FINITE DIMENSIONAL CONVOLUTION ALGEBRAS

BY

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Introduction

The notion of convolution (Ger. *Faltung*, Fr. *produit de composition*) is a venerable one in mathematical analysis. The convolution

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(\alpha)\frac{\sin\,(n+\frac{1}{2})\,(\alpha-x)}{2\,\sin\,\frac{1}{2}\,(\alpha-x)}\,d\alpha$$

is found in Dirichlet's original memoir [12] on Fourier series, and similar convolutions are extensively used in the classical literature on Fourier series and integrals (see for example Riemann [34]. Weierstrass's original proof [47] of the celebrated approximation theorem bearing his name utilizes a certain convolution. Fractional integration and differentiation are defined by means of convolutions (see for example Zygmund [51], pp. 222–225). A perusal of any adequate textbook on Fourier series or integrals will show the important place occupied by the notion of convolution in the field of harmonic analysis. The Hilbert transform is of course a convolution. The classical theory of this transform has recently been extended by Zygmund and Calderon [52].

More recently, it has been recognized that measures and certain classes of abstract linear functionals can be convolved. For functions of finite variation on $(-\infty, +\infty)$, for example, see Bochner [5], pp. 64–74, and, from another point of view, Beurling [4] and Gel'fand [15]. The notion is discussed and utilized *in extenso* by Jessen and Wintner [22]. L. Schwartz has studied convolutions for the class of linear functionals

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called distributions ([35], Ch. VI). Finally, we shall see that the algebras discussed in Bourbaki [6], pp. 110–115, can be considered to be convolution algebras.

The importance of convolutions in the theory of Lie groups was recognized by H. Weyl (Peter and Weyl [50] and Weyl [49], Ch. III, §§ 12–15). A very general description of convolution of linear *operators* is found in A. Weil's treatise on topological groups ([48], pp. 46–48), and it is to this source that the present paper owes its original inspiration. The notions sketched by Weil have been utilized and extended by many writers (see e.g. Segal [39], Godement [16], and Buck [7]).

In studying numbers of examples of convolutions and in particular analyzing the ideal structure of certain algebras in which multiplication is defined by a convolution, the authors have been led to formulate a very general, purely algebraic, definition of convolution algebra, This definition includes all of the examples of convolutions which we have found in the literature.[2] A preliminary announcement of a special case is found in Hewitt and Zuckerman [19]. Our definition includes as nontrivial cases both infinite and finite dimensional algebras, and in studying these two classes of convolution algebras, entirely different techniques are called for. Infinite dimensional convolution algebras are best treated by analytic and topological methods, while the tools of classical algebra are required in the finite dimensional case. The present paper is devoted primarily to a discussion of finite or at least finite dimensional objects, although when a theorem about an infinite situation can be obtained at no extra effort, we do not hesitate to state it. Some infinite dimensional examples are also included, in 1.4. A second communication will be devoted to infinite dimensional convolution algebras.

We use the following notation:

K denotes the complex number field;

$\mathfrak{M}_n$ denotes the algebra of all $n \times n$ complex matrices;

Z_p denotes the zero algebra over K of dimension p;

$A \oplus B$ denotes the direct sum of algebras A and B;

K_n denotes the direct sum $K \oplus \cdots \oplus K$ with n summands.

Throughout this paper, functions and linear functionals are always complex-valued. Linear spaces and algebras are always over the field K; and homomorphisms and ideals of algebras are taken in the algebra sense.

[2] It does not include the composition of functions of 2 variables discussed in Volterra and Pérès [45], but this composition in its general form can hardly be called a convolution.

§ 1. Basic definitions and theorems

We begin with a number of definitions.

1.1. Definition. A non-void set G is said to be a semigroup if there exists a binary operation defined for all $x, y \varepsilon G$ (usually written as xy) such that $x(yz) = (xy)z$ for all $x, y, z \varepsilon G$.[3] The cardinal number of a finite semigroup is called its order, and is often written as $o(G)$.

1.2. We list a few examples of semigroups.

1.2.1. Any group.

1.2.2. Any non-void set G, with $xy = y$ for all $x, y \varepsilon G$.

1.2.3. Any non-void set G; a a fixed element of G, and $xy = a$ for all $x, y \varepsilon G$.

1.2.4. Any non-void set G completely ordered under a relation $\leq$, with $xy = \max(x, y)$ for all $x, y \varepsilon G$.

1.2.5. The set $\{a, a+1, a+2, \ldots\}$, where a is a non-negative integer, and the semigroup operation is ordinary addition.

1.2.6. In Appendices 1 and 2, tables of all semigroups of orders 2 and 3 will be found. Such tables have been computed independently by Carman, Harden and Posey [8] and by Tamura [44]. Their results for orders 2 and 3 agree with ours. Carman, Harden, and Posey have an incomplete table of semigroups of order 4. G. E. Forsythe [14] has computed the semigroups of order 4 by mechanical means.

1.3. Definition. Let G be a semigroup, and let $\mathfrak{F}$ be a linear space of functions, with the usual definitions of sum and scalar multiplication, defined on G. We suppose that

1.3.1. for all $x \varepsilon G$ and $f \varepsilon \mathfrak{F}$, the function ${}_x f$, defined by the relation ${}_x f(y) = f(xy)$,[4] is an element of $\mathfrak{F}$.

Now let $\mathfrak{L}$ be a linear space of linear functionals defined on $\mathfrak{F}$. For $L \varepsilon \mathfrak{L}$, $f \varepsilon \mathfrak{F}$, and $x \varepsilon G$, let $L_y(f(xy))$ denote $L({}_x f)$. We suppose further that

1.3.2. for all $L \varepsilon \mathfrak{L}$ and $f \varepsilon \mathfrak{F}$, the function on G whose value at x is $L_y(f(xy))$, is an element of $\mathfrak{F}$;

1.3.3. for all $L, M \varepsilon \mathfrak{L}$, the linear functional N on $\mathfrak{F}$ defined by the relation $N(f) = M_x(L_y(f(xy)))$ is an element of $\mathfrak{L}$. Under these conditions, we write N as

[3] An extensive discussion of abstract semigroups and related systems may be found in Dubreil [13], Chapter II.

[4] The function f_x is defined analogously: it has the value $f(yx)$ at the point y.

$M\star L$, and call $M\star L$ the *convolution* of the linear functionals M and L. The linear space $\mathfrak{L}$ is said to be a *convolution algebra.*

Definition 1.3 includes all of the notions of convolution which the authors have found in the literature, whether of functions, measures, or distributions.[5]

1.4. Examples of convolution algebras. We now give a few examples of both finite and infinite dimensional convolution algebras, showing the diversity of structures included under our definition of convolution. A close study of these examples, however, is not essential for understanding the remainder of the present paper.

1.4.1. Let G be the semigroup defined in 1.2.3. Let $\mathfrak{F}$ be any linear space of functions on G containing the function identically equal to unity (written as 1). Let $\mathfrak{L}$ be any linear space of linear functionals on $\mathfrak{F}$ containing the functional λ_a such that $\lambda_a(f)=f(a)$ for all $f\,\varepsilon\,\mathfrak{F}$. It is easy to see that conditions 1.3.1–1.3.3 are satisfied and that $L\star M=L(1)\,M(1)\,\lambda_a$ for all $L, M\,\varepsilon\,\mathfrak{L}$.

1.4.2. Let G be a finite group. The group algebra of G is often described as the set of all formal complex linear combinations of elements of G, $\sum_{x\,\varepsilon\,G}\alpha_x x$, with termwise addition and scalar multiplication and with product defined by $(\sum_{x\,\varepsilon\,G}\alpha_x x)\star(\sum_{y\,\varepsilon\,G}\beta_y y)=$ $=\sum_{x\,\varepsilon\,G}\sum_{y\,\varepsilon\,G}\alpha_x\beta_y\,xy$. This algebra is isomorphic to the convolution algebra consisting of all linear functionals on the space of all functions on G. To see this, let λ_a be the functional such that $\lambda_a(f)=f(a)$, for all $a\,\varepsilon\,G$. Then $\lambda_a\star\lambda_b(f)=f(ab)$, as we shall show in the proof of Theorem 1.7. Hence $\lambda_a\star\lambda_b=\lambda_{ab}$. Since the linear functionals $\lambda_a\,(a\,\varepsilon\,G)$ form a basis for all linear functionals in question, the asserted isomorphism is established. It may also be noted that $\sum_{x\,\varepsilon\,G}\sum_{y\,\varepsilon\,G}\alpha_x\beta_y\,xy=\sum_{z\,\varepsilon\,G}(\sum_{y\,\varepsilon\,G}\alpha_{zy^{-1}}\beta_y)\,z$; this identity shows the isomorphism of the algebra defined here with the algebra $\mathfrak{L}_1(G)$ described in 1.4.6 *infra.*

1.4.3. A quite different example is provided by the algebra of all sequences of complex numbers $\{a_n\}_{n=0}^{\infty}$. We write $a=\{a_n\}_{n=0}^{\infty}$, $b=\{b_n\}_{n=0}^{\infty}$, and so on. The element $a+b$ is $\{a_n+b_n\}_{n=0}^{\infty}$, $ta=\{ta_n\}_{n=0}^{\infty}$, for all $t\,\varepsilon\,K$, and the product $a\star b$ of a and b is defined by the relation $a\star b=\left\{\sum_{k=0}^{n}a_k\,b_{n-k}\right\}_{n=0}^{\infty}$. This algebra is a convolution algebra in the sense of 1.3. Let $\mathfrak{F}$ be the space of all functions on $N_0=\{0, 1, 2, 3, \ldots\}$ which vanish except on finite subsets of N_0. As noted in 1.2.5, N_0 is a semigroup under

[5] Group algebras of finite groups have of course been studied for coefficients lying in arbitrary fields (see for example v. D. WAERDEN [46], Ch. XVII). Other generalizations are mentioned in BERMAN [2] and PAIGE [28].

addition. Let $\mathfrak{C}$ be the space of all linear functionals on $\mathfrak{F}$. It is clear that for all $A \varepsilon \mathfrak{C}$, there is a unique sequence $\{a_n\}_{n=0}^{\infty}$ as above such that $A(f) = \sum_{n=0}^{\infty} a_n f(n)$ for all $f \varepsilon \mathfrak{F}$. Conversely, every sequence $\{a_n\}_{n=0}^{\infty}$ defines a linear functional on $\mathfrak{F}$. It is easy to verify 1.3.1 for the present function space $\mathfrak{F}$. To verify 1.3.2, let f be an arbitrary element $\neq 0$ in $\mathfrak{F}$, and let $p-1$ be the greatest integer such that $f(p-1) \neq 0$. Then $f(m+n) = 0$ for all $n \varepsilon N_0$ and $m \geq p$. Therefore $A_n f(m+n) = \sum_{n=0}^{\infty} a_n f(m+n) = 0$ for all $m \geq p$, and hence 1.3.2 holds. Condition 1.3.3 is automatically satisfied, since $\mathfrak{C}$ here consists of all linear functionals on $\mathfrak{F}$. Let e_n be the function such that $e_n(m) = \delta_{nm}$ $(n, m \varepsilon N_0)$. Then clearly $A(e_n) = a_n$. For elements A and B of $\mathfrak{C}$, we therefore obtain $A \star B$ by computing $A \star B(e_n)$ for all $n \varepsilon N_0$. We have

$$A \star B(e_n) = A_k(B_l(e_n(k+l))) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_k b_l \delta_{n,k+l} = \sum_{k+l=n} a_k b_l = \sum_{k=0}^{n} a_k b_{n-k}.$$

Therefore the multiplication defined above for sequences $\{a_n\}_{n=0}^{\infty}$ is actually convolution in the sense of 1.3.

1.4.4. Consider the space $\mathfrak{L}_1(T)$, consisting of all Lebesgue integrable functions on the circle group T. For $f, g \varepsilon \mathfrak{L}_1(T)$, the integral

$$f \star g(x) = \int_0^{2\pi} f(e^{i(x-y)}) g(e^{iy}) \, dy,$$

the convolution of f and g, defines a function which is again in $\mathfrak{L}_1(T)$.

1.4.5. For $f, g \varepsilon \mathfrak{L}_1(R)$, where R denotes the additive group of real numbers, we have

$$f \star g(x) = \int_{-\infty}^{+\infty} f(x-y) g(y) \, dy,$$

and this function $f \star g$ is again in $\mathfrak{L}_1(R)$.

1.4.6. To show that the operations described in 1.4.4 and 1.4.5 are convolutions in the sense of 1.3, consider an arbitrary locally compact group G. As $\mathfrak{F}$, we take $\mathfrak{C}_\infty(G)$, the space of all continuous functions on G which are arbitrarily small outside of compact sets, and normed by $\|f\| = \max_{x \varepsilon G} |f(x)|$. As $\mathfrak{C}$, we take the space $\check{\mathfrak{C}}_\infty(G)$, consisting of all bounded linear functionals on $\mathfrak{C}_\infty(G)$. It is obvious that 1.3.1 holds for $\mathfrak{C}_\infty(G)$: if $|f(y)| < \varepsilon$ for y non εA, then $|{}_x f(y)| < \varepsilon$ for y non $\varepsilon\, x^{-1}A$, and $x^{-1}A$ is compact if A is compact. To verify 1.3.2 for the present case, we must use

F. Riesz's representation theorem for elements L of $\tilde{\mathfrak{C}}_\infty(G)$: $L(f)=\int_G f(x)\,d\lambda(x)$, where λ is a complex-valued, countably additive, bounded, regular Borel measure on G.[6] Consider the total variation $|\lambda|$ of λ (see Hewitt [18] for a definition). Since $|\lambda|$ is regular, for every $\eta>0$ there exists a compact subset B of G such that $|\lambda|(B')<\eta$. For an arbitrary $f\,\varepsilon\,\mathfrak{C}_\infty(G)$ and $\eta>0$, let A be a compact subset of G such that $|f(y)|<\eta$ for all $y\,\varepsilon\,A'$. Then we have

$$\int_G f(xy)\,d\lambda(y)=\int_B f(xy)\,d\lambda(y)+\int_{B'} f(xy)\,d\lambda(y).$$

It is plain that

$$\left|\int_{B'} f(xy)\,d\lambda(y)\right|\le\|f\|\cdot|\lambda|(B')<\eta\,\|f\|.$$

If x is not in the compact set AB^{-1}, then we have

$$\left|\int_B f(xy)\,d\lambda(y)\right|\le\int_B |f(xy)|\,d|\lambda|(y)\le\eta\,|\lambda|(G),$$

since $xy\,\varepsilon\,A$ and $y\,\varepsilon\,B$ imply that $x\,\varepsilon\,AB^{-1}$. It follows that $\int_G f(xy)\,d\lambda(y)$ becomes arbitrarily small outside of properly chosen compact sets. To show that $\int_G f(xy)\,d\lambda(y)$ is continuous as a function of x, it is necessary only to note that f is uniformly continuous. To verify 1.3.3 for the present case, we observe that

$$\begin{aligned}|M\star L(f)|&=\left|\int_G\int_G f(xy)\,d\lambda(y)\,d\mu(x)\right|\le\int_G\int_G|f(xy)|\,d|\lambda|(y)\,d|\mu|(x)\\&\le\|f\|\,|\lambda|(G)\cdot|\mu|(G)=\|f\|\cdot\|L\|\cdot\|M\|.\end{aligned}$$

Hence $M\star L$ is a bounded linear functional and $\|M\star L\|\le\|M\|\cdot\|L\|$. The convolution algebra $\tilde{\mathfrak{C}}_\infty(G)$ will be studied in detail in a second communication. This algebra has a subalgebra (actually a 2-sided ideal) consisting of all functionals for which the corresponding measures are absolutely continuous with respect to right Haar measure on G. As is well known, for such measures μ, we have $d\mu(x)=m(x)\,dx$, where $m\,\varepsilon\,\mathfrak{L}_1(G)$ and dx is the differential of right Haar measure. Let μ and ν be two such measures, with $d\nu(x)=n(x)\,dx$. Then, writing the functionals in question as M and N, respectively, we have

[6] For all special terms used in this paragraph, see Loomis [24].

$$M \star N(f) = \int_G \int_G f(xy)\, n(y)\, dy\, m(x)\, dx = \int_G \int_G f(xy)\, m(x)\, dx\, n(y)\, dy$$
$$= \int_G \int_G f(x)\, m(xy^{-1})\, dx\, n(y)\, dy = \int_G f(x) \int_G m(xy^{-1})\, n(y)\, dy\, dx.$$

That is, the measure corresponding to the functional $M \star N$ is absolutely continuous, and the differential of this measure is $\int_G m(xy^{-1})\, n(y)\, dy\, dx$. This shows that the classical convolutions mentioned in 1.4.4 and 1.4.5 are actually convolutions in our present sense.

1.4.7. Let the semigroup G be the additive group of real numbers, R. As the function space $\mathfrak{F}$, we take the space of all continuous functions f on R such that $\lim_{t \to \infty} f(t)$ and $\lim_{t \to -\infty} f(t)$ exist and are finite. Denote these limits by $E_+(f)$ and $E_-(f)$, respectively. The space $\mathfrak{F}$ is a Banach space under the usual addition and scalar multiplication and with $\|f\| = \sup_{t \varepsilon R} |f(t)|$. It is clear that E_+ and E_- are bounded linear functionals on $\mathfrak{F}$. As the space $\mathfrak{L}$, we take all bounded linear functionals on $\mathfrak{F}$. It is not difficult to show that every $L \varepsilon \mathfrak{L}$ has a unique representation of the form

$$L(f) = \int_{-\infty}^{\infty} f(x)\, d\lambda(x) + \alpha E_-(f) + \beta E_+(f),$$

where λ is a countably additive, complex-valued, bounded Borel measure on R and $\alpha, \beta \varepsilon K$. Conditions 1.3.1–1.3.3 are established by a routine calculation, which we omit. Hence we are in possession of another convolution algebra. Interesting features of this algebra are that is has non-zero Jacobson radical (Jacobson [21]) and is non-commutative, in spite of the fact that the basic group R is commutative. In fact,

$$E_+ \star E_-(f) = \lim_{x \to \infty} \left(\lim_{y = -\infty} (f(x+y)) \right) = \lim_{x \to \infty} (E_-(f)) = E_-(f),$$

while

$$E_- \star E_+(f) = E_+(f).$$

We now give a few simple but basic results.

1.5. Theorem. Every convolution algebra is associative.

Proof. Let L, M, N be elements of the convolution algebra $\mathfrak{L}$, and let $f \varepsilon \mathfrak{F}$. Then

$$L \star (M \star N)(f) = L_x(M \star N_y(f(xy))) = L_x(M_u(N_v(f(x(uv)))))$$
$$= L_x(M_u(N_v(f((xu)v)))).$$

On the other hand,

$$(L\star M)\star N(f)=L\star M_w(N_v(f(wv)))=L_x(M_u(N_v(f((xu)v)))).$$

This proves the present theorem.

1.6. Definition. Let G be an arbitrary finite semigroup, with elements $x_1, x_2, \ldots, x_n$. Let $\mathfrak{F}_1(G)$ denote the linear space of all functions defined on G, and let $\mathfrak{L}_1(G)$ denote the space of all linear functionals defined on $\mathfrak{F}_1(G)$.

Let φ_i be the function in $\mathfrak{F}_1(G)$ such that $\varphi_i(x_j)=\delta_{ij}$ $(i, j=1, 2, \ldots, n)$ and let λ_i be the element of $\mathfrak{L}_1(G)$ such that $\lambda_i(\varphi_j)=\delta_{ij}$ $(i, j=1, 2, \ldots, n)$. (Note the slight change in notation from 1.4.2.) Let $[i, j]$ be the integer such that $x_i x_j = x_{[i,j]}$ $(i, j=1, 2, \ldots, n)$.

1.7. Theorem. $\mathfrak{L}_1(G)$ is a convolution algebra. It is isomorphic to the algebra of all formal complex linear combinations $\sum_{x\varepsilon G}\alpha_x x$, where

$$\Big(\sum_{x\varepsilon G}\alpha_x x\Big)+\Big(\sum_{x\varepsilon G}\beta_x x\Big)=\sum_{x\varepsilon G}(\alpha_x+\beta_x)x,\quad \gamma\Big(\sum_{x\varepsilon G}\alpha_x x\Big)=\sum_{x\varepsilon G}(\gamma\alpha_x)x,$$

and

$$\Big(\sum_{x\varepsilon G}\alpha_x x\Big)\Big(\sum_{y\varepsilon G}\beta_y y\Big)=\sum_{x\varepsilon G}\sum_{y\varepsilon G}\alpha_x\beta_y xy.$$

Proof. The first statement of the present theorem is obvious. To prove the second, we observe first that the functions φ_i form a basis for $\mathfrak{F}_1(G)$: if $f\,\varepsilon\,\mathfrak{F}_1(G)$, then $f=\sum_{i=1}^{n}f(x_i)\varphi_i$. Next, the functionals λ_i form a basis for $\mathfrak{L}_1(G)$: if $L\,\varepsilon\,\mathfrak{L}_1(G)$, then $L=\sum_{i=1}^{n}L(\varphi_i)\lambda_i$. It is clear that $\lambda_i(f)=f(x_i)$ for all $f\,\varepsilon\,\mathfrak{F}_1(G)$ and $i=1, 2, \ldots, n$. We shall now show that $\lambda_i\star\lambda_j=\lambda_{[i,j]}$ $(i, j=1, 2, \ldots, n)$. We use the identity

$$f(xy)=\sum_{k,l=1}^{n}f(x_k x_l)\varphi_k(x)\varphi_l(y),$$

which is valid for all $f\,\varepsilon\,\mathfrak{F}_1(G)$ and $x, y\,\varepsilon\,G$. We then see that

$$\begin{aligned}\lambda_{i,x}(\lambda_{j,y}(f(xy)))&=\lambda_{i,x}\Big(\sum_{l,k=1}^{n}f(x_k x_l)\varphi_k(x)\lambda_j(\varphi_l)\Big)\\&=\sum_{l,k=1}^{n}f(x_k x_l)\lambda_i(\varphi_k)\lambda_j(\varphi_l)=\sum_{l,k=1}^{n}f(x_k x_l)\delta_{ik}\delta_{jl}=f(x_i x_j).\end{aligned}$$

It follows at once that $\lambda_i\star\lambda_j=\lambda_{[i,j]}$. Therefore, under the mapping $\sum_{i=1}^{n}\alpha_i\lambda_i\to\sum_{i=1}^{n}\alpha_i x_i$, we have an isomorphism between the algebra $\mathfrak{L}_1(G)$ and the algebra described in the present theorem.

