ON NON-LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER: III. THE EQUATION $\ddot{y} - k(1 - y^2)\dot{y} + y = b\mu k \cos(\mu t + \alpha)$ FOR LARGE k, AND ITS GENERALIZATIONS

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Introduction[1]

1. We are concerned with equations in real variables of the form

$$\ddot{y} + f(y)\dot{y} + g(y) = p(t),$$

where f, g, p are smooth functions of their arguments, and p has period $\lambda = 2\pi/\mu$ in t. About f we suppose that $\underline{\lim} f > 0$ as $y \to \pm\infty$; that is to say, we suppose the "damping" to be positive for large $|y|$. About g we suppose that it has a "restoring" effect, i.e. has the sign of y. The simplest case, and a specially important one, to be covered in any generalization, is $g = ay$ for positive a. We do in fact assume always that $g(0) = 0$, and that g' exists and has a positive lower bound.

There is some general theory of such equations. A trajectory (or "motion") with initial conditions $y(t_0) = \xi$, $\dot{y}(t_0) = \eta$ (ξ, η real) at some fixed $t = t_0$ is said to have the point $P = (\xi, \eta)$ as "representative point". If ξ', η' are the values of y, $\dot{y}$ at $t = t_0 + \lambda$ the transformation T from P to $P' = (\xi', \eta') = TP = T(\xi, \eta)$ is $1-1$ and continuous.

With the condition $\underline{\lim} f > 0$ and suitable conditions on g (fulfilled for $g = y$), every trajectory is bounded as $t \to \infty$, and T transforms a suitable large domain in the P space into a domain contained in the original one. Further, the vector V, or $P \to TP$, makes exactly one revolution as P moves positively round the boundary. Then a "fixed point" theorem holds, and the "index number" proof of it is valid.[2]

[1] This paper is based on joint work with M. L. CARTWRIGHT.

A paper I, with the same general title, was published in the *Journal London Math. Soc.*, 20 (1945), 180–189, jointly with M. L. CARTWRIGHT. This was written with the same aims as the present Introduction, but in drastically condensed form. We have borrowed some passages from it.

[2] N. LEVINSON, *Journal of Math. and Physics*, 22 (1943), 41–48, and *Annals of Math.*, 45 (1945), 723–727.

There is at least one "fixed point" (f.p. for short) of the transformation T, and there corresponds a *periodic motion* (p.m. for short) *of period* λ; this need not, however, be stable.

We must define our vocabulary in the matter of stability. A f.p. P_0 (and its corresponding p.m.) is called stable if $T^m P$ converges to P_0 as $m \to \infty$ for all P of a neighbourhood of P_0. It is called totally unstable if the reversed trajectories are stable: this is equivalent to the statement that P's near and not identical with P_0 recede from P_0 under iteration of T. f.p. and p.m. that are neither stable nor totally unstable we shall call non-stable. This is not a term in general use; but the strange fact is that our "non-stable" p.m. are for the most part infinitely singular (or "multiple"), and there is no point in classifying them. (The ordinary "col" does just occur and receive a passing mention.)

A periodic trajectory or motion (p.m.), of least period $n\lambda$, with $n > 1$, is called a *subharmonic of order n*. We are interested in the class K of "limiting trajectories",[1] the class whose representative points are the set of limit points of $T^m(\xi, \eta)$ as $m \to \infty$; this class as a whole is invariant under T, and its trajectories are all bounded in $-\infty < t < \infty$. The simplest possible case is that in which K is a single point; there is then a stable p.m. of order 1 and every trajectory converges to it. The next simplest case is that of a finite number of p.m., to some one of which every trajectory converges. In the general topological theory, however, other possibilities, indeed very "bad" ones, have to be contemplated, and it can be very difficult in a given case to rule them out.

2. The two simplest cases (in respect of the functions f and g) are:

(i) $f > c > 0$ for all y, together with suitable conditions on g, valid when $g = y$;

(ii) *small* departure from linearity, $f = \varepsilon F$, $g = y + \varepsilon G$, with fixed F, G, and small ε.[2]

We must not discuss here what is known about these cases, except to observe that they differ from each other, and are both completely unlike anything in the present paper. We now ask: what is the simplest equation not included in either of these cases?

Since f is not to be of constant sign, and since it is positive for large $|y|$, it must have two zeros at least; if we add symmetry about $y = 0$, and take the simplest possible g and p, our search leads to an equation that can be normalized as

[1] "Recurrent motions" of BIRKHOFF, *Dynamical Systems* (New York, 1927), 198, and G. A. HEDLUND, *Amer. Journal of Maths.*, 66 (1941), 605–620.

[2] $f = a + \varepsilon F$ for $a > 0$ is ruled out as a special case of (i).

$$\ddot{y} - k(1-y^2)\dot{y} + y = b\mu k \cos(\mu t + \alpha), \quad \mu = 2\pi/\lambda; \tag{E}$$

"van der Pol's equation". k must not be small, or we are in case (ii), the next possibility of simplification is to suppose it large, which gives us the equation of the title.

3. The equation has a considerable literature,[1] and experiments have suggested very interesting behaviour, especially in the case of large k. Stable motions occur which are subharmonics of large odd order (comparable with k), decreasing as b increases. Further, as b increases we have alternately *one* set of periodic stable motions, of order $2n+1$, and *two* sets, of orders $2n+1$ and $2n-1$, the shorter growing at the expense of the longer. In the experiments n can be of the order of 100 or 200.

It can be foreseen that the theoretical state of things behind these findings of experiment (two distinct stable periods neither a multiple of the other) must be of considerable complexity. If there is a subharmonic motion of order m there will be a "set" of m distinct periodic trajectories, with distinct representative P, obtained by shifting the original one $0, 1, \ldots, (m-1)$-periods forward. Now between two adjacent "stables" (p.m., or their representative P) there must exist some sort of "water shed". A stable f.p. P of order m has a "sphere of influence", any point of which converges to P under iteration of T^m; the sphere of influence is a domain bounded by "lines" of points corresponding to watershed behaviour. Under iteration of T (as opposed to T^m) P and its sphere of influence circulate through a set m in number, the picture as a whole being invariant under T. It is now a question of two such sets, circulating at different rates; what is the nature of the watershed lines and the non-stable f.p. that lie on them? Our methods are non-topological, and raise, not solve, topological questions.[2] We find, however, that there is a fine structure of limiting motions, some periodic non-stable and some non-periodic, of fantastic complexity, though fortunately describable in fairly simple general terms.[3]

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[1] See B. van der Pol, *Proc. Inst. Radio Engineers*, 22 (1934), 1051–1086, where further references are given. See also D. L. Herr, *Proc. Inst. Radio Engineers*, 27 (1939), 396–402, for graphical solutions, and B. van der Pol and J. van der Mark, *Nature*, 120 (1927), 363, for experimental results. The experiments, however, are actually concerned with an extremely unsymmetrical system.

[2] Cp. § 31 below, and M. L. Cartwright and J. E. Littlewood, *Annals of Maths.*, 54 (1951), 1–37.

[3] Since our preliminary announcement of our results, N. Levinson, *Annals of Mathematics*, 50 (1949), 127–153, has published a study of the equation $\ddot{y} + kf(y)\dot{y} + y = bk \sin t$, and $f(y) = +1$ or -1 according as $|y| > 1$ or $|y| < 1$. The solutions are piecings together of expressions in finite terms, and he is able to show, in reasonable compass, that for some intervals of b there are two sets of stable periodic solutions. There is also a fine structure, and we may refer to his paper for further comments on its nature.

The lesson would seem to be that the worse possibilities envisaged in the general theory not merely *can* occur, but are normal occurrences (once we depart from cases (i) and (ii) of § 2). It would be fairly obvious in any case that the general nature of the results is not bound up with the particularities of van der Pol's equation, but in our detailed account we do in fact consider an equation with generalized f, g, p;

$$\ddot{y} + k f(y) \dot{y} + g(y) = b k p(t), \qquad f, g, p \text{ independent of } k, k \text{ large.}$$

We retain only such features of the original as are necessary to keep the behaviour within a single category: the chief of these are (i) f must have only one pair of zeros (see § 33 for the drastic changes if this is dropped); (ii) $p_1(t) = \int p\,dt$ must not attain its upper or its lower bound more than once in a period; and (iii) we retain symmetry (f is even, g odd, $p(\frac{1}{2}\lambda + t) = -p(t)$, and p has mean value 0): if (ii) or (iii) is dropped fresh complications are involved.[1] In the account that follows in this Introduction, however, we keep to the special van der Pol's equation to avoid a multiplicity of constants and other distractions.

For the rest we try, in the Introduction, to give an account intelligible to a reader who is prepared to take the proofs of much of the complicated detail for granted: given certain key-lemmas (reasonably plausible in themselves), the main lines of the argument, and especially those leading to the most striking results, are fairly clear and unencumbered.

4. We proceed to describe the general nature of the limiting motions. Either one or two sets of stable subharmonics, as suggested by the experiments, do in fact exist, but there is much to add.[2]

If $b > \frac{2}{3}$, and $k > k_0(b, \lambda)$,[3] (E) shows the simplest possible behaviour: there is a stable p.m. of order 1, period λ, to which every trajectory converges.

In the case $b < \frac{2}{3}$ we restrict ourselves to the interval $\mathcal{B}$ defined by $\frac{1}{100} \leq b \leq \frac{2}{3} - \frac{1}{100}$. We have now to exclude certain intervals of b, in all a small proportion of $\mathcal{B}$. There then exist $\varepsilon = \varepsilon(\lambda)$, small for $k_0 = k_0(\lambda)$ large, with the following properties. If $k \geq k_0$ there is a set of excluded intervals in $\mathcal{B}$ of total length ε at most. The remainder

[1] The new features involved when one of them (especially, perhaps, (ii)) is dropped might be very interesting.

[2] Experiment could not be expected to do more than hint at the crudest of the non-stable p.m.

[3] This is the simplest instance of a point that should be emphasized. We never assert that behaviour is more and more nearly such and such as k increases, always that it is *exactly* such and such so soon as k exceeds a certain k_0 [here $k_0(b, \lambda)$]. In fact k is not "large", but only "large enough".

of $\mathcal{B}$ is also a set of intervals, $\mathcal{B}$, say; this varies with k, but has length at least $\frac{2}{3}-\varepsilon$. $\mathcal{B}$ divides into two parts, $\mathcal{B}_1$ and $\mathcal{B}_2$, comparable in size.

When b belongs to an interval I_1 of $\mathcal{B}_1$, (E) has a set of stable subharmonics of order $2n+1$ and most[1] trajectories converge each to some one of these. n is constant in I_1 and is of order $(\frac{2}{3}-b)k$.

When b belongs to an interval I_2 of $\mathcal{B}_2$, (E) has a set of stable subharmonics of order $2n+1$, and another of order $2n-1$; most trajectories converge each to some member of one of the two sets. It possesses a further set Σ, infinite in number, of non-stable p.m. of a great variety of "structures" described in more detail later. It possesses further a set X, of the power of the continuum, of non-periodic limiting trajectories, of the type described as "discontinuous recurrent". If we denote the sets of representative points P in the (ξ, η) plane also by Σ and X, then every point of Σ is a limit point of points of Σ, and also a limit point of points of X. A point of Σ is thus non-stable, and is clearly a highly singular, or multiple, f.p. The number n (which is of the order of k) is constant in I_2. Moreover the set K and its subsets Σ, X remain topologically equivalent throughout I_2. Thus a point of Σ remains "infinitely multiple" for all b of the interval I_2, contrary to the natural expectation that multiplicity would be confined to isolated values of b.

For b of an I_1 (of $\mathcal{B}_1$) there is a set of non-stable subharmonics of order $2n+1$.

For all b of $(\frac{1}{100}, \frac{2}{3}-\frac{1}{100})$ there is a single f.p. of order 1, and it is totally unstable. We shall call its representative point P_u, and denote by K_0 the set K less the point P_u.

As b increases in $\mathcal{B}$ (from $\frac{1}{100}$ to $\frac{2}{3}-\frac{1}{100}$), jumping the excluded intervals, the number n decreases. We have nothing to say about the *transitions* from one stable period $(2n+1)$ to two stable periods $(2n\pm 1)$ and *vice versa*; these take place in the excluded intervals, but we add finally that for any b of $\mathcal{B}$ there are always subharmonics of *some* kind, of order comparable with k.

5. We shall be mainly concerned in the rest of the Introduction with the range $\mathcal{B}$, i. e. $\frac{1}{100}\le b\le\frac{2}{3}-\frac{1}{100}$, of b: the case of small b is a separate (and interesting) problem that we do not discuss, and the case $b>\frac{2}{3}$, in which behaviour is simple, is treated by adaptations of the arguments we shall be describing.

By $A(x, y)$ we denote a constant depending only on λ and the parameters shown explicitly; by L a constant depending only on λ, by D a constant $A(\delta)$: all of these

[1] The general sense of "most" is fairly obvious: to define it precisely would occupy too much space.

to be positive.[1] The constant of an O is to be of type L. We use suffixes to identify constants; generally only for the moment, and starting afresh at 1 on the next occasion. Dependence on b always reduces to dependence on the lower bound of b (i.e. $\frac{1}{100}$) or the nearness of b to the critical $\frac{2}{3}$. Our present restriction to B makes dependence on b "uniform": O's and D's are "uniform in b". The reader may therefore treat it lightly. We shall sometimes allow ourselves the licence of using $o(1)$ without precise definition.[2] The lower bound k_0, which is ultimately a D, is to be re-chosen at any moment when the run of the argument requires it.

We use the symbol Q for the current point (t, y) of a trajectory Γ, φ for the phase $\mu t + \alpha$, η for $y-1$, y_1 for $\int y\, dt$. Let

$$F(y) = \int_0^y f(y)\, dy = \tfrac{1}{3} y^3 - y = \eta^2 + \tfrac{1}{3}\eta^3 - \tfrac{2}{3}.$$

Then we have an identity[3] (which we shall call the "$\dot{y}$-identity") with the alternative forms

$$\left.\begin{aligned} \dot{y} - \dot{y}_0 &= -k\,[F(y)]_{t_0}^{t} + b\,k\,(\sin\varphi - \sin\varphi_0) - [y_1]_{t_0}^{t}, \\ F(y) &= b\sin\varphi - (y_1 + y)/k + C. \end{aligned}\right\} \tag{1}$$

6. It is easy to show that for $0 \le b \le 2$ every trajectory Γ eventually satisfies $|y| < L_1$, $|\dot{y}| < L_1 k$; and if a trajectory is started subject to these inequalities at t_0, then $|y| < L_2$, $|\dot{y}| < L_2 k$ for $t \ge t_0$, where we may further suppose (to cover small accidents) that $L_2 > L_1 > 20$, say. We will call a trajectory so started (extending the natural meaning of the adjective) an "eventual" one. If $b > 0$ it crosses $y = 0$ infinitely often. The corresponding results for the general equation are, for once, considerably more difficult; but we can fortunately appeal to a proof we have given elsewhere.[4]

Let us now try to look ahead. The regions $y > 1 + \delta$, $y < -1 - \delta$ have large positive damping. We expect of a trajectory in $y > 1 + \delta$, once it has "settled down", and so long as it remains in the region, first that it is very stable; we return to this point presently. We expect, secondly, that both its velocity and its acceleration will be bounded: $\dot{y}, \ddot{y} = O(D)$.[5] Granted this, it follows from (E) itself that

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[1] The constant of an inequality "$< L$" or "$< D$" is to be chosen "large enough", that of "$> L$" or "$> D$", "small enough".

[2] Actually all the $o(1)$'s are $O(k^{-A})$'s.

[3] In which, the origin $t = 0$ being at our disposal, we generally take $t_0 = 0$, $\dot{y}_0 = \dot{y}_0(0)$.

