ON THE MAXIMUM TERM AND THE RANK OF AN ENTIRE FUNCTION

BY

S. K. SINGH

1. Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be an entire function and let $\mu(r)=\mu(r,f)$ be the maximum term of the series for $|z|=r$ and $\nu(r)$ the rank of the maximum term. Let R_n be the points of discontinuity of $\nu(r)$. Let $\mu'(r)$ and $\nu'(r)$ correspond to $f'(z)$ and in general $\mu^k(r)$ and $\nu^k(r)$ correspond to $f^k(z)$.

In section 1 we prove results concerning the maximum term $\mu(z)$ and in sections 2, 3, and 4, results concerning $\mu(r)$ and $\nu(r)$.

THEOREM 1. If $f(z)$ be an entire function of order $\varrho<1$, then

$$\frac{r^p\,\mu^k(r)}{\mu(r)}\to 0 \text{ as } r\to\infty$$

for
$$k=1,2,3,\ldots$$

and
$$p<k(1-\varrho).$$

REMARK. If $p=k(1-\varrho)$, then the result is not necessarily true. Take $k=1$ and consider $f(z)=\cos\sqrt{z}$.

$$\varrho(f)=\tfrac{1}{2}.$$

For
$$(2n-1)\,2n\le r<(2n+1)(2n+2)$$

$$\mu(r,f)=\frac{r^n}{(2n)!}\sim\frac{r^n}{(2n)^{2n+\frac{1}{2}}\,e^{-2n}\sqrt{2\pi}}\sim\frac{e^{\sqrt{r}}}{\sqrt{2\pi}\,r^{\frac{1}{4}}}.$$

Similarly, we can show that

$$\mu(r,f')\sim\frac{1}{2\sqrt{2\pi}}\,\frac{e^{\sqrt{r}}}{r^{\frac{3}{4}}}.$$

Hence
$$\frac{\mu(r,f')\,r^{\frac{1}{2}}}{\mu(r,f)}\sim\tfrac{1}{2}.$$

1 – 553810. *Acta Mathematica*. 94. Imprimé le 29 octobre 1955.

PROOF OF THEOREM 1. $f(z)=\sum_{n=0}^{\infty} a_n z^n$.

Set $$|a_n|=C_n.$$

Then for $R_n \leq r < R_{n+1}$

$$\mu^k(r)=\nu^k(r)(\nu^k(r)-1)\cdots(\nu^k(r)-k+1)\,C_{\nu^k(r)}\,r^{\nu^k(r)-k}$$
$$\leq \mu^{k-1}(r)\left(\frac{\nu^k(r)-k+1}{r}\right),$$

and by a repeated application of the above

$$\mu^k(r)\leq \frac{(\nu^k(r)-k+1)(\nu^{k-1}(r)-k+2)\cdots\nu'(r)}{r^k}\mu(r).$$

Also [1]

$$\limsup_{r\to\infty}\frac{\log \nu(r)}{\log r}=\varrho,$$

so for all $r\geq r_0$ we get

$$\nu(r)<r^{\varrho+\varepsilon}. \tag{1.1}$$

By taking r sufficiently large to ensure the inequality (1.1) for every $\nu(r)$, we have

$$\frac{\mu^k(r)}{\mu(r)}\leq r^{k\varrho+\varepsilon'-k}. \tag{1.2}$$

Taking $$\varrho<1 \text{ and } p<k(1-\varrho)$$

we get the result.

COROLLARY (i). If $f(z)$ is an entire function of order ϱ $(0\leq\varrho\leq\infty)$, then

$$\limsup_{r\to\infty}\frac{\log\{r\mu'(r)/\mu(r)\}}{\log r}=\varrho. \tag{1.3}$$

Putting $k=1$ in (1.2), we get

$$\mu'(r)\leq\mu(r)\,r^{\varrho-1+\varepsilon'}. \tag{1.4}$$

Hence

$$\limsup_{r\to\infty}\frac{\log\{r\mu'(r)/\mu(r)\}}{\log r}\leq\varrho;$$

further

$$\mu'(r)=\nu'(r)\,C_{\nu'(r)}\,r^{\nu'(r)-1}\geq\frac{\nu(r)}{r}\mu(r),$$

so

$$\limsup_{r\to\infty} \frac{\log \{r\,\mu'(r)/\mu(r)\}}{\log r} \geq \limsup_{r\to\infty} \frac{\log \nu(r)}{\log r} = \varrho.$$

Thus we get

$$\limsup_{r\to\infty} \frac{\log \{r\,\mu'(r)/\mu(r)\}}{\log r} = \varrho.$$

COROLLARY (ii). If $0 \leq \varrho \leq \infty$, then

$$\limsup_{r\to\infty} \frac{\log \{r\,M'(r)/M(r)\}}{\log r} = \varrho. \tag{1.5}$$

For $0 \leq \varrho < \infty$ it follows from (1.3) because for functions of finite order $\log \mu(r) \sim \sim \log M(r)$. That the result is also true for $\varrho = \infty$ follows from the inequality

$$M'(r) > \frac{M(r)}{r} \frac{\log M(r)}{\log r} \tag{1.6}$$

for $r \geq r_0(f)$, see T. Vijayraghavan [2].

