ON THE PRINCIPLE OF SUBORDINATION IN THE THEORY OF ANALYTIC DIFFERENTIAL EQUATIONS

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I. Preliminaries

1. According to Cauchy, the initial value problem

$$dw/dz = f(z, w), \quad w(0) = 0 \tag{1}$$

has a unique (regular) solution $w = w(z)$ in a neighborhood of $z = 0$ whenever f is a function of two complex variables, z and w, which is regular in a neighborhood of $(z, w) = (0,0)$. What is more, if $a > 0$ and $b > 0$ are chosen so small that $f(z, w)$ is regular on the dicylinder

$$|z| < a, \qquad |w| < b \tag{2}$$

(that is, if a convergent expansion

$$f(z, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{mn} z^m w^n \tag{3}$$

holds on (2)), and if, without loss of generality, $f(z, w)$ is supposed to be bounded on (2), say

$$|f(z, w)| < M \quad \text{on} \quad (2) \qquad (M < \infty), \tag{4}$$

then there exists a $p > 0$ which depends only on the three values a, b, M and which has the property that the solution $w(z)$ of (1) exists (as a regular function) on the circle $|z| < p$. In fact, if $\|f\|$ denotes the radius of convergence of the expansion, say

$$w(z) = \sum_{n=1}^{\infty} a_n z^n, \tag{5}$$

of the solution $w(z)$ of (1), then it is known (cf. [8], pp. 127–128) that

$$\|f\| \geqq \min(a, b/M). \tag{6}$$

There is an extensive literature (cf. [4], pp. 169–172), initiated by a paper of Painlevé ([6]; not quoted in [4]), which aims at an improvement of (6). I noticed however (cf. [8],

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pp. 128–129) that (6) is the *best* estimate of its kind (and that (6) cannot be improved in terms of *universal* constants even if $f(z, w)$ is restricted to be independent of z, that is, even if (1) and (6) reduce to

$$dw/dz = f(w), \quad w(0) = 0 \tag{7}$$

and $\|f\| \geqq b/M$ respectively).

From the methodical point of view, it is worth mentioning that (6) has never been proved by the procedure which seems to be the most natural one, namely, by inserting (3) and (5) in (1) and comparing the coefficients of like powers of z. The trouble is that the assumption (4) fails to imply (with reference to the *fixed* constants a, b, M which belong to f) the inequality which results if $f(z, w)$ is replaced by $f^*(z, w)$ in (4), where $f^*(z, w)$ denotes the "best majorant" of (3),

$$f^*(z, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |c_{mn}| z^m w^n \tag{8}$$

(even though (8) is convergent on (2) whenever (3) is).

By avoiding (8), and using Fejér's estimate ([2], pp. 22–24) of the Cesàro sums in connection with (3) itself, I found in a more general context [9] that the method of the comparison of coefficients is capable of supplying something like (6), namely,

$$\|f\| \geqq \min (a, \tfrac{1}{2}b/M). \tag{9}$$

But (6) shows that the $\frac{1}{2}$ is superfluous in (9), and this has thus far been proved only by methods adapted from the real field, such as the method of successive approximations or the "polygonal" methods of Cauchy–Lipschitz or Peano. On the other hand, a glance at the proof in [9] shows that the $\frac{1}{2}$ of (9) is just a manifestation of the $\frac{1}{2}$ of Rogosinski (concerning power series in general; cf. [2], pp. 25–26) and, correspondingly, it can be expected that, in order to obtain (6) on the basis of compared coefficients, the Cesàro sums, rather than the partial sums, of (5) must be used.

2. In order to simplify the notations, it will, without loss of generality, be assumed that $a = 1$ and $b = 1$. The resulting dicylinder (2) will be denoted by D:

$$D = (|z| < 1, |w| < 1). \tag{10}$$

Finally, it will be convenient to choose M to be its least possible value, that is, to replace (4) by

$$M = \sup_D |f(z, w)|; \quad \text{so that} \quad M = M_f < \infty \tag{11}$$

by assumption and, if the trivial case $f(z, w) \equiv 0$ is disregarded, $M > 0$.

