HYPERGEOMETRIC FUNCTIONS

BY

N. E. NØRLUND

in Copenhagen

Introduction

1. In this paper we shall consider hypergeometric series of the form

$$F\left(\begin{matrix}\alpha_1\ \alpha_2 \dots \alpha_n \\ \gamma_1\ \gamma_2 \dots \gamma_{n-1}\end{matrix}\middle| z\right) = \sum_{\nu=0}^{\infty} \frac{(\alpha_1)_\nu (\alpha_2)_\nu \dots (\alpha_n)_\nu}{\nu!\,(\gamma_1)_\nu \dots (\gamma_{n-1})_\nu} z^\nu, \tag{1}$$

where

$$(\alpha)_\nu = \alpha(\alpha+1)\dots(\alpha+\nu-1), \qquad (\alpha)_0 = 1.$$

When the argument z is omitted, it will be assumed that $z=1$. If $n=2$, it is usual to write

$$F(a, b, c; z) = \sum_{\nu=0}^{\infty} \frac{(a)_\nu (b)_\nu}{\nu!\,(c)_\nu} z^\nu. \tag{2}$$

This particular hypergeometric function has played an important role in the development of analysis due to the classical works of Euler, Gauss, Riemann and Kummer. The general case, where n is an arbitrary integer >2, has first been considered by Thomae [60], who showed that the series (1) satisfy a linear differential equation of the order n, which in the vicinity of $z=0$ has a fundamental system of solutions, represented by hypergeometric series multiplied by a power of z, provided none of the differences between the numbers $0, \gamma_1, \gamma_2 \dots \gamma_{n-1}$ is an integer. Goursat [15] has shown how the definition of the hypergeometric functions by means of their analytic properties, given by Riemann in the case $n=2$, may be extended to functions of an arbitrary order. Furthermore Goursat has considered multiple integrals of the order $n-1$, which represent the hypergeometric functions of the order n, when the parameters satisfy certain conditions. Similar multiple integrals have been used by Pochhammer [54] to represent all solutions y of the hypergeometric differential equation of the order n. He shows that these may be written in the form

$$y = \int_g^h (t-z)^\alpha f(t)\, dt, \tag{3}$$

$f(t)$ being a solution of a hypergeometric differential equation of the order $n-1$. Pochhammer's investigations have been extended and simplified by Winkler [63].

An integral representation of quite another character has been given by Pincherle [48, 50], who has shown, among other things, that the integral

$$\frac{1}{2\pi i} \int_{\varkappa - i\infty}^{\varkappa + i\infty} (-z)^x \frac{\Gamma(x+\alpha_1)\Gamma(x+\alpha_2)\dots\Gamma(x+\alpha_n)\Gamma(-x)}{\Gamma(x+\gamma_1)\Gamma(x+\gamma_2)\dots\Gamma(x+\gamma_{n-1})} dx \qquad \begin{matrix} 0 > \varkappa > -\Re(\alpha_s) \\ s = 1, 2, \dots, n \end{matrix} \tag{4}$$

for $|\arg(-z)| < \pi$ apart from a constant factor represents the function defined by (1). Furthermore Mellin in several important papers [30–37] has considered a similar more general class of Laplace integrals extended over a product of gamma-functions and trigonometric functions. E. W. Barnes [3] has used such integral representations in the special case $n=2$, and he has considered [2] the general confluent hypergeometric function, too. Barnes' papers have contributed much to make these integrals familiar in analysis and they are often referred to as integrals of Barnes' type. A summary of the results on this subject and references to the literature before 1935 have been given by Bailey in *Cambridge Tracts* No. 32.

In recent years Meijer [27–29] has published an extensive investigation of integrals of the Mellin-Barnes type.

A very valuable handbook on the subject is due to Bateman and Erdélyi [11]. This work contains a full account of the results hitherto found.

On the following pages we consider particularly the solutions in the vicinity of the singular point $z=1$. If the differential equation is of the order two, the solutions, as shown by Gauss, may be represented by hypergeometric series having $1-z$ as argument. But if the order of the differential equation is higher than two this is not the case. We first investigate the solution, singular at $z=1$, and in § 2 and § 4 we give integral representations of it. In § 1 we consider its expansion in power series, multiplied by a power of $1-z$, which is of a remarkably simple form. For the solutions regular at $z=1$ the power series, on the other hand, is complicated, but in § 5 we show how these solutions can be represented by series of hypergeometric polynomials. In § 3 we have considered very simple integral representations of the solutions with a logarithmic singularity at the origin and we have given a discussion of the exceptional cases which can occur. In § 6 we consider the case where there is a logarithmic singularity at the point $z=1$ and we give new and convenient representations of the solutions in this case.

§ 1. Solutions in Power-Series by the Method of Frobenius

2. The hypergeometric differential equation of order n

$$z^n (1-z) \frac{d^n y}{d z^n} + \sum_{\nu=0}^{n-1} (a_\nu - b_\nu z) z^\nu \frac{d^\nu y}{d z^\nu} = 0 \tag{1.1}$$

has three singularities $z=0, 1$, and ∞. All of them are regular singularities. If we use the operator $\vartheta y = z \frac{d y}{d z}$, it may also be written in the form

$$(\vartheta - \gamma_1)(\vartheta - \gamma_2) \dots (\vartheta - \gamma_n) y - z(\vartheta + \alpha_1)(\vartheta + \alpha_2) \dots (\vartheta + \alpha_n) y = 0 \tag{1.2}$$

because we have

$$(\vartheta - \gamma_1)(\vartheta - \gamma_2) \dots (\vartheta - \gamma_n) y = \sum_{\nu=0}^{n} a_\nu z^\nu \frac{d^\nu y}{d z^\nu}.$$

Putting $y = z^x$ we get from this

$$(x-\gamma_1)(x-\gamma_2) \dots (x-\gamma_n) = \sum_{\nu=0}^{n} a_\nu x (x-1) \dots (x-\nu+1),$$

which shows that $\nu ! a_\nu$ is the difference of order ν of the polynomial on the left-hand side. If we put

$$Q(x) = (x-\gamma_1)(x-\gamma_2) \dots (x-\gamma_n), \tag{1.3}$$

$$R(x) = (x+\alpha_1)(x+\alpha_2) \dots (x+\alpha_n), \tag{1.4}$$

we may write (1.2) more briefly as

$$Q(\vartheta) y - z R(\vartheta) y = 0 \tag{1.5}$$

and (1.1) may be written

$$\sum_{\nu=0}^{n} [\Delta^\nu Q(0) - z \Delta^\nu R(0)] \frac{z^\nu}{\nu !} \frac{d^\nu y}{d z^\nu} = 0. \tag{1.6}$$

If we substitute $z^\varrho y$ for y, where ϱ is an arbitrary constant, the differential equation will get the form

$$Q(\vartheta + \varrho) y - z R(\vartheta + \varrho) y = 0 \tag{1.7}$$

or the equivalent form

$$\sum_{\nu=0}^{n} [\Delta^\nu Q(\varrho) - z\,\Delta^\nu R(\varrho)] \frac{z^\nu}{\nu!} \frac{d^\nu y}{dz^\nu} = 0. \tag{1.8}$$

Thus the form of the differential equation does not change by this substitution.

The differential equation (1.2) depends on $2n$ parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\gamma_1, \gamma_2, \ldots, \gamma_n$. It is symmetric in $\alpha_1, \alpha_2, \ldots, \alpha_n$ and likewise in $\gamma_1, \gamma_2, \ldots, \gamma_n$. We are able to choose ϱ so that one of the parameters assumes an arbitrary value, for instance zero, but doing so we lose the symmetry.

If we put

$$z = \frac{1}{z_1}, \qquad \vartheta_1 y = z_1 \frac{dy}{dz_1},$$

we have

$$(\vartheta + \alpha)\,y = -(\vartheta_1 - \alpha)\,y.$$

It follows from this that the differential equation (1.2) does not change, if we replace z by $1/z$ and interchange α and γ. Thus from any solution one can derive a new solution by interchanging α and γ and replacing z by $1/z$.

3. We shall now use the method of Frobenius to find a solution of the form

$$y = \sum_{\nu=0}^{\infty} g_\nu(\varrho)\, z^{\varrho+\nu}. \tag{1.9}$$

If we differentiate this series with respect to z, we get

$$Q(\vartheta)\,y = \sum_{\nu=0}^{\infty} Q(\varrho+\nu)\, g_\nu(\varrho)\, z^{\varrho+\nu},$$

$$R(\vartheta)\,y = \sum_{\nu=0}^{\infty} R(\varrho+\nu)\, g_\nu(\varrho)\, z^{\varrho+\nu}.$$

If we substitute these two series in (1.5), we see that (1.9) is a solution of the inhomogeneous equation

$$Q(\vartheta)\,y - z\,R(\vartheta)\,y = g_0(\varrho)\, Q(\varrho)\, z^\varrho, \tag{1.10}$$

if g_ν is determined by

$$Q(\varrho+\nu)\, g_\nu(\varrho) = R(\varrho+\nu-1)\, g_{\nu-1}(\varrho). \tag{1.11}$$

From this follows

$$g_\nu(\varrho) = g_0(\varrho) \prod_{s=1}^{n} \frac{(\varrho+\alpha_s)_\nu}{(\varrho-\gamma_s+1)_\nu}. \tag{1.12}$$

If we suppose that none of the differences between the numbers $\gamma_1, \gamma_2, \ldots, \gamma_n$ is zero or an integer, we get by putting $\varrho = \gamma_s$ and $g_0 = 1$ that the hypergeometric series

$$y_s(z) = z^{\gamma_s} F\begin{pmatrix} \alpha_1+\gamma_s & \alpha_2+\gamma_s & \dots & \alpha_n+\gamma_s \\ \gamma_s-\gamma_1+1 & \gamma_s-\gamma_2+1 & \dots & \gamma_s-\gamma_n+1 \end{pmatrix}\!\!\left|\, z \right) \qquad s=1, 2, \dots, n \tag{1.13}$$

satisfy the differential equation (1.2), and as $z=0$ is a regular singularity, the series is convergent for $|z|<1$. Letting s assume the values $1, 2, \dots, n$, we get n linearly independent solutions.

If we suppose that none of the differences between the numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ is zero or an integer, we see in replacing z by $1/z$ and interchanging α and γ that the differential equation (1.2) has n linearly independent solutions of the form

$$\bar{y}_s(z) = z^{-\alpha_s} F\begin{pmatrix} \alpha_s+\gamma_1 & \alpha_s+\gamma_2 & \dots & \alpha_s+\gamma_n \\ \alpha_s-\alpha_1+1 & \alpha_s-\alpha_2+1 & \dots & \alpha_s-\alpha_n+1 \end{pmatrix}\!\!\left|\, \frac{1}{z} \right) \qquad s=1, 2, \dots, n \tag{1.14}$$

and that these series converge for $|z|>1$. If we put

$$\beta_n = n-1-\sum_{i=1}^{n}(\alpha_i+\gamma_i), \tag{1.15}$$

it follows from Weierstrass's test for convergence that the series (1.13) and (1.14) are absolutely convergent on the circle $|z|=1$ if $\Re(\beta_n)>0$; they converge except at $z=1$ if $0 \geq \Re(\beta_n) > -1$, and both series diverge on $|z|=1$ if $\Re(\beta_n) \leq -1$. The solutions (1.13) and (1.14) have first been given by Thomae [60] (see also Mellin [36]).

4. If some of the differences $\gamma_s-\gamma_j$ are integers, some of the series (1.13) coincide or are without meaning. Therefore, following Frobenius, we order the numbers γ_s in groups so that each group comprises all γ_s the mutual differences of which are integers. Let $\gamma_1, \gamma_2, \dots, \gamma_q$ be such a group and $\Re(\gamma_1) \geq \Re(\gamma_2) \geq \dots \geq \Re(\gamma_q)$. Putting $\varrho=\gamma_1$ in (1.9) again we get the solution

$$y_1(z) = z^{\gamma_1} F\begin{pmatrix} \alpha_1+\gamma_1 & \alpha_2+\gamma_1 & \dots & \alpha_n+\gamma_1 \\ \gamma_1-\gamma_2+1 & \gamma_1-\gamma_3+1 & \dots & \gamma_1-\gamma_n+1 \end{pmatrix}\!\!\left|\, z \right). \tag{1.16}$$

Furthermore, when $1<s\leq q$, differentiating (1.9) with respect to ϱ and choosing $g_0(\varrho)$ conveniently, as shown by Frobenius in a more general case, we get the linearly independent solutions

$$y_s(y) = z^{\gamma_s}\sum_{\nu=0}^{\infty}\left[g_\nu^{(s-1)}(\gamma_s) + \binom{s-1}{1} g_\nu^{(s-2)}(\gamma_s)\log z + \dots + g_\nu(\gamma_s)(\log z)^{s-1}\right] z^\nu, \tag{1.17}$$

where $g_0(\gamma_s), g_0^{(1)}(\gamma_s), \dots, g_0^{(s-1)}(\gamma_s)$ do not all vanish and the series is convergent for $|z|<1$.

It may happen that the terms in (1.17) containing logarithms all vanish. For this it is necessary primarily that all γ_s are different, i.e. $\Re(\gamma_1) > \Re(\gamma_2) > \cdots > \Re(\gamma_q)$. If in (1.11) we put $\varrho = \gamma_q$, we may choose $g_{\gamma_1 - \gamma_q}$ arbitrarily, and if

$$\prod_{i=0}^{\gamma_1-\gamma_q-1} R(\gamma_q + i) \neq 0,$$

then $g_\nu = 0$ for $\nu < \gamma_1 - \gamma_q$. Thus we find again the solution (1.16) so that the solutions belonging to the exponents γ_1 and γ_q are identical. But if

$$\prod_{i=0}^{\gamma_{s-1}-\gamma_s-1} R(\gamma_s + i) = 0 \qquad s = 2, 3, \ldots, q, \tag{1.18}$$

we may in (1.11), when $\varrho = \gamma_q$, choose $g_{\gamma_1-\gamma_q}, g_{\gamma_2-\gamma_q}, \ldots, g_{\gamma_{q-1}-\gamma_q}$ and g_0 arbitrary. Then we get a solution containing q arbitrary constants, and the coefficient of each of them is a solution. This means that besides (1.16) the following by z^{γ_s} multiplied hypergeometric polynomials

$$y_s(z) = z^{\gamma_s} \sum_{\nu=0}^{\gamma_{s-1}-\gamma_s-1} z^\nu \prod_{i=1}^{n} \frac{(\alpha_i + \gamma_s)_\nu}{(\gamma_s - \gamma_i + 1)_\nu} \qquad s = 2, 3, \ldots, q \tag{1.19}$$

are solutions of (1.2). These q solutions are linearly independent.

5. The above-mentioned series in powers of z can be transformed into series in powers of $z/(z-1)$. Euler has given the following transformation:

$$\sum_{\nu=0}^{\infty} \frac{(\alpha)_\nu}{\nu!} a_\nu z^\nu = (1-z)^{-\alpha} \sum_{\nu=0}^{\infty} \frac{(\alpha)_\nu}{\nu!} \Delta^\nu a_0 \left(\frac{z}{1-z}\right)^\nu. \tag{1.20}$$

If we put

$$a_\nu = \frac{(\alpha_2)_\nu (\alpha_3)_\nu \ldots (\alpha_n)_\nu}{(\gamma_1)_\nu (\gamma_2)_\nu \ldots (\gamma_{n-1})_\nu} x^\nu,$$

we get for the difference of order ν the hypergeometric polynomial

$$\Delta^\nu a_0 = (-1)^\nu F\left(\begin{matrix} -\nu & \alpha_2 & \alpha_3 \ldots \alpha_n \\ \gamma_1 & \gamma_2 & \ldots \quad \gamma_{n-1} \end{matrix} \middle| x\right).$$

Euler's transformation then gives the following relation

$$F\left(\begin{matrix} \alpha_1 & \alpha_2 \ldots \alpha_n \\ \gamma_1 & \gamma_2 \ldots \gamma_{n-1} \end{matrix} \middle| x z\right) = (1-z)^{-\alpha_1} \sum_{\nu=0}^{\infty} \frac{(\alpha_1)_\nu}{\nu!} F\left(\begin{matrix} -\nu & \alpha_2 & \alpha_3 \ldots \alpha_n \\ \gamma_1 & \gamma_2 & \ldots \quad \gamma_{n-1} \end{matrix} \middle| x\right) \left(\frac{z}{z-1}\right)^\nu, \tag{1.21}$$

or, if we replace z by $z/(z-1)$,

$$(1-z)^{-\alpha_1} F\left(\begin{matrix}\alpha_1 & \alpha_2 \dots \alpha_n \\ \gamma_1 & \gamma_2 \dots \gamma_{n-1}\end{matrix} \middle| \frac{xz}{z-1}\right) = \sum_{\nu=0}^{\infty} \frac{(\alpha_1)_\nu}{\nu!} F\left(\begin{matrix}-\nu & \alpha_2 & \alpha_3 \dots \alpha_n \\ \gamma_1 & \gamma_2 & \dots \gamma_{n-1}\end{matrix} \middle| x\right) z^\nu. \tag{1.22}$$

This may be proved in the following way. From Weierstrass's double series theorem it follows that (1.22) is true if $|z| < 1/(1+|x|)$. The function on the left has the singularities $z=1$ and $z=1/(1-x)$. The power-series on the right-hand side is therefore convergent if $|z|<1$ and $|z(1-x)|<1$. If we suppose that $|x-1|<1$, it follows that the series on the right of (1.21) is convergent in the half-plane $\Re(z) < \frac{1}{2}$. Putting $x=1$ we see that any hypergeometric series in powers of z can be transformed into a series in powers of $z/(z-1)$ which is convergent for $\Re(z)<\frac{1}{2}$, and this in n different ways if the order of the series is n. If $n=2$, the relation (1.21) reduces to Euler's formula

$$F(a, b, c; z) = (1-z)^{-a} F\left(a, c-b, c; \frac{z}{z-1}\right) = (1-z)^{-b} F\left(c-a, b, c; \frac{z}{z-1}\right).$$

(1.22) has been given by Chaundy [6] and by Meijer [29] (cf. also [43] and [43 a]).

6. We shall now consider the singularity at $z=1$. If we put $y = z^{\gamma_1}\mathfrak{y}$, then $\mathfrak{y}$ must satisfy the differential equation

$$\sum_{\nu=1}^{n} \left[\frac{\Delta^{\nu-1} Q(\gamma_1+1)}{(\nu-1)!} - z\frac{\Delta^\nu R(\gamma_1)}{\nu!}\right] z^{\nu-1}\frac{d^\nu \mathfrak{y}}{dz^\nu} - R(\gamma_1)\mathfrak{y} = 0, \tag{1.23}$$

where $R(x) = (x+\alpha_1)(x+\alpha_2)\dots(x+\alpha_n)$ and $Q(x) = (x-\gamma_2)(x-\gamma_3)\dots(x-\gamma_n)$.

If we substitute the series

$$\mathfrak{y} = \sum_{\nu=0}^{\infty} g_\nu(\varrho)(1-z)^{\varrho+\nu} \tag{1.24}$$

in (1.23), we see that it is a solution of the inhomogeneous differential equation

$$\sum_{\nu=1}^{n} \left[\frac{\Delta^{\nu-1} Q(\gamma_1+1)}{(\nu-1)!} - z\frac{\Delta^\nu R(\gamma_1)}{\nu!}\right] z^{\nu-1}\frac{d^\nu \mathfrak{y}}{dz^\nu} - R(\gamma_1)\mathfrak{y} \\ = g_0\varrho(\varrho-1)\dots(\varrho-n+2)(\varrho-\beta_n)(1-z)^{\varrho-n+1}, \tag{1.25}$$

provided that g_ν is determined by

$$\sum_{i=1}^{n-1}(-1)^{n-i}(\varrho+\nu+1)_i\left(\frac{\Delta^{i-1}Q(\varrho+\gamma_1+\nu+1)}{(i-1)!} - \frac{\Delta^i R(\varrho+\gamma_1+\nu)}{i!}\right] g_{\nu+i}(\varrho) + \\ + (-1)^{n-1}R(\varrho+\gamma_1+\nu)g_\nu(\varrho) = 0 \tag{1.26}$$

for $\nu = 0, 1, 2, \dots$ and by the $n-2$ equations, which we get if in (1.26) we put $\nu = -1, -2, \cdots, -n+2$ and $g_{-1} = g_{-2} = \cdots = g_{-(n-2)} = 0$.

The coefficient of $g_{\nu+n-1}$ in (1.26) may be reduced as

$$\frac{\Delta^{n-1} R(\varrho+\gamma_1+\nu)}{(n-1)!} - \frac{\Delta^{n-2} Q(\varrho+\gamma_1+\nu+1)}{(n-2)!} = \varrho+\nu+n-1-\beta_n,$$

where β_n is defined by (1.15). If we put $\varrho=\beta_n$, the right-hand side of (1.25) vanishes, and if we assume that β_n is not a negative integer, we may put

$$g_\nu = \frac{c_\nu}{(\beta_n+1)_\nu}$$

in (1.26). Then we get for the determination of the numbers c_ν the following equations, which we first write for the case $n=3$ for the sake of lucidity

$$\begin{gathered} c_1+[Q(\beta_3+\gamma_1)-\Delta R(\beta_3+\gamma_1-1)]\,c_0=0 \\ 2\,c_2+[Q(\beta_3+\gamma_1+1)-\Delta R(\beta_3+\gamma_1)]\,c_1+R(\beta_3+\gamma_1)\,c_0=0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (\nu+2)\,c_{\nu+2}+[Q(\beta_3+\gamma_1+\nu+1)-\Delta R(\beta_3+\gamma_1+\nu)]\,c_{\nu+1}+R(\beta_3+\gamma_1+\nu)\,c_\nu=0, \end{gathered} \tag{1.27}$$

and for all values of n we get

$$\begin{aligned} & c_1+\left[\frac{\Delta^{n-3}Q(\beta_n+\gamma_1-n+3)}{(n-3)!}-\frac{\Delta^{n-2}R(\beta_n+\gamma_1-n+2)}{(n-2)!}\right]c_0=0 \\ & 2\,c_2+\left[\frac{\Delta^{n-3}Q[\beta_n+\gamma_1-n+4)}{(n-3)!}-\frac{\Delta^{n-2}R(\beta_n+\gamma_1-n+3)}{(n-2)!}\right]c_1- \\ & \qquad -\left[\frac{\Delta^{n-4}Q(\beta_n+\gamma_1-n+4)}{(n-4)!}-\frac{\Delta^{n-3}R(\beta_n+\gamma_1-n+3)}{(n-3)!}\right]c_0=0 \qquad (1.28) \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & (\nu+n-1)\,c_{\nu+n-1}+\sum_{i=1}^{n-2}(-1)^{n-i}\left[\frac{\Delta^{i-1}Q(\beta_n+\gamma_1+\nu+1)}{(i-1)!}-\frac{\Delta^{i}R(\beta_n+\gamma_1+\nu)}{i!}\right]c_{\nu+i}+ \\ & \qquad +(-1)^{n-1}R(\beta_n+\gamma_1+\nu)\,c_\nu=0. \end{aligned}$$

We choose $c_0=1$. These equations then determine $c_1, c_2, c_3 \ldots$ for all values of the parameters. It follows that the series

$$z^{\gamma_1}(1-z)^{\beta_n}\sum_{\nu=0}^{\infty}\frac{c_\nu}{(\beta_n+1)_\nu}(1-z)^\nu \tag{1.29}$$

formally satisfies the differential equation (1.2) if β_n is not a negative integer, and as $z=1$ is a regular singularity, (1.29) is convergent in the circle $|z-1|<1$. If we divide (1.29) by $\Gamma(\beta_n+1)$, we get

$$z^{\gamma_1} \sum_{\nu=0}^{\infty} \frac{c_\nu}{\Gamma(\beta_n + \nu + 1)} (1-z)^{\beta_n + \nu}. \tag{1.30}$$

