MEAN VALUES OVER THE SPACE OF LATTICES

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1. Various methods have been used for calculating the mean value of a function, defined for all lattices of determinant 1, over some or all the lattices of determinant 1. It is accepted that the most natural way of calculating such an average is in terms of the invariant measure used by Siegel.[1] However, the averaging methods used by Rogers[2] and by Cassels[3] are more convenient to use, while the methods used by Mahler,[4] by Davenport and Rogers[5] and by Macbeath and Rogers[6] are more appropriate for the special problems considered. The method used recently by Rogers[7] has proved to be particularly convenient. The first object of this paper is to establish a close connection between this averaging method and Siegel's method, but while we will confine our attention to the relationship between this particular averaging method and Siegel's method, it will be clear from the nature of our proofs that the averaging methods mentioned above, except those used by Mahler, and by Davenport and Rogers, will stand in a similar relationship to Siegel's method.

Let $\Lambda = \Lambda(\theta_1, \theta_2, \ldots, \theta_{n-1}, \omega)$ denote the lattice generated by the points

$$\begin{aligned} \boldsymbol{A}_1 &= (\omega, 0, \ldots, 0, \theta_1 \omega^{-n+1}), \\ \boldsymbol{A}_2 &= (0, \omega, \ldots, 0, \theta_2 \omega^{-n+1}), \\ &\cdots\cdots\cdots\cdots \\ \boldsymbol{A}_{n-1} &= (0, 0, \ldots, \omega, \theta_{n-1} \omega^{-n+1}), \\ \boldsymbol{A}_n &= (0, 0, \ldots, 0, \omega^{-n+1}). \end{aligned}$$

[1] C. L. Siegel, *Annals of Math.*, 46 (1945), 340–347. We assume that the reader is familiar at least with pages 341 and 342 of Siegel's paper.

[2] C. A. Rogers, *Annals of Math.*, 48 (1947), 994–1002, and a paper to be published in the *Phil. Trans. Royal Soc.* (1955).

[3] J. W. S. Cassels, *Proc. Cambridge Phil. Soc.*, 49 (1953), 165–166.

[4] K. Mahler, *Duke Math. Jour.*, 13 (1946), 611–621.

[5] H. Davenport and C. A. Rogers, *Duke Math. Jour.*, 14 (1947), 367–375.

[6] A. M. Macbeath and C. A. Rogers, to appear in the *Proc. Cambridge Phil. Soc.*

[7] *loc. cit.* (1955).

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Let $\varrho(\Lambda)$ be a function defined for all lattices Λ of determinant 1. Then the mean value in question is defined to be the limit

$$\lim_{\omega \to +0} \int_0^1 \cdots \int_0^1 \varrho(\Lambda(\theta_1, \ldots, \theta_{n-1}, \omega))\, d\theta_1 \ldots d\theta_{n-1}, \tag{1}$$

if this limit exists, and will be denoted by

$$M_\Lambda[\varrho(\Lambda)]. \tag{2}$$

Siegel's mean value for such a function $\varrho(\Lambda)$ is taken to be

$$\int_F \varrho(\Omega \Lambda_0)\, d\mu(\Omega),$$

where Λ_0 denotes the lattice of points with integral coordinates, Ω denotes a linear transformation of determinant 1, F is a certain fundamental region in the space of linear transformations of determinant 1, defined by use of the Minkowski theory of the reduction of positive definite quadratic forms, and $\mu(\Omega)$ is the invariant measure on the space of linear transformations with determinant 1, normalized so that

$$\int_F d\mu(\Omega) = 1. \tag{3}$$

In a manuscript (which I do not intend to publish) I have shown that, if $\varrho(\Lambda)$ is a continuous function of Λ vanishing outside a compact set of lattices, then the mean value $M_\Lambda[\varrho(\Lambda)]$ will exist and will have the value

$$\int_F \varrho(\Omega \Lambda_0)\, d\mu(\Omega); \tag{4}$$

the proof leads to a determination of the normalizing factor for the measure $\mu(\Omega)$ independent of Siegel's. But, in this paper, it is more convenient to confine our attention to non-negative functions $\varrho(\Lambda)$ which can be proved directly to have a certain invariance property. We shall consider a function $\varrho(\Lambda)$ with the property that the mean values

$$M_\Lambda[\varrho(\Omega \Lambda)] \tag{5}$$

exist and have the same value for all linear transformations Ω of determinant 1. It will not be difficult to prove that, if $\varrho(\Lambda)$ is such a function and is Borel measurable in the space of lattices of determinant 1, then

$$M_\Lambda[\varrho(\Lambda)] = \int_F \varrho(\Omega \Lambda_0)\, d\mu(\Omega). \tag{6}$$

Since the mean value (5) is often easier to evaluate than the integral (4), the result (6) is useful for evaluating the integral (4). We use this method to evaluate the integral (4) when $\varrho(\Lambda)$ is taken to be a sum of the form

$$\sum_{X_1, \dots, X_m} \varrho(X_1, \dots, X_m), \tag{7}$$

where $1 \leq m \leq n-1$, and $X_1, X_2, \dots, X_m$ are restricted to be linearly independent points of Λ, and perhaps also restricted so that certain rational linear combination of $X_1, X_2, \dots, X_m$ are also points of Λ. In the special case, when the only restriction on $X_1, X_2, \dots, X_m$ is that they should be linearly independent points of Λ, the result reduces to the formula

$$\int_F \sum_{X_1, \dots, X_m} \varrho(\Omega X_1, \dots, \Omega X_m)\, d\mu(\Omega) = \int \dots \int \varrho(X_1, \dots, X_m)\, dX_1 \dots dX_m, \tag{8}$$

stated by Siegel[1] without proof. It seems likely that the more general result could also be proved by Siegel's methods, but this is beyond me.

By taking suitable combinations of sums of the type (7), where $X_1, \dots, X_m$ are restricted in suitable ways, we can build up a sum of the type (7) where $X_1, \dots, X_m$ run independently over all the points of Λ. This process enables us to prove the following theorem.

THEOREM 4. *Let $\varrho(X_1, \dots, X_k)$ be a non-negative Borel measurable function in the nk-dimensional space of points $(X_1, \dots, X_k)$. Then, if $1 \leq k \leq n-1$,*

$$\int_F \sum_{X_1 \in \Omega\Lambda_0, \dots, X_k \in \Omega\Lambda_0} \varrho(X_1, \dots, X_k)\, d\mu(\Omega)$$
$$= \varrho(O, \dots, O) + \int \cdots \int \varrho(X_1, \dots, X_k)\, dX_1 \dots dX_k +$$
$$+ \sum_{(\nu;\mu)} \sum_{q=1}^{\infty} \sum_{D} \left(\frac{e_1}{q} \cdot \frac{e_2}{q} \cdots \frac{e_m}{q}\right)^n \int \cdots \int \varrho\left(\sum_{i=1}^{m} \frac{d_{i1}}{q} X_i, \dots, \sum_{i=1}^{m} \frac{d_{ik}}{p} X_i\right) dX_1 \dots dX_m, \tag{9}$$

both sides perhaps having the value $+\infty$, where on the right the outer sum is over all divisions $(\nu; \mu) = (\nu_1, \dots, \nu_m; \mu_1, \dots, \mu_{k-m})$ of the numbers $1, 2, \dots, k$ into two sequences $\nu_1, \dots, \nu_m$ and $\mu_1, \dots, \mu_{k-m}$, with $1 \leq m \leq k-1$,

$$\begin{aligned} &1 \leq \nu_1 < \nu_2 < \cdots < \nu_m \leq k, \\ &1 \leq \mu_1 < \mu_2 < \cdots < \mu_{k-m} \leq k, \\ &\nu_i \neq \mu_j, \text{ if } 1 \leq i \leq m,\ 1 \leq j \leq k-m, \end{aligned} \tag{10}$$

[1] *loc. cit.* (347).

where the inner sum is over all $m \times k$ *matrices* D, *with integral elements, having highest common factor relatively prime to* q, *and with*

$$\left.\begin{aligned} &d_{i\nu_j} = q\,\delta_{ij}, \quad i=1, \ldots, m, \; j=1, \ldots, m, \\ &d_{i\mu_j} = 0, \text{ if } \mu_j < \nu_i, \quad i=1, \ldots, m, \; j=1, \ldots, k-m, \end{aligned}\right\} \tag{11}$$

and where

$$e_i = (\varepsilon_i, q), \quad i=1, \ldots, m,$$

$\varepsilon_1, \ldots, \varepsilon_m$ *being the elementary divisors of the matrix* D.

If in addition $\varrho(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k)$ *is bounded and vanishes outside a bounded region of space, and if also* $n \geq [\frac{1}{4}k^2]+2$, *then both sides of* (9) *are finite.*

This result is clearly an improvement on the more restricted result[1] I obtained previously using the mean value M_Λ. Unfortunately the right hand side of (9) may be expected to diverge when n is too small in comparison to k. In particular the right hand side of (9) will diverge when $n=2$, $k=2$ and $\varrho(\boldsymbol{X}_1, \boldsymbol{X}_2) \geq 1$ for all $\boldsymbol{X}_1, \boldsymbol{X}_2$ sufficiently close to $\boldsymbol{O}$. In this case the divergence can be eliminated if $\boldsymbol{X}_1, \boldsymbol{X}_2$ are restricted to be primitive points of the lattice $\Omega\Lambda_0$. We prove the following result.

THEOREM 5. *Let* $\varrho(\boldsymbol{X}_1, \boldsymbol{X}_2)$ *be a non-negative Borel measurable function in the* $2n$*-dimensional space of points* $(\boldsymbol{X}_1, \boldsymbol{X}_2)$. *Then, if* $n \geq 2$,

$$\begin{aligned} \int_F \sum_{\substack{\boldsymbol{X}_1 \in \Omega\Lambda_0^* \\ \boldsymbol{X}_2 \in \Omega\Lambda_0^*}} \varrho(\boldsymbol{X}_1, \boldsymbol{X}_2)\, d\mu(\Omega) \\ = \frac{1}{(\zeta(n))^2} \int\int \varrho(\boldsymbol{X}_1, \boldsymbol{X}_2)\, d\boldsymbol{X}_1\, d\boldsymbol{X}_2 + \\ + \frac{1}{\zeta(n)} \int \varrho(\boldsymbol{X}_1, \boldsymbol{X}_1)\, d\boldsymbol{X}_1 + \frac{1}{\zeta(n)} \int \varrho(\boldsymbol{X}_1, -\boldsymbol{X}_1)\, d\boldsymbol{X}_1, \end{aligned} \tag{12}$$

where Λ_0^* *is the set of primitive points of* Λ_0.

We use this result to prove the following theorem closely related to well-known results due to Khintchine and others in the metrical theory of Diophantine approximation.[2] We say that a result holds for almost all linear transformations Ω (now not necessarily of determinant 1), if the corresponding matrices (ω_{ij}) form a

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[1] *loc. cit.* (1955), Theorem 3.

[2] See J. K. KOKSMA, *Diophantische Approximationen* (Berlin, 1936), Chapter III, 30 and Chapter V, 11; see also J. W. S. CASSELS, *Proc. Camb. Phil. Soc.*, 46 (1950), 209–218, Theorem II. These results may be regarded as modified forms of Theorem 6 where a different measure is used and the linear transformation Ω is restricted to be of a special type.

set of measure zero, when regarded as points Ω with coordinates $(\omega_{11}, \omega_{12}, \ldots, \omega_{nn})$ in n^2-dimensional Euclidean space.

THEOREM 6. *Let $\varrho(\mathbf{X})$ be a bounded non-negative Borel measurable function defined for all points $\mathbf{X}$ of n-dimensional space $(n \geq 2)$. Then the sum*

$$\sum_{\mathbf{X} \in \Omega \Lambda_0} \varrho(\mathbf{X}) \tag{13}$$

is finite for almost all Ω, if the integral

$$\int \varrho(\mathbf{X})\, d\mathbf{X} \tag{14}$$

is finite; and the sum

$$\sum_{\mathbf{X} \in \Omega \Lambda_0^*} \varrho(\mathbf{X}), \tag{15}$$

where Λ_0^* is the set of primitive points of Λ_0, is infinite for almost all Ω, if the integral (14) *is infinite.*

2. Every lattice Λ of determinant 1 can be expressed uniquely in the form $\Omega\Lambda_0$, where Λ_0 is the lattice of points with integral coordinates and Ω is a linear transformation or matrix $\Omega = (\omega_{ij})$ of determinant 1 in the fundamental region F. If G is any set of matrices Ω with determinant 1, we use G^* to denote the corresponding set of all matrices $\Omega^* = \lambda\Omega$ where Ω is in G and $\frac{1}{2} \leq \lambda \leq 1$. We call G a Borel set if G^* is a Borel set, regarded as a sub-set of the n^2-dimensional space of points $\Omega^* = (\omega_{11}^*, \omega_{12}^*, \ldots, \omega_{nn}^*)$, and in this case, following Siegel, we take the measure $\mu(G)$ to be the product of the Lebesgue measure of G^*, regarded as a sub-set of the n^2-dimensional space, with a normalizing factor $\varkappa$. We say that a set of lattice Λ is a Borel set, if the set of corresponding matrices $\Omega = (\omega_{ij})$ in F is a Borel set according to this definition, and we define the concept of a Borel measurable function in the space of the lattices in the corresponding way. With these definitions and the notation introduced in § 1, we can prove our first result.