Accordingly, (6) means that

$$\|f\| \geqq 1 \quad \text{if} \quad M \leqq 1, \quad \text{and} \quad \|f\| \geqq 1/M \quad \text{if} \quad 1 < M < \infty. \tag{12}$$

Actually, the $\geqq$ in the second part of (12) can be improved to a $>$, that is,

$$\|f\| > 1/M \quad \text{if} \quad M > 1; \tag{13}$$

cf. [12].

If (1) is of the particular form (7) (so that (11) reduces to

$$M = \sup_{|w|<1} |f(w)|, \tag{14}$$

where $0 < M < \infty$ by assumption), then the a of (2) can be chosen arbitrarily large, hence the alternative (12) reduces simply to

$$\|f\| \geqq 1/M \qquad (\text{for } 0 < M < \infty). \tag{15}$$

Actually, (15) can be improved to

$$\|f\| > 1/M \qquad (\text{for } 0 < M < \infty), \tag{16}$$

since, in the proof of (12), the assumption, $M > 1$, of (13) is used in the form $b/M < a$, whereas a is arbitrarily large (and b is 1) in the present case.

3. Cauchy's local existence theorem, referred to at the beginning of Section 1, merely states that $\|f\| > 0$. He proved this by showing that

$$\|f\| \geqq 1 - \exp(-2M)^{-1} \qquad (\text{for } 0 < M < \infty). \tag{17}$$

Today it seems to be curious that Cauchy himself never applied his own "real" methods (the "Cauchy–Picard" or the "Cauchy–Lipschitz" method), methods which improve (17) to (12), in order to obtain $\|f\| > 0$.

Cauchy's proof of (17) is based on what he called *calcul des limites*. The latter comes from two sources, which will be referred to as (A) and (B):

(A) *Cauchy's coefficient estimate.* In view of (11) and (10), the estimate in question states that

$$|c_{mn}| \leqq M \qquad (m, n = 0, 1, 2, \ldots) \tag{18}$$

holds for the coefficients of (3).

(B) *Cauchy's principle of majorants.* This principle (the truth of which follows by comparing coefficients in a recursive way) can be formulated as follows: If, besides (1) with (3), another initial value problem, say

$$dW/dz = F(z, W), \qquad W(0) = 0, \tag{19}$$

with

$$F(z, W) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_{mn} z^m W^n, \tag{20}$$

is considered, and if (5) and

$$W(z) = \sum_{n=1}^{\infty} A_n z^n \tag{21}$$

represent the (in case of divergence for $z \neq 0$, just formal) solutions of (1) and (19) respectively, then the assumption

$$|c_{mn}| \leqq C_{mn} \qquad (m, n = 0, 1, 2, \ldots) \tag{22}$$

implies that

$$|a_n| \leqq A_n \qquad (n = 1, 2, \ldots), \tag{23}$$

and therefore, in particular, that

$$\|f\| \geqq \|F\|. \tag{23 bis}$$

It follows from (A) that the assumption of (B) is satisfied if $F(z, W)$ is chosen as follows:

$$F(z, W) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} M z^m W^n \equiv M/(1-z)(1-W) \tag{24}$$

(where $|z| < 1$ and $|W| < 1$). But the case (24) of (19) can be solved explicitly (by separating the variables), and the resulting explicit form of the function $W(z)$ (cf. [8], pp. 119–121) shows that $\|F\| = q$ (hence, $\|f\| \geqq q$) holds for the positive number q which is the difference on the right of the inequality (17). Hence (17) follows from (23 bis).

This is Cauchy's proof of (17). As observed by Stäckel (cf. [8], p. 120), the assumption (10), which requires more than (18), is not used in the proof of (17); in fact, (18) alone suffices. But (18) is satisfied even by the $F = F(z, w)$ defined by (24) (where $w = W$), and this $f = F$ fails to satisfy (11) by any $M < \infty$, since it has a pole on the boundary of D.