[4] *Annals of Math.*, 48 (1947), 472–494.

[5] The D arises from the δ just above. The fact is that we shall have to consider a number of different restrictions, each with a "δ": in the interest of simplicity we make all δ's the *same*.

The grounds for the "expectation" itself are further considered in § 7 below.

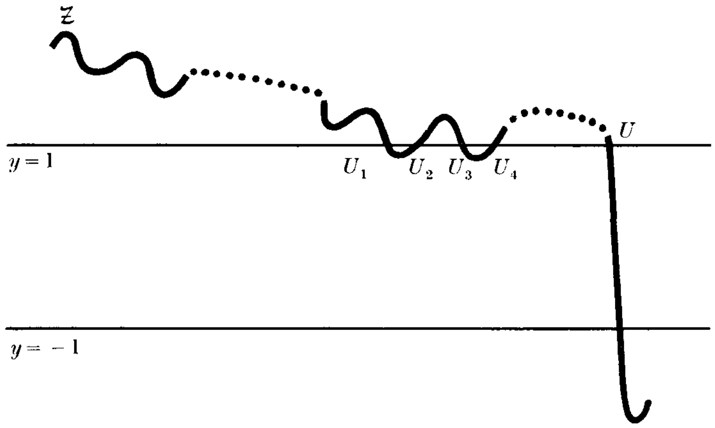

Fig. 1. The dips $U_1 U_2$, $U_3 U_4$ are much exaggerated.

$$(y^2-1)\,\dot{y} = b\mu\cos\varphi + O(D/k), \tag{1}$$

and from the $\dot{y}$-identity that

$$F(y) = C + b\sin\varphi - \frac{1}{k}[y_1]_{t_0}^{t} + O\left(\frac{D}{k}\right). \tag{2}$$

Of these (1) implies an approximate "linkage" between y and $\dot{y}$; it also implies that the velocity adjusts itself to make the damping balance the forcing-term; a "settling" of the trajectory: the regions of positive damping have a strongly "settling" effect. We shall find both "linkage" and "settling" very useful conceptions. (2) implies that

$$\tfrac{1}{3}\eta^3 + \eta^2 - \tfrac{2}{3} = F(y) = C + b\sin\varphi, \tag{3}$$

with error $O(D/k)$, over any interval of time of length $O(1)$. The curve (3) has period λ, and supposing $C > b - \frac{2}{3}$, it lies in $y > 1$ (it lies in $y > 1 + \delta$ if $C > b - \frac{2}{3} + A\delta$). It has maxima at $\varphi \equiv \frac{1}{2}\pi$ and minima at $\varphi \equiv -\frac{1}{2}\pi$. The effect of the term in y_1, on the right-hand side of (2), is to make a trajectory satisfying (2) consist, to error $O(D/k)$ throughout, of successive λ-waves of the form (3), the successive constants C diminishing by $[y_1]/k$, the increment $y_1 = \int y\,dt$ being taken over the preceding wave: the trajectory (with $C > b - \frac{2}{3} + A\delta$) gradually approaches $y = 1$ from above.

The neighbourhood of $y = 1$ (where the damping changes sign) is naturally highly critical, and we have to spend much time discussing it. But for clearness we state now what in fact next happens to the trajectory we have been following.

It continues its downward trend until one of its λ-waves reaches $y = 1$; this happens, as we should expect from the foregoing account, at a point U_1 where φ is approximately $-\frac{1}{2}\pi \pmod{2\pi}$. It then (if b is not too large) makes a small "dip" (depth of order $k^{-\frac{1}{2}}$) below $y=1$, emerges at U_2, and executes another λ-wave of type (3), followed, perhaps, by another "dip" $U_3 U_4$, and so on. After $O(1)$ dips $U_{2n-1} U_{2n}$ at most, the velocities remaining $O(1)$ throughout, the trajectory arrives at U of the diagram, and, unless by coincidence it belongs to a narrow exceptional class[1], it decisively enters $|y|<1$, is violently accelerated by the negative damping, acquiring a velocity of order k which takes it right through $|y| \leq 1$ to a *minimum* $\mathfrak{Z}'$ at a depth approximately $y=-2$, still with φ approximately $-\frac{1}{2}\pi \pmod{2\pi}$ ($U\mathfrak{Z}'$ occupies only a small time). The trajectory is slightly unsettled at $\mathfrak{Z}'$ (the region $|y|<1$ is "unsettling"), after a short time it is again settled, and it then repeats, in inverted form, its earlier behaviour in $y>1+\delta$.

We stop at this point in our description, observing that so far we have to cover three stages; the "settled" long descent to $y=1$, the stage of dips, during which downward velocity at the successive dips is appreciably increasing (this is a subtlety, and not *really* intuitive[2]), and finally a "shoot-through" from U to $\mathfrak{Z}'$.

7. We have said that the part of the trajectory above $y=1+\delta$ is "very stable". We do not here intend a very precise sense of "stable", and the matter is most intelligible if we consider the reversed motion (r.m.). This is "very unstable": if it is slightly disturbed, then (except for a coincidence) it "slices" or "pulls" away from the original. (See § 32, Fig. 7.)

It is instructive (both here and in later contexts) to recall a number of platitudes about the solution of the linear equations

$$\ddot{y} \pm k\dot{y} + y = k\mu \cos \mu t, \tag{1}$$

where k is large (and positive). The upper sign gives positive damping and stability, the lower sign corresponds to the reversed motion of this (reckoned from $t=0$), with negative damping and instability. The complete direct motions (d.m.) with the positive sign are

$$y = Y(t) + C_1 e^{-\alpha k t} + C_2 e^{-\beta t/k},$$

where

$$Y = \frac{1}{1+(1-\mu^2)^2/(k^2\mu^2)} \left\{ \sin \mu t + \frac{1-\mu^2}{k\mu} \cos \mu t \right\},$$

$$\alpha = \tfrac{1}{2}(1+(1-4k^{-2})^{\frac{1}{2}}) = 1 + O(k^{-2}),$$

$$\beta = 1/\alpha = 1 + O(k^{-2}),$$

[1] There is, naturally, a borderland between trajectories that make one more dip and those that do not: we shall have much to do with it.

[2] See § 15, f.n.

and C_1, C_2 are arbitrary constants which, for an "eventual" trajectory, may be supposed $O(1)$. A further time of the order of 1 now reduces the term $C_1 e^{-\alpha k t}$ to exponential smallness; this is "settling"; *exact* settlement corresponds to $C_1 = 0$, but when C_1 is not 0 there is a small "play" (deviation from exact settlement) depending on the past history of the trajectory. It is often very convenient to speak of settlement as if it were exact, ignoring the "play".[1] There is a similar "play" in "linkage" of y, $\dot{y}$ (for given t), and we sometimes speak of linkage also as if it were exact.

Ignoring the "play" in settlement, i.e. making $C_1 = 0$, gives us the 1-parameter family

$$y = Y + C_2 e^{-\beta t/k}. \tag{2}$$

Its members converge to the p.m. $y = Y$, but slowly, the time of half-decay being of the order k.

A small disturbance of (2) can be of two very different kinds; the one involves a coincidence that keeps $C_1 = 0$ and slightly alters C_2 only; here the new trajectory runs nearly parallel to the old and slowly converges to it; the other kind creates in addition a small C_1; here the new trajectory quickly arrives within an exponentially small distance of the trajectory (2) with the new C_2 (or of the original trajectory itself in the case where C_2 is not disturbed).

The r.m. (solutions of (1) with the negative sign) have their separate lesson. There is a 1-parameter family whose equation is

$$y = Y(-t) + C_2 e^{\beta t/k}. \tag{3}$$

The trajectory (3) is approximately periodic over a time of the order 1, and for $C_2 > 0$ the mean heights of the successive waves increase, very slowly. A slight disturbance *can* be to a neighbouring member of the family, but apart from this coincidence will introduce a term $C_1 e^{\alpha k t}$ involving a violent "pull" or "slice". Note that the d.m. (2) are "normal", if we ignore play; their reversals (3) are abnormal.

8. The foregoing remarks about the linear equation illustrate what we said about the "settled" motion of trajectories of (E) in $y > 1 + \delta$. We now briefly consider motion in $|y| < 1 - \delta$, which we do first in reversed form, with its positive damping. So long as the (reversed) motion stays in $|y| < 1 - \delta$, the conditions apply that led to (1) and (2) of § 6. The trajectory is approximately the periodic curve

$$\tfrac{1}{3} y^3 - y = F(y) = C + b \sin \varphi \tag{1}$$

[1] Exact settlement in the case of equation (E) could only be defined at the very end of the story, not at the beginning.

over a single λ-wave at a time, the successive C's increasing by $[y_1]/k$, where $[y_1]$ is the increment of $\int y\,dt$ over the preceding wave. For the curve (1) to lie in $|y|<1$ it is necessary and sufficient that $C\pm b$ both lie in $(-\frac{2}{3},\frac{2}{3})$ (the relevant range of $F(y)$), or that

$$-(\tfrac{2}{3}-b)<C<\tfrac{2}{3}-b;$$

this requires $b<\frac{2}{3}$ (and is one aspect of the critical value $\frac{2}{3}$ for b), and then the inequality for C can be satisfied. The r.m., and also the d.m., of period λ (the latter totally unstable when $b<\frac{2}{3}$), have each the same equation $y=y(\varphi)$; this is of the form

$$\tfrac{1}{3}y^3-y=b\sin\varphi+O(k^{-1}), \tag{2}$$

for *all* t, but φ has the values $\varphi=\alpha\pm\mu t$ for d.m. and r.m. respectively.

It is fairly clear that if C is initially, say, positive, then in the r.m. (where $\varphi=\alpha-\mu t$) the $[y_1]$ are negative, and C decreases (as long as it is positive). There is in fact convergence of the r.m. to the r.m. (2). To the convergence of a r.m. by decrease of C to 0 corresponds a d.m. whose successive λ-waves gradually rise until their maxima are all near $y=1$, when fresh complications naturally arise into which we need not go. But it is to be observed that, in accordance with the end of § 7, the slowly rising d.m. is abnormal and unstable, and if slightly disturbed will "pull" or "slice" away. Incidentally, *all* prolonged direct motion in $|y|\leq 1$ is abnormal and unstable.

9. We now proceed, after this partly tentative survey, to follow more closely the strict account of the course of the trajectory of § 6 and Fig. 1 up to its arrival at $\mathfrak{Z}'$.

The discussion of the first two stages (the first to U_1, the second from U_1 to U) is based on two key lemmas whose proofs we take for granted.

The first of these is

LEMMA A. (i) *Let $0<b<2$ (say), and let d be a non-negative and d' a positive constant. Then there is a $k_0(\lambda,d,d')$ such that when $k\geq k_0$ the following properties hold.*

Suppose that an eventual trajectory Γ (for which consequently $|y|<L$) has a piece XYZ $(t_X<t_Y<t_Z)$ lying entirely in $y\geq 1-dk^{-\frac{1}{3}}$; suppose also that (a) *XY has a time length at least d';* (b) *YZ contains a point at which $\varphi\equiv-\frac{1}{2}\pi$;* (c) *$YZ$ has time length at least $k^{-\frac{1}{3}}\log k$.*[1] *Then for any Q of YZ*

[1] (1) Conditions (a), (b), (c) are all fulfilled, with $d'=L$, if XZ has time length at least $\frac{3}{2}\lambda$, say (take Y so that XY has time length $\frac{1}{2}\lambda$).

It would be sufficient (as the proof shows), so far as the inequality for $|\dot{y}|$ in (1) below is

$$|\dot{y}| < A(d, d'), \ |\ddot{y}| < A(d, d') k^{\frac{1}{2}}, \ |\dddot{y}| < A(d, d') k; \tag{1}$$

$$\dot{y}(y^2 - 1) = b\mu \cos\varphi + O(A(d, d') k^{-\frac{1}{2}}). \tag{2}$$

In the $\dot{y}$-identity[1]

$$\tfrac{1}{3}y^3 - y = F(y) = C + b(1 + \sin\varphi) - y_1/k - \dot{y}/k \tag{3}$$

we may use any convenient point of XY to determine the constant C, and in the stretch YZ we may thus substitute $\dot{y} = O(A(d, d'))$.

(ii) *Suppose that $0 < b < 2$ and that d is a positive constant. At a Q that has been preceded by a piece of an eventual trajectory Γ, lying above $y = 1 + d$ and lasting a time $k^{-1}\log^2 k$ at least, we have, provided that $k \geq k_0(\lambda, d)$,*

$$|\dot{y}| < A(d), \ |\ddot{y}| < A(d), \ \ldots, \tag{4}$$

with various consequences, e.g. (2) *is valid with error-term improved to $O(A(d, d')k^{-1})$.*

The d in each part of Lemma A is chosen in different ways on different occasions, generally as a $1/D$ in part (i), and generally either as an A or else as δ in part (ii). (d and d' are blank cheques.) It will be observed that Lemma A (i) is specially designed to deal with the question of "dips" (it follows the facts very closely). The hypothesis in (i) that YZ contains a point $\varphi \equiv -\frac{1}{2}\pi$ should be noted; the result is false without it[2] (in fact all the apparent complications follow the actual facts very closely).

Part (ii) is the precise expression of the "settling" effect of a region $y > 1 + d$.

10. We shall now follow the course of a Γ starting at a vertex[3] $\mathfrak{Z}$, with

$$\dot{y}_0 = 0, \quad |y_0 - 2| < D_0 k^{-1}, \quad |\varphi_0 - \tfrac{1}{2}\pi| < D_0 k^{-\frac{1}{2}}.$$

D_0 is ultimately going to be a particular D. We shall find that (subject to an exception) Γ presently arrives at an inverted vertex $\mathfrak{Z}'$ satisfying

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concerned, to replace (a) by the hypothesis (in effect more general) that XY contains a point R for which $|\dot{y}_R| < d'$. But the corresponding results for $|\ddot{y}|$, $|\dddot{y}|$ would be less simple.

(2) We may fix ideas by supposing that Z is *on* $y = 1 - dk^{-\frac{1}{2}}$ (an important case in any event); for if Z is above this line we can prolong the trajectory until it meets the line for the first time (all conditions remaining fulfilled *a fortiori*).

[1] The right-hand side of (3) can take various forms; it may begin, e.g. $C + b\sin\varphi$, $C + b(1 + \sin\varphi)$, or $C - b(1 - \sin\varphi)$ according to convenience.

[2] This is a sophisticated affair; the examples are certain of the trajectories rising slowly off the unstable p.m. in $|y| < 1$. These can end with a piece above $y = 1$ of time-interval $\lambda + o(1)$, and a downward velocity of arrival at $y = 1$ greater than any assigned L.

[3] $\mathfrak{Z}$ is the initial of "zenith".

$$\dot{y}_0' = 0, \qquad |y_0' + 2| < D' k^{-1}, \qquad |\varphi_0' + \tfrac{1}{2}\pi| < D' k^{-\frac{1}{2}},$$

where, *provided* $k \geq k_0(\lambda, D_0)$, *D' is a particular D that does not depend on the choice of the original D_0.*[1] This independence on the part of D', achieved by the *dependence* of k_0, is completely vital to the argument. It enables us, breaking a vicious circle, to *choose* $D_0 = D'$ (k_0 then becoming a particular D); then $\mathfrak{Z}'$ satisfies the same condition (inverted) as $\mathfrak{Z}$, and we can start afresh from $\mathfrak{Z}'$. We mention all this as an advance warning, because much happens (and the "circle" is a long one) before we reach $\mathfrak{Z}'$ (in § 15).