1.8. Remark. It is obvious from the preceding theorem that $\mathfrak{L}_1(G)$ for a finite semigroup G is commutative if and only if G is commutative. This property no longer holds in the infinite dimensional case, as example 1.4.7 shows.

1.9. Theorem. Let G be a finite semigroup and let $f \varepsilon \mathfrak{F}_1(G)$, and $M, L \varepsilon \mathfrak{L}_1(G)$. Then

$$M_x(L_y(f(yx))) = L_x(M_y(f(xy))).$$ [7]

Proof. Since

$$f(xy) = \sum_i \sum_j f(x_i x_j)\, \varphi_i(x)\, \varphi_j(y),$$

we have

$$L_x(M_y(f(xy))) = \sum_i \sum_j f(x_i x_j)\, L(\varphi_i)\, M(\varphi_j).$$

On the other hand, we also have

$$f(yx) = \sum_u \sum_v f(x_u x_v)\, \varphi_u(y)\, \varphi_v(x),$$

and hence it follows that

$$M_x(L_y(f(yx))) = \sum_u \sum_v f(x_u x_v)\, L(\varphi_u)\, M(\varphi_v) = \sum_i \sum_j f(x_i x_j)\, L(\varphi_i)\, M(\varphi_j) = L_x(M_y(f(xy))).$$

1.10. Theorem. A finite dimensional algebra A is isomorphic to an algebra $\mathfrak{L}_1(G)$ for some finite semigroup G if and only if A has a basis which is closed under multiplication in A.

Proof. If A is isomorphic to $\mathfrak{L}_1(G)$ for some finite semigroup G under an isomorphism μ, then the elements $\mu^{-1}(\lambda_1), \ldots, \mu^{-1}(\lambda_n)$ are a basis for A of the kind required. Conversely, if A possesses a basis which is closed under multiplication, say $a_1, a_2, \ldots, a_n$, then the elements $a_1, a_2, \ldots a_n$, form a semigroup under the multiplication operation in A. It is clear that A is the set of all sums $\sum_{i=1}^{n} \alpha_i a_i$, and that $\left(\sum_{i=1}^{n} \alpha_i a_i\right)\left(\sum_{j=1}^{n} \beta_j a_j\right) = \sum_{i,j=1}^{n} \alpha_i \beta_j a_i a_j$. Hence A is an algebra of the sort described in Theorem 1.7 and is accordingly isomorphic to an $\mathfrak{L}_1$-algebra.

1.10.1. The preceding theorem shows that the algebras described by Bourbaki [6], pp. 110–115 are convolution algebras, for finite semigroups. An extension to the infinite case offers no difficulties.

1.11. Theorem. Every finite dimensional algebra A is a convolution algebra.

Proof. Adjoining a unit to A if necessary and using the regular representation, we obtain a faithful representation of A by complex $p \times p$ matrices, where

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[7] This theorem is due to Dr. THELMA CHANEY.

$p \leq \dim A + 1$. Thus we may regard A as a subalgebra of $\mathfrak{M}_p$. Now, $\mathfrak{M}_p$ is not an $\mathfrak{L}_1$-algebra, as we shall show in 4.3, but $\mathfrak{M}_p \oplus K$ is an $\mathfrak{L}_1$-algebra (this will be shown in Theorem 4.2). Therefore A is isomorphic to a subalgebra of an $\mathfrak{L}_1$-algebra. Definition 1.3 makes it clear that every subalgebra of a convolution algebra is a convolution algebra, and this completes the present proof.

1.12. Remark. Theorem 1.11 shows that finite dimensional convolution algebras are too general to be of any interest from our present point of view. Therefore we shall limit ourselves here to a study of algebras of the form $\mathfrak{L}_1(G)$ for finite semigroups G. These algebras are not nearly so general a class as one might at first suppose, and a certain amount of success has been obtained in determining their possible structures. From time to time, we shall permit ourselves to make statements regarding infinite semigroups, but we shall restrict ourselves to facts which can be obtained by essentially finite arguments.

§ 2. Finite semigroups

2.1. In spite of the large literature devoted to the algebraic theory of semigroups, we have found a number of apparently new theorems (in particular structure theorems), which are useful in classifying finite dimensional $\mathfrak{L}_1$-algebras. In addition, we find it convenient to reformulate a few well-known ideas. The present section is devoted to this program.

We first give 5 simple theorems showing various methods of adjoining new elements to an arbitrary semigroup.

2.2. Theorem. Let G be a semigroup. Let z be an object not in G. Then $G \cup \{z\}$ (also written as G_z), with the multiplication rules

$$xy = xy \text{ as in } G \text{ for all } x, y \in G,$$

$$xz = zx = z \text{ for all } x \in G_z,$$

is a semigroup. (G_z is said to be obtained from G by adjoining a zero.)

Proof. If $u, v, w \in G$, then $u(vw) = (uv)w$ by hypothesis. If $u, v, w \in G_z$ and at least one of u, v, and w is z, then $u(vw) = (uv)w = z$.

A slight variant on 2.2 is the following.

2.2.1. Let G be a semigroup with a zero a. Let b be an object not in G. Then $G \cup \{b\}$ is a semigroup under the multiplication rules

$$xy = xy \text{ as in } G \text{ for all } x, y \varepsilon G;$$

$$xb = bx = b \text{ for all } x \varepsilon G;$$

$$b^2 = a.$$

2.3. Theorem. Let G be a semigroup and let e be an object not in G. Then $G \cup \{e\}$ (also written G_e), with the multiplication rules

$$xy = xy \text{ as in } G \text{ for all } x, y \varepsilon G,$$

$$ex = xe = x \text{ for all } x \varepsilon G_e,$$

is a semigroup. (G_e is said to be obtained from G by adjoining a unit.)

We omit the proof.

2.4. Theorem. Let G be a semigroup. Let x_1 be an arbitrary element of G, and let a be an object not in G. Then $G \cup \{a\}$, with the multiplication rules

$$xy = xy \text{ as in } G \text{ for all } x, y \varepsilon G,$$

$$xa = xx_1 \text{ for all } x \varepsilon G,$$

$$ax = x_1 x \text{ for all } x \varepsilon G,$$

$$aa = x_1 x_1,$$

is a semigroup. (This semigroup is said to be obtained from G by adjoining a repeat element.)

Proof. For all $u \varepsilon G \cup \{a\}$, let $u' = u$ if $u \varepsilon G$, and let $u' = x_1$ if $u = a$. Then $uv = u'v'$ for all $u, v \varepsilon G \cup \{a\}$, and consequently $u(vw) = u'(vw)' = u'(v'w') = (u'v')w' = (uv)w$.

2.5. Theorem. Let G be a semigroup containing an idempotent element x_1 ($x_1^2 = x_1$). Let a be an object not in G. Then $G \cup \{a\}$, with the multiplication rules

$$xy = xy \text{ as in } G \text{ for all } x, y \varepsilon G,$$

$$xa = xx_1 \text{ for all } x \varepsilon G,$$

$$ax = x_1 x \text{ for all } x \varepsilon G,$$

$$a^2 = a,$$

is a semigroup. (The semigroup $G \cup \{a\}$ is said to be obtained from G by idempotent adjunction.)

Proof. Let u' be defined as in the proof of the preceding theorem. Then $uv=a$ if $u=v=a$, and $uv=u'v'$ otherwise. Since $a'=x_1=x_1x_1=a'a'$ and $u'v'\varepsilon G$, we have $(uv)'=u'v'$ for all $u,v\varepsilon G\cup\{a\}$. Then $u(vw)=u'(vw)'=u'(v'w')=(u'v')w'=(uv)'w'=(uv)w$ unless $u=v=w=a$. Since $a(aa)=(aa)a=a$, the proof is complete.

The remainder of the present section is devoted to a study of the algebra of semigroups and of the structure of certain classes of semigroups. A few of the results set forth here are to be found in one or another form in papers of Clifford [10], Poole [30], Rees [31, 32], Schwarz [36, 37, 38], and Suškevič [42]. For reasons of notation and substance, we find it wise to give a complete discussion.

2.6. Definition. Let x be an element of a semigroup. We shall say that x is of finite order[8] if there exist 2 integers $k\geq l$ and $l\geq 1$ such that $x^{k+l}=x^l$. It is easy to see that all elements of a finite semigroup are of finite order. We now list some simple properties of elements of finite order.

2.6.1. If x is of finite order, the sequence $x, x^2, x^3, \ldots$ contains at most $k+l-1$ distinct elements. If r is the smallest integer such that $x^r=x^s$, $1\leq s<r$, we let $l_x=s$, $k_x=r-s$. Then $x^p=x^q$, $p>q$, if and only if $q\geq l_x$ and $p=q+jk_x$, for some integer j.[9]

2.6.2. If x is of finite order, then $(x^m)^2=x^m$ if and only if the conditions $m=jk_x\geq l_x$ hold, for some positive integer j. This follows at once from 2.6.1

2.6.3. If $(x^m)^2=x^m$ for some $m\geq 1$, then x is of finite order.

2.6.4. If $(x^m)^2=x^m$ and $(x^r)^2=x^r$, then $x^m=(x^m)^r=(x^r)^m=x^r$.

2.6.5. If x is of finite order and $l_x=1$, then $(x^{k_x})^2=x^{k_x}$.

2.6.6. If G is a semigroup all of whose elements are of finite order and if, furthermore, G has a left unit e, and G has just one idempotent element, then G is a group. This follows from 2.6.2, since we then have $(x^m)^2=x^m$, $e^2=e$, and hence $x^m=e$. Then we have $xe=xx^m=x^mx=ex=x$ and $xx^{2m-1}=x^{2m-1}x=x^{2m}=e$; this shows that the semigroup G is a group. (In the finite case, this follows easily from Theorem 39 of Schwarz [37].)

2.7. Theorem. Let G be a finite or infinite, commutative, idempotent semigroup. Then G has a concrete representation as a system of subsets of the elements of G.

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[8] For an extensive utilization of this notation, see SCHWARZ [36].

[9] An interesting theorem concerning certain semigroups in which $l_x=1$ for all x is to be found in GREEN and REES [17].

Proof. For each $x \varepsilon G$, we let $M_x = \{zx; z \varepsilon G\}$, the set of all multiples of x. We have $x \varepsilon M_x$ and $xy \varepsilon M_x \cap M_y$, so that all of the sets M_x and all finite intersections of sets M_x are non-void. If $M_x = M_y$, then $x = uy$ and $y = vx$ for some $u, v \varepsilon G$. We thus find $x = uy = uyy = uyvx = xvx = vx = y$. Furthermore, if $z \varepsilon M_x \cap M_y$, then $z = ux = vy$ for some $u, v \varepsilon G$ and we have $z = z^2 = uvxy \varepsilon M_{xy}$. Conversely, if $z \varepsilon M_{xy}$, then $z = uxy = (ux)y = (uy)x$ and $z \varepsilon M_x \cap M_y$. Therefore $M_{xy} = M_x \cap M_y$ and the mapping $x \to M_x$ is an isomorphism of G onto the semigroup of sets $\{M_x\}_{x \varepsilon G}$, the semigroup operation being set-theoretic intersection.

2.7.1. As a converse to Theorem 2.7, we have: If S is any set and G is a set of subsets of S such that the intersection of every pair of subsets belonging to G is in G, then G is a commutative idempotent semigroup under the operation of intersection.

2.7.2. Remark. Theorem 2.7 and its converse provide an interesting sidelight to Stone's representation theorem [41] for Boolean rings. Let A be a Boolean ring, i.e., an associative ring in which $x^2 = x$ for all $x \varepsilon A$. It is an elementary fact that A is commutative. Thus A, under multiplication alone, forms an idempotent commutative semigroup with a zero; A may or may not have a unit. It is easy to show that if A contains more than 2 elements, then the zero is not adjoined: A contains divisors of zero. Stone's theorem asserts that A admits a concrete representation as a ring $\mathcal{A}$ of subsets of a certain set, the operation $a + b$ of A becoming $(A \cap B') \cup (A' \cap B)$ in $\mathcal{A}$ and $a \cdot b$ in A becoming $A \cap B$ in $\mathcal{A}$. Theorem 2.7 shows that if we wish merely to represent the multiplication operation in A, we can use the simple mapping $a \to \{za; z \varepsilon A\}$ to obtain a faithful representation of A by sets. However, this mapping shows no tendency at all to preserve the operation $+$, and we have no hope of obtaining by this method an elementary proof of Stone's theorem.

2.8. Theorem. Suppose that G is a finite or infinite commutative semigroup, all of whose elements are of finite order. If $l_x = 1$ for all $x \varepsilon G$, then G consists of a set of disjoint groups.

Proof. For each idempotent element $a \varepsilon G$, we take $S_a = \{x; x^m = a\}$, the set of all $x \varepsilon G$ such that $x^m = a$ for some m. From 2.6.4, we see that the S_a are disjoint and from 2.6.2, we see that every $x \varepsilon G$ is in some S_a. If $x, y \varepsilon S_a$, then, using 2.6.5 and 2.6.4, we have

$$(xy)^{k_x k_y} = (x^{k_x})^{k_y} (y^{k_y})^{k_x} = x^{k_x} y^{k_y} = a^2 = a$$

and hence $xy \varepsilon S_a$. We also have $ax = x^{k_x+1} = x$ and $xx^{2k_x-1} = x^{2k_x} = a$; thus a is the unit element of S_a and x^{2k_x-1} is the inverse of x in S_a.

If G is a semigroup satisfying the conditions of Theorem 2.8, then the set H of idempotent elements of G is itself a semigroup. Each $a \varepsilon H$ is the unit of a group S_a. If the idempotent semigroup H and the corresponding groups S_a are known, then G would be completely determined if the products of elements of different S_a were known. We will not attempt to characterize these semigroups completely[9a]. However, the following results will be useful.

2.9. Theorem. If G is as in 2.8 and $x \varepsilon S_a$, $y \varepsilon S_b$, then $xy \varepsilon S_{ab}$.

Proof. We have $(xy)^{k_x k_y} = x^{k_x} y^{k_y} = ab$, and therefore $xy \varepsilon S_{ab}$.

2.10.1. If $x \varepsilon S_a$ and $y \varepsilon S_b$, then xy, bx, and ay all belong to S_{ab}. Therefore we have $xy = abxy = (bx)(ay)$, so that the products xy are determined once we know the products bx for $x \varepsilon S_a$ and ay for $y \varepsilon S_b$, together with multiplication in the group S_{ab}.

2.10.2. If $x \varepsilon S_a$, $b^2 = b$, and $ab = a$, then $bx = b(ax) = ax = x$. In other words, the idempotent b is not only the unit for the group S_b, but is also a unit for $S = \bigcup_{ab=a} S_a$. It is easy to see that S is a semigroup.

2.11. We now consider a finite or infinite commutative semigroup G all of whose elements are of finite order. Let us denote the set $\{x;\, x \varepsilon G,\, l_x = 1\}$ by the symbol G°. If $x, y \varepsilon G^\circ$, then, by 2.6.5, we have

$$(xy)^{k_x k_y + 1} = (x^{k_x})^{k_y} (y^{k_y})^{k_x} xy = x^{k_x} y^{k_y} xy = x^{k_x+1} y^{k_y+1} = xy.$$

This implies that $l_{xy} = 1$, in view of 2.6.1. Therefore G° is a semigroup and it satisfies the conditions of Theorem 2.8. If H denotes as above the set of idempotent elements of G, then $H \subset G^\circ$. For each $a \varepsilon H$, we take $T_a = \{x;\, x \varepsilon G,\, x^m = a$ for some integer $m\}$. From 2.6.4 and 2.6.2, we see that the sets T_a are pairwise disjoint and that every $x \varepsilon G$ lies in some T_a. Each T_a is a semigroup and $T_a \cap G^\circ$ is a group: it is simply the S_a of Theorem 2.8 applied to G°. If $x \varepsilon T_a$ and $y \varepsilon T_b$, then $x^m = a$, $y^r = b$, and consequently $(xy)^{rm} = (x^m)^r (y^r)^m = a^r b^m = ab$. It follows that $xy \varepsilon T_{ab}$.

We thus see that T_a is a semigroup containing the group $T_a \cap G^\circ$. It is of interest to note that $aT_a = T_a \cap G^\circ$. To prove this, let x be any element of T_a. Then we have $x^m = a$ and hence $(ax)^{m+1} = ax^{m+1} = aax = ax$, so that $l_{ax} = 1$ by 2.6.1.

In the preceding few paragraphs, we have dealt only with commutative semigroups. We now drop that restriction but add another strong condition. We consider the effect of imposing a one-sided cancellation law.[10] Since it is immaterial whether

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[9a] But see CLIFFORD [10], Theorem 3.

[10] Other structural properties of semigroups with a one-sided cancellation law are given in TAMARI, *Bull. de la Soc. Math. de France*, 82 (1954), 53–96.

it be left- or right-sided, we confine our attention to the left cancellation law: $xy = xz$ implies $y = z$, for all $x, y, z \varepsilon G$.

2.12. Theorem. If G is a finite or infinite semigroup all of whose elements are of finite order, and if G obeys the left cancellation law, then G is the direct product of a group J and a semigroup H such that $xy = y$ for all $x, y \varepsilon H$.[11]

Proof. If $x \varepsilon G$ and $x^{k+l} = x^l$, $l > 1$, then $x^{l-1} x^{k+1} = x^{l-1} x$, and we cancel the factor x^{l-1} to obtain the equality $x^{k+1} = x$. Therefore $l_x = 1$, by 2.6.1, for all $x \varepsilon G$.

Next, let a be any fixed idempotent element of G and let $J = \{x;\, x \varepsilon G,\, xa = x\}$, $H = \{x;\, x \varepsilon G,\, x^2 = x\}$. If $x, y \varepsilon J$, then $xya = xy$ and $xy \varepsilon J$. Furthermore, $ax = aax$, so that $x = ax$ and a is a (two-sided) unit in J. Also $x x^{k_x} = x^{k_x+1} = x = xa$. Hence $x^{k_x} = a$. It follows that $x^{2k_x - 1} x = x x^{2k_x - 1} = x^{2k_x} = a^2 = a$; hence x has an inverse in J. Accordingly, J is a group. We next make the following observations.

2.12.1. If $x \varepsilon H$ and $y \varepsilon G$, then $xxy = xy$ and therefore $xy = y$. This makes it obvious that H is a semigroup.

If $x \varepsilon G$, we let $u_x = xa$ and $v_x = x^{k_x}$. Then $u_x a = xaa = xa = u_x$, so that $u_x \varepsilon J$. We have $v_x \varepsilon H$ from 2.6.5. Since $a \varepsilon H$, we can use 2.12.1 to find the following relation:

2.12.2 $$u_x v_x = x a x^{k_x} = x^{k_x+1} = x.$$

We next note that if $u \varepsilon J$, $v \varepsilon H$, and $w = uv$, then $u_w = wa = uva = ua = u$, by 2.12.1. Also, $v_w = w^{k_w} = (uv)^{k_w} = u^{k_w} v$ by 2.12.1. Therefore $u v_w = u^{k_w+1} v = (uv)^{k_w+1} = uv$. We cancel the factor u and obtain $v_w = v$.

We have now established the existence of a one-to-one correspondence

$$x \leftrightarrow (u_x, v_x) \qquad (x \varepsilon G,\ u_x \varepsilon J,\ v_x \varepsilon H).$$

If $z \leftrightarrow (u_x u_y,\, v_x v_y)$, then, in view of 2.12.2 and 2.12.1, we have $z = u_x u_y v_x v_y = u_x u_y v_y$ and $xy = u_x v_x u_y v_y = u_x u_y v_y$. It follows that $z = xy$, and hence the one-to-one correspondence just established is an isomorphism of G onto the direct product $J \times H$. This completes the present proof.

2.12.3. Note. According to a theorem of Clifford[10a], every semigroup G in which there exists a left unit e and in which, for all $x \varepsilon G$, an x' exists with $xx' = e$, is the direct product of a group and a semigroup in which $xy = y$ identically. Combining this result with Theorem 2.12, we have the following theorem. Let G be

[10a] *Ann. of Math.*, 2 Ser., 34 (1933), 865–871.

[11] This theorem, for finite semigroups G, follows immediately from results of Suškevič [42]. His methods are not, however, applicable in the general case treated here.

a semigroup in which every element has finite order. Then G obeys the left cancellation law if and only if G has a left unit e and right inverses relative to e.

2.12.4. Note. Theorem 2.12 fails for semigroups containing elements of infinite order, as the non-negative integers under addition show.

We turn now to the case of an idempotent semigroup that is not necessarily commutative.[12]

2.13. If G is a finite or infinite idempotent semigroup, then G consists of a set of disjoint semigroups, as follows. We take $S_a = \{x;\, x \varepsilon G,\, axa = a,\, xax = x\}$, for each $a \varepsilon G$. Since $aaa = a$, we have $a \varepsilon S_a$ so that no S_a is void and every $x \varepsilon G$ lies in some S_a.

If $x, y \varepsilon S_a$, we have $xyax = (xax)yax = x(aya)xyax = xa(yax)(yax) = xa(yax) = x(aya)x = xax = x$, and hence $xyx = xy(xyax) = (xy)(xy)ax = xyax = x$. Then we also have $yxy = y$ and therefore $x \varepsilon S_y$, $y \varepsilon S_x$. This shows that $x \varepsilon S_a$ implies $S_x = S_a$ and that the sets S_a are pairwise disjoint. Also if $x, y \varepsilon S_a$, we now have $x(xy)x = xyx = x$ and $(xy)x(xy) = xyxy = xy$, so that $xy \varepsilon S_x = S_a$. It follows that S_a is a semigroup.

