CONVERGENT SOLUTIONS OF ORDINARY LINEAR HOMOGENEOUS DIFFERENTIAL EQUATIONS IN THE NEIGHBORHOOD OF AN IRREGULAR SINGULAR POINT

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§ 1. Introduction

In this paper it will be shown that certain of the divergent asymptotic series which represent solutions of ordinary linear homogeneous differential equations in the neighborhood of an irregular singular point can be summed and replaced by convergent generalized factorial series. These results extend the earlier work of Horn [1][2], W. J. Trjitzinsky [2], and R. L. Evans [3].

In Evans' paper [3], the existence of integral (8) on page 91 is questionable because the function $\Psi_\nu(\xi)$ may increase more rapidly than any exponential func-

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[2] All references are listed at the end of this paper.

tion $e^{c|\xi|}$ as $\xi \to \infty$. The appearance of functions of such rapid growth has blocked the treatment of the most general case in the past and indeed has blocked the present author in his attempt to sum *all* the divergent asymptotic series solutions. However, considerable progress has been made, as the reader may see by glancing at the summary at the end of this paper.

The analysis begins in section § 2 with a detailed step-by-step procedure for calculating formal series solutions of a system of linear homogeneous differential equations. These solutions are analogous to those obtained by E. Fabry [4] for a single equation of the nth order. The steps in the calculations parallel closely a procedure used by the author in his 1952 paper [2] relating to expansions of solutions of a differential equation in powers of a parameter. The author wishes to take this occasion to direct the reader's attention to M. Hukuhara's [6] solution of the same problem in 1937 by another method.

When the procedure for computing the formal solution has been given in full detail, it becomes evident that in the neighborhood of an irregular singular point any given ordinary linear homogeneous differential equation can be reduced to a certain convenient canonical form. This canonical form, introduced in section § 3, is a refinement of the forms previously obtained by M. Hukuhara [7] and G. D. Birkhoff [13].

With the refined canonical form as a starting point, the analysis then proceeds in steps paralleling those used by W. J. Trjitzinsky [2]; however the computations in the present paper are carried out in matrix form to abbreviate at least to some extent the unavoidable algebraic complications. Formal Laplace integral representations of the solutions are introduced. The rate of growth of analytic solutions of a related system of integral equations is established and the Laplace integral representation of solutions is thereby rigorously justified. Finally the convergence of the factorial series representation of solutions is established by using certain theorems of N. E. Nörlund [8]. This means that Borel exponential summability, if properly applied, will sum at least certain of the formal, i.e. asymptotic, series solutions which are associated with an irregular singular point.

Once the Laplace integral representation has been substantiated, one can establish rigorously either a factorial series representation or an asymptotic series representation of solutions. It is believed that the factorial series representation is to be preferred; for once a value of the independent variable is fixed, the accuracy that can be attained in computing the corresponding value of a solution is definitely limited when the asymptotic series representation is used, while any desired degree of accuracy can be attained by using the convergent factorial series solution.

To be more precise we shall be concerned with solutions valid in the neighborhood of the origin $\tau=0$ of the system of n linear differential equations of the form

$$\tau^{g}\frac{dx_i}{d\tau}=\sum_{j=1}^{n}\sum_{k=0}^{\infty}a_{ijk}\,\tau^{k}x_j \qquad (i=1,\ldots,n), \tag{1}$$

where the integer $g\geq 0$ and the complex constants a_{ijk} are known. By hypothesis the series

$$\sum_{k=0}^{\infty}a_{ijk}\,\tau^{k} \qquad (i,j=1,\ldots,n)$$

all converge for $|\tau|<\tau_0$.

The matrix differential equation which corresponds to system (1) takes the form

$$\tau^{g}\frac{dX}{d\tau}=AX \tag{2}$$

where the matrix

$$A=\sum_{k=0}^{\infty}A_k\,\tau^{k} \tag{3}$$

and the element in the ith row and jth column of matrix A_k is a_{ijk}. Both matrices A and X are square and of order n.

A solution $X(\tau)$ of equation (2) is said to be *independent* if the determinant $|X(\tau)|$ is not zero in some domain $0<|\tau|<\tau_1<\tau_0$. If $X(\tau)$ is an independent solution of equation (2), the elements in any one column of $X(\tau)$ form a set of solutions for system (1) and all the columns together provide a fundamental set of solutions for (1) corresponding to the singular point $\tau=0$.

A precise statement of our conclusions is found in Theorems I, II, and III. To introduce the necessary symbolism for the statement of these theorems, let us proceed to the detailed analysis and begin by computing *formal* independent series solutions of equation (2) running in either full or fractional powers of τ.

§ 2. Formal Series Solutions

Once a matrix differential equation of type (2) is given, formal series solutions can always be obtained by carrying out the computational procedure which will now be outlined. Nine special cases are considered in turn and the series solutions computed in each case. It will then become clear how the formal solutions in the general case can be obtained.

Case I: $g=0$

If $g=0$, equation (2) takes the special form

$$\frac{dX}{d\tau}=(A_0+A_1\tau+\cdots)X. \tag{4}$$

In this event a formal independent series solution of the type

$$X(\tau)=\sum_{k=0}^{\infty}H_k\tau^k, \tag{5}$$

can be found by substituting series (5) into equation (4). When this has been done and the coefficients of like powers of τ equated, it is found that

$$H_1=A_0H_0=A_0;\quad H_2=(A_0H_1+A_1H_0)/2;$$

and in general that

$$H_k=(A_0H_{k-1}+A_1H_{k-2}+\cdots+A_{k-1}H_0)/k,\qquad (k=1,2,\ldots),$$

where $H_0=I$, the identity matrix. Thus all the coefficients in series (5) can be computed in succession. The solution (5) obtained in this fashion is a formally independent solution, for the lead term in the series is the identity matrix I.

Case II: The scalar case, $g>0$, $n=1$

If the matrices X and A in (2) are of order $n=1$, both X and A are scalars. In this event set

$$X=\exp\left\{A_{g-1}\log\tau-\frac{A_{g-2}}{\tau}-\frac{A_{g-3}}{2\tau^2}-\cdots-\frac{A_0}{(g-1)\tau^{g-1}}\right\}Y \tag{6}$$

and this substitution will reduce equation (2) to the form

$$\tau^g\frac{dY}{d\tau}=(A_g\tau^g+A_{g+1}\tau^{g+1}+\cdots)Y.$$

If $g=1$, only the log term is to be used in (6). Divide the reduced equation by τ^g and it then takes on a form treated in case I. Therefore in case II a formal independent series solution can also be found.

If neither $n=1$ nor $g=0$, make the *normalizing transformation*

$$X=PY \tag{7}$$

where P is a constant non-singular matrix. This substitution will then change (2) into the equation

$$\tau^g \frac{dY}{d\tau} = BY = \left(\sum_{k=0}^{\infty} B_k \tau^k\right) Y$$

where $B = P^{-1}AP$ and in particular

$$B_0 = P^{-1}A_0 P.$$

It is presumed that P has been so chosen that B_0 takes on the classical Jordan canonical form; or better yet, without loss of generality, assume at the outset that the lead coefficient A_0 in (3) is in this canonical form; i.e. assume that

(8) $$A_0 = \left\| \begin{matrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & M_m \end{matrix} \right\|$$

where

$$M_i = \left\| \begin{matrix} \varrho_i & 0 & 0 & \cdots & 0 \\ \beta_i & \varrho_i & 0 & & 0 \\ 0 & \beta_i & \varrho_i & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \beta_i & \varrho_i \end{matrix} \right\| \qquad (i = 1, \ldots, m)$$

and β_i is either zero or one.

Case III: $m = 1$ and $\beta_1 = 0$

If in the canonical form (8) there is but a single M, say M_1, and no 1's appear on the subdiagonal, A_0 has the special form

$$A_0 = \varrho_1 I$$

where I is the identity matrix. In this event make the following *exponential transformation*

(9) $$X = Y \exp\{-\varrho_1/(g-1)\tau^{g-1}\} \quad \text{if } g > 1$$

or

(10) $$X = Y \exp\{\varrho_1 \log \tau\} \quad \text{if } g = 1.$$

Such a substitution reduces (2) to the form

$$\tau^g \frac{dY}{d\tau} = (A_1\tau + A_2\tau^2 + \cdots)\, Y.$$

Divide out a τ and again obtain an equation of type (2) with the g lowered a unit. Repeat this process using once again a normalizing transformation of type (7) to throw the new lead coefficient A_1 into canonical form and then, if possible, use an exponential transformation of type (9) or (10) to lower g. Once g is reduced to zero the procedure given in case I is applicable and formal independent series solutions can be computed as indicated. This process fails only if two or more distinct roots appear in some one of the successive Jordan canonical forms or, if 1's appear on the first subdiagonal at some stage in the process. The situation when two or more distinct roots appear will be treated first.

Case IV: Distinct characteristic roots

Let A_0 once again have the canonical form (8) and assume that, if $g>1$, there are at least two distinct characteristic roots and assume that, if $g=1$, there are at least two distinct characteristic roots not differing by an integer. If this be true, a sequence of *zero-inducing transformations*

$$X=(I+\tau^k Q_k)\,Y \tag{11}$$

is utilized to separate the distinct roots. More precisely use substitution (11) to reduce equation (2) to the form

$$\tau^g \frac{dY}{d\tau}=(A_0+\cdots+A_{k-1}\tau^{k-1}+C_k\tau^k+C_{k+1}\tau^{k+1}+\cdots)\,Y \tag{12}$$

where

$$C_k=A_k+A_0Q_k-Q_kA_0 \qquad \text{if } g>1$$

and

$$C_k=A_k+A_0Q_k-Q_kA_0-kQ_k \qquad \text{if } g=1. \tag{13}$$

Note that transformation (11) does not affect the first k matrices $A_0, A_1, \ldots, A_{k-1}$. In obtaining (12) the formal expansion

$$(I+\tau^k Q_k)^{-1}=I-\tau^k Q_k+\tau^{2k}Q_k^2-\cdots$$

has been used.