In the notation of Lemma A we take X (and $t=0$) at $\mathfrak{Z}$ and we take Y to be the time-point $\log^2 k/k$ later, or the first arrival of Γ at $y = 2 - \frac{1}{10}$, whichever happens first. Over XY we have, setting $\tau = k\int_0^t f\,dt$,

$$\frac{d}{dt}(-\dot{y}e^{\tau}) = u e^{\tau}, \qquad u = -bk\mu\cos\varphi + y,$$

$$-\dot{y} = \int_0^t u(t')\exp\{-\tau(t) + \tau(t')\}\,dt'.$$

Since $|u| < Lk$ and $\tau(t) - \tau(t') \geq Lk(t - t')$, this gives

$$-\dot{y} < Lk\int_0^t e^{-Lk(t-t')}\,dt' < L,$$

and so
$$y > y_0 - Lt > 2 - D_0 k^{-1} - Lk^{-1}\log^2 k > 2 - \tfrac{1}{10}.$$

Hence it is $t = \log^2 k/k$ that happens first, and XY has time-length $\log^2 k/k$.

Next we have (over the whole Γ)

$$F(y) - F(1) = C + b(1 + \sin\varphi) - k^{-1}y_1 - \dot{y}k^{-1}, \tag{1}$$

in which, on substituting $t=0$ (and $\dot{y}_0 = 0$), we have

$$C = F(y_0) - F(1) - b(1 + \sin\varphi_0).$$

Here
$$F(2) = -F(1) = \tfrac{2}{3}, \qquad F(y_0) = F(2) + O(y_0 - 2) = \tfrac{2}{3} + O(D_0 k^{-1}),$$

$$\sin\varphi_0 = 1 + O((\varphi_0 - \tfrac{1}{2}\pi)^2) = 1 + O(D_0^2 k^{-1}),$$

[1] "D'" is a momentary notation, not to upset a chain $D_1, D_2, \ldots$ we shall presently need.

and so $$C = \tfrac{4}{3} - 2b + O(D_0^2 k^{-1}).[1]$$

Thus (1) becomes

$$F(y) - F(1) = (\tfrac{4}{3} - 2b) + b(1 + \sin\varphi) - k^{-1} y_1 - \dot{y} k^{-1} + O(D_0^2 k^{-1}). \tag{2}$$

Consider now the stretch from Y to the first arrival at $y = 1 + \frac{1}{10}$. Lemma A (ii) *is valid for this, with* $d = \frac{1}{10}$, *and consequently* $|\dot{y}| < L$.

Substituting this in (2) for the special time, t say, of arrival at $y = 1 + \frac{1}{10}$, we have

$$\begin{aligned} k^{-1} y_1 &> (\tfrac{4}{3} - 2b) + 0 - Lk^{-1} - LD_0^2 k^{-1} - (F(1 + \tfrac{1}{10}) - F(1)) \\ &> \tfrac{2}{100} - LD_0^2 k^{-1} - (-\tfrac{11}{10} + \tfrac{1}{3}(\tfrac{11}{10})^3 + \tfrac{2}{3}) = \tfrac{29}{3000} - LD_0^2 k^{-1} > \tfrac{1}{200}, \end{aligned}$$

provided the k_0 of $k \geq k_0(\lambda, D_0)$ is suitably chosen. Then $y_1 > Lk$, and so $t > Lk$. *A fortiori* the stretch from Y to the first arrival, at U_1, say, at $y = 1$ has time-length at least $Lk\,(> 2\lambda)$, and contains points with $\varphi \equiv -\frac{1}{2}\pi$. *Lemma* A (i) [*with* U_1 *for* Z] *is accordingly valid, with* $d' = 1$, and we have $|\dot{y}| < L$ for the stretch YU_1, and in particular $|\dot{y}_{U_1}| < L$ (note that the constant is L, not D); and further we may substitute $|\dot{y}| < L$ in (2).

Supposing, then, that Γ arrives at $y = 1$ for the first time at U_1, let

$$\varphi_{U_1} \equiv -\tfrac{1}{2}\pi - \omega_1 \quad (|\omega_1| \leq \pi), \qquad v_1 = -\dot{y}_{U_1}, \tag{3}$$

and let $$V_1 = v_1 + bk(1 - \cos\omega_1). \tag{4}$$

We have just seen that $v_1 < L$. Next, there is an S of Γ, at most λ before U_1, where $\varphi \equiv -\frac{1}{2}\pi$; the equation of SU_1 is, by Lemma A (i) (3), of the form[2]

$$\tfrac{1}{3} y^3 - y + \tfrac{2}{3} = C' + b(1 + \sin\varphi) - y_1 k^{-1} + O(Lk^{-1}),$$

where we may reckon $y_1 = 0$ at S. Substituting $\varphi = \varphi_S \equiv -\frac{1}{2}\pi$, $y = y_S \geq 1$ in this, we have $C' \geq -Lk^{-1}$. Hence

$$b(1 - \cos\omega_1) = b(1 + \sin\varphi_{u_1}) \leq 0 - C' + Lk^{-1} < Lk^{-1};$$

and so also $|\omega_1| < Lb^{-\frac{1}{2}} k^{-\frac{1}{2}}$.[3] Summing up we have

$$0 \leq v_1 < L, \qquad 0 \leq V_1 < L, \qquad |\omega_1| < Lk^{-\frac{1}{2}}. \tag{5}$$

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[1] Supposing $D_0 \geq 1$ [so $D_0 = O(D_0^2)$].

[2] This, like (2), is the $\dot{y}$-identity with $O(1)$ substituted for $\dot{y}$, but this time with $t = 0$ at S.

[3] We can get the result $b|\sin\omega_1| < Lk^{-\frac{1}{2}}$ at once from A (i) (2); but a special argument is then needed to show that ω_1 is near 0 and not π.

11. We prove next that v and ω, and so all three of v, V, ω, are approximately "linked". For this we need the second key-lemma, which covers the three questions of linkage at u_1, "dips", and the final shoot-through from U. (It retains its form unaltered when we deal with generalized f, g, p.)

LEMMA B.[1] *Consider the (Riccati) equation, for* $x \geq 0$,

$$\frac{dz}{dx} = z^2 - x^2 + 1 + \alpha - 2\beta x, \qquad z(0) = 0,$$

where $\alpha \geq -1$, *and* β *further satisfies* $\beta < 0$ *when* $\alpha = -1$. z *is positive for small positive* x.

There is a $\beta_0(\alpha)$ *such that:*

(i) *if* $\beta > \beta_0$ [*or* $0 > \beta > \beta_0$ *when* $\alpha = -1$], *then* z *changes sign to negative at an* $x = x_0(\alpha, \beta) > 0$;

(ii) *if* $\beta < \beta_0$, *then* $z \to +\infty$ *at an asymptote* $x = x_0(\alpha, \beta) > 0$;

(iii) *if* $\beta = \beta_0$, *there is a solution in* $(0, \infty)$ *for which* $z \geq 0$ *and*

$$z = x + \beta_0 + F(x, \alpha),$$

where F *is continuous in* (x, α), *and* $F = O(1/x)$ *as* $x \to \infty$.

Further, $\beta_0(\alpha)$ *and* $\alpha + \beta_0^2(\alpha)$ *are continuous and monotonic increasing.* $\beta_0(\alpha)$ *is large with large positive* α.

Finally, $\beta_0(\alpha)$ *has the sign of* α.

Return now to Γ and its "arrival at U_1". If

$$v^* = \mu\left(\tfrac{1}{2}b\right)^{\frac{1}{2}},$$

and $\beta_0(\alpha)$ is the function of Lemma B, we find a linkage expressed "implicitly" by

$$\left.\begin{aligned} V_1 &= v_1 + bk(1 - \cos\omega_1), \quad \alpha + 1 = v_1/v^*, \\ V_1 &= v^*(1 + \alpha + \beta_0^2(\alpha)) + o(1) \end{aligned}\right\} \tag{1}$$

(which connect v_1 (or V_1) with ω_1 when the parameter α is "eliminated"). Equivalently we may take the first equation of (1) together with three equations

$$\beta = \mu^{-1}(v^* k)^{\frac{1}{2}} \sin\omega_1, \qquad \alpha + 1 = v_1/v^*, \qquad \beta = \beta_0(\alpha) + o(1), \tag{1$'$}$$

containing two parameters α, β. In the particular case $\alpha = 0$ equations (1) give for V_1 when $v_1 = v^*$,

$$V_1(v^*) = v^* + o(1), \tag{2}$$

[1] The results are mostly fairly intuitive, but they take a good deal of proving.

and, to error $o(1)$, $v_1 \gtrless v^*$ are respectively equivalent to $V_1 \gtrless v^*$. To error $o(1)$, v_1 and V_1 increase with ω_1.[1]

The result (1) depends on Lemma B in a manner that is readily intelligible. We consider the r.m. from U_1 over a time of slightly larger order than $k^{-\frac{1}{3}}$. Its equation (in integral form, and using $\tau = -t$ for the new time, reckoned from U_1 as origin) is

$$\frac{dy}{d\tau} - v_1 = k(F(y) - F(1)) - bk(\sin(-\mu\tau + \varphi_{U_1}) - \sin\varphi_{U_1}) - \int_0^\tau y\, d\tau.$$

We write $y = 1 + v^{*\frac{1}{3}} k^{-\frac{1}{3}} z$, $\tau = v^{*-\frac{1}{3}} k^{-\frac{1}{3}} x$, expand in powers of z and x, and reject terms with coefficients $o(1)$, in particular the one arising from $\int_0^\tau y\, d\tau$. The result of this process, when we write $\alpha = v_1/v^* - 1$, and when β has the value in (1)′, is the equation

$$\frac{dz}{dx} = z^2 - x^2 + 1 + \alpha - 2\beta x \tag{3}$$

of Lemma B. If we add a $o(1)$ to the right-hand side, this is actually valid for the y of Γ reversed, over a range of x of length $\log k$, say. Moreover the error is of the form $O(A(D_0)k^{-A})$, and by slightly diminishing the index A, *and rechoosing* $k_0(L, D_0)$, it becomes $O(k^{-A})$, *independent of* D_0.[2] Now according to Lemma B, if we have $\beta > \beta_0(\alpha)$, then z becomes negative at an $x = x_0(\alpha, \beta)$; if $\beta < \beta_0(\alpha)$, then $z \to \infty$ at an asymptotic $x = x_0'(\alpha, \beta)$. But the z corresponding to Γ reversed certainly does neither of these things; we infer (restoring a $o(1)$) that α and β must be connected by

$$\beta = \beta_0(\alpha) + o(1). \tag{4}$$

The result (4) is in fact true, with a precise error-term in place of $o(1)$ that we need not particularize, and, like that in (3), independent of D_0 (since the bounds for v_1, ω_1 are).

12. We shall in future work mostly in terms, not of v_1's, but of the V_1's "linked" with them; there is more than one reason for preferring V_1, as will appear later. This being so, we need (to avoid mixing symbols later) a symbol for the V_1

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[1] In the case of V_1 because $\alpha + \beta_0^2(\alpha)$ is increasing.

[2] (1) We mean, of course, more precisely, that the *upper bound* implied in the error term is independent of D_0 (and similarly in future).

(2) A's occurring as indices are always absolute constants.

linked with $v_1 = v^*$; and (in the light of (2) of § 11) we *define* V^* to be v^*, so that $V_1(v^*) = v^* + o(1) = V^* + o(1)$. There are now three possibilities about V_1:

$$\text{(i)}\ V_1 > V^* + \delta; \qquad \text{(ii)}\ V_1 < V^* - \delta; \qquad \text{(iii)}\ |V_1 - V^*| \leq \delta.$$

The third we describe as a "gap" case; V lies in a critical gap round V^*.

Consider now the d.m. from u_1 over a range of slightly larger order than $k^{-\frac{1}{2}}$, and write $y = 1 - v^{*\frac{1}{2}} k^{-\frac{1}{2}} \zeta$, $t = v^{*-\frac{1}{2}} k^{-\frac{1}{2}} x$ (ζ with a minus sign since y is moving below $y = 1$); again we expand in ζ and x, and ignore terms with coefficients $o(1)$: the result of the changes of sign as against the (z, x) r.m. equation is

$$\frac{d\zeta}{dx} = \zeta^2 - x^2 + 1 + \alpha - 2(-\beta)x, \tag{1}$$

that is to say the equation (3) of § 11 with ζ for z and $-\beta$ for β.

Now with error $o(1)$ we have $\beta = \beta_0(\alpha)$; if further we are in case (i), then $\alpha > 0$,[1] and β having (by Lemma B) the sign of α, is positive. Then $-\beta < 0 < \beta_0(\alpha)$. By Lemma B this implies[2] that a ζ satisfying (1) tends to $+\infty$ at a finite asymptote $x = x_0'(\alpha, \beta)$. Corresponding to this we expect the $y - 1$ of Γ to become a large negative multiple of $k^{-\frac{1}{2}}$, and $\dot{y}$ to become large and negative, at a time comparable with $k^{-\frac{1}{2}}$. This does happen (when we go properly into the error-terms); the negative damping then takes hold (as the reader will easily believe: the arguments become cruder), and it is the fact that y acquires a downward velocity of order k and passes through $|y| \leq 1$ in a short time.

Consider now case (ii).[3] Here $\alpha < 0$, and β has the sign of α and is negative; $-\beta > 0 > \beta_0(\alpha)$. A ζ satisfying (1) therefore changes sign at $x = x_0(\alpha, \beta)$. The corresponding behaviour of the y of Γ is to descend below $y = 1$ to a depth $D_1 k^{-\frac{1}{2}}$ at most, and emerge upwards, at U_2 say, after time $D k^{-\frac{1}{2}}$ at most. Observe that, in accordance with what was said at the end of § 11, *D_1 and the D in $D k^{-\frac{1}{2}}$ above do not depend on D_0.*

Suppose now the next return of Γ to $y = 1$ is at U_2. Between U_1 and U_2, Γ *conforms to the hypothesis of Lemma A with $d = D_1$, provided k_0 is re-chosen to depend*

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[1] More precisely $\alpha > D$ (D arising from the δ of the gap $V^* \pm \delta$).

[2] Note the double use of Lemma B; first to establish an approximate equality $\beta = \beta_0(\alpha)$ because "neither event happens"; then, because the equality is upset by changing the sign of β, to infer that one or other event *does* happen in the new case.

[3] We shall here ignore the case $v_1 = 0$, or $\alpha = -1$, which would need a special treatment in Lemma B. Its apparently critical character is spurious; since a small dip is found to turn upwards, it is obvious that a graze will do so also.

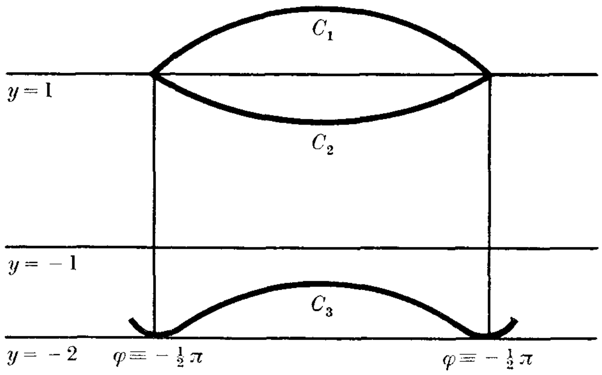

Fig. 2.

on D_1. The equation of Γ between U_1U_3 is accordingly given by Lemma A (3), slightly rearranged;

$$\tfrac{1}{3}y^3-y+\tfrac{2}{3}=C'+b(1+\sin\varphi)+o(1).$$

In this, since $y_{U_1}=1$ and $\varphi_{U_1}\equiv-\frac{1}{2}\pi+o(1)$, the constant C' is $o(1)$. Hence the curve U_1U_3 satisfies

$$\tfrac{1}{3}\eta^3+\eta^2=\tfrac{1}{3}y^3-y+\tfrac{2}{3}=b(1+\sin\varphi)+o(1). \tag{2}$$

13. We must now enter on something of a digression. When $b<\frac{2}{3}$ the locus

$$\tfrac{1}{3}y^3-y+\tfrac{2}{3}=b(1+\sin\varphi),$$

over a period λ, consists of three separate C_1, C_2, C_3, as shown in Fig. 2. In accordance with what was said earlier, C_3 and C_1, which are in $|y|\geq 1$, are approximations to stable trajectories, and C_2, which is in $|y|\leq 1$, is an approximation to a stable r.m. or unstable d.m. We shall have more to say about these curves later. Meanwhile we expect Γ to follow approximately the curve C_1, and to reach $y=1$ at a U_3 near the right-hand end of C_1. But two points remain to be established.