THEOREM 1. *Let $\varrho(\Lambda)$ be a non-negative Borel measurable function in the space of lattices of determinant* 1. *Suppose that the mean values*

$$M_\Lambda[\varrho(\Omega\Lambda)]$$

exist and have the same (finite) value for all linear transformations Ω of determinant 1. *Then*

$$\int_F \varrho(\Omega\Lambda_0)\, d\mu(\Omega) = M_\Lambda[\varrho(\Lambda)].$$

PROOF. The assumption that the $\varrho(\Lambda)$ is a Borel measurable function in the space of lattices of determinant 1 means that the function $\varrho(\Omega\Lambda_0)$ is a Borel measurable function of Ω defined for Ω in F. The function $\varrho(\Lambda)$ need not be defined for lattices Λ of determinant other than 1; we can define or re-define $\varrho(\Lambda)$ for such Λ by

$$\varrho(\Lambda) = \varrho(\{d(\Lambda)\}^{-1/n}\Lambda),$$

where $d(\Lambda)$ denotes the determinant of Λ. Then it is clear that $\varrho(\Omega\Lambda_0)$ will be Borel measurable for Ω in F, if and only if $\varrho(\Omega^*\Lambda_0)$ is Borel measurable for Ω^* in F^*. Now the set Γ^* of all linear transformations Ω^* with a positive determinant in the closed interval $[\frac{1}{2}, 1]$ can be expressed as the union

$$\Gamma^* = \bigcup F^*\Psi,$$

where Ψ runs through all matrices with integral elements and determinant 1. Since

$$\Psi\Lambda_0 = \Lambda_0$$

for any such Ψ, it follows that $\varrho(\Omega^*\Lambda_0)$ is Borel measurable for Ω^* in Γ^*.

For any fixed positive value of ω, consider the function

$$\varrho(\Omega^*\Lambda(\vartheta, \omega))$$

defined on the Cartesian product $\Gamma^* \times C$ where C is the set of all $\vartheta = (\theta_1, \dots, \theta_{n-1})$ with

$$0 \le \theta_1 \le 1, \dots, 0 \le \theta_{n-1} \le 1.$$

We regard $\Gamma^* \times C$ as a subset of the (n^2+n-1)-dimensional space of all points

$$(\Omega^*, \vartheta) = (\omega_{11}^*, \omega_{12}^*, \dots, \omega_{nn}^*, \theta_1 \dots, \theta_{n-1}).$$

Write

$$\Theta = \Theta(\vartheta, \omega) = \begin{pmatrix} \omega & 0 & \dots & 0 & 0 \\ 0 & \omega & \dots & 0 & 0 \\ \dots & & & & \\ 0 & 0 & \dots & \omega & 0 \\ \omega^{-n+1}\theta_1 & \omega^{-n+1}\theta_2 \dots & & \omega^{-n+1}\theta_{n-1} & \omega^{-n+1} \end{pmatrix},$$

so that

$$\Lambda(\vartheta, \omega) = \Theta(\vartheta, \omega)\Lambda_0.$$

Now as (Ω^*, ϑ) varies continuously in $\Gamma^* \times C$ the product $\Omega^*\Theta(\vartheta, \omega)$ varies continuously in Γ^*. So, if E is a closed sub-set of Γ^*, then the set of points (Ω^*, ϑ) in $\Gamma^* \times C$, such that $\Omega^*\Theta(\vartheta, \omega)$ is in E, is closed. It follows that, if E is any Borel set of points of Γ^*, then the set of points (Ω^*, ϑ) of $\Gamma^* \times C$ such that $\Omega^*\Theta(\vartheta, \omega)$ is in E is a Borel set. Now, as $\varrho(\Omega^*\Lambda_0)$ is a Borel measurable function of Ω^* in Γ^*, for each c the set of Ω^* in Γ^*, for which

$$\varrho(\Gamma^* \Lambda_0) < c,$$

is a Borel set. Thus the set of points (Ω^*, ϑ) of $\Gamma^* \times C$, for which

$$\varrho(\Omega^* \Theta(\vartheta, \omega) \Lambda_0) < c,$$

is a Borel set. Thus the function

$$\varrho(\Omega^* \Lambda(\vartheta, \omega)) = \varrho(\Omega^* \Theta(\vartheta, \omega) \Lambda_0)$$

is a Borel measurable function in $\Gamma^* \times C$, regarding this set as a subset of (n^2+n-1)-dimensional space.

Consider the function $\varrho_h(\Lambda)$ defined for all lattices Λ and for all positive integers h by

$$\varrho_h(\Lambda) = \begin{cases} \varrho(\Lambda), & \text{if } \varrho(\Lambda) \le h, \\ h \;\;, & \text{if } \varrho(\Lambda) \ge h. \end{cases}$$

As $\varrho(\Omega^* \Lambda(\vartheta, \omega))$ is Borel measurable in $\Gamma^* \times C$, it follows that $\varrho_h(\Omega^* \Lambda(\vartheta, \omega))$ is also Borel measurable in $\Gamma^* \times C$. Further $\varrho_h(\Omega^* \Lambda(\vartheta, \omega))$ is non-negative and bounded in $\Gamma^* \times C$. Again as F has measure 1 the set F^* has Lebesgue measure $\varkappa^{-1}$. Also C has Lebesgue measure 1. Thus, by Fubini's theorem, the set $F^* \times C$ has the finite Lebesgue measure $\varkappa^{-1}$. Hence $\varrho_h(\Omega^* \Lambda(\vartheta, \omega))$ is integrable in the Lebesgue sense over the set $F^* \times C$, and, by Fubini's theorem, the (n^2+n-1)-dimensional integral

$$\iint_{F^* \times C} \varrho_h(\Omega^* \Lambda(\vartheta, \omega))\, d\Omega^*\, d\vartheta$$
$$= \int \cdots \int_{F^* \times C} \varrho_h(\Omega^* \Lambda(\vartheta, \omega))\, d\omega_{11}^*\, d\omega_{12}^* \ldots d\omega_{nn}^*\, d\theta_1 \ldots d\theta_{n-1}$$

can be expressed in the alternative forms

$$\int_{F^*} \left\{ \int_C \varrho_h(\Omega^* \Lambda(\vartheta, \omega))\, d\vartheta \right\} d\Omega^*, \tag{16}$$

$$\int_C \left\{ \int_{F^*} \varrho_h(\Omega^* \Lambda(\vartheta, \omega))\, d\Omega^* \right\} d\vartheta. \tag{17}$$

Now for any ϑ in C, we have

$$\int_{F^*} \varrho_h(\Omega^* \Lambda(\vartheta, \omega))\, d\Omega^*$$
$$= \int_{F^*} \varrho_h(\Omega^* \Theta \Lambda_0)\, d\Omega^*$$
$$= \int_{F^* \Theta^{-1}} \varrho_h(\Omega^* \Lambda_0)\, d\Omega^*,$$

on changing the variable and remembering that $\Theta = \Theta(\vartheta, \omega)$ has determinant 1. But it follows, from known properties of the fundamental domain F, that F, and similarly F^*, can be split up into a finite number of disjoint Borel measurable sets

$$F_1, F_2, \ldots, F_k, \text{ and } F_1^*, F_2^*, \ldots, F_k^*,$$

such that

$$F\,\Theta^{-1} = \bigcup_{i=1}^{k} F_i \Psi_i,$$

$$F^*\,\Theta^{-1} = \bigcup_{i=1}^{k} F_i^* \Psi_i,$$

where $\Psi_1, \Psi_2, \ldots, \Psi_k$ are matrices with integral elements and determinant 1. So

$$\begin{aligned}\int_{F^*\Theta^{-1}} \varrho_h(\Omega^* \Lambda_0)\, d\,\Omega^* &= \sum_{i=1}^{k} \int_{F_i^* \Psi_i} \varrho_h(\Omega^* \Lambda_0)\, d\,\Omega^* \\ &= \sum_{i=1}^{k} \int_{F_i^*} \varrho_h(\Omega^* \Psi_i \Lambda_0)\, d\,\Omega^* \\ &= \sum_{i=1}^{k} \int_{F_i^*} \varrho_h(\Omega^* \Lambda_0)\, d\,\Omega^* \\ &= \int_{F^*} \varrho_h(\Omega^* \Lambda_0)\, d\,\Omega^*.\end{aligned}$$

Combining these results, we see that

$$\int_{F^*} \varrho_h(\Omega^* \Lambda_0)\, d\,\Omega^* = \int_{F^*} \varrho_h(\Omega^* \Lambda(\vartheta, \omega))\, d\,\Omega^*,$$

for any ϑ in C, so that

$$\begin{aligned}\int_{F^*} \varrho_h(\Omega^* \Lambda_0)\, d\,\Omega^* &= \int_C \left\{ \int_{F^*} \varrho_h(\Omega^* \Lambda_0)\, d\,\Omega^* \right\} d\,\vartheta \\ &= \int_C \left\{ \int_{F^*} \varrho_h(\Omega^* \Lambda(\vartheta, \omega))\, d\,\Omega^* \right\} d\,\vartheta \\ &= \int_{F^*} \left\{ \int_C \varrho_h(\Omega^* \Lambda(\vartheta, \omega))\, d\,\vartheta \right\} d\,\Omega^*, \qquad (18)\end{aligned}$$

using the equality of (16) and (17).

This result (18) holds for all $\omega > 0$. Also

$$0 \leq \int_C \varrho_h (\Omega^* \Lambda (\vartheta, \omega))\, d\vartheta \leq \int_C h\, d\vartheta = h.$$

So

$$\int_C \varrho_h (\Omega^* \Lambda (\vartheta, \omega))\, d\vartheta \leq \Big[\int_C \varrho (\Omega^* \Lambda (\vartheta, \omega))\, d\vartheta\Big]_h,$$

where we use the expression on the right hand side to denote

$$\min \Big\{h, \int_C \varrho (\Omega^* \Lambda (\vartheta, \omega))\, d\vartheta\Big\}.$$

Since the function

$$\varrho (\Omega^* \Lambda (\vartheta, \omega))$$

is Borel measurable in $F^* \times C$, it follows that

$$\Big[\int_C \varrho (\Omega^* \Lambda (\vartheta, \omega))\, d\vartheta\Big]_h$$

is Borel measurable in F^*. So

$$\int_{F^*} \varrho_h (\Omega^* \Lambda_0)\, d\Omega^* \leq \int_{F^*} \Big[\int_C \varrho (\Omega^* \Lambda (\vartheta, \omega))\, d\vartheta\Big]_h d\Omega^*, \tag{19}$$

for all $\omega > 0$.

Now, writing

$$\Omega = \{d (\Omega^*)\}^{-1/n}\, \Omega^*,$$

where $d(\Omega^*)$ is the determinant of Ω^*, we have

$$\int_C \varrho (\Omega^* \Lambda (\vartheta, \omega))\, d\vartheta = \int_C \varrho (\Omega \Lambda (\vartheta, \omega))\, d\vartheta.$$

By hypothesis, the mean value

$$M_\Lambda [\varrho (\Omega \Lambda)] = \lim_{\omega \to +0} \int_C \varrho (\Omega \Lambda (\vartheta, \omega))\, d\vartheta$$

exists and has the value

$$M_\Lambda [\varrho (\Lambda)].$$

So, provided $h > M_\Lambda [\varrho (\Lambda)]$, we have

$$\Big[\int_C \varrho (\Omega^* \Lambda (\vartheta, \omega))\, d\vartheta\Big]_h \to M_\Lambda [\varrho (\Lambda)]$$

as $\omega \to +0$. By the theorem of bounded convergence, it follows that

$$\int_{F^*} \left[\int_C \varrho(\Omega^* \Lambda(\vartheta, \omega))\, d\vartheta\right]_h d\Omega^* \to \int_{F^*} M_\Lambda[\varrho(\Lambda)]\, d\Omega^*$$

$$= \varkappa^{-1} M_\Lambda[\varrho(\Lambda)]$$

as $\omega \to +0$, provided $h > M_\Lambda[\varrho(\Lambda)]$. It follows from (19) that

$$\int_{F^*} \varrho_h(\Omega^* \Lambda_0)\, d\Omega^* \le \varkappa^{-1} M_\Lambda[\varrho(\Lambda)] \tag{20}$$

for all $h \ge 0$. This shows that the function $\varrho(\Omega^* \Lambda_0)$ is integrable over F^* and that

$$\int_{F^*} \varrho(\Omega^* \Lambda_0)\, d\Omega^* \le \varkappa^{-1} M_\Lambda[\varrho(\Lambda)]. \tag{21}$$

Since this integral is finite, we have, as in proof of (18),

$$\int_{F^*} \varrho(\Omega^* \Lambda_0)\, d\Omega^* = \int_C \left\{\int_{F^*} \varrho(\Omega^* \Lambda_0)\, d\Omega^*\right\} d\vartheta$$

$$= \int_C \left\{\int_{F^*} \varrho(\Omega^* \Lambda(\vartheta, \omega))\, d\Omega^*\right\} d\vartheta.$$

As $\varrho(\Omega^* \Lambda(\vartheta, \omega))$ is Borel measurable over $F^* \times C$ and is non-negative, it follows from Fubini's theorem that

$$\int_C \left\{\int_{F^*} \varrho(\Omega^* \Lambda(\vartheta, \omega))\, d\Omega^*\right\} d\vartheta$$

$$= \int_{F^*} \left\{\int_C \varrho(\Omega^* \Lambda(\vartheta, \omega))\, d\vartheta\right\} d\Omega^*,$$

both integrals being finite. But

$$\int_C \varrho(\Omega^* \Lambda(\vartheta, \omega))\, d\vartheta \to M_\Lambda[\varrho(\Lambda)]$$

as $\omega \to 0$. So, by Fatou's lemma,

$$\varkappa^{-1} M_\Lambda[\varrho(\Lambda)] = \int_{F^*} M_\Lambda[\varrho(\Lambda)]\, d\Omega^*$$

$$\le \lim_{\omega \to +0} \int_{F^*} \left\{\int_C \varrho(\Omega^* \Lambda(\vartheta, \omega))\, d\vartheta\right\} d\Omega^*$$

$$= \int_{F^*} \varrho(\Omega^* \Lambda_0)\, d\Omega^*. \tag{22}$$

Combining the results (21) and (22), we have

$$\int_{F^*} \varrho(\Omega^* \Lambda_0)\, d\Omega^* = \varkappa^{-1} M_\Lambda[\varrho(\Lambda)].$$

So, by definition of the measure $\mu(\Omega)$, we have

$$\int_F \varrho(\Omega\Lambda_0)\,d\mu(\Omega) = \varkappa \int_{F^*} \varrho(\Omega^*\Lambda_0)\,d\Omega^*$$
$$= M_\Lambda[\varrho(\Lambda)],$$

as required.