COROLLARY (iii). If $f(z)$ is an entire function of order ϱ, then for all $r \geq r_0$

$$M'(r) < M(r)\, r^{\varrho-1+\varepsilon}. \tag{1.7}$$

This follows easily from (1.5).

The inequality (1.7) is due to G. Valiron [3], where he mentions it without proof.

For an alternative proof of (1.5) and (1.7), see S. M. Shah [4]. We observe that (1.4) is a result analogous to (1.7) for $\mu(r)$ and $\mu'(r)$.

COROLLARY (iv). If $\varrho < 1$, then

$$\frac{M^k(r)\, r^p}{M(r)} \to 0 \quad \text{as} \quad r \to \infty$$

for $k = 1, 2, \ldots$ and $p < k(1-\varrho)$.

The proof can easily be supplied by a repeated application of (1.7).

THEOREM 2. If $f(z)$ be an entire function of lower order $\lambda > 1$, then

$$\frac{\mu^k(r)}{\mu(r) r^p} \to \infty \quad \text{as} \quad r \to \infty$$

for $k = 1, 2, \ldots$ and $p < k(\lambda - 1)$.

PROOF.

$$\mu'(r) = \nu'(r)\, C_{\nu'(r)}\, r^{\nu'(r)-1} \geq \frac{\nu(r)}{r} \mu(r).$$

Proceeding in this way, we get

$$\frac{\mu^k(r)}{\mu(r)} \geq \frac{\nu(r)(\nu'(r)-1)\cdots(\nu^{k-1}(r)-k+1)}{r^k}$$

and since [5]

$$\liminf_{r\to\infty} \frac{\log \nu(r)}{\log r} = \lambda$$

$$\nu(r) > r^{\lambda-\varepsilon} \text{ for all } r \geq r_0$$

and as usual taking r sufficiently large, we get

(1.8)
$$\frac{\mu^k(r)}{\mu(r)} \geq r^{k\lambda-\varepsilon'-k}$$

and taking $\lambda > 1$ and $p < k(\lambda-1)$ we get the result.

COROLLARY (i). If $f(z)$ be an entire function of lower order λ $(0 \leq \lambda \leq \infty)$, then

(1.9)
$$\liminf_{r\to\infty} \frac{\log \{r\mu'(r)/\mu(r)\}}{\log r} = \lambda.$$

Putting $k=1$ in (1.8), we get

$$\mu'(r) \geq \mu(r)\, r^{\lambda-1-\varepsilon'}$$

hence

(1.10)
$$\liminf_{r\to\infty} \frac{\log \{r\mu'(r)/\mu(r)\}}{\log r} \geq \lambda.$$

Further

$$\mu(r) = C_{\nu(r)}\, r^{\nu(r)} \geq \frac{\mu'(r)\, r}{\nu'(r)}$$

$$\frac{r\mu'(r)}{\mu(r)} \leq \nu'(r) < r^{\lambda+\varepsilon}$$

for an infinity of r.

Hence

(1.11)
$$\liminf_{r\to\infty} \frac{\log \{r\mu'(r)/\mu(r)\}}{\log r} \leq \lambda.$$

(1.10) and (1.11) give the result.

REMARK. The results (1.3) and (1.9) are known to be still true, if $\mu'(r)$ instead of standing for the maximum term of $f'(z)$ stands simply for the derivative of $\mu(r)$. See S. K. Singh [6].

2. Let

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mu(r)}{r^{\varrho}} = \begin{matrix}\alpha\\ \beta\end{matrix}; \quad \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\nu(r)}{r^{\varrho}} = \begin{matrix}\gamma\\ \delta\end{matrix}.$$

S. M. Shah [7] has proved that

$$\gamma \leq e \varrho \alpha \tag{2.1}$$

$$\delta \leq \varrho \alpha. \tag{2.2}$$

We prove here

THEOREM 3. (i)

$$\gamma + \delta \leq e \varrho \alpha \tag{2.3}$$

and that

(ii) equality cannot hold simultaneously in (2.2) and (2.3).