(i) The approximation (2) of § 12 shews that Γ is always near *one or other* of C_1, C_2, but not that it cannot shift from C_1 to C_2 while still near U_1. To settle this we need to go more closely into the actual error-terms we have presented as $o(1)$'s: when we do this (and the extensions are natural ones) we find that Γ keeps above $y=1$ for a time long enough after U_2 for C_1 and C_2 to have diverged more than the error-term in y; a shift is impossible. So Γ continues along C_1, *approximately*, as far as the right-hand end of C_1.

(ii) We have to show that Γ does not turn upwards near this point before actually reaching $y=1$. To deal with this[1] we need the following general principle.[2]

14. LEMMA C. *Suppose y_1, y_2 are respectively solutions of*

$$\dot{y} = \Phi(y, t) + R_{1,2},$$

where Φ is continuous in (y, t), $R_{1,2}$ are continuous, and $R_1 > R_2$ for $t \geq t_0$.

(i) *If now $y_1(t_0) \geq y_2(t)$, then $y_1 > y_2$ for $t > t_0$.*

(ii) *The conclusion is true if $R_1 > R_2$ for $t > t_0$ only, provided we know independently that $y_1 > y_2$ for small positive $t - t_0$.*

We state the complete truth. There is a touch of subtlety in that (ii) is not true without the final proviso.

We give the easy proof. For $t > t_0$ the y_1 curve is initially above the y_2 one, by hypothesis in (ii), and for slightly different reasons in (i) according as $y_1(t_0) > y_2(t_0)$ or $y_1(t_0) = y_2(t_0)$. If ever there is an intersection, let Q be the first one after $t = t_0$. At Q we have, as a matter of geometry, $\dot{y}_1 \leq \dot{y}_2$, contrary to $\dot{y}_1 - \dot{y}_2 = R_1 - R_2 > 0$.

Returning to Γ, let t_0 be a time long enough before t_{u_1} for y to be greater than 1 for several λ-periods after t_0. For $t_0 \leq t \leq t_{u_1}$, let $y(t)$ be the y of Γ, and let $y_1 = y(t)$, $y_2 = y(t+\lambda)$. The $\dot{y}$-identities for $y_{1,2}$ are of the form

$$\dot{y} = -kF(y) + R_{1,2},$$

where [see § 5, (1)] we have

$$R_1 - R_2 = \int_t^{t+\lambda} y\, dt,$$

and this is certainly positive until $y(t+\lambda)$ is 0 for the first time, at $t = \tau$, say. Next, it is possible to find a t_0 as above such that $y_1(t_0) > y_2(t_0)$; for over a *sufficiently* long time Γ certainly descends. It follows from Lemma C that $y_1 > y_2$ for $t_0 \leq t \leq \tau$. Since $y_1(t_{U_1}) = 1$, we cannot have $y(t) \geq 1$ for all t of $t_{U_1} \leq t \leq t_{U_1} + \lambda$ [which would imply both $\tau \geq t_{U_1}$ and $y_2(t_{U_1}) > 1 = y_1(t_{U_1})$]; and this is the desired result.

15. We find, then, that Γ arrives downwards at $y = 1$ for the first time after U_1 at a U_3, near $U_1 + \lambda$, and conforming to the hypothesis of Lemma A, with $d = D_1$ (the number associated with the dip at $U_1 U_2$, § 12). *There is a linkage of v_2, V_2, ω_2 at U_3; further*[3]

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[1] The subtleties involved are curious, but we do not see how to avoid them.

[2] Constantly used in the full account.

[3] The identity in (1) is one reason for working with V rather than v.

$$V_2-V_1=[y_1]_{u_1}^{u_2}=M+o(1), \tag{1}$$

where $M=\int_0^\lambda Y_1\,dt$, *and* $y=Y_1(t)$ *is the equation of* C_1; M *is of the form* $A(\lambda)$.

The three alternatives concerning "V_2" and the gap $V^*\pm\delta$ arise again at U_3; *either* V_2 *is in the gap; or* $V_2>V^*+\delta$ *and the velocity* $\dot{y}$ *becomes (negatively) large; or, finally,* $V_2<V^*-\delta$ *and there is another dip, of depth* $D_2k^{-\frac{1}{2}}$ *at most,* Γ *conforms to Lemma A with* $d=\operatorname{Max}(D_1, D_2)$, *and, if* k_0 *is rechosen to depend on* D_2, *there is a fresh arrival at* $U_5=U_3+\lambda+o(1)$, *with* $V_3-V_2=M+o(1)$;[1] *and so on.*

Now, by (1), if we do not arrive at a V in the gap, we must have $V_n>V^*+\delta$ after $n=1+(V^*+\delta)/(M+o(1))$ dips at most. And then *the final* D_n *and* d *are* D's, *independent of* D_0, *and the final* k_0 *is a* $k_0(\lambda,\delta,D_0)$.

Let us sum up, with partly changed notation,[2] what has been so far proved. There is a $k_0(\lambda,\delta,D_0)$; *if* $k\geq k_0$, Γ *starting at a vertex* of type $\dot{y}_0=0$, $|y-2|<D_0k^{-1}$, $|\varphi_0-\frac{1}{2}\pi|<D_0k^{-\frac{1}{2}}$, descends in accordance with the description of § 6; after time at least Lk it arrives, after possible dips, at U, on $y=1$, with

$$-\dot{y}_U=v,\ \varphi_U=-\tfrac{1}{2}\pi-\omega,\ V=v+bk(1-\cos\omega),$$

v, V, ω linked by (1) of § 11, the errors having bounds independent of D_0, and finally either (i) $V^*-\delta\leq V\leq V^*+\delta$, or else (ii) $V^*+\delta<V<V^*+\delta+M+o(1)$.

Suppose the gap case (i) does not occur, so that case (ii) does. Then, as we said in § 12, the motion becomes comparatively crude, and it is fairly easy to show that Γ descends in time $O(Dk^{-\frac{1}{2}})$ to an inverted vertex $\mathfrak{Z}'$ with $\dot{y}_{\mathfrak{Z}'}=0$, at which[3]

$$|y+2|<A_1(\delta)k^{-1},\ |\varphi+\tfrac{1}{2}\pi|<A_2(\delta)k^{-1},$$

where A_1 *and* A_2 *are independent of* D_0. *We are now able, without a vicious circle, to choose* $D_0=\operatorname{Max}(A_1,A_2)$. Then Γ, restricted by the D_0 inequalities at the starting vertex $\mathfrak{Z}$,

[1] We can now, but only now, infer the increase of v, linked with V, at successive dips.

[2] We do not consider dips again, and the $U_1, U_2, \ldots$ associated with them do not appear again, nor do the associated V_1, etc. It will therefore not cause confusion if, from now on, we use v, V for the things associated with the "final" U, and dashes (U', v', V') for "inverted" things, associated with $y=-1$.

[3] We first show that the time is $O(Dk^{-\frac{1}{2}})$, so that $|\varphi_{\mathfrak{Z}'}+\frac{1}{2}\pi|<Dk^{-\frac{1}{2}}$; and then we have, by the $\dot{y}$-identity between U and $\mathfrak{Z}'$

$$0=\dot{y}_{\mathfrak{Z}'}=bk(1+\sin\varphi_{\mathfrak{Z}'})-V-k(F(y_{\mathfrak{Z}'})-(F(1))-\int_0^{\mathfrak{Z}'}y\,dt$$

$$=O(k(Dk^{-\frac{1}{2}})^2)+O(1)-k(F(y_{\mathfrak{Z}'})-F(-2));$$

$$F(y_{\mathfrak{Z}'})-F(-2)=O(Dk^{-1}),\quad\text{and so}\quad |y_{\mathfrak{Z}'}+2|=O(Dk^{-1}).$$

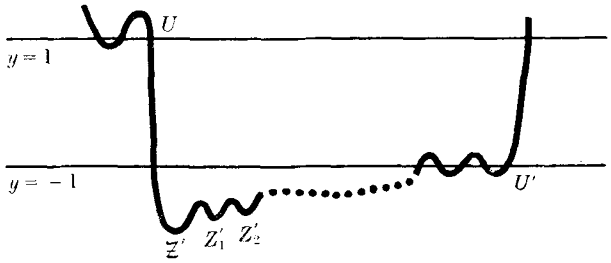

Fig. 3.

has arrived at an inverted vertex $\mathfrak{Z}'$ obeying the same restriction (in appropriate inverted form).[1] *It therefore repeats in inverted form its former behaviour, arriving, after inverted dips, at U' on $y=-1$ with*

$$\dot{y}_{U'}=v',\ \ \varphi_{U'}=\tfrac{1}{2}\pi-\omega',\ \ V'=v'+bk(1-\cos\omega'),$$

v', V', ω' are linked by (1) of § 11 (with dashed letters), and either V' is in the gap $V^*\pm\delta$, or else

$$V^*+\delta<V'<V^*+\delta+M+o(1).$$

16. We denote by $Z_1, Z_2, \ldots$ the time-points after $\mathfrak{Z}$ where $\varphi\equiv\frac{1}{2}\pi$; $N_1, N_2, \ldots$ those where $\varphi\equiv-\frac{1}{2}\pi$, N_1 being between Z_1 and Z_2, etc. These are approximately maxima and minima in the long descent.[2] The corresponding Z', N' similarly succeed $\mathfrak{Z}'$, and have $\varphi\equiv-\frac{1}{2}\pi, \frac{1}{2}\pi$ respectively. The stretch of a Γ from Z_1 to the ensuing Z'_1, or Z'_1 to the ensuing Z_1; or, again from U to the ensuing U' (or U' to U), we call a half-cycle.

We call a Γ "gap-free" in a range if it has, in the range, no V or V' in the gap $V^*\pm\delta$.

The $\dot{y}$-identity for Γ, between a U with its $V>V^*+\delta$ to the ensuing U' is

$$V'+V=-(\tfrac{4}{3}-2b)-\int_U^{U'} y\,dt. \tag{1}$$

Suppose now that $\Gamma_{1,2}$ "start" in the stream at Z_0 subject to the "D_0-inequalities" of § 15 (or that they join this stream after an earlier start). Suppose further that $\Gamma_{1,2}$ are gap-free, and, further again, that they "miss the same gaps", i.e. have

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[1] Inverted behaviour may be inferred from direct by changing y into $-y$, and φ into $\pi+\varphi$.

Our arguments implicitly prove, of course, that a trajectory, if started with not too violent initial velocity, and if remaining an unbroken time $\frac{3}{2}\lambda$ above $y=1$, will behave similarly, i.e. will arrive, after possible dips, at $y=1$ with a "gap" V, or else pass through an inverted vertex of the special kind.

[2] $\mathfrak{Z}$ is a *real* vertex, with exceptional behaviour; it is near Z_0, where $\varphi\equiv\frac{1}{2}\pi$.

their $U_{1,2}$ near together and $U'_{1,2}$ near together. These simplifying assumptions, which have to be justified in the strict account, enable us to steer clear of some complications.[1]

It is now the case that $\Gamma_{1,2}$ converge. We aim first, however, at their quasi-convergence, namely convergence (of y_1-y_2 and $\dot{y}_1-\dot{y}_2$ to 0) with error $O(\zeta)$, where ζ stands, throughout what follows, for things of the form $\exp(-Dk)$ (exponentially small).

17. Let $\Delta V=V_2-V_1$, and similarly for $\Delta\omega$, and other things associated with $\Gamma_{1,2}$. Let $w=\Delta y=y_2-y_1$. We have from (1) of § 16

$$\Delta V'=-\Delta V-\int_{U_1}^{U_1'} w\,dt-R,\quad R=\Big(\int_{U_1'}^{U_2'}-\int_{U_1}^{U_2}\Big)\,y_2\,dt.$$

The ranges of the integrals in R have lengths $\Delta\omega'$ and $\Delta\omega$.

So
$$\Delta V'=-\Delta V-\int_{U_1}^{U_1'} w\,dt+O(\Delta\omega)+O(\Delta\omega'). \tag{1}$$

The key-result (important also for results other than the convergence) is

LEMMA D. *To error $O(\zeta)$, ΔV and $\Delta V'$ have opposite signs, and*

$$L|\Delta V|-\zeta<|\Delta V'|<\alpha|\Delta V|+\zeta,$$

where α is an L satisfying $0<\alpha<1$.

We shall see in § 23 that $\Delta\omega=o(\Delta V)+O(\zeta)$, and similarly for $\Delta\omega'$; thus $\Delta\omega$, $\Delta\omega'$ are negligible in (1), and to get Lemma D from (1) is then equivalent to proving that, to error $O(\zeta)$, and normalizing to $(-\Delta V)>0$, $\int_{U_1}^{U_1'} w\,dt$ lies between multiples L and $\alpha<1$ of $(-\Delta V)$. The half-cycle U_1U_1' has a length of order Lk.

The number w satisfies the equations (the three alternatives have each their rôle)

$$\left.\begin{aligned}\dot{w}&=-\dot{T}w+c_0-w_1\\ \frac{d}{dt}(e^T w)&=(c_0-w_1)\,e^T,\\ w&=w_0e^{-T}+e^{-T}\int_0^t(c_0-w_1)\,e^T\,dt.\end{aligned}\right\}\tag{W}$$

[1] It is actually the case that when b belongs to $\mathcal{B}_1$ or $\mathcal{B}_2$ Γ exist that are gap-free for all positive time, but they need not exist for b of the excluded intervals. (And the very introduction of $\mathcal{B}_{1,2}$ comes naturally only *after* the work we are describing.) Further, it is a troublesome complication to work into the rest of the argument the proof that $\Gamma_{1,2}$ have their $U_{1,2}$ ($U'_{1,2}$) near together at the *next* place under consideration. (We allow, in short, vicious circles that the strict account, with difficulty, has to avoid.)

In these the origin $t=0$ is arbitrary, and $c_0=\dot{w}_0+\dot{T}_0 w_0$; the function $T=T(t)$ is $k\int_0^t u\,dt$, where $u=\Delta F(y)/\Delta y$; and $w_1=\int_0^t w\,dt$. The equations (W) are for the d.m. $\Gamma_{1,2}$ (d.m.'s are always understood unless the contrary is stated). The corresponding equations for r.m. $\Gamma_{1,2}$ are also very important; they are

$$\left.\begin{aligned} \dot{w} &= \dot{T}w+\bar{c}_0-w_1, \\ \frac{d}{dt}(e^{-T}w) &= (\bar{c}_0-w_1)\,e^{-T}, \\ w &= e^{T}w_0+e^{T}\int_0^t(\bar{c}_0-w_1)\,e^{-T}\,dt, \end{aligned}\right\} \qquad (\bar{\mathrm{W}})$$

where $\bar{c}_0=\dot{w}_0-\dot{T}_0 w_0$, *and* T *is* $k\int_0^t(\Delta F/\Delta y)\,dt$ *as before.*[1]

For d.m. and r.m. with the same $t=0$ we have $\bar{c}_0=-c_0$.