2.14. Theorem. Let G, S_a, and S_b be as in 2.13. If $x \varepsilon S_a$ and $y \varepsilon S_b$, then $xy \varepsilon S_{ab}$ and $S_{ba} = S_{ab}$.

Proof. If $x \varepsilon S_a$, then $(ba)(xb)(ba) = baxba = baxb(axa) = (bax)(bax)a = (bax)a = b(axa) = ba$. Furthermore, $(xb)(ba)(xb) = xbaxb = (xax)baxb = x(axb)(axb) = x(axb) = (xax)b = xb$. Therefore $xb \varepsilon S_{ba}$. This implies that $ab \varepsilon S_{ba}$; since we have $ab \varepsilon S_{ab}$, it follows that $S_{ba} = S_{ab}$.

If $x \varepsilon S_a$ and $y \varepsilon S_b$, we have $b \varepsilon S_b = S_y$ and then, from what we just proved above, we see that $xy \varepsilon S_{ay}$ and $ba \varepsilon S_{ya}$. But this implies that $xy \varepsilon S_{ya} = S_{ba} = S_{ab}$. This completes the present proof.

We note also that if H is the set of distinct S_a, then H is a commutative, idempotent semigroup under the operation $S_a S_b = S_{ab}$.

2.15. Theorem. Each S_a of 2.13 is isomorphic with a semigroup of pairs (y, z) in which the semigroup operation is defined by $(y_1, z_1)(y_2, z_2) = (y_1, z_2)$.

Proof. We take $S'_a = \{xa;\, x \varepsilon S_a\}$, $S''_a = \{ax;\, x \varepsilon S_a\}$. Then $S'_a \subset S_a$ and $S''_a \subset S_a$. Also $y \varepsilon S'_a$ implies that $ay = a$, $ya = y$; similarly, $z \varepsilon S''_a$ implies that $az = z$, $za = a$.

If $x \varepsilon S_a$, then $y = xa \varepsilon S'_a$ and $z = ax \varepsilon S''_a$. Also $yz = (xa)(ax) = xax = x$.

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[12] In connection with this topic, see also McLean [27] and Clifford [10]

If $y \varepsilon S'_a$ and $z \varepsilon S''_a$, then we have $x = yz \varepsilon S_a$ and $xa = yza = ya = y$, $ax = ayz = = az = z$.

We have thus established the existence of a one-to-one correspondence $x \leftrightarrow (xa, ax)$ between S_a and the set T_a of pairs (y, z) for $y \varepsilon S'_a$, $z \varepsilon S''_a$. If $x_1 \leftrightarrow (y_1, z_1)$ and $x_2 \leftrightarrow (y_2, z_2)$, then we have $x_1 x_2 = y_1 z_1 y_2 z_2 = (y_1 a) z_1 (y_2 a) z_2 = y_1 (a z_1 y_2 a) z_2 = y_1 a z_2 = y_1 z_2$. The correspondence becomes an isomorphism when we define $(y_1, z_1)(y_2, z_2)$ as (y_1, z_2). This completes the proof.

It is of interest to observe that for $x_1, x_2 \varepsilon S_a$, the equality $x_1 x_2 = x_2 x_1$ obtains if and only if $x_1 = x_2$.

The semigroups S_a are completely determined by the sets S'_a and S''_a. For example, if S_a is finite, we have $x \leftrightarrow (i, j)$, $1 \leq i \leq k$, $1 \leq j \leq l$, where k is the number of elements in S'_a and l the number in S''_a.

If $y \varepsilon S'_a$ and $z \varepsilon S''_a$, then $y \leftrightarrow (ya, ay) = (y, a)$ and $z \leftrightarrow (za, az) = (a, z)$. We also have $y_1 y_2 = y_1$ if $y_1, y_2 \varepsilon S'_a$ and $z_1 z_2 = z_2$ if $z_1, z_2 \varepsilon S''_a$. Furthermore, if $y \varepsilon S'_a$ and $x \varepsilon S_a$, then $xy \leftrightarrow (xa, ax)(y, a) = (xa, a)$, so that $xy \varepsilon S'_a$. Similarly, we find that $zx \varepsilon S''_a$ if $z \varepsilon S''_a$ and $x \varepsilon S_a$. We will say that S'_a is a left ideal of S_a and S''_a is a right ideal of S_a.

2.16. Definition. A non-void subset I of a semigroup G is said to be a left ideal of G if $xy \varepsilon I$ for all $x \varepsilon G$ and $y \varepsilon I$. Right and 2-sided ideals are defined similarly.[13]

§ 3. Representations of semigroups

We use the term *representation* in connection with a semigroup G to mean a homomorphism of G into the multiplicative semigroup $\mathfrak{M}_n$ for some $n \geq 1$. We begin with a study of 1-dimensional representations.

3.1. Definition. Let G be a semigroup. A complex function χ defined on G is said to be a semicharacter of G if $\chi(x) \neq 0$ for some $x \varepsilon G$ and $\chi(xy) = \chi(x)\chi(y)$ for all $x, y \varepsilon G$.[14]

We list a few simple properties of semicharacters. Proofs are left to the reader.

3.1.1. If x is an element of finite order belonging to a semigroup G and if χ is a semicharacter of G, then $\chi(x)$ is either 0 or a root of unity.

3.1.2. If x is an idempotent element and χ is a semicharacter, then $\chi(x) = 0$ or 1.

3.1.3. If G contains a unit, e, and if χ is a semicharacter of G, then $\chi(e) = 1$.

[13] This widely used concept goes back at least to SUŠKEVIČ.

[14] S. SCHWARZ [38] has used a slightly different definition of semicharacter and has obtained a number of interesting results paralleling ours. We are also indebted to Dr. SCHWARZ for personal conversations on this topic.

3.1.4. If G contains a zero, z, and if χ is a semicharacter of G not identically 1, then $\chi(z)=0$.

3.1.5. A semicharacter of a group is a character in the usual sense.

It is essential for our purposes that a semicharacter be allowed to assume the value 0. In fact:

3.1.6. Let G be a finite semigroup such that for every pair of distinct elements $x, y \varepsilon G$, there exists a semicharacter χ vanishing nowhere such that $\chi(x) \neq \chi(y)$. Then G is a commutative group.

3.2.1. For the moment, we consider a finite commutative semigroup G and suppose that χ is a semicharacter of G. We will use the notation and results of 2.11. For some $x \varepsilon G$, we have $\chi(x) \neq 0$. Then 2.6.2 shows that $x(a) \neq 0$ for some $a \varepsilon H$, and hence $\chi(a)=1$, by 3.1.2. We take $a_0 = \prod_{a \varepsilon H, \chi(a)=1} a$. It is clear that $\chi(a_0)=1$ and that for $a \varepsilon H$, we have $\chi(a)=1$ if $a_0 a = a_0$ and $\chi(a)=0$ if $a_0 a \neq a_0$. For all $x \varepsilon G$, we have $\chi(x)=\chi(a_0)\chi(x)=\chi(a_0 x)$. If $x \varepsilon T_a$ and $a_0 a = a_0$, then $a_0 x \varepsilon T_{a_0 a} = T_{a_0}$ and $a_0 x = a_0(a_0 x) \varepsilon a_0 T_{a_0} = T_{a_0} \cap G^{\circ}$. If $x \varepsilon T_a$ and $a_0 a \neq a_0$, then $x^m = a$ for some m and hence $\chi(x)^m = \chi(a) = 0$, which implies that $\chi(x)=0$. Now $T_{a_0} \cap G^{\circ}$ is a group and $\chi(a_0) \neq 0$; hence χ, with its domain restricted to the group $T_{a_0} \cap G^{\circ}$, is a character of that group. We have therefore proved the following theorem.

3.2. Theorem. If G is a finite commutative semigroup and χ is a semicharacter of G, then there is an element $a_0 \varepsilon H$ and a character χ_{a_0}[15] of the group $T_{a_0} \cap G^{\circ}$ such that

$$\chi(x)=\begin{cases} 0 \text{ if } a_0 a \neq a_0 \text{ for the element } a \text{ such that } x \varepsilon T_a, \\ \chi_{a_0}(a_0 x) \text{ if } a_0 a = a_0 \text{ for the element } a \text{ such that } x \varepsilon T_a. \end{cases}$$

3.2.2. We point out that Theorem 3.2 remains valid for commutative semigroups G in which all elements have finite order and in which H is finite.

3.3. Theorem. If G is a finite or infinite commutative semigroup, all of whose elements have finite order, if $a_0 \varepsilon H$, and if χ_{a_0} is a character of the group $T_{a_0} \cap G^{\circ}$, then the function χ, defined by the relations

$$\chi(x)=\begin{cases} 0 \text{ if } a_0 a \neq a_0 \text{ for the element } a \text{ such that } x \varepsilon T_a, \\ \chi_{a_0}(a_0 x) \text{ if } a_0 a = a_0 \text{ for the element } a \text{ such that } x \varepsilon T_a, \end{cases}$$

is a semicharacter of G.

[15] For typographical reasons, we violate here the convention of footnote 4.

Proof. Since $\chi(a_0) \neq 0$, we have only to prove $\chi(xy) = \chi(x)\chi(y)$. If $x \varepsilon T_a$ and $y \varepsilon T_b$, then $xy \varepsilon T_{ab}$. If $a_0 ab = a_0$, then $a_0 a = a_0 aba = a_0 ab = a_0$ and $a_0 b = a_0 abb = a_0 ab = a_0$. If $a_0 a = a_0 b = a_0$, then $a_0 ab = a_0 a a_0 b = a_0$. Therefore we have $\chi(xy) \neq 0$ if and only if $\chi(x) \neq 0$ and $\chi(y) \neq 0$. If $\chi(x) \neq 0$ and $\chi(y) \neq 0$, we then have $\chi(x)\chi(y) = \chi_{a_0}(a_0 x)\chi_{a_0}(a_0 y) = \chi_{a_0}(a_0 x a_0 y) = \chi_{a_0}(a_0 xy) = \chi(xy)$.

We have also the following simple consequences of Theorems 3.2 and 3.3.

3.3.1. Corollary. Let G be a finite commutative semigroup. Then the semicharacters of G form a linearly independent set of functions.

Proof. If $\chi_1, \ldots, \chi_m$ are semicharacters of G, if $\alpha_1, \ldots, \alpha_m \varepsilon K$, and $\sum_{j=1}^{m} \alpha_j \chi_j = 0$, then consider any idempotent $a \varepsilon G$ and the group $T_a \cap G^{\circ}$. On this group, every semicharacter χ_j is either identically 0 or is a character. Since characters of a finite group are linearly independent, we see that $\alpha_j = 0$ for all j such that $\chi_j(a) \neq 0$. Since a is arbitrary, it follows that all $\alpha_j = 0$ $(j = 1, 2, \ldots, m)$.

3.3.2. Corollary. Let G be a finite commutative semigroup admitting m distinct semicharacters. Then $m \leq o(G)$.

3.3.3. Note. Corollaries 3.3.1 and 3.3.2 remain valid for noncommutative finite semigroups G: one can show this by mapping G homomorphically onto a commutative semigroup H admitting just the same semicharacters as G. We omit the details.

3.4. Theorem. Let G be a semigroup all of whose elements are of finite order and having the property that for all $x, y \varepsilon G$ such that $x \neq y$, there is a semicharacter χ such that $\chi(x) \neq \chi(y)$. Then G is commutative and $l_x = 1$ for all $x \varepsilon G$.

Proof. For all semicharacters χ, we have $\chi(xy) = \chi(x)\chi(y) = \chi(y)\chi(x) = \chi(yx)$. It follows that $xy = yx$. Also $\chi(x)^{k_x + l_x} = \chi(x^{k_x + l_x}) = \chi(x^{l_x}) = \chi(x)^{l_x}$ and hence $\chi(x)^{k_x + 1} = \chi(x)$ for all χ: thus we have $x^{k_x + 1} = x$, and by 2.6.1, $l_x = 1$.

3.5. Theorem. Let G be a finite or infinite commutative semigroup all of whose elements are of finite order and for which $l_x = 1$ for all $x \varepsilon G$. Then, for every $x, y \varepsilon G$ such that $x \neq y$, there is a semicharacter χ such that $\chi(x) \neq \chi(y)$.

Proof. If $\chi(x) = \chi(y)$ for all of the semicharacters χ described in Theorem 3.3, we first take $a_0 = x^{k_x}$ and obtain $\chi(x) = \chi_{a_0}(a_0 x) = \chi_{a_0}(x^{k_x} x) = \chi_{a_0}(x)$. Since $\chi_{a_0}(x) \neq 0$, we have $\chi(y) = \chi(x) \neq 0$ and hence $a_0 y^{k_y} = a_0$, $\chi(y) = \chi_{a_0}(x^{k_x} y)$. Thus we have $\chi_{a_0}(x) = \chi_{a_0}(x^{k_x} y)$; but χ_{a_0} can be any character of the commutative group $T_{a_0} \cap G^{\circ}$ (which in this case is just T_{a_0}), so we have $x = x^{k_x} y$. This follows from the well-known theorem that distinct elements of any commutative group can be separated by characters. See for example Weil [48], p. 99. We also have $x^{k_x} y^{k_y} = x^{k_x}$ since $a_0 y^{k_y} = a_0$. Since x and y are quite interchangeable in this argument, we also have $y = y^{k_y} x$ and $y^{k_y} x^{k_x} = y^{k_y}$. Therefore $x^{k_x} = x^{k_x} y^{k_y} = y^{k_y}$ and $x = x^{k_x} y = y^{k_y} y = y$. This completes the proof.

3.6.1. Returning to the case of a finite commutative semigroup G, we see by Theorem 3.2 that all semicharacters of G are given by Theorem 3.3. Each semicharacter of G, is, of course, a semicharacter of G°. In the other direction, it follows from Theorem 3.3 that each semicharacter of G° can be extended to be a semicharacter of G in one and only one way. In fact, if $x \,\varepsilon\, T_a$ and x non $\varepsilon\, G^\circ$, then $ax \,\varepsilon\, G^\circ$ and $x^m = a$ for some $m \geq 1$. Therefore $\chi(x) = \chi(a)\chi(x) = \chi(ax)$ if $\chi(a) = 1$ and $\chi(x) = 0$ if $\chi(a) = 0$, or more shortly, $\chi(x) = \chi(a)\chi(ax)$.

3.6.2. We also notice that if G is a finite commutative semigroup, then the number of distinct semicharacters of G is just the number of elements of G°. This follows from Theorems 3.2 and 3.3, since the number of characters of a finite commutative group is just the number of its elements.

3.7. Let G be a semigroup. Let $\hat{G}$ denote the set of all semicharacters of G. For $\chi, \psi \,\varepsilon\, \hat{G}$, the product $\chi\psi$ is the function on G such that $\chi\psi(x) = \chi(x)\psi(x)$ for all $x \,\varepsilon\, G$.

We list a number of simple results, leaving the proofs to the reader.

3.7.1. The semicharacters of a semigroup either form a semigroup by themselves or they form a semigroup if an additional element 0 is supplied. See semigroups 1 and 5, Appendix 2.

3.7.2. If G is a semigroup, then $\hat{G}$ has a unit.

3.7.3. If G is a semigroup with a unit, then $\hat{G}$ is a semigroup.

3.7.4. If G and $\hat{G}$ are semigroups, then, in the notation of Theorems 2.2 and 2.3, $\hat{G}_e = (\hat{G})_z$ and $\hat{G}_z = (\hat{G})_e$.

3.7.5. If χ is a semicharacter of a semigroup G and if $\varepsilon(x) = 1$ for $\chi(x) \neq 0$ and $\varepsilon(x) = 0$ for $\chi(x) = 0$, then ε is a semicharacter of G.

3.7.6. Let G be a semigroup and let χ be a semicharacter of G. Let $\psi(x) = [\chi(x)]^{-1}$ for $\chi(x) \neq 0$ and $\psi(x) = 0$ for $\chi(x) = 0$. Then $\psi(x)$ is also a semicharacter of G.

3.7.7. If χ is a semicharacter, then so is $\bar{\chi}$.

3.8. Let H be a finite commutative idempotent semigroup. If $a \,\varepsilon\, H$, then we say that a is a prime element of H if the equality $a = bc \; (b, c \,\varepsilon\, H)$ implies $b = c = a$.

3.8.1. An element a of H is a prime element if and only if $a = ab$ and $b \varepsilon H$ imply $b = a$.

3.9. Theorem. Every finite commutative idempotent semigroup contains at least one prime element.

Proof. Let a_1 be any element of H. If a_1 is not a prime element, then there is an element $a_2 \varepsilon H$ such that $a_1 = a_1 a_2$ and $a_2 \neq a_1$. Repeating this step, we obtain a sequence $a_1, a_2, \ldots, a_m$ such that $a_i = a_i a_{i+1}$ and $a_{i+1} \neq a_i$ $(1 \leq i \leq m-1)$. If $h \leq j \leq m$, we have $a_h = a_h a_{h+1} a_{h+2} \ldots a_j$ and hence $a_h a_j = a_h$. If $h < j \leq m$ and $a_h = a_j$, then $a_h = a_h a_{h+1} = a_j a_{h+1} = a_{h+1}$, and this is a contradiction. Therefore the elements a_i are all distinct, and, H being finite, the sequence $a_1, a_2, \ldots, a_m$ will eventually end with a prime element a_m.

3.9.1. It is now clear that for every element $a_1 \varepsilon H$, there is a prime element a_m of H such that $a_1 = a_1 a_m$.

3.9.2. If H has just one prime element p, then $a = ap$ for all $a \varepsilon H$ and hence p is a unit of H. It is easy to see that if H has a unit e, then H has just the one prime element e.

3.10. Theorem. Let G be a commutative semigroup all of whose elements are of finite order and let H be the semigroup of idempotent elements of G. If H is finite, then $\hat{G}$ is a semigroup if and only if H has a unit.

Proof. The elements of $\hat{G}$ are the semicharacters of G, as constructed in Theorem 3.3. Let χ_1 be any of the semicharacters determined by $a_0 = b$, χ_2 a semicharacter determined by $a_0 = c$, where b and c are elements of H. Then $\chi_1 \chi_2$ fails to be a semicharacter if and only if $\chi_1(x)\chi_2(x) = 0$ for all $x \varepsilon G$; that is, if and only if $ba \neq b$ or $ca \neq c$ for every $a \varepsilon H$. If H has a unit e, then $be = b$ and $ce = c$ and $\chi_1 \chi_2$ is a semicharacter of G. If H does not have a unit, then 3.9 and 3.9.2 imply that H has at least two prime elements. If b and c are distinct prime elements of H, then $ba \neq b$ if $a \neq b$ and $ca \neq c$ if $a = b$, and therefore the product $\chi_1 \chi_2$ is identically 0 and is not a semicharacter of G.

3.11. Theorem. Let G be a commutative semigroup all of whose elements are of finite order, and let H be as usual the semigroup of idempotent elements of G. If H is finite, then[16] $\hat{G} \cong \widehat{G^\circ}$.

[16] The symbol "$\cong$" is taken to mean the existence of a one-to-one correspondence τ such that $\tau(xy) = \tau(x)\tau(y)$ whenever either side exists.

Proof. This follows at once from 3.2.2 and 3.3.

3.12. In some of the following paragraphs, we shall restrict the semigroup G a little more. We shall consider a commutative semigroup G all of whose elements are of finite order, such that the subsemigroup H of idempotent elements is finite, and such that the integers $k_x\,(x \varepsilon G)$ have a finite upper bound. These conditions are obviously satisfied by all finite commutative semigroups G. Since the k_x are bounded, they have a least common multiple k. It is clear from 2.6.1 that we have $x^{k+l_x} = x^{l_x}$ for all $x \varepsilon G$.

3.13. Theorem. If G satisfies 3.12 and H has a unit, then $\hat{G}$ satisfies 3.12 and $(\hat{G})^\circ = \hat{G}$. Also the set of idempotent elements of $\hat{G}$ is isomorphic with $\hat{H}$.

Proof. Theorem 3.10 shows that $\hat{G}$ is a semigroup. If $\chi \varepsilon \hat{G}$, then $\chi(x)^{k+l_x} = \chi(x^{k+l_x}) = \chi(x^{l_x}) = \chi(x)^{l_x}$, and hence $\chi(x)^{k+1} = \chi(x)$ for all $x \varepsilon G$. It follows that $\chi^{k+1} = \chi$ for all $\chi \varepsilon \hat{G}$. This, together with 2.6.1, shows that every element of $\hat{G}$ is of finite order, that the integers k_χ have the upper bound k, and that $(\hat{G})^\circ = \hat{G}$. From 3.1.2, 3.2.2, and 3.3, we see that if $\chi^2 = \chi$, then there is an $a_0 \varepsilon H$ such that

3.13.1. $$\chi(x) = \begin{cases} 0 \text{ if } a_0 a \neq a_0 \text{ for the element } a \text{ such that } x \varepsilon T_a, \\ 1 \text{ if } a_0 a = a_0 \text{ for the element } a \text{ such that } x \varepsilon T_a. \end{cases}$$

If 2 of these semicharacters χ are distinct, they take on different values for some $x \varepsilon H$. For $x \varepsilon H$, these are just the semicharacters of H. This proves the last statement of the theorem and completes the proof of the fact that $\hat{G}$ satisfies 3.12, since it is clear that $\hat{H}$ is finite.

3.14. Theorem. Let G satisfy 3.12 and let H have a unit. For each $x \varepsilon G$, the function $\psi_x(\chi) = \chi(x)$ is a semicharacter ψ_x of $\hat{G}$. Furthermore, the ψ_x are all distinct if and only if $G = G^\circ$.