It should be mentioned that if (1) is of the particular form (7), then Cauchy's own method can be applied to improve (17) to

$$\|f\| \geqq (2M)^{-1} \qquad (\text{for } 0 < M < \infty) \tag{25}$$

and, incidentally, to nothing better than (25) (note that (25) is twice as weak as the estimate (16) which, when improved to (16), is the *best* result on (7); cf. the parenthetical remark made in connection with (7) in Section 1). In fact, if $f(z, w) = f(w)$, then (24) can be replaced by

$$F(z, W) = F(W), \text{ where } F(W) = \sum_{m=0}^{\infty} M W^m \equiv M/(1-W) \tag{26}$$

(where $|W| < 1$). But the case (26) of (19) can be solved by the inversion of a quadrature, and the resulting explicit form of the function $W(z)$ shows that $1/\|F\| = 2M$ (hence, $1/\|f\| \leqq 2M$). Consequently, (25) follows from (23 bis).

II. The "best majorant" equation of Lindelöf

4. It is possible to apply (B) without involving (A) at all. In fact, the most favorable choice of C_{mn} in (22) is not Cauchy's choice, $C_{mn} = M$, but the choice $C_{mn} = |c_{mn}|$, first considered by Lindelöf [3]. Then (20) shows that (19) reduces to

$$dw/dz = f^*(z, w), \quad w(0) = 0 \tag{27}$$

(if w is written in place of W), where $f^*(z, w)$ is defined by (8), and it follows from (23 bis) that

$$\|f\| \geqq \|f^*\|. \tag{28}$$

But the circle $|z| < \|f^*\|$ is, in general, smaller than the circle $|z| < \|f\|$. As emphasized already in [10], this shows the disadvantage of treating (1) by *any* majorant method, since (27) is the *best* majorant of (1).

It is clear from (28) that more than (17), where $f = f(z, w)$, and (25), where $f = f(w)$, is contained in

$$\|f^*\| \geqq 1 - \exp(-2M)^{-1} \quad (\text{for } 0 < M < \infty) \tag{29}$$

and

$$\|f^*\| \geqq (2M)^{-1} \quad (\text{for } 0 < M < \infty) \tag{30}$$

respectively. But since (17) and (25) depended only on (18), and since (18) remains true if c_{mn} is replaced by $|c_{mn}|$, the *calcul des limites* actually proves (29) or (30) when it proves (17) or (25).

Since (1), (8) and (10) imply that $f^*(z, w)$ is regular on D if (and only if) $f(z, w)$ is, it is possible to introduce, corresponding to the definition (11) of M, a norm M^* for $f(z, w)$ by placing

$$M^* = \sup_D |f^*(z, w)|. \tag{31}$$

But it is well known that (even if $f(z, w)$ is a function of a single variable and is, in addition, uniformly continuous within the unit circle; cf. [2], pp. 29–31) the assumption, $M < \infty$, of (11) does not imply that $M^* < \infty$. On the other hand, if only those functions $f(z, w)$ are considered on D which happen to satisfy the condition $M^* < \infty$ (rather than just the condition $M < \infty$; for the treatment of the general case, the case in which not even $M < \infty$ holds, cf. [12]), then

$$\|f^*\| \geqq 1 \quad \text{if} \quad M^* \leqq 1, \quad \text{and} \quad \|f^*\| \geqq 1/M^* \quad \text{if} \quad 1 < M^* < \infty. \tag{32}$$

In fact, (32) follows if (12) is applied to (27), rather than to (1) itself.

The estimate of $\|f\|$ which results from (28) and (32) is the result of Lindelöf's paper [3]. He obtained (32) not as a consequence of (12) but by a peculiar combination of the method of comparison of coefficients with the method of successive approximations (this method of Lindelöf remained unnoticed in the literature until it was put to further use in [9]; subsequently, it was taken over by Kamke's *Differentialgleichungen* (1930) and it thus became generally known).

5. The estimate (29) of $\|f^*\|$ has a structure quite different from the structure of (32), since (29) does not assume that $M^* < \infty$ (but is, nevertheless, an estimate of $\|f^*\|$, rather than of $\|f\|$). But (29) is bound to be quite rough in every other respect, since it does not depend on anything like (11), but merely on (18). It is therefore worth showing that it is possible to deduce from the "sharp" estimate (12) of $\|f\|$ an estimate of $\|f^*\|$ which, like (29), does not involve M^* but (except for a universal constant factor) has the same structure as (12) itself.