PROOF (i).

$$\log \mu(kr) = A + \int_{r_0}^{r} \frac{\nu(t)}{t} dt + \int_{r}^{kr} \frac{\nu(t)}{t} dt \qquad (k > 1)$$

$$\geq A^{\dagger} + \frac{(\delta - \varepsilon) r^{\varrho}}{\varrho} + \nu(r) \log k$$

$$\frac{\log \mu(kr)}{(kr)^{\varrho}} \geq o(1) + \frac{(\delta - \varepsilon)}{\varrho} \frac{1}{k^{\varrho}} + \frac{\nu(r)}{r^{\varrho}} \frac{1}{k^{\varrho}} \log k$$

so

$$\alpha \geq \frac{\delta + \varrho \gamma \log k}{\varrho k^{\varrho}}. \tag{2.4}$$

The right hand side of (2.4) is a maximum when $k = e^{\frac{\gamma - \delta}{\gamma \varrho}}$, hence

$$\alpha \geq \frac{\gamma}{e \varrho} e^{\frac{\delta}{\gamma}}$$

$$= \frac{\gamma}{e \varrho}\left(1 + \frac{\delta}{\gamma} + \cdots\right).$$

Hence $$e \alpha \varrho \geq \gamma + \delta$$

which proves (2.3).

---

† A is not necessarily the same at each occurrence.

PROOF (ii). Let $\delta = \varrho\alpha$, then from (2.4)

$$\alpha \geq \frac{\varrho\alpha + \varrho\gamma \log k}{\varrho k^{\varrho}}$$

hence

$$\gamma \leq \frac{\alpha(k^{\varrho} - 1)}{\log k}.$$

Put

$$k = (1+\eta)^{\frac{1}{\varrho}} \quad \text{where} \quad \eta \to 0$$

so

$$\gamma \leq \frac{\varrho\alpha\eta}{\eta + O(\eta^2)} \leq \varrho\alpha.$$

Further

$$\delta \leq \gamma.$$

Hence

$$\gamma = \varrho\alpha$$

so

$$\delta + \gamma = 2\varrho\alpha < e\varrho\alpha.$$

Next suppose that

$$\gamma + \delta = e\varrho\alpha,$$

then δ will be less than $\varrho\alpha$, for if it were equal to $\varrho\alpha$, then by the above $\gamma + \delta$ will have to be less than $e\varrho\alpha$.

REMARK. The inequality in (2.2) and (2.3) can simultaneously hold, for instance consider

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{R_1 R_2 \cdots R_n}$$

where

$$R_n = n^{1/\varrho} e^{S_n} \quad \text{for} \quad n \geq n_0$$
$$R_n = 1 \qquad \text{for} \quad n < n_0$$

where S_n satisfies the following conditions

$$\liminf_{n\to\infty} S_n = -\frac{1}{\varrho}\log\gamma; \ \limsup_{n\to\infty} S_n = -\frac{1}{\varrho}\log\delta, \ (\delta < \gamma)$$

$$(S_{n+1} - S_n) = O\left(\frac{1}{\log n}\right)$$

$$\left(S_n - \frac{S_1 + \cdots + S_n}{n}\right) = O\left(\frac{1}{\log n}\right).$$

(The above example is that of S. M. Shah [8], where he constructs it for another purpose. The choice of such a sequence S_n is possible. See his lemma therein.)

With a little calculation we can easily show that for the above function $f(z)$

$$\alpha = \limsup_{r\to\infty} \frac{\log \mu(r)}{r^{\varrho}} = \frac{\gamma}{\varrho}$$

and

$$\limsup_{r\to\infty} \frac{\nu(r)}{r^{\varrho}} = \gamma; \quad \liminf_{r\to\infty} \frac{\nu(r)}{r^{\varrho}} = \delta.$$

Hence for the above function

$$\delta < \varrho\,\alpha$$

$$\gamma = \varrho\,\alpha.$$

Hence $$\gamma + \delta < 2\,\varrho\,\alpha.$$

and a fortiori $$\gamma + \delta < e\,\varrho\,\alpha.$$

3. S. M. Shah [9] has proved that if

$$\log\log \mu(r) = (1 + o(1)) \log\log r \tag{3.1}$$

for a sequence of values of r tending to infinity, then

$$\limsup_{r\to\infty} \frac{\log \mu(r)}{\nu(r)\log r} = 1. \tag{3.2}$$

(3.1) implies that the function is of zero order.

We prove below

THEOREM 4. (i) There exist entire functions of order ϱ $(0 < \varrho \leq \infty)$ for which (3.2) holds.