Thus this series represents a solution for all β_n which are not negative integers. If β_n converges towards a negative integer, say $\beta_n = -p$, it converges to

$$z^{\gamma_1} \sum_{\nu=0}^{\infty} \frac{c_{\nu+p}}{\nu!} (1-z)^\nu, \tag{1.31}$$

and it may be presumed that (1.31) represents a solution of (1.2). This can be verified in the following way. If we put $\varrho = 0$, the right-hand side of (1.25) vanishes, and therefore (1.2) has a solution of the form

$$y = z^{\gamma_1} \sum_{\nu=0}^{\infty} g_\nu (1-z)^\nu. \tag{1.32}$$

If β_n is not a positive integer, then $g_0, g_1, g_2, \ldots, g_{n-2}$ are arbitrary, whereas g_ν for $\nu \geq n-1$ is determined by (1.26), where $\varrho = 0$. Supposing that β_n is a negative integer, $\beta_n = -p$, and putting $g_\nu = b_{\nu+p}/\nu!$, we get from (1.26) a system of equations for the determination of the coefficients b_ν, and if we replace ν by $\nu - p$, these equations become identical with (1.28). The coefficients b_ν are determined uniquely when $b_p, b_{p+1}, \ldots, b_{p+n-2}$ are fixed. Now we choose these so that

$$b_p = c_p, \quad b_{p+1} = c_{p+1}, \ \ldots, \ b_{p+n-2} = c_{p+n-2}.$$

We then have $b_{\nu+p} = c_{\nu+p}$ for $\nu \geq 0$ and (1.32) reduces to (1.31), which consequently is a solution of (1.2) regular in the circle $|z-1| < 1$. We shall denote the solutions defined by the series (1.29) and (1.31) with $\xi_n(z)$ and $\eta_n(z)$, respectively. From (1.28) it appears that the coefficients c_ν are integral rational functions of the parameters, symmetric in $\alpha_1, \alpha_2, \ldots, \alpha_n$ and symmetric in $\gamma_2, \gamma_3, \ldots, \gamma_n$. In what follows, instead of c_ν, we write more elaborately $c_{\nu,n}^{(1)}$ to indicate that they belong to a differential equation of the order n. Let $c_{\nu,n}^{(i)}$ denote that function, which is obtained from $c_{\nu,n}^{(1)}$ when γ_1 and γ_i are interchanged. Then we have, when β_n is not a negative integer, the solution

$$\xi_n(z) = z^{\gamma_i} (1-z)^{\beta_n} \sum_{\nu=0}^{\infty} \frac{c_{\nu,n}^{(i)}}{(\beta_n + 1)_\nu} (1-z)^\nu \quad i = 1, 2, \ldots, n, \tag{1.33}$$

where $c_{0,n}^{(i)} = 1$, and when β_n is equal to the negative integer $-p$, we have the solution

$$\eta_n(z) = z^{\gamma_i} \sum_{\nu=0}^{\infty} \frac{c_{\nu+p,n}^{(i)}}{\nu!} (1-z)^\nu \quad i = 1, 2, \ldots, n. \tag{1.34}$$

These series are convergent in the circle $|z-1|<1$, and the n series (1.33) all represent the same solution. This is evident when β_n is not integral, because there is only one solution with the first coefficient 1 belonging to the exponent β_n. Thus we have, when β_n is not integral,

$$\sum_{\nu=0}^{\infty} \frac{c_{\nu,n}^{(i)}}{(\beta_n+1)_\nu}(1-z)^\nu = z^{\gamma_j-\gamma_i} \sum_{\nu=0}^{\infty} \frac{c_{\nu,n}^{(j)}}{(\beta_n+1)_\nu}(1-z)^\nu$$

$$= \sum_{\nu=0}^{\infty} \frac{(\gamma_i-\gamma_j)_\nu}{\nu!}(1-z)^\nu \sum_{\nu=0}^{\infty} \frac{c_{\nu,n}^{(j)}}{(\beta_n+1)_\nu}(1-z)^\nu$$

$$= \sum_{\nu=0}^{\infty} (1-z)^\nu \sum_{s=0}^{\nu} \frac{(\gamma_i-\gamma_j)_{\nu-s}}{(\nu-s)!} \frac{c_{s,n}^{(j)}}{(\beta_n+1)_s}.$$

It follows from this that

$$c_{\nu,n}^{(i)} = \sum_{s=0}^{\nu} \frac{(\gamma_i-\gamma_j)_{\nu-s}}{(\nu-s)!}(\beta_n+s+1)_{\nu-s}\, c_{s,n}^{(j)}. \tag{1.35}$$

(1.35) is proved if β_n is not an integer, but as (1.35) is a relation between integral rational functions of the parameters $\alpha_1, \ldots, \alpha_n$ and $\gamma_1, \ldots, \gamma_n$, it must be valid for all values of the parameters, in particular also if β_n is an integer or zero. Hence the n series (1.33) represent the same solution $\xi_n(z)$ also if β_n is a non-negative integer. If $\beta_n = -p$, the first terms in (1.35) vanish, when $\nu \geq p$, and we get

$$c_{\nu+p,n}^{(i)} = \sum_{s=0}^{\nu} \binom{\nu}{s} (\gamma_i-\gamma_j)_{\nu-s}\, c_{s+p,n}^{(j)} \qquad \nu = 0, 1, 2, \ldots. \tag{1.36}$$

Thus we find that the n series (1.34) represent the same solution $\eta_n(z)$.

7. Let $\bar{c}_{\nu,n}^{(i)}$ denote the polynomial obtained from $c_{\nu,n}^{(i)}$ when α and γ are interchanged. If in $\xi_n(z)$ we replace z by $1/z$ and interchange α and γ, we get a solution which we denote by $\bar{\xi}_n(z)$. From (1.33) then follows

$$\bar{\xi}_n(z) = z^{-\alpha_i}\left(\frac{z-1}{z}\right)^{\beta_n} \sum_{\nu=0}^{\infty} \frac{\bar{c}_{\nu,n}^{(i)}}{(\beta_n+1)_\nu}\left(\frac{z-1}{z}\right)^\nu \qquad i = 1, 2, \ldots, n, \tag{1.37}$$

and this series is convergent in the half-plane $\Re(z) > \frac{1}{2}$. But $\bar{\xi}_n(z)$ can only differ from $\xi_n(z)$ by a constant factor $e^{\pm\pi i\beta_n}$. So we have in the half-plane $\Re(z) > \frac{1}{2}$

$$\xi_n(z) = z^{-\alpha_i}\left(\frac{1-z}{z}\right)^{\beta_n} \sum_{\nu=0}^{\infty} \frac{\bar{c}_{\nu,n}^{(i)}}{(\beta_n+1)_\nu}\left(\frac{z-1}{z}\right)^\nu \qquad i = 1, 2, \ldots, n. \tag{1.38}$$

From (1.33) and (1.38) follows

$$\sum_{\nu=0}^{\infty} \frac{c_{\nu,n}^{(j)}}{(\beta_n+1)_\nu}(1-z)^\nu = z^{-\alpha_i-\beta_n-\gamma_j} \sum_{\nu=0}^{\infty} \frac{\bar{c}_{\nu,n}^{(i)}}{(\beta_n+1)_\nu}\left(\frac{z-1}{z}\right)^\nu \qquad i, j=1, 2, \ldots, n. \tag{1.39}$$

(1.38) may also be derived from (1.33) by Euler's transformation:

$$\sum_{\nu=0}^{\infty} \frac{(\alpha)_\nu}{\nu!} a_\nu (1-z)^\nu = z^{-\alpha} \sum_{\nu=0}^{\infty} \frac{(\alpha)_\nu}{\nu!} \Delta^\nu a_0 \left(\frac{1-z}{z}\right)^\nu. \tag{1.40}$$

If we put $\alpha=\alpha_i+\beta_n+\gamma_j$, it follows from (1.39) that

$$\bar{c}_{\nu,n}^{(i)} = \sum_{s=0}^{\nu} \frac{(-1)^s}{(\nu-s)!} c_{s,n}^{(j)} (\beta_n+1+s)_{\nu-s} (\alpha_i+\beta_n+\gamma_j+s)_{\nu-s} \qquad i, j=1, 2, \ldots, n. \tag{1.41}$$

If we interchange α and γ, we get the inverse relation

$$c_{\nu,n}^{(i)} = \sum_{s=0}^{\nu} \frac{(-1)^s}{(\nu-s)!} \bar{c}_{s,n}^{(j)} (\beta_n+1+s)_{\nu-s} (\alpha_j+\beta_n+\gamma_i+s)_{\nu-s} \qquad i, j=1, 2, \ldots, n. \tag{1.42}$$

If $\beta_n=-p$, (1.41) reduces to

$$\bar{c}_{\nu+p,n}^{(i)} = (-1)^p (\alpha_i+\gamma_j)_\nu \sum_{s=0}^{\nu} (-1)^s \binom{\nu}{s} \frac{c_{s+p,n}^{(j)}}{(\alpha_i+\gamma_j)_s} \qquad i, j=1, 2, \ldots, n. \tag{1.43}$$

If we apply Euler's transformation to (1.34), it follows from (1.43) that $\eta_n(z)$ also may be represented by the series

$$\eta_n(z) = (-1)^p z^{-\alpha_i} \sum_{\nu=0}^{\infty} \frac{\bar{c}_{\nu+p,n}^{(i)}}{\nu!}\left(\frac{z-1}{z}\right)^\nu \qquad i=1, 2, \ldots, n, \tag{1.44}$$

which are convergent in the half-plane $\Re(z) > \frac{1}{2}$.

Later we shall see that $\eta_n(z)$ can be identically zero, but only in a very special case. This can never happen to $\xi_n(z)$, because the first coefficient in the series (1.33) is 1.

8. It is readily seen how $c_{\nu,n}^{(i)}$ behave asymptotically for large positive values of ν. If we first suppose that no two of the γ_i differ by an integer, then $\xi_n(z)$ and $\eta_n(z)$ are linear functions with constant coefficients of the hypergeometric series $y_1, y_2, \ldots, y_n$. As $z=0$ is the only singularity situated on the periphery of the circle of convergence for (1.33), it follows[1] that

$$\frac{c_{\nu,n}^{(i)}}{\Gamma(\beta_n+\nu+1)} = \sum_{\substack{s=1 \\ s\neq i}}^{n} [k_s \nu^{\gamma_i-\gamma_s-1} + O(|\nu^{\gamma_i-\gamma_s-2}|)], \tag{1.45}$$

[1] See [39] pp. 21–22 and [47] p. 7.

where the k_s are constants. From (3.44) we see that

$$k_s = \frac{\prod_{r=1}^{n}{}' \Gamma(\gamma_r - \gamma_s)}{\prod_{r=1}^{n} \Gamma(1 - \alpha_r - \gamma_s)}, \tag{1.46}$$

where the dash means that in the product $r = s$ and $r = i$ are to be omitted. Interchanging α and γ in (1.45) we get, if none of the differences between the parameters α_i is an integer,

$$\frac{\bar{c}_{\nu, n}^{(i)}}{\Gamma(\beta_n + \nu + 1)} = \sum_{\substack{s=1 \\ s \neq i}}^{n} [\bar{k}_s \nu^{\alpha_i - \alpha_s - 1} + O(|\nu^{\alpha_i - \alpha_s - 2}|)], \tag{1.47}$$

where

$$\bar{k}_s = \frac{\prod_{r=1}^{n}{}' \Gamma(\alpha_r - \alpha_s)}{\prod_{r=1}^{n} \Gamma(1 - \alpha_s - \gamma_r)}. \tag{1.48}$$

If any of the differences between the parameters γ_i is an integer or zero, it follows in the same way from a theorem by Perron [46] p. 368 that for large positive ν we have

$$\frac{c_{\nu, n}^{(i)}}{\Gamma(\beta_n + \nu + 1)} \sim \sum_{s=1}^{n} K_s \nu^{\gamma_i - \gamma_s - 1} (\log \nu)^{r_s}, \tag{1.49}$$

where r_s are non-negative integers and K_s are functions of the parameters independent of ν. These functions are of a more complicated form than (1.46). If any of the differences between the α_i is an integer or zero, we have in the same way

$$\frac{\bar{c}_{\nu, n}^{(i)}}{\Gamma(\beta_n + \nu + 1)} \sim \sum_{s=1}^{n} \bar{K}_s \nu^{\alpha_i - \alpha_s - 1} (\log \nu)^{r_s}. \tag{1.50}$$

These asymptotic expressions are valid for all β_n, also in particular if β_n is a negative integer.

§ 2. The Solution $\xi_n(z)$

9. Let us consider the integral

$$\Phi(z) = \int_{1}^{\frac{1}{z}} t^{\alpha_n - 1} (t - 1)^{-\alpha_n - \gamma_n} \xi_{n-1}(zt) \, dt. \tag{2.1}$$

If $0 < z < 1$, we suppose that $\arg t$ and $\arg (t - 1)$ are zero. For simplicity we assume

$$\Re(\alpha_n + \gamma_n) < 0 \quad \text{and} \quad \Re(\beta_{n-1}) > n - 1. \tag{2.2}$$

Then the integral (2.1) converges absolutely. Differentiating with respect to z, we get

$$\Phi'(z) = \int_1^{\frac{1}{z}} t^{\alpha_n} (t-1)^{-\alpha_n-\gamma_n} \xi'_{n-1}(zt)\, dt.$$

Integration by parts gives

$$\vartheta \Phi(z) = \int_1^{\frac{1}{z}} t^{\alpha_n} (t-1)^{-\alpha_n-\gamma_n} \frac{d\,\xi_{n-1}(zt)}{dt} dt$$

$$= [t^{\alpha_n}(t-1)^{-\alpha_n-\gamma_n} \xi_{n-1}(zt)]_1^{\frac{1}{z}} + \int_1^{\frac{1}{z}} t^{\alpha_n-1}(t-1)^{-\alpha_n-\gamma_n-1}(\alpha_n+\gamma_n t)\, \xi_{n-1}(zt)\, dt.$$

The first term on the right-hand side vanishes in both limits, and so we have

$$(\vartheta+\alpha_n)\,\Phi(z) = (\alpha_n+\gamma_n)\int_1^{\frac{1}{z}} t^{\alpha_n}(t-1)^{-\alpha_n-\gamma_n-1}\xi_{n-1}(zt)\, dt$$

$$(\vartheta-\gamma_n)\,\Phi(z) = (\alpha_n+\gamma_n)\int_1^{\frac{1}{z}} t^{\alpha_n-1}(t-1)^{-\alpha_n-\gamma_n-1}\xi_{n-1}(zt)\, dt.$$

From this we get by continued differentiation

$$(\vartheta-\gamma_1)(\vartheta-\gamma_2)\dots(\vartheta-\gamma_n)\,\Phi(z) - z(\vartheta+\alpha_1)(\vartheta+\alpha_2)\dots(\vartheta+\alpha_n)\,\Phi(z)$$

$$= (\alpha_n+\gamma_n)\int_1^{\frac{1}{z}} t^{\alpha_n-1}(t-1)^{-\alpha_n-\gamma_n-1}[(\vartheta-\gamma_1)\dots(\vartheta-\gamma_{n-1})\,\xi_{n-1}(zt) -$$

$$- zt(\vartheta+\alpha_1)\dots(\vartheta+\alpha_{n-1})\,\xi_{n-1}(zt)]\, dt.$$

If in (1.2) we substitute $n-1$ for n, then $\xi_{n-1}(z)$ satisfies the resulting equation and so the squared bracket under the sign of the integration vanishes. $\Phi(z)$ therefore must be a solution of (1.2). Substituting t/z for t we get

$$\Phi(z) = z^{\gamma_n}\int_z^1 t^{\alpha_n-1}(t-z)^{-\alpha_n-\gamma_n}\,\xi_{n-1}(t)\, dt. \tag{2.3}$$

We have now

$$t^{-\gamma_i}\,\xi_{n-1}(t) = \sum_{\nu=0}^{\infty} \frac{c_{\nu,\,n-1}^{(i)}}{(\beta_{n-1}+1)_\nu}(1-t)^{\beta_{n-1}+\nu} \tag{2.4}$$

$$t^{\alpha_n+\gamma_i-1} = \sum_{\nu=0}^{\infty} \frac{(1-\alpha_n-\gamma_i)_\nu}{\nu!}(1-t)^\nu. \tag{2.5}$$

The product of these two series is given by

$$t^{\alpha_n - 1}\,\xi_{n-1}(t) = \sum_{\nu=0}^{\infty} (1-t)^{\beta_{n-1}+\nu} \sum_{s=0}^{\nu} \frac{(1-\alpha_n-\gamma_i)_{\nu-s}}{(\nu-s)!} \frac{c_{s,n-1}^{(i)}}{(\beta_{n-1}+1)_s}. \tag{2.6}$$

Furthermore

$$z^{\gamma_n}\int_z^1 (t-z)^{-\alpha_n-\gamma_n}(1-t)^{\beta_{n-1}+\nu}\,dt = z^{\gamma_n}(1-z)^{\beta_n+\nu}\int_0^1 (1-t)^{-\alpha_n-\gamma_n} t^{\beta_{n-1}+\nu}\,dt$$

$$= z^{\gamma_n}(1-z)^{\beta_n+\nu}\frac{\Gamma(1-\alpha_n-\gamma_n)\,\Gamma(\beta_{n-1}+\nu+1)}{\Gamma(\beta_n+\nu+1)}.$$

We now assume that $|z-1|<1$, insert (2.6) in (2.3), and integrate term-by-term, which is evidently justified. Then we get

$$\Phi(z) = z^{\gamma_n}\,\Gamma(1-\alpha_n-\gamma_n)\sum_{\nu=0}^{\infty}(1-z)^{\beta_n+\nu}\frac{\Gamma(\beta_{n-1}+\nu+1)}{\Gamma(\beta_n+\nu+1)}\sum_{s=0}^{\nu}\frac{(1-\alpha_n-\gamma_i)_{\nu-s}}{(\nu-s)!}\frac{c_{s,n-1}^{(i)}}{(\beta_{n-1}+1)_s}.$$

From this follows $\Phi(z) = A\,\xi_n(z)$, where A is a constant, and as $c_{0,n}^{(n)} = 1$, we have

$$A = \frac{\Gamma(1-\alpha_n-\gamma_n)\,\Gamma(\beta_{n-1}+1)}{\Gamma(\beta_n+1)}$$

and

$$c_{\nu,n}^{(n)} = \sum_{s=0}^{\nu}\frac{(1-\alpha_n-\gamma_i)_{\nu-s}}{(\nu-s)!}(\beta_{n-1}+s+1)_{\nu-s}\,c_{s,n-1}^{(i)} \qquad i=1,2,\ldots,n-1. \tag{2.7}$$

Therefore equation (2.3) may be written

$$\xi_n(z) = \frac{\Gamma(\beta_n+1)\,z^{\gamma_n}}{\Gamma(1-\alpha_n-\gamma_n)\,\Gamma(\beta_{n-1}+1)}\int_z^1 t^{\alpha_n-1}(t-z)^{-\alpha_n-\gamma_n}\,\xi_{n-1}(t)\,dt. \tag{2.8}$$

The last integral converges if

$$\Re(\beta_n) > \Re(\beta_{n-1}) > -1. \tag{2.9}$$

By analytic continuation it is seen that the relation (2.8) holds when the condition (2.9) is satisfied. The recurrence formula (2.7) is a relation between polynomials, and so it is valid for any values of the parameters. It is very convenient for successive calculation of $c_{\nu,n}^{(n)}$. As $\xi_1(z) = z^{\gamma_1}(1-z)^{-\alpha_1-\gamma_1}$, we have $c_{\nu,1}^{(1)} = 0$ when $\nu>0$, and we get from (2.7)

$$c_{\nu,2}^{(2)} = \frac{(1-\alpha_1-\gamma_1)_\nu\,(1-\alpha_2-\gamma_1)_\nu}{\nu!} \qquad c_{\nu,2}^{(1)} = \frac{(1-\alpha_1-\gamma_2)_\nu\,(1-\alpha_2-\gamma_2)_\nu}{\nu!} \tag{2.10}$$

$$c_{\nu,3}^{(3)} = \frac{(\alpha_1+\beta_3+\gamma_3)_\nu\,(\alpha_2+\beta_3+\gamma_3)_\nu}{\nu!}\,F\begin{pmatrix}-\nu & 1-\alpha_3-\gamma_1 & 1-\alpha_3-\gamma_2\\ \alpha_1+\beta_3+\gamma_3 & \alpha_2+\beta_3+\gamma_3 & \end{pmatrix}$$

$$= \frac{(\alpha_1+\beta_3+\gamma_3)_\nu\,(\alpha_3+\beta_3+\gamma_3)_\nu}{\nu!} F\begin{pmatrix} -\nu & 1-\alpha_2-\gamma_1 & 1-\alpha_2-\gamma_2 \\ \alpha_1+\beta_3+\gamma_3 & \alpha_3+\beta_3+\gamma_3 & \end{pmatrix}$$

$$= \frac{(\alpha_2+\beta_3+\gamma_3)_\nu\,(\alpha_3+\beta_3+\gamma_3)_\nu}{\nu!} F\begin{pmatrix} -\nu & 1-\alpha_1-\gamma_1 & 1-\alpha_1-\gamma_2 \\ \alpha_2+\beta_3+\gamma_3 & \alpha_3+\beta_3+\gamma_3 & \end{pmatrix}.$$

By induction it is shown that for all positive n

$$c_{\nu,n}^{(n)} = \sum_{s_{n-2}=0}^{\nu} \cdots \sum_{s_2=0}^{s_3} \sum_{s_1=0}^{s_2} \frac{(\beta_1+1)_{s_1}(\beta_2+s_1+1)_{s_2-s_1}\cdots(\beta_{n-1}+s_{n-2}+1)_{\nu-s_{n-2}}}{s_1!\,(s_2-s_1)!\,(s_3-s_2)!\dots(\nu-s_{n-2})!} H, \qquad (2.11)$$

where

$$H = (1-\alpha_2-\gamma_1)_{s_1}(1-\alpha_3-\gamma_2)_{s_2-s_1}(1-\alpha_4-\gamma_3)_{s_3-s_2}\cdots(1-\alpha_n-\gamma_{n-1})_{\nu-s_{n-2}}.$$

10. If β_{n-1} is a negative integer, say $\beta_{n-1}=-p$, the first terms in (2.7) vanish, and we get

$$c_{\nu+p,n}^{(n)} = \sum_{s=0}^{\nu} \binom{\nu}{s} (1-\alpha_n-\gamma_i)_{\nu-s}\, c_{s+p,n-1}^{(i)} \qquad i=1,2,\dots,n-1. \qquad (2.12)$$

If $n>2$, we have

$$t^{-\gamma_i}\,\eta_{n-1}(t) = \sum_{\nu=0}^{\infty} \frac{c_{\nu+p,n-1}^{(i)}}{\nu!}(1-t)^\nu.$$

Multiplying this series by (2.5) and using (2.12), we get

$$t^{\alpha_n-1}\,\eta_{n-1}(t) = \sum_{\nu=0}^{\infty} \frac{c_{\nu+p,n}^{(n)}}{\nu!}(1-t)^\nu. \qquad (2.13)$$

We now assume that $\Re(\alpha_n+\gamma_n)<1$ and consider the integral

$$\int_z^1 t^{\alpha_n-1}(t-z)^{-\alpha_n-\gamma_n}\,\eta_{n-1}(t)\,dt. \qquad (2.14)$$

If in this we substitute the series (2.13) and assume that $|z-1|<1$, we may integrate term-by-term, and we find that the integral (2.14) equals

$$\sum_{\nu=0}^{\infty} \frac{c_{\nu+p,n}^{(n)}}{(1-\alpha_n-\gamma_n)_\nu}(1-z)^{1-\alpha_n-\gamma_n+\nu} = (\beta_n+1)_p \sum_{\nu=0}^{\infty} \frac{c_{\nu+p,n}^{(n)}}{(\beta_n+1)_{\nu+p}}(1-z)^{\beta_n+p+\nu}.$$

From this it follows that if $\beta_{n-1}=-p$ but β_n is not a negative integer and $\Re(\alpha_n+\gamma_n)<1$, (2.8) must be replaced by

$$\xi_n(z) = z^{\gamma_n}(1-z)^{\beta_n} \sum_{\nu=0}^{p-1} \frac{c_{\nu,n}^{(n)}}{(\beta_n+1)_\nu}(1-z)^\nu + \frac{\Gamma(\beta_n+1)\,z^{\gamma_n}}{\Gamma(1-\alpha_n-\gamma_n)} \int_z^1 t^{\alpha_n-1}(t-z)^{-\alpha_n-\gamma_n}\,\eta_{n-1}(t)\,dt. \qquad (2.15)$$

11. Let us now consider the integral $\int_0^1 z^{x-1}\xi_n(z)\,dz$. As $\xi_n(z)$ is a linear function of the solutions $y_1, y_2, \dots, y_n$ considered in § 1, the integral converges if $\Re(\beta_n) > -1$ and $\Re(x+\gamma_s)>0$, $s=1,2,\dots,n$. If $n=1$, it reduces to the Eulerian integral

$$\int_0^1 z^{x+\gamma_1-1}(1-z)^{-\alpha_1-\gamma_1}\,dz = \frac{\Gamma(x+\gamma_1)\,\Gamma(1-\alpha_1-\gamma_1)}{\Gamma(x-\alpha_1+1)} \qquad \Re(x+\gamma_1)>0, \quad \Re(\alpha_1+\gamma_1)<1. \tag{2.16}$$

Assuming that $\Re(\beta_{n-1}) > -1$, $\Re(\alpha_n+\gamma_n)<1$, and $\Re(x+\gamma_s)>0$, $s=1,2,\dots,n$, we get from (2.8)

$$\int_0^1 z^{x-1}\xi_n(z)\,dz = \frac{\Gamma(\beta_n+1)}{\Gamma(\beta_{n-1}+1)\,\Gamma(1-\alpha_n-\gamma_n)}\int_0^1 z^{x+\gamma_n-1}\int_z^1 t^{\alpha_n-1}(t-z)^{-\alpha_n-\gamma_n}\xi_{n-1}(t)\,dt\,dz.$$

From a theorem of W. A. Hurwitz[1] it follows that we may interchange the order of integration on the right-hand side. This integral then equals

$$\frac{\Gamma(\beta_n+1)}{\Gamma(\beta_{n-1}+1)\,\Gamma(1-\alpha_n-\gamma_n)}\int_0^1 t^{\alpha_n-1}\xi_{n-1}(t)\int_0^t z^{x+\gamma_n-1}(t-z)^{-\alpha_n-\gamma_n}\,dz\,dt$$

$$= \frac{\Gamma(\beta_n+1)}{\Gamma(\beta_{n-1}+1)\,\Gamma(1-\alpha_n-\gamma_n)}\int_0^1 t^{x-1}\xi_{n-1}(t)\,dt\int_0^1 z^{x+\gamma_n-1}(1-z)^{-\alpha_n-\gamma_n}\,dz.$$

The last integral on the right-hand side is an Eulerian integral and so this equation reduces to

$$\int_0^1 z^{x-1}\xi_n(z)\,dz = \frac{\Gamma(\beta_n+1)\,\Gamma(x+\gamma_n)}{\Gamma(\beta_{n-1}+1)\,\Gamma(x-\alpha_n+1)}\int_0^1 z^{x-1}\xi_{n-1}(z)\,dz. \tag{2.17}$$

Assuming that $\Re(\beta_s) > -1$ for $s=1,2\dots,n$, we get from (2.17) and (2.16)

$$\int_0^1 z^{x-1}\xi_n(z)\,dz = \Gamma(\beta_n+1)\prod_{s=1}^{n}\frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)}. \tag{2.18}$$

As mentioned above, the integral on the left converges if $\Re(\beta_n) > -1$ and $\Re(x+\gamma_s)>0$, $s=1,2,\dots,n$. Analytic continuation shows that this is also the condition of the validity of (2.18). Hj. Mellin ([30] p. 147 and [32] pp. 83–85) has given a similar formula, leaving a constant factor undetermined.

