for which $\int_C h\,ds < \infty$. It was shown above that such curves exist, and it follows from the definition of h that the line integral is independent of the choice of C. In this way a function u has been defined everywhere in $X - E$, and the equation (1) is

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(1) A Baire function $h_1 \in L^p(R^n)$, $h_1 \geq 0$ possessing this latter property, exists according to Theorems 10 and 2. If $p < n$, take $h = h_1$; if $p \geq n$, take $h(x) = (1 + |x|)^{-\alpha} h_1(x)$, where $\alpha > 1 - n/p$. It follows then from Hölder's inequality that

$$\int_{R^n} (1 + |x|)^{1-n} h(x)\,dx = \int_{R^n} (1 + |x|)^{1-n-\alpha} h_1(x)\,dx < \infty,$$

and hence $U_1^h \not\equiv \infty$ in view of the remark following the formulation of Theorem 6. Clearly, $\int_C h\,ds < \infty$ implies $\int_C h_1\,ds < \infty$ since C is bounded.

easily verified for any curve $C \subset X$ such that $\int_C h\,ds < \infty$. When $p > n$, U_1^h is finite and continuous everywhere, and it may be shown that the primitive u constructed above is defined and continuous everywhere in X.

THEOREM 12. *If u is a primitive function of a differential form $\sum_i f_i\,dx_i$, $f_i \in L^p(X)$, then the partial derivatives of order 1 of u exist almost everywhere in X, and the equation* $\operatorname{grad} u = f$, *that is,*

$$\frac{\partial u}{\partial x_i} = f_i(x), \quad i = 1, \ldots, n,$$

holds almost everywhere in X.

Proof. Consider a straight line $L = L(x_2, \ldots, x_n)$ parallel to the x_1-axis, given by the constant values of $x_2, \ldots, x_n$. It follows easily from Theorem 2 and Fubini's theorem that (1) holds along $X \cap L(x_2, \ldots, x_n)$ for almost every choice of $(x_2, \ldots, x_n)$ in the projection of X on the hyperplane given by $x_1 = 0$. Hence u is absolutely continuous, as a function of x_1, on $X \cap L$, with the derivative $\partial u / \partial x_1 = f_1(x)$ for almost every value of x_1 in the projection of $X \cap L$ on the x_1-axis. In particular, $\partial u / \partial x_1 = f_1(x)$ a.e. in X, and similarly for the other coordinates.

THEOREM 13. *Let u be a primitive function of a differential form $f \cdot dx$, $f \in L^p(X)$.*

(a) *In order that u be likewise a primitive of $g \cdot dx$, $g \in L^p(X)$, it is necessary and sufficient that $f(x) = g(x)$ almost everywhere in X.*

(b) *In order that v be, likewise, a primitive of $f \cdot dx$, it is necessary and, provided $p > 1$, sufficient that $u(x) - v(x)$* be constant (exc $\mathfrak{E}^p$).

Proof. (a) If $f(x) = g(x)$ a.e. in X, it follows from Theorem 3, (d), that $f(x) = g(x)$ almost everywhere on p-a.e. curve $C \subset X$, and hence $\int_a^b f \cdot dx = \int_a^b g \cdot dx$ along p-a.e. curve $C \subset X$, a and b denoting arbitrary points of C. This implies the sufficiency of the condition. The necessity is contained in Theorem 12, which states that f is determined uniquely a.e. in X by the equation $f = \operatorname{grad} u$. (b) The general case may be easily reduced to the case $u = f = 0$. If v is a primitive of $0 \cdot dx$, then $v(x) =$ constant in X (exc $\mathfrak{E}^p$). In fact, v is constant along p-a.e. curve $C \subset X$. Choose a Baire function $h \in L^p(R^n)$, $h \geq 0$, so that $U_1^h \not\equiv +\infty$, and v is constant along every curve $C \subset X$ for which $\int_C h\,ds < \infty$. (Cf. the proof of Theorem 11.) The set