The constant matrices C_k, A_k, and Q_k can now be subdivided into smaller blocks in just the same way that matrix A_0 is subdivided in (8). After this subdivision is made, denote the block in the rth row of blocks and the sth column of blocks respectively by C_{rs}, A_{rs}, Q_{rs}, and also let the elements in the ith row and jth column of these blocks be respectively c_{ij}, a_{ij}, q_{ij} with $i=1,\ldots,u$ and $j=1,\ldots,v$.

If $g > 1$, a judicious choice of the q_{ij} can and will be made so that all the elements c_{ij} in C_{rs} are zero ($i = 1, \ldots, u$; $j = 1, \ldots, v$) provided $\varrho_r \neq \varrho_s$. If $g = 1$, this statement is still correct provided that in addition to

$$\varrho_r \neq \varrho_s, \quad k \neq \varrho_r - \varrho_s.$$

A proof of this statement when $g > 1$ is to be found in reference [4], pp. 86–88. However the details when $g = 1$ require attention. Assuming then that $g = 1$, note first that the main diagonal matrix M_r in (8) can be written in the form

$$M_r = \varrho_r I_r + E_r \qquad (r = 1, \ldots, m)$$

where I_r is an identity matrix of the same order as M_r and E_r is a square matrix made up of zero elements except for the β_r running down the first subdiagonal. Likewise

$$M_s = \varrho_s I_s + E_s \qquad (s = 1, \ldots, m).$$

Then from (13) it is clear that

$$C_{rs} = A_{rs} + (\varrho_r I_r + E_r) Q_{rs} - Q_{rs} (\varrho_s I_s + E_s) - k Q_{rs}$$

and therefore, if every element c_{ij} in C_{rs} is to be zero, we must have

$$a_{ij} + (\varrho_r - \varrho_s) q_{ij} + \beta_r q_{i-1,j} - \beta_s q_{i,j+1} - k q_{ij} = 0 \tag{14}$$

for $i = 1, \ldots, u$; $j = 1, \ldots, v$ where $q_{0j} = 0$ for all j and $q_{i,v+1} = 0$ for all i. When the simultaneous system of equations (14) is solved for the q_{ij} it is found that, if $(\varrho_s - \varrho_r + k) \neq 0$,

$$q_{1v} = a_{1v} / (\varrho_s - \varrho_r + k)$$

and

$$q_{i+1,v} = (a_{i+1,v} + \beta_r q_{i,v}) / (\varrho_s - \varrho_r + k), \qquad (i = 1, \ldots, u-1).$$

Thus the elements q_{ij} in the last column of Q_{rs} can be calculated by working from the top down. Likewise working from right to left the q_{ij}'s in successive columns can be evaluated beginning with the top element and working down the successive columns.

Thus, the statement that, when $g = 1$ and $\varrho_r \neq \varrho_s$, a judicious choice of Q_{rs} will make C_{rs} zero, is correct provided $k \neq \varrho_r - \varrho_s$. It is presumed then that whenever $k \neq \varrho_r - \varrho_s$ the appropriate values of Q_{rs} are chosen to throw all zeros into the C_{rs} matrices. To complete the evaluation of the Q_{rs}, if $\varrho_r = \varrho_s$ arbitrarily set $Q_{rs} = 0$; also, if $g = 1$, $\varrho_r \neq \varrho_s$, and $k = \varrho_r - \varrho_s$ set $Q_{rs} = 0$.

By choosing the Q_k in this fashion many zeros are thrown into the C_k matrix in (12). This is done first using transformation (11) with $k=1$, then $k=2, 3, \ldots$ and so on to infinity.

Thus if A_0 is in the Jordan canonical form, a non-singular formal zero-inducing transformation

$$X = [(I + \tau Q_1)(I + \tau^2 Q_2)(I + \tau^3 Q_3) \ldots] Y \tag{15}$$

has been found which reduces equation (2) *to a new equation*

$$\tau^g \frac{dY}{d\tau} = G\, Y \tag{16}$$

where

$$G = \| \delta_{ij}\, G_i \| \qquad (i, j = 1, \ldots \sigma \le m). \tag{17}$$

Here and in subsequent formulas δ_{ij} is the well-known Kronecker delta. The elements G_i in matrix G are themselves submatrices. Each G_i represents an expansion

$$G_i = G_{i0} + G_{i1}\tau + G_{i2}\tau^2 + \cdots \qquad (i = 1, \ldots, \sigma),$$

where each lead matrix G_{i0} is itself in a Jordan canonical form of type (8).

If $g > 1$, the same root ϱ_i appears in each of the diagonal blocks in G_{i0} and furthermore the ϱ_i's corresponding to the various G_{i0} in (17) *will all be distinct. On the other hand, if $g = 1$, different ϱ's may appear in the same G_{i0} provided these ϱ's all differ by integers. Indeed, if $g = 1$, all the characteristic roots differing one from the other by integers appear in the same G_{i0} and any two distinct characteristic roots which do not differ in value by an integer appear in different G_{i0}'s.*

More details will be given presently relating to the structure of the G_i's when $g = 1$.

In any case equation (16) splits into σ separate equations

$$\tau^g \frac{dY_i}{d\tau} = G_i\, Y_i; \qquad (i = 1, \ldots, \sigma) \tag{18}$$

where

$$Y = \| \delta_{ij}\, Y_i \| \qquad (i, j = 1, \ldots, \sigma);$$

and again δ_{ij} is the Kronecker delta. Since by hypothesis there are at least two G_i's, the order of each matrix G_i is less than n the order of A. Usually the infinite product in (15) diverges.

Case IV covers specifically the following two subcases:

(a) If $g > 1$ and all the characteristic roots are distinct, equation (2) at this stage splits into n separate equations all of order one and each of the type considered in case II. Therefore in this special situation a formal independent series solution of equation (2) can be found.

(b) If $g=1$ and all the characteristic roots are distinct and no two of them differ by an integer, equation (2) splits into n separate equations each of order one and each of the type considered in case II. Therefore in this special situation formal independent series solutions can also be found.

Case V: A single characteristic root and $g=1$

If the procedure outlined up to this point has failed to produce the formal independent solutions, return to the equations given in (18). Once again the entire procedure described in cases I–IV is applied to each of these equations individually. Either the formal independent series solutions corresponding to each of these equations is eventually obtained by applying the prescribed procedure or the procedure is blocked temporarily by one of the following three contingencies:

(1°) A single multiple characteristic root appears in a canonical lead matrix: $g=1$; and 1's are present.

(2°) Two or more distinct characteristic roots differing by integers appear in a canonical lead matrix and $g=1$.

(3°) A single multiple characteristic root appears in the canonical lead matrix; 1's are present; and $g>1$.

These three contingencies will be treated in turn.

As case V take contingency 1° and thus consider an equation of the form

$$\tau \frac{dX}{d\tau} = (A_0 + A_1 \tau + \cdots) X \tag{19}$$

where

$$A_0 = \varrho I + E; \quad E = \| \delta_{ij} E_i \|, \qquad (i, j = 1, \ldots, m),$$

and again E_i is a square matrix made up of β_i's running down the secondary diagonal with all other elements zeros. By hypothesis at least one of these β_i's is a 1.

A formal solution of the form

$$X(\tau) = (H_0 + H_1 \tau + \cdots + H_k \tau^k + \cdots) \exp \{(\varrho I + E) \log \tau\} \tag{20}$$

where $H_0 = I$ can now be found. To obtain it, note that[1]

[1] Details relating to matrix manipulations are given in S. LEFSCHETZ's text [9].

$$\frac{dX}{d\tau} = \sum_{k=0}^{\infty} [(k+1) H_{k+1} \tau^k + H_k \tau^{k-1} (\varrho I + E)] \exp \{(\varrho I + E) \log \tau\} \tag{21}$$

and then substitute expansions (20) and (21) into (19) and equate coefficients of like powers of τ. When this is done it turns out that

$$H_1 + H_1 E = E H_1 + A_1 \tag{22}$$

$$2 H_2 + H_2 E = E H_2 + A_1 H_1 + A_2 \tag{23}$$

$$\cdots$$

$$k H_k + H_k E = E H_k + A_1 H_{k-1} + A_2 H_{k-2} + \cdots + A_{k-1} H_1 + A_k$$

where $k = 1, 2, \ldots$. Thus the elements in H_1 are uniquely determined by solving equation (22).

In carrying out this computation the element in the upper right-hand corner of H_1 is computed first. The other elements running down the last column of H_1 are then evaluated in turn. The next to the last column of H_1 can then be evaluated working from the top down and so on across the matrix working with the successive column from right to left. Once H_1 is known equation (23) is solved for the elements in H_2 and so on; thus $H_1, H_2, H_3, \ldots$ are all determined in succession. In this way a formal series solution (20) for equation (19) can be computed. Note that since $H_0 = I$ a formal independent series solution has again been found.