18. Lemma D is concerned with the half-cycle U to U'.[2] There are two possibilities: (i) $\Gamma_{1,2}$ intersect somewhere in the half-cycle; (ii) they do not so intersect. In (ii) there are several things to our advantage: w and w_1 are of constant sign, and (W), ($\bar{\mathrm{W}}$) become much easier to handle; further, as a matter of the mere geometry, $\Delta\omega$ and $\Delta\omega'$ have opposite signs.[3] In (i) the actual facts are that if, with a small reservation, $\Gamma_{1,2}$ have an intersection in a half-cycle, then their y, $\dot{y}$ differ by $O(\zeta)$ throughout the half-cycle. From this it naturally follows that ΔV, $\Delta V'$ (and $\Delta\omega$, $\Delta\omega'$) are $O(\zeta)$; and Lemma D is true in a degenerate form.

It is instructive to make the separation (i), (ii), but actually a whole group of results and arguments are closely interwoven and cannot be separated, and we follow a different division of cases. Moreover we begin at Z_1, and consider the (near) whole cycle from Z_1, through the ensuing $U_{1,2}$, to the ensuing $U'_{1,2}$. The results in the two new cases are (α) and (β) following.[4]

(α) *There is a certain* ζ_1, *with the following properties: If* $w(Z_1)>\zeta_1$,[5] *then* $\Gamma_{1,2}$

[1] The increment of T over a given stretch is the same for the d.m. PQ as it is for the r.m. QP. But e.g. the two $\int_0^t e^{-T}dt$ are *not* the same.

[2] Here we use, e.g., "U" for the general neighbourhood of $U_{1,2}$, and there is a similar slight looseness in speaking of the "half-cycle".

[3] Since V and ω are "linked", we expect as a consequence that ΔV and $\Delta V'$ will, at least approximately, have opposite signs, and this appears in due course.

[4] (α), (β), ..., (ζ) are stages of one sort or another in the journey to Lemma D.

[5] We can normalize $|w(Z_1)|>\zeta_1$ to $w(Z_1)>\zeta_1$ (altering the numbering of $\Gamma_{1,2}$ if necessary). When there is no intersection we then have $U_1<U_2$, $U'_2<U'_1$.

do not intersect before U_2'; *and the ratio of any two of* $k\,w(Z_1)$, $k\,w(Z_1')$, $c(Z_1)$, $c(Z_1')$, $k^{\frac{1}{2}} w(U_1)$, $k^{\frac{1}{2}} w(U_2')$, $c(U_1)$, $c(U_2')$, *lies between two L's.*[1]

(β) *If* $|w(Z_1)| \le \zeta_1$, *then* w, $\dot{w} = O(\zeta)$ *in* (Z_1, U_2'), *and* $\Delta V'$ *and* ΔV *are* $O(\zeta)$ [*so that Lemma D is true in this case*].

Once (α) has been proved, (β) is easily proved by similar and much easier arguments, and we will take its truth for granted. (α) and (β) together lead quickly to Lemma D (the final steps occupy only the short § 23). The main problem is accordingly (α), but this is formidable.

To attack it we introduce a mesh of Γ's of the stream intermediate, at Z_1, between $\Gamma_{1,2}$, and such that for every pair of consecutives of the mesh we have $o < w < k^{-10}$.[2] Since $k^{10}\zeta$ is still a ζ, *is is enough to prove* (α) *for an arbitrary pair of consecutives of the mesh.* This is because the extreme $\Gamma_{1,2}$ do not intersect if the consecutives do not, and further the numbers w, c, whose ratios are considered in (α) are all additive over a non-intersecting mesh,[3] and have the various properties of (α) if each pair of consecutives have them.

We may suppose, then, that $\Gamma_{1,2}$ are consecutives of the mesh, with w satisfying $w(Z_1) < k^{-10}$ as well as $w(Z_1) > \zeta_1$. It is then the fact, as one could expect, that $|w| < k^{-5}$ throughout (Z_1, U_2'). The effect of this is that $T = k\int_0^t (\Delta F/\Delta y)\,dt$ differs negligibly from $\tau = k\int_0^t f(y_1)\,dt$. Now τ depends only on the single (gap-free) Γ_1, and, in spite of the complications of the unsettling region $|y| \le 1$, we are in fact able to calculate τ, from a sufficiently approximate knowledge of the behaviour of Γ, so as to control the equations (W), ($\overline{\mathrm{W}}$) that govern the behaviour of w.[4] We will take the complicated details for granted and suppose that we know a complete "dictionary" of τ. The full account is extremely involved, but we will try to give some idea of

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[1] We use, e.g., Z_1, U_2' as names of time-points. $c(t_0)$ means the c_0 of (W) for a time origin at t_0.

[2] By continuity there exists a Γ of the stream through any point of the ordinate at Z_1 between those of $\Gamma_{1,2}$.

[3] Note that $c(t)$ is $\Delta(\dot{y} + kF(y))$. Actually there is additivity for w, c over a mesh, intersecting or not.

[4] The mesh argument is accordingly very powerful. We have, moreover, found ourselves quite unable to dispense with it. The original pair $\Gamma_{1,2}$ have $w = O(k^{-1})$ in $|y| > 1 + L$, and $w = O(Dk^{-\frac{1}{2}})$ in $|y| \ge 1$, except near a $\mathfrak{Z}$ or $\mathfrak{Z}'$; but in $|y| < 1$ w is in general comparable with 1, and $\dot{T}$ is subject to an uncertainty factor comparable with k, as $\Gamma_{1,2}$ perform their unrelated staggerings. Since the time concerned is of order $k^{-\frac{1}{2}}$, the e^T of (W) has an uncertainty factor like $\exp(k^{\frac{1}{2}})$, which is quite ruinous.

the leading arguments.[1] We start from (W), $(\bar{\mathrm{W}})$. In these we suppose, always, that T is replaced by the "known" τ; we include, however, what are essentially strict proofs that the justifying "$w < k^{-5}$" is really true.

19. In (W), $(\bar{\mathrm{W}})$ we generally normalize to w being initially positive; i.e. either $w_0 > 0$, or, if $w_0 = 0$, then $\dot{w}_0 > 0$. With normalized w we have, up to the next[2] intersection (if any)

$$\left.\begin{aligned} &w \le w_0 e^{-\tau} + c_0\,\varphi(t) \quad \text{(d.m. up to intersection)} \\ &\varphi(t) = e^{-\tau}\int_0^t e^{\tau}\,dt. \end{aligned}\right\} \qquad (1)^3$$

Substituting this in (W_3) we have

$$\left.\begin{aligned} &w e^{\tau} \ge w_0 - w_0\,\psi + c_0\left(\int_0^t e^{\tau}\,dt - \vartheta J\right) \text{ (d.m. to intersection)}, \\ &\psi = \iint\limits_{0\le\xi\le\eta\le t} e^{\tau_\eta - \tau_\xi}\,d\xi\,d\eta, \qquad J = \iiint\limits_{0\le\xi\le\eta\le\zeta\le t} e^{\tau_\zeta - \tau_\eta + \tau_\xi}\,d\xi\,d\eta\,d\zeta, \end{aligned}\right\} \qquad (2)$$

and we use ϑ's always for numbers satisfying $0 \le \vartheta \le 1$.[4] If we drop w_0 from the right-hand side in (2) we obtain the two further inequalities (for normalized w)

$$\left.\begin{aligned} &w \ge -w_0 e^{-\tau}\psi + c_0\,\varphi\left(1 - \vartheta J\Big/\int_0^t e^{\tau}\,dt\right), \\ &w \ge c_0\,\varphi\left(1 - \vartheta J\Big/\int_0^t e^{\tau}\,dt - (w_0/c_0)\,\psi\Big/\int_0^t e^{\tau}\,dt\right), \end{aligned}\right\} \text{ (d.m. to intersection)} \qquad (3)$$

of which the second (through true generally) is appropriate only when $c_0 > 0$.

For r.m., and normalized w we have, by similar considerations,

$$w e^{-\tau} \le w_0 + \bar{c}_0 \int_0^t e^{-\tau}\,dt \quad \text{(r.m. to intersection)}, \qquad (4)$$

$$\left.\begin{aligned} &w e^{-\tau} \ge w_0(1 - \bar{\psi}) + \bar{c}_0\left(\int_0^t e^{-\tau}\,dt\right)\left(1 - \vartheta \bar{J}\Big/\int_0^t e^{-\tau}\,dt\right) \text{ (r.m. to intersection)}, \\ &\bar{\psi} = \iint\limits_{0\le\xi\le\eta\le t} e^{-\tau_\eta + \tau_\xi}\,d\xi\,d\eta, \qquad \bar{J} = \iiint\limits_{0\le\xi\le\eta\le\zeta\le t} e^{-\tau_\xi + \tau_\eta - \tau_\zeta}\,d\xi\,d\eta\,d\zeta. \end{aligned}\right\} \qquad (5)$$

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[1] We do this at considerable length and in considerable detail because this is perhaps the most central part of the work, and the arguments the most characteristic.

[2] That is, the next after $t = 0$ if that is one.

[3] The numbering is continuous till further notice.

[4] We write τ_ξ for $\tau(\xi)$ etc. The ϑ is needed in (2_1) because c_0 may be negative.

The special value $t = \frac{1}{4}\lambda$ is important here, and we define the constant μ_0, associated with the origin $t=0$, by

$$\mu_0 = \int_0^{\frac{1}{4}\lambda} e^{-\tau} dt \quad \text{(r.m.)}.^1 \tag{6}$$

With $t = \frac{1}{4}\lambda$, (4) and (5), translated into d.m. language, become: *if there is no intersection of the d.m. for time* $\frac{1}{4}\lambda$ *before* $t=0$, *and if* $w_0 > 0$, *then*

$$\mu_0 c_0 \le w_0,\ \ \mu_0 c_0 (1 - \vartheta \bar{J}(\tfrac{1}{4}\lambda)/\mu_0) > w_0 (1 - \bar{\psi}(\tfrac{1}{4}\lambda)) - \zeta.^2 \tag{7}$$

We set out next part of the τ-dictionary.[3]

$$\left.\begin{aligned} &\iint\limits_{0\le\xi\le\eta\le t} e^{-\tau_\eta + \tau_\xi} d\xi\, d\eta < D k^{-\frac{1}{2}} \quad (\text{d.m.},\ \ 0 \le t \le \tfrac{3}{2}\lambda), \\ \bar{\psi} = &\iint\limits_{0\le\xi\le\eta\le t} e^{-\tau_\eta + \tau_\xi} d\xi\, d\eta < D k^{-\frac{1}{2}} \quad (\text{r.m.},\ \ 0 \le t \le \tfrac{3}{2}\lambda). \end{aligned}\right\} \tag{8}$$

If $(0, t)$ *is contained in* (Z_1, U_1) *and if* $t \le L_1 k$, *where* L_1 *is a certain* L, *then for the d.m. we have*

$$\iint\limits_{0\le\xi\le\eta\le t} e^{-\tau_\eta + \tau_\xi} d\xi\, d\eta < \tfrac{1}{2}, \tag{9}$$

and

$$\psi / \int_0^t e^{\tau} dt < D k^{-\frac{1}{2}}. \tag{10}$$

When $t=0$ *is the special point* U_1, *and* $t \le \frac{3}{2}\lambda$, *then, for the d.m*

$$\psi = \iint\limits_{0\le\xi\le\eta\le t} e^{\tau_\eta - \tau_\xi} d\xi\, d\eta < D \int_0^t e^{\tau} dt. \tag{11}$$

Since

$$J \le \Big(\int_0^t e^{\tau_\zeta} d\zeta\Big) \Big(\iint\limits_{0\le\xi\le\eta\le t} e^{-\tau_\eta + \tau_\xi} d\xi\, d\eta \Big),$$

we have, from (8) and (9),

$$J / \int_0^t e^{\tau} dt < D k^{-\frac{1}{2}} \qquad (\text{d.m.},\ 0 \le t \le \tfrac{3}{2}\lambda), \tag{12}$$

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[1] We do not give μ_0 the bar associated with r.m. because it will appear frequently in d.m. results.

[2] We translate $\bar{c}_0$ into $-c_0$. The ζ comes from $w_0 \exp(-\tau(\frac{1}{4}\lambda))$ in (5).

[3] τ (and for that matter T for the original $\Gamma_{1,2}$) increases by an amount of order Lk for unit increment of t, and e^{-T} acquires a factor like e^{-Lk}. e^{-T} is a "modified e^{-Lkt}".

$$J/\int_0^t e^{\tau}\,dt < \tfrac{1}{2} \quad (0 \le t \le L_1 k, \text{ and subject to the conditions for (9)}). \tag{13}$$

Since[1]
$$\bar{J} \le \Big(\int_0^t e^{-\tau_\xi}\,d\xi\Big)\Big(\iint_{0\le\eta\le\zeta\le t} e^{-\tau_\zeta+\tau_\eta}\,d\eta\,d\zeta\Big) = \Big(\int_0^t e^{-\tau}\,dt\Big)\,\bar{\psi}, \tag{14}$$

we have, from (8), $\quad \bar{J}/\int_0^t e^{-\tau}\,dt < D k^{-\frac{1}{2}} \quad$ (r.m., $0 \le t \le \frac{3}{2}\lambda$).

We write $\mu(t_0)$ for μ_0 with $t=0$ taken at t_0.

If t_0 is preceded by a stretch of Γ_1, of time-length $\frac{1}{4}\lambda$ at least, that has $y > 1+L$, then

$$\mu(t_0) \lessgtr L k^{-1}. \tag{15}$$

In particular (15) is true for any Z of (Z_1, U_1).

If t_0 is in (Z_1, U_1), then

$$L k^{-1} < \mu(t_0) < D k^{-\frac{1}{2}}. \tag{16}$$

For the special point U_1 $\quad \mu(U_1) \lessgtr L k^{-\frac{1}{2}}$.[2] $\quad$ (17)

The function $\varphi(t) = e^{-\tau}\int_0^t e^{\tau}\,dt$, formed (for the d.m.) with $t=0$ at t_0, we call $\varphi_{t_0}(t)$, and we write $\varphi_{Z_1}(t)$, etc. We have (as part of the τ-dictionary)

$\varphi_{t_0}(t) < D k^{-\frac{1}{2}}$ *if (t_0, t) is contained in (Z_1, U_1) and $t > t_0$*.[3] $\quad$ (18)

$$\left.\begin{aligned} &\varphi_Z(t) < L k^{-1} \text{ for a } Z \text{ in } (Z_1, U_1) \text{ and } t \ge Z;\\ &\varphi_Z(t) > L k^{-1} \text{ if } t - Z \ge \tfrac{1}{4}\lambda.\end{aligned}\right\} \tag{19}$$

$\varphi_{t_0}(t) = \mu(t) + \vartheta\zeta$, *provided* $t - t_0 \ge \frac{1}{4}\lambda$.[4] $\quad$ (20)

We conclude our collection of results with some identities and near-identities.

$$c(t_2) - c(t_1) = -\int_{t_1}^{t_2} w\,dt \tag{21}$$

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[1] We separate out the ξ-integration here, as against the ζ one for J.

[2] (15), (16), (17) are of course true also in inverted form (for Γ_1 suitably gap-free).

[3] For quite general t_0 we have $\varphi_{t_0}(t) < D k^{\frac{1}{2}}$.