Proof. If $\chi_1, \chi_2 \varepsilon \hat{G}$ and $x \varepsilon G$, then $\psi_x(\chi_1\chi_2) = \chi_1\chi_2(x) = \chi_1(x)\,\chi_2(x) = \psi_x(\chi_1)\,\psi_x(\chi_2)$. Since the function $\chi_0(x) = 1$ is a semicharacter of G, we have $\psi_x(\chi_0) = 1$ and $\chi_0 \varepsilon \hat{G}$. Therefore $\psi_x(\chi)$ is not identically 0 and hence is a semicharacter of $\hat{G}$. The second part of the theorem follows at once from 3.4 and 3.5.

3.15. Theorem. Let G satisfy 3.12 and let H have a unit. Then the set of all distinct ψ_x of Theorem 3.14 are just those ψ_x for which $x \varepsilon G^\circ$.

Proof. Suppose that $x_1 \varepsilon G$ and x_1 non εG°. Let $a \varepsilon H$ be the a such that $x_1 \varepsilon T_a$ and let $x_2 = a x_1$. Then $x_2 \varepsilon G^\circ$ and $x_2 \varepsilon T_a$. If $\chi \varepsilon \hat{G}$, we use 3.2 and 3.3 to establish the equalities

$$\chi(x_1) = \begin{cases} 0 & \text{if } a_0 a \neq a_0 \\ \chi_{a_0}(a_0 x_1) & \text{if } a_0 a = a_0 \end{cases},$$

$$\chi(x_2) = \begin{cases} 0 & \text{if } a_0 a \neq a_0 \\ \chi_{a_0}(a_0 x_2) & \text{if } a_0 a = a_0 \end{cases}.$$

Since $a_0 x_2 = a_0 a x_1 = a_0 x_1$ when $a_0 a = a_0$, we have $\chi(x_1) = \chi(x_2)$ and therefore $\psi_{x_1} = \psi_{x_2}$, with $x_2 \varepsilon G^\circ$. Furthermore, if $x_1, x_2 \varepsilon G^\circ$ and $\psi_{x_1} = \psi_{x_2}$, then $\chi(x_1) = \chi(x_2)$ for all $\chi \varepsilon \hat{G}$. This implies that $x_1 = x_2$, in view of 2.11 and 3.5.

3.16. Theorem. If G satisfies 3.12 and H has a unit, then the set of distinct semicharacters ψ_x of 3.14 is isomorphic with G°.

Proof. Theorem 3.15 shows that the correspondence $x \leftrightarrow \psi_x$ is a one-to-one correspondence between G° and the set of distinct ψ_x. Furthermore, if $x_1, x_2 \varepsilon G^\circ$, we have $\psi_{x_1 x_2}(\chi) = \chi(x_1 x_2) = \chi(x_1)\chi(x_2) = \psi_{x_1}(\chi)\psi_{x_2}(\chi)$, and therefore this correspondence is an isomorphism.

3.17. Theorem. Let G be a finite commutative semigroup. Then $\hat{\hat{G}} \cong G$ if and only if G has a unit and $G = G^\circ$.

Proof. From 3.7.2 and 3.11, we see that the conditions enunciated are necessary. From 3.16, we see that we shall have proved their sufficiency as soon as we have shown that $\hat{G}$ has no semicharacters other than the ψ_x of Theorem 3.14. Since, as Theorem 3.13 shows, $(\hat{G})^\circ = \hat{G}$, we can apply 3.6.2 to find that G, $\hat{G}$, and $\hat{\hat{G}}$ all have the same number of elements. But the number of distinct ψ_x is, in the present case, the number of elements of G.

3.17.1. It is of interest to note that if G is a finite commutative semigroup, then $\hat{\hat{G}} \cong G^\circ$ if and only if H has a unit.

3.17.2. It is also of interest to note that a finite commutative semigroup G is isomorphic to $\hat{L}$ for some finite semigroup L if and only if $G = G^\circ$ and G has a unit. In connection with this problem, see Schwarz [38].

3.18. In the preceding paragraphs, the requirement that H have a unit was needed to make $\hat{G}$ a semigroup. If $\hat{G}$ is not a semigroup, we can make it one if we supply it with a zero. We suppose that G is a finite commutative semigroup such that $\hat{G}$ is not a semigroup. We let $\dot{G}$ be the set consisting of all the χ of $\hat{G}$ and one other element ω. We define multiplication in $\dot{G}$ as follows:

$$\omega \cdot \omega = \omega \cdot \chi = \chi \cdot \omega = \omega\,;$$

$$\chi_1 \cdot \chi_2 = \chi_1 \chi_2 \quad \text{if } \chi_1 \chi_2 \text{ is not identically } 0 \text{ on } G;$$

$$\chi_1 \cdot \chi_2 = \omega \qquad \text{if } \chi_1 \chi_2 \text{ is identically } 0 \text{ on } G.$$

Here $\chi_1 \chi_2$ denotes the function $\chi_1 \chi_2(x) = \chi_1(x)\chi_2(x)$. Now $\dot{G}$ is a semigroup with a unit. If k is the least common multiple of all of the k_x for $x \varepsilon G$, we have $\chi(x)^{k+l_x} = \chi(x^{k+l_x}) = \chi(x^{l_x}) = \chi(x)^{l_x}$ and hence $\chi(x)^{k+1} = \chi(x)$ and $\chi^{k+1} = \chi$. Since we also have $\omega^2 = \omega$, we see that $\dot{G}^\circ = \dot{G}$. Therefore, according to 3.6.2, $\dot{G}$ and $\hat{\dot{G}}$ have the same numbers of elements.

If $x \varepsilon G$, we let $\psi_x(\omega) = 0$, $\psi_x(\chi) = \chi(x)$. Then we have

$$\psi_x(\omega \cdot \omega) = \psi_x(\omega) = 0 = \psi_x(\omega)\,\psi_x(\omega),$$

$$\psi_x(\omega \cdot \chi) = \psi_x(\omega) = 0 = \psi_x(\omega)\,\psi_x(\chi),$$

$$\psi_x(\chi_1 \cdot \chi_2) = \psi_x(\chi_1\chi_2) = \chi_1(x)\,\chi_2(x) = \psi_x(\chi_1)\,\psi_x(\chi_2) \quad \text{if } \chi_1\chi_2 \text{ is not identically } 0,$$

$$\psi_x(\chi_1 \cdot \chi_2) = \psi_x(\omega) = 0 = \chi_1(x)\,\chi_2(x) = \psi_x(\chi_1)\,\psi_x(\chi_2) \quad \text{if } \chi_1\chi_2 \text{ is identically } 0.$$

It follows that ψ_x is a semicharacter of $\dot{G}$ if it is not identically 0. But χ_0, the function identically 1, is a semicharacter of G. Therefore we have $\psi_x(\chi_0) = \chi_0(x) = 1$ and ψ_x is accordingly a semicharacter of $\dot{G}$. We can produce one additional semicharacter of $\dot{G}$, namely, the function ψ_0 which is identically 1 on $\dot{G}$.

As before, we obtain the set of all distinct ψ_x as x ranges through G°. It is plain that ψ_0 is distinct from all of the ψ_x. If G° has n elements, then $\hat{G}$ has n elements, by 3.6.2. Then $\dot{G}$ and $\hat{\dot{G}}$ both have $n+1$ elements. Therefore $\hat{\dot{G}}$ is the set consisting of ψ_0 and the ψ_x with $x \varepsilon G^\circ$.

If $x_1, x_2 \varepsilon G^\circ$, then $\psi_{x_1}(\chi)\,\psi_{x_2}(\chi) = \chi(x_1)\,\chi(x_2) = \chi(x_1 x_2) = \psi_{x_1 x_2}(\chi)$ and $\psi_{x_1}(\omega)\,\psi_{x_2}(\omega) = 0 = \psi_{x_1 x_2}(\omega)$. Therefore we have $\psi_{x_1}\psi_{x_2} = \psi_{x_1 x_2}$. We also have $\psi_0 \psi_{x_1} = \psi_{x_1}$ and $\psi_0 \psi_0 = \psi_0$.

Bringing these results together, we see that we have proved that if G is a finite commutative semigroup such that $\hat{G}$ is not a semigroup, then $\hat{\dot{G}} \cong (G^\circ)_e$.

3.19. Another possible way to take care of the eventuality that $\hat{G}$ fails to be a semigroup is to include the function identically 0 among the semicharacters of G. Again we suppose that G is a finite commutative semigroup, and we let $\bar{G}$ be the set consisting of all the semicharacters of G and the function ω such that $\omega(x) = 0$ for all $x \varepsilon G$. We define multiplication in $\bar{G}$ in the usual way.

If $\hat{G}$ is not a semigroup, it is easy to show that we have $\bar{G} \cong G$. If $\hat{G}$ is a semigroup, we have $\bar{G} \cong (\hat{G})_z$. In both cases, we see that $\bar{G}$ is a semigroup, and $\bar{\bar{G}} = (\hat{\bar{G}})_z$. From 3.18, we have $\hat{\bar{G}} \cong (G^\circ)_e$ if $\hat{G}$ is not a semigroup. If $\hat{G}$ is a semigroup, we have $\hat{\hat{G}} \cong G^\circ$ by 3.10 and 3.17.1. Hence $\widehat{(\hat{G})_z} \cong (\hat{\hat{G}})_e \cong (G^\circ)_e$ by 3.7.4. Therefore in both cases, we have $\bar{\bar{G}} \cong ((G^\circ)_e)_z$. We note, finally, the following equalities:

$$((G^\circ)_e)_z = ((G^\circ)_z)_e = ((G_e)^\circ)_z = ((G_z)^\circ)_e = ((G_e)_z)^\circ = ((G_z)_e)^\circ.$$

3.20. If G is a finite commutative semigroup such that $\hat{\hat{G}} \cong G$, it is natural to ask about the relation of $\hat{G}$ to G. If the semigroup G is a group, then we have $\hat{G} \cong G$. If G is the semigroup of order 3 listed as No. 10 in Appendix 2, then $\hat{G}$ is the semigroup No. 8 of Appendix 2. Since G has a zero and $\hat{G}$ does not, we see that $\hat{G} \cong G$ is false in this case. Necessary and sufficient conditions for $\hat{G} \cong G$ appear to be quite complicated; however, we give a brief sketch of certain conditions which are necessary for this isomorphism to obtain.

Suppose, then, that $\hat{\hat{G}} \cong G$. This implies that $G^\circ = G$ and that G has a unit. For elements a_1, a_2 in H, the subsemigroup of idempotent elements of G, we write $a_1 < a_2$ if $a_1 a_2 = a_1$. It is easily seen that H is a lattice under the partial ordering just defined, that the meet $a_1 \wedge a_2$ is $a_1 a_2$, while the join $a_1 \vee a_2$ is the product of all $a \in H$ such that $a_1 a = a_1$ and $a_2 a = a_2$. Since H has a unit, the product defining $a_1 \vee a_2$ is never void. If χ is any semicharacter of G such that $\chi^2 = \chi$, then, by 3.13.1, we find that $\chi(x) = 0$ if $a \not< x^k$ and $\chi(x) = 1$ if $a < x^k$, where k is the least common multiple of all of the integers k_x $(x \in G)$, and where a is some element of H. If we call this semicharacter χ_a, we see that the χ_a are all distinct, even over the subsemigroup H. It is now clear that the set $\{\chi_a\}$ of all idempotent elements of $\hat{G}$ is isomorphic with the dual $\hat{H}$ of H.[17]

Now, $\chi_{a_1} < \chi_{a_2}$ if and only if $\chi_{a_1}\chi_{a_2} = \chi_{a_1}$; this equality is equivalent to the assertion $\chi_{a_1}(x) = 1$ implies $\chi_{a_2}(x) = 1$, which, in turn, is just $a_2 < a_1$. Therefore the lattice $\hat{H}$ is isomorphic to the lattice obtained from H by inverting the relation $<$. If the relation $\hat{G} \cong G$ is to obtain, we must have $\hat{H} \cong H$, or, in other words, H must be lattice-isomorphic with the lattice obtained by inverting it. The semigroup of order 5 with the table

[17] This idea has also been used by SCHWARZ [38].

	1	2	3	4	5
1	1	2	3	4	5
2	2	2	3	4	5
3	3	3	3	5	5
4	4	4	5	4	5
5	5	5	5	5	5

has the properties $G = H$ and $\hat{\hat{G}} \cong G$, but it does not have the property that $\hat{G} \cong G$, as a sketch of the lattice H will show.

Even if G has the property that H is isomorphic with its inverted lattice, the isomorphism $\hat{G} \cong G$ can still fail. An isomorphism between $\hat{G}$ and G implies an isomorphism $\tau(a) \leftrightarrow \chi_a \leftrightarrow a$ between H and its inverted lattice. It can be proved that $\hat{G} \cong G$ then implies an isomorphism between T_a and $T_{\tau(a)}$, which are groups in the present case. The semigroup No. 10 of Appendix 2 fails to satisfy $\hat{G} \cong G$ because $\tau(1) = 3$ and $T_1 = \{1, 2\} \not\cong T_3 = \{3\}$.

It is, finally, possible to have $T_a \cong T_{\tau(a)}$ for all $a \varepsilon H$ and still to have $\hat{G} \cong G$ fail because of the way elements from different T_a multiply. The relationships appear to be quite involved. The semigroup of order 6 with the following multiplication table will serve as an example.

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	3	4	5	6
3	3	3	3	4	5	6
4	4	4	4	3	6	5
5	5	5	5	6	5	6
6	6	6	6	5	6	5

Finally, we make some remarks concerning general representations of semigroups.

3.21. Theorem. Every semigroup G of finite order n, commutative or not, has a faithful representation by matrices of order not exceeding $n+1$.

Proof. Let $G = \{x_1, x_2, \ldots, x_n\}$. We adjoin a unit x_0 to G and consider the matrices $M_k = (c_{ij}^k)$, $k = 1, 2, \ldots, n$, where the element in the i-th row and j-th column is $c_{ij}^k = \varphi_i(x_k x_j)$. $(i, j = 0, 1, \ldots, n)$. If $M_k M_l = (d_{ij})$ and $x_k x_l = x_m$, we have

$$d_{ij} = \sum_{h=0}^{n} c_{ih}^{k} c_{hj}^{l} = \sum_{h=0}^{n} \varphi_i(x_k x_h)\, \varphi_h(x_l x_j) = \varphi_i(x_k x_l x_j) = \varphi_i(x_m x_j) = c_{ij}^{m},$$

and hence $M_k M_l = M_m$. Furthermore, if $M_k = M_l$, then $\varphi_i(x_k x_j) = \varphi_i(x_l x_j)$ for $i = 0, 1, \ldots, n$. This implies that $x_k x_j = x_l x_j$ for $j = 0, 1, \ldots, n$; but we have $x_k x_0 = x_k$ and $x_l x_0 = x_l$, and hence $k = l$. Therefore the correspondence $x_k \leftrightarrow M_k$ is an isomorphism.

The adjunction of a unit to G is done to assure that the correspondence $x_k \to M_k$ be one-to-one. It can be omitted if G contains no pair of elements x_k, x_l such that $x_k x_j = x_l x_j$ for all $x_j \varepsilon G$. If G contains no pair of elements x_k, x_l such that $x_j x_k = x_j x_l$ for all $x_j \varepsilon G$, then the adjunction can be omitted if c_{ij}^{k} is taken to be $\varphi_i(x_j x_k)$.

3.22. Theorem. Let $x \to B(x)$ be a homomorphism of a finite semigroup G into the semigroup $\mathfrak{M}_s$. Then the set of matrices $\{B(x)\}_{x \varepsilon G}$ is irreducible if and only if there are exactly s^2 linearly independent matrices in the set $\{B(x)\}_{x \varepsilon G}$.

This is essentially Burnside's theorem (v. d. Waerden [46], p. 197).

3.23. Irreducible representations of a finite semigroup by matrices need not be unitary. In fact, a semicharacter χ of a semigroup G such that $\chi(x) = 0$ for some $x \varepsilon G$ is a 1-dimensional representation of G which is plainly not unitary. As another example, consider the semigroup described in Theorem 4.2 *infra*. The mapping which sends each sequence 4.2.1 into its t-th coordinate $(t = 2, 3, \ldots, p+1$ and $s_t > 1)$ is an obviously irreducible representation of the semigroup in $\mathfrak{M}_{s_t}$ in which no image matrix can be made unitary under any inner product.

3.24. Note. Extensive discussions of representations from another point of view are found in Clifford [9] and Suškevič [43]. See also the very general notion of representation introduced by Lyapin [25].

§ 4. $\mathfrak{L}_1$ Algebras

In the present section, we apply the structure and representation theory for finite semigroups developed in §§ 2 and 3 to a study of algebras $\mathfrak{L}_1(G)$. We remark first that all $\mathfrak{L}_1$ algebras of dimensions 2 and 3 are listed in Appendices 1 and 2.

4.1. Theorem. Let $G = \{x_1, x_2, \ldots, x_n\}$ be a finite semigroup. The algebra $\mathfrak{L}_1(G)$ has a faithful representation as a subalgebra of $\mathfrak{M}_p$, where $p \le n+1$.

Proof. By Theorem 1.7, $\mathfrak{L}_1(G)$ is isomorphic to the algebra of all formal linear combinations $\sum_{j=1}^{n} \alpha_j x_j$. Adjoining a unit x_0 to G and defining M_j as in 3.21, we consider the mapping $\sum_{j=0}^{n} \alpha_j x_j \to \sum_{j=0}^{n} \alpha_j M_j$. It is easy to see that this produces an isomorphism of $\mathfrak{L}_1(G)$ into $\mathfrak{M}_{n+1}$. (It is in fact the regular representation of $\mathfrak{L}_1(G_e)$.)

4.1.1. Remark. The algebras G and H (Appendix 2), which can be shown to have no isomorphs contained in $\mathfrak{M}_3$, show that Theorem 4.1 cannot be strengthened. The algebra E (Appendix 2), on the other hand, shows that in some cases, $\mathfrak{L}_1(G)$ has an isomorph contained in $\mathfrak{M}_q$ with $q < o(G)$.

4.2. Theorem. Let A be a finite dimensional semisimple algebra over K. Then A is isomorphic to $\mathfrak{L}_1(G)$ for a finite semigroup G if and only if, in the representation of A as a direct sum of full matrix algebras, at least one summand is 1-dimensional.

Proof. Suppose that A is isomorphic to the algebra $A' = K \oplus \mathfrak{M}_{s_1} \oplus \mathfrak{M}_{s_2} \oplus \cdots \oplus \mathfrak{M}_{s_p}$, where $s_1, s_2 \ldots, s_p$ are positive integers. Then, writing $e_i^{(s)}$ as the element $(\delta_{ki}\delta_{lj})_{k,l=1}^{s}$ of $\mathfrak{M}_s$, we see that the elements

$$4.2.1 \qquad \begin{cases} \{1, 0, 0, \ldots, 0\} \\ \{1, 0, \ldots, 0, e_j^{(s_k)}, 0, \ldots, 0\} \qquad (i, j = 1, 2, \ldots, s_k;\ k = 1, 2, \ldots, p), \end{cases}$$

form a basis for A' which is closed under multiplication. We then apply Theorem 1.10.

To prove the converse, we note that the mapping $L \to L(1)$ (where we denote by 1 the function identically 1) of $\mathfrak{L}_1(G)$ onto K is obviously a linear functional on $\mathfrak{L}_1(G)$ and furthermore that $M \star L(1) = M_x(L_y(1(xy))) = M(1)L(1)$. Hence $\mathfrak{L}_1(G)$ admits at least one homomorphism onto K, and if $\mathfrak{L}_1(G)$ is semisimple, this property is reflected in the fact that some direct summand of $\mathfrak{L}_1(G)$ is isomorphic to K. This completes the proof.

4.3. Remark. Let G be a finite semigroup. Then $\mathfrak{L}_1(G)$ is not isomorphic to $\mathfrak{M}_p$ for any integer $p > 1$, nor is $\mathfrak{L}_1(G)$ a radical algebra. These observations follow at once from the fact, noted in the proof of Theorem 4.2, that $\mathfrak{L}_1(G)$ admits a homomorphism onto K. The fact that $\mathfrak{L}_1(G)$ is never a radical algebra is also a simple consequence of the existence of at least one idempotent in G. If a is an idempotent in G, then the functional λ_a is an idempotent in $\mathfrak{L}_1(G)$, and clearly $\mathfrak{L}_1(G)$ is not a nilpotent algebra. However, the radical of $\mathfrak{L}_1(G)$ can have any dimension from 0 to $o(G) - 1$, inclusive (see Theorems 4.2, 4.9, and 4.11).

4.4. Remark. It follows from 4.3 that no s^2 linearly independent $s \times s$ complex matrices $(s > 1)$ can form a semigroup; if so, their linear combinations would be an

$\mathfrak{L}_1$ algebra isomorphic to $\mathfrak{M}_s$. It follows from 4.3 and Theorem 3.22 that no semigroup of linearly independent complex $s \times s$ matrices ($s > 1$) can be irreducible.

4.5. Example. We see from 4.3 that not all finite dimensional algebras are $\mathfrak{L}_1$ algebras. Consider also the algebra over K with basis $\{e_1, e_2, u\}$ and the multiplication table

	e_1	e_2	u
e_1	e_1	0	u
e_2	0	e_1	0
u	0	u	0

(v. d. Waerden [46], p. 144). It is a routine matter to prove that this algebra is not an $\mathfrak{L}_1$ algebra.

4.6. The algebras $\mathfrak{M}_p$ ($p = 2, 3, \ldots$) are not $\mathfrak{L}_1$ algebras (4.3) but they do admit bases such that the product of 2 basis elements is either 0 or another basis element. The algebra over K with basis $\{y_1, y_2, y_3\}$ and the multiplication table

	y_1	y_2	y_3
y_1	0	y_3	0
y_2	$-y_3$	0	0
y_3	0	0	0

fails to have this property. We omit the verification.