[1] *Annals of Mathematics* (2), 9 (1908), 183–192.

If in (2.18) we interchange α and γ and replace z by $1/z$ and x by $-x$, we get

$$\int_1^\infty z^{x-1}\bar{\xi}_n(z)\,dz = \Gamma(\beta_n+1)\prod_{s=1}^{n}\frac{\Gamma(\alpha_s-x)}{\Gamma(1-\gamma_s-x)}, \tag{2.19}$$

provided that $\Re(\beta_n) > -1$ and $\Re(x) < \Re(\alpha_s)$, $s=1, 2, \ldots, n$. Naturally this relation may also be derived from the equation

$$\bar{\xi}_n(z) = \frac{\Gamma(\beta_n+1)\,z^{\gamma_n}}{\Gamma(1-\alpha_n-\gamma_n)\,\Gamma(\beta_{n-1}+1)}\int_1^z t^{\alpha_n-1}(z-t)^{-\alpha_n-\gamma_n}\,\bar{\xi}_{n-1}(t)\,dt \tag{2.20}$$

by a similar way of reasoning as used above.

12. If the series (1.33) is substituted in (2.18) and term-by-term integration performed, the justification for which is easily seen,[1] we get

$$\sum_{\nu=0}^{\infty}\frac{c_{\nu,n}^{(i)}}{(x+\gamma_i+\beta_n+1)_\nu} = \frac{\Gamma(x+\gamma_i+\beta_n+1)}{\Gamma(x+\gamma_i)}\prod_{s=1}^{n}\frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)} \qquad i=1, 2, \ldots, n. \tag{2.21}$$

From (1.49) it is seen that the series on the left-hand side converges absolutely when $\Re(x+\gamma_s) > 0$, $s=1, 2, \ldots, n$. Consequently the polynomials $c_{\nu,n}^{(i)}$ are the coefficients in the factorial series expansion of the fraction on the right-hand side, the nominator and denominator of which are products of gamma-functions. Interchange of α and γ together with substitution of x by $-x$ gives

$$\sum_{\nu=0}^{\infty}\frac{\bar{c}_{\nu,n}^{(i)}}{(\alpha_i+\beta_n+1-x)_\nu} = \frac{\Gamma(\alpha_i+\beta_n+1-x)}{\Gamma(\alpha_i-x)}\prod_{s=1}^{n}\frac{\Gamma(\alpha_s-x)}{\Gamma(1-\gamma_s-x)} \qquad i=1, 2, \ldots, n, \tag{2.22}$$

where using (1.50) we see that the factorial series on the left-hand side converges when $\Re(\alpha_s-x) > 0$, $s=1, 2, \ldots, n$. From this the points $x=\alpha_i+\beta_n+1$, $\alpha_i+\beta_n+2, \ldots$ must be excluded when they are situated in the half-plane of convergence. The same remarks apply to (2.21). The relation (2.22) naturally may also be derived from (2.19) by substitution of (1.37) for $\bar{\xi}_n$ and integration term-by-term. In the proof for (2.21) and (2.22) we have supposed that $\Re(\beta_n) > -1$, but by analytic continuation it is seen that the relations are valid for all values of β_n.

Taking (2.21) as a definition of the $c_{\nu,n}^{(i)}$ it would also be possible to prove (2.7) by the rule for multiplication of two factorial series, [41] p. 382. This proof is a little longer than the proof given above, but it is valid without any restriction upon the parameters.

[1] Bromwich [5], p. 497.

If none of the differences between the numbers γ_i are integers, it follows from (1.45) that the series on the left-hand side of (2.21) converges provided $\Re(x+\gamma_s)>0$, $s=1, 2, \ldots, i-1, i+1, \ldots, n$. If none of the differences between the numbers α_i are integers we see from (1.47) that the series on the left-hand side of (2.22) converges when $\Re(x-\alpha_s)<0$, $s=1, 2, \ldots, i-1, i+1, \ldots, n$.

Assuming that $\Re(\gamma_s-\gamma_i)>0$, $s=1, 2, \ldots, i-1, i+1, \ldots, n$ and putting $x=-\gamma_i$ in (2.21), we get

$$\sum_{\nu=0}^{\infty} \frac{c_{\nu,n}^{(i)}}{(\beta_n+1)_\nu} = \Gamma(\beta_n+1) \frac{\prod_{s=1}^{n}{}' \Gamma(\gamma_s-\gamma_i)}{\prod_{s=1}^{n} \Gamma(1-\alpha_s-\gamma_i)}, \tag{2.23}$$

provided that β_n is not a negative integer, and, if $\beta_n=-p$,

$$\sum_{\nu=0}^{\infty} \frac{c_{\nu+p,n}^{(i)}}{\nu!} = \frac{\prod_{s=1}^{n}{}' \Gamma(\gamma_s-\gamma_i)}{\prod_{s=1}^{n} \Gamma(1-\alpha_s-\gamma_i)}, \tag{2.24}$$

where the dash signifies that $s=i$ is to be omitted in the product.

If $n=2$, (2.23) with modified notation reduces to Gauss's theorem

$$F(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad \Re(c-a-b)>0.$$

Thus if $n=2$, the sum of the coefficients in the expansion of the hypergeometric function in powers of z may be expressed by gamma-functions, in case the parameters satisfy a certain inequality, which involves the convergence of the series. But we may equally well say that the sum of the coefficients in the expansion of this function in powers of $1-z$ may be expressed by gamma-functions, because this series is a hypergeometric one when $n=2$. So far (2.23) may be regarded as a natural extension of this important theorem.

13. In (2.18) we have supposed that $\Re(\beta_n)>-1$. But if we relate the above integral with a contour integral, this supposition will be unnecessary, and we get

$$\frac{1}{2\pi i}\int_0^{(1+)} z^{x-1}\xi_n(z)\,dz = \frac{1}{\Gamma(-\beta_n)} \prod_{s=1}^{n} \frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)} \qquad \Re(x+\gamma_s)>0 \quad s=1, 2, \ldots, n, \tag{2.25}$$

when β_n is not an integer or zero. In the same way (2.19) may be written in the form

$$\frac{1}{2\pi i}\int_{\varkappa-i\infty}^{\varkappa+i\infty} z^{x-1}\xi_n(z)\,dz = \frac{1}{\Gamma(-\beta_n)}\prod_{s=1}^{n}\frac{\Gamma(\alpha_s-x)}{\Gamma(1-\gamma_s-x)} \qquad 0<\varkappa<1 \tag{2.26}$$

if $\Re(x)<\Re(\alpha_s)$, $s=1,2,\ldots,n$ and β_n is not an integer or zero.

14. Let us now consider the integral $\int_0^1 z^{x-1}\eta_n(z)\,dz$. Inserting the series (1.34) and integrating term-by-term, which is evidently justified, we get

$$\int_0^1 z^{x-1}\eta_n(z)\,dz = \sum_{\nu=0}^{\infty} c_{\nu+p,n}^{(i)}\frac{\Gamma(x+\gamma_i)}{\Gamma(x+\gamma_i+\nu+1)} = \sum_{\nu=p}^{\infty} c_{\nu,n}^{(i)}\frac{\Gamma(x+\gamma_i)}{\Gamma(x+\gamma_i+\nu-p+1)}, \tag{2.27}$$

where the integral as well as the series converge for $\Re(x+\gamma_s)>0$, $s=1,2,\ldots,n$. From this, putting $\beta_n=-p$ in (2.21), we get

$$\int_0^1 z^{x-1}\eta_n(z)\,dz = \prod_{s=1}^{n}\frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)} - q(x), \tag{2.28}$$

$q(x)$ being a polynomial in x of the degree $p-1$

$$q(x) = \sum_{\nu=0}^{p-1} c_{\nu,n}^{(i)}\frac{\Gamma(x+\gamma_i)}{\Gamma(x+\gamma_i+\nu-p+1)} = \sum_{\nu=0}^{p-1} c_{p-1-\nu,n}^{(i)}(x+\gamma_i-\nu)_\nu \qquad i=1,2,\ldots,n. \tag{2.29}$$

From (1.44), integrating term-by-term, we get in the same way

$$\int_1^\infty z^{x-1}\eta_n(z)\,dz = (-1)^p\sum_{\nu=p}^{\infty}\bar{c}_{\nu,n}^{(i)}\frac{\Gamma(\alpha_i-x)}{\Gamma(\alpha_i-x+\nu-p+1)} \tag{2.30}$$

where both the integral and the series converge if $\Re(x-\alpha_s)<0$, $s=1,2,\ldots,n$. Putting $\beta_n=-p$ in (2.22) we see that

$$(-1)^p\int_1^\infty z^{x-1}\eta_n(z)\,dz = \prod_{s=1}^{n}\frac{\Gamma(\alpha_s-x)}{\Gamma(1-\gamma_s-x)} - \bar{q}(x), \tag{2.31}$$

where $\bar{q}(x)$ is the following polynomial in x of the degree $p-1$

$$\bar{q}(x) = \sum_{\nu=0}^{p-1}\bar{c}_{\nu,n}^{(i)}\frac{\Gamma(\alpha_i-x)}{\Gamma(\alpha_i-x+\nu-p+1)} = \sum_{\nu=0}^{p-1}\bar{c}_{p-1-\nu,n}^{(i)}(\alpha_i-x-\nu)_\nu \qquad i=1,2,\ldots,n. \tag{2.32}$$

Let us now assume that $\Re(\alpha_i+\gamma_j)>0$, $i,j=1,2,\ldots,n$. The integral $\int_0^\infty z^{x-1}\eta_n(z)\,dz$ then converges in the strip

$$-\Re(\gamma_j)<\Re(x)<\Re(\alpha_i) \qquad i,j=1,2,\ldots,n. \tag{2.33}$$

Adding (2.28) and (2.31) we get

$$\int_0^\infty z^{x-1}\eta_n(z)\,dz=\prod_{s=1}^n\frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)}+(-1)^p\prod_{s=1}^n\frac{\Gamma(\alpha_s-x)}{\Gamma(1-\gamma_s-x)}-q(x)-(-1)^p\bar{q}(x). \quad (2.34)$$

But this formula may be reduced. From (2.21) and (2.22) we get for $\beta_n=-p$

$$\begin{aligned}\prod_{s=1}^n\frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)}+(-1)^p\prod_{s=1}^n\frac{\Gamma(\alpha_s-x)}{\Gamma(1-\gamma_s-x)}&\\ =q(x)+(-1)^p\bar{q}(x)+\sum_{\nu=0}^\infty\frac{c^{(i)}_{\nu+p,n}}{(x+\gamma_i)_{\nu+1}}+(-1)^p\sum_{\nu=0}^\infty\frac{\bar{c}^{(i)}_{\nu+p,n}}{(\alpha_i-x)_{\nu+1}}&.\end{aligned} \quad (2.35)$$

The left-hand side of (2.35) equals

$$g(x)\frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)},$$

where

$$g(x)=1+(-1)^p\prod_{s=1}^n\frac{\sin\pi(\gamma_s+x)}{\sin\pi(\alpha_s-x)}.$$

Putting $x=\sigma+i\tau$ we have

$$\lim_{\tau\to\pm\infty}\prod_{s=1}^n\frac{\sin\pi(x+\gamma_s)}{\sin\pi(x-\alpha_s)}=e^{\mp\pi i\sum_{s=1}^n(\alpha_s+\gamma_s)}=(-1)^{n-1+p}. \quad (2.36)$$

Consequently

$$\lim_{\tau\to\pm\infty}g(x)=0.$$

But as $g(x)$ is a rational function of $e^{2\pi i x}$,

$$g(x)=O(e^{-2\pi|\tau|}). \quad (2.37)$$

The left-hand side of (2.35) is a meromorphic function of x, which is regular in the strip (2.33). In this strip it tends towards zero when $\tau\to\pm\infty$. The two factorial series on the right-hand side of (2.35) both converge in the strip (2.33) and they tend there uniformly towards zero when $\tau\to\pm\infty$. From this it follows that the polynomials $q(x)$ and $\bar{q}(x)$ satisfy the relation

$$q(x)+(-1)^p\bar{q}(x)=0 \quad (2.38)$$

and so (2.34) reduces to

$$\int_0^\infty z^{x-1}\eta_n(z)\,dz=\prod_{s=1}^n\frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)}+(-1)^p\prod_{s=1}^n\frac{\Gamma(\alpha_s-x)}{\Gamma(1-\gamma_s-x)}. \quad (2.39)$$

From this relation we get, using Mellin's theorem of inversion,

$$\eta_n(z)=\frac{1}{2\pi i}\int_{\varkappa-i\infty}^{\varkappa+i\infty} z^{-x}\left[\prod_{s=1}^{n}\frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)}+(-1)^p\prod_{s=1}^{n}\frac{\Gamma(\alpha_s-x)}{\Gamma(1-\gamma_s-x)}\right]dx, \tag{2.40}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ must be to the right and $-\gamma_1, -\gamma_2, \ldots, -\gamma_n$ to the left of the contour of integration. Consequently $\varkappa$ is situated in the strip (2.33). From (2.37) it follows that the integral (2.40) converges for $2\pi > \arg z > -2\pi$. Putting $z=1$ in (2.40), we get

$$c_{p,n}^{(j)}=\frac{1}{2\pi i}\int_{\varkappa-i\infty}^{\varkappa+i\infty}\left[\prod_{s=1}^{n}\frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)}+(-1)^p\prod_{s=1}^{n}\frac{\Gamma(\alpha_s-x)}{\Gamma(1-\gamma_s-x)}\right]dx \qquad j=1,2,\ldots,n. \tag{2.41}$$

From this it is seen that $c_{p,n}^{(j)}$ is symmetric in $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\gamma_1, \gamma_2, \ldots, \gamma_n$. Multiplying both sides of (2.40) by $z^{-\gamma_1}$ and differentiating ν times with respect to z and then putting $z=1$, we get

$$c_{\nu+p,n}^{(1)}=\frac{1}{2\pi i}\int_{\varkappa-i\infty}^{\varkappa+i\infty}\left[\frac{\Gamma(x+\gamma_1+\nu)\,\Gamma(x+\gamma_2)\ldots\Gamma(x+\gamma_n)}{\Gamma(x-\alpha_1+1)\,\Gamma(x-\alpha_2+1)\ldots\Gamma(x-\alpha_n+1)}+\right.$$
$$\left.+(-1)^{p+\nu}\frac{\Gamma(\alpha_1-x)\,\Gamma(\alpha_2-x)\ldots\Gamma(\alpha_n-x)}{\Gamma(1-\gamma_1-\nu-x)\,\Gamma(1-\gamma_2-x)\ldots\Gamma(1-\gamma_n-x)}\right]dx.$$

If $\beta_n=-p$, we thus get $c_{\nu+p,n}^{(j)}$ from $c_{p,n}^{(j)}$ when we replace γ_j by $\gamma_j+\nu$.

If we suppose none of the differences between the parameters γ_s to be integers and $|z|<1$, then by putting the integral (2.40) equal to $2\pi i$ times the sum of the residues of the integrand at the poles $x=-\gamma_s-\nu$ on the left of contour, we get

$$\eta_n(z)=\sum_{s=1}^{n}\frac{\prod_{\nu=1}^{n}{}'\,\Gamma(\gamma_\nu-\gamma_s)}{\prod_{\nu=1}^{n}\Gamma(1-\alpha_\nu-\gamma_s)}\,z^{\gamma_s}\,F\left(\begin{matrix}\alpha_1+\gamma_s & \alpha_2+\gamma_s & \ldots & \alpha_n+\gamma_s\\ \gamma_s-\gamma_1+1 & \gamma_s-\gamma_2+1 & \ldots & \gamma_s-\gamma_n+1\end{matrix}\middle|\, z\right), \tag{2.42}$$

where $|\arg z|<2\pi$. Supposing none of the differences between the parameters α_s to be integers and $|z|>1$, then by interchanging α and γ and replacing z by $1/z$ in (2.42), we get

$$\eta_n(z)=(-1)^p\sum_{s=1}^{n}\frac{\prod_{\nu=1}^{n}{}'\,\Gamma(\alpha_\nu-\alpha_s)}{\prod_{\nu=1}^{n}\Gamma(1-\alpha_s-\gamma_\nu)}\,z^{-\alpha_s}\,F\left(\begin{matrix}\alpha_s+\gamma_1 & \alpha_s+\gamma_2 & \ldots & \alpha_s+\gamma_n\\ \alpha_s-\alpha_1+1 & \alpha_s-\alpha_2+1 & \ldots & \alpha_s-\alpha_n+1\end{matrix}\middle|\,\frac{1}{z}\right). \tag{2.43}$$

15. Next we consider the function $\xi_n(z)$. We suppose $\Re(\beta_n)>-1$ and $\Re(x+\gamma_s)>0$, $s=1,2,\ldots,n$. Then the integral on the left-hand side of (2.18) converges absolutely and $\xi_n(z)$ is regular in the interval $0<z<1$. We may imagine the integration extended from 0 to ∞ of a function which vanishes for $z>1$. The conditions for application of Fourier's integral formula are satisfied; hence from (2.18) we get

$$\xi_n(z)=\frac{\Gamma(\beta_n+1)}{2\pi i}\int_{\varkappa-i\infty}^{\varkappa+i\infty} z^{-x}\prod_{s=1}^{n}\frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)}\,dx,\qquad 0<z<1, \tag{2.44}$$

and the integral is vanishing for $z>1$. Here $\Re(\varkappa+\gamma_s)$ must be >0, $s=1, 2, 3, \dots, n$. The contour must be chosen in such a way that all the poles are on the left. From the factorial series (2.21) follows

$$\prod_{s=1}^{n}\frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)}=\frac{1}{(x+\gamma_i)^{\beta_n+1}}+\frac{\mu(x)}{(x+\gamma_i)^{\beta_n+2}},$$

where $|\mu(x)|$ is bounded in the half-plane $\Re(x)\geq\varkappa$. The integral on the right-hand side of (2.44) therefore converges in the usual sense and exists not only as Cauchy's principal value as in the general Fourier formula (see e.g. Doetsch [7] p. 115).

If we assume that $\Re(\beta_n)>-1$ and $\Re(x)<\Re(\alpha_s)$, $s=1, 2, \dots, n$, then the integral on the left-hand side of (2.19) converges absolutely and $\bar{\xi}_n(z)$ is regular for $1<z<\infty$. Hence by Fourier's integral formula we get from (2.19)

$$\bar{\xi}_n(z)=\frac{\Gamma(\beta_n+1)}{2\pi i}\int_{\varkappa-i\infty}^{\varkappa+i\infty} z^{-x}\prod_{s=1}^{n}\frac{\Gamma(\alpha_s-x)}{\Gamma(1-\gamma_s-x)}\,dx\qquad z>1, \tag{2.45}$$

and the integral is vanishing if $0<z<1$. Here we must have $\varkappa<\Re(\alpha_s)$, $s=1, 2, \dots, n$. Therefore the contour must be chosen so as to make all the poles lie on the right.

If the z-plane is cut from 0 to $-\infty$ and from 1 to $+\infty$ and if $\arg z$ and $\arg(1-z)$ vanish for $0<z<1$, then $\xi_n(z)$ is uniquely defined in the cut plane by the series (1.33) and its analytic continuation.

If we cut the z-plane from 1 through the origin to $-\infty$ and suppose $\arg z=\arg(z-1)=0$ for $z>1$, then $\bar{\xi}_n(z)$ is defined by the series (1.37) and its analytic continuation.

We have $\bar{\xi}_n(z)=e^{\pm\pi i\beta_n}\xi_n(z)$, where the upper (lower) sign is to be applied if z is above (below) the real axis.

For a moment we suppose $\Re(\alpha_i+\gamma_j)>0$, $i, j=1, 2, \dots, n$ and choose $\varkappa$ in such a way that

$$-\Re(\gamma_j)<\varkappa<\Re(\alpha_i)\qquad i, j=1, 2, \dots, n.$$

The integrals (2.44) and (2.45) only converge if z is positive and $\Re(\beta_n)>-1$. By addition of these two integrals we get

$$\frac{\xi_n(z)}{\Gamma(\beta_n+1)}=\frac{1}{2\pi i}\int_{\varkappa-i\infty}^{\varkappa+i\infty} z^{-x}\left[\prod_{s=1}^{n}\frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)}+e^{\mp\pi i\beta_n}\prod_{s=1}^{n}\frac{\Gamma(\alpha_s-x)}{\Gamma(1-\gamma_s-x)}\right]dx, \qquad (2.46)$$

where for $z>1$ $\xi_n(z)$ is to be taken on the upper (lower) edge of the cut from 1 to ∞ if we choose the upper (lower) sign in (2.46). Equally we have the equivalent formula

$$\frac{\bar{\xi}_n(z)}{\Gamma(\beta_n+1)}=\frac{1}{2\pi i}\int_{\varkappa-i\infty}^{\varkappa+i\infty} z^{-x}\left[e^{\pm\pi i\beta_n}\prod_{s=1}^{n}\frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)}+\prod_{s=1}^{n}\frac{\Gamma(\alpha_s-x)}{\Gamma(1-\gamma_s-x)}\right]dx, \qquad (2.47)$$

where for $0<z<1$ $\bar{\xi}_n(z)$ is to be taken on the upper (lower) edge of the cut line from 0 to 1 if the upper (lower) sign in (2.47) is chosen.