3. In this section we investigate the mean value $M_\Lambda[\varrho(\Lambda)]$ in the case when

$$\varrho(\Lambda) = \sum_{X_1,\ldots,X_m} \varrho(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m)$$

and $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ are restricted to be linearly independent points of Λ, perhaps satisfying certain auxiliary conditions. It will be convenient to write conditions of summation in brackets to the right of the summation sign, and to use dim $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m)$ to denote the dimension of the linear space generated by the points $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$.

THEOREM 2. *Let $\varrho(\mathfrak{X}) = \varrho(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m)$ be a Riemann integrable function over the nm-dimensional space of points*

$$\mathfrak{X} = (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m) = (x_1^{(1)}, \ldots, x_n^{(1)}, \ldots, x_1^{(m)}, \ldots, x_n^{(m)}),$$

where $1 \le m \le n-1$. Let q be a positive integer and let C be an integral $m \times h$ matrix

$$C = (c_{ij}),$$

where $h \ge 0$. Then the mean value $M_\Lambda[\varrho(\Lambda)]$ of the function

$$\varrho(\Lambda) = \sum \left[\begin{array}{c} \boldsymbol{X}_1, \ldots, \boldsymbol{X}_m \text{ in } \Lambda \\ \dim(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m) = m \\ \sum_{i=1}^{m} (c_{ij}/q)\,\boldsymbol{X}_i \text{ in } \Lambda \end{array}\right] \varrho(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m) \tag{23}$$

exists and has the value

$$M_\Lambda[\varrho(\Lambda)] = \left(\frac{N(C,q)}{q^m}\right)^n \int \varrho(\mathfrak{X})\,d\mathfrak{X}$$

where $N(C, q)$ is the number of sets of integers $(a_1, \ldots, a_m)$ with

$$0 \le a_1 < q, \ \ldots, \ 0 \le a_m < q,$$

$$\sum_{i=1}^{m} c_{ij}\,a_i \equiv 0 \pmod q, \quad j = 1, \ldots, h.$$

Here $N(C, q)$ is to be given the value q^m in the case when $h = 0$.

PROOF.[1] The condition that $\varrho(\mathfrak{X})$ is a Riemann integrable function over the whole space implies that there is a number R so large that $\varrho(\mathfrak{X})=0$ unless

$$|x_r^{(i)}|<R, \quad i=1, \ldots, m, \quad r=1, \ldots, n.$$

We investigate the mean value

$$\int_0^1 \cdots \int_0^1 \varrho(\Lambda(\theta_1, \ldots, \theta_{n-1}, \omega))\, d\theta_1 \ldots d\theta_{n-1}$$

on the assumption that ω is small in comparison to R^{-n+1}. The general point of the lattice $\Lambda(\theta_1, \ldots, \theta_{n-1}, \omega)$ has coordinates

$$(\omega u_1, \ldots, \omega u_{n-1}, \omega^{-n+1}\{\theta_1 u_1 + \cdots + \theta_{n-1} u_{n-1} + u_n\})$$

where $u_1, u_2, \ldots, u_n$ are integers. It is convenient to write

$$\boldsymbol{u}=(u_1, \ldots, u_{n-1}),$$

$$\vartheta=(\theta_1, \ldots, \theta_{n-1}),$$

$$\vartheta\, \boldsymbol{u}=\theta_1 u_1 + \cdots + \theta_{n-1} u_{n-1},$$

so that the general point of $\Lambda(\theta_1, \ldots, \theta_{n-1}, \omega)$ takes the form

$$(\omega\, \boldsymbol{u}, \omega^{-n+1}\{\vartheta\, \boldsymbol{u}+u_n\})$$

where $\boldsymbol{u}$ belongs to the lattice L of points in $(n-1)$-dimensional space with integral coordinates and u_n is an integer. With this notation the contributions to the sum (23) may be split up into two types: the contributions from points $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$, such that the corresponding points $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ are linearly independent; and the contributions from points $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$, which are linearly independent, although $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ are linearly dependent. Our first object is to prove that there are no non-zero contributions of the second type.

Suppose that $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ are points of the lattice $\Lambda=\Lambda(\theta_1, \ldots, \theta_{n-1}, \omega)$ such that

$$\varrho(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m)\neq 0,$$

and the corresponding points $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ are linearly dependent. We proceed to prove that $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ are linearly dependent. If the rank of the matrix

$$(u_r^{(i)})_{\substack{1\leq i\leq m \\ 1\leq r\leq n-1}} \tag{24}$$

[1] This proof is very similar to a proof I have given before for a similar result *loc. cit.* (1955); but there are a number of differences which make it best to give all the details up to a certain point.

were less than $m-1$, then the rank of the matrix

$$(x_r^{(i)})_{\substack{1\le i\le m\\1\le r\le n}}=\begin{pmatrix}\omega u_1^{(1)} & \omega u_1^{(2)} & \dots & \omega u_1^{(m)}\\ \dots & \dots & & \dots\\ \omega u_{n-1}^{(1)} & \omega u_{n-1}^{(2)} & \dots & \omega u_{n-1}^{(m)}\\ \omega^{-n+1}\{\vartheta \boldsymbol{u}_1+u_n^{(1)}\} & \omega^{-n+1}\{\vartheta \boldsymbol{u}_2+u_n^{(2)}\}\dots & & \omega^{-n+1}\{\vartheta \boldsymbol{u}_m+u_n^{(m)}\}\end{pmatrix}$$

would be less than m and the points $\boldsymbol{X}_1, \dots, \boldsymbol{X}_m$ would be linearly dependent as required. So we may suppose that the rank of the matrix (24) is $m-1$. We may suppose, without loss of generality, that the minor

$$(u_r^{(i)})_{\substack{1\le i\le m-1\\1\le r\le m-1}}$$

is non-singular. Let $b_1, \dots, b_m$ be the cofactors of the elements $u_r^{(1)}, \dots, u_r^{(m)}$ in the matrix

$$\begin{pmatrix}u_1^{(1)} & u_1^{(2)} & \dots & u_1^{(m)}\\ \dots & \dots & & \dots\\ u_{m-1}^{(1)} & u_{m-1}^{(2)} & \dots & u_{m-1}^{(m)}\\ u_r^{(1)} & u_r^{(2)} & \dots & u_r^{(m)}\end{pmatrix}.$$

Then, as $b_1, \dots, b_m$ are independent of r, while this matrix is singular for $r=1, 2, \dots, n-1$, it follows that

$$\sum_{i=1}^{m} b_i u_r^{(i)}=0, \quad r=1, 2, \dots, n-1.$$

Thus

$$\sum_{i=1}^{m} b_i \boldsymbol{u}_i=\boldsymbol{o}. \tag{25}$$

Further $b_1, \dots, b_m$ are integers with $b_m\neq 0$ and using Hadamard's inequality

$$|b_i|\le (m-1)^{(m-1)/2}\big[\max_{\substack{1\le i\le m\\1\le r\le n-1}}|u_r^{(i)}|\big]^{m-1}.$$

Since we are supposing that

$$\varrho(\boldsymbol{X}_1, \dots, \boldsymbol{X}_m)\neq 0,$$

it follows that

$$|x_r^{(i)}|=|\omega u_r^{(i)}|<R, \quad i=1, \dots, m, \ r=1, \dots, n-1.$$

Thus

$$|b_i|<(m-1)^{(m-1)/2} R^{m-1}\omega^{-m+1}, \quad i=1, \dots, m.$$

Further the integers $u_n^{(1)}, \dots, u_n^{(m)}$ must be such that

$$|\omega^{-n+1}\{\vartheta \boldsymbol{u}_i+u_n^{(i)}\}|<R, \quad i=1, \dots, m.$$

Consequently, using (25)

$$\left|\sum_{i=1}^{m} b_i\, u_n^{(i)}\right| = \left|\sum_{i=1}^{m} b_i\, \{\vartheta\, \boldsymbol{u}_i + u_n^{(i)}\}\right|$$
$$< \sum_{i=1}^{m} |b_i|\, \omega^{n-1} R$$
$$< m\,(m-1)^{(m-1)/2} R^m\, \omega^{n-m}$$
$$< 1,$$

provided $m < n$ and ω is sufficiently small. Hence

$$\sum_{i=1}^{m} b_i\, u_n^{(i)},$$

being an integer, is zero. This, together with (25), implies that the points $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ are linearly dependent.

It follows from the result of the last paragraph that, provided ω is sufficiently small,

$$\varrho(\Lambda) = \sum \begin{bmatrix} \boldsymbol{u}_1, \ldots, \boldsymbol{u}_m \text{ in } L \\ \dim(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m) = m \end{bmatrix} \sum \begin{bmatrix} u_n^{(1)}, \ldots, u_n^{(m)} \text{ integers} \\ \sum_{i=1}^{m} (c_{ij}/q)\, X_i \text{ in } \Lambda \end{bmatrix} \times$$
$$\times\, \varrho\,(\omega\, \boldsymbol{u}_1,\, \omega^{-n+1}\{\vartheta\, \boldsymbol{u}_1 + u_n^{(1)}\}, \ldots, \omega\, \boldsymbol{u}_m,\, \omega^{-n+1}\{\vartheta\, \boldsymbol{u}_m + u_n^{(m)}\}).$$

It is convenient to use

$$\varrho_{1\le i\le m}(\boldsymbol{X}_i)$$

to denote

$$\varrho\,(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m)$$

so that

$$\varrho\,(\omega\, \boldsymbol{u}_1,\, \omega^{-n+1}\{\vartheta\, \boldsymbol{u}_1 + u_n^{(1)}\}, \ldots, \omega\, \boldsymbol{u}_m,\, \omega^{-n+1}\{\vartheta\, \boldsymbol{u}_m + u_n^{(m)}\})$$
$$= \varrho_{1\le i\le m}(\omega\, \boldsymbol{u}_i,\, \omega^{-n+1}\{\vartheta\, \boldsymbol{u}_i + u_n^{(i)}\}).$$

The condition that

$$\sum_{i=1}^{m} (c_{ij}/q)\, \boldsymbol{X}_i$$

should be in Λ for $j = 1, \ldots, h$ is equivalent to the conditions that

$$\sum_{i=1}^{m} c_{ij}\, u_r^{(i)} \equiv 0 \pmod{q}$$

for $r = 1, 2, \ldots, n$ and $j = 1, \ldots, h$. So we have

$$\varrho(\Lambda) = \sum \begin{bmatrix} \boldsymbol{u}_1, \ldots, \boldsymbol{u}_m \text{ in } L \\ \dim(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m) = m \\ \sum_{i=1}^{m} (c_{ij}/q)\, \boldsymbol{u}_i \text{ in } L \end{bmatrix} \sum \begin{bmatrix} u_n^{(1)}, \ldots, u_n^{(m)} \text{ integers} \\ \\ \sum_{i=1}^{m} c_{ij}\, u_n^{(i)} \equiv 0 \pmod q \end{bmatrix} \varrho_{1\le i\le m}(\omega\, \boldsymbol{u}_i,\ \omega^{-n+1}\{\vartheta\, \boldsymbol{u}_i + u_n^{(i)}\}).$$

Provided ω is sufficiently small, the integrand here will be zero, unless $u_n^{(1)}, \ldots, u_n^{(m)}$ are the integers chosen so that

$$-\tfrac{1}{2} < \vartheta\, \boldsymbol{u}_i + u_n^{(i)} \le \tfrac{1}{2}, \quad i = 1, \ldots, m.$$

Using $\|x\|$ to denote $x + n(x)$ where $n(x)$ is the integer such that

$$-\tfrac{1}{2} < x + n(x) \le \tfrac{1}{2},$$

this implies that

$$\vartheta\, \boldsymbol{u}_i + u_n^{(i)} = \|\vartheta\, \boldsymbol{u}_i\|, \quad i = 1, \ldots, m.$$

Consequently

$$\varrho(\Lambda) = \sum \begin{bmatrix} \boldsymbol{u}_1, \ldots, \boldsymbol{u}_m \text{ in } L \\ \dim(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m) = m \\ \sum_{i=1}^{m} (c_{ij}/q)\, \boldsymbol{u}_i \text{ in } L \\ \sum_{i=1}^{m} c_{ij}\{\|\vartheta\, \boldsymbol{u}_i\| - \vartheta\, \boldsymbol{u}_i\} \equiv 0 \pmod q \end{bmatrix} \varrho_{1\le i\le m}(\omega\, \boldsymbol{u}_i,\ \omega^{-n+1}\|\vartheta\, \boldsymbol{u}_i\|).$$

We now consider the integral

$$\int \varrho(\Lambda)\, d\vartheta = \int_0^1 \cdots \int_0^1 \varrho(\Lambda(\theta_1, \ldots, \theta_{n-1}, \omega))\, d\theta_1 \ldots d\theta_{n-1}.$$

We adopt the convention that conditions of integration may be placed in brackets after the integral sign. Since the sum has only a finite number of non-zero terms, we have

$$\int \varrho(\Lambda)\, d\vartheta = \sum \begin{bmatrix} \boldsymbol{u}_1, \ldots, \boldsymbol{u}_m \text{ in } L \\ \dim(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m) = m \\ \sum_{i=1}^{m} (c_{ij}/q)\, \boldsymbol{u}_i \text{ in } L \end{bmatrix} I(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m), \tag{26}$$

where

$$I(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m) = \int \begin{bmatrix} 0 \le \theta_i \le 1 \\ \sum_{i=1}^{m} c_{ij}\{\|\vartheta\, \boldsymbol{u}_i\| - \vartheta\, \boldsymbol{u}_i\} \equiv 0 \pmod q \end{bmatrix} \varrho_{1\le i\le m}(\omega\, \boldsymbol{u}_i,\ \omega^{-n+1}\|\vartheta\, \boldsymbol{u}_i\|)\, d\vartheta. \tag{27}$$

We first suppose that $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ are linearly independent points of L and that the points

$$\sum_{i=1}^{m} (c_{ij}/q)\, \boldsymbol{u}_i, \quad j=1, \ldots, h,$$

are points of L; and we prove that, provided ω is sufficiently small,

$$I(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m) = \omega^{m(n-1)} \frac{N(C, q)}{q^m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varrho^{1 \le i \le m} (\omega\, \boldsymbol{u}_i, \xi_i)\, d\xi_1 \ldots d\xi_m. \tag{28}$$