Case VI: Characteristic roots differing by integers; $g = 1$

Begin with an equation (19) with A_0 not only in the Jordan canonical form, but also with all its characteristic roots differing by integers. Then when the successive zero-inducing transformations (11) are applied to (19) zeros are thrown into the C_{rs} matrices for all the various values of k except $k = \varrho_r - \varrho_s$. Despite this exception, (11) is applied first for $k = 1$, then $k = 2$, and so on to infinity; i.e. the formal transformation (15) is again used and this time because of the exceptional values of k, (14) is formally reduced to a system of the form

$$\tau \frac{dY}{d\tau} = (D_1 + \tau D_2 + K) Y \tag{24}$$

where

$$D_1 = \| \delta_{ij} (\varrho_i I_i + J_i) \|, \qquad (i, j = 1, \ldots, m),$$

and

$$J_k = \| \delta_{ij} E_{kij} \|, \qquad (k = 1, \ldots, m \text{ and } i, j = 1, \ldots, \sigma_k),$$

where each diagonal square matrix E_{kij} is made up of zeros except for the β_{kij}'s which run down the first sub-diagonal and equal zero or one.

Also since the ϱ_i's differ by integers we can arrange them without loss of generality in such a way that

$$\varrho_i = \varrho_m + k_{m-i}, \qquad (i = 1, \ldots, m-1),$$

where the integers k_{m-i} are ordered so that

$$0 < k_1 < k_2 < \cdots < k_{m-1}.$$

The matrix D_2 is likewise in diagonal form with

$$D_2 = \left\| \delta_{ij} \left(\sum_{k=0}^{\infty} D_{ik} \tau^k \right) \right\|, \qquad (i, j = 1, \ldots, m),$$

where the D_{ik} are constant matrices.

The elements in matrix K in (24) as in D_1 and D_2 are themselves blocks, i.e. submatrices. The number of rows of blocks and the number of columns of blocks in K is m, just as in D_1 and D_2. The main diagonal blocks in K are filled with zeros and all blocks below the main diagonal are likewise filled with zeros. The block above the diagonal in the rth row and sth column $(s > r)$ is a constant matrix K_{rs} multiplied by the scalar $\tau^{(k_{m-r} - k_{m-s})}$.

A *root-equalizing transformation*

$$Y = \| \delta_{ij} \tau^{\varrho_i} I_i \| Z \qquad (i, j = 1, \ldots, m),$$

where the I_i's are identity matrices, is applied to equation (24). When this is done (24) takes the new form

$$\tau \frac{dZ}{d\tau} = (D_3 + \tau D_2) Z \tag{25}$$

where the constant matrix

$$D_3 = \begin{Vmatrix} J_1, & K_{12}, & K_{13}, & \cdots & K_{1m} \\ 0, & J_2, & K_{23}, & & \vdots \\ & & J_3 & & \\ \vdots & & & \ddots & \\ & & & J_{m-1}, & K_{m-1,m} \\ 0 & \cdots & & 0, & J_m \end{Vmatrix}$$

The important feature in equation (25) is that the roots of the characteristic equation $|D_3 - \varrho I| = 0$ *are all zero.* Thus one more normalizing transformation $Z = PW$ will reduce equation (25) to a new equation in W of precisely the type considered under case V with $\varrho = 0$. Therefore in case VI there will also be a formal independent series solution.

The procedure outlined in cases I–VI will always yield the desired formal independent series solutions if at the outset $g=1$. We are now ready to examine the situation when 1's appear on the subdiagonal and $g>1$.

Case VII: Single characteristic root, ones on the subdiagonal, $g>1$, $\mu \geq 1$

The details relating to this case are almost precisely those given in the author's 1952 paper, pp. 89–97 and need not be repeated in full here. Only the main features will be pointed out. Remembering that $g>1$, reconsider an equation of type (2) where A_0 is in the Jordan classical canonical form (8). In this particular case it is assumed however that all the characteristic roots are equal; i.e. that $\varrho_i=\varrho_1$ for $i=1,\dots,m$. In this event an exponential substitution (9) will reduce all the characteristic roots to zero; so without loss of generality assume that in (2) the A_0 has the special form

$$A_0=\|\delta_{ij}E_i\|, \qquad (i,j=1,\dots,m),$$

where every element in E_1 is either zero or every element is zero except for 1's running down the first subdiagonal. Each matrix $E_2,\dots,E_m$ is made up of zeros except for the 1's which run down the first subdiagonal. By hypothesis at least one E matrix is present with 1's on its subdiagonal. There is no loss of generality in assuming, as we do, that the E_i-matrices with 1's in them are arranged in order of size, the largest, if there is one, at the bottom.

Again a zero-inducing transformation of type (15) is used to throw as large a number of zeros as possible into the matrix coefficients of the successive powers of τ. For example, if $g>1$, and

$$A_0=\left\|\begin{array}{cc|cc|cccc}
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0\\
\hline
0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&1&0
\end{array}\right\|$$

the corresponding zero-inducing transformation (15) reduces equation (2) to the form

$$\tau^g \frac{dY}{d\tau} = \left(\sum_{k=0}^{\infty} C_k \tau^k \right) Y = \| c_{ij} \| Y \tag{26}$$

where $C_0 = A_0$ and

$$C^k = \left\| \begin{array}{cc|cc|cccc} x & x & 0 & x & 0 & 0 & 0 & x \\ x & x & 0 & x & 0 & 0 & 0 & x \\ \hline x & x & 0 & x & 0 & 0 & 0 & x \\ x & x & 0 & x & 0 & 0 & 0 & x \\ \hline x & x & x & x & 0 & 0 & 0 & x \\ x & x & x & x & 0 & 0 & 0 & x \\ x & x & x & x & 0 & 0 & 0 & x \\ x & x & x & x & 0 & 0 & 0 & x \end{array} \right\|, \quad (k = 1, 2, \ldots). \tag{27}$$

In (27) the x indicates an element which may not have been reduced to zero.

Let the order of equation (26) be n. In the illustration (27) the $n = 8$ and $m = 3$. Equation (26) is now ready for the *shearing transformation*

$$Y = \| \delta_{ij} \tau^{\mu(n-i)} \| Z \qquad (i, j = 1, \ldots, n), \tag{28}$$

which converts (26) into the new equation

$$\tau^g \frac{dZ}{d\tau} = DZ = \| d_{ij} \| Z = \left(\sum_{k=0}^{\infty} D_k \tau^k \right) Z \tag{29}$$

where the element

$$d_{ij} = c_{ij} \tau^{\mu(i-j)} - (n-i) \mu \delta_{ij} \tau^{g-1}.$$

The δ_{ij} in this equation is again the Kronecker delta.

The main purpose of the shearing transformation (28) is to induce *above the main diagonal*, when possible, non-zero elements into the lead matrix D_0 in (29). To select the appropriate positive value for μ for this purpose, note first that any c_{ij} in (26) which is not identically zero can be represented as a series

$$c_{ij} = \tau^{h_{ij}} \sum_{k=0}^{\infty} c_{ijk} \tau^k$$

where $c_{ijk} \neq 0$ and each positive integer h_{ij} is *at least one*, except for those *special* elements on the first subdiagonal whose expansion begins with a 1. For these special elements the $h_{ij} = 0$. By hypothesis at least one such special element is present. At the outset, i.e. when $\mu = 0$, the h_{ij} for the special elements are lower in value than those pertaining to any other element which is not identically zero; but after transformation (28) has been made the respective expansions of the elements off the main diagonal begin with the powers $\tau^{h_{ij}+\mu(i-j)}$, excluding from consideration elements which are identically zero. In particular the expansions for the special elements begin with the power μ. For sufficiently small positive μ, this power μ of τ will be less than $h_{ij}+\mu(i-j)$, the power of τ for an off-diagonal ordinary element; but as μ increases a stage will be reached when for the first time $\mu=(i-j)\mu+h_{ij}$ for some i and j, $j>i$, for one or more elements if there exists at least one such h_{ij}. Note this critical value μ_0 of μ.

A special case arises if all elements c_{ij} above the main diagonal are identically zero and all the $h_{ii}\geq g-1$. In this case set $\mu=g-1$ and equation (26) is reduced at once to an equation of type (2) with $g=1$ with its corresponding independent formal solutions.

If this special case does not occur and μ_0 is 1 or greater, set $\mu=1$ in (28) and as a consequence in (29) the expansion of each element d_{ij} begins with at least the first power of τ, if not a higher power. Therefore remove the common factor, say τ; and this in effect lowers the g by at least one unit in (29). We are then ready to repeat the entire process as described up to this point. If, in fact, at each repetition one fails to get the desired formal independent solutions by the methods described in cases I–VI and each time reaches a stage as described where $\mu_0\geq 1$, g is lowered again and again, and in a *finite number of stages* an equation with $g=0$ or 1 must be reached. One final application of the procedure thus far described therefore yields the formal independent solutions desired. If at any stage $\mu_0<1$, a more complicated situation arises.

Case VIII: $0<\mu_0<1$; $g>1$

If $\mu_0<1$, μ_0 is a fraction $\mu_0=q/p$ where q and p are positive integers, $q<p$, and q is prime to p. In this event set $\mu=\mu_0$ in (28) and again our equation takes the form (29). Fractional powers of τ have appeared of necessity for the first time. These fractional powers are removed by introducing a new independent variable

$$t=\tau^{1/p} \tag{30}$$

into equation (29). Before doing this note that, if $\mu=q/p$ in (29), the matrix D when expanded in ascending powers of $\tau^{1/p}$ will begin with the power $\tau^{q/p}$. Therefore divide out of (29) a factor t^q at the same time that transformation (30) is made and (29) becomes

$$t^h \frac{dZ}{dt} = FZ = \| f_{ij} \| Z = \left(\sum_{k=0}^{\infty} F_k t^k \right) Z \tag{31}$$

where

$$h = pq - p + 1 - q, \quad h > 1,$$

and

$$f_{ij} = p\, c_{ij}\, t^{q(i-j-1)} - q\, \delta_{ij}\, (n-i)\, t^{pq-p-q}$$

Equation (31) is precisely of type (2), but this time the power g has in general *increased*, say to h. The lead coefficient F_0 has zeros and ones on its first subdiagonal and all elements below this subdiagonal are zero. Also all elements on the main diagonal F_0 are zero and at least one non-zero element appears above the main diagonal.