(ii) There exist entire functions of zero order for which

$$\lim_{r\to\infty} \frac{\log \mu(r)}{\nu(r)\log r} = 1. \tag{3.3}$$

REMARK. For functions of non-zero order, (3.3) is never true, because for all entire functions of order ϱ $(0 \leq \varrho \leq \infty)$

$$\liminf_{r\to\infty} \frac{\log \mu(r)}{\nu(r)} \leq \frac{1}{\varrho}. \tag{A}$$

(We give a proof of (A) in Theorem 5 below.)

PROOF OF THEOREM 4 (i).

Consider $$f(z) = \sum_{n=0}^{\infty} \left(\frac{z}{a_n}\right)^{\lambda_n}$$

where a_n are positive real numbers $\overset{\infty}{\uparrow}$ and λ_n are integers such that $\lambda_{n+1} = 10^{\lambda_n}$.

Now
$$\mu(r)=\frac{r^{\lambda_n}}{a_n^{\lambda_n}}; \quad \nu(r)=\lambda_n$$
for
$$R_n \leq r < R_{n+1}$$
where
$$R_n=\exp\left\{\frac{\lambda_n \log a_n-\lambda_{n-1}\log a_{n-1}}{\lambda_n-\lambda_{n-1}}\right\}.$$

First take
$$a_n=\lambda_n;$$
then
$$\varrho=\limsup_{n\to\infty}\frac{\lambda_n \log \lambda_n}{\lambda_n \log a_n}=1$$
and
$$\frac{\log \mu(r)}{\nu(r)\log r}=1-\frac{\log \lambda_n}{\log r}$$
so
$$\frac{\log \mu(R_{n+1})}{\nu(R_{n+1})\log R_{n+1}}=1-\frac{(\log \lambda_n)(\lambda_{n+1}-\lambda_n)}{\lambda_{n+1}\log \lambda_{n+1}-\lambda_n \log \lambda_n}$$
$$=1-\frac{\lambda_{n+1}\log \lambda_n+o(\lambda_{n+1})}{\lambda_{n+1}\lambda_n \log 10+o(\lambda_{n+1})}\sim 1.$$
Hence

(3.4)
$$\limsup_{r\to\infty}\frac{\log \mu(r)}{\nu(r)\log r}\geq 1.$$

Further
$$\log \mu(r)=O(1)+\int_{r_0}^{r}\frac{\nu(t)}{t}\,dt$$
$$\leq O(1)+\nu(r)\log r.$$
Hence

(3.5)
$$\limsup_{r\to\infty}\frac{\log \mu(r)}{\nu(r)\log r}\leq 1$$

(3.4) and (3.5) give the result in the case when $\varrho=1$.

We observe that ϱ can be made to have any finite value by a proper choice of a_n.

Again let
$$a_n=\log \lambda_n;$$
then
$$\varrho=\infty$$
and
$$\frac{\log \mu(R_{n+1})}{\nu(R_{n+1})\log R_{n+1}}=1-\frac{(\log\log \lambda_n)(\lambda_{n+1}-\lambda_n)}{\lambda_{n+1}\log\log \lambda_{n+1}-\lambda_n \log\log \lambda_n}\sim 1.$$

Hence as usual

$$\limsup_{r\to\infty} \frac{\log \mu(r)}{\nu(r) \log r} = 1, \quad \text{when} \quad \varrho = \infty.$$

REMARK. Here in both the cases the function is of irregular growth. In the first case

$$\log \mu(R_{n+1}) = \lambda_n \left\{ \frac{\lambda_{n+1} \log \lambda_{n+1} - \lambda_n \log \lambda_n}{\lambda_{n+1} - \lambda_n} - \log \lambda_n \right\}$$
$$\sim (\log 10)\, \lambda_n^2$$

$$\log \log \mu(R_{n+1}) \sim 2 \log \lambda_n.$$

So

$$\frac{\log \log (\mu R_{n+1})}{\log R_{n+1}} \sim \frac{(2 \log \lambda_n)(\lambda_{n+1} - \lambda_n)}{\lambda_{n+1} \log \lambda_{n+1} - \lambda_n \log \lambda_n} \to 0 \text{ as } n \to \infty.$$

Hence

$$\lambda = \liminf_{r\to\infty} \frac{\log \log \mu(r)}{\log r} = 0$$

whereas

$$\varrho = 1.$$

In the second case, proceeding similarly, we get

$$\lambda = \liminf_{r\to\infty} \frac{\log \log \mu(r)}{\log r} \leq 1$$

whereas

$$\varrho = \limsup_{r\to\infty} \frac{\log \log \mu(r)}{\log r} = \infty.$$

PROOF OF (ii). Consider

$$f(z) = \left(\frac{z}{\Phi(n)}\right)^n$$

where

$$\Phi(n) = e^{e^n}.$$

Clearly $f(z)$ is an entire function of zero order.