The estimate in question is the following:

$$\|f^*\| \geqq \Theta \quad \text{if} \quad M \leqq 1, \quad \text{and} \quad \|f^*\| \geqq \Theta/M \quad \text{if} \quad 1 < M < \infty, \tag{33}$$

where Θ is a universal constant, defined as the (unique) positive root of the transcendental equation

$$\sum_{n=1}^{\infty} (n+1)^{\frac{1}{2}}\, \Theta^n = \tfrac{1}{2} \tag{34}$$

(incidentally, it will be clear from the proof that

$$0 < \Theta < \tfrac{1}{3}, \tag{35}$$

the $\frac{1}{3}$ being the universal constant of Bohr; cf. [2], pp. 32–34).

It will turn out (cf. Section 7) that (33) does not contain the best estimate of $\|f^*\|$ (in terms of M alone). But since (33), in contrast to the best estimate of $\|f^*\|$, has the same structure as (12) itself, it is worthwhile to prove (33) directly.

6. First, if a power series

$$\sum_{k=0}^{\infty} b_k\, t^k \tag{36}$$

converges within the unit circle and if there exists a constant M majorizing the absolute value of the function (36) if $|t| < 1$, then

$$|b_k| \leqq 2\,(M - |b_0|), \quad \text{where } k = 1, 2, \ldots .$$

This inequality, due to Carathéodory, is connected with his and Toeplitz' criterion for power series having a non-negative real part in the circle $|t| < 1$. For a short direct proof, cf. [2], pp. 33–34.

Next, if $f(z, w)$ is regular on D and satisfies (11), then, by replacing (z, w) by (tz, tw), where t varies on the circle $|t| < 1$ and (z, w) is any point of D, it is seen from (3) that the conditions required of (36) are satisfied if

$$b_k = \sum_{m+n=k} c_{mn}\, w^m z^n \qquad (k = 0, 1, \ldots).$$

But since (w, z) is any point of (z, w), the last two formula lines imply that

$$\sup_D \Big| \sum_{m+n=k} c_{mn}\, w^m z^n \Big| \leqq 2\,(M - |c_{00}|), \quad \text{where } k = 1, 2, \ldots . \tag{37}$$

7. In view of (11) and (37), Parseval's relation implies that

$$\Big(\sum_{m+n=k} |c_{mn}|^2\Big)^{\frac{1}{2}} \leqq 2\,(M - |c_{00}|),$$

and so, since there are $k+1$ terms in the sum on the left,

$$\sum_{m+n=k} |c_{mn}| \leqq 2\,(M - |c_{00}|)\,(k+1)^{\frac{1}{2}}$$

(Schwarz), where $k = 1, 2, \ldots$. It follows therefore from (8) that, if $0 < \Theta < 1$,

$$f^*(\Theta, \Theta) \leqq |c_{00}| + 2\,(M - |c_{00}|) \sum_{k=1}^{\infty} (k+1)^{\frac{1}{2}}\, \Theta^k,$$

and so, if Θ is chosen to be the positive root of the equation (34),

$$f^*(\Theta, \Theta) \leqq M. \tag{38}$$

What this actually proves is the following fact: If a function $f(z, w)$ is regular on D, then (11) implies (38). This fact can be thought of as the two-dimensional analogue of the result of Bohr, referred to in connection with (35). But it remains undecided whether the absolute constant Θ, defined by (35), is the best universal constant in (38).

It is clear from (8) that (4) (or, rather, that variant of (4) in which the $<$ is relaxed to a $\leqq$) is satisfied if f, a, b, M are replaced by $f^*, \Theta, \Theta, f^*(\Theta, \Theta)$ respectively. It follows therefore from (4) that

$$\|f^*\| \geqq \min\,(\Theta, \Theta / f^*(\Theta, \Theta)). \tag{39}$$

In view of (38), this proves (33).