Despite the fact that we are now dealing with a number h, usually larger than g, the entire procedure described up to this point is reapplied. *If on normalizing* F_0 *in* (31) *two or more distinct characteristic roots are found, the system is split by a zero-inducing transformation into two or more distinct systems of type* (2), *each of lower order than* (31). The procedure is again applied to each of these new systems. Either the desired formal solutions are found or new equations of type (31) of still lower order are reached with usually another increase in h. If the characteristic equations of the new F_0's always yield at least two distinct roots, finally we shall reach systems of order 1 with possibly very large h's. These are handled as in case II and the desired independent formal series are thus procured.

The only thing which could possibly block this process would be to reach an equation of type (31) where all the characteristic roots of F_0 are alike. This brings us to the last possible case.

Case IX: Roots of F_0 all alike

If the roots of F_0 are all alike, it is proved in reference [4], pp. 93–97 that, if the process outlined in cases I–VIII is repeatedly carried out, the process must necessarily terminate after a *finite* number of stages and yield the desired formal series solutions of equation (2).

Thus in all cases the desired independent formal series solutions can be found. The precise nature of these formal solutions can best be described by first introducing a canonical form for a given differential equation, as described in the next section.

§ 3. A Canonical Form

In order eventually to proceed rigorously and not just formally it is convenient to have

Theorem I. *Corresponding to a given differential equation of the form*

$$\tau^{g} \frac{dX}{d\tau} = \sum_{k=0}^{\infty} A_k \tau^k X, \tag{32}$$

where the A_k's are constant square matrices, there exists a transformation

$$X = P(\tau)\, Y = \left(\sum_{k=0}^{w} P_k \tau^{k/p} \right) Y \tag{33}$$

which reduces equation (32) to the canonical form

$$t^{h} \frac{dY}{dt} = \left\| \delta_{ij} \left(\varrho_i(t)\, I_i + J_i t^{h-1} \right) + \sum_{\nu=h}^{\infty} B_{ij\nu} t^{\nu} \right\| Y \tag{34}$$

where $i, j = 1, \dots, m$. In (33) the P_k's are appropriate constant square matrices; p and w are suitable positive integers; and the determinant $|P(\tau)|$ is not zero in some region $0 < |\tau| < \tau_2 < \tau_0$. In (34) the independent variable $t = \tau^{1/p}$; h is a non-negative integer; δ_{ij} is the Kronecker delta; the $B_{ij\nu}$'s are constant matrices; and the I_i's are identity matrices. If $h = 0$, the matrices J_i are all identically zero; if $h > 0$, the J_i's are square matrices with zeros, or 1's, or a mixture of zeros and 1's on the first subdiagonal while all other elements in J_i are zero. If $h = 0$ the polynomials $\varrho_i(t)$ are all identically zero; and, if $h > 0$, the polynomials

$$\varrho_i(t) = \varrho_{i0} + \varrho_{i1} t + \cdots + \varrho_{i,h-1} t^{h-1}, \tag{35}$$

where the coefficients are constants and no two of these polynomials are identical. In particular, if $i \neq j$, and

$$\varrho_{ik} = \varrho_{jk} \quad \textit{for} \quad k = 0, 1, \dots, h-2,$$

then the difference $\varrho_{i,h-1} - \varrho_{j,h-1}$ is not only not zero, but it also is not an integer. If the infinite series $\sum_{k=0}^{\infty} A_k \tau^k$ converges for all $|\tau| < \tau_0$, then the infinite series

$$\sum_{\nu=h}^{\infty} B_{ij\nu} t^{\nu} \qquad (i, j = 1, \dots, m) \tag{36}$$

also converge for sufficiently small $|t|$, say $|t| < t_0$.

Proof: We intend to show first of all that there exists a formal substitution

$$X = \mathfrak{P}(\tau)\, Y, \tag{37}$$

where

$$\mathfrak{P}(\tau) = \sum_{k=0}^{\infty} P_k\, \tau^{k/p}; \qquad t = \tau^{1/p}; \tag{38}$$

which will reduce equation (32) to the form

$$t^h \frac{dY}{dt} = \mathfrak{B}(t)\, Y \tag{39}$$

where

$$\mathfrak{B}(t) = \| \delta_{ij}\, \mathfrak{B}_i(t) \|, \qquad (i, j = 1, \ldots, m): \tag{40}$$

and

$$\mathfrak{B}_i(t) = \varrho_i(t)\, I_i + J_i\, t^{h-1} + \sum_{\nu=h}^{\infty} \mathfrak{B}_{i\nu}\, t^{\nu}. \tag{41}$$

Moreover it will become evident that transformation (37) is non-singular in the sense that formally the determinant $|\mathfrak{P}(\tau)|$ is not identically zero when expanded in powers of $\tau^{1/p}$. The polynomials $\varrho_i(t)$ in (41), $(i = 1, \ldots, m)$, will meet the specifications of Theorem I and, if $h = 0$, $\varrho_i(t) \equiv 0$ and $J_i = 0$.

These facts will be established by checking back through the details relating to the nine special cases considered in section § 2. In cases I and II, for example, equation (32) is in the desired canonical form (34) at the outset.

In case III a sequence of normalizing transformations of type (7) and exponential transformations of type (9) and (10) reduce (2) to an equation of type (4) where $g = 0$. If the exponential transformations are all omitted and the successive normalizing transformations are all used and incorporated into a single transformation, the resulting transformation is of type (37). Moreover this single transformation is obviously non-singular and reduces equation (2) to the desired form (39).

Next consider case IV-a where a sequence of normalizing and exponential transformations combined with one zero-inducing transformation (15) reduces equation (2) essentially to n distinct equations each of the first order. Observe that, if the exponential transformations are again omitted and all the other transformations used and combined into a single transformation, this transformation will be of type (37) and will reduce (2) to the desired form (39) with $m = n$. Since it is assumed that at some stage in the process, while g still exceeds one, n distinct characteristic roots appear, the $\varrho_i(t)$'s in (35) are sure to be distinct polynomials.

In case IV-b the situation is the same as in case IV-a except that the characteristic roots become all distinct only at the stage when $g=1$ and then they are not only distinct, but do not even differ one from the other by integers.

Note also in case IV that the determinant of the single combined transformation which reduces (2) to (39) is the product of a certain finite number of constant non-zero determinants multiplied by the determinant of a formal matrix

$$[(I+\tau Q_1)(I+\tau^2 Q_2)(I+\tau^3 Q_3)\ldots].$$

The product of all these determinants is obviously of the form $d_0+d_1\tau+d_2\tau^2+\cdots$ where the lead coefficient $d_0 \neq 0$ and hence the combined transformation is non-singular.

If the details given in cases I–IV fail to produce the desired formal series solution, one is driven to consider the solution of a set of equations of type (18). Up to this stage a sequence of normalizing and exponential transformations combined with one zero-inducing transformation has split equation (2) of order n into σ, $(\sigma>1)$, separate equations, each of lower order than n. If the exponential transformations are omitted and the other substitutions are all used and incorporated into a single transformation

$$X=\mathfrak{P}_0(\tau)\,Z, \tag{42}$$

this transformation is of type (37) where $p=1$ and the determinant $|\mathfrak{P}_0(\tau)|$ again is not identically zero, for the expansion of this determinant in powers of τ will begin with a non-zero constant. Furthermore substitution (42) reduces equation (2) to a new equation equivalent to σ, $(\sigma>1)$, separate systems of the form

$$\tau^g\frac{d Z_i}{d\tau}=L_i Z_i, \qquad (i=1,\ldots,\sigma), \tag{43}$$

where

$$Z=\|\delta_{ij} Z_i\|, \qquad (i,j=1,\ldots,\sigma);$$

i.e. despite the omission of the exponential terms splitting occurs. The matrix L_i in (43) has the same order as the corresponding matrix G_i in (18), and each of these orders is less than n.

If at this stage there exists a set of transformations

$$Z_i=\mathfrak{R}_i(\tau)\,Y_i, \qquad (i=1,\ldots,\sigma),$$

where each $\mathfrak{R}_i(\tau)$ has the same structure as the $\mathfrak{P}(\tau)$ in (38), which will reduce (43) to a set of equations

$$\tau^{g} \frac{d Y_i}{d \tau} = \mathfrak{C}_i(\tau)\, Y_i, \qquad (i = 1, \ldots, \sigma),$$

where each $\mathfrak{C}_i(\tau)$ has the same structure as the $\mathfrak{B}(t)$ in (40) with $t = \tau$, then the non-singular transformation

$$X = \mathfrak{P}_0(\tau)\, \| \delta_{ij}\, \mathfrak{R}_i(\tau) \|\, Y, \qquad (i, j = 1, \ldots, \sigma),$$

has the structure of (37) with $p = 1$ and it will reduce equation (2) to the desired form (39). This means that, if Theorem I can be shown to be applicable to each of the equations in (43), then Theorem I is likewise applicable to the original system (2). Therefore attention is focused on the individual equations in (43).