It is convenient to write

$$\begin{aligned} F(\vartheta) &= F(\theta_1, \ldots, \theta_{n-1}) \\ &= \begin{cases} \varrho_{1 \le i \le m} (\omega\, \boldsymbol{u}_1, \omega^{-n+1} \| \vartheta\, \boldsymbol{u}_i \|), & \text{if } \sum_{i=1}^{m} c_{ij} \{ \| \vartheta\, \boldsymbol{u}_i \| - \vartheta\, \boldsymbol{u}_i \} \equiv 0 \pmod q \\ 0 \qquad , & \text{otherwise,} \end{cases} \end{aligned}$$

so that

$$I(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m) = \int_{0 \le \theta_i \le 1} F(\vartheta)\, d\vartheta.$$

Now, if $\vartheta = (\theta_1, \ldots, \theta_{n-1})$ and $\vartheta' = (\theta_1', \ldots, \theta_{n-1}')$ differ by integers in each coordinate, it is clear that

$$\| \vartheta'\, \boldsymbol{u}_i \| = \| \vartheta\, \boldsymbol{u}_i \|, \quad i = 1, \ldots, m,$$

and also

$$\begin{aligned} &\sum_{i=1}^{m} c_{ij} \{ \| \vartheta'\, \boldsymbol{u}_i \| - \vartheta'\, \boldsymbol{u}_i \} \\ &\qquad = \sum_{i=1}^{m} c_{ij} \{ \| \vartheta\, \boldsymbol{u}_i \| - \vartheta\, \boldsymbol{u}_i \} + (\vartheta - \vartheta') \cdot \left(\sum_{i=1}^{m} c_{ij}\, \boldsymbol{u}_i \right) \\ &\qquad \equiv \sum_{i=1}^{m} c_{ij} \{ \| \vartheta\, \boldsymbol{u}_i \| - \vartheta\, \boldsymbol{u}_i \} \pmod q, \end{aligned}$$

as the coordinates of the points

$$\sum_{i=1}^{m} c_{ij}\, \boldsymbol{u}_i$$

are congruent to zero modulo q. Thus

$$F(\vartheta') = F(\vartheta).$$

This shows that $F(\vartheta)$ is periodic in $\theta_1, \ldots, \theta_{n-1}$ with period 1. It follows that $I(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m)$ is the limit of the mean value of $F(\vartheta)$ taken over any suitable region, in the space of points ϑ, which becomes large in an appropriate way.

Since $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ are linearly independent, we may suppose that the determinant of the matrix

$$(u_r^{(i)})_{\substack{1 \le i \le m \\ 1 \le r \le m}}$$

does not vanish. Consider the transformation from the variables $\theta_1, \ldots, \theta_{n-1}$ to the variables

$$\varphi_i = \vartheta\, \boldsymbol{u}_i, \quad i = 1, \ldots, m,$$
$$\varphi_i = \theta_i, \quad i = m+1, \ldots, n-1.$$

This is a non-singular transformation. So the region defined by the inequalities

$$|\varphi_i| < \Phi, \quad i = 1, \ldots, n-1,$$

is a parallepiped in the ϑ-space, which becomes large as Φ tends to infinity. Also

$$F(\theta_1, \ldots, \theta_m) = G(\varphi_1, \ldots, \varphi_m),$$

where

$$G(\varphi_1, \ldots, \varphi_m) = \left\{ \begin{array}{lll} \varrho_{1 \le i \le m}(\omega\, \boldsymbol{u}_i\, \omega^{-n+1} \| \varphi_i \|), & \text{if } \sum_{i=1}^{m} c_{ij} \{\| \varphi_i \| - \varphi_i\} \equiv 0 \pmod{q}, \\ 0 & , \quad \text{otherwise} \end{array} \right\}. \tag{29}$$

Thus

$$\begin{aligned} I(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m) &= \int_{0 \le \theta_i \le 1} F(\vartheta)\, d\vartheta \\ &= \lim_{\Phi \to \infty} \frac{\int_{|\varphi_i| \le \Phi} F(\vartheta)\, d\vartheta}{\int_{|\varphi_i| \le \Phi} d\vartheta} \\ &= \lim_{\Phi \to \infty} \frac{\int \cdots \int_{|\varphi_i| \le \Phi} G(\varphi_1, \ldots, \varphi_m)\, d\varphi_1 \ldots d\varphi_{n-1}}{\int \cdots \int_{|\varphi_i| \le \Phi} d\varphi_1 \ldots d\varphi_{n-1}} \\ &= \lim_{\Phi \to \infty} \frac{\int \cdots \int_{|\varphi_i| \le \Phi} G(\varphi_1, \ldots, \varphi_m)\, d\varphi_1 \ldots d\varphi_m}{\int \cdots \int_{|\varphi_i| \le \Phi} d\varphi_1 \ldots d\varphi_m}. \end{aligned}$$

But, it is clear from (29) that $G(\varphi_1, \ldots, \varphi_m)$ is periodic in $\varphi_1, \ldots, \varphi_m$ with period q. Hence

$$I(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m) = \frac{1}{q^m} \int_{-\frac{1}{2}}^{q-\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{q-\frac{1}{2}} G(\varphi_1, \ldots, \varphi_m)\, d\varphi_1 \ldots d\varphi_m. \tag{30}$$

Now, if $a_1, \dots, a_m$ are any integers and

$$\varphi_i = a_i + \alpha_i, \quad i = 1, \dots, m,$$

where

$$-\tfrac{1}{2} < \alpha_i < \tfrac{1}{2}, \quad i = 1, \dots, m,$$

then

$$\|\varphi_i\| = \alpha_i, \ \varphi_i - \|\varphi_i\| = a_i, \quad i = 1, \dots, m,$$

so that

$$G(\varphi_1, \dots, \varphi_m) = \varrho_{1 \le i \le m}(\omega \boldsymbol{u}_i, \omega^{-n+1} \alpha_i),$$

if

$$\sum_{i=1}^{m} c_{ij} a_i \equiv 0 \pmod q, \quad j = 1, \dots, h,$$

while

$$G(\varphi_1, \dots, \varphi_m) = 0$$

otherwise. Hence

$$I(\boldsymbol{u}_1, \dots, \boldsymbol{u}_m) = \sum \begin{bmatrix} a_1, \dots, a_m \text{ integers} \\ 0 \le a_i < q \\ \sum_{i=1}^{m} c_{ij} a_i \equiv 0 \pmod q \end{bmatrix} \frac{1}{q^m} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} \varrho_{1 \le i \le m}(\omega \boldsymbol{u}_i, \omega^{-n+1} \alpha_i) \, d\alpha_1 \dots d\alpha_m$$

$$= \omega^{m(n-1)} \frac{N(C, q)}{q^m} \int_{-1/(2\omega^{n-1})}^{1/(2\omega^{n-1})} \cdots \int_{-1/(2\omega^{n-1})}^{1/(2\omega^{n-1})} \varrho_{1 \le i \le m}(\omega \boldsymbol{u}_i, \xi_i) \, d\xi_1 \dots d\xi_m,$$

where $N(C, q)$ is the number of sets of integers $a_1, \dots, a_m$ with

$$0 \le a_1 < q, \dots, 0 \le a_m < q,$$

$$\sum_{i=1}^{m} c_{ij} a_i \equiv 0 \pmod q, \quad j = 1, \dots, h.$$

Since

$$\varrho_{1 \le i \le m}(\omega \boldsymbol{u}_i, \xi_i) = 0,$$

unless

$$|\xi_i| < R, \quad i = 1, \dots, m,$$

it follows that

$$I(\boldsymbol{u}_1, \dots, \boldsymbol{u}_m) = \omega^{m(n-1)} \frac{N(C, q)}{q^m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varrho_{1 \le i \le m}(\omega \boldsymbol{u}_i, \xi_i) \, d\xi_1 \dots d\xi_m,$$

as required.

It now follows, precisely as in § 5 of my previous paper,[1] using the lemmas of § 4 of that paper, that

[1] *loc. cit* (1955).

$$\int \varrho(\Lambda)\, d\vartheta = \sum \begin{bmatrix} \boldsymbol{u}_1, \dots, \boldsymbol{u}_m \text{ in } L \\ \dim(\boldsymbol{u}_1, \dots, \boldsymbol{u}_m) = m \\ \sum_{i=1}^{m} (c_{ij}/q)\, \boldsymbol{u}_i \text{ in } L \end{bmatrix} I(\boldsymbol{u}_1, \dots, \boldsymbol{u}_m)$$

converges to

$$\left(\frac{N(C, q)}{q^m}\right)^n \int \varrho(\mathfrak{X})\, d\,\mathfrak{X}$$

as ω tends to zero. It is perhaps worth explaining how this result is obtained. It is first shown that the sum

$$\sum \begin{bmatrix} \boldsymbol{u}_1, \dots, \boldsymbol{u}_m \text{ in } L \\ \dim(\boldsymbol{u}_1, \dots, \boldsymbol{u}_m) = m \\ \sum_{n=1}^{m} (c_{ij}/q)\, \boldsymbol{u}_i \text{ in } L \end{bmatrix} \omega^{m(n-1)} \frac{N(C, q)}{q^m} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \varrho_{1 \le i \le m}(\omega\, \boldsymbol{u}_i, \xi_i)\, d\xi_1 \dots d\xi_m$$

may be completed to the sum

$$\sum \begin{bmatrix} \boldsymbol{u}_1, \dots, \boldsymbol{u}_m \text{ in } L \\ \sum_{i=1}^{m} (c_{ij}/q)\, \boldsymbol{u}_i \text{ in } L \end{bmatrix} \omega^{m(n-1)} \frac{N(C, q)}{q^m} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \varrho_{1 \le i \le m}(\omega\, \boldsymbol{u}_i, \xi_i)\, d\xi_1 \dots d\xi_m$$

with an error of the order $O(\omega^{n-m})$. The conditions

$$\boldsymbol{u}_1, \dots, \boldsymbol{u}_m \text{ in } L,$$

$$\sum_{i=1}^{m} (c_{ij}/q)\, \boldsymbol{u}_i \text{ in } L, \quad j = 1, \dots, h$$

restrict the point

$$(\boldsymbol{u}_1, \dots, \boldsymbol{u}_m) = (u_1^{(1)}, \dots, u_{n-1}^{(1)}, \dots, u_1^{(m)}, \dots, u_{n-1}^{(m)})$$

in $(n-1)m$-dimensional space to a certain sub-lattice of the lattice of points with integral coordinates, which may be seen to have determinant

$$\left(\frac{q^m}{N(C, q)}\right)^{n-1}.$$

It follows, by the theory of Riemann integration, that the completed sum tends to the limit

$$\left(\frac{N(C, q)}{q^m}\right)^n \int \dots \int \left\{ \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \varrho_{1 \le i \le m}(\boldsymbol{x}_i, \xi_i)\, d\xi_1 \dots d\xi_m \right\} d\boldsymbol{x}_1 \dots d\boldsymbol{x}_m$$

$$= \left(\frac{N(C, q)}{q^m}\right)^n \int \varrho(\mathfrak{X})\, d\,\mathfrak{X},$$

as ω tends to zero. This proves the required result.

4. In this section we prove a simple lemma, expressing the number $N(C, q)$ introduced in the statement of Theorem 2 in terms of the elementary divisors of the matrix C.

LEMMA 1. *Let q be a positive integer and let C be an integral $m\times h$ matrix*

$$C=(c_{ij})$$

where $h>0$. Then the number $N(C, q)$ of sets of integers $(a_1, \dots, a_m)$ with

$$0\le a_1<q, \dots, 0\le a_m<q,$$

$$\sum_{i=1}^{m} c_{ij}\, a_i\equiv 0 \pmod q, \quad j=1, \dots, h,$$

is given by

$$N(C, q)=e_1 e_2 \dots e_r q^{m-r},$$

where $r=\min(m, h)$ and

$$e_i=(\varepsilon_i, q), \quad i=1, \dots, r,$$

$\varepsilon_1, \dots, \varepsilon_r$ being the elementary divisors of C.

PROOF. By the theory of elementary divisors, there are $m\times m$ and $h\times h$ integral unimodular matrices S and T such that

$$SCT=E, \tag{31}$$

where E is a diagonal matrix (not in general square) with diagonal elements $\varepsilon_1, \dots, \varepsilon_r$, where

$$r=\min(m, h).$$

Further we may suppose that S and T are chosen so that $\varepsilon_1, \dots, \varepsilon_r$ are the elementary divisors of C. Since T is an integral unimodular matrix the numbers $(a_1, \dots, a_m)$ satisfy the congruences

$$\sum_{i=1}^{m} c_{ij}\, a_i\equiv 0 \pmod q, \quad j=1, \dots, h,$$

if and only if they satisfy the congruences

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{h} c_{ij}\, t_{jk}\right) a_i\equiv 0 \pmod q, \quad k=1, \dots, h.$$

So $N(C, q)$ is the number of sets of integers $(a_1, \dots, a_m)$ with

$$0 \leq a_1 < q, \ldots, 0 \leq a_m < q,$$

$$\sum_{i=1}^{m} \left(\sum_{j=1}^{h} c_{ij} \, t_{jk} \right) a_i \equiv 0 \pmod{q}, \quad k = 1, \ldots, h.$$

Since S is an integral unimodular matrix, the system of congruences

$$a_i \equiv \sum_{g=1}^{m} s_{gi} \, b_g \pmod{q}, \quad i = 1, \ldots, m$$

establishes a one-one correspondence between the sets of integers $(a_1, \ldots, a_m)$ with

$$0 \leq a_1 < q, \ldots, 0 \leq a_m < q$$

and the sets of integers $(b_1, \ldots, b_m)$ with

$$0 \leq b_1 < q, \ldots, 0 \leq b_m < q.$$

Thus $N(C, q)$ is the number of sets of integers $(b_1, \ldots, b_m)$ with

$$0 \leq b_1 < q, \ldots, 0 \leq b_m < q,$$

$$\sum_{g=1}^{m} \left(\sum_{i=1}^{m} \sum_{j=1}^{m} s_{gi} \, c_{ij} \, t_{jk} \right) b_g \equiv 0 \pmod{q}, \quad k = 1, \ldots, h.$$

But, by (31), this system of congruences reduces to the system

$$\varepsilon_g \, b_g \equiv 0 \pmod{q}, \quad g = 1, \ldots, r.$$

Now it is clear that $N(C, q)$ is given by

$$N(C, q) = e_1 \, e_2 \ldots e_r \, q^{m-r},$$

where

$$e_i = (\varepsilon_i, q), \quad i = 1, 2, \ldots, r.$$

5. The aim of this section is to prove Theorem 3 below; we first show that the result follows from those of sections 2, 3 and 4 in the special case when $\varrho(\mathbf{X}_1, \ldots, \mathbf{X}_m)$ is integrable in the Riemann sense, and then show that this implies that the same result also holds for more general functions $\varrho(\mathbf{X}_1, \ldots, \mathbf{X}_m)$.