Now we have

$$\prod_{s=1}^{n}\frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)}+e^{\mp\pi i\beta_n}\prod_{s=1}^{n}\frac{\Gamma(\alpha_s-x)}{\Gamma(1-\gamma_s-x)}=g(x)\prod_{s=1}^{n}\frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)},$$

where

$$g(x)=1+e^{\mp\pi i\beta_n}\prod_{s=1}^{n}\frac{\sin\pi(x+\gamma_s)}{\sin\pi(\alpha_s-x)}. \qquad (2.48)$$

If we put $x=\varkappa+i\tau$, then

$$\lim_{\tau\to\pm\infty}\prod_{s=1}^{n}\frac{\sin\pi(x+\gamma_s)}{\sin\pi(\alpha_s-x)}=e^{\pm\pi i\left(n-\sum_{s=1}^{n}(\alpha_s+\gamma_s)\right)}=-e^{\pm\pi i\beta_n}.$$

Hence

$$\lim_{\tau\to\infty} g(x)=0 \quad \text{if the upper sign is chosen}$$

and

$$\lim_{\tau\to-\infty} g(x)=0 \quad \text{if the lower sign is chosen.}$$

$g(x)$ is a rational function of $e^{2\pi i x}$, consequently

$$g(x)=O(e^{-2\pi|\tau|}) \qquad (2.49)$$

for large positive (negative) values of τ if the upper (lower) sign is taken. Putting $z=re^{iv}$ we have

$$|z^{-x}|=r^{-\varkappa}e^{v\tau}.$$

Then the integral on the right-hand side of (2.46) is convergent in the angle $0\leq\arg z<2\pi$ if we choose the upper sign. If the lower sign is chosen, then the integral converges in the angle $0\geq\arg z>-2\pi$. Therefore the equations (2.46) and (2.47) are valid in these angles, respectively, when the upper or lower sign is chosen. From this the point $z=1$ must be excepted if $0\geq\Re(\beta_n)>-1$.

We have chosen the path of integration in such a way that all the poles $\alpha_i + \nu$ are situated on the right and the poles $-\gamma_i - \nu$ on the left. This will always be possible because we have assumed that $\Re(\alpha_i + \gamma_j) > 0$. If α_i and γ_j are permitted to vary in such a way that this condition is not satisfied, then (2.46) and (2.40) retain their validity if the rectilinear path of integration is deformed so as to make the poles $\alpha_i + \nu$ lie on the right and the poles $-\gamma_j - \nu$ on the left. Consequently the numbers $\alpha_i + \gamma_j$ must not be negative integers or zero.

If z is positive and different from 1 the integral (2.46) is convergent when $\Re(\beta_n) > -1$. But if we suppose $0 < \arg z < 2\pi$ or $0 > \arg z > -2\pi$, respectively, then the condition $\Re(\beta_n) > -1$ is unnecessary, because the function under the sign of integration in this case tends to zero of an exponential order in both directions.

If $n > 1$ and β_n is an integer or zero, the two expressions (2.48) are identical and (2.49) is valid for both positive and negative values of τ. Then the equations (2.46) and (2.47) are valid in the angle $2\pi > \arg z > -2\pi$. If β_n is a negative integer, we have proved (2.40), which may be looked upon as a special case of (2.46) and which may be derived by analytic continuation.

If $n = 1$, (2.44) reduces to

$$(1-z)^{\alpha-1} = \frac{\Gamma(\alpha)}{2\pi i} \int_{\varkappa - i\infty}^{\varkappa + i\infty} z^{-x} \frac{\Gamma(x)}{\Gamma(x+\alpha)} dx \qquad \varkappa > 0, \quad \Re(\alpha) > 0, \tag{2.50}$$

where $0 < z < 1$, while this integral vanishes for $z > 1$. (2.46) then reduces to Mellin's formula [34] p. 21

$$\frac{\Gamma(\alpha)}{(1+z)^{\alpha}} = \frac{1}{2\pi i} \int_{\varkappa - i\infty}^{\varkappa + i\infty} z^{-x} \Gamma(x) \Gamma(\alpha - x) dx \qquad 0 < \varkappa < \Re(\alpha), \tag{2.51}$$

where $\pi > \arg z > -\pi$. If z is negative and different from -1, (2.51) is valid when $0 < \Re(\alpha) < 1$, $(1+z)^{-\alpha}$ having the meaning mentioned above.

§ 3. Solutions at the Origin and the Point at Infinity

16. Again we suppose that none of the differences $\alpha_i - \alpha_j$ and $\gamma_i - \gamma_j$ is an integer and that the numbers $\alpha_i + \gamma_s$, $i = 1, 2, \ldots, n$, are not negative integers or zero. Furthermore we suppose that $\Re(\beta_n) > 0$. Then it is easy to establish relations between the solutions $y_s(z)$ and $\bar{y}_s(z)$ defined by (1.13) and (1.14). These series are convergent on the circle $|z| = 1$. Consider the function

$$\varphi(x)=\frac{z^x e^{-\pi i x}}{\sin\pi(x-\gamma_s)}\prod_{\nu=1}^{n}\frac{\Gamma(x+\alpha_\nu)}{\Gamma(x+1-\gamma_\nu)}, \tag{3.1}$$

where $z=e^{iv}$, $2\pi\geq v\geq 0$. $\varphi(x)$ is a meromorphic function of x having poles of the first order at the points

$$\gamma_s+\nu \quad\text{and}\quad -\alpha_i-\nu,\qquad i=1,2,\ldots,n,\quad \nu=0,1,2,\ldots. \tag{3.2}$$

Now we have

$$\prod_{\nu=1}^{n}\frac{\Gamma(x+\alpha_\nu)}{\Gamma(x+1-\gamma_\nu)}\sim\frac{1}{x^{\beta_{n+1}}}\qquad \pi-\varepsilon>|\arg x|.$$

Cutting out the points (3.2) by circles having these poles as centres and very small radii, we see without difficulty that $x\varphi(x)$ converges uniformly towards zero when $x\to\infty$ in the cut plane. From this it follows (see Lindelöf [22] chap. II) that the sum of the residues of $\varphi(x)$ equals zero. But the residues give us the series (1.13) and (1.14) multiplied by a constant factor. Thus we get

$$\prod_{\nu=1}^{n}\frac{\Gamma(\alpha_\nu+\gamma_s)}{\Gamma(\gamma_s-\gamma_\nu+1)}y_s(z)=\sum_{j=1}^{n}\bar{y}_j(z)\frac{\pi e^{\pi i(\alpha_j+\gamma_s)}}{\sin\pi(\alpha_j+\gamma_s)}\frac{\prod_{\nu=1}^{n}{}'\,\Gamma(\alpha_\nu-\alpha_j)}{\prod_{\nu=1}^{n}\Gamma(1-\alpha_j-\gamma_\nu)}, \tag{3.3}$$

where $s=1,2,\ldots,n$ and $2\pi\geq\arg z\geq 0$. If we cut the z-plane along the real axis from 0 to ∞, this relation is valid in the cut plane. It may often be convenient to multiply the solutions $y_s(z)$ and $\bar{y}_s(z)$ by certain constant factors. If we put

$$y_s^*(z)=y_s(z)\prod_{\nu=1}^{n}\frac{\Gamma(\alpha_\nu+\gamma_s)}{\Gamma(\gamma_s-\gamma_\nu+1)}\qquad \bar{y}_s^*(z)=\bar{y}_s(z)\prod_{\nu=1}^{n}\frac{\Gamma(\alpha_s+\gamma_\nu)}{\Gamma(\alpha_s-\alpha_\nu+1)}, \tag{3.4}$$

we get from (3.3)

$$y_s^*(z)=\sum_{j=1}^{n}\bar{y}_j^*(z)\frac{e^{\pi i(\alpha_j+\gamma_s)}}{\sin\pi(\alpha_j+\gamma_s)}\frac{\prod_{\nu=1}^{n}\sin\pi(\alpha_j+\gamma_\nu)}{\prod_{\nu=1}^{n}{}'\sin\pi(\alpha_\nu-\alpha_j)}, \tag{3.5}$$

and interchanging α and γ we get the inverse relation

$$\bar{y}_s^*(z)=\sum_{j=1}^{n}y_j^*(z)\frac{e^{\pi i(\alpha_s+\gamma_j)}}{\sin\pi(\alpha_s+\gamma_j)}\frac{\prod_{\nu=1}^{n}\sin\pi(\alpha_\nu+\gamma_j)}{\prod_{\nu=1}^{n}{}'\sin\pi(\gamma_\nu-\gamma_j)}. \tag{3.6}$$

These interrelations have first been given by Thomae [60] (see also Mellin [36], Winkler [63] and F. C. Smith [57]).

If $\alpha_r + \gamma_s = -p$, p being a positive integer or zero, the above proof is not valid, because two of the rows of poles partly coincide. (3.3) is without meaning because both sides of it contain the factor $\Gamma(\alpha_r + \gamma_s)$, which is infinite. But in this case y_s and $\bar{y}_r$ contain only a finite number of terms, and except for a constant factor they are identical, only written in reversed order. Consequently we have

$$y_s(z) = C\,\bar{y}_r(z) \tag{3.7}$$

By identification of the coefficients to the highest power of z we find

$$C = \prod_{\nu=1}^{n} \frac{(\alpha_\nu + \gamma_s)_p}{(\gamma_s - \gamma_\nu + 1)_p}.$$

17. We shall now consider the integral representations for the solutions. We put

$$y(z) = \int_L z^x\, \psi(x)\, dx. \tag{3.8}$$

If $\psi(x)$ is such that we may differentiate under the sign of integration, we get

$$(\vartheta - \gamma_1)(\vartheta - \gamma_2)\dots(\vartheta - \gamma_n)\, y(z) = \int_L z^x (x - \gamma_1)(x - \gamma_2)\dots(x - \gamma_n)\, \psi(x)\, dx \tag{3.9}$$

$$(\vartheta + \alpha_1)(\vartheta + \alpha_1)\dots(\vartheta + \alpha_n)\, y(z) = \int_L z^x (x + \alpha_1)(x + \alpha_2)\dots(x + \alpha_n)\, \psi(x)\, dx. \tag{3.10}$$

We shall take $\psi(x)$ as a solution of the difference equation

$$\psi(x+1) = \frac{(x + \alpha_1)(x + \alpha_2)\dots(x + \alpha_n)}{(x + 1 - \gamma_1)(x + 1 - \gamma_2)\dots(x + 1 - \gamma_n)}\, \psi(x). \tag{3.11}$$

Then from (3.10) we get

$$z(\vartheta + \alpha_1)(\vartheta + \alpha_1)\dots(\vartheta + \alpha_n)\, y(z) = \int_L z^{x+1} (x + 1 - \gamma_1)\dots(x + 1 - \gamma_n)\, \psi(x+1)\, dx. \tag{3.12}$$

If (3.9) and (3.12) are inserted in (1.2), it is seen that the integral (3.8) is a solution of the differential equation (1.2) if

$$\int_L z^x (x - \gamma_1)(x - \gamma_2)\dots(x - \gamma_n)\, \psi(x)\, dx = \int_L z^{x+1} (x + 1 - \gamma_1)(x + 1 - \gamma_2)\dots(x + 1 - \gamma_n)\, \psi(x+1)\, dx. \tag{3.13}$$

This condition may also be written

$$\int_L z^x (x + \alpha_1)(x + \alpha_2)\dots(x + \alpha_n)\, \psi(x)\, dx = \int_L z^{x-1} (x + \alpha_1 - 1)(x + \alpha_2 - 1)\dots(x + \alpha_n - 1)\, \psi(x-1)\, dx. \tag{3.14}$$

The difference equation (3.11) has the solution

$$\psi(x) = \frac{1}{2\pi i} \frac{\pi e^{\pi i(\gamma_s - x)}}{\sin \pi(\gamma_s - x)} f(x), \tag{3.15}$$

where

$$f(x) = \prod_{\nu=1}^{n} \frac{\Gamma(x+\alpha_\nu)}{\Gamma(x+1-\gamma_\nu)}. \tag{3.16}$$

Now we assume that none of the numbers $\alpha_i + \gamma_s$, $i = 1, 2, \ldots, n$, is a negative integer or zero and none of the differences $\gamma_s - \gamma_i$ is an integer. (3.15) has poles at $x = \gamma_s + \nu$, $\nu = 0, 1, 2, \ldots$ and at the points $-\alpha_i - \nu$, $i = 1, 2, \ldots, n$. We choose a path of integration from $\varkappa - i\infty$ to $\varkappa + i\infty$ in such a way that all the poles $\gamma_s + \nu$ are on the right and all the poles $-\alpha_i - \nu$ on the left. The integrand on the left-hand side of (3.13) has no pole at γ_s and the condition (3.13) is satisfied. Therefore the differential equation (1.2) has the solution

$$y_s^*(z) = \frac{1}{2\pi i} \int_{\varkappa - i\infty}^{\varkappa + i\infty} z^x \frac{\pi e^{\pi i(\gamma_s - x)}}{\sin \pi(\gamma_s - x)} f(x)\, dx, \tag{3.17}$$

this integral being convergent for $2\pi > \arg z > 0$.

If $\alpha_s + \gamma_s = -p$, p being a non-negative integer, while none of the numbers $\alpha_i + \gamma_s$, $i = 1, 2, \ldots s-1$, $s+1, \ldots, n$ is a negative integer or zero, then the above integral representation cannot be used, because in this case it is not possible to find a path of integration with the property mentioned. But denoting by l a closed contour traversed in the positive sense, which encloses the poles γ_s, $\gamma_s + 1, \ldots, \gamma_s + p$ and no other poles, we get the solution

$$\begin{aligned} y_s^*(z) &= \frac{1}{2\pi i} \int_l z^x f(x)\, dx \\ &= z^{\gamma_s} \frac{(-1)^p}{p!} {\prod_{\nu=1}^{n}}' \frac{\Gamma(\alpha_\nu + \gamma_s)}{\Gamma(\gamma_s - \gamma_\nu + 1)} F\left(\begin{matrix} \alpha_1 + \gamma_s & \alpha_2 + \gamma_s & \ldots & \alpha_n + \gamma_s \\ \gamma_s - \gamma_1 + 1 & \gamma_s - \gamma_2 + 1 & \ldots & \gamma_s - \gamma_n + 1 \end{matrix} \middle| z \right), \end{aligned} \tag{3.18}$$

where the dash signifies that $\nu = s$ is to be omitted in the product.

Next we assume that none of the numbers $\alpha_s + \gamma_i$, $i = 1, 2, \ldots, n$, is a negative integer or zero and that none of the differences between the numbers α_i is an integer. Putting

$$\bar{f}(x) = \prod_{\nu=1}^{n} \frac{\Gamma(\gamma_\nu - x)}{\Gamma(1 - \alpha_\nu - x)} \tag{3.19}$$

and choosing a path of integration such that all the poles $-\alpha_s - \nu$ lie on the left and the poles $\gamma_i + \nu$, $i = 1, 2, \ldots, n$ on the right, we see in the same way that (1.2) has the solution

$$\bar{y}_s^*(z) = \frac{1}{2\pi i} \int_{\varkappa - i\infty}^{\varkappa + i\infty} z^x \frac{\pi e^{-\pi i (x + \alpha_s)}}{\sin \pi (x + \alpha_s)} \bar{f}(x)\, dx, \tag{3.20}$$

which converges for $2\pi > \arg z > 0$.

If $\alpha_s + \gamma_s = -p$ and none of the numbers $\alpha_s + \gamma_i$, $i = 1, 2, \ldots, s-1, s+1, \ldots, n$ is a negative integer or zero, this integral is not applicable and must be replaced by

$$\begin{aligned}\bar{y}_s^*(z) &= \frac{1}{2\pi i} \int_l z^x \bar{f}(x)\, dx \\ &= z^{-\alpha_s} \frac{(-1)^p}{p!} \prod_{\substack{\nu=1 \\ \nu \neq s}}^{n} \frac{\Gamma(\alpha_s + \gamma_\nu)}{\Gamma(\alpha_s - \alpha_\nu + 1)} F\left(\begin{matrix} \alpha_s + \gamma_1 & \alpha_s + \gamma_2 & \ldots & \alpha_s + \gamma_n \\ \alpha_s - \alpha_1 + 1 & \alpha_s - \alpha_2 + 1 & \ldots & \alpha_s - \alpha_n + 1 \end{matrix} \middle| \frac{1}{z}\right),\end{aligned} \tag{3.21}$$

l being a closed contour traversed in the negative sense and enclosing the poles $-\alpha_s, -\alpha_s - 1, \ldots, -\alpha_s - p$ and no other poles.

The integral representations (3.17) and (3.20) are due to Pincherle [50] and Mellin [32, 36], who furthermore have proved, that if $|z| < 1$, then (3.17) equals minus the sum of the residues at the poles to the right of the path of integration and when $|z| > 1$, (3.20) equals the sum of the residues at the poles to the left of the path of integration. From this it follows that y_s^* and $\bar{y}_s^*$ have the same meaning as in (3.4). Putting s equal to $1, 2, \ldots, n$ we get two fundamental systems of solutions.

18. If $\gamma_1, \gamma_2, \ldots, \gamma_q$ form a group, then the integrals (3.17) corresponding to $s = 1, 2, \ldots, q$ are identical. In this case they must be chosen differently. Let us suppose $\Re(\gamma_1) \geq \Re(\gamma_2) \geq \cdots \geq \Re(\gamma_q)$ and that none of the numbers $\alpha_i + \gamma_s$, $i = 1, 2, \ldots, n$, is a negative integer or zero, s being one of the numbers $1, 2, \ldots, q$. Let $L_{\gamma_s, \infty}$ denote a (not necessarily rectilinear) path of integration from $\varkappa - i\infty$ to $\varkappa + i\infty$ chosen such that the poles $\gamma_s, \gamma_s + 1, \gamma_s + 2, \ldots$ lie on the right, the remaining poles on the left. Then we have the following solution of (1.2)

$$y_s^*(s) = \frac{1}{(2\pi i)^s} \int_{L_{\gamma_s, \infty}} z^x f(x) \prod_{\nu=1}^{s} \frac{\pi e^{\pi i (\gamma_\nu - x)}}{\sin \pi (\gamma_\nu - x)}\, dx, \tag{3.22}$$

because the condition (3.13) is satisfied. This integral converges for $2\pi s > \arg z > 0$. It may also be written in the form

$$y_s^*(z) = \int\limits_{L_{\gamma_s,\infty}} \frac{z^x f(x)\,dx}{[1-e^{2\pi i(x-\gamma_1)}]^s} \tag{3.23}$$

or, more explicitly,

$$y_s^*(z) = \frac{1}{(2\pi i)^s} \int\limits_{L_{\gamma_s,\infty}} z^x e^{\pi i(\sum\limits_1^s \gamma_\nu - s x)} \frac{\prod\limits_{\nu=1}^{s} \Gamma(\gamma_\nu - x) \prod\limits_{\nu=1}^{n} \Gamma(x+\alpha_\nu)}{\prod\limits_{\nu=s+1}^{n} \Gamma(x-\gamma_\nu+1)}\, dx. \tag{3.24}$$

Putting s equal to $1, 2, \ldots, q$, we get q different solutions. We shall now show that these are linearly independent. Let us suppose

$$\gamma_1 = \gamma_2 = \cdots = \gamma_{\lambda_1} = \varrho_1 \qquad \gamma_{\lambda_1+1} = \gamma_{\lambda_1+2} = \cdots = \gamma_{\lambda_1+\lambda_2} = \varrho_2$$

and generally

$$\gamma_{\lambda_1+\lambda_2+\cdots+\lambda_{r-1}+1} = \gamma_{\lambda_1+\lambda_2+\cdots\lambda_{r-1}+2} = \cdots = \gamma_{\lambda_1+\lambda_2+\cdots+\lambda_r} = \varrho_r \qquad r = 1, 2, \ldots, \mu,$$

where $\lambda_1 + \lambda_2 + \cdots + \lambda_\mu = q$. Then we have $\Re(\varrho_1) > \Re(\varrho_2) > \cdots > \Re(\varrho_\mu)$. The solutions may then be arranged in subgroups in such a way that we get the first subgroup by putting s equal to $1, 2, \ldots, \lambda_1$, the second by putting s equal to $\lambda_1+1, \lambda_1+2, \ldots, \lambda_1+\lambda_2$ etc. The function $f(x)$ then has the form

$$f(x) = \frac{\prod\limits_{\nu=1}^{n} \Gamma(x+\alpha_\nu)}{\prod\limits_{\nu=1}^{\mu} (\Gamma(x-\varrho_\nu+1))^{\lambda_\nu} \prod\limits_{\nu=q+1}^{n} \Gamma(x-\gamma_\nu+1)}.$$

For the first subgroup the integrand is

$$\frac{z^x e^{\pi i s(\varrho_1 - x)} (\Gamma(\varrho_1 - x))^s \prod\limits_{\nu=1}^{n} \Gamma(x+\alpha_\nu)}{(\Gamma(x-\varrho_1+1))^{\lambda_1 - s} \prod\limits_{\nu=\lambda_1+1}^{n} \Gamma(x-\gamma_\nu+1)} \qquad s = 1, 2, \ldots, \lambda_1,$$

for the second

$$\frac{z^x e^{\pi i(\lambda_1(\varrho_1 - x) + s(\varrho_2 - x))} (\Gamma(\varrho_1 - x))^{\lambda_1} (\Gamma(\varrho_2 - x))^s \prod\limits_{\nu=1}^{n} \Gamma(x+\alpha_\nu)}{(\Gamma(x-\varrho_2+1))^{\lambda_2 - s} \prod\limits_{\nu=\lambda_1+\lambda_2+1}^{n} \Gamma(x-\gamma_\nu+1)},$$

where $s = 1, 2, \ldots, \lambda_2$, etc. If (3.24) is expanded in powers of z, we get a series of the form (1.17). If only the first term in this series is calculated, we easily see that for the solutions in the first subgroup is

$$\lim_{z\to 0} \frac{y_s^*(z)}{z^{\varrho_1}(\log z)^{s-1}} = A_s^{(1)} \qquad s=1,2,\ldots,\lambda_1,$$

for the solutions in the second subgroup is

$$\lim_{z\to 0} \frac{y_{\lambda_1+s}^*(z)}{z^{\varrho_2}(\log z)^{s-1}} = A_s^{(2)} \qquad s=1,2,\ldots,\lambda_2$$

and for the solutions in rth subgroup

$$\lim_{z\to 0} \frac{y_{\lambda_1+\lambda_2+\cdots+\lambda_{r-1}+s}^*(z)}{z^{\varrho_r}(\log z)^{s-1}} = A_s^{(r)} \qquad \begin{matrix} s=1,2,\ldots,\lambda_r \\ r=1,2,\ldots,\mu, \end{matrix} \tag{3.25}$$

the $A_s^{(r)}$ being constants different from zero. From this it follows that $y_1^*, y_2^*, \ldots, y_q^*$ are linearly independent.

19. If $\alpha_1, \alpha, \ldots, \alpha_p$ form a group, then the solutions (3.20) corresponding to $s=1,2,\ldots p$ are identical. We suppose that $\Re(\alpha_1) \geq \Re(\alpha_2) \geq \cdots \geq \Re(\alpha_p)$ and that none of the numbers $\alpha_s+\gamma_i$, $i=1,2,\ldots,n$, is a negative integer or zero, s being one of the numbers $1,2,\ldots,p$. In the same way as in art. 18 we see that the differential equation (1.2) has the following linearly independent solutions

$$\bar{y}_s^*(z) = \frac{(-1)^{s-1}}{(2\pi i)^s} \int_{\varkappa-i\infty}^{\varkappa+i\infty} z^x \bar{f}(x) \prod_{\nu=1}^{s} \frac{\pi e^{-\pi i(\alpha_\nu+x)}}{\sin \pi(x+\alpha_\nu)} dx = -\int_{\varkappa-i\infty}^{\varkappa+i\infty} \frac{z^x \bar{f}(x)\, dx}{[1-e^{2\pi i(x+\alpha_1)}]^s}, \tag{3.26}$$

where $s=1,2,\ldots,p$ and the path of integration is chosen in such a way that the poles $-\alpha_s$, $-\alpha_s-1$, $-\alpha_s-2,\ldots$ lie on the left and the remaining poles on the right. More explicitly we may write

$$\bar{y}_s^*(z) = \frac{(-1)^{s-1}}{(2\pi i)^s} \int_{\varkappa-i\infty}^{\varkappa+i\infty} z^x e^{-\pi i(\sum_1^s \alpha_\nu + s x)} \frac{\prod_{\nu=1}^{s} \Gamma(x+\alpha_\nu) \prod_{\nu=1}^{n} \Gamma(\gamma_\nu - x)}{\prod_{\nu=s+1}^{n} \Gamma(1-\alpha_\nu - x)} dx. \tag{3.27}$$

These integrals all converge for $2\pi s > \arg z > 0$. Forming the difference $\bar{y}_{s+1}^* - \bar{y}_s^*$, we get

$$\bar{y}_{s+1}^*(z) - \bar{y}_s^*(z) = -\int_{\varkappa-i\infty}^{\varkappa+i\infty} \frac{z^x \bar{f}(x) e^{2\pi i(x+\alpha_1)}\, dx}{[1-e^{2\pi i(x+\alpha_1)}]^{s+1}} \qquad s=1,2,\ldots,p-1. \tag{3.28}$$

This integral converges for $2\pi s > \arg z > -2\pi$. The point $z=1$ is then situated in the interior of the region of convergence and the integral represents a function regular at the point $z=1$. Thus the solutions (3.26) in our group has the remarkable property that the difference between two consecutive solutions is a solution regular at the point $z=1$. Likewise we see from (3.23) that $y^*_{s+1}(z)-y^*_s(z)$ is regular at $z=1$, and is represented by an integral convergent for $2\pi s > \arg z > -2\pi$.