If we then attempt to show that Theorem I is applicable to some particular equation in (43), either we succeed by a repeated reapplication of the reasoning that has just been applied in cases I–IV or we fail because one of the three contingencies 1°, 2°, or 3° mentioned in case V temporarily block the way.

If contingency 1° occurs this brings us back to case V, where it may be assumed, without loss of generality, that the chosen equation in (43) is such that after an exponential transformation, followed by a normalizing transformation, equation (43) is reduced to form (19). But equation (19) is in the desired form (39) just as it stands. If in making this reduction, the exponential transformation is omitted, and the normalizing transformation alone is used, it is evident that we have found a transformation of the appropriate form (37) which reduces (43) to the desired form (39) and again Theorem I is applicable. We have spoken for brevity as though each equation in (43) is treated directly, while actually one or more of these equations may split under appropriate transformations of type (37) into equations of still lower order. If this is the case we would actually deal with these lower ordered equations instead of (43).

If contingency 2° occurs this brings us back to case VI, where it may be assumed, without loss of generality, that the chosen equation in (43) is such that it can be reduced to form (19) by the following sequences of five transformations: an exponential, a normalizing, a zero-inducing, a root-equalizing transformation, and finally another normalizing transformation. But (19) is of the desired form (39); hence, if again the exponential transformation is omitted and the other four successive transformations are combined into a single transformation, this transformation is of the appropriate type (37) and reduces equation (43) again to the desired form (39) and Theorem I is once again applicable.

If contingency 3° occurs, this brings us back to cases VII–IX, where it may be assumed without loss of generality that a finite sequence of exponential, normalizing, zero-inducing, shearing, and possibly a root-equalizing, transformations reduces equation (43) to either form (4) of case I or form (19) of case V, both forms being of the desired type (39). Once again, if all the exponential transformations are omitted and all the other transformations are combined into a single transformation, this transformation is of the appropriate type (38) and reduces (43) to the desired form (39).

Thus the desired substitution (37) exists in all cases. Moreover formally the determinant of this transformation can be expanded in a series running in powers of $\tau^{1/p}$, namely

$$|\mathfrak{P}(\tau)| = \tau^{\eta/p} \sum_{i=0}^{\infty} \alpha_i \tau^{i/p}$$

where the lead constant $\alpha_0 \neq 0$ and the η is a non-negative integer.

Once substitution (37) has been found one has merely to use the first $(w+1)$ terms in expansion (38) for matrix $P(\tau)$ in Theorem I, see (33), provided w is sufficiently large, say $w > pq + 2q + 1$. Theorem I has therefore been demonstrated.

It is evident from this proof that if the w were increased, more of the off-diagonal blocks in (34) could be made identically zero, but there seems to be no point in such a refinement, unless some large *finite* value of w by chance annuls enough off-diagonal blocks that the system can be split into two or more distinct systems of lower order. In general such a rigorous reduction of order does not occur.

If in the canonical form (34) all the lead coefficients ϱ_{i0}, $(i = 1, \ldots, m)$, in the polynomials are equal, an exponential transformation would remove the ϱ_{i0} from the canonical form and a division by t would lower the h in (34) a unit. *It will therefore be assumed without loss of generality that if $m > 1$ in (34) then at least two of the ϱ_{i0}'s have different values. Likewise, if $m = 1$, it is assumed that an exponential transformation has removed $\varrho_1(\tau) I_1 + J_1 t^{h-1}$ from (34) and a division by t^h has reduced h to zero.*

To proceed several new symbols are needed. If $i \neq j$, let

$$\varrho_i(t) - \varrho_j(t) = r_{\beta_{ij}, ij} t^{\beta_{ij}} + r_{\beta_{ij}+1, ij} t^{\beta_{ij}+1} + \cdots + r_{h-1, ij} t^{h-1},$$

where $r_{\beta_{ij}, ij} \neq 0$ and in particular, if $\beta_{ij} = h-1$, the corresponding $r_{\beta_{ij}, ij}$ is not an integer. If $i \neq j$, define the Γ_{ij} matrices by the equations

$$r_{\beta_{ij}, ij} \Gamma_{ij} + B_{ijh} = 0 \quad \text{if } \beta_{ij} \leq h-2 \tag{44}$$

and by

$$(r_{\beta_{ij}, ij} - 1) \Gamma_{ij} + J_i \Gamma_{ij} - \Gamma_{ij} J_j + B_{ijh} = 0 \quad \text{if } \beta_{ij} = h-1\,; \tag{45}$$

observing that when these equations are solved for the elements in the Γ_{ij}'s, these elements are uniquely determined. Set $\Gamma_{ii}=0$.

With this new notation in mind we can state

Theorem II. *A differential equation in canonical form* (34) *possesses a formal independent series solution of the form*

$$Y(t)=U(t)\,\|\delta_{ij}\exp\{f_i(t)\,I_i+J_i\log t\}\|,\qquad (i,j=1,\ldots,m),\tag{46}$$

where

$$f_i(t)=\varrho_{i,h-1}\log t-\frac{\varrho_{i,h-2}}{t}-\frac{\varrho_{i,h-3}}{2t^2}-\cdots-\frac{\varrho_{i0}}{(h-1)\,t^{h-1}};$$

and

$$U=U(t)=\|\delta_{ij}(I_i+U_{ii})+(1-\delta_{ij})\,t^{h-\beta_{ij}}(\Gamma_{ij}+U_{ij})\|,\tag{47}$$

$(i,j=1,\ldots,m)$. *The* U_{ij} *in* (47) *represent formal series*

$$U_{ij}=U_{ij}(t)=\sum_{k=1}^{\infty}U_{ijk}\,t^k,\qquad (i,j=1,\ldots,m)\tag{48}$$

where the U_{ijk} *are appropriate constant matrices.*

Proof: The fact that such formal independent series solutions exist is fairly evident from the procedure outlined in section § 2. However an independent proof is given in section § 5 where essentially a method is given for computing the successive U_{ijk} in (48) as k increases. The series in (48) usually diverge; nevertheless it is to be expected that the formal solution exhibited in (46) is in fact an asymptotic series representation of a true solution of (34) if t is restricted to an appropriate sector in the complex t-plane which has the origin $t=0$ as a boundary point (see for example Trjitzinsky [10]). The validity of this asymptotic representation will not be proved here in full generality, for the chief objective of this paper is to sum as many of the divergent series (48) as possible.

If $h=0$ in (34), the origin is a regular point and, if $h=1$, the origin is a regular singular point. In either of these two cases it is well known (see G. Ehlers [11] or H. Kneser [12]) that when the series (36) converge for $|t|<t_0$, the series in (48) also converge for the same values of t and (46) is a true convergent independent matrix solution of (34). Thus there is no need to sum series (48) in these cases; hence from this point forward it is assumed that $h\geq 2$ and $m\geq 2$ in (34).

§ 4. A Related Non-homogeneous System of Differential Equations

It is evident from Theorem II that, if one is eventually to find solutions of the canonical system (34) in a rigorous fashion of the type indicated in (46) and (47), it will be necessary to obtain and solve the system of differential equations satisfied by the unknown functions $U_{ij} = U_{ij}(t)$, $(i = 1, \ldots, m)$. In order to do this substitute (46) into (34); utilize (47); cancel a few terms by virtue of the relations (44) and (45); and then after dividing through by the appropriate power of t it will be found that the functions $U_{ij}(t)$ satisfy the following non-homogeneous system (49–51) of differential equations:

$$t\frac{dU_{jj}}{dt} = J_j U_{jj} - U_{jj} J_j + \sum_{\substack{k=1\\k\neq j}}^{m} \sum_{\nu=h}^{\infty} t^{1+\nu-\beta_{kj}} B_{jk\nu}(\Gamma_{kj} + U_{kj}) + \\ + \sum_{\nu=h}^{\infty} t^{\nu+1-h} B_{jj\nu} + \sum_{\nu=h}^{\infty} t^{\nu+1-h} B_{jj\nu} U_{jj}; \tag{49}$$

also

$$t^{h-\beta_{ij}} \frac{dU_{ij}}{dt} = (\beta_{ij} - h)\, t^{h-1-\beta_{ij}} (\Gamma_{ij} + U_{ij}) + \\ + (r_{\beta_{ij},ij} + r_{\beta_{ij}+1,ij}\, t + \cdots + r_{h-1,ij}\, t^{h-1-\beta_{ij}})\, U_{ij} + \\ + t^{h-1-\beta_{ij}} (J_i [\Gamma_{ij} + U_{ij}] - [\Gamma_{ij} + U_{ij}] J_i) + \\ + \sum_{\substack{k=1\\k\neq j}}^{m} \sum_{\nu=h}^{\infty} t^{\nu-\beta_{kj}} B_{ik\nu} (\Gamma_{kj} + U_{kj}) + \sum_{\nu=h+1}^{\infty} t^{\nu-h} B_{ij\nu} + \\ + \sum_{\nu=h}^{\infty} t^{\nu-h} B_{ij\nu} U_{jj} + (r_{\beta_{ij}+1,ij}\, t + \cdots + r_{h-1,ij}\, t^{h-1-\beta_{ij}})\, \Gamma_{ij} \tag{50}$$

if $i \neq j$ and $0 \leq \beta_{ij} \leq h-2$; and

$$t\frac{dU_{ij}}{dt} = (r_{h-1,ij} - 1)\, U_{ij} + J_i U_{ij} - U_{ij} J_j + \\ + \sum_{\substack{k=1\\k\neq j}}^{m} \sum_{\nu=h}^{\infty} t^{\nu-\beta_{kj}} B_{ik\nu} (\Gamma_{kj} + U_{kj}) + \\ + \sum_{\nu=h+1}^{\infty} t^{\nu-h} B_{ij\nu} + \sum_{\nu=h}^{\infty} t^{\nu-h} B_{ij\nu} U_{jj} \tag{51}$$

if $\beta_{ij} = h - 1$; where $i, j = 1, \ldots, m$.