THEOREM 3. *Let $\varrho(\mathfrak{X}) = \varrho(\mathbf{X}_1, \ldots, \mathbf{X}_m)$ be a Borel measurable function which is integrable in the Lebesgue sense over the whole $\mathfrak{X}$-space. Let q be a positive integer and let C be an integral $m \times h$ matrix $C = (c_{ij})$, where $h \geq 0$. Then the lattice function*

$$\varrho(\Lambda) = \sum \begin{bmatrix} \mathbf{X}_1, \ldots, \mathbf{X}_m \text{ in } \Lambda \\ \dim(\mathbf{X}_1, \ldots, \mathbf{X}_m) = m \\ \sum_{i=1}^{m} (c_{ij}/q) \, \mathbf{X}_i \text{ in } \Lambda \end{bmatrix} \varrho(\mathbf{X}_1, \ldots, \mathbf{X}_m) \tag{32}$$

is Borel measurable in the space of lattices Λ *of determinant* 1 *and*

$$\int_F \varrho(\Omega\Lambda_0)\, d\mu(\Omega) = \left(\frac{e_1}{q}\frac{e_2}{q}\cdots\frac{e_r}{q}\right)^n \int \varrho(\mathfrak{X})\, d\mathfrak{X},$$

where $r = \min(m, h)$ *and*

$$e_1 = (\varepsilon_i, q), \quad i = 1, \ldots, r,$$

$\varepsilon_1, \ldots, \varepsilon_r$ *being the elementary divisors of* C.

PROOF. We first show that the function $\varrho(\Lambda)$ is a Borel measurable function in the space of lattices Λ of determinant 1. We have to show that the function

$$\varrho(\Omega\Lambda_0) = \sum \begin{bmatrix} \boldsymbol{X}_1, \ldots, \boldsymbol{X}_m \text{ in } \Omega\Lambda_0 \\ \dim(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m) = m \\ \sum_{i=1}^m (c_{ij}/q)\boldsymbol{X}_i \text{ in } \Omega\Lambda_0 \end{bmatrix} \varrho(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m)$$

$$= \sum \begin{bmatrix} \boldsymbol{U}_1, \ldots, \boldsymbol{U}_m \text{ in } \Lambda_0 \\ \dim(\boldsymbol{U}_1, \ldots, \boldsymbol{U}_m) = m \\ \sum_{i=1}^m (c_{ij}/q)\boldsymbol{U}_i \text{ in } \Lambda_0 \end{bmatrix} \varrho(\Omega\boldsymbol{U}_1, \ldots, \Omega\boldsymbol{U}_m),$$

is Borel measurable in the space of linear transformations Ω of determinant 1. It clearly suffices to prove that, for every set of linearly independent points $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_m$, the function $\varrho(\Omega\boldsymbol{U}_1, \ldots, \Omega\boldsymbol{U}_m)$ is Borel measurable in this space. Now, as $\varrho(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m)$ in Borel measurable, for each c the set of linear transformations Ω^* for which

$$\varrho(\Omega^*\boldsymbol{U}_1, \ldots, \Omega^*\boldsymbol{U}_m) < c$$

is a Borel set n^2-dimensional space. Thus the set of all linear transformations Ω of determinant 1 with

$$\varrho(\Omega\boldsymbol{U}_1, \ldots, \Omega\boldsymbol{U}_m) < c \tag{33}$$

is a Borel set, when regarded as a subset of n^2-dimensional space; and so also is the set of all linear transformations

$$\Omega^* = \lambda\Omega,$$

where $\frac{1}{2} \le \lambda \le 1$ and Ω is a linear transformation of determinant 1 satisfying (33). It follows, from our definitions, that $\varrho(\Omega\boldsymbol{U}_1, \ldots, \Omega\boldsymbol{U}_m)$ is a Borel measurable function in the space of all linear transformations Ω of determinant 1. So $\varrho(\Lambda)$ is a Borel measurable function in the space of lattices of determinant 1.

Now, if $\varrho(X_1, \dots, X_m)$ is a Riemann integrable function, so is the function $\varrho(\Omega X_1, \dots, \Omega X_m)$ for any linear transformation Ω of determinant 1. So, by Theorem 2, the mean value $M_\Lambda[\varrho(\Omega\Lambda)]$ exists and has the value

$$M_\Lambda[\varrho(\Omega\Lambda)] = \left(\frac{N(C, q)}{q^m}\right)^n \int \varrho(\Omega \mathfrak{X})\, d\mathfrak{X},$$

$$= \left(\frac{N(C, q)}{q^m}\right)^n \int \varrho(\mathfrak{X})\, d\mathfrak{X},$$

where $N(C, q)$ is the number of sets of integers $(a_1, \dots, a_m)$ with

$$0 \le a_1 < q, \dots, 0 \le a_m < q,$$

$$\sum_{i=1}^{m} c_{ij} a_i \equiv 0 \pmod q, \quad j = 1, \dots, h.$$

Here, by the lemma of section 4,

$$N(C, q) = e_1 e_2 \dots e_r q^{m-r},$$

where $r = \min(m, h)$ and

$$e_i = (\varepsilon_i, q), \quad i = 1, \dots, r,$$

$\varepsilon_1, \dots, \varepsilon_r$ being the elementary divisors of C. So

$$M_\Lambda[\varrho(\Omega\Lambda)] = \left(\frac{e_1}{q} \cdot \frac{e_2}{q} \cdot \dots \cdot \frac{e_r}{q}\right)^n \int \varrho(\mathfrak{X})\, d\mathfrak{X}.$$

Now provided $\varrho(\mathfrak{X}_1, \dots, \mathfrak{X}_m)$ is non-negative and Borel measurable as well as being Riemann integrable,[1] it follows from Theorem 1 that

$$\int_F \varrho(\Omega\Lambda_0)\, d\mu(\Omega) = M_\Lambda[\varrho(\Lambda)]$$

$$= \left(\frac{e_1}{q} \cdot \frac{e_2}{q} \cdot \dots \cdot \frac{e_r}{q}\right)^n \int \varrho(\mathfrak{X})\, d\mathfrak{X},$$

as required.

Now consider a function $\varrho(\mathfrak{X})$ which is non-negative, bounded, Borel measurable and which vanishes outside a bounded region. More explicitly, suppose that M is a constant such that $0 \le \varrho(\mathfrak{X}) \le M$ for all $\mathfrak{X}$; and that $\sigma(\mathfrak{X})$ is the characteristic function of an open interval I chosen so that $\varrho(\mathfrak{X}) = 0$ for all $\mathfrak{X}$ which are not in a

[1] Note that a Riemann integrable function is not necessarily Borel measurable.

closed interval I' contained in I. Then, for each positive integer k, it is possible to choose a set E_k and a continuous function $\varrho_k(\mathfrak{X})$ vanishing outside I, such that

(a) $|\varrho(\mathfrak{X})-\varrho_k(\mathfrak{X})|<k^{-1}$, if $\mathfrak{X}$ is not in E_k,

(b) $m(E_k)<k^{-1}$, and

(c) $0\le\varrho_k(\mathfrak{X})\le M$, for all $\mathfrak{X}$.

Let $\tau_k(\mathfrak{X})$ be the characteristic function of the set E_k. Then, for each positive integer k,

$$|\varrho(\mathfrak{X})-\varrho_k(\mathfrak{X})|\le k^{-1}\sigma(\mathfrak{X})+2M\tau_k(\mathfrak{X}),$$

for all $\mathfrak{X}$.

Now E_k can be covered by an infinite sequence of intervals, such that the sum of their measures does not exceed

$$m(E_k)+k^{-1}.$$

So it is possible to choose continuous non-negative functions $\tau_{ks}(\mathfrak{X})$ each vanishing outside a bounded region, and such that

$$\sum_{n=1}^{\infty}\tau_{ks}(\mathfrak{X})\ge\tau_k(\mathfrak{X})$$

for all $\mathfrak{X}$, and

$$\sum_{s=1}^{\infty}\int\tau_{ks}(\mathfrak{X})\,d\mathfrak{X}\le m(E_m)+2k^{-1}.$$

Then

$$|\varrho(\mathfrak{X})-\varrho_k(\mathfrak{X})|\le k^{-1}\sigma(\mathfrak{X})+2M\sum_{s=1}^{\infty}\tau_{ks}(\mathfrak{X}),$$

for all points $(\mathfrak{X})$.

Define $\sigma(\Lambda)$ and $\tau_{ks}(\Lambda)$ by analogy with the definition (32) of $\varrho(\Lambda)$. Then clearly

$$|\varrho(\Lambda)-\varrho_k(\Lambda)|\le k^{-1}\sigma(\Lambda)+2M\sum_{s=1}^{\infty}\tau_{ks}(\Lambda), \tag{34}$$

and

$$0\le\varrho(\Lambda)\le\varrho_k(\Lambda)+k^{-1}\sigma(\Lambda)+2M\sum_{s=1}^{\infty}\tau_{ks}(\Lambda). \tag{35}$$

Now the functions $\varrho_k(\mathfrak{X})$, $\sigma(\mathfrak{X})$, $\tau_{ks}(\mathfrak{X})$ are all non-negative Riemann integrable functions which are also Borel measurable. So, by the special result we have already proved,

$$\int_F \varrho_k(\Omega\Lambda_0)\,d\mu(\Omega) = \left(\frac{e_1}{q}\cdot\frac{e_2}{q}\cdot\;\cdots\;\cdot\frac{e_r}{q}\right)^n \int \varrho_k(\mathfrak{X})\,d\mathfrak{X},$$

$$\int_F \sigma(\Omega\Lambda_0)\,d\mu(\Omega) = \left(\frac{e_1}{q}\cdot\frac{e_2}{q}\cdot\;\cdots\;\cdot\frac{e_r}{q}\right)^n \int \sigma(\mathfrak{X})\,d\mathfrak{X},$$

$$\int_F \tau_{ks}(\Omega\Lambda_0)\,d\mu(\Omega) = \left(\frac{e_1}{q}\cdot\frac{e_2}{q}\cdot\;\cdots\;\cdot\frac{e_r}{q}\right)^n \int \tau_{ks}(\mathfrak{X})\,d\mathfrak{X}.$$

Since

$$\sum_{s=1}^{\infty}\int \tau_{ks}(\mathfrak{X})\,d\mathfrak{X} \le m(E_k) + 2k^{-1} \le 3k^{-1},$$

the function

$$\varrho_k(\Omega\Lambda_0) + k^{-1}\sigma(\Omega\Lambda_0) + 2M\sum_{s=1}^{\infty}\tau_{ks}(\Omega\Lambda_0)$$

will be integrable with respect to the measure $\mu(\Omega)$ over F. As the function $\varrho(\Omega\Lambda_0)$ is Borel measurable with respect to the measure $\mu(\Omega)$ over F and satisfies

$$0 \le \varrho(\Omega\Lambda_0) \le \varrho_k(\Omega\Lambda_0) + k^{-1}\sigma(\Omega\Lambda_0) + 2M\sum_{s=1}^{\infty}\tau_{ks}(\Omega\Lambda_0),$$

it follows that $\varrho(\Omega\Lambda_0)$ is integrable with respect to $\mu(\Omega)$ over F. Further

$$\begin{aligned}
&\left|\int_F \varrho(\Omega\Lambda_0)\,d\mu(\Omega) - \left(\frac{e_1}{q}\cdot\frac{e_2}{q}\cdot\;\cdots\;\cdot\frac{e_r}{q}\right)^n \int \varrho(\mathfrak{X})\,d\mathfrak{X}\right| \\
&\quad\le \left|\int_F \varrho(\Omega\Lambda_0)\,d\mu(\Omega) - \left(\frac{e_1}{q}\cdot\frac{e_2}{q}\cdot\;\cdots\;\cdot\frac{e_r}{q}\right)^n \int \varrho_k(\mathfrak{X})\,d\mathfrak{X}\right| + \\
&\quad + \left(\frac{e_1}{q}\cdot\frac{e_2}{q}\cdot\;\cdots\;\cdot\frac{e_r}{q}\right)^n \left|\int \varrho(\mathfrak{X})\,d\mathfrak{X} - \int \varrho_k(\mathfrak{X})\,d\mathfrak{X}\right| \\
&\quad\le \int_F \left|\varrho(\Omega\Lambda_0) - \varrho_k(\Omega\Lambda_0)\right| d\mu(\Omega) + \left(\frac{e_1}{q}\cdot\frac{e_2}{q}\cdot\;\cdots\;\cdot\frac{e_r}{q}\right)^n \int \left|\varrho(\mathfrak{X}) - \varrho_k(\mathfrak{X})\right| d\mathfrak{X} \\
&\quad\le \int_F \left\{k^{-1}\sigma(\Omega\Lambda_0) + 2M\sum_{s=1}^{\infty}\tau_{ks}(\Omega\Lambda_0)\right\} d\mu(\Omega) + \\
&\quad + \left(\frac{e_1}{q}\cdot\frac{e_2}{q}\cdot\;\cdots\;\cdot\frac{e_r}{q}\right)^n \int \left\{k^{-1}\sigma(\mathfrak{X}) + 2M\sum_{s=1}^{\infty}\tau_{ks}(\mathfrak{X})\right\} d\mathfrak{X} \\
&\quad = 2\left(\frac{e_1}{q}\cdot\frac{e_2}{q}\cdot\;\cdots\;\cdot\frac{e_r}{q}\right)^n \left\{k^{-1}\int \sigma(\mathfrak{X})\,d\mathfrak{X} + 2M\sum_{s=1}^{\infty}\int \tau_{ks}(\mathfrak{X})\,d\mathfrak{X}\right\} \\
&\quad \le 2\left(\frac{e_1}{q}\cdot\frac{e_2}{q}\cdot\;\cdots\;\cdot\frac{e_r}{q}\right)^n \left\{\int \sigma(\mathfrak{X})\,d\mathfrak{X} + 6M\right\}\frac{1}{k}.
\end{aligned}$$