4.7. Remark. Algebras having the property mentioned in 4.6 can be used to construct $\mathfrak{L}_1$ algebras, as follows. Let A be an algebra with basis $\{y_1, y_2, \ldots, y_m\}$ such that $y_i y_j = 0$ or some y_k ($i, j = 1, 2, \ldots, m$), and let $B \cong \mathfrak{L}_1(G)$, where $G = \{x_1, x_2, \ldots x_n\}$ is a semigroup, and $x_1^2 = x_1$. Then the direct sum $A \oplus B$ has a basis

$$\{(y_1, x_1),\ (y_2, x_1), \ldots, (y_m, x_1),\ (0, x_1),\ (0, x_2), \ldots, (0, x_n)\}.$$

This basis is a semigroup and hence, by Theorem 1.10, $A \oplus B$ is an $\mathfrak{L}_1$ algebra.

4.8. Remark. An important class of algebras are those which are linear space sums of their minimal left ideals (see for example Hopkins [20] or Dieudonné [11]). $\mathfrak{L}_1$ algebras need not have this property. Consider, for example, the $\mathfrak{L}_1$ algebra formed with semigroup 9 of Appendix 2. As noted in Appendix 2, this algebra is isomorphic to the algebra of all 2×2 matrices of the form

4.8.1
$$A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}.$$

This algebra is not the sum of its minimal left or right ideals. A simple calculation shows that if $a_{22} \neq 0$, for a matrix A of the form 4.8.1, then every left ideal containing A must contain all matrices of the form

4.8.2
$$\begin{pmatrix} 0 & b_{12} \\ 0 & b_{22} \end{pmatrix};$$

and the left ideal of all matrices of the form 4.8.2 is not minimal, the set of all matrices of the form

4.8.3
$$\begin{pmatrix} 0 & b_{12} \\ 0 & 0 \end{pmatrix}$$

being a proper sub-left ideal. A similar argument shows that no element 4.8.1 with $a_{11} \neq 0$ is contained in a minimal right ideal.

We next give concrete representations for $\mathfrak{L}_1(G)$ for a few finite semigroups G.

4.9. Theorem. Let $o(G) = n$ and let $xy = y$ for all $x, y \in G$. Then $\mathfrak{L}_1(G)$ is isomorphic to the algebra of all $n \times n$ matrices $(a_{ij})_{i,j=1}^{n}$ such that $a_{ij} = 0$ for $2 \leq i \leq n$.

Proof. Let e_{ij} $(i, j = 1, 2, \ldots, n)$ be as in the proof of Theorem 4.2, and let $x_1, x_2, \ldots, x_n$ be the elements of G, written in any order. Then the mapping $x_1 \to e_{11}$, $x_i \to e_{11} + e_{1i}$ $(i = 2, 3, \ldots, n)$ is an isomorphism of G into the multiplicative semigroup $\mathfrak{M}_n$, with the image matrices linearly independent. The linear combinations of the matrices $e_{11}, e_{11} + e_{12}, \ldots, e_{11} + e_{1n}$ being just the matrices $(a_{ij})_{i,j=1}^{n}$ with $a_{ij} = 0$ for $i > 1$, the theorem follows.

4.10. Theorem. Let $G = \{x_1, x_2, \ldots, x_n\}$ and let $x_i x_j = x_1$ $(i, j = 1, 2, \ldots, n)$. Then $\mathfrak{L}_1(G)$ is isomorphic to $K \oplus Z_{n-1}$.

Proof. We first represent G faithfully by linearly independent elements of $\mathfrak{M}_{n+1}$, as follows (notation analogous to that in 4.2): $x_1 \to e_{11}$, $x_i \to e_{11} + e_{2,i+1}$ $(i = 2, 3, \ldots, n)$. It is easy to see that this mapping has the properties required. It follows that $\mathfrak{L}_1(G)$ is isomorphic to the algebra of all linear combinations of the matrices $e_{11}, e_{2,3}, e_{2,4}, \ldots, e_{2,n+1}$ in $\mathfrak{M}_{n+1}$; and this algebra is obviously isomorphic to $K \oplus Z_{n-1}$.

4.11. Theorem. Let G be a finite semigroup with a single generator x, for which $l_x = l$ and $k_x = k$ (see 2.6.1) are arbitrary. Then $\mathfrak{L}_1(G)$ is isomorphic to the direct sum $A \oplus K_k$, where A is an algebra of dimension $l-1$ having a single generator u for which $u, u^2, \ldots, u^{l-1}$ are linearly independent and $u^l = 0$.

Proof. Consider the mapping of G into $A \oplus K_k$ defined by

$$x^j \to \left\{u^j;\ 1, \exp\left(\frac{2\pi i j}{k}\right),\ \exp\left(\frac{2\pi i 2 j}{k}\right), \ldots, \exp\left(\frac{2\pi i (k-1) j}{k}\right)\right\}$$

$$\text{for } j = 1, 2, \ldots, l-1$$

and by

$$x^j \to \left\{0;\ 1, \exp\left(\frac{2\pi i j}{k}\right),\ \exp\left(\frac{2\pi i 2 j}{k}\right), \ldots, \exp\left(\frac{2\pi i (k-1) j}{k}\right)\right\}$$

$$\text{for } j = l, l+1, \ldots, l+k-1$$

$(i^2 = -1)$. It is clear that this is an isomorphism of G into $A \oplus K_k$, and that the images of $x, x^2, \ldots, x^{l+k-1}$ are a linearly independent basis in $A \oplus K_k$. This establishes the present theorem.

4.12. Theorem. Let $G = \{x_1, x_2, \ldots, x_n\}$ with the operation $x_i x_j = x_{\max(i,j)}$. The algebra $\mathfrak{L}_1(G)$ is isomorphic to K_n.

Proof. The mapping $x_j \to \{0, 0, \ldots, 0_{(i-1)}, 1_{(i)}, \ldots, 1_{(n)}\}$ $(i = 1, 2, \ldots, n)$ is an isomorphism carrying G onto a basis for K_n.

4.13. Remark. The semigroup described in the preceding theorem is a simple example of a non-group whose $\mathfrak{L}_1$ algebra is isomorphic to that of an Abelian group of order n. In Theorem 5.21 *infra* we shall determine all finite semigroups having this property.

We next consider an arbitrary finite semigroup $G = \{x_1, x_2, \ldots, x_n\}$. We wish to discuss the effect upon $\mathfrak{L}_1(G)$ of certain adjunctions of an element to G. We make use of the fact that there is a representation $x_i \to \tau(x_i)$ of G by linearly independent matrices $\tau(x_i)$ and a corresponding representation $L \to \tau(L) = \sum_{i=1}^{n} L(\varphi_i)\,\tau(x_i)$ of $\mathfrak{L}_1(G)$ (see Theorem 4.1). We will form new matrices $\tau'(x_i)$ and $\tau'(a)$ to represent $G \cup \{a\}$, and these will determine a representation of $\mathfrak{L}_1(G \cup \{a\})$, from which we shall be able to determine the structure of $\mathfrak{L}_1(G \cup \{a\})$ in terms of the structure of $\mathfrak{L}_1(G)$.

4.14. Theorem. If G is a finite semigroup, then $\mathfrak{L}_1(G_z) \cong \mathfrak{L}_1(G) \oplus K$.

Proof. If z is the adjoined zero, we take

$$\tau'(x_i) = \begin{pmatrix} 0 & 0 \\ 0 & \tau(x_i) \end{pmatrix}, \quad \tau'(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

where by $\begin{pmatrix} 0 & 0 \\ 0 & \tau(x_i) \end{pmatrix}$ we mean the matrix $\tau(x_i)$ bordered above by a row of 0's and on the left by a column of 0's. It is clear that the matrices $\tau'(x_i)$, $\tau'(z)$ are linearly independent and that they represent G_z. Taking linear combinations of the matrices $\tau'(x_i)$, $\tau'(z)$, we see that $\mathfrak{L}_1(G_z)$ is isomorphic to the set of matrices

$$\begin{pmatrix} \alpha & 0 \\ 0 & \tau(L) \end{pmatrix}, \quad \alpha \,\varepsilon\, K, \ L \,\varepsilon\, \mathfrak{L}_1(G).$$

4.15. Theorem. If $G \cup \{a\}$ is obtained from G by idempotent adjunction (see Theorem 2.5), then $\mathfrak{L}_1(G \cup \{a\}) \cong \mathfrak{L}_1(G) \oplus K$.

Proof. If $a x_i = x_1 x_i$, $x_i a = x_i x_1$ $(i = 1, 2, \ldots, n)$, and $a^2 = a$, we take

$$\tau'(x_i) = \begin{pmatrix} 0 & 0 \\ 0 & \tau(x_i) \end{pmatrix}, \quad \tau'(a) = \begin{pmatrix} 1 & 0 \\ 0 & \tau(x_1) \end{pmatrix}.$$

4.16. Theorem. If $G \cup \{a\}$ is obtained from G by the adjunction of a repeat element (see Theorem 2.4), then $\mathfrak{L}_1(G \cup \{a\}) \cong \mathfrak{L}_1(G) \oplus Z_1$.

Proof. If $a x_i = x_1 x_i$, $x_i a = x_i x_1$ $(i = 1, 2, \ldots, n)$, and $a^2 = x_1^2$, we take

$$\tau'(x_i) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tau(x_i) \end{pmatrix}, \quad \tau'(a) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tau(x_1) \end{pmatrix}.$$

4.17. Theorem. Let G be a finite semigroup with a zero, a. Let $G \cup \{b\}$ be obtained from G as in 2.2.1. Then $\mathfrak{L}_1(G \cup \{b\}) \cong \mathfrak{L}_1(G) \oplus K$.

Proof. We take

$$\tau'(x_i) = \begin{pmatrix} 1 & 0 \\ 0 & \tau(x_i) \end{pmatrix}, \quad \tau'(b) = \begin{pmatrix} -1 & 0 \\ 0 & \tau(a) \end{pmatrix}.$$

The relation of $\mathfrak{L}_1(G_e)$ to $\mathfrak{L}_1(G)$ is different from the foregoing. We can take

$$\tau'(x_i) = \begin{pmatrix} 0 & 0 \\ 0 & \tau(x_i) \end{pmatrix}, \quad \tau'(e) = \begin{pmatrix} 1 & 0 \\ 0 & I \end{pmatrix},$$

where I is the unit matrix of the same order as the matrices $\tau(x_i)$, but this in general does not allow us to write $\mathfrak{L}_1(G_e)$ as a direct sum. We see that $\mathfrak{L}_1(G)$ is isomorphic to the set of all linear combinations of $\tau'(x_i)$, and hence $\mathfrak{L}_1(G_e)$ is the algebra obtained by adjoining a unit to $\mathfrak{L}_1(G)$.

4.18. Theorem. If $\mathfrak{L}_1(G)$ has a unit, then $\mathfrak{L}_1(G_e) \cong \mathfrak{L}_1(G) \oplus K$.

Proof. Let U be the unit of $\mathfrak{L}_1(G)$. Let E be the unit adjoined to $\mathfrak{L}_1(G)$ to obtain $\mathfrak{L}_1(G_e)$. For all $L \varepsilon \mathfrak{L}_1(G)$, we have $U \star L = L \star U = E \star L = L \star E = L$, and we also have $E \star E = E$. If $M \varepsilon \mathfrak{L}_1(G_e)$, we have $M = \alpha E + N$ for some $\alpha \varepsilon K$ and some $N \varepsilon \mathfrak{L}_1(G)$. We can write $M = \alpha(E - U) + (N + \alpha U)$. Now $N + \alpha U \varepsilon \mathfrak{L}_1(G)$, $(E-U) \star (E-U) = E - U$, $(E-U) \star L = L \star (E-U) = 0$ if $L \varepsilon \mathfrak{L}_1(G)$. This implies, clearly enough, that $\mathfrak{L}_1(G_e) \cong \mathfrak{L}_1(G) \oplus K$.

4.19. Theorem. If $\mathfrak{L}_1(G_e) \cong \mathcal{A} \oplus \mathcal{B}$, where $\mathcal{A} \subset \mathfrak{L}_1(G_e)$ and $\mathcal{B} \subset \mathfrak{L}_1(G_e)$, then, except for a possible interchange of $\mathcal{A}$ and $\mathcal{B}$, we have $\mathcal{A} \subset \mathfrak{L}_1(G)$, $\mathcal{A}$ has a unit, and $\mathfrak{L}_1(G) = \mathcal{A}$ or $\mathfrak{L}_1(G) \cong \mathcal{A} \oplus (\mathcal{B} \cap \mathfrak{L}_1(G))$.

Proof. If E is the unit adjoined to $\mathfrak{L}_1(G)$, we have $E = A_1 + B_1$ for some $A_1 \varepsilon \mathcal{A}$ and $B_1 \varepsilon \mathcal{B}$; furthermore, $A_1 = \alpha E + M$ and $B_1 = (1-\alpha) E - M$ for some $\alpha \varepsilon K$ and some $M \varepsilon \mathfrak{L}_1(G)$. Now $A_1 \star B_1 = 0$, so we have $\alpha(1-\alpha) E + (1 - 2\alpha) M - M^2 = 0$ and hence $\alpha(1-\alpha) = 0$. After a possible interchange of $\mathcal{A}$ and $\mathcal{B}$, we can suppose $\alpha = 0$, and we thus have $A_1 = M$ and $B_1 = E - M$. If $\beta E + L \varepsilon \mathcal{A}$, where $L \varepsilon \mathfrak{L}_1(G)$, we have $0 = B_1 \star (\beta E + L) = (E - M) \star (\beta E + L) = \beta E + L - \beta M - M \star L$, from which it follows that $\beta = 0$ and $L = M \star L$. Therefore we have $\mathcal{A} \subset \mathfrak{L}_1(G)$ and $M \star L = L$ for all $L \varepsilon \mathcal{A}$. If $L \varepsilon \mathcal{A}$, we also have $0 = L \star (E - M) = L - L \star M$, so we see that M is a unit of $\mathcal{A}$. If $N \varepsilon \mathfrak{L}_1(G)$, then $N \varepsilon \mathfrak{L}_1(G_e)$ also, and we have $N = A_2 + B_2$ for some $A_2 \varepsilon \mathcal{A}$ and $B_2 \varepsilon \mathcal{B}$. Since $B_2 = N - A_2 \varepsilon \mathfrak{L}_1(G)$, we have $\mathfrak{L}_1(G) = \mathcal{A}$ if $\mathcal{B} \cap \mathfrak{L}_1(G) = 0$ and $\mathfrak{L}_1(G) \cong \mathcal{A} \oplus (\mathcal{B} \cap \mathfrak{L}_1(G))$ otherwise. This completes the present proof.

4.20. Theorem. If $\mathfrak{L}_1(G) \cong \mathcal{A} \oplus \mathcal{B}$, where $\mathcal{A} \subset \mathfrak{L}_1(G)$ and $\mathcal{B} \subset \mathfrak{L}_1(G)$, and if $\mathcal{A}$ has a unit, then $\mathfrak{L}_1(G_e) \cong \mathcal{A} \oplus \mathcal{C}$, where $\mathcal{C}$ is an algebra described below.

Proof. If U is the unit of $\mathcal{A}$ and E is the unit adjoined to $\mathfrak{L}_1(G)$ to obtain $\mathfrak{L}_1(G_e)$, then an arbitrary element of $\mathfrak{L}_1(G_e)$ can be written as $L + \alpha E = A_1 + B_1 + \alpha E = (A_1 + \alpha U) + (B_1 + \alpha(E - U)) = A_2 + (B_1 + \alpha(E - U))$, where $L \varepsilon \mathfrak{L}_1(G)$, $\alpha \varepsilon K$, $A_1 \varepsilon \mathcal{A}$, $B_1 \varepsilon \mathcal{B}$, and $A_2 \varepsilon \mathcal{A}$. If $\mathcal{C}$ consists of all elements of the form $B_2 + \beta(E - U)$ with $B_2 \varepsilon \mathcal{B}$ and $\beta \varepsilon K$, then it is easy to verify that $\mathcal{C}$ is an algebra and that $\mathfrak{L}_1(G_e) \cong \mathcal{A} \oplus \mathcal{C}$.

4.21. If G_1 and G_2 are finite semigroups and $G \cong G_1 \times G_2$, we can obtain a matrix representation of G from matrix representations of G_1 and G_2. Let $x_i \to \tau_1(x_i)$

be a matrix representation of G_1. We write $\tau_1(x_i) = (\sigma_{rs}(x_i))$, where $\sigma_{rs}(x_i)$ is the element of $\tau_1(x_i)$ in the r-th row and s-th column. If $y_j \to \tau_2(y_j)$ is a matrix representation of G_2, we consider the matrix $\tau(x_i, y_j) = (\sigma_{rs}(x_i)\,\tau_2(y_j))$, meaning by this the matrix whose order is the product of the orders of $\tau_1(x_i)$ and $\tau_2(y_j)$, formed by replacing each element $\sigma_{rs}(x_i)$ of $\tau_1(x_i)$ by the block of elements given by $\sigma_{rs}(x_i)\,\tau_2(y_j)$. For all $x_g,\ x_i \,\varepsilon\, G_1$ and $y_h,\ y_j \,\varepsilon\, G_2$, we have

$$\tau(x_g, y_h)\,\tau(x_i, y_j) = (\sigma_{rs}(x_g)\,\tau_2(y_h))\,(\sigma_{rs}(x_i)\,\tau_2(y_j)) = (\sum_k \sigma_{rk}(x_g)\,\tau_2(y_h)\,\sigma_{ks}(x_i)\,\tau_2(y_j)) =$$

$$= (\sum_k \sigma_{rk}(x_g)\,\sigma_{ks}(x_i)\,\tau_2(y_h)\,\tau_2(y_j)) = (\sigma_{rs}(x_g\, x_i)\,\tau_2(y_h\, y_j)) = \tau(x_g\, x_i,\ y_h\, y_j).$$

The elements of G can be taken to be (x_i, y_j) with $(x_g, y_h)\,(x_i, y_j) = (x_g\, x_i,\ y_h\, y_j)$, so we see from the above computation that the mapping $(x_i, y_j) \to \tau(x_i, y_j)$ is a matrix representation of G.

If the matrices $\tau(x_i, y_j)$ are linearly independent, then they generate in the usual fashion a faithful representation of $\mathfrak{L}_1(G)$. Since we can find representations of both G_1 and G_2 by linearly independent matrices (Theorem 4.1), we want to show that the matrices $\tau(x_i, y_j)$ are linearly independent if the matrices $\tau_1(x_i)$ are linearly independent and the matrices $\tau_2(y_j)$ are linearly independent. We establish this fact as follows. If $\sum_{i,j} \alpha_{ij}\,\tau(x_i, y_j) = 0$, then we have $\sum_{i,j} \alpha_{ij}\,\sigma_{rs}(x_i)\,\tau_2(y_j) = 0$ for all r, s. If the matrices $\tau_2(y_j)$ are linearly independent, the last equality implies that $\sum_i \alpha_{ij}\,\sigma_{rs}(x_i) = 0$ for all j, r, s. However, this equality is equivalent to the equality $\sum_i \alpha_{ij}\,\tau_1(x_i) = 0$ for all j. If the matrices $\tau_1(x_i)$ are linearly independent, the last equality implies that $\alpha_{ij} = 0$ for all i, j.

The algebra $\mathfrak{L}_1(G)$ is the Kronecker product $\mathfrak{L}_1(G_1) \wedge \mathfrak{L}_1(G_2)$ of $\mathfrak{L}_1(G_1)$ and $\mathfrak{L}_1(G_2)$.

4.22. If G is a finite semigroup with the left cancellation law, then, as shown in Theorem 2.12, we have $G \cong G_1 \times G_2$, where $x_g\, x_i = x_i$ for all $x_g,\ x_i \,\varepsilon\, G_1$ and where G_2 is a group. From the proof of Theorem 4.9, we see that we can take

$$\tau_1(x_1) = e_{11},\ \ \tau_1(x_i) = e_{11} + e_{1i}, \quad \text{for } i \geq 2.$$

Since G_2 is a finite group, it admits a matrix representation of the form

$$y_j \to \tau_2(y_j) = \begin{pmatrix} A_1(y_j) & 0 & 0 & \dots 0 \\ 0 & A_2(y_j) & 0 & \dots 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots A_m(y_j) \end{pmatrix},$$

where the $A_k(y_j)$ represent square blocks of elements and the 0's represent rectangular blocks of zeros. Furthermore, if $B_1, B_2, \ldots, B_m$ is an arbitrary set of matrices such that the order of B_k is equal to the order of $A_k(y_j)$ for $1 \leq k \leq m$, then there exist complex numbers α_j such that

$$\sum_j \alpha_j \tau_2(y_j) = \begin{pmatrix} B_1 & 0 & 0 \ldots 0 \\ 0 & B_2 & 0 \ldots 0 \\ \multicolumn{3}{c}{\dotfill} \\ 0 & 0 & 0 \ldots B_m \end{pmatrix}.$$

(See for example v. d. Waerden [46], p. 182.)

If we let $\{B_k^h, 1 \leq h \leq p = o(G_1), 1 \leq k \leq m\}$ be an arbitrary set of matrices such that the order of B_k^h is equal to the order of $A_k(y_j)$, then it is not hard to see that every matrix $\tau(x_i, y_j)$ is of the form

4.22.1 $$\begin{pmatrix} \begin{matrix} B_1^1 & 0 & 0 \ldots 0 \\ 0 & B_2^1 & 0 \ldots 0 \\ \ldots & \ldots & \ldots \\ 0 & 0 & 0 \cdots B_m^1 \end{matrix} & \begin{matrix} B_1^2 & 0 & 0 \ldots 0 \\ 0 & B_2^2 & 0 \ldots 0 \\ \ldots & \ldots & \ldots \\ 0 & 0 & 0 \ldots B_m^2 \end{matrix} & \cdots & \begin{matrix} B_1^p & 0 & 0 \ldots 0 \\ 0 & B_2^p & 0 \ldots 0 \\ \ldots & \ldots & \ldots \\ 0 & 0 & 0 \cdots B_m^p \end{matrix} \\ 0 & 0 & \cdots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

It is also not hard to see that every matrix of the form 4.22.1 can be obtained as a linear combination of the matrices $\tau(x_i, y_j)$. Therefore, if G is a finite semigroup with the left cancellation law, then, for some m and p, $\mathfrak{L}_1(G)$ is isomorphic to the algebra of all matrices of the form 4.22.1.