20. Now we consider again the case where $\gamma_1\, \gamma_2, \ldots, \gamma_q$ form a group. In art. 2 we have seen that the form of the differential equation does not change if we multiply y by a power of z. Therefore without loss of generality we may suppose $\gamma_1, \gamma_2, \ldots, \gamma_q$ to be integers. We use the same notation as in art. 18. Then $\varrho_1, \ldots, \varrho_\mu$ are integers and $\varrho_1 > \varrho_2 > \cdots > \varrho_\mu$. Again we consider the integral

$$y^*_s(z) = \int\limits_L \frac{z^x f(x)\, dx}{(1-e^{2\pi i x})^s}. \tag{3.29}$$

If $L=L_{\varrho_1,\infty}$ and none of the numbers α_i is an integer $\leq -\varrho_1$, it represents the solutions in the first subgroup for $0 < s \leq \lambda_1$. These solutions belong to the exponent ϱ_1. If $L=L_{\varrho_r,\infty}$ and none of the numbers α_i is an integer $\leq -\varrho_r$, (3.29) represents the solutions in the rth subgroup for $\sum\limits_{i=1}^{r-1}\lambda_i < s \leq \sum\limits_{i=1}^{r}\lambda_i$. These solutions belong to the exponent ϱ_r, and this is true for $r=1, 2, \ldots, \mu$.

If some of the parameters α_i do not satisfy the condition just mentioned, the condition (3.13) is not satisfied for some of our integrals. In this case we must choose them in a different way and allow s to take negative values.

Let ν_r be a positive integer or zero. Let us assume that ν_r of the α_i are integers situated in the interval $-\varrho_{r-1} < \alpha_i \leq -\varrho_r$ and that this is true for $r=1, 2, \ldots, \mu$. Here ϱ_0 shall be an integer $> \varrho_1$, such that none of the α_i is an integer $\leq -\varrho_0$. The function $f(x)$ is then regular and different from zero at the point ϱ_0. Let us put

$$\varepsilon_j = \sum_{i=1}^{j}(\lambda_i - \nu_i) \quad j>0 \text{ and } \varepsilon_0 = 0.$$

The function

$$\frac{f(x)}{(x-\varrho_r)^{\varepsilon_r-\lambda_r}} \quad r=1, 2, \ldots, \mu$$

is regular and different from zero at the point ϱ_r. Let $L_{\varrho_r\varrho_p}$, where $p<r$, denote a closed contour traversed in the positive sense and enclosing the poles $\varrho_r, \varrho_r+1, \ldots, \varrho_p-1$ and no other poles. If $\varepsilon_r-\lambda_r < s \leq \varepsilon_r$, the integrand in (3.29) has a pole of an order $\leq \lambda_r$ at $x=\varrho_r$. If $s \leq \varepsilon_p$, $x=\varrho_p$ is a pole of an order $\leq \lambda_p$ or a regular point. It follows that the condition (3.13) is satisfied for $L = L_{\varrho_r \varrho_p}$, $0 \leq p < r$, and the integral belongs to the exponent ϱ_r.

We get the solutions in the rth subgroup, when in (3.29) s is put equal to all integers in the interval $\varepsilon_r - \lambda_r < s \leq \varepsilon_r$.

Let ε_p be the greatest of the numbers $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{r-1}$.

If $\varepsilon_p \geq \varepsilon_r$, we put $L = L_{\varrho_r \varrho_p}$ for all solutions in the rth subgroup.

If $\varepsilon_p < \varepsilon_r$, we put $L = L_{\varrho_r \varrho_p}$ for $\varepsilon_r - \lambda_r < s \leq \varepsilon_p$

and $L = L_{\varrho_r, \infty}$ for $\varepsilon_p < s \leq \varepsilon_r$.

As $\varepsilon_r - \lambda_r = \varepsilon_{r-1} - \nu_r$ and as $\varepsilon_p \geq \varepsilon_{r-1}$, at least ν_r solutions having a finite path of integration belong to the rth subgroup. The entire group contains at least $\sum_{i=1}^{\mu} \nu_i$ solutions having a finite path of integration.

Thus in all cases we can find q linearly independent solutions of the form (3.29) belonging to the group and thus form a fundamental system in the vicinity of the singular point $z = 0$. The integral $y_s^*(z)$ is of the form (1.17), which more briefly may be written

$$y_s^*(z) = z^{\gamma_s} (\varphi_0(z) + \varphi_1(z) \log z + \cdots + \varphi_{s-1}(z) (\log z)^{s-1}). \tag{3.30}$$

In the general case considered in art. 18, $\varphi_0(z), \varphi_1(z), \ldots, \varphi_{s-1}(z)$ are power series convergent for $|z| < 1$. But if any of the parameters assume the here-mentioned exceptional values some of these power series reduce to a finite number of terms or to zero. Let us consider some particular cases.

EXAMPLE 1. If $\nu_r = \lambda_r$, $r = 1, 2, \ldots, \mu$ all solutions are of the form

$$y_s^*(z) = \frac{1}{2\pi i} \int_L z^x f(x) (1 - e^{2\pi i x})^s \, dx. \tag{3.31}$$

We get the rth subgroup by putting s equal to $0, 1, 2, \ldots, \lambda_r - 1$ and $L = L_{\varrho_r \varrho_{r-1}}$, $r = 1, 2, \ldots, \mu$. These q solutions are linearly independent and regular at $z = 1$. Especially, if $\lambda_i = 1$, $i = 1, 2, \ldots, \mu$, the solutions in our group are

$$y_r^*(z) = \frac{1}{2\pi i} \int_{L_{\varrho_r \varrho_{r-1}}} z^x f(x) \, dx = \sum_{\nu=\varrho_r}^{\varrho_{r-1}-1} A_\nu z^\nu,$$

A_ν being the residue of $f(x)$ at the point $x = \nu$.

EXAMPLE 2. Now we assume that all the numbers $\nu_1, \nu_2, \ldots, \nu_\mu$ are zero except one, say ν_r, which is positive. The first $r - 1$ subgroups are determined as in the initial part of this article. The solutions in the rth subgroup are found by putting s equal to all integers in the interval

$$-\nu_r + \sum_1^{r-1} \lambda_i < s \leq -\nu_r + \sum_1^{r} \lambda_i.$$

If $\nu_r \geq \lambda_r$, we put $L = L_{\varrho_r\, \varrho_{r-1}}$ for all solutions in the rth subgroup. If $\nu_r < \lambda_r$, we put

$$L = L_{\varrho_r\, \varrho_{r-1}} \quad \text{for} \quad -\nu_r + \sum_1^{r-1} \lambda_i < s \leq \sum_1^{r-1} \lambda_i$$

and

$$L = L_{\varrho_r,\, \infty} \quad \text{for} \quad \sum_1^{r-1} \lambda_i < s \leq -\nu_r + \sum_1^{r} \lambda_i.$$

Next we consider the mth subgroup, where $r < m \leq \mu$. In this we get all solutions by putting s equal to all integers in the interval

$$-\nu_r + \sum_1^{m-1} \lambda_i < s \leq -\nu_r + \sum_1^{m} \lambda_i.$$

If $\nu_r \leq \sum_r^{m-1} \lambda_i$, we put $L = L_{\varrho_m,\, \infty}$ for all solutions.

If $\nu_r \geq \sum_r^{m} \lambda_i$, we put $L = L_{\varrho_m\, \varrho_{r-1}}$ for all solutions.

If

$$\sum_r^{m-1} \lambda_i < \nu_r < \sum_r^{m} \lambda_i, \text{ we put } L = L_{\varrho_m\, \varrho_{r-1}} \quad \text{for } -\nu_r + \sum_1^{m-1} \lambda_i < s \leq \sum_1^{r-1} \lambda_i$$

$$\text{and } L = L_{\varrho_m,\, \infty} \qquad \text{for } \sum_1^{r-1} \lambda_i < s \leq -\nu_r + \sum_1^{m} \lambda_i.$$

In particular, if $r = 1$ and $\nu_1 = q$, all solutions in our group are of the form (3.31), where $s = 0, 1, 2, \ldots, q-1$. The function $f(x)$ is regular at $x = \varrho_0$, and at the point ϱ_r it has a pole of the order $\sum_r^{\mu} \lambda_i$, $r = 1, 2, \ldots, \mu$.

We get the first subgroup by putting $s = q-1, q-2, \ldots, q-\lambda_1$, and $L = L_{\varrho_1\, \varrho_0}$; the second subgroup by putting $s = q - \lambda_1 - 1,\ q - \lambda_1 - 2, \ldots, q - \lambda_1 - \lambda_2$ and $L = L_{\varrho_2\, \varrho_0}$, etc., the μth subgroup by putting $s = 0, 1, 2, \ldots, \lambda_\mu - 1$ and $L = L_{\varrho_\mu\, \varrho_0}$.

These solutions are linearly independent and all regular at $z = 1$, which is an apparent singularity.

EXAMPLE 3. If $\alpha_i + \gamma_i = 1$, $i = 1, 2, \ldots, n$, then $f(x) = 1$ and the differential equation (1.2) has the following n solutions all linearly independent

$$\frac{z^{\varrho_i}}{1-z}(\log z)^{s-1} \qquad \begin{array}{l} s=1, 2, \ldots, \lambda_i \\ i=1, 2, \ldots, \mu. \end{array}$$

This holds whether all γ_i belong to the same group or not.

What has been said here concerning the solutions at the origin, is also valid for the solutions at $z=\infty$. If $\alpha_1, \alpha_2, \ldots, \alpha_p$ form a group, and any of the parameters γ_i do not satisfy the conditions in art. 19, then the solutions mentioned there reduce in the same way as above. It is only necessary to interchange α and γ together with z and $1/z$.

21. The relations between the solutions about $z=0$ and $z=\infty$ can be found in the following manner. We suppose first that no two of the α_i or γ_i are equal or differ by an integer and that $\alpha_i+\gamma_s$ is not zero or a negative integer. Let us consider the decomposition into partial fractions

$$\frac{1}{\sin \pi(\gamma_s - x)} \prod_{\nu=1}^{n} \frac{\sin \pi(\gamma_\nu - x)}{\sin \pi(x+\alpha_\nu)} = \sum_{j=1}^{n} \frac{b_j}{\sin \pi(x+\alpha_j)}. \tag{3.32}$$

If we multiply both sides by $\sin \pi(x+\alpha_j)$ and then put $x=-\alpha_j$, we get the following expression for the constants b_j

$$b_j = \frac{1}{\sin \pi(\alpha_j+\gamma_s)} \frac{\prod\limits_{\nu=1}^{n} \sin \pi(\alpha_j+\gamma_\nu)}{\prod\limits_{\nu=1}^{n}{}' \sin \pi(\alpha_\nu - \alpha_j)}. \tag{3.33}$$

Multiplying both sides of (3.32) by $\pi \bar{f}(x)$ we get

$$\frac{\Gamma(\gamma_s - x) \prod\limits_{\nu=1}^{n} \Gamma(x+\alpha_\nu)}{\prod\limits_{\substack{\nu=1 \\ \nu \neq s}}^{n} \Gamma(x+1-\gamma_\nu)} = \sum_{j=1}^{n} b_j \frac{\Gamma(x+\alpha_j) \prod\limits_{\nu=1}^{n} \Gamma(\gamma_\nu - x)}{\prod\limits_{\substack{\nu=1 \\ \nu \neq j}}^{n} \Gamma(1-\alpha_\nu - x)}, \tag{3.34}$$

where s is one of the numbers $1, 2, \ldots, n$. If we multiply both sides of this relation by $\frac{1}{2\pi i} z^x e^{-\pi i x}$ and integrate from $\varkappa - i\infty$ to $\varkappa + i\infty$ in such a way that all poles $\gamma_\nu, \gamma_\nu+1, \gamma_\nu+2, \ldots$ lie on the right and the poles $-\alpha_\nu, -\alpha_\nu-1, -\alpha_\nu-2, \ldots$ ($\nu=1, 2, \ldots, n$) on the left of the path of integration, we get (3.5), which may be written

$$e^{-\pi i \gamma_s} y_s^*(z) = \sum_{j=1}^{n} b_j e^{\pi i \alpha_j} \bar{y}_j^*(z) \qquad s=1, 2, \ldots, n, \tag{3.35}$$

and this relation is valid for $2\pi > \arg z > 0$.

22. We now suppose that $\alpha_1, \alpha_2, \ldots, \alpha_p$ form a group and that the same is the case for $\gamma_1, \gamma_2, \ldots, \gamma_q$. We assume them to be ranged in a descending order of the real parts. Further we suppose that $\alpha_i + \gamma_s$ is not zero or a negative integer. Let r be one of the numbers $1, 2, \ldots, q$. By decomposition into partial fractions we get

$$e^{-\pi i r x} \frac{\prod_{\nu=r+1}^{n} \sin \pi (\gamma_\nu - x)}{\prod_{\nu=1}^{n} \sin \pi (x+\alpha_\nu)}$$

$$= (2i)^r \sum_{s=1}^{p} \frac{\varphi^{(p-s)}(e^{-2\pi i \alpha_1})\, e^{\pi i (\sum_1^s \alpha_\nu - s x)}}{(p-s)!\,(2i)^s \prod_{\nu=1}^{s} \sin \pi (x+\alpha_\nu)} + \tag{3.36}$$

$$+ \sum_{s=p+1}^{n} \frac{\prod_{\nu=r+1}^{n} \sin \pi (\alpha_s + \gamma_\nu)}{\prod_{\nu=1}^{n}{}' \sin \pi (\alpha_\nu - \alpha_s)} e^{\pi i r \alpha_s} \frac{e^{-\pi i (x+\alpha_s)}}{\sin \pi (x+\alpha_s)},$$

where $\varphi(x)$ is the rational function

$$\varphi(x) = (-1)^{n-r} e^{-\pi i (\sum_1^n \alpha_\nu + \sum_{r+1}^n \gamma_\nu)} \frac{\prod_{r+1}^{n} (x - e^{2\pi i \gamma_\nu})}{\prod_{p+1}^{n} (x - e^{-2\pi i \alpha_\nu})}. \tag{3.37}$$

Multiplying both sides of (3.36) by $\dfrac{\bar{f}(x)}{(2i)^r}$, we get

$$\frac{e^{-\pi i r x}}{(2\pi i)^r} \frac{\prod_1^r \Gamma(\gamma_\nu - x) \prod_1^n \Gamma(x+\alpha_\nu)}{\prod_{r+1}^{n} \Gamma(x+1-\gamma_\nu)}$$

$$= \sum_{s=1}^{p} \frac{\varphi^{(p-s)}(e^{-2\pi i \alpha_1})\, e^{\pi i (\sum_1^s \alpha_\nu - s x)}}{(p-s)!\,(2\pi i)^s} \frac{\prod_1^s \Gamma(x+\alpha_\nu) \prod_1^n \Gamma(\gamma_\nu - x)}{\prod_{s+1}^{n} \Gamma(1-\alpha_\nu - x)} +$$

$$+ \frac{1}{(2i)^{r-1}} \sum_{s=p+1}^{n} \frac{\prod_{\nu=r+1}^{n} \sin \pi (\alpha_s + \gamma_\nu)}{\prod_{\nu=1}^{n} \sin \pi (\alpha_\nu - \alpha_s)} e^{\pi i r \alpha_s} \frac{e^{-\pi i (x+\alpha_s)}}{2\pi i} \frac{\Gamma(x+\alpha_s) \prod_1^n \Gamma(\gamma_\nu - x)}{\prod_{\substack{\nu=1 \\ \nu \neq s}}^{n} \Gamma(1-\alpha_\nu - x)}.$$

If we multiply both sides of this equation by z^x and integrate from $\varkappa - i\infty$ to $\varkappa + i\infty$ in the same way as above, we get, using the notations in art. 18 and 19,

$$e^{-\pi i \sum_1^r \gamma_\nu} y_r^*(z) = \sum_{s=1}^{p} (-1)^{s-1} \frac{\varphi^{(p-s)}(e^{-2\pi i \alpha_1})}{(p-s)!} e^{2\pi i s \alpha_1} \bar{y}_s^*(z) + \\ + \frac{1}{(2i)^{r-1}} \sum_{s=p+1}^{n} \frac{\prod_{\nu=r+1}^{n} \sin \pi (\alpha_s + \gamma_\nu)}{\prod_{\nu=1}^{n}{}' \sin \pi (\alpha_\nu - \alpha_s)} e^{\pi i r \alpha_s} \bar{y}_s^*(z), \tag{3.38}$$

and this relation is valid for $2\pi > \arg z > 0$.

Especially, if we put $r = 1$ and interchange γ_1 and γ_j, where $j > q$, we get the relation between $y_j^*(z)$ and $\bar{y}_1^*(z)$, $\bar{y}_2^*(z), \ldots, \bar{y}_n^*(z)$. If the α_i or the γ_i form several groups, we may treat each of these groups in the same way.

To get the inverse relation we only need to decompose

$$e^{-\pi i r x} \frac{\prod_{r+1}^{n} \sin \pi (x + \alpha_\nu)}{\prod_{1}^{n} \sin \pi (\gamma_\nu - x)}$$

into partial fractions, where r now is one of the numbers $1, 2, \ldots, p$; let $\varphi_1(x)$ denote the rational function

$$\varphi_1(x) = (-1)^n e^{\pi i (\sum_{r+1}^{n} \alpha_\nu + \sum_{1}^{n} \gamma_\nu)} \frac{\prod_{r+1}^{n} (x - e^{-2\pi i \alpha_\nu})}{\prod_{q+1}^{n} (x - e^{2\pi i \gamma_\nu})}. \tag{3.39}$$

Thus we find

$$(-1)^{r-1} e^{\pi i \sum_1^r \alpha_\nu} \bar{y}_r^*(z) = \sum_{s=1}^{q} (-1)^s \frac{\varphi_1^{(q-s)}(e^{2\pi i \gamma_1})}{(q-s)!} e^{-2\pi i s \gamma_1} y_s^*(z) + \\ + \frac{1}{(2i)^{r-1}} \sum_{s=q+1}^{n} \frac{\prod_{\nu=r+1}^{n} \sin \pi (\alpha_\nu + \gamma_s)}{\prod_{\nu=1}^{n}{}' \sin \pi (\gamma_\nu - \gamma_s)} e^{-\pi i r \gamma_s} y_s^*(z), \tag{3.40}$$

where $2\pi > \arg z > 0$.

The interrelations of the logarithmic solutions about the origin and the point at infinity have been considered by Lindelöf [21] and Mehlenbacher [26] in the case $n = 2$; further by F. C. Smith [59] in the general case; but the above demonstration is much simpler.

23. Now we shall consider the relations between $\xi_n(z)$ and the above-mentioned solutions. We suppose, first, that no two of the γ_i are equal or differ by an integer and that $\alpha_i+\gamma_s$ is not zero or an integer. By decomposition into partial fractions we get

$$\prod_{\nu=1}^{n} \frac{\sin \pi(x+\alpha_\nu)}{\sin \pi(\gamma_\nu - x)} = -e^{\mp \pi i \beta_n} + \sum_{s=1}^{n} B_s \frac{e^{\pm \pi i(\gamma_s - x)}}{\sin \pi(\gamma_s - x)}, \tag{3.41}$$

where

$$B_s = \frac{\prod\limits_{\nu=1}^{n} \sin \pi(\alpha_\nu + \gamma_s)}{\prod\limits_{\nu=1}^{n}{}' \sin \pi(\gamma_\nu - \gamma_s)}. \tag{3.42}$$

Multiplying both sides of (3.41) by $f(x)$, we get

$$\bar{f}(x) + e^{\mp \pi i \beta_n} f(x) = \sum_{s=1}^{n} B_s \frac{e^{\pm \pi i(\gamma_s - x)}}{\sin \pi(\gamma_s - x)} f(x).$$

If we multiply by $z^x/2\pi i$ and integrate from $\varkappa - i\infty$ to $\varkappa + i\infty$, we get by (2.46) and (3.17)

$$\frac{\xi_n(z)}{\Gamma(\beta_n+1)} = \frac{1}{\pi} \sum_{s=1}^{n} B_s y_s^*(z) \tag{3.43}$$

$$= \sum_{s=1}^{n} \frac{\prod\limits_{\nu=1}^{n}{}' \Gamma(\gamma_\nu - \gamma_s)}{\prod\limits_{\nu=1}^{n} \Gamma(1-\alpha_\nu-\gamma_s)} y_s(z). \tag{3.44}$$

If we suppose that no two of the α_i are equal or differ by an integer, we may interchange α and γ and we get

$$\frac{\bar{\xi}_n(z)}{\Gamma(\beta_n+1)} = \frac{1}{\pi} \sum_{s=1}^{n} \bar{B}_s \bar{y}_s^*(z) \tag{3.45}$$

$$= \sum_{s=1}^{n} \frac{\prod\limits_{\nu=1}^{n}{}' \Gamma(\alpha_\nu - \alpha_s)}{\prod\limits_{\nu=1}^{n} \Gamma(1-\alpha_s-\gamma_\nu)} \bar{y}_s(z), \tag{3.46}$$

where

$$\bar{B}_s = \frac{\prod\limits_{\nu=1}^{n} \sin \pi(\alpha_s + \gamma_\nu)}{\prod\limits_{\nu=1}^{n}{}' \sin \pi(\alpha_\nu - \alpha_s)}. \tag{3.47}$$

24. Next we suppose that $\gamma_1, \gamma_2, \ldots, \gamma_q$ form a group. If we put

$$\psi(x) = -e^{-\pi i \beta_n} \frac{\prod_1^n (x - e^{-2\pi i \alpha_\nu})}{\prod_{q+1}^n (x - e^{2\pi i \gamma_\nu})}, \tag{3.48}$$

we have by decomposition into partial fractions

$$\prod_{\nu=1}^n \frac{\sin \pi (x + \alpha_\nu)}{\sin \pi (\gamma_\nu - x)} = -e^{-\pi i \beta_n} + \sum_{s=q+1}^n B_s \frac{e^{\pi i (\gamma_s - x)}}{\sin \pi (\gamma_s - x)} + \\ + \sum_{s=1}^q \frac{(-1)^s \psi^{(q-s)}(e^{2\pi i \gamma_1}) e^{-2\pi i s \gamma_1}}{(q-s)!\ (2i)^s} \frac{e^{\pi i (\sum_1^s \gamma_\nu - s x)}}{\prod_{\nu=1}^s \sin \pi (\gamma_\nu - x)}. \tag{3.49}$$

Multiplying by $f(x)$, we get

$$\bar{f}(x) + e^{-\pi i \beta_n} f(x) = \sum_{s=q+1}^n B_s \frac{e^{\pi i (\gamma_s - x)}}{\sin \pi (\gamma_s - x)} f(x) + \\ + \sum_{s=1}^q \frac{(-1)^s \psi^{(q-s)}(e^{2\pi i \gamma_1}) e^{-2\pi i s \gamma_1}}{(q-s)!\ (2\pi i)^s} e^{\pi i (\sum_1^s \gamma_\nu - s x)} \frac{\prod_{\nu=1}^s \Gamma(\gamma_\nu - x) \prod_{\nu=1}^n \Gamma(x + \alpha_\nu)}{\prod_{s+1}^n \Gamma(x - \gamma_\nu + 1)}.$$

If we multiply both sides of this equation by z^x and integrate from $\varkappa - i\infty$ to $\varkappa + i\infty$, in the same manner as above, we find

$$\frac{\xi_n(z)}{\Gamma(\beta_n + 1)} = \sum_{s=1}^q \frac{(-1)^s \psi^{(q-s)}(e^{2\pi i \gamma_1}) e^{-2\pi i s \gamma_1}}{(q-s)!\ 2\pi i} y_s^*(z) + \frac{1}{\pi} \sum_{s=q+1}^n B_s y_s^*(z). \tag{3.50}$$

If we finally suppose that $\alpha_1, \alpha_2, \ldots, \alpha_p$ form a group, we have in the same way

$$\frac{\bar{\xi}_n(z)}{\Gamma(\beta_n + 1)} = \sum_{s=1}^p \frac{(-1)^s \psi_1^{(p-s)}(e^{-2\pi i \alpha_1}) e^{2\pi i s \alpha_1}}{(p-s)!\ 2\pi i} \bar{y}_s^*(z) + \frac{1}{\pi} \sum_{s=p+1}^n \bar{B}_s \bar{y}_s^*(z), \tag{3.51}$$

where

$$\psi_1(x) = e^{\pi i \beta_n} \frac{\prod_1^n (x - e^{2\pi i \gamma_\nu})}{\prod_{p+1}^n (x - e^{-2\pi i \alpha_\nu})}. \tag{3.52}$$

Thus all these relations between the different solutions follow immediately from the elementary formula for decomposition of a rational function into partial fractions.