$$\mu(r) = \left(\frac{r}{\Phi(n)}\right)^n, \quad \nu(r) = n$$

for

$$R_n \leq r < R_{n+1}$$

where

$$R_n = \exp\{n \log \Phi(n) - (n-1) \log \Phi(n-1)\}.$$

Now

$$\frac{\log \mu(r)}{\nu(r) \log r} = 1 - \frac{\log \Phi(n)}{\log r}.$$

Hence

$$\frac{\log \mu(R_n)}{\nu(R_n) \log R_n} = 1 - \frac{\log \Phi(n)}{n \log \Phi(n) - (n-1) \log \Phi(n-1)}$$

$$= 1 - \frac{e^n}{n e^n - (n-1) e^{n-1}} \to 1.$$

Hence

$$\liminf_{r \to \infty} \frac{\log \mu(r)}{\nu(r) \log r} = 1,$$

and since $\limsup\limits_{r \to \infty} \dfrac{\log \mu(r)}{\nu(r) \log r} \leq 1$ always the result follows.

4. Next we prove

THEOREM 5. (i) $$\limsup_{r \to \infty} \frac{\log \mu(r)}{\nu(r)} \geq \frac{1}{\lambda} \quad (0 \leq \lambda \leq \infty)$$

(ii) $$\liminf_{r \to \infty} \frac{\log \mu(r)}{\nu(r)} \leq \frac{1}{\varrho} \quad (0 \leq \varrho \leq \infty).$$

For an alternative proof see S. M. Shah [10].

PROOF (i). Let

$$\limsup_{r \to \infty} \frac{\log \mu(r)}{\nu(r)} = \varkappa.$$

Then

$$\log \mu(r) < (\varkappa + \varepsilon) \nu(r) \quad \text{for} \quad r \geq r_0.$$

Also

$$\log \mu(r) = \log \mu(r_0) + \int_{r_0}^{r} \frac{\nu(t)}{t} \, dt$$

so

$$\frac{\mu'(r)}{\mu(r)} = \frac{\nu(r)}{r}$$

except for a set of values of r of measure zero. Here $\mu'(r)$ means the derivative of $\mu(r)$. So

$$\frac{\mu'(r)}{\mu(r) \log \mu(r)} > \frac{1}{(\varkappa + \varepsilon) r}.$$

Thus

$$\log \log \mu(r) > \frac{1}{\varkappa + \varepsilon} \log r + A.$$

Hence

$$\liminf_{r \to \infty} \frac{\log \log \mu(r)}{\log r} \geq \frac{1}{\varkappa}.$$

so

$$\lambda \geq \frac{1}{\varkappa}.$$

Proof of (ii) is similar, for if we set

$$\liminf_{r\to\infty} \frac{\log \mu(r)}{\nu(r)} = \varkappa_1,$$

then as usual

$$\frac{\mu'(r)}{\mu(r)\log\mu(r)} < \frac{1}{r}\,\frac{1}{\varkappa_1-\varepsilon}$$

so

$$\log\log\mu(r) < \frac{\log r}{\varkappa_1-\varepsilon} + A.$$

$$\varrho = \limsup_{r\to\infty} \frac{\log\log\mu'(r)}{\log r} \le \frac{1}{\varkappa_1}.$$

Finally I thank Dr. S. M. Shah for his valuable criticism of this paper.

Dharma Samaj College, Aligarh (India).

References

[1]. G. Valiron, *General Theory of Integral Functions.* Chelsea Pub. Co., 1949.
[2]. T. Vijayraghavan, *Journal Lond. Math. Soc.*, 10 (1935), 116–117.
[3]. G. Valiron, *Fonctions entières et fonctions méromorphes d'une variable*, Paris, 1925, p. 6.
[4]. S. M. Shah, *Journal of Univ. of Bombay*, XIII, part 3 (1944), 1–3.
[5]. J. M. Whittaker, *Journal Lond. Math. Soc.* (1933), 20 ff.
[6]. S. K. Singh, *Journal Univ. of Bombay*, XX, part 5 (1952), 1–7.
[7]. S. M. Shah, *Quart. Journ. of Math.* (Oxford), 19 (1948), 220–223.
[8]. ——, *Ganit Lucknow* (India), I, No. 2 (1950), 82–85.
[9]. ——, *Quart. Journ. of Math.* (Oxford), (2), 1 (1950), 112–116.
[10]. ——, *Math. Student* (Madras) 10 (1942), 80–82.