We now take up uniqueness theorems for $\mathfrak{L}_1(G)$: under what conditions does $\mathfrak{L}_1(G)$ determine G? That $\mathfrak{L}_1(G)$ as an algebra does not determine G, in general, is proved by the fact that for all Abelian groups G of order n, $\mathfrak{L}_1(G)$ is isomorphic to K_n (see also Theorems 4.12 and 5.21.)[18] Nevertheless, an analogue of Kawada's theorem [23] can be proved, as follows.

4.23. Definition. A functional $L \varepsilon \mathfrak{L}_1(G)$, where $G = \{x_1, x_2, \ldots, x_n\}$ is a finite semigroup, is said to be non-negative if $L(\varphi_j) \geq 0$ for $j = 1, 2, \ldots, n$. If some $L(\varphi_j)$ is positive and L is non-negative, then L is said to be a positive functional.

4.24. Theorem. Let G and G^* be finite semigroups, $G = \{x_1, x_2, \ldots, x_n\}$ and $G^* = \{y_1, y_2, \ldots, y_m\}$, and suppose that there exists an isomorphism Δ of $\mathfrak{L}_1(G)$ onto

[18] For discussions of uniqueness in rather different contexts, see BERMAN [3] and PERLIS and WALKER [29].

$\mathfrak{L}_1(G^*)$ such that ΔL is positive if and only if L is positive. Then G and G^* are isomorphic semigroups.

Proof. Consider the set $\mathfrak{P}(G)$ consisting of all non-negative $L \varepsilon \mathfrak{L}_1(G)$. Let $\mathcal{E}(G)$ be the set of elements $M \varepsilon \mathfrak{P}(G)$ which are extreme in $\mathfrak{P}(G)$ in the sense that if $L_1, L_2 \varepsilon \mathfrak{P}(G)$ and $M = \alpha L_1 + (1-\alpha) L_2$ $(0 \leq \alpha \leq 1)$, then L_1 and L_2 are both non-negative real multiples of M.[19] It is simple to show that a functional M is in $\mathcal{E}(G)$ if and only if it has the form $\alpha \lambda_i$ for some $\alpha \geq 0$ and $x_i \varepsilon G$. We leave the details of this argument to the reader. The analogous set $\mathcal{E}(G^*) \subset \mathfrak{P}(G^*)$ is characterized in just the same way. The isomorphism Δ being a linear space isomorphism carrying $\mathfrak{P}(G)$ onto $\mathfrak{P}(G^*)$, it follows that a set of $o(G)$ linearly independent elements of $\mathcal{E}(G)$ must map onto a set of $o(G)$ linearly independent elements of $\mathcal{E}(G^*)$ under the isomorphism Δ. Using the argument set forth above, we see that any set of n linearly independent elements of $\mathcal{E}(G)$ must have the form $\{\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n\}$, where all α_j are positive. (By a slight abuse of notation, we write $\alpha_j \lambda_j$ as $\alpha_j x_j$; similarly with elements of $\mathfrak{L}_1(G^*)$.) It is clear that $m = n$ and that $\Delta(\alpha_j x_j) = \beta_r y_r$ for $1 \leq j \leq n$, some r such that $1 \leq r \leq n$, and $\beta_r > 0$. Thus $\Delta(x_j) = \gamma_r y_r$ $(\gamma_r > 0)$. Since Δ is an algebra isomorphism, we have $\Delta(x_j^s) = \gamma_r^s y_r^s$ for all positive integers s. If $x_j^k = x_j^l$, then we have $\gamma_r^l y_r^l = \gamma_r^k y_r^k$, and consequently $y_r^l = y_r^k$, and $\gamma_r = 1$. It follows that $\Delta(x_j) = y_r$. The mapping Δ of G onto G^* is obviously an isomorphism.

For a few finite semigroups G, the algebras $\mathfrak{L}_1(G)$ determine G completely, as we now show.

4.25. Theorem. Let G be as in Theorem 4.9. Suppose that $\mathfrak{L}_1(G)$ is isomorphic to $\mathfrak{L}_1(H)$ for a semigroup H. Then G is isomorphic to H.

Proof. If $\mathfrak{L}_1(G)$ is isomorphic to $\mathfrak{L}_1(H)$, then the algebra of all $n \times n$ matrices $(a_{ij})_{i,j=1}^n$ with $a_{ij} = 0$ for $i > 1$ admits a basis $A, B, C, \ldots$ forming a semigroup isomorphic with H under matrix multiplication. Using an obvious abbreviation, we write $A = \{a_1, a_2, \ldots, a_n\}$. Then $A^k = \{a_1^k, a_1^{k-1} a_2, \ldots, a_1^{k-1} a_n\}$ $(k = 1, 2, \ldots)$. The elements A, $A^2, A^3, \ldots$ are only finite in number and none of them is 0. Hence a_1 is a root of unity, say $a_1^p = 1$ and $a_1^j \neq 1$ for $0 < j < p$. Then the elements $A, A^2, \ldots, A^p$ are all distinct and hence must be linearly independent. However, we have

$$A + A^2 + A^3 + \cdots + A^p = \{a_1 + a_1^2 + \cdots + a_1^p,\ a_2(1 + a_1 + \cdots + a_1^{p-1}), \ldots, a_n(1 + a_1 + \cdots + a_1^{p-1})\} = 0$$

if $p > 1$, since in this case we have $1 + a_1 + a_1^2 + \cdots + a_1^{p-1} = 0$. Hence $a_1 = 1$. This being

[19] This notion was suggested to us by Dr. V. L. KLEE.

so, we have $AB = B$, and so on, and the basis $A, B, C, \ldots$ under multiplication is isomorphic to the semigroup of order n in which $xy = y$ for all x and y.

4.26. Theorem. Let G be the semigroup of Theorem 4.10. If H is a semigroup such that $\mathfrak{L}_1(G)$ is isomorphic to $\mathfrak{L}_1(H)$, then G is isomorphic to H.

Proof. By Theorem 4.10, $\mathfrak{L}_1(G)$ is representable as $K \oplus Z_{n-1}$. If $\mathfrak{L}_1(H)$ is isomorphic to $K \oplus Z_{n-1}$, then $K \oplus Z_{n-1}$ has a basis isomorphic to H under multiplication. Let (α, b) $(\alpha \varepsilon K,\ b \varepsilon Z_{n-1})$ be any element of this basis. Since $(\alpha, b)^2 = (\alpha^2, 0)$, it is clear that $\alpha \neq 0$. Since $(\alpha, b)^k = (\alpha^k, 0)$ $(k > 1)$, it is established, just as in the proof of Theorem 4.25, that $\alpha = 1$. Therefore $(1, 0)$ is an element of the basis, and the product of any two elements of the basis is $(1, 0)$.

4.27. Remark. Let G be a finite group such that $o(G) > 1$. There exists a non-group H such that $\mathfrak{L}_1(G) \cong \mathfrak{L}_1(H)$. We construct H by the device used in Theorem 4.2, in connection this time with the semisimple algebra $\mathfrak{L}_1(G)$, which of course has a 1-dimensional direct summand. The semigroup H is obviously a non-group.

§ 5. Ideals in $\mathfrak{L}_1$ algebras

Much of the preceding work culminates in the present section, where we identify certain classes of ideals in $\mathfrak{L}_1(G)$, characterize the radical of $\mathfrak{L}_1(G)$ (Theorem 5.20), and obtain a particularly simple criterion for semisimplicity of $\mathfrak{L}_1(G)$ for commutative G (Theorem 5.21). We begin with 3 relevant definitions.

5.1. Definition. For $\mathfrak{A} \subset \mathfrak{F}_1(G)$, let $\mathfrak{N}(\mathfrak{A})$ be the set of all $L \varepsilon \mathfrak{L}_1(G)$ such that $L(f) = 0$ for all $f \varepsilon \mathfrak{A}$.

5.2. Definition. For $\mathfrak{B} \subset \mathfrak{L}_1(G)$, let $\mathfrak{N}(\mathfrak{B})$ be the set of all $f \varepsilon \mathfrak{F}_1(G)$ such that $L(f) = 0$ for all $L \varepsilon \mathfrak{B}$.

5.3. Definition. For $f \varepsilon \mathfrak{F}_1(G)$ and $x \varepsilon G$, let f_x and ${}_xf$ be as in 1.3.1. A linear subspace $\mathfrak{A}$ of $\mathfrak{F}_1(G)$ such that $f \varepsilon \mathfrak{A}$ implies $({}_xf \varepsilon \mathfrak{A};\ f_x \varepsilon \mathfrak{A};\ {}_xf, f_x \varepsilon \mathfrak{A})$ is said to be a (left, right, 2-sided) invariant subspace of $\mathfrak{F}_1(G)$.

5.4. Theorem. Let $\mathfrak{J}$ be a (left, right, 2-sided) ideal in $\mathfrak{L}_1(G)$, where $G = \{x_1, x_2, \ldots, x_n\}$. Then $\mathfrak{N}(\mathfrak{J})$ is a (left, right, 2-sided) invariant subspace of $\mathfrak{F}_1(G)$.

Proof. Changing slightly the notation introduced in 1.7, we write λ_u for the element of $\mathfrak{L}_1(G)$ such that $\lambda_u(f) = f(u)$ $(u \varepsilon G,\ f \varepsilon \mathfrak{F}_1(G))$. If $\mathfrak{J}$ is a right ideal in $\mathfrak{L}_1(G)$, $M \varepsilon \mathfrak{J}$, $u \varepsilon G$, and $f \varepsilon \mathfrak{N}(\mathfrak{J})$, then $M \star \lambda_u(f) = 0$. That is, we have $M_x(\lambda_{u,y}(f(xy))) = 0$ To compute $\lambda_{u,y}(f(xy))$ as a function of x, we have

$$\lambda_{u,y}(f(xy)) = \lambda_{u,y}(\sum_{i,j=1}^{n} f(x_i x_j)\varphi_i(x)\varphi_j(y)) = \sum_{i,j=1}^{n} f(x_i x_j)\varphi_i(x)\varphi_j(u)$$

$$= \sum_{i=1}^{n} f(x_i u)\varphi_i(x) = f(xu).$$

It follows that $f_u \varepsilon \mathfrak{N}(\mathfrak{J})$ if $f \varepsilon \mathfrak{N}(\mathfrak{J})$, and as $\mathfrak{N}(\mathfrak{J})$ is trivially a linear subspace of $\mathfrak{F}_1(G)$, the theorem follows in this case.

If $\mathfrak{J}$ is a left ideal in $\mathfrak{L}_1(G)$, then we have $\lambda_u \star M(f) = 0$; using Theorem 1.9, this equality gives us $0 = \lambda_{u,x}(M_y(f(xy))) = M_x(\lambda_{u,y}(f(xy)))$. Now $\lambda_{u,y}(f(yx)) = f(ux)$, as above, and it follows that $\mathfrak{N}(\mathfrak{J})$ is a left invariant subspace.

5.5. Theorem. Let $\mathfrak{S}$ be a (left, right, 2-sided) invariant subspace of $\mathfrak{F}_1(G)$. Then $\mathfrak{N}(\mathfrak{S})$ is a (left, right, 2-sided) ideal in $\mathfrak{L}_1(G)$.

Proof. If $\mathfrak{S}$ is left invariant, and if $M \varepsilon \mathfrak{N}(\mathfrak{S})$, then $M_y(f(xy)) = 0$ for all $f \varepsilon \mathfrak{S}$ and all $x \varepsilon G$. Hence, if $L \varepsilon \mathfrak{L}_1(G)$, we have $L \star M(f) = 0$ for all $f \varepsilon \mathfrak{S}$; as $\mathfrak{N}(\mathfrak{S})$ is trivially a linear subspace of $\mathfrak{L}_1(G)$, the theorem follows in this case. If $\mathfrak{S}$ is right invariant, $M \varepsilon \mathfrak{N}(\mathfrak{S})$, and $L \varepsilon \mathfrak{L}_1(G)$, then we have, using Theorem 1.9 again, that $M \star L(f) = L_x(M_y(f(yx))) = 0$. This completes the proof.

5.6. Theorem. Let $\mathcal{A}$ be a linear subspace of $\mathfrak{L}_1(G)$. Then $\mathcal{A} = \mathfrak{N}(\mathfrak{N}(\mathcal{A}))$.

Proof. It is clear that $\mathcal{A} \subset \mathfrak{N}(\mathfrak{N}(\mathcal{A}))$. Conversely, suppose that L non $\varepsilon\, \mathcal{A}$. Then there exists a linear functional ψ on $\mathfrak{L}_1(G)$ such that $\psi(N) = 0$ for all $N \varepsilon \mathcal{A}$ and $\psi(L) = 1$. Since $\mathfrak{F}_1(G)$ is finite dimensional, we have $\psi(M) = M(f)$ for some $f \varepsilon \mathfrak{F}_1(G)$ and all $M \varepsilon \mathfrak{L}_1(G)$. Hence $f \varepsilon \mathfrak{N}(\mathcal{A})$ and $L(f) \neq 0$: hence L non $\varepsilon\, \mathfrak{N}(\mathfrak{N}(\mathcal{A}))$.

Combining Theorems 5.4, 5.5, and 5.6, we have the following result.

5.7. Theorem. A subset $\mathcal{A}$ of $\mathfrak{L}_1(G)$ is a (left, right, 2-sided) ideal in $\mathfrak{L}_1(G)$ if and only if $\mathcal{A} = \mathfrak{N}(\mathfrak{S})$, where $\mathfrak{S}$ is a (left, right, 2-sided) invariant subspace of $\mathfrak{F}_1(G)$.

In the remainder of the present paper, the word "ideal" means 2-sided ideal, and the term "invariant subspace" means 2-sided invariant subspace.

5.8. Theorem. A subset $\mathfrak{J}$ of $\mathfrak{L}_1(G)$ is a maximal ideal if and only if $\mathfrak{J} = \mathfrak{N}(\mathfrak{S})$, where $\mathfrak{S}$ is a minimal invariant subspace of $\mathfrak{F}_1(G)$.

Proof. This assertion follows at once from Theorem 5.7 and the fact that the correspondence $\mathcal{A} \to \mathfrak{N}(\mathcal{A})$ for subspaces of $\mathfrak{L}_1(G)$ inverts inclusion: $\mathcal{A}_1 \subset \mathcal{A}_2$ if and only if $\mathfrak{N}(\mathcal{A}_1) \supset \mathfrak{N}(\mathcal{A}_2)$.

We now consider in more detail homomorphisms of $\mathfrak{L}_1(G)$ onto simple algebras.

5.9. Theorem. A linear functional ψ on $\mathfrak{L}_1(G)$ is a homomorphism of $\mathfrak{L}_1(G)$ onto K if and only if $\psi(M) = M(\chi)$ for a semicharacter χ of G and all $M \varepsilon \mathfrak{L}_1(G)$.

Proof. The sufficiency of the condition stated being obvious, we consider only its necessity. Any linear functional ψ has the form $\psi(M) = M(f)$ for some $f \varepsilon \mathfrak{F}_1(G)$ and all $M \varepsilon \mathfrak{L}_1(G)$. If ψ is a homomorphism onto K, then $M \star L(f) = M(f)\, L(f)$ for all $M, L \varepsilon \mathfrak{L}_1(G)$. This equality may be rewritten as $0 = M_x(L_y(f(xy))) - M(f)\, L(f) = M_x[L_y(f(xy)) - L_y(f(y))\, f(x)] = M_x[L_y\{f(xy) - f(x)\, f(y)\}]$. Since M is an arbitrary linear functional, it follows that $L_y\{f(xy) - f(x)\, f(y)\} = 0$ for all $x \varepsilon G$. Since L is an arbitrary linear functional, it follows that $f(xy) - f(x)\, f(y) = 0$ for all $x, y \varepsilon G$. Since $f \neq 0$, the theorem follows.

5.10. For a finite semigroup G, let $\mathfrak{T}(G)$ denote the subspace of $\mathfrak{F}_1(G)$ consisting of all f such that ${}_x f = 0$ for all $x \varepsilon G$. It is clear that $\mathfrak{T}(G) \neq 0$ if and only if there are elements $u \varepsilon G$ such that no product xy is equal to u. Then $\mathfrak{T}(G)$ consists of all f which vanish except at these elements u.

5.11. Theorem. A linear functional ψ on $\mathfrak{L}_1(G)$ is a homomorphism of $\mathfrak{L}_1(G)$ onto Z_1 if and only if $\psi(M) = M(\sigma)$ $(M \varepsilon \mathfrak{L}_1(G))$, where σ is a function in $\mathfrak{T}(G)$.

The proof is similar to that of Theorem 5.9 and is omitted.

5.12. Theorem. Let $\mathfrak{S}$ be a 1-dimensional invariant subspace of $\mathfrak{F}_1(G)$. Then $\mathfrak{S}$ is spanned by a semicharacter or by a function $\sigma \varepsilon \mathfrak{T}(G)$.

Proof. The maximal ideal $\mathfrak{N}(\mathfrak{S})$ being $(n-1)$-dimensional $(n = o(G))$, the difference algebra $\mathfrak{L}_1(G) - \mathfrak{N}(\mathfrak{S})$ is 1-dimensional and is isomorphic to K or to Z_1. The mapping of $\mathfrak{L}_1(G)$ onto this difference algebra is in any case a linear functional $L \to L(f)$, for some $f \varepsilon \mathfrak{F}_1(G)$. The kernel $\mathfrak{N}(\mathfrak{S})$ consists of all $L \varepsilon \mathfrak{L}_1(G)$ for which $L(f) = 0$; and it remains only to apply Theorems 5.9 and 5.11.

5.13. Theorem. $\mathfrak{L}_1(G)$ admits a homomorphism onto K_m if and only if there are m distinct semicharacters of G.

Proof. If $\chi_1, \chi_2, \ldots, \chi_m$ are distinct semicharacters of G, then, in view of Theorem 5.9 and Corollary 3.3.1, the mapping $L \to \{L(\chi_1), L(\chi_2), \ldots, L(\chi_m)\}$ is a homomorphism of $\mathfrak{L}_1(G)$ onto K_m. Conversely, given a homomorphism $L \to \{a_1(L), a_2(L), \ldots, a_m(L)\}$ of $\mathfrak{L}_1(G)$ onto K_m, the mappings $L \to a_j(L)$ $(j = 1, 2, \ldots, m)$ are homomorphisms onto K and are generated by (necessarily distinct) semicharacters, in accordance with Theorem 5.9.

5.14. Theorem. $\mathfrak{L}_1(G)$ admits a homomorphism onto the algebra Z_m if and only if $\mathfrak{T}(G)$ has dimensionality $\geq m$.

Proof. If $\mathfrak{A}$ is any m-dimensional subspace of $\mathfrak{T}(G)$, choose an arbitrary basis $\{f_1, f_2, \ldots, f_m\}$ in $\mathfrak{A}$. The mapping $L \to \{L(f_1), L(f_2), \ldots, L(f_m)\}$ is a homomorphism of $\mathfrak{L}_1(G)$ onto Z_m, if we define $\{\alpha_1, \alpha_2, \ldots, \alpha_m\} \cdot \{\beta_1, \beta_2, \ldots, \beta_m\}$ as $\{0, 0, \ldots, 0\}$ $(\alpha_j, \beta_j \, \varepsilon \, K)$: since $M \star L(f_j) = M_x(L_y(f_j(xy))) = M_x(L_y(0)) = 0$ for all $M, L \, \varepsilon \, \mathfrak{L}_1(G)$, and the functions $f_1, f_2, \ldots, f_m$ are linearly independent.

Conversely, given a homomorphism $L \to \Delta(L)$ of $\mathfrak{L}_1(G)$ onto Z_m, let $u_1, \ldots, u_m$ be a basis in Z_m. Then we have $L \to \Delta(L) = \sum_{j=1}^{m} \alpha_j(L)\, u_j$, and the mapping $L \to \alpha_j(L)$ is a linear functional on $\mathfrak{L}_1(G)$. We can therefore write $\alpha_j(L) = L(f_j)$ for some $f_j \, \varepsilon \, \mathfrak{F}_1(G)$. Since the numbers $\alpha_j(L)$ are capable of assuming arbitrary values, the functions f_j are linearly independent $(j = 1, 2, \ldots, m)$. Since $\alpha_j(L \star M) = 0$ for all $L, M \, \varepsilon \, \mathfrak{L}_1(G)$, and $1 \leq j \leq m$, it follows from Theorem 5.11 that $f_j \, \varepsilon \, \mathfrak{T}(G)$.

5.15. Theorem. Let $\mathfrak{S}$ be a minimal invariant subspace of $\mathfrak{F}_1(G)$ with the property that $\mathfrak{S} \cap \mathfrak{T}(G) = 0$. Then $\mathfrak{S}$ admits a basis $\{\beta_{ij}\}_{i,j=1}^{s}$ such that $\beta_{ij}(xy) = \sum_{k=1}^{s} \beta_{ik}(x)\, \beta_{kj}(y)$ $(x, y \, \varepsilon \, G)$ and the matrices $B(x) = (\beta_{ij}(x))_{i,j=1}^{s}$ form an irreducible set in $\mathfrak{M}_s$ (all $x \, \varepsilon \, G$).