Since $\Theta < 1$, for no value of M can the result (33) assure for the solution $w(z)$ of (27) a circle

$$|z| < R = R_M \tag{40}$$

which comes close to the circle $|z| < 1$, the first factor of the product set (10) on which the $f^*(z, w)$ of (27) is given as regular. On the other hand, (29) is free of this shortcoming, since it assures that R_M can be brought *arbitrarily close to* 1 by choosing M small enough.

But is it true that the solution $w(z)$ of (27) must become regular on the *fixed* circle $|z| < 1$ whenever M is small enough? In other words, is it true that

$$\|f^*\| \geqq 1 \quad \text{whenever} \quad 0 < M \leqq \Theta_0, \tag{41}$$

where Θ_0 is a certain (sufficiently small) absolute constant? If $\|f^*\|$ is replaced by $\|f\|$, then (12) shows that the answer becomes affirmative (the best value of the corresponding absolute constant being 1). But (11) and (31) are so far apart that (41) will be expected to be false. It turns out, however, that this argument is misleading; in other words, that (41) happens to be true for a certain $\Theta_0 > 0$.

More than this will be shown by proving the following fact, which will be the main result of this paper:

$$\|f^*\| \geqq 1 \quad \text{if} \quad M \leqq \tfrac{1}{2}, \quad \text{and} \quad \|f^*\| \geqq \sin\left(\tfrac{1}{4}\pi/M\right) \quad \text{if} \quad M \geqq \tfrac{1}{2}. \tag{42}$$

The two assertions of (42) coincide if $M = \frac{1}{2}$. The first assertion of (42) implies (41), with $\Theta_0 = \frac{1}{2}$. It is easy to verify that (42) contains both (29) and (33) as corollaries.

Note that (42), in contrast to (29) and (33), where $\Theta < 1$, has the same structure as (12). There are indications (cf. Section 12 below) that (42) is the best possible result for every M (in the same sense in which (12) is sure to be the best possible result for every M; cf. the parenthetical remark made in connection with (7) in Section 1), but this will not be proved.

It will also be shown that if $f(z, w)$ or $f^*(z, w)$ is independent of z, as in (30), then (42) can be improved to

$$\|f^*\| \geqq \tfrac{1}{4}\pi/M \qquad (\text{for } 0 < M < \infty). \tag{43}$$

This, of course, is sharper than (30), and (43) seems to be final (in the same sense in which (15) is sure to be final).

The proofs will be adaptations of those developed in [13], where, however, the issues are disguised by the choice $M = 1$ and by the introduction of an erroneous "transcendental equation".

III. The Parseval subordination of the "best majorant" equation

8. Instead of Cauchy's principle of majorants, formulated under (B) in Section 3, recourse will have to be had to the following comparison theorem (C) (which is the particular case of a more general lemma on "subordination," a lemma in which the comparison function, the F of (C) below, need not be a power series but can be any continuous function of the non-negative variables $r = |w|$, $s = |z|$).

(C) *The principle of subordination.* If $f(z, w)$ and $F(z, w)$ are regular on a dicylinder D about $(0, 0)$, say on (10), and if

$$|f(z, w)| \leqq F(|z|, |w|) \tag{44}$$

holds at every point (z, w) of D, then

$$\|f\| \geqq \|F\|. \tag{45}$$

Note that the assertion, (45), of (C) is the same as the *corollary*, (23 bis), of the actual assertion, (23) of (B) (in this regard, cf. the concluding remark of [11]). But the point is that the assumption, (44), of (C) requires much less than the assumption, (22), of (B); cf., in fact, (3) and (22).

As pointed out in [11], the truth of (C) can be read off from methods of proof customary in the (local) existence proofs for the initial value problem of ordinary differential equations in the *real* field. In a slightly different version, and with a more elaborate proof, the assertion of (C) (and more) is contained in a note of Nakano ([5]; cf. [4], p. 170). A direct proof of (C) (and of much more) was based in [12] on the method of successive approximations.