Since the notation is becoming cumbersome, it is desirable to make a few simplifications in the symbolism, particularly in (49–51), and yet preserve the essential features. Note first of all that system (49–51) is actually not a single system, but m

distinct systems, one system corresponding to each choice of j, $(j=1, \dots, m)$. Attention can therefore be focused on one of these particular, yet representative, systems and the second subscript j omitted. Remembering then that $U_i = U_{ij}$; $\beta_i = \beta_{ij}$; $r_{\beta_i} = r_{\beta_{ij}, ij} \neq 0$; and $r_{h-1, i} = r_{h-1, ij}$; system (49–51) can be rewritten in the more convenient form

$$t \frac{d U_j}{d t} = J_j U_j - U_j J_j + B_{j1} t U_j + \sum_{i=1}^{m} \sum_{\nu=2}^{\infty} B_{i\nu} t^{\nu} U_i + \sum_{\nu=1}^{\infty} C_{\nu} t^{\nu}; \tag{52}$$

$$t^{h-\beta_i} \frac{d U_i}{d t} = r_{\beta_i} U_i + \sum_{\nu=1}^{h-\beta_i-1} B_{i\nu} t^{\nu} U_i + \sum_{\nu=0}^{h-1} B_{j\nu} t^{\nu} U_j + \\ + \sum_{k=1}^{m} \sum_{\nu=h}^{\infty} B_{k\nu} t^{\nu-\beta_k} U_k + \sum_{\nu=1}^{\infty} C_{\nu} t^{\nu}, \tag{53}$$

where $i \neq j$ and $\beta_i \leq h-2$; and

$$t \frac{d U_i}{d t} = (r_{h-1, i} - 1) U_i + J_i U_i - U_i J_j + B_{j0} U_j + \\ + \sum_{k=1}^{m} \sum_{\nu=1}^{\infty} B_{k\nu} t^{\nu} U_k + \sum_{\nu=1}^{\infty} C_{\nu} t^{\nu} \tag{54}$$

where $i \neq j$ and $\beta_i = h-1$. In these three equations and in subsequent equations the B's and C's with one or more subscripts are known constant matrices. There is no need here to give the precise interrelationship between these matrix coefficients and those in (49–51). The one essential fact to bear in mind is that here, and throughout the remainder of the paper, *all the B and C series running in powers of t, such as $\sum_{\nu=2}^{\infty} B_{i\nu} t^{\nu}$ and $\sum_{\nu=1}^{\infty} C_{\nu} t^{\nu}$ converge for $|t| < t_0$.* However no two such series are necessarily the same series in (52–54) or in any of the succeeding formulas.

System (52–54) is the desired *related non-homogeneous system of differential equations.*

§ 5. The Decomposed System of Differential Equations

Theorem II states essentially that system (49–51) is formally satisfied by the series (48). Dropping the second subscript j, this means that system (52–54) possesses a formal series solution of the form

$$U_i = \sum_{\nu=1}^{\infty} \mathfrak{U}_{i\nu} t^{\nu} \quad (i = 1, \dots, m), \tag{55}$$

where $U_i = U_{ij}$ and $\mathfrak{U}_{i\nu} = U_{ij\nu}$. The successive values of the $\mathfrak{U}_{i\nu}$ as ν increases can be readily computed by substituting the series (55) into (52–54) and equating coefficients of like powers of t. For example when the coefficients of t are equated, one obtains from (52) the equation

$$\mathfrak{U}_{j0} = J_j \mathfrak{U}_{j0} - \mathfrak{U}_{j0} J_j + C_0; \tag{56}$$

from (53) the equation

$$r_{\beta_i} \mathfrak{U}_{i0} + B_{j0} \mathfrak{U}_{j0} + C_0 = 0, \qquad (i \neq j,\ \beta_i \leq h-2), \tag{57}$$

and from (54) the equation

$$\mathfrak{U}_{i0} = (r_{h-1,i} - 1)\,\mathfrak{U}_{i0} + J_i \mathfrak{U}_{i0} - \mathfrak{U}_{i0} J_j + B_{j0} \mathfrak{U}_{j0} + C_0, \qquad (i \neq j,\ \beta_i = h-1). \tag{58}$$

The numerical value of each element in $\mathfrak{U}_{j0}$ is readily computed from (56) and then using (57) and (58) the elements in the $\mathfrak{U}_{i0}$, $i \neq j$, are computed; each element in these matrices being uniquely determined. In a similar fashion the coefficients of t^2 are equated and the elements in $\mathfrak{U}_{i1}$, $(i = 1, \dots, m)$, evaluated; then the coefficients of t^3 are equated and the $\mathfrak{U}_{i2}$ are evaluated, and so on. Thus all the matrices $\mathfrak{U}_{i\nu}$ in (55) are determined in succession as ν increases and Theorem II is demonstrated. From this point forward therefore the matrices $\mathfrak{U}_{i\nu}$ and $U_{ij\nu}$ are to be treated as known quantities. More details about the values of the elements in $\mathfrak{U}_{i\nu}$ are given in section § 6.

An estimate as to the rate of growth of the elements in the matrices $\mathfrak{U}_{i\nu}$ is needed as $\nu \to \infty$. To obtain such an estimate it is expedient to first split the related non-homogeneous system of differential equations into a new decomposed non-homogeneous system of equations consisting in general of a larger number of differential equations of simpler, but equivalent, structure. It is the object of this section to obtain such a decomposed system. However difficulties appear later on when estimating rates of growth of solutions of certain related integral equations. To avoid these future difficulties the following assumption is made:

(59) **Restrictive Hypothesis:** *In the canonical form* (34) *either* $h = 2$, *or, if* $h > 2$, *then for the chosen value of* j *under consideration all the corresponding* $\beta_i = \beta_{ij} = 0$ $(i = 1, \dots, m)$, *in system* (52–54).

This hypothesis dominates the remainder of this paper and because of it certain simplifications occur which would not be valid in the general case. Note that if $h > 2$, and $m > 3$, there may not always be a value of j on the range 1 to m which will satisfy hypothesis (59).

With this restrictive hypothesis in mind let us return to the problem of splitting the related non-homogeneous system of differential equations and begin by breaking up the functions

$$U_i(t) = \sum_{\eta=1}^{\infty} \mathfrak{U}_{i\eta} t^{\eta} \qquad (i = 1, \ldots, m)$$

into the sum of

$$r = h - 1$$

distinct new functions

$$T_{ik} = T_{ik}(t)$$

by writing

$$U_i(t) = \sum_{k=1}^{r} \{t^k \mathfrak{U}_{ik} + t^k T_{ik}\} \tag{60}$$

where the $\mathfrak{U}_{ik}$ are known constant matrices and formally

$$T_{ik}(t) = \sum_{\eta=1}^{\infty} \mathfrak{U}_{i,\eta r+k} t^{\eta r}, \qquad (k = 1, \ldots, r). \tag{61}$$

To obtain the differential system satisfied by the new functions $T_{ik}(t)$, begin by substituting the expression for U_i given in (60) into (52). Since the $\mathfrak{U}_{ik}$ have been chosen so as to satisfy equations (56–57), as well as all similar equations which arise when coefficients of higher powers of t are equated, it is evident the coefficients of $t, t^2, \ldots,$ and t^r all cancel out and (52) is reduced to the form

$$\begin{aligned}\sum_{k=1}^{r} \left\{k t^k T_{jk} + t^{k+1} \frac{d T_{jk}}{dt}\right\} &= J_j \sum_{k=1}^{r} t^k T_{jk} - \sum_{k=1}^{r} t^k T_{jk} J_j + \\ &+ B_{j1} \sum_{k=1}^{r} t^{k+1} T_{jk} + \sum_{i=1}^{m} \sum_{k=1}^{r} \sum_{\nu=2}^{\infty} B_{i\nu} t^{\nu+k} T_{ik} + \sum_{\nu=r+1}^{\infty} C_\nu t^\nu.\end{aligned} \tag{62}$$

The next step is to split (62) into r separate equations by retaining in any particular one of these equations only those terms which involve powers of t that are equal modulo r, treating all the T_{ik}'s as though they were constant matrices and all the products $t \dfrac{d T_{ik}}{dt}$'s as though they too were constant matrices. When this is done (62) is decomposed into r separate equations, namely

$$\begin{aligned}t^{\omega+1} \frac{d T_{j\omega}}{dt} + w t^{\omega} T_{j\omega} &= J_j t^{\omega} T_{j\omega} - t^{\omega} T_{j\omega} J_j + \sum_{\eta=1}^{\infty} C_\eta t^{\omega+\eta r} + \\ &+ B_{j1} t^{\omega+\delta r} T_{j,\omega-1+\delta r} + \sum_{i=1}^{m} \sum_{k=1}^{\omega-2+\delta r} \sum_{\eta=\delta}^{\infty} B_{ik\eta} t^{\omega+\eta r} T_{ik} + \\ &+ \sum_{i=1}^{m} \sum_{k=\omega-1+\delta r}^{r} \sum_{\eta=1+\delta}^{\infty} B_{ik\eta} t^{\omega+\eta r} T_{ik}\end{aligned} \tag{63}$$

where $\omega = 1, \ldots, r$. The new symbol

$$\delta = \begin{cases} 1 & \text{if } \omega = 1 \text{ and} \\ 0 & \text{if } \omega > 0 \end{cases}$$

in (63) and in subsequent formulas.