[4] $\varphi(t) = e^{-\tau}\int_0^t e^{\tau}\,dt = \int_0^t \exp\{-\tau(t) - \tau(t-x)\}\,dx = \int_0^{\frac{1}{4}\lambda} + \int_{\frac{1}{4}\lambda}^t .$

The first term is μ_0; the second is a $\vartheta\zeta$. We include this to illustrate "translation" from d.m. to r.m. language.

$$\left.\begin{aligned} c(U_1) &= -\Delta V+\int_{U_1}^{U_2} y_2\,dt,\ c(U_{1,2})=-\Delta V+O(\Delta\omega),\\ c(U_2') &= \Delta V'-\int_{U_1'}^{U_2'} y_1\,dt,\ c(U_{1,2}')=\Delta V'+O(\Delta\omega'). \end{aligned}\right\} \tag{22}$$

$$\left.\begin{aligned} &\textit{If}\ -\Delta\omega>0,\ \textit{then}\ w(U_1)\lesssim L(-\Delta\omega);\\ &\textit{if}\ \Delta\omega'>0,\ \textit{then}\ w(U_2')\lesssim L(\Delta\omega'). \end{aligned}\right\} \tag{23}$$

(21) is the difference of the $\dot{y}$-identities for $\Gamma_{1,2}$ (rather disguised). It shows that when $w>0$ $c(t)$ is decreasing. The identities in (22) are easily verified; the other results are trivial consequences of the identities. (23) follows from the fact that before and near a $U\,[U']$ the $\dot{y}$ for a gap-free Γ satisfies $|\dot{y}|\lesssim L$.

20. We begin by proving (α) "up to U_1"; i.e. so far as its results relate to the range (Z_1, U_1). The proof proceeds by successive stages of two kinds, embodied in (ε), (ζ) below, with (γ) to provide a starting point [(δ) is part of the proof of (ζ)]. We are supposing always, and tacitly, that $w(Z_1)>\zeta_1$, where ζ_1 is a "suitable" ζ, which we rechoose as the argument proceeds. We set out (γ) to (ζ) before beginning proofs.

(γ). *If* Ω *is in* $(Z_1-\frac{1}{4}\lambda,\ Z_1+\frac{1}{4}\lambda)$ *there is a "w, c linkage" at* Ω:

$$w(\Omega)\lesssim L\mu(\Omega)\,c(\Omega)\pm\zeta\lesssim Lk^{-1}c(\Omega)\pm\zeta. \tag{24}$$

In particular $$c(Z_1)\lesssim Lk\,w(Z_1). \tag{25}$$

In (δ), (ε), (ζ) we suppose that t_0 is in (Z_1, U_1), and we give the short name $H(t_0)$ to the hypothesis, or proposition (as the case may be); "there is no intersection,[1] and $w<k^{-5}$, in (Z_1, t_0)". We replace T by τ throughout, dealing, as we shall do, only with ranges where $0<w<k^{-5}$.

(δ). *If* $H(t_0)$ *is true, then*

$$w(t_0)<L\mu(t_0)\,k\,w(Z_1). \tag{26}$$

(ε). *If* $H(t_0)$ *is true, then there is a w, c linkage at* t_0:

$$w(t_0)\lesssim L\mu(t_0)\,c(t_0)\pm\zeta. \tag{27}$$

In this $\mu(t_0)$ *satisfies* [*after* (16) and (17)]

$$Lk^{-1}<\mu(t_0)<Dk^{-\frac{1}{3}};\ \mu(U_1)\lesssim Lk^{-\frac{1}{3}}. \tag{28}$$

[1] So that $w>0$.

(ζ). *Let Z be a Z of (Z_1, U_1) $[Z = Z_1$ is permitted], and suppose $H(Z)$ is true, and $c(Z) > \zeta'$, where ζ' is a certain definite ζ. If now*

$$\tfrac{1}{4}\lambda \leq t_1 - Z \leq L_1 k \qquad [L_1 \text{ of } (10)],$$

then $H(t_1)$ is true. Also

$$w(t_1) \leq L\mu(t_1)\, c(Z). \tag{29}$$

We postpone the proof of (γ), which is more easily explained a little later: we assume its results provisionally.

In (δ), (ε) we may suppose $t_0 \geq Z_1 + \frac{1}{4}\lambda$, since the range $(Z_1, Z_1 + \frac{1}{4}\lambda)$ is covered in each case by $(\gamma)\,[=(24)+(25)]$.[1]

For (δ) we have, by (1) and (20),

$$\begin{aligned} w(t_0) &\leq w(Z_1)\, e^{-(\tau_{t_0} - \tau_{Z_1})} + c(Z_1)\, \varphi_{Z_1}(t_0) < \zeta + \mu(t_0)\, c(Z_1) \\ &< \zeta + \mu(t_0)\, L k\, w(Z_1), \end{aligned}$$

by (γ), and we may drop the ζ.

In (ε), consider the r.m. from t_0, taken as origin, over $t \leq \frac{1}{4}\lambda$. This lies in a range with no intersection and $w < k^{-5}$. We can apply (7), of which the first part is $\mu_0 c_0 \leq w_0$. In the second we have, by (8), and (14) with $t = \frac{1}{4}\lambda$, $\bar{\psi}(\frac{1}{4}\lambda) < \frac{1}{2}$, $\bar{J}(\frac{1}{4}\lambda)/\mu_0 < \frac{1}{2}$, so that

$$\mu_0 c_0 (1 - \tfrac{1}{2}\vartheta) > w_0 (1 - \tfrac{1}{2}\vartheta') - \zeta,$$

and dividing by $1 - \frac{1}{2}\vartheta'$ we get the remaining inequality of (ε).

For (ζ), consider the d.m. from Z till the first intersection, if any, or till $w = k^{-5}$, or till $t = t_1$, whichever happens first. In this range, with $t = 0$ at Z, we have (3_2), i.e.

$$w > c_0 \varphi \left(1 - \vartheta J \Big/ \int_0^t e^{\tau}\, dt - (w_0/c_0)\, \psi \Big/ \int_0^t e^{\tau}\, dt\right). \tag{30}$$

Since $t \leq L_1 k$, the factor of ϑ is $< \frac{1}{2}$, by (13), and that of w_0/c_0 is $D k^{-\frac{1}{2}}$, by (10). Also $w_0/c_0 = w(Z)/c(Z)$, and by (ε)

$$w(Z) < L\mu(Z)\, c(Z) + \zeta < L k^{-1} c(Z) + \zeta,$$

and for suitable ζ' this gives $w(Z)/c(Z) < L k^{-1}$. Hence (30) gives

$$w > L c(Z)\, \varphi_Z(t) > 0, \tag{31}$$

and incidentally an intersection is not the first event, and (since $w < k^{-5}$ in the range up to t) $H(t)$ is true. Then, by (δ),

---

[1] Note that $c(t) \leq c(Z_1)$ for $t \geq Z_1$, w being positive in virtue of $H(t_0)$ in (21); also that we can drop the ζ in $w(t_0) < L\mu(t_0)\, k\, w(Z_1) + \zeta$ when ζ_1 is suitable.

$$w<L\mu(t)\,k\,w(Z_1)<L\,k^{\frac{1}{2}-10}<k^{-5},$$

and $w=k^{-5}$ is not the first event. So $t=t_1$ is the first event, and $H(t_1)$ is true. Since $\varphi_z(t_1)\geq\mu(t_1)$, (31) gives $w(t_1)>L\mu(t_1)c(Z)$. Also, by (1) with $t=0$ at Z,

$$w(t_1)<\zeta+c(Z)\varphi_Z(t_1)<\zeta+c(Z)(\mu(t_1)+\zeta),$$

by (20), and for suitable ζ' this gives $w(t_1)<L\mu(t_1)c(Z)$. This completes the proof of (ζ).

21. Return now to the postponed (γ). This is in point of fact true for *any* pair $\Gamma_{1,2}$ of the stream, and since the mesh does not conveniently help us *backwards* from Z_1, we prove it in this form, retaining the T in (W) and $(\overline{\text{W}})$ (instead of τ). Let $\mathcal{R}$ be the range $(Z_1-\frac{3}{4}\lambda,\ Z_1+\frac{3}{4}\lambda)$; this includes both the d.m. and the r.m. from Ω to time $2.\frac{1}{4}\lambda$, and on the other hand it is to the right of Z_0+L (for some L). In $\mathcal{R}$ we have, for each of $\Gamma_{1,2}$, $y>1+L$. By Lemma A we have $\dot{y},\ \ddot{y}=O(1)$, and by its (3), together with the initial conditions for the stream, we have $\Delta F=O(Dk^{-1})$.

Next, we have $\dot{T}>Lk$, $T>Lkt$ in $\mathcal{R}$, for d.m. or r.m. It follows by straightforward calculations that, for origin Ω, and $t\leq\frac{1}{4}\lambda$,

$$\bar{\psi}<Lk^{-1},\ \bar{J}/\int_0^t e^{-T}\,dt<Lk^{-1}, \tag{32}$$

and also that

$$\mu(\Omega)\lesssim Lk^{-1}. \tag{33}$$

Consider now the r.m. from Ω to $t=\frac{1}{4}\lambda$, or the first intersection (if any), whichever happens first. There are two cases: (i) an intersection happens first, (ii) no intersection before $t=\frac{1}{4}\lambda$.

Case (i). Consider the r.m. from the intersection as new origin, for a (further) time $\frac{1}{4}\lambda$, or to the next intersection, whichever happens first. We have $\dot{w}_0<0$ and $w\leq 0$ in this range. (5) is valid, with w changed to $-w$ to normalize, and $w_0=0$, $\bar{c}_0=-\dot{w}_0=|\dot{w}_0|$; and it gives [for the unchanged w], after (32),

$$-w\,e^{-T}\geq\tfrac{1}{2}|\dot{w}_0|\int_0^t e^{-T}\,dt. \tag{34}$$

This shows, first that there is no (second) intersection, and so, secondly, that (34) is valid at $t=\frac{1}{4}\lambda$, when it gives $\dot{w}_0=O(\zeta)$.

Consider now the d.m. from the intersection up to Ω. There is no intersection inside the range, and (1) is valid with $w_0=0$, $c_0=|\dot{w}_0|$, so that $0\leq w\leq|\dot{w}_0|\varphi<\zeta$, and $w_1 = O\left(\int_0^{\frac{1}{4}\lambda} |w|\,dt\right) = O(\zeta)$. In particular $w(\Omega) = O(\zeta)$; also, taking $t = \Omega$ in $\dot{w} = -\dot{T} w + w_0 - w_1$, we have $\dot{w}(\Omega) = O(\zeta)$, and so finally $c(\Omega) = O(\zeta)$. (γ) is accordingly true in degenerate form.

Case (ii). In this case (4) and (5) are valid at $t = \frac{1}{4}\lambda$ of the r.m., with $w_0 = w(\Omega)$, $\bar{c}_0 = -c(\Omega)$; $\bar{\psi}$ and $\bar{J}$ behave as before at $t = \frac{1}{4}\lambda$, and by an argument now familiar we get the w, c linkage at Ω, in case (ii) as well as in case (i).

Finally, to prove (25), we observe that for suitable ζ_1 we can drop the $\pm\zeta$ in (24) when $\Omega = Z_1$. This completes the proof of (γ).

22. Having now (γ) to (ζ) at our disposal, we divide the range (Z_1, U_1) at $Z_1, t_1, t_2, \ldots, t_\nu, U_1$, where t_n are at Z's, and the time-lengths s of the successive steps satisfy $L_2 k < s < L_1 k$, where L_1 is the L_1 of (10) and L_2 is suitably small. This is clearly possible, and with $\nu + 1 < L$. Since the t_n are Z_n, we have $\mu(t_n) \gtrless L k^{-1}$.

At Z_1, we have $c(Z_1) > L k w(Z_1) > \zeta'$ for suitable ζ_1; hence, by (ζ) with $Z = Z_1$, $H(t_1)$ is true. Also $w(t_1) \lesssim L \mu(t_1) c(Z_1)$, and so, by (25) and (15),

$$w(t_1) \lesssim L w(Z_1). \tag{35}$$

By (ε), with its $t_0 = t_1$, $c(t_1) \lesssim L \mu^{-1} |(t_1)| w(t_1) \pm \zeta$; therefore, by (15) and (35), $c(t_1) \lesssim L k w(Z_1) \pm \zeta$, and so, for ζ_1 suitable,

$$c(t_1) \lesssim L k w(Z_1) > \zeta'. \tag{36}$$

Since t_1 is a Z we can apply (ζ) with its $t_1 = t_2$, and (ε) with its $t_0 = t_2$; $H(t_2)$ is true, $w(t_2) \lesssim L \mu(t_2) c(t_1) \leq L k^{-1} c(t_1)$ and $c(t_2) \lesssim L k w(t_2) \pm \zeta$, which gives, after (36), and for suitable ζ_1, $w(t_2) \lesssim L w(Z_1)$, and

$$c(t_2) \lesssim L k w(Z_1) \pm \zeta \lesssim L k w(Z_1) > \zeta'.$$

The process can evidently be repeated (L steps at most are needed); $H(t_\nu)$ is true, and $c(t_\nu) \lesssim L k w(t_\nu) \lesssim L k w(Z_1) > \zeta'$.

In the final step to U_1, (ζ) shows that $H(U_1)$ is true [so no intersection in (Z_1, U_1)] and

$$w(U_1) \lesssim L \mu(U_1) c(t_\nu) \lesssim L k^{\frac{1}{2}} w(Z_1),$$

by (28); and from (ε) with its $t_0 = U_1$, and for suitable ζ_1,

$$c(U_1) \lesssim L \mu^{-1}(U_1) w(U_1) \pm \zeta \lesssim L k w(Z_1) \pm \zeta \lesssim L k w(Z_1).$$

We have now proved (α) up to U_1.

We have now to make the step from U_1 to Z_1'. The nature of the reasoning is much the same; the difficult region $|y|<1$ is involved, but the step is less than $\frac{3}{2}\lambda$ (as against, Lk), and the results (11), (12), (14) are available, and adequate to deal with the new situation.[1] The upshot is (for suitable ζ_1) no intersection before Z_1', and we can add

$$c(Z_1)\lesssim Lkw(Z_1'),\ w(Z_1')\lesssim Lw(Z_1),$$

to our previous results.

We can now make a fresh start at Z_1', and proceed to U_2', except for the detail that we have only $w(Z_1')>L\zeta_1$ instead of $w(Z_1')>\zeta_1$; this can be adjusted by a final re-choice of the original ζ_1. The results of (α) are now all accounted for.

After (α) we have agreed to take (β) for granted.

23. We can now prove Lemma D. After (β) we may suppose that $w(Z_1)>\zeta_1$; and it is enough to show that for consecutives of the mesh satisfying $w(Z_1)>\zeta_1$ we have $-\Delta V,\ \Delta V'>0$, and

$$L(-\Delta V)<\Delta V'<\alpha\Delta V;$$

for these results are additive.

Since $$w(U_1)<Dk^{-\frac{1}{3}}c(U_1),\ w(U_2')<Dk^{-\frac{1}{3}}c(U_2'),$$

(22) and (23) give

$$-\Delta\omega=O(Dk^{-\frac{1}{3}})(-\Delta V),\ \Delta\omega'=O(Dk^{-\frac{1}{3}})\Delta V',$$

so that, by (21),

$$\Delta V'(1+o(1))+\Delta V(1+o(1))=-\int_{U_1}^{U_2'}w\,dt. \tag{37}$$

Also $$\Delta V'=c(U_2')(1+o(1))\lesssim Lkw(Z_1),$$
$$-\Delta V=c(U_1)(1+o(1))\lesssim Lkw(Z_1),$$

by (α), and we infer that $\Delta V'$, $(-\Delta V)>0$, and $\Delta V'>L(-\Delta V)$. It remains only, after (37), to prove that $\int_{U_1}^{U_2'}w\,dt>Lkw(Z_1)$. And since $w>0$, it is enough to prove that $w>Lw(Z_1)$ over a range from $Z_1'+\frac{1}{4}\lambda$ to a further distance L_1k, with L_1 conveniently small. Now with origin Z_1', (3) [in inverted form] gives, for this range,

[1] Compare the proof of (ζ): the last term in (3_2) is small with w_0/c_0, though its other factor is no longer small.