Since this holds for all positive integers k, it follows that

$$\int_F \varrho(\Omega \Lambda_0)\, d\mu(\Omega) = \left(\frac{e_1}{q}\cdot\frac{e_2}{q}\cdot \dots \cdot\frac{e_r}{q}\right)^n \int \varrho(\mathfrak{X})\, d\mathfrak{X}.$$

Finally we consider the general case when $\varrho(\mathfrak{X})$ is a Borel measurable function which is integrable in the Lebesgue sense over the whole $\mathfrak{X}$-space. Then it is possible to express $\varrho(\mathfrak{X})$ in the form

$$\varrho(\mathfrak{X}) = \sum_{s=1}^{\infty} \sigma_s(\mathfrak{X}) - \sum_{s=1}^{\infty} \tau_s(\mathfrak{X}),$$

where each of the functions $\sigma_s(\mathfrak{X})$, $\tau_s(\mathfrak{X})$ are non-negative, bounded, Borel measurable functions, which vanish outside bounded regions, and where the sums

$$\sum_{s=1}^{\infty} \int \sigma_s(\mathfrak{X})\, d\mathfrak{X}, \quad \sum_{s=1}^{\infty} \int \tau_s(\mathfrak{X})\, d\mathfrak{X}$$

are convergent. Then, using the natural notation, by the result we have already proved

$$\int_F \sigma_s(\Omega \Lambda_0)\, d\mu(\Omega) = \left(\frac{e_1}{q}\cdot\frac{e_2}{q}\cdot \dots \cdot\frac{e_r}{q}\right)^n \int \sigma_s(\mathfrak{X})\, d\mathfrak{X},$$

$$\int_F \tau_s(\Omega \Lambda_0)\, d\mu(\Omega) = \left(\frac{e_1}{q}\cdot\frac{e_2}{q}\cdot \dots \cdot\frac{e_r}{q}\right)^n \int \tau_s(\mathfrak{X})\, d\mathfrak{X},$$

for $s = 1, 2, \dots$. Since the functions $\sigma_s(\Omega \Lambda_0)$, $\tau_s(\Omega \Lambda_0)$ are Borel measurable and non-negative, while the sums

$$\sum_{s=1}^{\infty} \int_F \sigma_s(\Omega \Lambda_0)\, d\mu(\Omega), \quad \sum_{s=1}^{\infty} \int_F \tau_s(\Omega \Lambda_0)\, d\mu(\Omega)$$

are convergent, it follows that the function $\varrho(\Omega \Lambda_0)$ is integrable with respect to $\mu(\Omega)$ over F and that

$$\begin{aligned}
&\int_F \varrho(\Omega \Lambda_0)\, d\mu(\Omega) \\
&= \sum_{s=1}^{\infty} \int_F \sigma_s(\Omega \Lambda_0)\, d\mu(\Omega) - \sum_{s=1}^{\infty} \int_F \tau_s(\Omega \Lambda_0)\, d\mu(\Omega) \\
&= \left(\frac{e_1}{q}\cdot\frac{e_2}{q}\cdot \dots \cdot\frac{e_r}{q}\right)^n \left\{\sum_{s=1}^{\infty} \int \sigma_s(\mathfrak{X})\, d\mathfrak{X} - \sum_{s=1}^{\infty} \tau_s(\mathfrak{X})\, d\mathfrak{X}\right\} \\
&= \left(\frac{e_1}{q}\cdot\frac{e_2}{q}\cdot \dots \cdot\frac{e_r}{q}\right)^n \int \varrho(\mathfrak{X})\, d\mathfrak{X}.
\end{aligned}$$

This completes the proof.

COROLLARY. *If $\varrho(\mathfrak{X})$ is a non-negative Borel measurable function, which is not integrable in the Lebesgue sense over the whole $\mathfrak{X}$-space, then the integral*

$$\int_F \varrho(\Omega \Lambda_0)\, d\mu(\Omega)$$

diverges.

PROOF. As $\varrho(\mathfrak{X})$ is non-negative and Borel measurable but is not Lebesgue integrable, it follows that

$$\int \varrho(\mathfrak{X})\, d\mathfrak{X} = +\infty.$$

So it is possible to choose non-negative, Borel measurable, Lebesgue integrable functions, $\sigma(\mathfrak{X})$ with

$$0 \leq \sigma(\mathfrak{X}) \leq \varrho(\mathfrak{X})$$

for all $\mathfrak{X}$ and with

$$\int \sigma(\mathfrak{X})\, d\mathfrak{X}$$

arbitarily large. Since, then

$$\int_F \varrho(\Omega \Lambda_0)\, d\mu(\Omega) \geq \int_F \varrho(\Omega \Lambda_0)\, d\mu(\Omega)$$

$$= \left(\frac{e_1}{q} \cdot \frac{e_2}{q} \cdot \ldots \cdot \frac{e_r}{q}\right)^n \int \varrho(\mathfrak{X})\, d\mathfrak{X},$$

it follows that

$$\int_F \varrho(\Omega \Lambda_0)\, d\mu(\Omega) = +\infty.$$

6. In this section we show how results of the type given by Theorem 3 can be combined to give a

PROOF OF THEOREM 4. We first prove that for any lattice the terms of the sum

$$\sum_{\boldsymbol{X}_1 \in \Lambda, \ldots, \boldsymbol{X}_k \in \Lambda} \varrho(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k) \tag{36}$$

can be rearranged in the form

$$\varrho(\boldsymbol{O}, \ldots, \boldsymbol{O}) +$$

$$+ \sum_{(\nu;\mu)} \sum_{q=1}^{\infty} \sum_{D} \sum \begin{bmatrix} \boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_m \text{ in } \Lambda \\ \dim(\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_m) = m \\ \sum_{i=1}^{m} (d_{ij}/q)\, \boldsymbol{Y}_i \text{ in } \Lambda \end{bmatrix} \varrho\left(\sum_{i=1}^{m} \frac{d_{i1}}{q} \boldsymbol{Y}_i, \ldots, \sum_{i=1}^{m} \frac{d_{ik}}{q} \boldsymbol{Y}_i\right) +$$

$$+ \sum \begin{bmatrix} \boldsymbol{X}_1, \ldots, \boldsymbol{X}_k \text{ in } \Lambda \\ \dim(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k) = k \end{bmatrix} \varrho(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k). \tag{37}$$

Corresponding to each set of points $X_1, \ldots, X_k$ of Λ there will be just one term $\varrho(X_1, \ldots, X_k)$ in the sum (36). If $X_1, \ldots, X_k$ are linearly independent the term $\varrho(X_1, \ldots, X_k)$ will occur just once in the sum

$$\sum \begin{bmatrix} X_1, \ldots, X_k \text{ in } \Lambda \\ \dim(X_1, \ldots, X_k) = k \end{bmatrix} \varrho(X_1, \ldots, X_k), \tag{38}$$

and will not occur elsewhere in the sum (37), as the points

$$\sum_{i=1}^{m} \frac{d_{i1}}{q} Y_i, \ldots, \sum_{i=1}^{m} \frac{d_{ik}}{q} Y_i$$

are necessarily linearly dependent since $m < k$.

If $X_1, \ldots, X_k$ are linearly dependent but not all zero, there will be a unique division $(\nu; \mu) = (\nu_1, \ldots, \nu_m; \mu_1, \ldots, \mu_{k-m})$ of the numbers $1, 2, \ldots, k$ into two sequences $\nu_1, \ldots, \nu_m$ and $\mu_1, \ldots, \mu_{k-m}$, with $1 \le m \le k-1$, satisfying the conditions (10) such that the points

$$X_{\nu_1}, \ldots, X_{\nu_m}$$

are linearly independent, while, for each j, the point X_{μ_j} is linearly dependent on

$$X_1, X_2, \ldots, X_{\mu_j - 1}.$$

Then for each j, the point X_{μ_j} will be linearly dependent on the points

$$X_{\nu_1}, \ldots, X_{\nu_i},$$

where i is chosen so that

$$\nu_i < \mu_j < \nu_{i+1}.$$

So, for each j, the point X_{μ_j} can be expressed uniquely in the form

$$X_{\mu_j} = \sum_{i=1}^{m} c_{ij} X_{\nu_i}$$

where

$$c_{ij} = 0, \text{ if } \nu_i \ge \mu_j.$$

Here the numbers c_{ij} will be uniquely determined rational numbers. Let q be the lowest common denominator of these rational numbers. Define integers d_{ij} for

$$1 \le i \le m, \; 1 \le j \le k$$

by the equations

$$d_{i\nu_j} = q\,\delta_{ij}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, m,$$

$$d_{i\mu_j} = q\,c_{ij}, \quad i = 1, \ldots, m, \quad j = 1, \ldots, k-m.$$

Then the matrix $D = (d_{ij})$ has integral elements having highest common factor relatively prime to q and satisfying (11). Further, if we take

$$\boldsymbol{Y}_1 = \boldsymbol{X}_{\nu_1}, \ldots, \boldsymbol{Y}_m = \boldsymbol{X}_{\nu_m},$$

we see that $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_m$ are linearly independent points of Λ and that the points

$$\boldsymbol{X}_1 = \sum_{i=1}^{m} \frac{d_{i1}}{q} \boldsymbol{Y}_i, \ldots, \boldsymbol{X}_k = \sum_{i=1}^{m} \frac{d_{ik}}{q} \boldsymbol{Y}_i \tag{39}$$

are points of Λ. Thus, there is a term

$$\varrho\left(\sum_{i=1}^{m} \frac{d_{i1}}{q} \boldsymbol{Y}_1, \ldots, \sum_{i=1}^{m} \frac{d_{ik}}{q} \boldsymbol{Y}_i\right) = \varrho(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k) \tag{40}$$

in the sum (37) corresponding in this way to the points $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k$. Further, it is clear that the points $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k$ and the equations (39), together with the conditions on the division $(\nu; \mu)$, the matrix D and the points $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_m$, determine uniquely the division $(\nu; \mu)$, the positive integer q, the matrix D and the points $\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_m$. So corresponding to each set of points $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k$, which are linearly dependent but not all zero, there will be just one term of the form (40) in the sum (37). Similarly, the term $\varrho(\boldsymbol{O}, \ldots, \boldsymbol{O})$ is the only term in the sum (37) corresponding to the set of points $\boldsymbol{O}, \ldots, \boldsymbol{O}$. This shows that each term in the sum (36) occurs just once in the sum (37).

We now have to show that each term in the sum (37) corresponds to just one term in the sum (36). This is clear for the term $\varrho(\boldsymbol{O}, \ldots, \boldsymbol{O})$ and for the terms of the sum (38). Consider a term

$$\varrho\left(\sum_{i=1}^{m} \frac{d_{i1}}{q} \boldsymbol{Y}_i, \ldots, \sum_{i=1}^{m} \frac{d_{ik}}{q} \boldsymbol{Y}_i\right) \tag{41}$$

from the sum

$$\sum_{(\nu;\mu)} \sum_{q=1}^{\infty} \sum_{D} \sum \begin{bmatrix} \boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_m \text{ in } \Lambda \\ \dim(\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_m) = m \\ \sum_{i=1}^{m} (d_{ij}/q)\, \boldsymbol{Y}_i \text{ in } \Lambda \end{bmatrix} \varrho\left(\sum_{i=1}^{m} \frac{d_{i1}}{q} \boldsymbol{Y}_i, \ldots, \sum_{i=1}^{m} \frac{d_{ik}}{q} \boldsymbol{Y}_i\right).$$

The condition that the points

$$\sum_{i=1}^{m} (d_{ij}/q)\, \boldsymbol{Y}_i, \quad j = 1, 2, \ldots, k,$$

should be in Λ ensures that this term (41) corresponds to the unique term in (36) where $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k$ are defined by the equations (39). This establishes the required one-one correspondence between the terms of the sums (36) and (37). Since $\varrho(\mathfrak{X})$ is non-negative, it follows that

$$\sum_{\boldsymbol{X}_1 \in \Lambda, \ldots, \boldsymbol{X}_k \in \Lambda} \varrho(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k) = \varrho(\boldsymbol{O}, \ldots, \boldsymbol{O}) +$$

$$+ \sum_{(\nu;\mu)} \sum_{q=1} \sum_{D} \sum \begin{bmatrix} \boldsymbol{Y}_1, \ldots \boldsymbol{Y}_m \text{ in } \Lambda \\ \dim(\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_m) = m \\ \sum_{i=1}^{m} (d_{ij}/q)\, \boldsymbol{Y}_i \text{ in } \Lambda \end{bmatrix} \varrho\left(\sum_{i=1}^{m} \frac{d_{i1}}{q} \boldsymbol{Y}_i, \ldots, \sum_{i=1}^{m} \frac{d_{ik}}{q} \boldsymbol{Y}_i\right) +$$

$$+ \sum \begin{bmatrix} \boldsymbol{X}_1, \ldots, \boldsymbol{X}_k \text{ in } \Lambda \\ \dim(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k) = k \end{bmatrix} \varrho(\boldsymbol{X}_1 \ldots, \boldsymbol{X}_k), \tag{42}$$

with the convention that both sides may have the value $+\infty$.