In these sums, as well as in succeeding sums in subsequent formulas, it may be found that the upper limit for the range of summation is less than that for the lower limit in a particular sum; *in all such cases the particular sum concerned is to be omitted* in the formula under consideration. For example, if in the first triple sum in (63) the $\omega = 2$, the entire sum

$$\sum_{i=1}^{m} \sum_{k=1}^{\omega-2+\delta r} \sum_{\eta=\delta}^{\infty} B_{ik\eta} t^{\omega+\eta r} T_{ik}$$

is to be omitted from formula (63).

Divide each side of equation (63) by t^{ω} and change the independent variable to

$$s = t^{-r} \tag{64}$$

throwing the irregular singular point at the origin of the complex t-plane out to infinity in the complex s-plane. When these steps have been taken (63) takes the form

$$\begin{aligned} \omega T_{j\omega} - rs\frac{dT_{j\omega}}{ds} &= J_j T_{j\omega} - T_{j\omega} J_j + \sum_{\eta=1}^{\infty} C_{\eta} s^{-\eta} + \\ &+ B_{j1} s^{-\delta r} T_{j,\omega-1+\delta r} + \sum_{i=1}^{m} \sum_{k=1}^{\omega-2+\delta r} \sum_{\eta=\delta}^{\infty} B_{ik\eta} s^{-\eta} T_{ik} + \\ &+ \sum_{i=1}^{m} \sum_{k=\omega-1+r\delta}^{r} \sum_{\eta=1+\delta}^{\infty} B_{ik\eta} s^{-\eta} T_{ik} \end{aligned} \tag{65}$$

where $\omega = 1, \ldots, r$. In (65) and subsequent formulas it is to be emphasized that all the B and C series running in powers of $1/s$ converge for $|s|$ sufficiently large, say $|s| > s_0$. Equation (65) is the first of the three equations which make up the desired decomposed system.

To get the 2nd equation in the decomposed system, keep the restrictive hypothesis (59) in mind and split equation (53) into r separate equations similar to (65). To do this the first step is to substitute the right-hand member of (60) into (53) in place of each U_i. Again the coefficients of $t, t^2, \ldots, t^r$ all cancel out. Then the resulting equation is split into r separate equations by retaining in any particular one of these equations only those terms which involve powers of t that are equal modulo r, treating all the T_{ik}'s as though they were constants and all the $t\dfrac{d\,T_{ik}}{dt}$'s as though they also were constants. Then divide t^{ω} out of each equation resulting from the split and again change the independent variable to $s=t^{-r}$ with the result that

$$
\begin{aligned}
(66)\qquad \omega\, s^{-1}\, T_{i\omega} - r\frac{d\,T_{i\omega}}{d\,s} &= r_{\beta_i}\, T_{i\omega} + \sum_{k=1}^{w} B_{j,\omega-k}\, T_{jk} + \\
&+ \sum_{k=\omega}^{r} B_{j,\omega+r-k}\, s^{-1}\, T_{jk} + \sum_{k=1}^{w-1} B_{i,\omega-k}\, T_{ik} + \sum_{k=\omega}^{r} B_{i,\omega+r-k}\, s^{-1}\, T_{ik} + \\
&+ \sum_{k=1}^{m}\sum_{\mu=1}^{r}\sum_{\nu=1}^{\infty} B_{k\mu\nu}\, s^{-\nu}\, T_{k\mu} + \sum_{\nu=1}^{\infty} C_{\nu}\, s^{-\nu}
\end{aligned}
$$

where $\omega=1,\ldots,r$; $i\neq j$; and $\beta_i=0$.

Under the restrictive hypothesis (59), equations of type (54) are present only if $h=2$, and in this event $r=1$; ω equals only 1; $t=s^{-1}$; and, in terms of the new independent variable s, equation (54) becomes

$$
\begin{aligned}
(67)\qquad T_{i1} - s\frac{d\,T_{i1}}{d\,s} &= (r_{h-1,i}-1)\, T_{i1} + J_i\, T_{i1} - T_{i1}\, J_j + B_{j0}\, T_{j1} + \\
&+ \sum_{k=1}^{m}\sum_{\nu=1}^{\infty} B_{k\nu}\, s^{-\nu}\, T_{k1} + \sum_{\nu=1}^{\infty} C_{\nu}\, s^{-\nu}
\end{aligned}
$$

where $i\neq j$ and $\beta_i=1$.

Equations (65), (66), and (67) make up the *decomposed system of differential equations*. These equations are in a suitable form for estimating the growth of the coefficients $T_{ik\eta}$ in the formal expansions

$$
(68)\qquad T_{ik} = \sum_{\eta=1}^{\infty} T_{ik\eta}\, s^{-\eta}
$$

where

$$
T_{ik\eta} = \mathfrak{U}_{i,\eta r+k}, \qquad (i=1,\ldots,m;\ k=1,\ldots,r;\ \eta=1,2,\ldots);
$$

see equation (61).

§ 6. Rate of Growth of the Coefficients $T_{ik\eta}$ as $\eta\to\infty$

Substitute the series (68) into (65–67) and equate coefficients of successively higher powers of $1/s$. When this has been done, it is found from (65) that for all sufficiently large n,

$$(69)\qquad (rn+\omega)\,T_{j\,\omega\,n} = J_j\,T_{j\,\omega\,n} - T_{j\,\omega\,n}\,J_j + C_n + B_{j1}\,T_{j,\omega-1+r\delta,\,n-\delta} + \\ + \sum_{i=1}^{m}\sum_{k=1}^{\omega-2+r\delta}\sum_{\nu=1}^{n-\delta} B_{ik,\,n-\nu}\,T_{i\,k\,\nu} + \\ + \sum_{i=1}^{m}\sum_{k=\omega-1+r\delta}^{r}\sum_{\nu=1}^{n-1-\delta} B_{ik,\,n-\nu}\,T_{i\,k\,\nu}$$

where $\omega = 1, \ldots, r$.

Similarly from (66) for all sufficiently large n,

$$(70)\qquad r_{\beta_i}\,T_{i\,\omega n} = w\,T_{i\,\omega,\,n-1} + r\,(n-1)\,T_{i\,\omega,\,n-1} - \sum_{k=1}^{\omega} B_{j,\,\omega-k}\,T_{j\,k\,n} - \\ - \sum_{k=\omega}^{r} B_{j,\,\omega+r-k}\,T_{j\,k,\,n-1} - \sum_{k=1}^{w-1} B_{i,\,\omega-k}\,T_{i\,k\,n} - \\ - \sum_{k=\omega}^{r} B_{i,\,\omega+r-k}\,T_{i\,k,\,n-1} - \\ - \sum_{k=1}^{m}\sum_{\mu=1}^{r}\sum_{\nu=1}^{n-1} B_{k\mu,\,n-\nu}\,T_{k\,\mu\,\nu} + C_n$$

where $\omega = 1, \ldots, r$; $i \neq j$; and $\beta_i = 0$.

Likewise from (67) for all sufficiently large n

$$(71)\qquad (n+2-r_{h-1,\,i})\,T_{i\,1\,n} = J_i\,T_{i\,1\,n} - T_{i\,1\,n}\,J_j + B_{j0}\,T_{j\,1\,n} + \\ + \sum_{k=1}^{m}\sum_{\eta=1}^{n-1} B_{k,\,n-\eta}\,T_{k\,1\,\eta} + C_n$$

where $\beta_i = h-1$, $r = 1$, and all $\omega = 1$.

If all the $T_{j\,\omega\,k}$ and $T_{i\,\omega\,k}$ are known for $k = 1, \ldots, n-1$ and n is sufficiently large, it is clear from system (69–71) that one can first calculate $T_{j\,1\,n}$; next calculate all the $T_{i\,1\,n}$; after that $T_{j\,2\,n}$, then all the $T_{i\,2\,n}$, and so on in succession up to and including $T_{i\,r\,n}$.

To estimate the rate of growth of the elements of the $T_{i\,\omega\,n}$ matrices as $n \to \infty$ we shall introduce a system of equations similar to (67–71) which determines the successive values of certain *dominating* matrices $W_{i\,\omega\,n}$ as $n \to \infty$. The new system is so selected that the rate of growth of the elements in the $W_{i\,\omega\,n}$-matrices is relatively easy to estimate and yet the $W_{i\,\omega\,n}$ grow fast enough so that every element in every matrix $W_{i\,\omega\,n}$ will be positive and at least as large or larger than the absolute value of the corresponding element in matrix $T_{i\,\omega\,n}$.

Since all the B and C series in (65–67) have the typical power series forms $\sum_{\nu=0}^{\infty} B_\nu s^{-\nu}$ and $\sum_{\nu=0}^{\infty} C_\nu s^{-\nu}$ with possibly more subscripts attached to the B's and C's and since these series all simultaneously converge for $|s|>s_0$ it follows that there exist two sufficiently large positive constants θ and ζ such that any coefficient B_ν or C_ν is *dominated* by $\zeta^\nu \Theta$ where Θ is respectively either a matrix with the same number of rows and columns as B_ν or a vector with the same number of elements, i.e. components, as vector C_ν and each element in Θ is the constant θ. By "dominated" we mean that every element in all the various B_ν's and C_ν's is less in absolute value than the corresponding element $\theta \zeta^\nu$ in $\zeta^\nu \Theta$. If a particular matrix, say $B_{ik\nu}$, carries several subscripts the corresponding dominating Θ-matrix, say Θ_{ik}, will carry one less subscript.