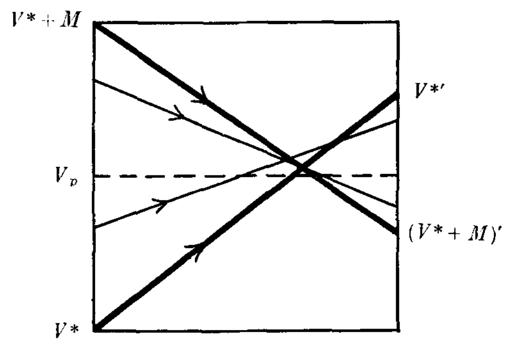

Fig. 4.

$$w > c(Z_1')\,\varphi_{Z_1'}(t)\left(1 - \vartheta J/\int_0 e^{\tau}\,dt - (w_0/c_0)\,\psi/\int_0 e^{\tau}\,dt\right).$$

By arguments now familiar the large bracket is $>\frac{1}{2}$; also, since

$$t \geq \tfrac{1}{4}\lambda,\quad \varphi_{Z_1'}(t) \geq \mu(Z_1')\,L k^{-1}.$$

Since $c(Z_1') > Lkw(Z_1)$, by (α), we get the inequality $w > Lw(Z_1)$ that we need.

24. We propose to exercise some licence in the explanations that follow. (i) We generally ignore δ's, representing them by 0's, and we ignore $O(\zeta)$; (ii) we use a symbolism "$V'(V)$" for the V' resulting from "V", as if there were a definite function $V'(V)$ of V over the range $V^*-0 \leq V \leq V^*+M-0$[1] (the 0's here representing suppressed δ's): this is in effect supposing "linkage" to be exact. We now use $(V^*)'$ to mean what can be indicated roughly as $V'(V^*+0)$. Our bases of argument are now: (a) *$V'(V)$ is defined over $V^*+0 \leq V \leq V^*+M-0$, and lies in the same interval*; (b) *$V'(V)$ varies in the opposite sense to V*; (c) *the interval (V_1, V_2) "shrinks"; (V_1', V_2') is smaller than (V_1, V_2) in a ratio $1-L$.*

Suppose (what will in fact normally happen, as we shall see later) that $V^{*\prime}$ is not near either end of the range (V^*, V^*+M). As V increases from V^* to V^*+M, V' decreases from $V^{*\prime}$: it may now happen that (i) there is a V_* such that $V_*' = V^*$, or again (ii) there may be no V_*, in which case $(V^*+M)' > V^*$; we shall find that cases (i), (ii) occur each for certain ranges of b.

Take first the simpler case (ii); the relations of V, V' are shown diagrammatically in Fig. 4. The lines from any two V's of the range (V^*, V^*+M) to their V' will

[1] This "functional" use of a dash produces a slight clash with the usual "inversion" meaning (though it originates from the latter meaning).

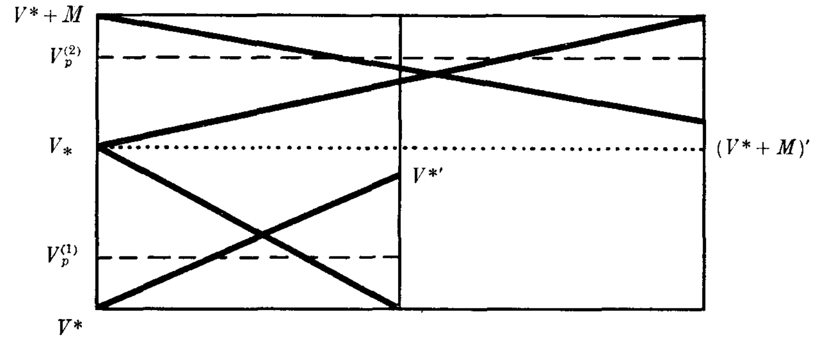

Fig. 5.

cross and the difference will shrink, and V, V' move in opposite senses. If we start from any V and form the sequence V', $V'' = V'(V')$, V''', ..., there will clearly be convergence, *in the diagram*, of $V^{(n)}$ to a limit, V_p say. An actual (settled)[1] Γ with a V not in the gap has its V', V'', ... clear of gaps, and $V^{(m)}$ *quasi-converges* to a V_p, i.e. converges except for error $O(\zeta)$. The Γ ultimately has its V's all of the form $V_p + O(\zeta)$, and the distance between successive U, U'[2] is of the form $(n-\frac{1}{2})\lambda + O(\zeta)$, with constant n. The ultimate Γ is therefore quasi-periodic, of quasi-half-period $(n-\frac{1}{2})\lambda$, and any pair $\Gamma_{1,2}$ starting near the same U quasi-converge to each other.

In case (i) we must build up the diagram, Fig. 5, in two stages. We have $(V_* - 0)' = V^* + 0$, by (b); hence the downward sloping line from V_*. On the other hand, again by (b), we have $(V_* + 0)' = V^* - 0$, and *this means a dip and* $V' = V^* + M$ *a λ-period later*,[3] a place represented by the right hand vertical line. Thus we have two lines from V_* as shown. Further, $(V^*)'$ is below V_*, by (c).

Consider now $(V^* + M)'$. It must not be placed on the middle line, since there it would, by (b), be below $V'_* = V^*$; hence it is placed on the right hand line, and, by (c), above V_*.

There is, in this case (i), a new type of gap, a V_*-gap, *since a V in $V_* \pm \delta$ leads to a V' in $V^* \pm \delta$ at the U' following U.* A V of the diagram strictly between V^* and V_* has its V', V'', ... converging to a $V_p^{(1)}$ in this range, one strictly between V_* and $V^* + M$ has its V', V'', ... converging to a $V_p^{(2)}$ in that range. An actual Γ with V in $V^* + \delta$, $V_* - \delta$ (so not in either kind of gap) ultimately has its V's of the

[1] In future we suppose that all Γ mentioned are settled.

[2] U, U' have approximate phases $-\frac{1}{2}\pi$, $+\frac{1}{2}\pi$ respectively.

[3] Note that we are using $V'(V)$ to mean the V' at the *true* U'; a $V'(V)$ has a discontinuity at V_*, $V'(V_* - 0)$ being $V^* + 0$ at the middle ordinate of the diagram, and $V'(V_* + 0)$ being $V^* + M$ at the end one.

form $V_p^{(1)}+O(\zeta)$, and its successive distances U, U' of the form $(n-\frac{1}{2})\lambda+O(\zeta)$, and any pair of Γ of the kind quasi-converge. A pair of Γ with V's in $V_*+\delta$, $V^*+M-\delta$ ultimately have V's of the form $V_p^{(2)}+O(\zeta)$, and U, U' of length $(n+\frac{1}{2})\lambda+O(\zeta)$, and they quasi-converge.

25. We have seen that, assuming $V^{*\prime}$ is not near V^* or V^*+M, then, if a trajectory starts with a V in neither the V^* gap nor the V_* gap (if there is a V_*), its $V', V'', \ldots$ stay clear of gaps. Further that there is either one group of which every pair quasi-converge, or two groups, such that any pair belonging to the same group quasi-converge. It remains to show that quasi-convergence (of the kind considered, clear of gaps) of a pair of trajectories involves their *exact* convergence (y_1-y_2 and $\dot{y}_1-\dot{y}_2$ tend to 0 as $t\to\infty$); and it is then an easy consequence that a group of quasi-convergent trajectories converge to a single trajectory, necessarily strictly periodic. The argument separates two cases, (i) when the pair ultimately do not intersect, (ii) where they have an infinity of intersections: it works in either case; but whether either case happens always, sometimes, or never, we do not know.

26. This time the approximation τ for T has error only $O(\zeta)$.

Case (i). Suppose w is ultimately positive. Then first, w_1, which is an increasing function, must be bounded. Otherwise, for an arbitrarily large G and $t>t_0(G)$, we should have $c_0-w_1<-G$ and so

$$w=w_0e^{-T}+e^{-T}\int_0^t\left(c_0-w_1\right)e^T\,dt<\left(w_0+c_0\int_0^{t_0}e^T\,dt\right)e^{-T}-G\,e^{-T}\int_{t_0}^t e^T\,dt.$$

By the τ-dictionary[1] the first term is less than a constant independent of G, and the factor of $-G$ is greater than Lk^{-1} for $t>t_0+1$. Hence w is ultimately negative, a contradiction. Since, then, (a) w_1 is bounded, (b) $w>0$, and (c) $\dot{w}$ is bounded, we must have $w\to 0$; and then $\dot{w}\to 0$ since $\ddot{w}$ is bounded.

Case (ii). Let the intersections be I_n, $n=1, 2, \ldots$, and $w_n=w(I_n)$. In the first place we have, for any w in $I_n I_{n+1}$,

$$|w|<D\,|\dot{w}_n|\,k^{\frac{1}{2}}. \tag{1}$$

For, taking $t=0$ at I_n, and normalizing to $\dot{w}_n$ and w non-negative, we have

$$w=e^{-T}\int_0^t(\dot{w}_n-w_1)\,e^T\,dt\le\dot{w}_n\,e^{-T}\int_0^t e^T\,dt<D\,k^{\frac{1}{2}}\,\dot{w}_n,\quad\text{by a }\tau\text{-result.}$$

[1] We are taking this as known.

Consider now the r.m. from I_{n+1}, taken as $t=0$, up to $t=\frac{1}{4}\lambda$, or till we reach I_n, whichever happens first. If w is, say, non-negative, we have from (5) of § 19 (with $w_0 = 0$, $\bar{c}_0 = \dot{w}_{n+1}$),

$$w e^{-T} \geq \dot{w}_{n+1} \int_0^t e^{-T} dt \left(1 - \iiint e^{-T_\zeta + T_\eta - T_\xi} d\xi\, d\eta\, d\zeta \Big/ \int_0^t e^{-T} dt\right) \tag{2}$$

and by considerations we used earlier the large bracket is greater than $\frac{1}{2}$. Hence

$$w \geq \tfrac{1}{2} \dot{w}_{n+1} e^{T} \int_0^t e^{-T} dt,$$

and in particular I_n is not reached before $t=\frac{1}{4}\lambda$. So we have

$$w(\tfrac{1}{4}\lambda) > \tfrac{1}{2}\dot{w}_{n+1} e^{T(\frac{1}{4}\lambda)} \int_0^{\frac{1}{4}\lambda} e^{-T} dt > \tfrac{1}{2}\dot{w}_{n+1} \cdot e^{Lk} \cdot L k^{-1}$$

by two τ-results. On the other hand, this $w(\frac{1}{4}\lambda)$, being a w of $I_n I_{N+1}$, satisfies (1), so that $|w(\frac{1}{4}\lambda)|w(\frac{1}{4}\lambda)| \leq L k^{-1} |\dot{w}_n|$. It follows that $|\dot{w}_{n+1}| \leq \frac{1}{2}|\dot{w}_n|$. Hence $\dot{w}_n \to 0$, and so, from (1), $w \to 0$ uniformly. Finally $\dot{w}$ must tend to 0 since $\ddot{w}$ is bounded.

27. It is possible to give a formula, correct to a factor $1+o(1)$, for the period P of a stable periodic Γ.[1]

Let $Y(x, b, \varphi)$ be defined by (see § 6 (3))

$$-F(Y) = \tfrac{2}{3} - Y + \tfrac{1}{3} Y^3 = x + b(1 + \sin\varphi)$$

for the range $0 \leq x \leq \frac{4}{3} - 2b$ of x, and then for the long descent

$$\psi(x, b) = \int_{-\pi}^{\pi} Y(x, b, \varphi)\, d\varphi$$

is k times the approximate change in the constant x over a period of φ. Hence

$$\frac{\pi P}{k} \sim \int_0^{4/3 - 2b} \frac{dx}{\psi(x, b)}.$$

The right-hand side is independent of λ (as we should expect).

28. For the next developments we need to show that, for a fixed V, V' varies smoothly with b. It is to be expected that a change of order k^{-1} in b corresponds to a change of order 1 in V', or of one λ-period in the period of a p.m. We prove

[1] Where there are two periods they differ by a factor $1+o(1)$.

that an increase of order, say, $(k \log k)^{-1}$ of b produces an increase of order $(\log k)^{-1}$ in V' for fixed V.[1] As b increases through $L \log k$ such steps, V' will run effectively through the whole range V^*, V^*+M,[2] which last is equivalent to V^* a λ-period earlier. It is further clear from the diagrams, and (a), (b), (c) of § 24, that when $V^{*\prime}$ is just above V^* there will be a V_*,[3] and that when it is just below V^*+M there will be no V_*. We see now the genesis of the excluded intervals of § 4, which are those for which $V^{*\prime}$ falls in the $V^* \pm \delta$ gap,[4] and of the alternation of the two cases $\mathcal{B}_1$, $\mathcal{B}_2$.

If a system has stable subharmonics of which the order decreases with increase of a parameter b, it is perhaps to be expected, as the most natural way of bridging the gap, that there will be this alternation of one stable period and of two stable periods (the shorter increasing its sphere of influence at the expense of the longer), with all the attendant consequences of the latter case.

29. However this may be, we are now in a position to discuss in more detail the structure of the non-stable trajectories when $b \,\varepsilon\, \mathcal{B}_2$. We have seen that a Γ arriving after a long descent ("settled") at $y=1$ has phase $-\frac{1}{2}\pi-\omega$, where ω lies in a range $(-Lk^{-\frac{1}{3}}, Lk^{-\frac{1}{3}})$. There are two V-gaps, namely $V^* \pm \delta$ round V^*, and $V_* \pm L_1 \delta$ round V_*, where L_1 will be explained presently. If we suppose a particular linkage of V and ω to be set up in some way, these gaps correspond to two "gateways" on $y=1$, both lying in the ω-range above; there is such a pair of gateways near every place of phase $-\frac{1}{2}\pi$, and similar ones on $y=-1$ near every place of phase $\frac{1}{2}\pi$. The V^* and V_* gateways we call G (suppressing an upper star) and G_* respectively when they are on $y=1$, G' and G'_* when they are on $y=-1$. A continuity argument (backed by some subsidiary considerations we will not go into) shows that a continuous stream S (starting at some Z'_0) exists, whose Γ arrive near a given place of phase $-\frac{1}{2}\pi$ on $y=-1$, with V' having any assigned value in $V^* - 2\delta \leq V' \leq V^*+M-\delta$; they arrive after a long ascent (settled), and with $|\dot{y}|>L$. We may take as S the representative points (at Z'_0) of whose Γ lie on a segment of a straight line in the space of r.p. There is a sub-segment with a corresponding sub-stream going

[1] The increment is moreover approximately independent of the given V. This less "obvious" fact is used, so that the long proof is not merely a tiresome necessity.

[2] V^* and M are functions of b, but vary so slowly that they may be treated as locally constant.

[3] This depends on the inequality $|V'_2 - V'_1| > L(V_2 - V_1) + O(\zeta)$. Nor can the L be a D! It is this that calls for the simplifying $b > \frac{1}{100}$, without which there would be much further complication.