Now for any division $(\nu; \mu)$, positive integer q and matrix D we consider the sum

$$\varrho(\nu; \mu; q; D; \Lambda) = \sum \begin{bmatrix} \boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_m \text{ in } \Lambda \\ \dim(\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_m) = m \\ \sum_{i=1}^{m} (d_{ij}/q)\, \boldsymbol{Y}_i \text{ in } \Lambda \end{bmatrix} \varrho\left(\sum_{i=1}^{m} \frac{d_{i1}}{q} \boldsymbol{Y}_i, \ldots, \sum_{i=1}^{m} \frac{d_{ik}}{q} \boldsymbol{Y}_i\right). \tag{43}$$

Since $\varrho(\mathfrak{X})$ is non-negative and Borel measurable, the function

$$\varrho\left(\sum_{i=1}^{m} \frac{d_{i1}}{q} \boldsymbol{Y}_i, \ldots, \sum_{i=1}^{m} \frac{d_{ik}}{q} \boldsymbol{Y}_i\right)$$

is a non-negative Borel measurable function of the point $(\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_m)$ in mn-dimensional space. So, by Theorem 3 and its Corollary

$$\int_F \varrho(\nu; \mu; q; D; \Omega \Lambda_0)\, d\mu(\Omega)$$

$$= \left(\frac{e_1}{q} \cdot \frac{e_2}{q} \cdot \ldots \cdot \frac{e_m}{q}\right)^n \int \cdots \int \varrho\left(\sum_{i=1}^{m} \frac{d_{i1}}{q} \boldsymbol{Y}_i, \ldots, \sum_{i=1}^{m} \frac{d_{ik}}{q} \boldsymbol{Y}_i\right) d\boldsymbol{Y}_1 \ldots d\boldsymbol{Y}_m, \tag{44}$$

with the convention that both sides may have the value $+\infty$, where

$$e_i = (\varepsilon_i, q) \quad i = 1, \ldots, m,$$

and $\varepsilon_1, \ldots, \varepsilon_m$ are the elementary divisors of the matrix D. Similarly

$$\int_F \sum \begin{bmatrix} \boldsymbol{X}_1, \ldots, \boldsymbol{X}_k \text{ in } \Omega \Lambda_0 \\ \dim(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k) = k \end{bmatrix} \varrho(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k)\, d\mu(\Omega)$$

$$= \int \cdots \int \varrho(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k)\, d\boldsymbol{X}_1 \ldots d\boldsymbol{X}_k. \tag{45}$$

We also have
$$\int_F \varrho(\boldsymbol{O}, \ldots, \boldsymbol{O})\, d\mu(\Omega) = \varrho(\boldsymbol{O}, \ldots, \boldsymbol{O}). \tag{46}$$

Thus, combining the results (44), (45), (46), and using (43) and (42) we obtain the required formula (9) with the convention that both sides may have the value $+\infty$.

To complete the proof, it remains to show that the right hand side of (9) is finite when $\varrho(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k)$ is bounded and vanishes outside a bounded region. But I have shown, in another paper,[1] using a slightly different notation, and making additional assumptions about the function $\varrho(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k)$, which are clearly not relevant to the proof of convergence, that the sum of the right hand side of (9) is convergent, in this case, provided
$$n \geq \max_{1 \leq m < k} m(k-m) + 2 = [\tfrac{1}{4}k^2] + 2.$$

7. In this section we show how results of the type given by Theorem 3, in the special case when $h=0$ can be combined to give a

PROOF OF THEOREM 5. If $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are primitive points of a lattice Λ and $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are linearly dependent, then it is clear that either $\boldsymbol{X}_2 = \boldsymbol{X}_1$ or $\boldsymbol{X}_2 = -\boldsymbol{X}_1$. So, if Λ^* is the set of primitive points of a lattice Λ, then the sum
$$\sum_{\boldsymbol{X}_1 \in \Lambda^*, \boldsymbol{X}_2 \in \Lambda^*} \varrho(\boldsymbol{X}_1, \boldsymbol{X}_2) \tag{47}$$
can be rearranged in the form
$$\sum \begin{bmatrix} \boldsymbol{X}_1, \boldsymbol{X}_2 \text{ in } \Lambda^* \\ \dim(\boldsymbol{X}_1, \boldsymbol{X}_2) = 2 \end{bmatrix} \varrho(\boldsymbol{X}_1, \boldsymbol{X}_2) + \sum [\boldsymbol{X}_1 \text{ in } \Lambda^*]\, \varrho(\boldsymbol{X}_1, \boldsymbol{X}_1) + \\ + \sum [\boldsymbol{X}_1 \text{ in } \Lambda^*]\, \varrho(\boldsymbol{X}_1, -\boldsymbol{X}_1), \tag{48}$$
the rearrangement being justified since each sum has only a finite number of non-zero terms.

Let P denote the product of the first k primes $p_1, p_2, \ldots, p_k$. Let Λ_0^P denote the set of all points $\boldsymbol{U} = (u_1, \ldots, u_n)$ of the lattice Λ_0 with
$$(u_1, \ldots, u_n, P) = 1.$$

Then Λ_0^* is a subset of Λ_0^P which is a subset of Λ_0. Further, as $\varrho(\boldsymbol{X}_1, \boldsymbol{X}_2) = \varrho(\boldsymbol{\mathfrak{X}})$ vanishes outside a bounded region of $\boldsymbol{\mathfrak{X}}$-space, for each linear transformation Ω of determinant 1, we have

[1] *loc. cit.* (1955), see the footnote in § 9.

$$\sum \begin{bmatrix} X_1, X_2 \text{ in } \Omega\Lambda_0^P \\ \dim (X_1, X_2) = 2 \end{bmatrix} \varrho (X_1, X_2) = \sum \begin{bmatrix} X_1, X_2 \text{ in } \Omega\Lambda_0^* \\ \dim (X_1, X_2) = 2 \end{bmatrix} \varrho (X_1, X_2),$$

provided k is sufficiently large. Thus

$$\lim_{k\to+\infty} \sum \begin{bmatrix} X_1, X_2 \text{ in } \Omega\Lambda_0^P \\ \dim (X_1, X_2) = 2 \end{bmatrix} \varrho (X_1, X_2) = \sum \begin{bmatrix} X_1, X_2 \text{ in } \Omega\Lambda^* \\ \dim (X_1, X_2) = 2 \end{bmatrix} \varrho (X_1, X_2).$$

Also the function

$$\sum \begin{bmatrix} X_1, X_2 \text{ in } \Omega\Lambda_0^P \\ \dim (X_1, X_2) = 2 \end{bmatrix} \varrho (X_1, X_2)$$

is dominated for all Ω by the function

$$\sum \begin{bmatrix} X_1, X_2 \text{ in } \Omega\Lambda_0 \\ \dim (X_1, X_2) = 2 \end{bmatrix} \varrho (X_1, X_2),$$

which is integrable over the region F by Theorem 3. So, by the theory of dominated convergence, we have

$$\int_F \sum \begin{bmatrix} X_1, X_2 \text{ in } \Omega\Lambda_0^* \\ \dim (X_1, X_2) = 2 \end{bmatrix} \varrho (X_1, X_2)\, d\mu(\Omega)$$
$$= \lim_{k\to+\infty} \int_F \sum \begin{bmatrix} X_1, X_2 \text{ in } \Omega\Lambda_0^P \\ \dim (X_1, X_2) = 2 \end{bmatrix} \varrho (X_1, X_2)\, d\mu(\Omega). \tag{49}$$

If X is any point of Λ_0, other than O, write

$$\lambda(X) = (x_1, \ldots, x_n),$$

so that $\lambda(X)$ is the highest common factor of the coordinates of the point X. Then it is easy to verify that, if μ denotes the Möbius function,

$$\sum \begin{bmatrix} r|P,\ s|P \\ r>0,\ s>0 \end{bmatrix} \mu(r)\,\mu(s) \sum \begin{bmatrix} Y_1, Y_2 \text{ in } \Omega\Lambda_0 \\ \dim (Y_1, Y_2) = 2 \end{bmatrix} \varrho (r Y_1, s Y_2)$$
$$= \sum \begin{bmatrix} X_1, X_2 \text{ in } \Omega\Lambda_0 \\ \dim (X_1, X_2) = 2 \end{bmatrix} \sum \begin{bmatrix} r|P,\ s|P \\ X_1/r,\ X_2/s \text{ in } \Omega\Lambda_0 \end{bmatrix} \mu(r)\,\mu(s)\,\varrho (X_1, X_2)$$
$$= \sum \begin{bmatrix} X_1, X_2 \text{ in } \Omega\Lambda_0 \\ \dim (X_1, X_2) = 2 \end{bmatrix} \left\{ \sum \begin{bmatrix} r|P \\ r|\lambda(\Omega^{-1} X_1) \end{bmatrix} \mu(r) \right\} \times$$
$$\times \left\{ \sum \begin{bmatrix} s|P \\ s|\lambda(\Omega^{-1} X_2) \end{bmatrix} \mu(s) \right\} \varrho (X_1, X_2). \tag{50}$$

But, if $\boldsymbol{U}$ is any point of Λ_0,

$$\sum \begin{bmatrix} r \mid P \\ r \mid \lambda(\boldsymbol{U}) \end{bmatrix} \mu(r) = \sum_{r \mid (\lambda(\boldsymbol{U}),\, P)} \mu(r) = \begin{cases} 1, & \text{if } (\lambda(\boldsymbol{U}), P) = 1, \\ 0, & \text{if } (\lambda(\boldsymbol{U}), P) > 1. \end{cases}$$

So

$$\sum \begin{bmatrix} r \mid P \\ r \mid \lambda(\boldsymbol{U}) \end{bmatrix} \mu(r) = \begin{cases} 1, & \text{if } \boldsymbol{U} \text{ is in } \Lambda_0^P, \\ 0, & \text{otherwise.} \end{cases}$$

Thus the sum (50) reduces to

$$\sum \begin{bmatrix} \boldsymbol{X}_1, \boldsymbol{X}_2 \text{ in } \Omega \Lambda_0^P \\ \dim(\boldsymbol{X}_1, \boldsymbol{X}_2) = 2 \end{bmatrix} \varrho(\boldsymbol{X}_1, \boldsymbol{X}_2).$$

Hence, using Theorem 3,

$$\begin{aligned} &\int_F \sum \begin{bmatrix} \boldsymbol{X}_1, \boldsymbol{X}_2 \text{ in } \Omega \Lambda_0^P \\ \dim(\boldsymbol{X}_1, \boldsymbol{X}_2) = 2 \end{bmatrix} \varrho(\boldsymbol{X}_1, \boldsymbol{X}_2)\, d\mu(\Omega) \\ &= \int_F \sum \begin{bmatrix} r \mid P,\ s \mid P \\ r > 0,\ s > 0 \end{bmatrix} \mu(r)\,\mu(s) \sum \begin{bmatrix} \boldsymbol{Y}_1, \boldsymbol{Y}_2 \text{ in } \Omega \Lambda_0 \\ \dim(\boldsymbol{Y}_1, \boldsymbol{Y}_2) = 2 \end{bmatrix} \varrho(r\boldsymbol{Y}_1, s\boldsymbol{Y}_2)\, d\mu(\Omega) \\ &= \sum \begin{bmatrix} r \mid P,\ s \mid P \\ r > 0,\ s > 0 \end{bmatrix} \mu(r)\,\mu(s) \int_F \sum \begin{bmatrix} \boldsymbol{Y}_1, \boldsymbol{Y}_2 \text{ in } \Omega \Lambda_0 \\ \dim(\boldsymbol{Y}_1, \boldsymbol{Y}_2) = 2 \end{bmatrix} \varrho(r\boldsymbol{Y}_1, s\boldsymbol{Y}_1)\, d\mu(\Omega) \\ &= \sum \begin{bmatrix} r \mid P,\ s \mid P \\ r > 0,\ s > 0 \end{bmatrix} \mu(r)\,\mu(s) \int\!\!\int \varrho(r\boldsymbol{Y}_1, s\boldsymbol{Y}_2)\, d\boldsymbol{Y}_1\, d\boldsymbol{Y}_2 \\ &= \sum \begin{bmatrix} r \mid P,\ s \mid P \\ r > 0,\ s > 0 \end{bmatrix} \mu(r)\,\mu(s)\, r^{-n} s^{-n} \int\!\!\int \varrho(\boldsymbol{X}_1, \boldsymbol{X}_2)\, d\boldsymbol{X}_1\, d\boldsymbol{X}_2 \\ &= \left\{ \sum \begin{bmatrix} r \mid P \\ r > 0 \end{bmatrix} \mu(r)\, r^{-n} \right\}^2 \int\!\!\int \varrho(\boldsymbol{X}_1, \boldsymbol{X}_2)\, d\boldsymbol{X}_1\, d\boldsymbol{X}_2. \qquad (51) \end{aligned}$$

Here

$$\sum \begin{bmatrix} r \mid P \\ r > 0 \end{bmatrix} \mu(r)\, r^{-n} \to \sum_{r=1}^{\infty} \mu(r)\, r^{-n} = \frac{1}{\zeta(n)}, \qquad (52)$$

as $k \to +\infty$. So, by (49), (51) and (52)

$$\int_F \sum \begin{bmatrix} \boldsymbol{X}_1, \boldsymbol{X}_2 \text{ in } \Omega \Lambda_0^* \\ \dim(\boldsymbol{X}_1, \boldsymbol{X}_2) = 2 \end{bmatrix} \varrho(\boldsymbol{X}_1, \boldsymbol{X}_2)\, d\mu(\Omega) = \frac{1}{(\zeta(n))^2} \int\!\!\int \varrho(\boldsymbol{X}_1, \boldsymbol{X}_2)\, d\boldsymbol{X}_1\, d\boldsymbol{X}_2. \qquad (53)$$

A precisely similar argument shows that

$$\int_F \sum [\boldsymbol{X}_1 \text{ in } \Omega \Lambda_0^*]\, \varrho(\boldsymbol{X}_1, \boldsymbol{X}_1)\, d\mu(\Omega) = \frac{1}{\zeta(n)} \int \varrho(\boldsymbol{X}_1, \boldsymbol{X}_1)\, d\boldsymbol{X}_1, \qquad (54)$$

$$\int_F \sum [X_1 \text{ in } \Omega \Lambda_0^*] \, \varrho(X_1, -X_1) \, d\mu(\Omega) = \frac{1}{\zeta(n)} \int \varrho(X_1, -X_1) \, dX_1. \tag{55}$$

Alternatively these results may be deduced from the result

$$\int_F \sum [X \text{ in } \Omega \Lambda_0^*] \, \varrho(X) \, d\mu(\Omega) = \frac{1}{\zeta(n)} \int \varrho(X) \, dX \tag{56}$$

proved by Siegel[1] for a Riemann integrable function, first extending the result to functions, which are not necessarily Riemann integrable, by the standard method used in the proof of Theorem 3.