§ 4. New Integral Representations of the Solution $\xi_n(z)$

25. Let us suppose $\Re(\beta_n) > -1$ and choose a number α so that $\Re(\alpha+\beta_n+\gamma_s)>0$ $(s=1, 2, \ldots, n)$. Then the integral

$$\int_0^1 t^{x-1}(1-t)^{\alpha-x-1}\xi_n(t)\,dt \tag{4.1}$$

is convergent in the strip

$$-\Re(\gamma_s)<\Re(x)<\Re(\alpha+\beta_n) \qquad s=1, 2, \ldots, n. \tag{4.2}$$

If we expand $(1-t)^{\alpha-x-1}$ by the binomial theorem, term-by-term integration is permissible (see Bromwich [5] p. 497), because the integral

$$\int_0^1 |t^{x-1}\xi_n(t)|\sum_0^\infty\left|\frac{(x+1-\alpha)_\nu}{\nu!}t^\nu\right|dt$$

is convergent. Then we have by (2.18) that the integral (4.1) is equal to

$$\Gamma(\beta_n+1)\prod_1^n\frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)}F\begin{pmatrix}x-\alpha+1 & x+\gamma_1 & x+\gamma_2 & \ldots & x+\gamma_n \\ x-\alpha_1+1 & x-\alpha_2+1 & \ldots & & x-\alpha_n+1\end{pmatrix} \tag{4.3}$$

and this series converges in the half-plane $\Re(x)<\Re(\alpha+\beta_n)$. Putting $t=z/(1+z)$ in (4.1), we get

$$\begin{aligned}&\int_0^\infty z^{x-1}(1+z)^{-\alpha}\xi_n\left(\frac{z}{1+z}\right)dz\\ &\quad=\Gamma(\beta_n+1)\prod_1^n\frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)}F\begin{pmatrix}x-\alpha+1 & x+\gamma_1 & x+\gamma_2 & \cdots & x+\gamma_n \\ x-\alpha_1+1 & x-\alpha_2+1 & & \cdots & x-\alpha_n+1\end{pmatrix}.\end{aligned} \tag{4.4}$$

The right-hand side is regular in the strip (4.2) and the integral on the left-hand side is absolutely convergent in this strip. The integrand is regular for all finite values of z, different from 0 and -1, and from (1.33) we know how it behaves in the vicinity of the point at infinity. It is easily seen that the integral on the left does not change if we rotate the path of integration and integrate from 0 to $\infty e^{i\vartheta}$, where $\pi>\vartheta>-\pi$. It follows that we can apply Mellin's inversion theorem (see Doetsch [7] p. 115) and we get from (4.4)

$$\begin{aligned}&\xi_n(z)=\\ &\frac{\Gamma(\beta_n+1)}{2\pi i}z^{-\alpha}\int_{\varkappa-i\infty}^{\varkappa+i\infty}\left(\frac{1-z}{z}\right)^{x-\alpha}\prod_1^n\frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)}F\begin{pmatrix}x-\alpha+1 & x+\gamma_1 & x+\gamma_2\ldots x+\gamma_n \\ x-\alpha_1+1 & x-\alpha_2+1\ldots x-\alpha_n+1\end{pmatrix}dx,\end{aligned} \tag{4.5}$$

where $-\Re(\gamma_s) < \varkappa < \Re(\alpha + \beta_n)$, and this integral is convergent for $\pi > \arg\left(\frac{1}{z} - 1\right) > -\pi$, i.e. in the z-plane cut from 1 to $+\infty$ and from 0 to $-\infty$.

If in the integral (4.1) we substitute the series (1.33), term-by-term integration is permissible because the integral

$$\int_0^1 \left| t^{x+\gamma_i-1}(1-t)^{\alpha+\beta_n-x-1} \right| \sum_{\nu=0}^{\infty} \left| \frac{c_{\nu,n}^{(i)}}{(\beta_n+1)_\nu} \right| (1-t)^\nu \, dt$$

is convergent, as is easily seen from (1.49). Thus we get

$$\prod_{s=1}^{n} \frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)} F\begin{pmatrix} x-\alpha+1 & x+\gamma_1 & x+\gamma_2 \dots x+\gamma_n \\ x-\alpha_1+1 & x-\alpha_2+1 & \dots x-\alpha_n+1 \end{pmatrix} = \frac{\Gamma(x+\gamma_i)\,\Gamma(\alpha+\beta_n-x)}{\Gamma(\beta_n+1)\,\Gamma(\alpha+\beta_n+\gamma_i)} \sum_{\nu=0}^{\infty} \frac{c_{\nu,n}^{(i)}}{(\beta_n+1)_\nu} \frac{(\alpha+\beta_n-x)_\nu}{(\alpha+\beta_n+\gamma_i)_\nu}. \tag{4.6}$$

The series on the left is convergent in the half-plane $\Re(x) < \Re(\alpha+\beta_n)$, and the series on the right is convergent in the half-plane $\Re(x+\gamma_s) > 0$ $(s = 1, 2, \dots, n)$, thus both series are convergent in the strip (4.2). This remarkable transformation formula gives the analytic continuation in the entire plane of any hypergeometric series with $z = 1$, and it shows that the hypergeometric series on the left-hand side has poles of the first order at the points $x = \alpha + \beta_n$, $\alpha + \beta_n + 1$, $\alpha + \beta_n + 2, \dots$. In the special case $n = 2$ (4.6) reduces to a formula given by Thomae [61] p. 33 and later by Barnes [4].

α is a parameter, which we have introduced to secure the convergence in all cases. The right-hand side of (4.5) only apparently depends on α. If we put $\alpha = \alpha_i$, then F reduces to a hypergeometric series of the order n, but in that case we must assume that the parameters satisfy the inequalities $\Re(\alpha_i + \beta_n + \gamma_s) > 0$, $s = 1, 2, \dots, n$. If we give i the values $1, 2, \dots, n$, from (4.5), we get n equivalent integral representations valid under the last-mentioned assumption.

26. Now we choose a number γ so that $\Re(\alpha_s + \beta_n + \gamma) > 0$ $(s = 1, 2, \dots, n)$ and assume $\Re(\beta_n) > -1$. The integral

$$\int_1^\infty t^{-\gamma}(t-1)^{\gamma-x-1}\,\xi(t)\,dt = \int_1^\infty t^{-x-1}\left(1-\frac{1}{t}\right)^{\gamma-x-1} \xi_n(t)\,dt$$

is convergent in the strip

$$-\Re(\alpha_s) < \Re(x) < \Re(\beta_n + \gamma) \qquad s = 1, 2, \dots, n. \tag{4.7}$$

Expanding by the binomial theorem and integrating term-by-term we get by (2.19)

$$\begin{aligned}\int_1^\infty t^{-\gamma}(t-1)^{\gamma-x-1}\,\xi_n(t)\,dt &= \sum_{\nu=0}^{\infty}\frac{(x-\gamma+1)_\nu}{\nu!}\int_1^\infty t^{-x-\nu-1}\,\xi_n(t)\,dt \\ &= \Gamma(\beta_n+1)\prod_1^n \frac{\Gamma(x+\alpha_s)}{\Gamma(x-\gamma_s+1)}\,F\begin{pmatrix}x-\gamma+1 & x+\alpha_1 & x+\alpha_2 \dots x+\alpha_n\\ x-\gamma_1+1 & x-\gamma_2+1 & \dots x-\gamma_n+1\end{pmatrix},\end{aligned} \tag{4.8}$$

where the series on the right-hand side is convergent in the half-plane $\Re(x) < \Re(\beta_n+\gamma)$. Putting $t=(1+z)/z$ we get in the strip (4.7)

$$\begin{aligned}&\int_0^\infty z^{x-1}(1+z)^{-\gamma}\,\xi_n\left(\frac{1+z}{z}\right)dz \\ &\qquad = \Gamma(\beta_n+1)\prod_1^n \frac{\Gamma(x+\alpha_s)}{\Gamma(x-\gamma_s+1)}\,F\begin{pmatrix}x-\gamma+1 & x+\alpha_1 & x+\alpha_2 \dots x+\alpha_n\\ x-\gamma_1+1 & x-\gamma_2+1 & \dots x-\gamma_n+1\end{pmatrix}.\end{aligned} \tag{4.9}$$

The right-hand side is regular in the strip (4.7) and the integral on the left-hand side is absolutely convergent in this strip. The integrand is regular for all finite z different from 0 and -1; from (1.37) we know how it behaves in the vicinity of the point at infinity. The integral on the left-hand side does not change if we rotate the path of integration and integrate from the origin to $\infty e^{i\vartheta}$ where $\pi > \vartheta > -\pi$. It follows that we can apply Mellin's inversion theorem to (4.9) and we get

$$\begin{aligned}&\xi_n(z) = \\ &\frac{\Gamma(\beta_n+1)}{2\pi i}z^\gamma\int_{\varkappa-i\infty}^{\varkappa+i\infty}(z-1)^{x-\gamma}\prod_1^n\frac{\Gamma(x+\alpha_s)}{\Gamma(x-\gamma_s+1)}\,F\begin{pmatrix}x-\gamma+1 & x+\alpha_1 & x+\alpha_2 \dots x+\alpha_n\\ x-\gamma_1+1 & x-\gamma_2+1 & \dots x-\gamma_n+1\end{pmatrix}dx,\end{aligned} \tag{4.10}$$

where $-\Re(\alpha_s) < \varkappa < \Re(\beta_n+\gamma)$, and this integral is convergent for $\pi > \arg(z-1) > -\pi$. If we cut the z plane from the point 1 through the origin to $-\infty$, then the integral (4.10) is convergent in the cut plane.

If we substitute the series (1.37) into (4.8) and integrate term-by-term, we get

$$\begin{aligned}&\prod_{s=1}^n \frac{\Gamma(x+\alpha_s)}{\Gamma(x-\gamma_s+1)}\,F\begin{pmatrix}x-\gamma+1 & x+\alpha_1 & x+\alpha_2 \dots x+\alpha_n\\ x-\gamma_1+1 & x-\gamma_2+1 & \dots x-\gamma_n+1\end{pmatrix} \\ &\qquad = \frac{\Gamma(x+\alpha_i)\,\Gamma(\beta_n+\gamma-x)}{\Gamma(\beta_n+1)\,\Gamma(\alpha_i+\beta_n+\gamma)}\sum_{\nu=0}^{\infty}\frac{\bar c_{\nu,n}^{(i)}}{(\beta_n+1)_\nu}\,\frac{(\beta_n+\gamma-x)_\nu}{(\alpha_i+\beta_n+\gamma)_\nu},\end{aligned} \tag{4.11}$$

where the series on the left is convergent in the half-plane $\Re(x) < \Re(\beta_n+\gamma)$ and the series on the right is convergent in the half-plane $\Re(x+\alpha_s) > 0$, $(s=1, 2, \dots, n)$. Thus both series are convergent in the strip (4.7) and one of them gives the analytic continuation of the other.

If we put $\gamma = \gamma_i$ in (4.10), then F reduces to a hypergeometric series of the order n, but in that case we must assume that the parameters satisfy the inequalities $\Re(\alpha_s + \beta_n + \gamma_i) > 0$, $(s = 1, 2, \ldots, n)$. If we give i the values $1, 2, \ldots, n$, we get n equivalent integral representations for $\bar{\xi}_n(z)$.

Naturally we may obtain (4.11) from (4.6) by interchanging the α_i and the γ_i. In the same way (4.10) may be obtained from (4.5) by interchanging letters.

We have supposed $\Re(\beta_n) > -1$. By analytic continuation we see that (4.5) and (4.6) are valid for all β_n which are not negative integers.

27. If β_n is a negative integer, say $\beta_n = -p$, we may for the solution $\eta_n(z)$ regular at the point $z = 1$ repeat the preceding argument. Then we must use the relations (2.28) and (2.31) instead of (2.18) and (2.19) and take into consideration that the series

$$\sum_{\nu=1}^{\infty} \frac{(x)_\nu}{\nu!} \nu^r,$$

convergent in the half-plane $\Re(x) < -r$, where r is a non-negative integer, is identically zero for all x in the half-plane of convergence (see [42] p. 105). Then we get

$$\int_0^1 t^{x-1}(1-t)^{\alpha-x-1}\eta_n(t)\,dt = \prod_{s=1}^{n} \frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)} F\begin{pmatrix} x-\alpha+1 & x+\gamma_1 & x+\gamma_2 & \ldots & x+\gamma_n \\ x-\alpha_1+1 & x-\alpha_2+1 & \ldots & x-\alpha_n+1 & \end{pmatrix}. \tag{4.12}$$

This relation is valid in the strip (4.2). The integral on the left-hand side converges in the wider strip $-\Re(\gamma_s) < \Re(x) < \Re(\alpha)$, but the series on the right-hand side is only convergent for $\Re(x) < \Re(\alpha - p)$. Likewise we see that in the strip (4.7) we have

$$(-1)^p \int_1^\infty t^{-\gamma}(t-1)^{\gamma-x-1}\eta_n(t)\,dt = \prod_{s=1}^{n} \frac{\Gamma(x+\alpha_s)}{\Gamma(x-\gamma_s+1)} F\begin{pmatrix} x-\gamma+1 & x+\alpha_1 & x+\alpha_2 & \ldots & x+\alpha_n \\ x-\gamma_1+1 & x-\gamma_2+1 & \ldots & x-\gamma_n+1 & \end{pmatrix}, \tag{4.13}$$

where the series on the right is convergent in the half-plane $\Re(x) < \Re(\gamma - p)$. It follows that when one omits the factor $\Gamma(\beta_n + 1)$, the right-hand side of (4.5) represents the function $\eta_n(z)$ and the right-hand side of (4.10) represents the function $(-1)^p \eta_n(z)$. The relation (4.6) reduces to

$$\prod_{s=1}^{n} \frac{\Gamma(x+\gamma_s)}{\Gamma(x-\alpha_s+1)} F\begin{pmatrix} x-\alpha+1 & x+\gamma_1 & x+\gamma_2 \dots x+\gamma_n \\ x-\alpha_1+1 & x-\alpha_2+1 & \dots x-\alpha_n+1 \end{pmatrix} = \frac{\Gamma(x+\gamma_i)\,\Gamma(\alpha-x)}{\Gamma(\alpha+\gamma_i)} \sum_{\nu=0}^{\infty} \frac{c_{\nu+p,\,n}^{(i)}}{\nu!} \frac{(\alpha-x)_\nu}{(\alpha+\gamma_i)_\nu}, \tag{4.14}$$

where both series converge in the strip (4.2). The series on the left-hand side converges in the half-plane $\Re(x) < \Re(\alpha - p)$ and the series on the right-hand side converges in the half-plane $\Re(x+\gamma_s) > 0$, $s = 1, 2, \dots, n$. The function represented by these two series has poles of the first order in the points $x = \alpha, \alpha+1, \alpha+2, \dots$, but it is regular in the points $x = \alpha - 1, \alpha - 2, \dots, \alpha - p$. Thus we see that when β_n is not a negative integer, there is always a singularity situated on the line, which bounds the half-plane of convergence for the series on the left-hand side of (4.6), but when β_n is a negative integer this is not true. Then there is a strip of the width p to the right of the half-plane of convergence, in which the function in question is regular.

If we suppose that none of the differences between the γ_i is zero or an integer, the condition of convergence of the series on the right-hand side of (4.6) and (4.14) may be given a little more precisely. From (1.45) it appears that these series then converge if $\Re(x+\gamma_s) > 0$, $s = 1, 2, \dots, i-1, i+1, \dots, n$.

§ 5. The Solutions Regular at $z = 1$

28. We suppose now that none of the differences between the γ_i is zero or an integer. If we form the difference between two solutions of the form (3.17), we get

$$y_s^*(z) - y_h^*(z) = \frac{\sin \pi(\gamma_h - \gamma_s)}{2i} \int_{\varkappa - i\infty}^{\varkappa + i\infty} \frac{z^x f(x)\,dx}{\sin \pi(\gamma_s - x)\,\sin \pi(\gamma_h - x)}.$$

This integral is convergent for $2\pi > \arg z > -2\pi$ and therefore represents a solution regular at $z = 1$. For the sake of brevity we put

$$y_s^*(z) - y_h^*(z) = \frac{\sin \pi(\gamma_h - \gamma_s)}{\pi} y_{h,s}(z). \tag{5.1}$$

Then we have

$$y_{h,s}(z) = \frac{1}{2\pi i} \int_{\varkappa - i\infty}^{\varkappa + i\infty} \frac{\pi^2 z^x f(x)\,dx}{\sin \pi(\gamma_s - x)\,\sin \pi(\gamma_h - x)} \tag{5.2}$$

and consequently $y_{h,s}$ is symmetric in γ_h and γ_s. Here we may give h and s any two distinct of the values 1, 2, ..., n. All these solutions are not linearly independent. It is immediately seen that there exist between them relations of the form

$$\sin \pi (\gamma_h - \gamma_i)\, y_{h,i} + \sin \pi (\gamma_i - \gamma_j)\, y_{i,j} + \sin (\gamma_j - \gamma_h)\, y_{j,h} = 0, \tag{5.3}$$

but the solutions $y_{i,j}$ $(j = 1, 2, \ldots, i-1, i+1, \ldots, n)$ are linearly independent and together with ξ_n they form a fundamental system if β_n is not an integer.

If no two of the α_i are equal or differ by an integer, we see in the same way, if we form the difference between two solutions of the form (3.20) and put

$$\bar{y}_s^*(z) - \bar{y}_h^*(z) = \frac{\sin \pi (\alpha_h - \alpha_s)}{\pi}\, \bar{y}_{h,s}(z), \tag{5.4}$$

that $\bar{y}_{h,s}(z)$ is regular at $z = 1$, and we have

$$\bar{y}_{h,s}(z) = \frac{1}{2\pi i} \int_{\varkappa - i\infty}^{\varkappa + i\infty} \frac{\pi^2 z^x \bar{f}(x)\, dx}{\sin \pi (x + \alpha_s) \sin \pi (x + \alpha_h)}. \tag{5.5}$$

Consequently $\bar{y}_{h,s}$ is symmetric in α_h and α_s and we have the relations

$$\sin \pi (\alpha_h - \alpha_i)\, \bar{y}_{h,i} + \sin \pi (\alpha_i - \alpha_j)\, \bar{y}_{i,j} + \sin \pi (\alpha_j - \alpha_h)\, \bar{y}_{j,h} = 0. \tag{5.6}$$

The solutions $\bar{y}_{i,j}$ $(j = 1, 2, \ldots, i-1, i+1, \ldots, n)$ are linearly independent and together with ξ_n they form a fundamental system, if β_n is not an integer.

It is sufficient to consider one of these solutions regular at $z = 1$, for instance $y_{1,2}(z)$, as the others may be derived from this by interchanging letters. Thus we have for $2\pi > \arg z > -2\pi$

$$y_{1,2}(z) = \frac{1}{2\pi i} \int_{\varkappa - i\infty}^{\varkappa + i\infty} \frac{z^x \Gamma(\gamma_1 - x)\, \Gamma(\gamma_2 - x) \prod_1^n \Gamma(x + \alpha_\nu)}{\prod_3^n \Gamma(x + 1 - \gamma_\nu)}\, dx, \tag{5.7}$$

where the contour is curved to separate the increasing and decreasing sequences of poles. We shall give a representation of these solutions in series of hypergeometric polynomials. By this we shall need the following lemma.

29. In (3.3) we replace n by $n+1$ and let $z \to 1$ on both edges of the cut from 0 to ∞. Then we get two equations between which we eliminate the last term on the right-hand side. If in addition we replace α_{n+1} by $-x$, we get the following relation

$$\frac{\prod_{\nu=1}^{n}\Gamma(\alpha_\nu+\gamma_1)}{\prod_{\nu=1}^{n+1}\Gamma(\gamma_1-\gamma_\nu+1)} F\begin{pmatrix}\alpha_1+\gamma_1 & \alpha_2+\gamma_1 & \dots & \alpha_n+\gamma_1 & \gamma_1-x \\ \gamma_1-\gamma_2+1 & \gamma_1-\gamma_3+1 & \dots & \gamma_1-\gamma_n+1 & \end{pmatrix}$$
$$= \sum_{i=1}^{n} A_i \frac{\Gamma(x-\gamma_1+1)}{\Gamma(x+\alpha_i+1)} F\begin{pmatrix}\alpha_i+\gamma_1 & \alpha_i+\gamma_2 & \dots & \alpha_i+\gamma_{n+1} \\ \alpha_i-\alpha_1+1 & \dots & \alpha_i-\alpha_n+1 & x+\alpha_i+1\end{pmatrix}, \tag{5.8}$$

where the A_i are independent of x and have the following values:

$$A_i = \Gamma(\alpha_i+\gamma_1) \frac{\prod_{\nu=1}^{n}{}' \Gamma(\alpha_\nu-\alpha_i)}{\prod_{\nu=2}^{n+1}\Gamma(1-\alpha_i-\gamma_\nu)}.$$

Here all series are convergent if the real part of x is greater than the real part of $-n+\sum_1^n \alpha_i + \sum_1^{n+1}\gamma_i$. The hypergeometric series on the right-hand side are factorial series in x, which converge uniformly towards 1, when x increases towards infinity in the half-plane of convergence. Therefore (5.8) shows immediately how the function of x defined by the series on the left-hand side behaves asymptotically in the half-plane of convergence. This may be expressed more briefly in the following way. For any hypergeometric series with $z=1$ we have

$$F\begin{pmatrix}\alpha_1 & \alpha_2 & \dots & \alpha_n & -x \\ \gamma_1 & \gamma_2 & \dots & & \gamma_n\end{pmatrix} \sim \sum_{i=1}^{n}\frac{C_i}{x^{\alpha_i}}, \tag{5.9}$$

where C_i are constants independent of x. This series converges if the real part of x is greater than the real part of $\sum_1^n(\alpha_i-\gamma_i)$ and the asymptotic expression is valid when x approaches the point at infinity in the half-plane of convergence. By this we must suppose that none of the differences between the α_i is zero or integers and that the α_i and the γ_i are not zero or negative integers. Moreover, it can easily be proved that for the analytic continuation of the function of x defined by the series we have the same asymptotic expression in the angle $\pi-\varepsilon > \arg x > -\pi+\varepsilon$, but here we do not need this extension.

One may derive a similar and in a certain respect more general theorem from the transformation formula (1.22), which can be written

$$(1-x)^{-1} F\left(\begin{matrix}1 & \alpha_1 & \dots & \alpha_n \\ \gamma_1 & \gamma_2 & \dots & \gamma_n\end{matrix}\middle| z\frac{x}{x-1}\right) = \sum_{\nu=0}^{\infty} F\left(\begin{matrix}\alpha_1 & \alpha_2 & \dots & \alpha_n & -\nu \\ \gamma_1 & \gamma_2 & \dots & & \gamma_n\end{matrix}\middle| z\right) x^\nu.$$

Here we think of z as a fixed number which satisfies the inequality $|z-1|<1$. Then the point $x=1$ is the singularity nearest to the origin, and the left-hand side may by (3.3) be represented as a sum of $n+1$ power series in $1-x$ with the first exponents 0, α_1-1, α_2-1, ..., α_n-1. From a previously proved theorem ([39] pp. 21–22) it follows that for large positive values of ν we have

$$F\begin{pmatrix}\alpha_1 & \alpha_2 & \dots & \alpha_n & -\nu \\ \gamma_1 & \gamma_2 & \dots & & \gamma_n\end{pmatrix}\bigg|\, z\bigg) = \sum_{i=1}^{n} C_i\, \nu^{-\alpha_i} + O\left(\left|\nu^{-\alpha_i-1}\right|\right), \tag{5.10}$$

where C_i does not depend on ν. This relation is valid under the assumptions mentioned above for (5.9). But if any of the differences between the α_i is zero or an integer, it is seen in the same way by (3.38), using the theorem of Perron [46] p. 368 already mentioned in art. 8, that

$$F\left(\begin{matrix}\alpha_1 & \alpha_2 & \dots & \alpha_n & -\nu \\ \gamma_1 & \gamma_2 & \dots & & \gamma_n\end{matrix}\,\middle|\, z\right) \sim \sum_{i=1}^{n} C_i\, \nu^{-\alpha_i} (\log \nu)^{r_i}, \tag{5.11}$$

where r_i are non-negative integers and $|z-1|<1$.