Proof. By Theorem 5.8, $\mathfrak{N}(\mathfrak{S})$ is a maximal ideal in $\mathfrak{L}_1(G)$. The difference algebra $\mathfrak{D} = \mathfrak{L}_1(G) - \mathfrak{N}(\mathfrak{S})$ must accordingly be isomorphic to Z_1, K, or $\mathfrak{M}_s$ $(s = 2, 3, \ldots)$. The case $\mathfrak{D} \cong Z_1$ is ruled out by the construction given in the proof of Theorem 5.14 and the hypothesis $\mathfrak{S} \cap \mathfrak{T}(G) = 0$. If $\mathfrak{D} \cong K$, we may appeal to the construction used to prove Theorem 5.13. It remains to consider the case $\mathfrak{D} \cong \mathfrak{M}_s$ $(s = 2, 3, \ldots)$. The natural homomorphism of $\mathfrak{L}_1(G)$ onto $\mathfrak{M}_s$ with kernel $\mathfrak{N}(\mathfrak{S})$ may be denoted by Δ. For all $L \, \varepsilon \, \mathfrak{L}_1(G)$, we have $\Delta(L) = (a_{ij}(L))_{i,j=1}^{s} \, \varepsilon \, \mathfrak{M}_s$; and the mapping $L \to a_{ij}(L)$ is clearly a linear functional on $\mathfrak{L}_1(G)$ $(i, j = 1, 2, \ldots, s)$. Hence we have $a_{ij}(L) = L(\beta_{ij})$ for some $\beta_{ij} \, \varepsilon \, \mathfrak{F}_1(G)$, and we have, for all $x \, \varepsilon \, G$, $\lambda_x \to (\beta_{ij}(x))_{i,j=1}^{s}$. Since $\lambda_x \star \lambda_y = \lambda_{xy}$ and since Δ is a homomorphism, it follows that $\beta_{ij}(xy) = \sum_{k=1}^{s} \beta_{ik}(x)\, \beta_{kj}(y)$. Since Δ is a mapping onto $\mathfrak{M}_s$, the functions β_{ij} are linearly independent. Since every matrix in $\Delta(\mathfrak{L}_1(G))$ is a linear combination $\sum_{x \varepsilon G} \alpha_x B(x)$, it follows that the set $\{B(x)\}_{x \varepsilon G}$ is irreducible. Since $L \, \varepsilon \, \mathfrak{N}(\mathfrak{S})$ if and only if $L(\beta_{ij}) = 0$ for $i, j = 1, 2, \ldots, s$, it follows that the functions β_{ij} form a basis for $\mathfrak{S}$.

5.16. Theorem. Let $(\beta_{ij})_{i,j=1}^{s}$ be a matrix of functions in $\mathfrak{F}_1(G)$ such that the mapping $x \to B(x) = (\beta_{ij}(x))_{i,j=1}^{s}$ is an irreducible representation of G in the multiplicative semigroup $\mathfrak{M}_s$. Then the mapping $L \to \sum_{x \varepsilon G} L(\varphi_x)\, B(x) = \Delta(L)$ is a homomorphism of $\mathfrak{L}_1(G)$ onto $\mathfrak{M}_s$, and the functions β_{ij} are a basis for a minimal invariant subspace of $\mathfrak{F}_1(G)$ which intersects $\mathfrak{T}(G)$ in 0 alone.[20]

Proof. It follows from Theorem 3.22 that every matrix in $\mathfrak{M}_s$ is a linear combination of the matrices $B(x)$ $(x \varepsilon G)$, and the fact that the mapping Δ is a homomorphism is obvious. The last assertion follows readily from Theorem 5.8.

5.17. Theorem. A subset $\mathfrak{J}$ of $\mathfrak{L}_1(G)$ is a regular maximal ideal if and only if $\mathfrak{J} = \mathfrak{N}(\mathfrak{S})$, where $\mathfrak{S}$ is a subspace of $\mathfrak{F}_1(G)$ spanned by functions β_{ij} $(i, j = 1, 2, \ldots, s)$, with the properties set forth in Theorem 5.16.

This assertion is obvious from the foregoing discussion.

If $\mathfrak{T}(G)$ has dimension >1, then $\mathfrak{F}_1(G)$ admits a continuum of 1-dimensional and hence minimal invariant subspaces; these, however, produce only homomorphisms of $\mathfrak{L}_1(G)$ onto Z_1 (see Theorem 5.14) and are of negligible interest. Minimal invariant subspaces intersecting $\mathfrak{T}(G)$ in 0 alone, on the other hand, exist only in finite numbers, and have other special properties as set forth in the following theorem.

5.18. Theorem. There are only a finite number of minimal invariant subspaces $\mathfrak{S}$ of $\mathfrak{F}_1(G)$ such that $\mathfrak{S} \cap \mathfrak{T}(G) = 0$: we call them $\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_l$. These subspaces are all linearly independent in the sense that $\mathfrak{S}_i \cap (\mathfrak{S}_{j_1} + \mathfrak{S}_{j_2} + \cdots + \mathfrak{S}_{j_k}) = 0$ for $i \neq j_1, \cdots, j_k$. Furthermore, $(\mathfrak{S}_1 \dotplus \mathfrak{S}_2 \dotplus \cdots \dotplus \mathfrak{S}_l) \cap \mathfrak{T}(G) = 0$.

Proof. Consider a maximal set of linearly independent minimal invariant subspaces $\mathfrak{S}$ for which $\mathfrak{S} \cap \mathfrak{T}(G) = 0$. Plainly there are only a finite number of subspaces in such a maximal set: say, $\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_l$. Let $\mathfrak{U}$ be any minimal invariant subspace such that $\mathfrak{U} \cap \mathfrak{T}(G) = 0$. If $\mathfrak{U} \cap (\mathfrak{S}_1 \dotplus \mathfrak{S}_2 \dotplus \cdots \dotplus \mathfrak{S}_l) = 0$, then the family $\{\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_l\}$ is not maximal. If $\mathfrak{U} \cap (\mathfrak{S}_1 \dotplus \mathfrak{S}_2 \dotplus \cdots \dotplus \mathfrak{S}_l) \neq 0$, then $\mathfrak{U} \subset \mathfrak{S}_1 \dotplus \mathfrak{S}_2 \dotplus \dotplus \cdots \dotplus \mathfrak{S}_l$, since $\mathfrak{U}$ is a minimal invariant subspace. This implies that $\mathfrak{N}(\mathfrak{U}) \supset \bigcap_{i=1}^{l} \mathfrak{N}(\mathfrak{S}_i)$. Let $\mathfrak{Z}$ denote the radical of $\mathfrak{L}_1(G)$; as is well known, this is the intersection of all regular maximal (2-sided) ideals. It is clear that $\mathfrak{N}(\mathfrak{U}) - \mathfrak{Z} \supset \bigcap_{i=1}^{l} [\mathfrak{N}(\mathfrak{S}_i) - \mathfrak{Z}]$. In the semisimple algebra $\mathfrak{L}_1(G) - \mathfrak{Z}$, $\mathfrak{N}(\mathfrak{U}) - \mathfrak{Z}$ and $\mathfrak{N}(\mathfrak{S}_i) - \mathfrak{Z}$ are again regular maximal ideals, and the last inclusion written down implies that $\mathfrak{N}(\mathfrak{U}) - \mathfrak{Z}$ is actually one of the ideals $\mathfrak{N}(\mathfrak{S}_{i_0}) - \mathfrak{Z}$. To see this, compute the maximal ideals in a semisimple algebra (over K) and their intersections: it is easy to see that the intersection of a

[20] Here φ_x is the function in $\mathfrak{F}_1(G)$ such that $\varphi_x(y) = \delta_{xy}$ for all $y \varepsilon G$; this notation violates for typographical reasons the convention of footnote 4.

family of maximal ideals completely determines the maximal ideals used in forming it. Since $\mathfrak{J}$ is a subset of $\mathfrak{N}(\mathfrak{U}) \cap (\bigcap_{i=1}^{l} \mathfrak{N}(\mathfrak{S}_i))$, it follows that $\mathfrak{N}(\mathfrak{U}) = \mathfrak{N}(\mathfrak{S}_{i_0})$. Thus the first 2 assertions of the present theorem are established. To prove the last one, suppose that $f \,\varepsilon\, (\mathfrak{S}_1 \dotplus \mathfrak{S}_2 \dotplus \cdots \dotplus \mathfrak{S}_l) \cap \mathfrak{T}(G)$. Let $\{\beta_{ij}^{(k)}\}_{i,j=1}^{s_k}$ $(k = 1, 2, \ldots, l)$ be a basis in $\mathfrak{S}_k$ of the kind described in Theorem 5.15, and let $f = \sum_{i,j,k} a_{ijk} \beta_{ij}^{(k)}$. Then

$$0 = f(xy) = \sum_{i,j,k} a_{ijk} \beta_{ij}^{(k)}(xy) = \sum_{i,j,k,m} a_{ijk} \beta_{im}^{(k)}(x) \beta_{mj}^{(k)}(y).$$

Since the functions $\beta_{mj}^{(k)}$ are linearly independent, we infer that the coefficient of each $\beta_{mj}^{(k)}(y)$ is 0. As these are linear combinations of functions $\beta_{im}^{(k)}(x)$, it follows that all a_{ijk} are 0. This completes the present proof.

5.19. Theorem. The set of all functions on G which are coefficients of irreducible matrix representations of G (only one representation being admitted from each equivalence class) are linearly independent elements of $\mathfrak{F}_1(G)$.

This generalization of Corollary 3.3.1 follows at once from Theorems 5.16 and 5.18.

5.20. Theorem. Let $\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_l$ be as in Theorem 5.18. The radical of $\mathfrak{L}_1(G)$ consists of all L such that $L(f) = 0$ for all $f \,\varepsilon\, \mathfrak{S}_1 \dotplus \mathfrak{S}_2 \dotplus \cdots \dotplus \mathfrak{S}_l$.

Proof. This result follows readily from Theorems 5.15, 5.17, and 5.18, and the fact that the radical of $\mathfrak{L}_1(G)$ is the intersection of all regular maximal ideals of $\mathfrak{L}_1(G)$.

5.21. Theorem. Let G be a commutative semigroup of order n. Then $\mathfrak{L}_1(G)$ is semisimple (and hence isomorphic to K_n) if and only if $l_x = 1$ for all $x \,\varepsilon\, G$ (see 2.6.1). This condition is equivalent to the equality $G^\circ = G$.

Proof. It follows from Theorem 5.20 that $\mathfrak{L}_1(G)$ is semisimple if and only if $\mathfrak{S}_1 \dotplus \mathfrak{S}_2 \dotplus \cdots \dotplus \mathfrak{S}_l = \mathfrak{F}_1(G)$. It is easy to see that every $\mathfrak{S}_i$ must be 1-dimensional if G is commutative, and in this case $\mathfrak{S}_i$ is generated by a semicharacter, χ_i. Hence there are n distinct semicharacters of G if and only if $\mathfrak{L}_1(G)$ is semisimple, and $l_x = 1$ for all $x \,\varepsilon\, G$ is a necessary and sufficient condition for this to occur, as 3.6.2 states.

The radical of $\mathfrak{L}_1(G)$ can be very simply characterized if G is commutative, as follows.

5.22. Theorem. Let G be a finite commutative semigroup such that $\mathfrak{L}_1(G)$ has a non-zero radical. For each $x_i \,\varepsilon\, G$ such that x_i non $\varepsilon\, G^\circ$, let m be a positive integer such that $(x_i^m)^2 = x_i^m$, and let $x_i^{m+1} = x_{j(i)}$. Then the radical of $\mathfrak{L}_1(G)$ consists of all functionals $L = \Sigma\, \alpha_i\, (\lambda_i - \lambda_{j(i)})$, where the sum is taken over all i such that x_i non $\varepsilon\, G^\circ$.

Proof. For simplicity, we can suppose that m has been determined so that $(x^m)^2 = x^m$ for all $x \,\varepsilon\, G$. For x_i non $\varepsilon\, G^\circ$ and $j = j(i)$, we have $x_j^m = x_i^{m^2+m} = x_i^m$, $x_j^{m+1} = x_i^{m^2+2m+1} = x_i^{m+1} = x_j$ and $x_i\, x_j^m = x_i^{1+m} = x_j$. Now

$$(\lambda_i - \lambda_j)^{m+1} = \lambda_i^{m+1} + \sum_{h=1}^{m} \binom{m+1}{h} (-1)^{m+1-h}\, \lambda_i^h \star \lambda_j^{m+1-h} + (-1)^{m+1}\, \lambda_j^{m+1},$$

but

$$x_i^{m+1} = x_j \text{ and } x_i^h\, x_j^{m+1-h} = x_i^h\, x_j^{hm+m+1-h} = (x_i\, x_j^m)^h\, x_j^{m+1-h} =$$
$$= x_j^h\, x_j^{m+1-h} = x_j^{m+1} = x_j \text{ for } 1 \leq h \leq m,$$

so we have

$$(\lambda_i - \lambda_j)^{m+1} = \lambda_j + \sum_{h=1}^{m} \binom{m+1}{h} (-1)^{m+1-h}\, \lambda_j + (-1)^{m+1}\, \lambda_j = (1-1)^{m+1}\, \lambda_j = 0.$$

Therefore $\lambda_i - \lambda_j$ is a nilpotent element of $\mathfrak{L}_1(G)$.

Since $x_j^{m+1} = x_j$, we see that every x_j is an element of G° while no x_i is. Therefore the functionals $\lambda_i - \lambda_j$ are linearly independent and span a space of dimension $o(G) - o(G^\circ)$. However, this is just the dimension of the radical of $\mathfrak{L}_1(G)$. This follows from Theorem 5.20 and the fact that there are exactly $o(G^\circ)$ distinct semicharacters on G. Therefore this space is the radical of $\mathfrak{L}_1(G)$.

5.22.1. It may be of interest to note that Theorem 5.21 can be proved without the use of the apparatus introduced in § 5. As in the proof of Theorem 5.22, we see that $\mathfrak{L}_1(G)$ contains a nilpotent element, and hence is not semisimple, if $l_x > 1$ for some $x \,\varepsilon\, G$. If $l_x = 1$ for all $x \,\varepsilon\, G$, then $o(G^\circ) = o(G)$. Writing the semicharacters of G as $\chi_1, \chi_2, \ldots, \chi_n$, we see that the mapping $L \to \{L(\chi_1), L(\chi_2), \ldots, L(\chi_n)\}$ is an isomorphism of $L_1(G)$ onto K_n.

5.23. Theorem. The algebra $\mathfrak{L}_1(G)$ is semisimple if and only if coefficients of irreducible matrix representations of G span the space $\mathfrak{F}_1(G)$.

This follows at once from Theorem 5.20.

5.24. Examples. We now give some examples to illustrate the preceding theorems.

5.24.1. Let Γ be an arbitrary field, and let G be the semigroup of all matrices $\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$ $(a, b \,\varepsilon\, \Gamma)$. The elements $\begin{pmatrix} 0 & 0 \\ b & 1 \end{pmatrix}$ form a 2-sided ideal H in G, and in fact $\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix}$. Let μ be an irreducible representation of G in $\mathfrak{M}_s$. Suppose that $\mu(h) \neq 0$ for some $h \varepsilon H$. Then, for any s-dimensional vector ξ such that $\mu(h)\xi \neq 0$ and any $x \varepsilon G$, we have $\mu(x)\mu(h)\xi = \mu(h)\xi$, and we see that the vector $\mu(h)\xi$ spans a 1-dimensional invariant subspace of the s-dimensional representation space. Hence $s = 1$, and it is easy to show that $\mu(x) = 1$ for all $x \varepsilon G$. Thus an irreducible representation μ with $s > 1$ has the property that $\mu(h) = 0$ for all $h \varepsilon H$; on the set $G \cap H'$, which is a group, μ can be an arbitrary representation. For Γ finite, we have of course $o(\Gamma) = p^m$ for a prime p and a positive integer m; and the subspace of $\mathfrak{F}_1(G)$ spanned by coefficients of irreducible representations is easily seen to be $p^{2m} - p^m + 1$. Hence the radical of $\mathfrak{L}_1(G)$ has dimension $p^m - 1$.

5.24.2. In dealing with irreducible measurable representations of compact groups, it suffices to know that these representations are capable of distinguishing between arbitrary pairs of points in the group, in order to show that coefficients of these representations span the space of all continuous functions under the uniform topology. This is, basically, because all such representations are unitary. (See Stone [40] for a complete discussion.) In finite semigroups G, on the other hand, it is quite possible for irreducible representations to separate points and yet for the dimensionality of the space spanned by coefficients of these representations to be less than $o(G)$; in this case, $\mathfrak{L}_1(G)$ fails to be semisimple, by Theorem 5.23. Consider as an example the matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix},$$

together with $A_{4+i} = -A_i$ $(i = 1, 2, 3, 4)$. These matrices form a semigroup G under multiplication, and the identity mapping is evidently an irreducible representation of G in $\mathfrak{M}_2$. However, this and the mapping $A_i \to 1$ $(i = 1, \ldots, 8)$ are the only irreducible representations of G. Accordingly, the radical of $\mathfrak{L}_1(G)$ is 3-dimensional. It is an elementary, if lengthy, exercise to show that the radical of $\mathfrak{L}_1(G)$ consists of all linear functionals of the form

$$\alpha(\lambda_1 + \lambda_5) + \beta(\lambda_2 + \lambda_6) + \gamma(\lambda_3 + \lambda_7) - (\alpha + \beta + \gamma)(\lambda_4 + \lambda_8)$$
$$(\alpha, \beta, \gamma \varepsilon K;\ \lambda_i = \lambda_{A_i}\ (i = 1, \ldots, 8)).$$

5.24.3. Let G be the semigroup described in 4.2, whose elements are $\{1, 0, 0, \ldots, 0\}$ and all $\{1, 0, \ldots, 0, e_{ij}^{(s_k)}, 0, \ldots, 0\}$. As noted in 4.2, $\mathfrak{L}_1(G)$ is semisimple, The irreducible representations of G are the mappings which carry these sequences into their r-th components $(r = 1, 2, \ldots, p+1)$. Plainly the coefficients of these representations span $\mathfrak{F}_1(G)$.

5.24.4. Appendices 3 and 4 list the semigroups of orders 5 and 6, respectively, having semisimple $\mathfrak{L}_1$ algebras. In each case, it is easy to see that the coefficients of irreducible representations span the space $\dot{\mathfrak{F}}_1$.

We close with 2 rather special theorems.

5.25. Theorem. If G and H are finite semigroups, then $\mathfrak{L}_1(G \times H)$ is semisimple if and only if $\mathfrak{L}_1(G)$ and $\mathfrak{L}_1(H)$ are semisimple.

Proof. Suppose that $\mathfrak{L}_1(G)$ and $\mathfrak{L}_1(H)$ are semisimple. We then have the following one-to-one correspondences:

$$x_i \leftrightarrow \{\tau_1(x_i),\ \tau_2(x_i),\ \ldots,\ \tau_l(x_i)\},$$

$$y_j \leftrightarrow \{\varrho_1(y_j),\ \varrho_2(y_j),\ \ldots,\ \varrho_m(y_j)\},$$

where $x_i \in G$, $y_j \in H$; the $\tau_g(x_i) = (\sigma^g_{rs}(x_i))$ are matrices of order, say, a_g; the $\varrho_h(y_j)$ are matrices of order b_h; the $\{\tau_1(x_i), \ldots, \tau_l(x_i)\}$ for $i = 1, 2, \ldots, o(G)$ are linearly independent; the $\{\varrho_1(y_j), \ldots, \varrho_m(y_j)\}$ for $j = 1, 2, \ldots, o(H)$ are linearly independent; and $\Sigma a_g^2 = o(G)$, $\Sigma b_h^2 = o(H)$. These facts follow from Wedderburn's theorems and the fact that the functionals λ form a basis in $\mathfrak{L}_1$. Now the mapping $x_i \to \tau_g(x_i)$ is a representation of G by matrices, and the mapping $y_j \to \varrho_h(y_j)$ is a representation of H by matrices. From 4.21, with a slight change of notation, we see that the mapping $(x_i, y_j) \to \tau_g(x_i) \wedge \varrho_h(y_j)$ is a representation of $G \times H$ by matrices. Here $\tau_g(x_i) \wedge \varrho_h(y_j)$ denotes the Kronecker product of the matrices $\tau_g(x_i)$ and $\varrho_h(y_j)$. Therefore

5.25.1 $$(x_i, y_j) \to \{\tau_1(x_i) \wedge \varrho_1(y_j),\ \tau_1(x_i) \wedge \varrho_2(y_j),\ \ldots,\ \tau_g(x_i) \wedge \varrho_h(y_j),\ \ldots,\ \tau_l(x_i) \wedge \varrho_m(y_j)\}$$

is a representation of $G \times H$. As in 4.21, we can show that the expressions on the right hand side of 5.25.1 are linearly independent for $1 \leq i \leq o(G)$, $1 \leq j \leq o(H)$. Therefore the representation of $\mathfrak{L}_1(G \times H)$ defined in 5.25.1 is faithful. Furthermore, the algebra $\mathfrak{L}_1(G \times H)$ has dimension $o(G) \cdot o(H) = (\Sigma a_g^2)(\Sigma b_h^2) = \Sigma\Sigma(a_g b_h)^2$, which is the same as the dimension of the space of all sequences $\{M_{11}, M_{12}, \ldots, M_{gh}, \ldots, M_{lm}\}$, where each M_{gh} is an arbitrary matrix of order $a_g b_h$ (note that $a_g b_h$ is just the order of the matrix $\tau_g(x_i) \wedge \varrho_h(y_j)$). Therefore the faithful representation of $\mathfrak{L}_1(G \times H)$ generated by 5.25.1 yields an isomorph of $\mathfrak{L}_1(G \times H)$ which is a direct sum of full matrix algebras; *i.e.*, $\mathfrak{L}_1(G \times H)$ is semisimple.