Since the sums (47) and (48) are equal, the required result (12) follows from (53), (54) and (55).

8. Before we proceed to the proof of Theorem 6, it is convenient to give in this section a simple lemma connecting certain integrals taken over the space of all linear transformations and over the space of all linear transformations of determinant 1. Here it is convenient to use C to denote the cone of all linear transformations Ω, such that the transformation $\{d(\Omega)\}^{-1/n}\Omega$ of determinant 1 lies in the fundamental region F.

LEMMA 2. *Let $\varrho(\Lambda)$ be a non-negative lattice function such that the function $\varrho(\Omega\Lambda_0)$ is Borel measurable in the space of linear transformations Ω. Then for any positive number N*

$$\int \begin{bmatrix} \Omega \text{ in } C \\ 0 \le d(\Omega) \le N \end{bmatrix} \varrho(\Omega\Lambda_0) \, d\Omega$$
$$= n\varkappa^{-1}\{1 - (\tfrac{1}{2})^n\}^{-1} \int_0^N \nu^{n-1} \left\{ \int_F \varrho(\nu^{1/n}\Omega\Lambda_0) \, d\mu(\Omega) \right\} d\nu, \tag{57}$$

with the convention that both sides may have the value $+\infty$.

PROOF. We obtain the result, by using the definition for

$$\int_F \varrho(\nu^{1/n}\Omega\Lambda_0) \, d\mu(\Omega),$$

changing the order of integration in the right hand side of (57), making the substitution $\nu = \lambda d(\Omega)$, changing the order of integration, making the substitution

[1] *loc. cit.*

$\Omega = \lambda^{-1/n}\,\Omega^*$, and changing the order of integration again. It is easy to justify these operations as the function $\varrho(\Omega\Lambda_0)$ is non-negative and Borel measurable; in particular all the integrals will be properly defined, although they may have the value $+\infty$. Carrying out this programme, we have

$$\int_0^N \nu^{n-1}\left\{\int_F \varrho(\nu^{1/n}\,\Omega\Lambda_0)\,d\mu(\Omega)\right\}d\nu$$

$$= \int_0^N \left\{\nu^{n-1}\,\varkappa\int\begin{bmatrix}\Omega \text{ in } C\\ \tfrac{1}{2}\le d(\Omega)\le 1\end{bmatrix}\varrho(\nu^{1/n}\{d(\Omega)\}^{-1/n}\,\Omega\Lambda_0)\,d\Omega\right\}d\nu$$

$$= \varkappa\int\begin{bmatrix}\Omega \text{ in } C\\ \tfrac{1}{2}\le d(\Omega)\le 1\end{bmatrix}\left\{\int_0^N \nu^{n-1}\varrho(\nu^{1/n}\{d(\Omega)\}^{-1/n}\,\Omega\Lambda_0)\,d\nu\right\}d\Omega$$

$$= \varkappa\int\begin{bmatrix}\Omega \text{ in } C\\ \tfrac{1}{2}\le d(\Omega)\le 1\end{bmatrix}\left\{\int_0^{N/d(\Omega)} \lambda^{n-1}\{d(\Omega)\}^n\,\varrho(\lambda^{1/n}\,\Omega\Lambda_0)\,d\lambda\right\}d\Omega$$

$$= \varkappa\int_0^{2N}\left\{\int\begin{bmatrix}\Omega \text{ in } C\\ \tfrac{1}{2}\le d(\Omega)\le \min\{1, N/\lambda\}\end{bmatrix}\lambda^{n-1}\{d(\Omega)\}^n\,\varrho(\lambda^{1/n}\,\Omega\Lambda_0)\,d\Omega\right\}d\lambda$$

$$= \varkappa\int_0^{2N}\left\{\int\begin{bmatrix}\Omega^* \text{ in } C\\ \tfrac{1}{2}\lambda\le d(\Omega^*)\le \min\{\lambda, N\}\end{bmatrix}\lambda^{-n-1}\{d(\Omega^*)\}^n\,\varrho(\Omega^*\Lambda_0)\,d\Omega^*\right\}d\lambda$$

$$= \varkappa\int\begin{bmatrix}\Omega^* \text{ in } C\\ 0\le d(\Omega^*)\le N\end{bmatrix}\left\{\int_{d(\Omega^*)}^{2d(\Omega^*)} \lambda^{-n-1}\{d(\Omega^*)\}^n\,d\lambda\right\}\varrho(\Omega^*\Lambda_0)\,d\Omega^*$$

$$= \frac{\varkappa}{n}\{1-(\tfrac{1}{2})^n\}\int\begin{bmatrix}\Omega^* \text{ in } C\\ 0\le d(\Omega^*)\le N\end{bmatrix}\varrho(\Omega^*\Lambda_0)\,d\Omega^*.$$

This proves the lemma.

9. In this section we show how Theorem 5 can be combined with a result of Siegel (the special case of Theorem 4 when $k=1$) to give a

PROOF OF THEOREM 6. First suppose that the integral

$$\int \varrho(\mathbf{X})\,d\mathbf{X}$$

is finite. Then, by Siegel's result[1] (which is also the special case $k=1$ of Theorem 4), we have

$$\begin{aligned}\int_F \sum_{X\in\Omega\Lambda_0} \varrho(\nu^{1/n} X)\, d\mu(\Omega) &= \varrho(O) + \int \varrho(\nu^{1/n} X)\, dX \\ &= \varrho(O) + \nu^{-1}\int \varrho(X)\, dX,\end{aligned}$$

for all $\nu > 0$. It follows, by Lemma 2, that

$$\int \begin{bmatrix}\Omega \text{ in } C \\ 0 \le d(\Omega) \le N\end{bmatrix} \sum_{X\in\Omega\Lambda_0} \varrho(X)\, d\Omega = \frac{n}{\varkappa}\{1-(\tfrac{1}{2})^n\}^{-1}\left\{\frac{N^n}{n}\varrho(O) + \frac{N^{n-1}}{n-1}\int \varrho(X)\, dX\right\},$$

for each $N > 0$. So, the right hand side being finite, it follows that the sum

$$\sum_{X\in\Omega\Lambda_0} \varrho(X) \tag{58}$$

is finite for almost all Ω in C with

$$0 \le d(\Omega) \le N.$$

Since this holds for all $N > 0$, the sum will be finite for almost all Ω in C. But every Ω which is not singular is of the form $\Omega'\Psi$ where Ψ is an integral unimodular matrix and Ω' is in C. Since, then

$$\sum_{X\in\Omega\Lambda_0} \varrho(X) = \sum_{X\in\Omega'\Lambda_0} \varrho(X),$$

while there are only a countable number of integral unimodular matrices Ψ, it follows that the sum (58) is finite for almost all Ω.

Now suppose that the integral

$$\int \varrho(X)\, dX$$

has the value $+\infty$. Since $\varrho(X)$ is bounded, it is clear from homogeneity considerations, that we may suppose that

$$0 \le \varrho(X) \le 1 \tag{59}$$

for all points X. We consider a modified function $\varrho_R(X)$ defined by

$$\begin{aligned}&\varrho_R(X) = \varrho(X), \text{ if } |x_i| \le R, \quad i = 1, \dots, n,\\ &\varrho_R(X) = 0, \text{ otherwise.}\end{aligned}$$

[1] *loc. cit.*

Write

$$m_R = \frac{1}{\zeta(n)} \int \varrho_R(\mathbf{X})\, d\mathbf{X}.$$

Then,

$$m_R \to +\infty \quad \text{as} \quad R \to +\infty.$$

For each positive ν and R we consider the integral

$$\begin{aligned}
&\int_F \Big(\sum_{\mathbf{X} \in \Omega \Lambda_0^*} \varrho_R(\nu^{1/n} \mathbf{X}) - \nu^{-1} m_R \Big)^2 d\mu(\Omega) \\
&\quad = \int_F \sum_{\substack{\mathbf{X}_1 \in \Omega \Lambda_0^* \\ \mathbf{X}_2 \in \Omega \Lambda_0^*}} \varrho_R(\nu^{1/n} \mathbf{X}_1)\, \varrho_R(\nu^{1/n} \mathbf{X}_2)\, d\mu(\Omega) - \\
&\quad - 2\nu^{-1} m_R \int_F \sum_{\mathbf{X} \in \Omega \Lambda_0^*} \varrho_R(\nu^{1/n} \mathbf{X})\, d\mu(\Omega) + \\
&\quad + \nu^{-2} m_R^2.
\end{aligned}$$

Now, by the extention of Siegel's result (56), used in the proof of Theorem 5, to the case of a Borel measurable function, we have

$$\begin{aligned}
&\int_F \sum_{\mathbf{X} \in \Omega \Lambda_0^*} \varrho_R(\nu^{1/n} \mathbf{X})\, d\mu(\Omega) \\
&\quad = \frac{1}{\zeta(n)} \int \varrho_R(\nu^{1/n} \mathbf{X})\, d\mathbf{X} \\
&\quad = \nu^{-1} m_R.
\end{aligned}$$

Also, by Theorem 5,

$$\begin{aligned}
&\int_F \sum_{\substack{\mathbf{X}_1 \in \Omega \Lambda_0^* \\ \mathbf{X}_2 \in \Omega \Lambda_0^*}} \varrho_R(\nu^{1/n} \mathbf{X}_1)\, \varrho_R(\nu^{1/n} \mathbf{X}_2)\, d\mu(\Omega) \\
&\quad = \left(\frac{1}{\zeta(n)}\right)^2 \int\int \varrho_R(\nu^{1/n} \mathbf{X}_1)\, \varrho_R(\nu^{1/n} \mathbf{X}_2)\, d\mathbf{X}_1\, d\mathbf{X}_2 + \\
&\quad + \frac{1}{\zeta(n)} \int \varrho_R(\nu^{1/n} \mathbf{X})\, \varrho_R(\nu^{1/n} \mathbf{X})\, d\mathbf{X} + \\
&\quad + \frac{1}{\zeta(n)} \int \varrho_R(\nu^{1/n} \mathbf{X})\, \varrho_R(-\nu^{1/n} \mathbf{X})\, d\mathbf{X} \\
&\quad = \nu^{-2} m_R^2 + \frac{1}{\nu \zeta(n)} \int \varrho_R(\mathbf{X}) \{\varrho_R(\mathbf{X}) + \varrho_R(-\mathbf{X})\}\, d\mathbf{X}.
\end{aligned}$$

Thus, combining these results and using (59), we obtain

$$\int_F \Big(\sum_{X \in \Omega \Lambda_0^*} \varrho_R (\nu^{1/n} X) - \nu^{-1} m_R \Big)^2 d\mu(\Omega)$$

$$= \frac{1}{\nu \zeta(n)} \int \varrho_R(X) \{\varrho_R(X) + \varrho_R(-X)\} dX$$

$$\leq 2\nu^{-1} m_R.$$

Now, using Lemma 2, we have

$$\int \begin{bmatrix} \Omega \text{ in } C \\ 0 \leq d(\Omega) \leq N \end{bmatrix} \left(\sum_{X \in \Omega \Lambda_0^*} \varrho_R(X) - \frac{m_R}{d(\Omega)} \right)^2 d\Omega$$

$$= n \varkappa^{-1} \{1 - (\tfrac{1}{2})^n\}^{-1} \int_0^N \nu^{n-1} \left\{ \int_F \left(\sum_{X \in \Omega \Lambda_0^*} \varrho_R(\nu^{1/n} X) - \nu^{-1} m_R \right)^2 d\mu(\Omega) \right\} d\nu$$

$$\leq n \varkappa^{-1} \{1 - (\tfrac{1}{2})^n\}^{-1} \int_0^N 2\nu^{n-2} m_R \, d\nu$$

$$= 2n \varkappa^{-1} \{1 - (\tfrac{1}{2})^n\}^{-1} \frac{N^{n-1}}{n-1} m_R,$$

for all $N > 0$.

Now, suppose that for some positive numbers ε, N, M there is a measurable subset C_ε of the intersection of C with the set of Ω with

$$0 \leq d(\Omega) \leq N,$$

with measure ε and such that

$$\sum_{X \in \Omega \Lambda_0^*} \varrho(X) \leq M,$$

for all Ω in C_ε. Provided R is so large that

$$m_R / N > M$$

it follows that, for all Ω in C_ε,

$$\frac{m_R}{d(\Omega)} - \sum_{X \in \Omega \Lambda_0^*} \varrho_R(X) \geq \frac{m_R}{N} - M > 0,$$

so that

$$\int \begin{bmatrix} \Omega \text{ in } C \\ 0 \leq d(\Omega) \leq N \end{bmatrix} \left(\sum_{X \in \Omega \Lambda_0^*} \varrho_R(X) - \frac{m_R}{d(\Omega)} \right)^2 d\Omega$$

$$\geq \varepsilon \left(\frac{m_R}{N} - M \right)^2.$$

Thus

$$\varepsilon\left(\frac{m_R}{N} - M\right)^2 \leq 2\,n\,\varkappa^{-1}\{1-(\tfrac{1}{2})^n\}^{-1}\frac{N^{n-1}}{n-1}\,m_R.$$

But this cannot hold for large values of m_R. This contradiction shows that the set of Ω in C, such that

$$0 \leq d(\Omega) \leq N,$$

$$\sum_{\boldsymbol{X}\in\Omega\Lambda_0^*} \varrho(\boldsymbol{X}) \leq M,$$

is of measure zero for all positive numbers N, M. Consequently, the set of all Ω in C, for which the sum

$$\sum_{\boldsymbol{X}\in\Omega\Lambda_0^*} \varrho(\boldsymbol{X})$$

is finite, is of measure zero. It follows, by the argument used before, that the set of all Ω, for which this sum is finite, is of measure zero. This completes the proof.