30. We suppose now that no two of the γ_i are equal or differ by an integer and we shall, for $n>2$, consider the solution $y_{1,2}(z)$, regular at $z=1$. If $n=2$, we have the well-known relation

$$\begin{aligned}\frac{\Gamma(\alpha_1+\gamma_2)\,\Gamma(\alpha_2+\gamma_2)}{\Gamma(\gamma_2-\gamma_1+1)}\, z^{\gamma_2} F\left(\begin{matrix}\alpha_1+\gamma_2 & \alpha_2+\gamma_2 \\ \gamma_2-\gamma_1+1 & \end{matrix}\,\middle|\, z\right) - \\ - \frac{\Gamma(\alpha_1+\gamma_1)\,\Gamma(\alpha_2+\gamma_1)}{\Gamma(\gamma_1-\gamma_2+1)}\, z^{\gamma_1} F\left(\begin{matrix}\alpha_1+\gamma_1 & \alpha_2+\gamma_1 \\ \gamma_1-\gamma_2+1 & \end{matrix}\,\middle|\, z\right) \\ = \frac{\sin \pi(\gamma_1-\gamma_2)}{\pi}\, \frac{\Gamma(\alpha_1+\gamma_1)\,\Gamma(\alpha_2+\gamma_1)\,\Gamma(\alpha_1+\gamma_2)\,\Gamma(\alpha_2+\gamma_2)}{\Gamma(1-\beta_2)}\, z^{\gamma_1} F\left(\begin{matrix}\alpha_1+\gamma_1 & \alpha_2+\gamma_1 \\ 1-\beta_2 & \end{matrix}\,\middle|\, 1-z\right)\end{aligned} \tag{5.12}$$

given by Gauss in a non-symmetric form [14] p. 213.

The relation (2.19) can be written

$$\int_0^z t^{x-1}\, \xi_n\left(\frac{z}{t}\right) dt = z^x\, \Gamma(\beta_n+1) \prod_{s=1}^{n} \frac{\Gamma(x+\alpha_s)}{\Gamma(x+1-\gamma_s)}, \tag{5.13}$$

where $\Re(\beta_n) > -1$ and $\Re(x+\alpha_s)>0$, $s=1, 2, \dots, n$. The function $\xi_n(z)$ depends on the parameters $\alpha_1, \dots, \alpha_n$ and $\gamma_1, \dots, \gamma_n$, and we can more elaborately write

$$\xi_n\left(\begin{matrix}\alpha_1 & \alpha_2 & \dots & \alpha_n \\ \gamma_1 & \gamma_2 & \dots & \gamma_n\end{matrix}\,\middle|\, z\right).$$

If in (5.13) we replace n by $n-p$, where $p<n$, we get

$$\int_0^z t^{x-1}\,\xi_{n-p}\left(\begin{matrix}\alpha_{p+1}\ \ldots\ \alpha_n\\ \gamma_{p+1}\ \ldots\ \gamma_n\end{matrix}\,\middle|\,\frac{z}{t}\right)dt = z^x\,\Gamma(\beta_n-\beta_p)\prod_{p+1}^{n}\frac{\Gamma(x+\alpha_s)}{\Gamma(x+1-\gamma_s)}, \tag{5.14}$$

where $\Re(\beta_n-\beta_p)>0$ and $\Re(x+\alpha_s)>0$, $s=p+1, p+2, \ldots, n$.

We shall now consider the integral

$$\int_0^z t^{\gamma_1-1}F\left(\begin{matrix}\alpha_1+\gamma_1\ \ldots\ \alpha_p+\gamma_1\\ \gamma_1-\gamma_2+1\ \ldots\ \gamma_1-\gamma_p+1\end{matrix}\,\middle|\,t\right)\xi_{n-p}\left(\begin{matrix}\alpha_{p+1}\ \ldots\ \alpha_n\\ \gamma_{p+1}\ \ldots\ \gamma_n\end{matrix}\,\middle|\,\frac{z}{t}\right)dt, \tag{5.15}$$

where we suppose $|z|<1$, $\Re(\beta_n-\beta_p)>0$ and $\Re(\alpha_s+\gamma_1)>0$, $s=p+1, p+2, \ldots, n$. Then the integral is convergent. We expand the hypergeometric function under the sign of integration in powers of t. This series is uniformly convergent in the interval of integration and the integral (5.14) is absolutely convergent for $x=\gamma_1$. Therefore we can integrate term-by-term, and if for the sake of brevity we denote the integral (5.15) by $g(z)$, we have by (5.14)

$$\Gamma(\beta_n-\beta_p)\,y_1^*(z)=\prod_{s=1}^{p}\frac{\Gamma(\alpha_s+\gamma_1)}{\Gamma(\gamma_1-\gamma_s+1)}\,g(z). \tag{5.16}$$

If we put $p=2$, interchange γ_1 and γ_2 and form the difference between the two expressions thus obtained, we have by (5.12)

$$y_{1,2}(z)=K\int_0^z t^{\gamma_1}F\left(\begin{matrix}\alpha_1+\gamma_1\ \ \alpha_2+\gamma_1\\ 1-\beta_2\end{matrix}\,\middle|\,1-t\right)\xi_{n-2}\left(\begin{matrix}\alpha_3\ \ldots\ \alpha_n\\ \gamma_3\ \ldots\ \gamma_n\end{matrix}\,\middle|\,\frac{z}{t}\right)\frac{dt}{t}, \tag{5.17}$$

where

$$K=\frac{\Gamma(\alpha_1+\gamma_1)\,\Gamma(\alpha_2+\gamma_1)\,\Gamma(\alpha_1+\gamma_2)\,\Gamma(\alpha_2+\gamma_2)}{\Gamma(\beta_n-\beta_2)\,\Gamma(1-\beta_2)}.$$

We suppose now that

$$|z-1|<1,\ \Re(\beta_n-\beta_2)>0,\ \Re(\gamma_2)>\Re(\gamma_1)>-\Re(\alpha_s),\qquad s=1, 2, \ldots, n. \tag{5.18}$$

We expand the hypergeometric function under the sign of integration in powers of $1-t$. This series is convergent for $t=0$ and consequently uniformly convergent in the interval of integration and the integral (5.14) is absolutely convergent for $x=\gamma_1$. Therefore we can integrate term-by-term and we get

$$y_{1,2}(z)=K\sum_{\nu=0}^{\infty}\frac{(\alpha_1+\gamma_1)_\nu(\alpha_2+\gamma_1)_\nu}{\nu!\ (1-\beta_2)_\nu}\int_0^z t^{\gamma_1-1}(1-t)^\nu\,\xi_{n-2}\left(\begin{matrix}\alpha_3\ \ldots\ \alpha_n\\ \gamma_3\ \ldots\ \gamma_n\end{matrix}\,\middle|\,\frac{z}{t}\right)dt. \tag{5.19}$$

From (5.14) follows now

$$\int_0^z t^{\gamma_1-1}(1-t)^\nu \xi_{n-2}\left(\frac{z}{t}\right)dt = \sum_{s=0}^{\nu}\frac{(-\nu)_s}{s!}\int_0^z t^{\gamma_1+s-1}\xi_{n-2}\left(\frac{z}{t}\right)dt$$

$$= z^{\gamma_1}\Gamma(\beta_n-\beta_2)\prod_3^n \frac{\Gamma(\alpha_s+\gamma_1)}{\Gamma(\gamma_1-\gamma_s+1)} F\left(\begin{matrix}-\nu & \alpha_3+\gamma_1 & \alpha_4+\gamma_1 & \dots & \alpha_n+\gamma_1\\ & \gamma_1-\gamma_3+1 & \gamma_1-\gamma_4+1 & \dots & \gamma_1-\gamma_n+1\end{matrix}\middle| z\right).$$

Substituting this expression in (5.19) we get

$$y_{1,2}(z) = C_1 z^{\gamma_1}\sum_{\nu=0}^{\infty}\frac{(\alpha_1+\gamma_1)_\nu(\alpha_2+\gamma_1)_\nu}{\nu!\,(\alpha_1+\alpha_2+\gamma_1+\gamma_2)_\nu} F\left(\begin{matrix}-\nu & \alpha_3+\gamma_1 & \alpha_4+\gamma_1 & \dots & \alpha_n+\gamma_1\\ & \gamma_1-\gamma_3+1 & \gamma_1-\gamma_4+1 & \dots & \gamma_1-\gamma_n+1\end{matrix}\middle| z\right), \quad (5.20)$$

where

$$C_1 = \frac{\Gamma(\alpha_1+\gamma_2)\,\Gamma(\alpha_2+\gamma_2)}{\Gamma(\alpha_1+\alpha_2+\gamma_1+\gamma_2)}\,\frac{\prod\limits_{s=1}^{n}\Gamma(\alpha_s+\gamma_1)}{\prod\limits_{s=3}^{n}\Gamma(\gamma_1-\gamma_s+1)}. \quad (5.21)$$

From (5.11) it follows that the series on the right-hand side of (5.20) is convergent if $|z-1|<1$ and $\Re(\alpha_s+\gamma_2)>0$, $s=3, 4, \dots, n$. By analytic continuation it is seen that (5.20) is valid if these conditions are fulfilled and none of the numbers $\alpha_1+\gamma_2$, $\alpha_2+\gamma_2$, $\alpha_s+\gamma_1$, $s=1, 2, \dots, n$ is zero or negative integers.

Putting $p=1$ in (5.15) we have in the same way

$$y_{1,2}(z) = K\int_0^z t^{\gamma_2-1} F\left(\begin{matrix}1 & \alpha_1+\gamma_2\\ & \alpha_1+\gamma_1+1\end{matrix}\middle| 1-t\right)\xi_{n-1}\left(\begin{matrix}\alpha_2 & \dots & \alpha_n\\ \gamma_2 & \dots & \gamma_n\end{matrix}\middle|\frac{z}{t}\right)dt$$

$$= K\int_0^z t^{\gamma_1-1} F\left(\begin{matrix}\alpha_1+\gamma_1 & \gamma_1-\gamma_2+1\\ & \alpha_1+\gamma_1+1\end{matrix}\middle| 1-t\right)\xi_{n-1}\left(\begin{matrix}\alpha_2 & \dots & \alpha_n\\ \gamma_2 & \dots & \gamma_n\end{matrix}\middle|\frac{z}{t}\right)dt,$$

where

$$K = \frac{\Gamma(\alpha_1+\gamma_2)\,\Gamma(\gamma_1-\gamma_2+1)}{(\alpha_1+\gamma_1)\,\Gamma(\beta_n-\beta_1)}.$$

Expanding in powers of $1-t$ and integrating term-by-term we get in the same way as above the following two new expansions

$$y_{1,2}(z) = C_2 z^{\gamma_1}\sum_{\nu=0}^{\infty}\frac{(\alpha_1+\gamma_1)_\nu}{(\alpha_1+\gamma_2)_{\nu+1}} F\left(\begin{matrix}-\nu & \alpha_2+\gamma_1 & \alpha_3+\gamma_1 & \dots & \alpha_n+\gamma_1\\ 1 & \gamma_1-\gamma_3+1 & \gamma_1-\gamma_4+1 & \dots & \gamma_1-\gamma_n+1\end{matrix}\middle| z\right) \quad (5.22)$$

$$y_{1,2}(z) = C_3 z^{\gamma_1}\sum_{\nu=0}^{\infty}\frac{(\gamma_1-\gamma_2+1)_\nu}{\nu!\,(\alpha_1+\gamma_1+\nu)} F\left(\begin{matrix}-\nu & \alpha_2+\gamma_1 & \alpha_3+\gamma_1 & \dots & \alpha_n+\gamma_1\\ & \gamma_1-\gamma_2+1 & \gamma_1-\gamma_3+1 & \dots & \gamma_1-\gamma_n+1\end{matrix}\middle| z\right), \quad (5.23)$$

where

$$C_2 = \Gamma(\gamma_2 - \gamma_1 + 1) \frac{\prod_{s=1}^{n} \Gamma(\alpha_s + \gamma_1)}{\prod_{s=3}^{n} \Gamma(\gamma_1 - \gamma_s + 1)} \tag{5.24}$$

$$C_3 = \Gamma(\alpha_1 + \gamma_2) \frac{\prod_{s=2}^{n} \Gamma(\alpha_s + \gamma_1)}{\prod_{s=3}^{n} \Gamma(\gamma_1 - \gamma_s + 1)}. \tag{5.25}$$

These series are convergent and represent $y_{1,2}(z)$ if $|z-1|<1$ and $\Re(\alpha_s+\gamma_2)>0$ $s=2, 3, \ldots, n$. Furthermore we must assume that none of the numbers $\alpha_1+\gamma_2$, $\alpha_s+\gamma_1$, $s=1, 2, \ldots, n$ is zero or a negative integer.

Next we shall consider

$$\int_0^z t^{\gamma_1-1} \xi_n\left(\frac{z}{t}\right) \frac{dt}{1-t},$$

where $|z|<1$, $\Re(\beta_n)>-1$ and $\Re(\alpha_s+\gamma_1)>0$, $s=1, 2, \ldots, n$. Then the integral is convergent and if we replace $(1-t)^{-1}$ by $\sum_0^\infty t^\nu$ term-by-term integration is permissible for the same reason as above and we get from (5.13)

$$y_1^*(z) = \frac{1}{\Gamma(\beta_n+1)} \int_0^z t^{\gamma_1-1} \xi_n\left(\frac{z}{t}\right) \frac{dt}{1-t}. \tag{5.26}$$

From this follows again

$$y_{1,2}(z) = \frac{\pi}{\sin \pi(\gamma_1-\gamma_2)} \frac{1}{\Gamma(\beta_n+1)} \int_0^z \frac{t^{\gamma_2}-t^{\gamma_1}}{1-t} \xi_n\left(\frac{z}{t}\right) \frac{dt}{t}. \tag{5.27}$$

We assume now in (5.27)

$$|z-1|<1,\ \Re(\beta_n)>-1,\ \Re(\gamma_2)>\Re(\gamma_1)>-\Re(\alpha_s) \qquad s=1, 2, \ldots, n. \tag{5.28}$$

From the binomial theorem we get

$$\frac{t^{\gamma_2}-t^{\gamma_1}}{1-t} = t^{\gamma_1} \sum_{\nu=0}^{\infty} \frac{(\gamma_1-\gamma_2)_{\nu+1}}{(\nu+1)!} (1-t)^\nu. \tag{5.29}$$

The series on the right-hand side is uniformly convergent in the interval of integration and the integral (5.13) is absolutely convergent for $x=\gamma_1$. Inserting the series (5.29) in the integral (5.27) we are allowed to integrate term-by-term and we get

$$y_{1,2}(z)=\frac{\pi}{\sin\pi(\gamma_1-\gamma_2)}\frac{1}{\Gamma(\beta_n+1)}\sum_{\nu=0}^{\infty}\frac{(\gamma_1-\gamma_2)_{\nu+1}}{(\nu+1)!}\int_0^z t^{\gamma_1-1}(1-t)^{\nu}\,\xi_n\left(\frac{z}{t}\right)dt. \qquad (5.30)$$

From (5.13) it now follows

$$\int_0^z t^{\gamma_1-1}(1-t)^{\nu}\,\xi_n\left(\frac{z}{t}\right)dt=\sum_{s=0}^{\nu}\frac{(-\nu)_s}{s!}\int_0^z t^{\gamma_1+s-1}\,\xi_n\left(\frac{z}{t}\right)dt$$

$$=z^{\gamma_1}\Gamma(\beta_n+1)\prod_{s=1}^{n}\frac{\Gamma(\alpha_s+\gamma_1)}{\Gamma(\gamma_1-\gamma_s+1)}F\left(\begin{matrix}-\nu & \alpha_1+\gamma_1 & \alpha_2+\gamma_1 & \dots & \alpha_n+\gamma_1\\ 1 & \gamma_1-\gamma_2+1 & \gamma_1-\gamma_3+1 & \dots & \gamma_1-\gamma_n+1\end{matrix}\middle| z\right).$$

Inserting this expression in (5.30) we get

$$y_{1,2}(z)=C\,z^{\gamma_1}\sum_{\nu=0}^{\infty}\frac{(\gamma_1-\gamma_2+1)_{\nu}}{(\nu+1)!}F\left(\begin{matrix}-\nu & \alpha_1+\gamma_1 & \alpha_2+\gamma_1 & \dots & \alpha_n+\gamma_1\\ 1 & \gamma_1-\gamma_2+1 & \gamma_1-\gamma_3+1 & \dots & \gamma_1-\gamma_n+1\end{matrix}\middle| z\right), \qquad (5.31)$$

where

$$C=\frac{\pi(\gamma_1-\gamma_2)}{\sin\pi(\gamma_1-\gamma_2)}\prod_{s=1}^{n}\frac{\Gamma(\alpha_s+\gamma_1)}{\Gamma(\gamma_1-\gamma_s+1)}. \qquad (5.32)$$

Thus this relation is proved when the conditions (5.28) are fulfilled, but from (5.11) it appears that the last series converges when $|z-1|<1$ and $\Re(\alpha_s+\gamma_2)>0$, $s=1, 2, \dots, n$. By analytic continuation we see that (5.31) is true when the last-mentioned conditions are fulfilled and none of the numbers $\alpha_s+\gamma_1$, $s=1, 2, \dots, n$, is zero or negative integers.

As $y_{1,2}(z)$ is symmetric in γ_1 and γ_2, we can interchange γ_1 and γ_2 in these four series of hypergeometric polynomials. We then get new series which represent $y_{1,2}(z)$. Furthermore we may permute the α_i and still the series represent the same solution.

If we interchange α and γ and replace z by $1/z$, we get the series which represent the solution $\bar{y}_{1,2}(z)$ regular at $z=1$. These series are convergent in the half-plane $\Re(z)>\frac{1}{2}$.

In this connection we shall in addition mention a remarkable expansion. The transformation formula (1.22) can be written

$$(1-x)^{-\alpha}F\left(\begin{matrix}\alpha & \alpha_1 & \alpha_2 & \dots & \alpha_n\\ & \gamma_1 & \gamma_2 & \dots & \gamma_n\end{matrix}\middle| z\frac{x}{x-1}\right)=\sum_{\nu=0}^{\infty}\frac{(\alpha)_\nu}{\nu!}F\left(\begin{matrix}-\nu & \alpha_1 & \alpha_2 & \dots & \alpha_n\\ & \gamma_1 & \gamma_2 & \dots & \gamma_n\end{matrix}\middle| z\right)x^{\nu}. \qquad (5.33)$$

If $|z-1|<1$ and $\Re(\alpha)<\Re(\alpha_s)$, $s=1, 2, \dots, n$, the series on the right-hand side is convergent for $x=1$ and from (3.3) it follows that the left-hand side converges towards a limit when $x\to1$. Then this limit is equal to the sum of the series in virtue of Abel's theorem on continuity of power-series and we have

$$\sum_{\nu=0}^{\infty} \frac{(\alpha)_\nu}{\nu!} F\left(\begin{matrix} -\nu & \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \gamma_1 & \gamma_2 & & \dots & \gamma_n \end{matrix} \middle| z\right) = z^{-\alpha} \prod_{s=1}^{n} \frac{\Gamma(\alpha_s - \alpha)\,\Gamma(\gamma_s)}{\Gamma(\gamma_s - \alpha)\,\Gamma(\alpha_s)}. \tag{5.34}$$

This series can then be evaluated in terms of gamma-functions. In particular if α is a negative integer $-p$, we have the elementary relation

$$\sum_{\nu=0}^{p} (-1)^\nu \binom{p}{\nu} F\left(\begin{matrix} -\nu & \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \gamma_1 & \gamma_2 & & \dots & \gamma_n \end{matrix} \middle| z\right) = z^p \prod_{s=1}^{n} \frac{(\alpha_s)_p}{(\gamma_s)_p}.$$

It is easily seen that all these relations may be obtained by convolution, but the above proof is simpler.

31. If in (5.7) we differentiate s times with respect to z and then put $z=1$, we get

$$(-1)^s \left(\frac{d^s\,(z^{-\gamma_2}\, y_{1,2}(z))}{d z^s}\right)_{z=1} = \frac{1}{2\pi i} \int_{\varkappa - i\infty}^{\varkappa + i\infty} \frac{\Gamma(\gamma_1 - x)\,\Gamma(\gamma_2 + s - x) \prod_{1}^{n} \Gamma(x + \alpha_\nu)}{\prod_{3}^{n} \Gamma(x + 1 - \gamma_\nu)}\, dx.$$

The integral on the right-hand side is equal to the series which is obtained from (5.20) by replacing γ_2 by $\gamma_2 + s$ and putting $z=1$. From Taylor's formula we then get

$$y_{1,2}(z) = C_1 z^{\gamma_2} \sum_{s=0}^{\infty} \frac{(\alpha_1 + \gamma_2)_s\,(\alpha_2 + \gamma_2)_s}{s!\,(\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2)_s} A_s\,(1-z)^s, \tag{5.35}$$

where C_1 is the constant (5.21) and

$$A_s = \sum_{\nu=0}^{\infty} \frac{(\alpha_1 + \gamma_1)_\nu\,(\alpha_2 + \gamma_1)_\nu}{\nu!\,(\alpha_1 + \alpha_2 + \gamma_1 + \gamma_2 + s)_\nu} F\left(\begin{matrix} -\nu & \alpha_3 + \gamma_1 & \alpha_4 + \gamma_1 & \dots & \alpha_n + \gamma_1 \\ \gamma_1 - \gamma_3 + 1 & \gamma_1 - \gamma_4 + 1 & \dots & \gamma_1 - \gamma_n + 1 \end{matrix}\right).$$

The power series (5.35) is convergent if $|z-1|<1$ and $\Re(\alpha_s + \gamma_2) > 0$, $s = 3, 4, \dots, n$. From (5.22) we get in the same way

$$y_{1,2}(z) = C_2 z^{\gamma_2} \sum_{s=0}^{\infty} \frac{(\gamma_2 - \gamma_1 + 1)_s}{s!} B_s\,(1-z)^s, \tag{5.36}$$

where C_2 is the constant (5.24) and

$$B_s = \sum_{\nu=0}^{\infty} \frac{(\alpha_1 + \gamma_1)_\nu}{(\alpha_1 + \gamma_2 + s)_{\nu+1}} F\left(\begin{matrix} -\nu & \alpha_2 + \gamma_1 & \alpha_3 + \gamma_1 & \dots & \alpha_n + \gamma_1 \\ 1 & \gamma_1 - \gamma_3 + 1 & \gamma_1 - \gamma_4 + 1 & \dots & \gamma_1 - \gamma_n + 1 \end{matrix}\right).$$

The last series is convergent if $\Re(\alpha_i + \gamma_2) > -s$, $i = 2, 3, \dots, n$. A_s and B_s converge towards 1 and zero respectively when $s \to \infty$. The series (5.36) is convergent for $|z-1|<1$. If we multiply both sides of (5.7) by z^{α_1} and differentiate with respect to $1/z$, we get in the same way from Taylor's formula

$$y_{1,2}(z) = C_1 z^{-\alpha_1} \sum_{s=0}^{\infty} \frac{(\alpha_1+\gamma_1)_s (\alpha_1+\gamma_2)_s}{s!\,(\alpha_1+\alpha_2+\gamma_1+\gamma_2)_s} E_s \left(\frac{z-1}{z}\right)^s, \tag{5.37}$$

where C_1 is the constant (5.21) and

$$E_s = \sum_{\nu=0}^{\infty} \frac{(\alpha_1+\gamma_1+s)_\nu (\alpha_2+\gamma_1)_\nu}{\nu!\,(\alpha_1+\alpha_2+\gamma_1+\gamma_2+s)_\nu} F\begin{pmatrix} -\nu & \alpha_3+\gamma_1 & \alpha_4+\gamma_1 & \dots & \alpha_n+\gamma_1 \\ \gamma_1-\gamma_3+1 & \gamma_1-\gamma_4+1 & \dots & \gamma_1-\gamma_n+1 & \end{pmatrix}.$$

The series (5.37) is convergent in the half-plane $\Re(z) > \frac{1}{2}$, if $\Re(\alpha_i+\gamma_2) > 0$, $i = 3, 4, \dots, n$. The series (5.35), (5.36) and (5.37) have the remarkable property that we are able to interchange γ_1 and γ_2 and permute all the α_i without changing the sum of the series. If $n = 3$, then (5.37) reduces to

$$y_{1,2}(z) = C_4 z^{-\alpha_1} \sum_{s=0}^{\alpha} \frac{(\alpha_1+\gamma_1)_s (\alpha_1+\gamma_2)_s}{s!\,(1-\beta_3)_s} F\begin{pmatrix} 1-\alpha_1-\gamma_3-s & 1-\alpha_2-\gamma_3 & 1-\alpha_3-\gamma_3 \\ & \gamma_1-\gamma_3+1 & \gamma_2-\gamma_3+1 \end{pmatrix} \left(\frac{z-1}{z}\right)^s,$$

and (5.35) reduces to

$$y_{1,2}(z) = C_4 z^{\gamma_1} \sum_{s=0}^{\infty} \frac{(\alpha_1+\gamma_1)_s (\alpha_2+\gamma_1)_s (\alpha_3+\gamma_1)_s}{s!\,(1-\beta_3)_s (\gamma_1-\gamma_3+1)_s} F\begin{pmatrix} 1-\alpha_1-\gamma_3 & 1-\alpha_2-\gamma_3 & 1-\alpha_3-\gamma_3 \\ & \gamma_1-\gamma_3+1+s & \gamma_2-\gamma_3+1 \end{pmatrix} (1-z)^s,$$

where
$$C_4 = \frac{\prod_{\nu=1}^{3} (\Gamma(\alpha_\nu+\gamma_1)\,\Gamma(\alpha_\nu+\gamma_2))}{\Gamma(1-\beta_3)\,\Gamma(\gamma_1-\gamma_3+1)\,\Gamma(\gamma_2-\gamma_3+1)}.$$

Here we must assume that $\Re(\beta_3) < 1$.