[4] The V_* gap can only be reached from a V in a V^* gap; with such V's almost anything can happen.

through G' (and "filling" it); let $\Gamma_{1,2}$ be the Γ's through the $V^* \pm \delta$ ends of G' respectively. Γ_1, 1 of Fig. 6, shoots through $|y| \leq 1$ and begins converging to a p.m. of half-period $(n-\frac{1}{2})\lambda$. Γ_2, 2 of Fig. 6, makes a dip, and then begins converging to a p.m. of half-period $(n+\frac{1}{2})\lambda$. U_1 is at a time $(n-\frac{1}{2})\lambda$ later than U_1' (approximately, understood); on the other hand U_2' is time λ later than U_1'; since further V_2' is approximately $V^* + M$, U_2 is $(n+\frac{1}{2})\lambda$ later than U_2', with $V_2 = (V^* + M)'$ approximately. U_1 and U_2 are accordingly *two* λ-periods apart, and $V_1 = (V^* + 0)'$, $V_2 = (V^* + M)'$ approximately. These extremes $\Gamma_{1,2}$ of Fig. 6 (i) bound a "delta" where they arrive at $y = 1$.[1] This delta contains two G's, G_1 and G_2, and one G_*. The stream through the initial G' will contain sub-streams through each of these, again arriving (settled) after a long descent and with $|\dot{y}| > L$. Those through $G_{1,2}$ are scattered similarly to the original one through G'; that through G_* is not immediately scattered, but leads, after time $(n-\frac{1}{2})\lambda$, to a new G', and if L_1 is suitably large it contains a sub-stream through and filling this G'.[2] We have now three sub-streams, each ending by going through a G or G', and the process repeats.

We note some further details. The extremes $\Gamma_{1,2}$, after leaving $y = -1$, cross, approximately at height $y = 2$, and at time approximately λ after U_1'; other Γ's cross similarly. The stream becomes a narrow channel, of width $O(k^{-1})$, for the long descent. Γ's of the stream starting near Γ_1 stay below Γ_1, those starting near Γ_2 stay above Γ_2; both are presently lost for good from the long descent "tube" enclosed by $\Gamma_{1,2}$;[3] it is in fact only a central core of the initial stream (with width probably of order about k^{-1}) that provides the stream through the tube.[4]

30. We proceed to analyse the possible "structures" of streams flowing only through gateways at their U, U' (we reject the parts going outside). We have seen that the three possibilities following a G' are: (a) a "short" sub-stream, s [length $(n-\frac{1}{2})\lambda$], to G_1; (b) a "long" one, l [length $(n+\frac{1}{2})\lambda$], to G_2; (c) a long one to G_*, followed immediately by a short one to a G'; the combination we will call C. l', s', C' have the inverse senses.

We next define a "unit" undashed "structure" u to be a stream (through gateways) starting from a G and ending at a G, the latter being the first G after the

[1] Distances of order $k^{-\frac{1}{3}}$, $\delta k^{-\frac{1}{3}}$ are of course grossly magnified in the figures. Fig. 6 (v) is drawn for a case in which Γ_2 makes one dip at $y = 1$; $(V^* + M)' - M$ lies between 0 and M. The V's linked with various points on $y = 1$ are also shown.

[2] By the left hand inequality of Lemma D. (By the right hand one we must have $L_1 > 1$).

[3] Γ's starting just outside Γ_1 or Γ_2 *enter* the tube, but we are not concerned with these.

[4] Some Γ's of the "core", e.g. 7, 8, of Fig. 6 (i) go below Γ_2 before ending up in the tube.

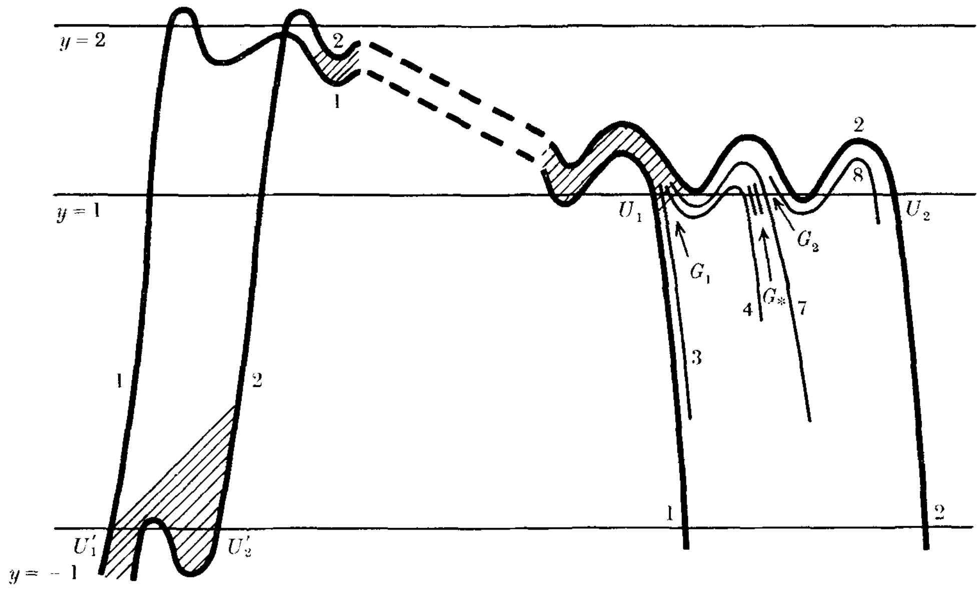

Fig. 6 (i).

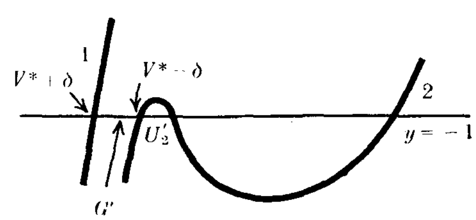

Fig. 6 (ii). Start at a G'.

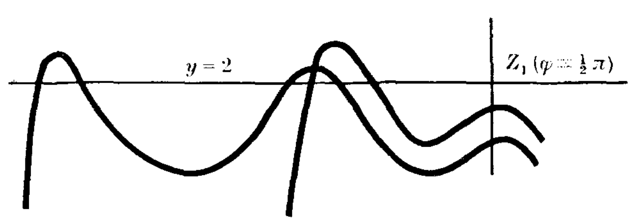

Fig. 6 (iii). Arrivals at $y=2$.

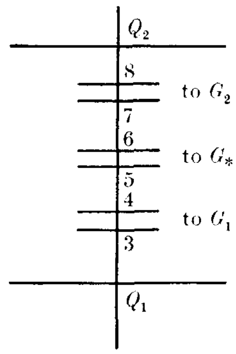

Fig. 6 (iv). Intersections with Z_1 of the substreams.

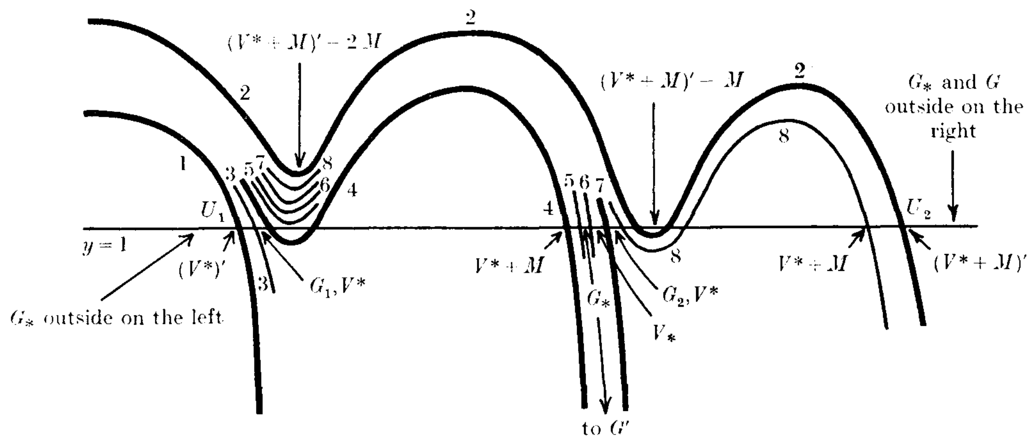

Fig. 6 (v). Arrivals at $y=1$. $(V^*+M)'-M<V^*<(V^*)'<V_*<V^*+M$ by Fig. 5.

former. The possible u's are now as follows, where each of x and y may be either s or l:

(i) C; (ii) $x+y'$; (iii) $x+mC'+y'$, m being any positive integer.

The most general structure, from $-\infty$ to ∞, say, is an arbitrary succession of u's, or else the inverse of such a structure. [It is generally *both*, but, e.g., a succession of C's only is an exception. And a structure can be identical with a dashed structure suitably translated.] Given any structure there exists a continuous stream through any finite section $-T<t<T$ of it; and by a limit argument there exist trajectories through all the gaps of any possible structure. These provide the trajectories X of § 4; and when the structure is periodic they provide quasi-periodic trajectories having the structure. The final step to the existence of a strictly periodic

trajectory having the structure requires an index-number argument. This last is the only topological argument we have occasion to use.[1]

31. It follows from the famous "last geometrical theorem" of Poincaré,[2] *inter alia*, that of a transformation T, which is $1-1$, continuous and area-preserving in the annulus between two curves, has f.p. of order n_1 on one curve and f.p. of a different order n_2 on the other, such that the points go round the curves once in n_1 and n_2 transformations respectively, then it has f.p. in the annulus of every order N such that m/N lies between $1/n_1$ and $1/n_2$ for some integer m; if $n_1 = 2n+1$ and $n_2 = 2n-1$, it has f.p. of orders $2n$, $4n \pm 1$, $6n \pm 1$, $8n \pm 3$. As a matter of fact an *annulus* is not essential to such behaviour: in our case of stable periods $(2n \pm 1)\lambda$ there is no annulus; all the points of K_0 lie on the frontier of a simply-connected domain containing the point at infinity. There would seem, moreover, to be a much richer "fine structure" of non-stables than is provided by the annulus. For example, to take only multiples of $2n\lambda$, there exist 4 distinct kinds of set of least period $2n\lambda$, 6 of least period $4n\lambda$, and a considerable number of least period $6n\lambda$.[3] The annulus may have as few as 2 of least period $2n\lambda$ and none of a least periods $4n\lambda$, $6n\lambda$.

Since a trajectory exists conforming to any possible structure of gaps from $-\infty$, to $+\infty$, cases arise that seem curious at first sight. We may instance that of a pair of trajectories running close together through the same system of gaps for large positive and large negative t, but running apart over a stretch in the middle; the gap-structure towards $t = +\infty$ and $t = -\infty$ can, again, be periodic of one period in one direction and of another in the other.

32. In the case $b \varepsilon \mathcal{B}_2$ a non-stable p.m., or a trajectory of the set X has at every critical crossing of $y = \pm 1$ its V either V^* or V_*, with error $o(1)$. While it is not necessary to the results to explain, in detail, how this can happen, the reader may feel some curiosity on the point, and we shall say something about it.[4]

Any trajectory satisfies an identity

$$\tfrac{1}{3}y^3 - y + \tfrac{2}{3} = C + b(1 + \sin\varphi) - (\dot{y} + y_1)/k. \tag{1}$$

[1] And except for it all our analysis is entirely "low-brow".

[2] See G. D. Birkhoff, *Dynamical Systems* (New York, 1927), 165.

[3] Owing to possibilities of symmetry, and permutations of constituent units that may be like or unlike, the exact number of distinct kinds is not at all easy to determine.

[4] There are other types of "odd" behaviour, arising mainly from the critical nature of the lines $y = \pm 1$, for trajectories emerging *slowly* from $|y| < 1$: it is *necessary* to the proof of strictly periodic non-stables to discuss these, but it would take too much space to describe them here.

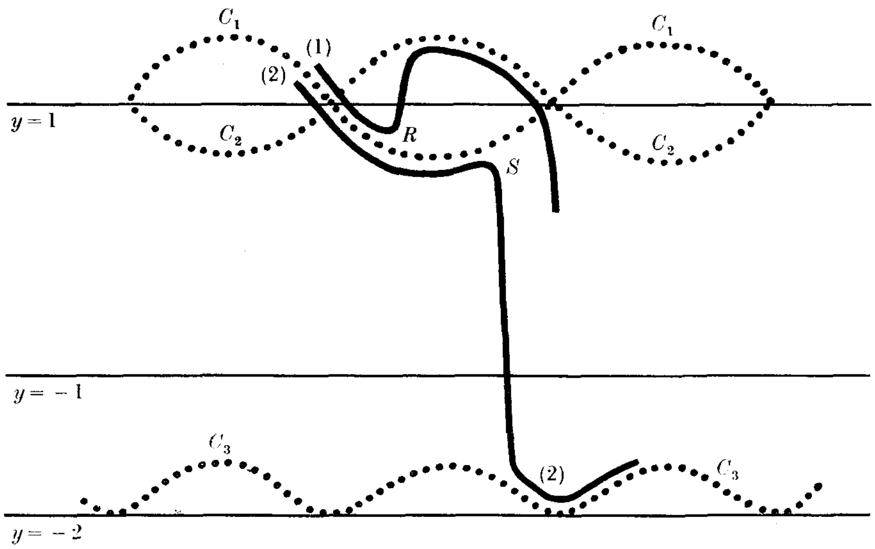

Fig. 7 (i). Trajectory "pulled" at R.
Fig. 7 (ii). Trajectory "sliced" at S.

During a "quiet" interval, of duration not more than $o(k)$, in which $\dot{y}$ is $O(1)$, this is approximately of the form

$$\tfrac{1}{3}y^3 - y + \tfrac{2}{3} = C + b(1 + \sin\varphi). \tag{2}$$

If two such quiet intervals are separated by an interlude of duration $o(k)$ [it is in practice $o(1)$], the interlude contributes $o(1)$ to y_1/k and to error $o(1)$ *the C of the second quiet interval is the same as that of the first.* Now the locus (2) can (for certain C, e.g. $C=0$) consist of as many as three distinct branches; most of the time a trajectory is pursuing one or other of these, but it can jump from one to another in a short interlude in which $\dot{y}$ is large. The value $C=0$ is particularly important since the last stages of a "long descent" to $y=1$ are given, with error $o(1)$, by (2) with $C=0$.[1] As we saw in § 13, the locus (2) has then three branches C_1, C_2, C_3, shown dotted in Fig. 7.

It is now possible for a trajectory to behave approximately as follows: (i) arrive at X on $y=1$ along a C_1, (ii) pursue the unstable C_2 to some point, (iii) either (1) "pull" up to the point of C_1 vertically above and follow C_1, or (2) "slice" down to the point of C_3 vertically below, and follow C_3. In case (iii) (1) it is possible, by varying the moment of the "pull", to make the value of y_1 at Y, the next $y=1$ point,

[1] This applies only to y, not to $\dot{y}$; we may not "differentiate" (2) near $y=1$.

vary up to the amount of the area contained by a C_1 and C_2. Since, identically, $V_Y = V_X + [y_1]_X^Y$, this makes possible a variation, of the order of 1, in the value of V at Y. Most of the delta is of course free of "normal" trajectories: the delta is none the less "filled" by the stream, and in fact mostly by the nearly vertical trajectories in the act of pulling or slicing.

33. We conclude with the obvious remark that if f is allowed to have more than one pair of zeros the situation can become very different. Consider for example the equation

$$\ddot{y} + k(1 - y^2 + e^{-k} y^4)\,\dot{y} + y = b\mu k \cos(-\mu t + \alpha).$$

If the term in e^{-k} is omitted this is (E) with the sign of t changed, and the trajectories are the r.m. of (E) (and include, for $b < \frac{2}{3}$, a *stable* p.m. of order 1). The non-stable p.m.'s and the trajectories of X, which are bounded for all t, positive and negative, continue to exist (reversed[1], and slightly modified) when the new term is present; the original stable subharmonics become totally unstable ones. Groups of trajectories will try to escape to ∞, but the positive damping then takes charge, and all trajectories are ultimately bounded[2] (the bounds of y, $\dot{y}$ are of course exponentially large in k).

[1] The reverse of a non-stable is a non-stable.

[2] N. Levinson, *Journal of Math. and Physics*, 22 (1943), 41–48.