In what follows, (C) will be needed only in the particular case in which the f of (C) is an f^*; cf. (3) and (8). Clearly, (C) then reduces to the following assertion (C bis):

(C bis) *If $f(z, w)$ and $g(z, w)$ are regular on D, and if*

$$f^*(r, s) \leqq g(r, s) \tag{46}$$

holds on the square

$$0 \leqq r < 1, \qquad 0 \leqq s < 1 \tag{47}$$

(*which, if $r = |z|$ and $s = |w|$, corresponds to the dicylinder* (10)), *then*

$$\|f^*\| \geqq \|g\|. \tag{48}$$

Needless to say, (46) implies that $g(r, s) \geqq 0$ on (47).

9. In order to deduce (42) from (C bis), suppose that $f(z, w)$ is regular on D and satisfies (11) (for a fixed M). Then it follows from (3) and (10) that, by virtue of Parseval's relation (or just of Bessels's inequality), the sum of all squares $|c_{mn}|^2$ does not exceed M^2. Hence it is seen from (8) and (47) that, in view of Schwarz' inequality, condition (46) is satisfied by

$$g(r, s) = M\left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} r^{2m} s^{2n}\right)^{\frac{1}{2}} \equiv M/(1-r^2)^{\frac{1}{2}}(1-s^2)^{\frac{1}{2}}. \tag{49}$$

It follows therefore from (48) that (42) will be proved if it is shown that (42) is true when $\|f\|$ is read in place of $\|f^*\|$, that is, if (19) is replaced by

$$dw/dz = g(z, w) \qquad w(0) = 0, \tag{50}$$

where

$$g(z, w) = M/(1-z^2)^{\frac{1}{2}}(1-w^2)^{\frac{1}{2}} \tag{51}$$

on D.

Incidentally,

$$b_{mn} \geqq 0 \tag{52}$$

if

$$g(z, w) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{mn} w^m z^n. \tag{53}$$

In fact, it is clear from (51) and (53), that, if

$$(1-s^2)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \beta_k s^{2k} \tag{54}$$

then b_{mn} is $M\beta_{\frac{1}{2}m}\beta_{\frac{1}{2}n}$ or 0 according as m *and* n are even or not. Since $M > 0$, and since (54), being equivalent to

$$\beta_k = \prod_{j=1}^{k} (2j-1)/(2j) \qquad (\beta_0 = 1), \tag{55}$$

implies that $\beta_k > 0$, the assertion (52) follows. But it is clear from the comments made on (C) in Section 7 that (52) will be irrelevant, since, even though (52) happens to be true, it is *not* true that the assumption, (22), of (B) becomes satisfied by $C_{mn} = b_{mn}$.

This is precisely the reason why (C bis) will, whereas (B) cannot possibly, lead to the proof of anything like (42). Correspondingly, the best that (B), when combined with (A), could supply was Cauchy's majorant, the case

$$g(z, w) = M/(1-z)(1-w)$$

of (50); cf. Section 3. But the point is that, at every point $(z, w) = (r, s) \neq (0, 0)$ of (47), this g of Cauchy is *greater* than the g assigned by (51) (in fact, the greater the closer is (r, s) to the common singularity at $(r, s) = (1, 1)$).

10. The only use to which (52) will be put will be a slight (and actually unnecessary) simplification of the discussion of the solution $w(z)$ of (51). The simplification consists of the following circumstance:

If (5) is the solution of (50) (for small $|z|$), and if both (53) and (5) are inserted in (50), then comparison of like powers of z shows that $a_n \geqq 0$ holds by virtue of (52). But $a_n \geqq 0$ is known to imply (Vivanti–Pringsheim; cf. [2], pp. 72–73) that, since $\|g\|$ denotes the radius of convergence of (5), the function $w(z)$ must become singular at $z = \|g\|$ (if $\|g\| < \infty$). Hence, instead of discussing the case (51) of (50) on the dicylinder (10), it is sufficient to discuss

$$ds/dr = M/(1-r^2)^{\frac{1}{2}}(1-s^2)^{\frac{1}{2}}, \quad s(0) = 0 \tag{56}$$

on the square (47). But this will readily lead to the following determination (instead of just an estimate) of the value of $\|g\|$:

$$\|g\| = 1 \quad \text{if} \quad M \leqq \tfrac{1}{2}, \quad \text{and} \quad \|g\| = \sin(\tfrac{1}{4}\pi/M) \quad \text{if} \quad M > \tfrac{1}{2}. \tag{57}$$

In view of (48), this will prove more than what is needed for the completion of the proof of (42).