$$E = \{x \in X : U_1^h(x) = +\infty\}$$

belongs to $\mathfrak{E}^p$, and any two points a and x of $X-E$ may be joined by a polygonal line $L\subset X$ such that $\int\limits_L h\,ds<\infty$. Consequently $v(x)=v(a)$ for every $x\in X-E$. Conversely, let $v(x)=c$ (a constant) for every $x\in X-E$, E being exceptional of order p. When $p>1$, it follows from Theorem 6 that p-a.e. curve $C\subset X$ is contained in $X-E$ and hence $v=c$ on the curve. Thus v is a primitive of $0\cdot dx$.

Let $1\le p<\infty$. Any function u which is a primitive of some differential form $f\cdot dx$, $f\in L^p(X)$, is called a *Beppo Levi function* (of order p) and we write

$$\operatorname{grad} u = f.$$

The class of all Beppo Levi functions of order p in X will be denoted by $BL^p(X)$. The intersection $S^p(X)=BL^p(X)\cap C^1(X)$ consists of all functions $u\in C^1(X)$ for which $\operatorname{grad} u\in L^p(X)$.

THEOREM 14. *Let $u^{(\nu)}\in BL^p(X)$, $\nu=1,2,\dots$, and assume that* grad $u^{(\nu)}$ *converges in the mean of order p over X. Then there exist a function $u\in BL^p(X)$, a subsequence $\{\nu_q\}$, and a corresponding sequence of constants c_q, such that*

$$u^{(\nu_q)}-c_q\to u \quad \textit{pointwise in } X \ (\text{exc } \mathfrak{E}^p),$$

and
$$\operatorname{grad} u^{(\nu)}\to \operatorname{grad} u \quad \textit{in the mean of order } p \textit{ over } X.$$

Proof. Since the class of irrotational fields in $L^p(X)$ is, by definition, closed in $L^p(X)$, the limit in mean f of the sequence of irrotational fields $f^{(\nu)}=\operatorname{grad} u^{(\nu)}$ is irrotational and hence of the form $f=\operatorname{grad} u$, $u\in BL^p(X)$, by Theorem 11. In view of Theorem 3, (b) and (f), there exist a subsequence $\{\nu_q\}$ and a Baire function $h\in L^p(R^n)$, $h\ge 0$, such that $U_1^h\not\equiv\infty$ (cf. the proof of Theorem 11) and the following two statements hold for any bounded curve $C\subset X$ for which $\int\limits_C h\,ds<\infty$:

(i) $$u^{(\nu)}(x)-u^{(\nu)}(a)=\int_a^x f^{(\nu)}\cdot dx;\qquad u(x)-u(a)=\int_a^x f\cdot dx,$$

the integration being performed along C, on which a and x are arbitrary points.

(ii) $f^{(\nu_q)}\to f$ in the mean of order 1 on C.

The set $E=\{x\in R^n: U_1^h(x)=+\infty\}$ belongs to $\mathfrak{E}^p$. Select a point $a\in X-E$. An arbitrary point $x\in X-E$ may be connected with a by a polygonal line $L\subset X$ for which $\int\limits_L h\,ds<\infty$ (cf. the proof of Theorem 11). Writing $c_q=u^{(\nu_q)}(a)-u(a)$, we obtain from (i) and (ii)

$$u^{(\nu_q)}(x)-c_q\to u(x) \quad \text{as } q\to\infty,\ x\in X-E.$$

The class $BL^p(X)$ is a completion of the above class $S^p(X)$ of "smooth" Beppo Levi functions, in the sense described in the following theorem, which is a consequence of Theorems 10 and 14.

THEOREM 15. *If* $u \in BL^p(X)$, *there exists a sequence of functions* $u^{(\nu)} \in C^1(X)$ *such that* $\operatorname{grad} u^{(\nu)} \in L^p(X)$, *and* $\operatorname{grad} u^{(\nu)} \to \operatorname{grad} u$ *in the mean of order* p *over* X, *while* $u^{(\nu)} \to u$ *pointwise in* X (exc $\mathfrak{E}^p$).