32. From (5.26) we get for all values of z which are not positive and ≥ 1

$$y_s^*(z) = \frac{z^{\gamma_s}}{\Gamma(\beta_n+1)} \int_1^{\infty} t^{-\gamma_s} \xi_n(t) \frac{dt}{t-z}, \tag{5.38}$$

where $\Re(\alpha_i+\gamma_s) > 0$, $i = 1, 2, \dots, n$ and $\Re(\beta_n) > -1$. We can get rid of the last condition by relating the above integral with a contour integral. Thus we find, if β_n is not an integer or zero,

$$y_s^*(z) = \Gamma(-\beta_n) \frac{z^{\gamma_s}}{2\pi i} \int_{\varkappa-i\infty}^{\varkappa+i\infty} t^{-\gamma_s} \frac{\xi_n(t)}{t-z}\, dt \qquad 0 < \varkappa < 1, \tag{5.39}$$

where z lies on the left of the path of integration. Letting z pass through this we get

$$y_s^*(z) = \Gamma(-\beta_n)\, \xi_n(z) + \varphi_s(z), \tag{5.40}$$

where $\varphi_s(z)$ is a solution regular in $z = 1$, which may be represented by the integral

$$\varphi_s(z) = \Gamma(-\beta_n)\frac{z^{\gamma_s}}{2\pi i}\int_{\varkappa-i\infty}^{\varkappa+i\infty} t^{-\gamma_s}\frac{\xi_n(t)}{t-z}\,dt \qquad 0<\varkappa<1, \tag{5.41}$$

where z lies on the right of the path of integration.

If we assume

$$\varphi_s(z) = z^{\gamma_s}\sum_{\nu=0}^{\infty} b_\nu (z-1)^\nu, \tag{5.42}$$

this series is convergent for $|z-1|<1$, and for the coefficients we have the following simple expression

$$b_\nu = \frac{\Gamma(-\beta_n)}{2\pi i}\int_{\varkappa-i\infty}^{\varkappa+i\infty} t^{-\gamma_s}\frac{\xi_n(t)}{(t-1)^{\nu+1}}\,dt. \tag{5.43}$$

Putting the expressions (3.17) and (2.46) into (5.40), we get the following integral representation of the solution $\varphi_s(z)$

$$\varphi_s(z) = \frac{1}{2i\sin\pi\beta_n}\int_{\varkappa-i\infty}^{\varkappa+i\infty} z^x\left[f(x)\frac{\sin\pi(\beta_n+\gamma_s-x)}{\sin\pi(\gamma_s-x)} + \bar{f}(x)\right]dx, \tag{5.44}$$

which is convergent for $2\pi > \arg z > -2\pi$. We have assumed that the poles γ_i are on the right and the poles $-\alpha_i$ on the left of the contour. The solution $\varphi_s(z)$ is of course a linear function of $y_{s,1}, y_{s,2}, \ldots, y_{s,n}$. If we give $y_{s,s}$ the meaning zero, we easily find from (3.43) and (5.1)

$$\varphi_s(z) = \sum_{i=1}^{n}\frac{\sin\pi(\gamma_s-\gamma_i)\prod_{\nu=1}^{n}\sin\pi(\alpha_\nu+\gamma_i)}{\pi\sin\pi\beta_n\prod_{\nu=1}^{n}{}'\sin\pi(\gamma_\nu-\gamma_i)}\,y_{s,i}(z). \tag{5.45}$$

§ 6. Logarithmic Singularities

33. We shall now consider the case where the number β_n defined by (1.15) is an integer. For the sake of brevity we omit in what follows the index and write β for β_n. If $\beta \geq 0$, the integral (5.38) is applicable, when $\Re(\alpha_i+\gamma_s)>0$, $i=1, 2, \ldots, n$. We have now

$$\frac{1}{t-z} = \sum_{\nu=0}^{p-1}\frac{(z-1)^\nu}{(t-1)^{\nu+1}} + \frac{(z-1)^p}{(t-1)^p\,(t-z)}.$$

Putting $p=\beta$, we get

$$\int_1^\infty t^{-\gamma_s}\,\xi_n(t)\,\frac{dt}{t-z}=\sum_{\nu=0}^{\beta-1}(z-1)^\nu\int_1^\infty\frac{t^{-\gamma_s}\,\xi_n(t)}{(t-1)^{\nu+1}}\,dt+(z-1)^\beta\int_1^\infty\frac{t^{-\gamma_s}\,\xi_n(t)}{(t-1)^\beta\,(t-z)}\,dt.$$

From (4.8) we see that the first term on the right-hand side equals

$$\Gamma(\beta+1)\sum_{\nu=0}^{\beta-1}(z-1)^\nu\prod_{i=1}^{n}\frac{\Gamma(\alpha_i+\gamma_s+\nu)}{\Gamma(\gamma_s-\gamma_i+1+\nu)}\cdot$$
$$\cdot F\begin{pmatrix}\alpha_1+\gamma_s+\nu & \alpha_2+\gamma_s+\nu & \dots & \alpha_n+\gamma_s+\nu\\ \gamma_s-\gamma_1+1+\nu & \gamma_s-\gamma_2+1+\nu & \dots & \gamma_s-\gamma_n+1+\nu\end{pmatrix}.$$

If this is substituted in (5.38), we get

$$y_s^*(z)-z^{\gamma_s}\sum_{\nu=0}^{\beta-1}(z-1)^\nu\prod_{i=1}^{n}\frac{\Gamma(\alpha_i+\gamma_s+\nu)}{\Gamma(\gamma_s-\gamma_i+1+\nu)}F\begin{pmatrix}\alpha_1+\gamma_s+\nu & \dots & \alpha_n+\gamma_s+\nu\\ \gamma_s-\gamma_1+1+\nu & \dots & \gamma_s-\gamma_n+1+\nu\end{pmatrix}$$
$$=\frac{z^{\gamma_s}(z-1)^\beta}{\Gamma(\beta+1)}\int_1^\infty\frac{t^{-\gamma_s}\,\xi_n(t)}{(t-1)^\beta\,(t-z)}\,dt. \tag{6.1}$$

If we replace the integral on the right-hand side by a contour integral, we get

$$-\frac{z^{\gamma_s}(z-1)^\beta}{\Gamma(\beta+1)}\,\frac{1}{2\pi i}\int_{\varkappa-i\infty}^{\varkappa+i\infty}t^{-\gamma_s}\,(1-t)^{-\beta}\log(1-t)\frac{\xi_n(t)}{t-z}\,dt\qquad 0<\varkappa<1,$$

where z is to the left of the contour. If we let z cross the contour, we get

$$-\frac{(-1)^\beta}{\Gamma(\beta+1)}\xi_n(z)\log(1-z)-\frac{z^{\gamma_s}(z-1)^\beta}{\Gamma(\beta+1)}\,\frac{1}{2\pi i}\int_{\varkappa-i\infty}^{\varkappa+i\infty}t^{-\gamma_s}(1-t)^{-\beta}\log(1-t)\frac{\xi_n(t)}{t-z}\,dt,$$

where z is on the right of the contour. If this is substituted in (6.1), we get the remarkable equation

$$y_s^*(z)=z^{\gamma_s}\sum_{\nu=0}^{\beta-1}(z-1)^\nu\prod_{i=1}^{n}\frac{\Gamma(\alpha_i+\gamma_s+\nu)}{\Gamma(\gamma_s-\gamma_i+1+\nu)}F\begin{pmatrix}\alpha_1+\gamma_s+\nu & \dots & \alpha_n+\gamma_s+\nu\\ \gamma_s-\gamma_1+1+\nu & \dots & \gamma_s-\gamma_n+1+\nu\end{pmatrix}-$$
$$-\frac{(-1)^\beta}{\Gamma(\beta+1)}\xi_n(z)\log(1-z)+\varphi_s(z), \tag{6.2}$$

$\varphi_s(z)$ being a function regular at $z=1$ and having the integral representation

$$\varphi_s(z)=\frac{(-1)^{\beta+1}}{\Gamma(\beta+1)}z^{\gamma_s}(1-z)^\beta\frac{1}{2\pi i}\int_{\varkappa-i\infty}^{\varkappa+i\infty}t^{-\gamma_s}(1-t)^{-\beta}\log(1-t)\,\frac{\xi_n(t)}{t-z}\,dt\qquad 0<\varkappa<1. \tag{6.3}$$

$\varphi_s(z)$ may be expanded in a power series convergent in the circle $|z-1|<1$ and having the form

$$\varphi_s(z)=\frac{(-1)^{\beta}}{\Gamma(\beta+1)}z^{\gamma_s}(1-z)^{\beta}\sum_{\nu=0}^{\infty}b_\nu(1-z)^\nu, \tag{6.4}$$

where the coefficients b_ν have the simple integral representation

$$b_\nu=\frac{1}{2\pi i}\int_{\varkappa-i\infty}^{\varkappa+i\infty}\frac{t^{-\gamma_s}\xi_n(t)}{(1-t)^{\beta+\nu+1}}\log(1-t)\,dt. \tag{6.5}$$

We have supposed

$$\Re(\alpha_i+\gamma_s)>0 \qquad i=1,2,\dots,n. \tag{6.6}$$

But the integrals (6.3) and (6.5) converge for

$$\Re(\alpha_i+\beta+\gamma_s)>0 \qquad i=1,2,\dots,n. \tag{6.7}$$

By analytic continuation it is seen that (6.6) may be replaced by the less restricting condition (6.7) provided none of the numbers $\alpha_i+\gamma_s$ is a negative integer or zero.

34. Now we suppose β to be a negative integer, say $-p$. From (2.31) and (2.38) we get

$$(-1)^p\int_0^z t^{x-1}\eta_n\left(\frac{z}{t}\right)dt=z^x\prod_{\nu=1}^{n}\frac{\Gamma(x+\alpha_\nu)}{\Gamma(x-\gamma_\nu+1)}+(-1)^p z^x\sum_{\nu=0}^{p-1}c^{(s)}_{p-1-\nu,n}(\gamma_s-x-\nu)_\nu, \tag{6.8}$$

where $\Re(x+\alpha_i)>0$, $i=1,2,\dots n$. Now we consider the integral

$$\int_0^z t^{\gamma_s-1}\eta_n\left(\frac{z}{t}\right)\frac{dt}{1-t}$$

supposing $\Re(\alpha_i+\gamma_s)>0$, $i=1,2,\dots,n$. The integral is convergent and if we suppose $|z|<1$, we get, using (6.8),

$$\begin{aligned}\int_0^z t^{\gamma_s-1}\eta_n\left(\frac{z}{t}\right)\frac{dt}{1-t}&=\sum_{i=0}^{\infty}\int_0^z t^{\gamma_s+i-1}\eta_n\left(\frac{z}{t}\right)dt\\&=(-1)^p y_s^*(z)+z^{\gamma_s}\sum_{\nu=0}^{p-1}(-1)^\nu c^{(s)}_{p-1-\nu,n}\sum_{i=0}^{\infty}(i+1)_\nu z^i.\end{aligned} \tag{6.9}$$

But as

$$\sum_{i=0}^{\infty}(i+1)_\nu z^i=\frac{\nu!}{(1-z)^{\nu+1}},$$

(6.9) reduces to

$$y_s^*(z)=\Gamma(p)z^{\gamma_s}\sum_{\nu=0}^{p-1}\frac{c^{(s)}_{\nu,n}}{(1-p)_\nu}(1-z)^{\nu-p}+(-1)^p\int_0^z t^{\gamma_s-1}\eta_n\left(\frac{z}{t}\right)\frac{dt}{1-t}.$$

Replacing p by $-\beta$ we get

$$y_s^*(z) = \Gamma(-\beta)\, z^{\gamma_s}(1-z)^{\beta} \sum_{\nu=0}^{-\beta-1} \frac{c_{\nu,n}^{(s)}}{(\beta+1)_\nu}(1-z)^\nu + (-1)^\beta z^{\gamma_s} \int_1^\infty t^{-\gamma_s} \frac{\eta_n(t)}{t-z}\, dt. \qquad (6.10)$$

This expression represents our solution $y_s^*(z)$ for all values of z which are not positive and ≥ 1. The integral on the right-hand side may also be written

$$(-1)^{\beta+1} \frac{z^{\gamma_s}}{2\pi i} \int_{\varkappa-i\infty}^{\varkappa+i\infty} t^{-\gamma_s} \log(1-t) \frac{\eta_n(t)}{t-z}\, dt \qquad 0<\varkappa<1,$$

where z is on the left of the contour, or

$$(-1)^{\beta+1} \eta_n(z) \log(1-z) + \frac{(-1)^{\beta+1}}{2\pi i} \int_{\varkappa-i\infty}^{\varkappa+i\infty} \left(\frac{z}{t}\right)^{\gamma_s} \log(1-t) \frac{\eta_n(t)}{t-z}\, dt,$$

where z is on the right of it. If this is substituted in (6.10), we get

$$y_s^*(z) = \Gamma(-\beta)\, z^{\gamma_s}(1-z)^{\beta} \sum_{\nu=0}^{-\beta-1} \frac{c_{\nu,n}^{(s)}}{(\beta+1)_\nu}(1-z)^\nu + (-1)^{\beta+1}\eta_n(z) \log(1-z) + \varphi_s(z), \qquad (6.11)$$

where $\varphi_s(z)$ is a function regular at $z=1$ having the integral representation

$$\varphi_s(z) = \frac{(-1)^{\beta+1}}{2\pi i} \int_{\varkappa-i\infty}^{\varkappa+i\infty} \left(\frac{z}{t}\right)^{\gamma_s} \log(1-t) \frac{\eta_n(t)}{t-z}\, dt \qquad 0<\varkappa<1, \qquad (6.12)$$

z lying on the right of the contour. The function $\varphi_s(z)$ may also be represented by a power series convergent in the circle $|z-1|<1$ having the form

$$\varphi_s(z) = (-1)^\beta z^{\gamma_s} \sum_{\nu=0}^{\infty} b_\nu (1-z)^\nu, \qquad (6.13)$$

where the coefficients are given by

$$b_\nu = \frac{1}{2\pi i} \int_{\varkappa-i\infty}^{\varkappa+i\infty} \frac{t^{-\gamma_s}\eta_n(t)}{(1-t)^{\nu+1}} \log(1-t)\, dt. \qquad (6.14)$$

If, as we suppose, no two of the γ_i are equal or differ by an integer, we can give s the values $1, 2, \ldots, n$. Then if β is an integer, we get n linearly independent solutions having logarithmic singularities at the point $z=1$. These are of the form (6.11), when $\beta<0$, and of the form (6.2) when $\beta \geq 0$. The difference between two of these solutions is regular at the point $z=1$ and of the form $y_{i,j}(z)$ considered in § 5.

Therefore we may also say that we have one solution logarithmic at the point $z=1$ and $n-1$ solutions $y_{i,j}(z)$ regular at the same point, which together form a fundamental system.

It may happen that the logarithmic term vanishes in all these solutions. If $\beta<0$, $\eta_n(z)$ must be equal to zero and all solutions have a pole of the order $-\beta$ at the point $z=1$. In this case the coefficient of $y_s(z)$ in (2.42) must be zero for $s=1, 2, \ldots, n$, because $y_1(z), \ldots, y_n(z)$ are linearly independent. It follows that the necessary and sufficient condition of vanishing of $\eta_n(z)$ is

$$\prod_{i=1}^{-\beta} R(\gamma_s-i)=0, \qquad s=1, 2, \ldots, n, \tag{6.15}$$

$R(x)$ being the polynomial defined by (1.4). Thus (6.11) reduces to

$$y_s^*(z)=\Gamma(-\beta)\, z^{\gamma_s}(1-z)^{\beta} \sum_{\nu=0}^{-\beta-1} \frac{c_{\nu,n}^{(s)}}{(\beta+1)_\nu}(1-z)^{\nu} \qquad s=1, 2, \ldots, n. \tag{6.16}$$

These n solutions are linearly independent. From (6.15) it follows that it is possible to number the parameters in such a way that $\alpha_s+\gamma_s=p_s+1$ for $s=1, 2, \ldots, n$, p^s being a non-negative integer, and we have $\beta=-1-\sum_1^n p_i$. Interchanging α and γ, we see that (6.16) may also be written in the form

$$y_s^*(z)=\Gamma(-\beta)\, z^{-\alpha_s}\left(\frac{1-z}{z}\right)^{\beta} \sum_{\nu=0}^{-\beta-1} \frac{\bar{c}_{\nu,n}^{(s)}}{(\beta+1)_\nu}\left(\frac{z-1}{z}\right)^{\nu} \qquad s=1, 2, \ldots, n. \tag{6.17}$$

If we omit the factor $\Gamma(-\beta)$, these solutions have the same form as $\xi_n(z)$ and they satisfy the relations of art. 13. From (2.7) it follows that $c_{\nu,n}^{(s)}=0$, if $\nu \geq -\beta-p_s$, and that $c_{\nu,n}^{(1)}$ may be represented by the hypergeometric series

$$c_{\nu,n}^{(1)}=(-1)^{\beta+p_1+\nu+1} \frac{\prod\limits_{\nu=1}^{n}(\alpha_1-\alpha_\nu+1)_{p_\nu}}{p_1!\,\Gamma(-\beta-p_1-\nu)} F\begin{pmatrix}\alpha_1+\gamma_1+\beta+\nu & \alpha_1+\gamma_2 & \alpha_1+\gamma_3 & \ldots & \alpha_1+\gamma_n \\ & \alpha_1-\alpha_2+1 & \alpha_1-\alpha_3+1 & \ldots & \alpha_1-\alpha_n+1\end{pmatrix},$$

if $\nu<-\beta-p_1$. From this we get $c_{\nu,n}^{(s)}$ by interchanging α_1 and α_s, γ_1 and γ_s together with p_1 and p_s.

If these expressions are substituted in (6.16) and (6.17), we get for the solutions defined by (1.14) and (1.13) the following expansions

$$\bar{y}_1(z)=z^{\gamma_1}(z-1)^{-p_1-1} \sum_{\nu=0}^{p_2+\cdots+p_n} \binom{p_1+\nu}{\nu} F\begin{pmatrix}-\nu & \alpha_1+\gamma_2 & \ldots & \alpha_1+\gamma_n \\ \alpha_1-\alpha_2+1 & \ldots & & \alpha_1-\alpha_n+1\end{pmatrix}(1-z)^{-\nu}$$

$$y_1(z)=z^{\gamma_1}(1-z)^{-p_1-1} \sum_{\nu=0}^{p_2+\cdots+p_n} \binom{p_1+\nu}{\nu} F\begin{pmatrix}-\nu & \alpha_2+\gamma_1 & \ldots & \alpha_n+\gamma_1 \\ \gamma_1-\gamma_2+1 & \ldots & & \gamma_1-\gamma_n+1\end{pmatrix}\left(\frac{z}{z-1}\right)^{\nu}.$$

These relations may also be derived by the transformation formula (1.21). The equation (2.21) reduces in this case to a special case of (5.34) and we have

$$\frac{\Delta^{\nu} \prod_{i=1}^{n} (x-\alpha_i+1)_{p_i}}{\prod_{i=1}^{n} (x-\alpha_i+1)_{p_i}} = (-1)^{\nu} F\begin{pmatrix} -\nu & x+\gamma_1 \;\; x+\gamma_2 \dots & x+\gamma_n \\ x-\alpha_1+1 & x-\alpha_2+1 \;\; \dots & x-\alpha_n+1 \end{pmatrix}.$$

In particular for $\nu = \sum_{1}^{n} p_i$ we get

$$F\begin{pmatrix} 1+\beta & x+\gamma_1 \;\; x+\gamma_2 \dots & x+\gamma_n \\ x-\alpha_1+1 & x-\alpha_2+1 \;\; \dots & x-\alpha_n+1 \end{pmatrix} = (-1)^{\beta+1} \Gamma(-\beta) \prod_{i=1}^{n} \frac{\Gamma(x-\alpha_i+1)}{\Gamma(x+\gamma_i)}.$$

35. Finally we suppose that $\beta \geq 0$. Then we have (6.2) which in a better way may be written

$$\begin{aligned} y_s(z) = z^{\gamma_s} \sum_{\nu=0}^{\beta-1} (z-1)^{\nu} \prod_{i=1}^{n} \frac{(\alpha_i+\gamma_s)_{\nu}}{(\gamma_s-\gamma_i+1)_{\nu}} F\begin{pmatrix} \alpha_1+\gamma_s+\nu & \dots & \alpha_n+\gamma_s+\nu \\ \gamma_s-\gamma_1+1+\nu & \dots & \gamma_s-\gamma_n+1+\nu \end{pmatrix} + \\ + \left[\varphi_s(z) - \frac{(-1)^{\beta}}{\Gamma(\beta+1)} \xi_n(z) \log(1-z)\right] \prod_{i=1}^{n} \frac{\Gamma(\gamma_s-\gamma_i+1)}{\Gamma(\alpha_i+\gamma_s)}, \end{aligned} \tag{6.18}$$

$s = 1, 2, \dots, n$. These relations are valid when the condition (6.7) is satisfied. If the logarithmic term vanishes, the coefficient of $\xi_n(z)$ must vanish, because $\xi_n(z)$ can never be identically zero. Especially, it may happen that the logarithmic term vanishes in all n solutions. The condition of this is

$$\prod_{i=1}^{\beta-n+1} R(\gamma_s+i) = 0 \qquad s = 1, 2, \dots, n. \tag{6.19}$$

In this case we may number the parameters α_s in such a way that $\alpha_s+\gamma_s = -p_s$, $s = 1, 2, \dots, n$, p_s being a non-negative integer, and we have $\beta = n-1+\sum_{1}^{n} p_i$. This case only occurs when $\beta \geq n-1$. Then the equation (6.18) reduces to

$$y_s(z) = z^{\gamma_s} \sum_{\nu=0}^{p_s} (z-1)^{\nu} \prod_{i=1}^{n} \frac{(\alpha_i+\gamma_s)_{\nu}}{(\gamma_s-\gamma_i+1)_{\nu}} F\begin{pmatrix} \alpha_1+\gamma_s+\nu & \dots & \alpha_n+\gamma_s+\nu \\ \gamma_s-\gamma_1+1+\nu & \dots & \gamma_s-\gamma_n+1+\nu \end{pmatrix}, \tag{6.20}$$

where $s = 1, 2, \dots, n$. All solutions are then regular at the point $z = 1$, which is an apparent singularity. From (1.21) it is seen that e.g. $y_1(z)$ may also be represented in the forms

$$\begin{aligned} y_1(z) &= z^{-\alpha_1} \sum_{\nu=0}^{p_1} \binom{p_1}{\nu} F\begin{pmatrix} \nu-p_1 & \alpha_2+\gamma_1 \;\; \alpha_3+\gamma_1 \dots & \alpha_n+\gamma_1 \\ \gamma_1-\gamma_2+1 & \gamma_1-\gamma_3+1 \;\; \dots & \gamma_1-\gamma_n+1 \end{pmatrix} \left(\frac{1-z}{z}\right)^{\nu} \\ &= \prod_{i=1}^{n} \frac{(\alpha_i+\gamma_1)_{p_1}}{(\gamma_1-\gamma_i+1)_{p_1}} z^{\gamma_1} \sum_{\nu=0}^{p_1} \binom{p_1}{\nu} F\begin{pmatrix} \nu-p_1 & \alpha_1+\gamma_2 \;\; \alpha_1+\gamma_3 \dots & \alpha_1+\gamma_n \\ \alpha_1-\alpha_2+1 & \alpha_1-\alpha_3+1 \;\; \dots & \alpha_1-\alpha_n+1 \end{pmatrix} (z-1)^{\nu}. \end{aligned}$$

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