Whether $t = r$ or $t = s$, let $\arcsin t$, where $0 \leqq t < 1$, denote the determination satisfying $0 \leqq \arcsin t < \frac{1}{2}\pi$. Then, since the square roots in (56) refer to their positive determinations, two quadratures show that the (differentiable) solution $s = s(r)$ of (56) (as far as it exists for $r \geqq 0$) is given by

$$h(s) = 2M \arcsin r, \tag{58}$$

where $h(s)$ denotes the integral of $2(1-t^2)^{\frac{1}{2}}$ over $0 \leqq t \leqq s\,(<1)$, that is,

$$h(s) = s(1-s^2)^{\frac{1}{2}} + \arcsin s. \tag{59}$$

But (59), where $0 \leqq s < 1$, is positive and increasing for $0 < s < 1$ (the derivative of (59) is $2(1-s^2)^{\frac{1}{2}} > 0$). Since (59) also shows that $h(0) = 0$ and $h(1-0) = \frac{1}{2}\pi$, it follows that (59) defines a one-to-one (continuous) correspondence between $0 \leq s < 1$ and $0 \leq h < \frac{1}{2}\pi$.

For every $M > 0$, let M_0 be defined by the property that $0 \leqq r < M_0$ is the greatest r-interval on which the solution $s = s(r)$ of (56), where $r \geqq 0$, exists (as a differentiable function). It is clear from (56) that $s(r)$ is positive and increasing on the interval $0 < r < M_0$ (while $s(0) = 0$), and that the value of M_0 can also be characterized by the following condition: $1 > s(r) \to 1$ if $0 \leqq r \to M_0$. If this is compared with (58) and with that property of $h(s)$ which was pointed out at the end of the preceding paragraph, then it is seen that, for every $M > 0$, the value of M_0 is given by the alternative condition which results if M_0 is read in place of $\|g\|$ in (57).

For the sake of clarity, let the function (51) be now denoted by $g_M(z, w)$. Thus $\|g_M\|$ is the radius of the greatest circle (about $z = 0$) within which the solution $w(z) = w_M(z)$ of the case $g = g_M$ of (50) is regular. It follows therefore from the fact which, just before the introduction of (56), was concluded from $a_n \geqq 0$, that the number $\|g_M\|$ is precisely the number M_0, defined at the beginning of the preceding paragraph. Consequently, the truth of (57) for $\|g\| = \|g_M\|$ follows from the alternative representation of M_0, found at the end of the preceding paragraph.

11. This completes the proof of (42), where $f = f(z, w)$. The proof of (43), where $f = f(w)$, requires only slight alterations in the proof of (42).

First, it is clear that (46) and (49) now reduce to $f^*(r) \leqq g(r)$ and

$$g(r) = M\left(\sum_{m=0}^{\infty} r^{2m}\right)^{\frac{1}{2}} \equiv M/(1-r^2)^{\frac{1}{2}}$$

respectively. Hence the case (51) of (50) reduces to the case $g(z, w) = g^M(w)$ of (50), where $g^M = M/(1-w^2)^{\frac{1}{2}}$, and (48) shows that (43) (and more) will be proved if it is verified that

$$\|g^M\| = \tfrac{1}{4}\pi/M. \tag{60}$$

Next, instead of the case $g(z, w) = g^M(z)$ of (50) on (10), it is sufficient to discuss the initial value problem

$$ds/dr = M/(1-s^2)^{\frac{1}{2}}, \quad s(0) = 0 \tag{61}$$

on (47). In fact, the possibility of this reduction follows, via the possibility of the coefficients, (55), of (54), in the same way as the reduction of (50)–(51) to (56) did.