We now suppose that $\mathfrak{L}_1(H)$ is not semisimple; let $\mathfrak{R}$ be its radical. Let $L \to \tau(L)$ be a faithful representation of $\mathfrak{L}_1(G)$ by matrices, and let $M \to \varrho(M)$ be a faithful representation of $\mathfrak{L}_1(H)$ by matrices. Then, by 4.21, $\mathfrak{L}_1(G \times H)$ is isomorphic to the set $\mathfrak{K}$ of all Kronecker products $\tau(L) \wedge \varrho(M)$ ($L \in \mathfrak{L}_1(G)$, $M \in \mathfrak{L}_1(H)$). Let $\mathfrak{S}$ be the set of all sums $\Sigma\tau(L_i)\wedge\varrho(R_i)$ with $L_i\,\varepsilon\,\mathfrak{L}_1(G)$ and $R_i\,\varepsilon\,\mathcal{R}$. Then if $L, L_i\,\varepsilon\,\mathfrak{L}_1(G)$, $R_i\,\varepsilon\,\mathcal{R}$, and $M\,\varepsilon\,\mathfrak{L}_1(H)$, we have[21]

$$(\tau(L)\wedge\varrho(M))\,(\Sigma\tau(L_i)\wedge\varrho(R_i))=\Sigma(\tau(L)\,\tau(L_i))\wedge(\varrho(M)\,\varrho(R_i))=\Sigma\tau(L\star L_i)\wedge\varrho(M\star R_i),$$

and this is in $\mathfrak{S}$ since $L\star L_i\,\varepsilon\,\mathfrak{L}_1(G)$ and $M\star R_i\,\varepsilon\,\mathcal{R}$. It is now clear that $\mathfrak{S}$ is a left ideal in $\mathfrak{L}$. We now show that $\mathfrak{S}$ is nilpotent. Any product of k elements of $\mathfrak{S}$ has the form

$$\prod_{j=1}^{k}\left(\sum_{i=1}^{a_j}\tau(L_{i,j})\wedge\varrho(R_{i,j})\right)=\sum_{i_1=1}^{a_1}\sum_{i_2=1}^{a_2}\cdots\sum_{i_k=1}^{a_k}\prod_{j=1}^{k}\tau(L_{i_j,j})\wedge\varrho(R_{i_j,j})$$
$$=\sum_{i_1=1}^{a_1}\cdots\sum_{i_k=1}^{a_k}\tau\left(\prod_{j=1}^{k}L_{i_j,j}\right)\wedge\varrho\left(\prod_{h=1}^{k}R_{i_h,h}\right).$$

Since $\mathcal{R}$ is nilpotent, we have $\prod_{h=1}^{k}R_{i_h,h}=0$ for $k\geq k_0$, and therefore the above product is 0 for $k\geq k_0$. Therefore $\mathfrak{S}$ is a non-zero, nilpotent left ideal of $\mathfrak{L}$, and hence is contained in the radical of $\mathfrak{L}$. This completes the present proof.

For our final theorem, we require a lemma.

5.26. Lemma. Let G be a finite semigroup such that $\mathfrak{L}_1(G)$ is semisimple, and let B be an ideal in G. Then $\mathfrak{L}_1(B)$ is semisimple.

Proof. Let $\mathcal{A}$ be the set of all $L=\sum_{x\,\varepsilon\,G}\alpha_x\lambda_x$ such that $\alpha_x=0$ for x non $\varepsilon\,B$. It is plain that $\mathcal{A}$ is isomorphic to $\mathfrak{L}_1(B)$ and that $\mathcal{A}$ is an ideal in the algebra $\mathfrak{L}_1(G)$. It is easy to show that every ideal in a semisimple algebra is itself semisimple, and hence $\mathfrak{L}_1(B)$ is semisimple.

5.27. Theorem. Let G be an idempotent finite semigroup. Then $\mathfrak{L}_1(G)$ is semisimple if and only if G is commutative.

Proof. We use finite induction to prove that if $\mathfrak{L}_1(G)$ is semisimple, then G is commutative. If G is of order 1, this is trivial, but true. We suppose it is true for $o(G)<n$ and consider a G of order n. Using the notation and results of 2.13 and 2.14, we consider the commutative semigroup H whose elements are the distinct sets $S_a\,(a\,\varepsilon\,G)$. If $b\,\varepsilon\,G$, we take $J_b=\{S_a;\,S_a\,\varepsilon\,H,\,S_a S_b=S_a\}$ and $I_b=\{x;\,x\,\varepsilon\,G,\,S_x\,\varepsilon\,J_b\}$. It is clear that J_b is an ideal of H. If $x\,\varepsilon\,I_b$ and $y\,\varepsilon\,G$, then $xy\,\varepsilon\,S_{xy}=S_x S_y$, $yx\,\varepsilon\,S_{yx}=S_y S_x$, and $S_x S_y\,\varepsilon\,J_b$. Therefore I_b is a two-sided ideal in G. Since $\mathfrak{L}_1(G)$

[21] We use the formulas $(A\wedge B)\cdot(C\wedge D)=(AC)\wedge(BD)$, $(A+B)\wedge C=A\wedge C+B\wedge C$, $A\wedge(B+C)=A\wedge B+A\wedge C$, $(\alpha A)\wedge(\beta B)=\alpha\beta\,(A\wedge B)$, where A, B, C, and D are matrices and $\alpha,\beta\,\varepsilon\,K$.

is semisimple, so is $\mathfrak{L}_1(I_b)$, as Lemma 5.26 shows. Since $S_b \subset I_b$ and since S_b is itself a semigroup that is not commutative if $o(S_b) \geq 2$ (see Theorem 2.15), we can use the induction hypothesis to see that $o(S_b)=1$ for all b such that $o(I_b)<n$. However, if $o(I_b)=n$, then $I_b=G$, $J_b=H$, and $S_a S_b = S_a$ for all $S_a \varepsilon H$; that is, S_b is a unit of H. If H does not have a unit, then we have $o(S_b)=1$ for all $b \varepsilon G$ and G is commutative.

There remains the case in which S_e, say, is the unit of H, and $o(S_b)=1$ if $S_b \neq S_e$. We can number the elements of G so that $S_e=\{x_1, x_2, \ldots, x_m\}$, where m is an integer such that $1 \leq m \leq n$. Then we have $S_{k_i}=\{x_i\}$ for $m<i \leq n$. Now, if

$$L=\sum_{i=1}^{m} \alpha_i \lambda_i \quad \text{and} \quad \sum_{i=1}^{m} \alpha_i = 0,$$

then

$$\lambda_j \star L = \sum_{i=1}^{m} \alpha_i \lambda_j \star \lambda_i = \sum_{i=1}^{m} \alpha_i \lambda_j = 0 \quad \text{and} \quad L \star \lambda_j = \sum_{i=1}^{m} \alpha_i \lambda_j = 0,$$

for $m<j \leq n$, since $x_i x_j \varepsilon S_e S_{x_j} = S_{x_j} = \{x_j\}$. Also, for $1 \leq j \leq m$, we have

$$\lambda_j \star L = \sum_{i=1}^{m} \alpha_i \lambda_j \star \lambda_i = \sum_{h=1}^{m} \alpha_h' \lambda_h,$$

where

$$\sum_{h=1}^{m} \alpha_h' = \sum_{h=1}^{m} \sum_{\substack{i \\ \lambda_j \star \lambda_i = h}} \alpha_i = \sum_{i=1}^{m} \alpha_i = 0.$$

In like fashion, we see that

$$L \star \lambda_j = \sum_{h=1}^{m} \alpha_h'' \lambda_h, \quad \text{where} \quad \sum_{h=1}^{m} \alpha_h'' = 0.$$

Therefore the set $\mathfrak{J}$ of all these L is, if not zero, a proper 2-sided ideal in $\mathfrak{L}_1(G)$. If $x_h, x_i, x_j \varepsilon S_e$, then $x_h x_i x_j = x_h x_j$ (Theorem 2.15) and hence

$$\lambda_h \star L \star \lambda_j = \sum_{i=1}^{m} \alpha_i \lambda_h \star \lambda_i \star \lambda_j = \sum_{i=1}^{m} \alpha_i \lambda_h \star \lambda_j = 0 \quad \text{if} \quad 1 \leq h \leq m,\ 1 \leq j \leq m.$$

From this it is evident that $\mathfrak{J}^3=0$. Since G is semisimple, $\mathfrak{L}_1(G)$ contains no proper nilpotent ideal, so we have $\mathfrak{J}=0$, $m=1$, $o(S_e)=1$, and, finally, G is commutative.

If G is commutative and idempotent, it is obvious that $G=G^\circ$, and, by Theorem 5.21, $\mathfrak{L}_1(G)$ is semisimple.

Appendix 1

Semigroups of order 2. Except for isomorphisms and anti-isomorphisms, the following is a complete list of the multiplication tables of all semigroups of order 2.

The letter in the upper left hand corner of each table designates the corresponding $\mathfrak{L}_1$ algebra, to be found below.

1.

A	1	2
1	1	2
2	2	1

2.

B	1	2
1	1	2
2	1	2

3.

C	1	2
1	1	1
2	1	1

4.

A	1	2
1	1	2
2	2	2

We express each $\mathfrak{L}_1$ algebra as a matrix algebra and display its general element. The letters represent arbitrary, independent, complex numbers. These three algebras are easily seen to be non-isomorphs.

$$A.\ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \qquad B.\ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \qquad C.\ \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$$

Appendix 2

Semigroups of order 3. Again, isomorphs and anti-isomorphs are omitted. The letters $A, \ldots, I$ refer to the corresponding $\mathfrak{L}_1$ algebras, which are listed below.

1.

A	1	2	3
1	1	2	3
2	2	3	1
3	3	1	2

2.

B	1	2	3
1	1	2	3
2	1	2	3
3	1	2	3

3.

C	1	2	3
1	1	2	3
2	2	2	2
3	2	2	2

4.

C	1	2	3
1	1	2	3
2	3	2	3
3	3	2	3

5.

A	1	2	3
1	1	1	1
2	1	2	1
3	1	1	3

6.

D	1	2	3
1	1	1	1
2	1	1	1
3	1	1	2

7.

A	1	2	3
1	1	2	3
2	2	2	3
3	3	3	3

8.

A	1	2	3
1	1	2	1
2	2	1	2
3	1	2	3

9.

E	1	2	3
1	1	2	1
2	1	2	2
3	1	2	3

10.

A	1	2	3
1	1	2	3
2	2	1	3
3	3	3	3

11.

F	1	2	3
1	1	1	1
2	1	1	2
3	1	2	3

12.

C	1	2	3
1	1	2	3
2	1	2	3
3	3	3	3

13.

G	1	2	3
1	1	1	1
2	1	1	1
3	1	1	1

14.

H	1	2	3
1	1	1	1
2	1	2	1
3	1	1	1

15.

H	1	2	3
1	1	1	1
2	1	2	2
3	1	2	2

16.

H	1	2	3
1	1	2	2
2	2	1	1
3	2	1	1

17.

H	1	2	3
1	1	2	1
2	2	1	2
3	1	2	1

18.

I	1	2	3
1	1	2	2
2	1	2	2
3	1	2	2

The $\mathfrak{C}_1$ algebras of dimension 3. These 9 algebras are written in the notation of Appendix 1. It can be shown that no 2 of these algebras are either isomorphic or anti-isomorphic. Furthermore, algebras D and F cannot be written as matrix algebras of any order in which all non-zero entries are independent. Finally, none of the algebras A–I has an isomorph which is a matrix algebra in which the order of the matrices is less than the order appearing in the representation shown. The proofs of these assertions are of only minor interest for our present purposes, and so are omitted.

A. $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ *B.* $\begin{pmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ *C.* $\begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & 0 & 0 \end{pmatrix}$

D. $\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & b & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix}$ *E.* $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ *F.* $\begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & 0 & b \end{pmatrix}$

G. $\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & b & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *H.* $\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{pmatrix}$ *I.* $\begin{pmatrix} a & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$

Appendix 3

All semigroups of order 5 whose $\mathfrak{L}_1$ algebra is semisimple. In this table, we write the 2 semigroups in question as subsemigroups of $K \oplus \mathfrak{M}_2$ under multiplication.

1. $\left\{1, \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\right\}, \left\{1, \begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}\right\}, \left\{1, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right\}.$

2. $\left\{1, \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\}, \left\{1, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right\}, \left\{1, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right\}.$

Appendix 4

All semigroups of order 6 whose $\mathfrak{L}_1$ algebra is semisimple. In this table, we write the 13 semigroups in question as subsemigroups of $K \oplus K \oplus \mathfrak{M}_2$ under multiplication. The number ω in 1. is a primitive cube root of unity.

1. $\left\{1, 1, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}\omega & 0\\ 0 & \omega^2\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}\omega^2 & 0\\ 0 & \omega\end{pmatrix}\right\}, \left\{1, -1, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\}, \left\{1, -1, \begin{pmatrix}0 & \omega\\ \omega^2 & 0\end{pmatrix}\right\}, \left\{1, -1, \begin{pmatrix}0 & \omega^2\\ \omega & 0\end{pmatrix}\right\}.$

2. $\left\{1, -1, \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right\}.$

3. $\left\{1, -1, \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right\}.$

4. $\left\{1, -1, \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\}, \left\{1, -1, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right\}, \left\{1, -1, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right\}.$

5. $\left\{1, 0, \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}, \left\{1, 0, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 0, \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 0, \begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}\right\}, \left\{1, 0, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right\}.$

6. $\left\{1, 0, \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}, \left\{1, 0, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 0, \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\}, \left\{1, 0, \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 0, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right\}.$

7. $\left\{1, 0, \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 0, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 0, \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 0, \begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}\right\}, \left\{1, 0, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right\}.$

8. $\left\{1, 0, \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right\}.$

9. $\left\{1, 0, \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 1, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 0, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 0, \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\right\}, \left\{1, 0, \begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}\right\}, \left\{1, 0, \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right\}.$

10. $\left\{ 1, 0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ 1, 0, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ 1, 1, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ 1, 0, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ 1, 0, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}, \left\{ 1, 0, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$

11. $\left\{ 1, 0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ 1, 1, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ 1, 0, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ 1, 0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \left\{ 1, 0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ 1, 0, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$

12. $\left\{ 1, 0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ 1, 1, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ 1, 1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ 1, 1, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \left\{ 1, 1, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ 1, 1, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$

13. $\left\{ 1, 0, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ 1, 1, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ 1, 0, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ 1, 0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}, \left\{ 1, 0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}, \left\{ 1, 0, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$

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References

[1]. I. V. ARNOL'D, Ideale in kommutativen Halbgruppen. *Mat. Sbornik*, 36 (1929), 401–407.

[2]. S. D. BERMAN, O nekotoryh svoĭstvah celočislennyh gruppovyh kolec. *Doklady Akad. Nauk SSSR (N. S.)*, 91 (1953), 7–9.

[3]. ——, Ob isomorfizme centrov gruppovyh kolec p-grupp. *Doklady Akad. Nauk SSSR (N. S.)*, 91 (1953), 185–187.

[4]. ARNE BEURLING, Sur les intégrales de Fourier absolument convergentes et leur application à une transformation fonctionnelle. *Nionde Skandinaviska Matematikerkongressen, 1938*, 345–366.

[5]. S. BOCHNER, *Vorlesungen über Fouriersche Integrale.* Akademische Verlagsgesellschaft, Leipzig, 1932. (Reprint 1948, Chelsea Publishing Co., New York.)

[6]. N. BOURBAKI, *Éléments de mathématique.* Livre II, Ch. II. Algèbre linéaire. Act. Sci. et Ind. 1032, Hermann et Cie. Paris, 1947.

[7]. R. C. BUCK, Generalized group algebras. *Proc. Nat. Acad. Sci. U.S.A.*, 36 (1950), 747–749.

[8]. K. S. CARMAN, J. C. HARDEN, and E. E. POSEY, *Appendix* [To the authors' master's theses, University of Tenn., c. 1949].

[9]. A. H. CLIFFORD, Matrix representations of completely simple semigroups. *Amer. J. Math.*, 64 (1942), 327–342.

[10]. ——, Semigroups admitting relative inverses. *Ann. of Math. 2 Ser.*, 42 (1941), 1037–1049.

[11]. JEAN DIEUDONNÉ, Sur le socle d'un anneau et les anneaux simples infinis. *Bull. Soc. Math. France*, 70 (1942), 46–75.

[12]. P. G. L. DIRICHLET, Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites données. *Jour. für die reine u. angew. Math.*, 4 (1829), 157–169.

[13]. PAUL DUBREIL, *Algèbre,* Tome I. Cahiers Scientifiques publiés sous la direction de M. Gaston Julia, Fasc. XX. Gauthier-Villars et Cie., Paris, 1946.

[14]. GEORGE E. FORSYTHE, *SWAC computes all 126 distinct semigroups of order 4.* To appear.

[15]. I. M. GEL'FAND. Über absolut konvergente trigonometrische Reihen und Integrale. *Mat. Sbornik*, 9 (51) (1941), 51–66.

[16]. ROGER GODEMENT, Les fonctions de type positif et la théorie des groupes. *Trans. Amer. Math. Soc.*, 63 (1948), 1–84.

[17]. J. A. GREEN and DAVID REES, On semi-groups in which $x^r = x$. *Proc. Cambridge Phil. Soc.*, 48 (1952), 35–40.

[18]. EDWIN HEWITT, Remarks on the inversion of Fourier-Stieltjes transforms. *Ann. of Math. 2 Ser.*, 57 (1953), 458–474.

[19]. EDWIN HEWITT and HERBERT S. ZUCKERMAN. On convolution algebras. *Proc. Int. Congr. Math.*, 1950, Vol. I, p. 455.

[20]. CHARLES HOPKINS, Rings with minimal condition for left ideals. *Ann. of Math. 2 Ser.*, 40 (1939), 712–730.

[21]. NATHAN JACOBSON, The radical and semi-simplicity for arbitrary rings. *Amer. J. Math.*, 67 (1945), 300–320.

[22]. BØRGE JESSEN and AUREL WINTNER, Distribution functions and the Riemann zeta function. *Trans. Amer. Math. Soc.*, 38 (1935), 48–88.

[23]. YUKIYOSI KAWADA, On the group ring of a topological group. *Math. Japonicae*, 1 (1948), 1–5.

[24]. LYNN H. LOOMIS, *An introduction to abstract harmonic analysis*. D. Van Nostrand Company, New York, 1953.

[25]. E. S. LYAPIN, Polugruppy, vo vseh predstavleniyah kotoryh operatory imeyut nepodvižnye točki I. *Mat. Sbornik*, 34 (76) (1954), 289–306.

[26]. H. B. MANN, On certain systems which are almost groups. *Bull. Amer. Math. Soc.*, 50 (1944), 879–881.

[27]. DAVID MCLEAN, Idempotent semigroups. *Amer. Math. Monthly*, 61 (1954), 110–113.

[28]. L. J. PAIGE, Power associative loop algebras. *Bull. Amer. Math. Soc.* 60 (1954), 141 (abstract).

[29]. SAM PERLIS and GORDON L. WALKER, Abelian group algebras of finite order. *Trans. Amer. Math. Soc.*, 68 (1950), 420–426.

[30]. A. R. POOLE, Finite ova. *Amer. J. Math.*, 59 (1937), 23–32.

[31]. DAVID REES, Note on semi-groups. *Proc. Cambridge Phil. Soc.*, 37 (1941), 434–435.

[32]. ——, On semi-groups. *Proc. Cambridge Phil. Soc.*, 36 (1940), 387–400.

[33]. ——, On the ideal structure of a semi-group satisfying a cancellation law. *Quart. J. Math., Oxford*, Ser. 19 (1948), 101–108.

[34]. G. F. B. RIEMANN, *Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe*. Ges. Werke, repr. Dover Publications, New York, 1953, 227–271.

[35]. LAURENT SCHWARTZ, *Théorie des distributions*. Hermann et C^{ie}., Paris, Tome I, 1950, Tome II, 1951.

[36]. ŠTEFAN SCHWARZ, K teorii periodičeskih polugrupp. *Czech. Math. J.*, 3 (78) (1953), 7–21.

[37]. ——, *Téoria pologrup*. Sborník Prác Prírodoveckej Fakulty Slovenskej Univerzity v Bratislave no. 6 (1943), 64 p.

[38]. ——, Über Charaktere kommutativer Halbgruppen. *Proc. Int. Congr. Math. 1954*, Vol. I.

[39]. I. E. SEGAL, The group algebra of a locally compact group. *Trans. Amer. Math. Soc.*, 61 (1947), 69–105.

[40]. MARSHALL H. STONE, The generalized Weierstrass approximation theorem. *Math. Mag.*, 21 (1948), 167–183, 237–254.

[41]. ——, The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc.*, 40 (1936), 37–111.

[42]. A. Suškevič [Suschkewitsch], Über die endlichen Gruppen ohne das Gesetz der eindeutigen Umkehrbarkeit. *Math. Ann.*, 99 (1928), 30–50.

[43]. ——, Über die Matrizendarstellungen der verallgemeinerten Gruppen. *Comm. Soc. Math. Harkov*, IV Ser. 6 (1933), 27–38.

[44]. Takayuki Tamura, Some remarks on semi-groups and all types of semi-groups of order 2, 3. *Jour. Gakugei Tokushima Univ.*, 3 (1953), 1–11.

[45]. Vito Volterra and Joseph Pérès, *Leçons sur la composition des fonctions permutables.* Collection de monographies sur la théorie des fonctions publiée sous la direction de M. Émile Borel. Gauthier-Villars et C^{ie}., Paris, 1924.

[46]. B. L. van der Waerden, *Moderne Algebra.* II. Auflage. Teil II. Grundlehren der Math. Wiss., Springer, Berlin, 1937. Reprint Ungar Publishing Co., New York, 1943.

[47]. K. Weierstrass, *Über die analytische Darstellbarkeit sogenannter willkürlicher Funktionen reeller Argumente.* Sitzungsberichte der Königlichen Preussischen Akademie der Wiss. zu Berlin, 1885, 633–640, 789–806. Mat. Werke, Mayer u. Müller, Berlin, 1903.

[48]. André Weil, *L'intégration dans les groupes topologiques et ses applications.* Hermann et C^{ie}., Paris, 1940 (Act. Sci. et Ind. No. 869).

[49]. Hermann Weyl, *Gruppentheorie und Quantenmechanik.* S. Hirzel, Leipzig, 1928.

[50]. Hermann Weyl and F. Peter, Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe. *Math. Ann.*, 97 (1927), 737–755.

[51]. Antoni Zygmund, *Trigonometrical series.* Monografje Matematyczne, Warszawa-Lwów, 1935.

[52]. Antoni Zygmund and A. P. Calderon, On the existence of certain singular integrals. *Acta Math.*, 88 (1952), 85–139.