Now, let $p>1$. In the terminology of Aronszajn [2], Theorems 13, 14, and 15 above imply that the class $BL^p(X)$ is the *pseudofunctional completion* of the class $S^p(X)$ relative to the class $\mathfrak{E}^p$ of exceptional sets. (Since the L^p-norm $\|\operatorname{grad} u\|_p$ is an improper norm of u, one must first convert it into a proper norm in a well-known manner by adding a suitable expression which does not vanish when u is equal (exc $\mathfrak{E}^p$) to a constant $\neq 0$.) This completion is *perfect* because there corresponds to any given set $E \in \mathfrak{E}^p$, $E \subset X$, a family of functions $u^{(\varepsilon)} \in S^p(X)$ such that $\{u^{(\varepsilon_\nu)}\}$ is a Cauchy sequence (with respect to the above-mentioned proper norm) as $\varepsilon_\nu \to 0$, and yet $\lim_{\varepsilon \to 0} u^{(\varepsilon)}(x) = +\infty$ for every $x \in E$.(1)

In the particular case $p=2$ it is known that this perfect pseudofunctional completion of $S^p(X)$ is identical with the class of "fonctions (BL) précisées" in the sense of Deny and Lions [10]; see also Aronszajn and Smith [3]. These functions are characterized by the following properties:

(a) $\operatorname{grad} u \in L^2(X)$, interpreted in the sense of the theory of distributions,

(b) to every $\varepsilon>0$ there is an open set G with $\operatorname{cap}_2 G < \varepsilon$ such that the restriction of u to $X-G$ is continuous;

or, equivalently, by the following structure properties:

(a_1) u is absolutely continuous along almost every line parallel to one of the coordinate axes,

(a_2) the partial derivatives of u of order 1, which hence exist almost everywhere in X, belong to $L^2(X)$,

(b) as above.

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(1) It is easy to reduce the proof of this statement to the case where E is bounded. Then a function $f \in L^p(R^n)$, $f \geq 0$, may be so chosen that $u = U_1^f = +\infty$ everywhere in E, and f vanishes outside some bounded set. Following the procedure described in the proof of Theorem 6 (in the case $|\alpha| = k = 1$), we define

$$u^{(\varepsilon)}(x) = \int_{R^n} (|x-y|^2 + \varepsilon^2)^{\frac{1-n}{2}} f(y)\, dy.$$

This family of functions has the desired properties.

The former characterization (a), (b) shows that this class is independent of the choice of the coordinate system (it is even invariant under arbitrary Lipschitz transformations of the underlying region X). The structure of the functions in the class appear perhaps more clearly from the properties (a_1), (a_2), and (b).

The functions studied originally by B. Levi [25] were more special; in particular only continuous functions were admitted. The class of functions determined by the requirements (a_1), (a_2) alone was investigated by Nikodym [29], who introduced the name Beppo Levi functions. This wider class forms, likewise, a pseudofunctional completion of the class $S^2(X)$ (with larger exceptional sets than above). This completion is, however, not perfect. It depends, moreover, effectively on the choice of the coordinate system.—From the results of the present section follows, for any $p>1$, that the perfect pseudofunctional completion of $S^p(X)$, being identical with the class which we have denoted by $BL^p(X)$, is characterized by the following structure properties:

(A_1) u is absolutely continuous along p-a.e. curve in X,

(A_2) the partial derivatives of u of order 1, which hence exist almost everywhere in X, belong to the class $L^p(X)$.

(The associated class $\mathfrak{E}^p$ of exceptional sets was defined on p. 212.)—In this manner, the continuity assumption (b) has become superfluous, and at the same time the dependence of (a_1) on the coordinate system has disappeared. Furthermore, these properties (A_1), (A_2) exhibit clearly (cf. Theorem 4) the invariance of the class $BL^p(X)$ under Lipschitz transformations of the region X.

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