Clearly, the solution $s = s(r)$ of (61) is given by

$$h(s) = 2Mr, \tag{62}$$

if $h(s)$ is defined, as before, by (59). This means that (58) must now be replaced by (62). But if M^0 is defined, with reference to (61), in the same way in which M_0 was defined with reference to (56), then, on the one hand,

$$M^0 = \|g^M\| \tag{63}$$

(cf. the end of Section 10) and, on the other hand, M^0 is characterized by the property that $1 > s \to 1$ if $0 \leqq r \to M^0$, where $s = s(r)$ is the solution of (61). Clearly, this characterization of M^0, when compared with (62) and with the monotony of the function of (59) on $0 \leqq s < 1$, can be re-stated as follows: $2Mr \to h(1-0)$ as $r \to M^0 - 0$.

Since (59) shows that $h(1-0) = \frac{1}{2}\pi$, this means that $2M = \frac{1}{2}\pi/M_0$. In view of (63), this proves (60). Finally, (43) follows from (60) and (48), where $g = g^M$.

12. There remains to be decided whether or not the universal limitations (42), (43) of $\|f^*\|$ are final; final in the sense that, when nothing more than a fixed value M of the norm (11) of f is given, then, for this value of M, the limitation of $\|f^*\|$ which is supplied by (42) or (43) cannot be improved in terms of *absolute constants* (which, however, could depend on M).

Since (42) refers to any $f = f(z, w)$ but (43) only to the particular type $f = f(w)$, and since $0 < \sin \alpha < \alpha$ if $0 < \alpha < \frac{1}{2}\pi$, it is clear that, in the sense just specified, (42) must be final (for a given M) if (43) is final (for the same M). But there is an indication that (43) *is* final (for any given M).

The indication results if, on the one hand, (54) and (55) are taken into account in (61) and, on the other hand, the resulting partial sums (which are the polynomials

$$\sum_{k=0}^{n} \beta_k t^k, \quad \text{cf. (55)}, \tag{64}$$

if $M = 1$ and $t = s^2$) occur in Landau's determination of the extremal functions of his "coefficient problem"; cf. [2], pp. 26–28. In fact, it is precisely (61) that led, via (60), to (43). The suspected connection could perhaps be established by using, instead of Landau's own approach alone, a fact discovered by Schur [7], pp. 122–124; he embedded Landau's result into a general theory, from which he was able to conclude that the (rational) functions which are the solutions of Landau's extremal problem are (finite) Jacobi–Jensen products.

13. It was mentioned in Section 2 that, although the universal constants $(=1)$ which occur in (12) and (15) are final for every fixed M (cf. the parenthetical remark made in connection with (7) in Section 1), both (12) and (15) can be improved (in another sense), since both (13) and (16) are true. It turns out that (42) and (43) can be improved similarly.

It will be sufficient to consider the case of (43), since it will be clear that the proof applies to the case of (42) also. In the case of (43), the improvement in question states that the sign of equality cannot hold (for any M) in (43); so that

$$\|f^*\| > \tfrac{1}{4}\pi/M \qquad (\text{for } 0 < M < \infty). \tag{65}$$

This can be seen as follows:

If $f(w) = \sum_{n=0}^{\infty} c_n w^n$, hence $f^*(w) = \sum_{n=0}^{\infty} |c_n| w^n$ as well, is convergent for $|w| < 1$, and if $|f(w)| \leqq M$ for $|w| < 1$, then, according to Hardy, two *different* statements can be made on $f^*(s)$, where $s = |w| < 1$:

$$f^*(s) \leqq M/(1-s^2)^{\frac{1}{2}} \qquad \text{for } 0 \leqq s < 1 \tag{66}$$

and

$$f^*(s) = \varepsilon(s)/(1-s^2)^{\frac{1}{2}}, \quad \text{where} \quad \lim_{s\to 1} \varepsilon(s) = 0 \tag{67}$$

(cf. [1]; in [2], pp. 31–32, just (67) is reproduced explicitly, whereas (66) is between the lines of the proof). But only the straightforward inequality (66) (Parseval, Schwarz) was used, via (61) and (60), in the proof of (43). If (67), too, is used, then it is clear that the proof (43) leads to (65).

Hardy [1] refers to an example of Fabry, and gives another example, to the effect that (66)–(67) cannot be improved. But as the situation is somewhat involved in those examples, I could not decide whether the same examples also prove that (43) is final (in the sense specified at the beginning of Section 12 above).

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