With this notation in mind the desired dominant system can now be written. Equation (69) is to be compared with

$$\begin{aligned}(72)\qquad (rn+\omega)\, W_{j\,\omega n} &= J_j W_{j\,\omega n} + W_{j\,\omega n} J_j + \zeta^n \Theta + \zeta\, \Theta_j W_{j,\omega-1+r\delta,n-\delta} + \\ &+ \sum_{i=1}^{m} \sum_{k=1}^{\omega-2+r\delta} \sum_{\nu=1}^{n-\delta} \zeta^{n-\nu} \Theta_{ik} W_{ik\nu} + \\ &+ \sum_{i=1}^{m} \sum_{k=\omega-1+r\delta}^{r} \sum_{\nu=1}^{n-1-\delta} \zeta^{n-\nu} \Theta_{ik} W_{ik\nu}\end{aligned}$$

where $\omega=1,\dots,r$.

Similarly (70) is to be compared with

$$\begin{aligned}(73)\qquad |r_{\beta_i}|\, W_{i\,\omega n} &= \omega W_{i\,\omega,n-1} + r(n-1)\, W_{i\,\omega,n-1} + \sum_{k=1}^{\omega} \zeta^{\omega-k} \Theta_j W_{jkn} + \\ &+ \sum_{k=\omega}^{r} \zeta^{\omega+r-k} \Theta_i W_{ik,n-1} + \sum_{k=1}^{\omega-1} \zeta^{\omega-k} \Theta_i W_{ikn} + \\ &+ \sum_{k=\omega}^{r} \zeta^{\omega+r-k} \Theta_i W_{ik,n-1} + \sum_{k=1}^{m} \sum_{\mu=1}^{r} \sum_{\nu=1}^{n-1} \zeta^{n-\nu} \Theta_{k\mu} W_{k\mu\nu} + \\ &+ \zeta^n \Theta\end{aligned}$$

where $\omega=1,\dots,r$; $i \neq j$; and $\beta_i=0$.

Likewise (71) is to be compared with

$$\begin{aligned}(74)\qquad (n+2-r_{h-1,i})\, W_{i1n} &= J_i W_{i1n} + W_{i1n} J_j + \Theta_j W_{j1n} + \\ &+ \sum_{k=1}^{m} \sum_{\eta=1}^{n-1} \zeta^{n-\eta} \Theta_k W_{k1\eta} + \zeta^n \Theta\end{aligned}$$

where $\beta_i=h-1$; $i \neq j$; $r=1$, and all $\omega=1$.

The dominate system (72–74) determines the rate of growth of the matrices $W_{i\,w\,n}$ as $n \to \infty$, but to estimate this rate of growth it is helpful to replace system (72–74) by an equivalent homogeneous recurrent system. To get the first of these recurrent relations, replace n by $(n+1)$ in (72) and from the new resulting equation subtract ζ times (72) without stepping up the n a unit. The result of such a subtraction is that

$$\begin{aligned}(75)\qquad (n\,r + r + \omega)\, W_{j\,\omega,\,n+1} = {} & \zeta\,(n\,r + \omega)\, W_{j\,\omega\,n} + J_j\,(W_{j\,\omega,\,n+1} - \zeta\, W_{j\,\omega\,n}) + \\ & + (W_{j\,\omega,\,n+1} - \zeta\, W_{j\,\omega\,n})\, J_j + \zeta\,\Theta_j\,(W_{j,\,\omega-1+r\delta,\,n+1-\delta} - \zeta\, W_{j,\,\omega-1+r\delta,\,n-\delta}) + \\ & + \sum_{i=1}^{m} \sum_{k=1}^{\omega-2+r\delta} \zeta^{\delta}\,\Theta_{i\,k}\, W_{i\,k,\,n+1-\delta} \\ & + \sum_{i=1}^{m} \sum_{k=\omega-1+r\delta}^{r} \zeta^{1+\delta}\,\Theta_{i\,k}\, W_{i\,k,\,n-\delta}\end{aligned}$$

where $\omega = 1, \ldots, r$.

Similarly, if n is replaced by $(n+1)$ in (73) and from the new resulting equation ζ times (73) is subtracted, the result is that

$$\begin{aligned}(76)\qquad |r_{\beta_i}|\, W_{i\,\omega,\,n+1} = {} & \zeta\,|r_{\beta_i}|\, W_{i\,\omega\,n} + (\omega - r)\,(W_{i\,\omega\,n} - \zeta\, W_{i\,\omega,\,n-1}) + \\ & + r\,(n+1)\, W_{i\,\omega\,n} - \zeta\, r\, n\, W_{i\,\omega,\,n-1} + \sum_{k=1}^{\omega} \zeta^{\omega-k}\,\Theta_j\,(W_{j\,k,\,n+1} - \zeta\, W_{j\,k\,n}) + \\ & + \sum_{k=\omega}^{r} \zeta^{\omega+r-k}\,\Theta_j\,(W_{j\,k\,n} - \zeta\, W_{j\,k,\,n-1}) + \sum_{k=1}^{w-1} \zeta^{\omega-k}\,\Theta_i\,(W_{i\,k,\,n+1} - \zeta\, W_{i\,k\,n}) + \\ & + \sum_{k=\omega}^{r} \zeta^{\omega+r-k}\,\Theta_i\,(W_{i\,k\,n} - \zeta\, W_{i\,k,\,n-1}) + \sum_{k=1}^{m} \sum_{\mu=1}^{r} \zeta\,\Theta_{k\,\mu}\, W_{k\,\mu\,n}\end{aligned}$$

where $\omega = 1, \ldots, r$; $i \neq j$; and $\beta_i = 0$.

In a like fashion (74) is replaced by

$$\begin{aligned}(77)\qquad (n + 3 - r_{h-1,\,i})\, W_{i\,1,\,n+1} = {} & \zeta\,(n + 2 - r_{h-1,\,i})\, W_{i\,1\,n} + \\ & + J_i\,(W_{i\,1,\,n+1} - \zeta\, W_{i\,1\,n}) + (W_{i\,1,\,n+1} - \zeta\, W_{i\,1\,n})\, J_j + \\ & + \Theta\,(W_{j\,1,\,n+1} - W_{j\,1\,n}) + \sum_{k=1}^{m} \zeta\,\Theta_k\, W_{k\,1\,n}\end{aligned}$$

where $\beta_i = 1$; $i \neq j$; $r = 1$; and all $\omega = 1$.

The recurrence relations (75–77) are a system of simultaneous homogeneous linear difference equations satisfied by the $W_{i\,\omega\,n}$. The presence of the factor $(n+1)$ in the third term of the right member of equation (76) suggests the substitution

$$\begin{cases} W_{i\,\omega\,n} = (n-1)!\,\mathfrak{W}_{i\,\omega\,n} & \text{if } i \neq j \text{ and } \beta_i = 0; \\ W_{j\,\omega\,n} = (n-2)!\,\mathfrak{W}_{j\,\omega\,n}; & \text{and} \\ W_{i\,\omega\,n} = (n-3)!\,\mathfrak{W}_{i\,\omega\,n} & \text{if } i \neq j \text{ and } \beta_i = 1. \end{cases}$$

After this substitution has been made in (75–77), divide the first two equations by $n!$ and the third by $(n-1)!$ and it will be found that the

$$\mathfrak{W}_{i\,\omega\,n}, \qquad (i = 1, \ldots, m;\ \omega = 1, \ldots, n),$$

satisfy a system of linear homogeneous difference equations similar in structure to (75-77) with the special feature that, although in general the coefficients vary with n, nevertheless as $n \to \infty$ all the coefficients *uniformly* approach constant values. More specifically the system of difference equations for the $\mathfrak{W}_{i\,w\,n}$ is equivalent to a certain matrix difference equation of the form

$$\mathfrak{W}(n+1) = A(n)\,\mathfrak{W}(n) \tag{78}$$

where $A(n)$ uniformly approaches a constant matrix $A(\infty)$ as $n \to \infty$. For large n, difference equation (78) is approximated by the matrix equation

$$\mathfrak{W}_1(n+1) = A(\infty)\,\mathfrak{W}_1(n) \tag{79}$$

with constant coefficients. Any solution of such a difference equation as (79) cannot grow, as is well known, with more than exponential rapidity as $n \to \infty$. The same restriction on the rate of growth of solutions applies with equal force to system (78) and to the system for the $\mathfrak{W}_{i\,\omega\,n}$. This means there exist two positive constants c and q such that every element in matrix $\mathfrak{W}_{i\,\omega\,n}$ is less in absolute value than $c\,e^{q\,n}$ for $i = 1, \ldots, m$; $\omega = 1, \ldots, r$; and $n = 1, 2, \ldots$. This in turn implies that the absolute values of all the elements in $T_{i\,\omega\,n}$ are less than

$$(n-1)!\,c\,e^{q\,n} \qquad \text{for } i = 1, \ldots, m;\ \omega = 1, \ldots r;\ \text{and } n = 1, 2, \ldots.$$

§ 7. A Related System of Integral Equations

A simultaneous system of related integral equations will be introduced in this section. The system will be so constructed that, if its *formal* Laplace transform is taken, one obtains the decomposed system of differential equations (65–67). For this purpose let the new functions $V_{i\,\omega}(t)$ $(i = 1, \ldots, m;\ \omega = 1, \ldots, r)$ formally satisfy the equation

$$T_{i\,\omega}(t) = T_{i\,\omega}(s^{-1/r}) = \int_0^\infty e^{-s\,t}\, V_{i\,\omega}(t)\,d\,t. \tag{80}$$

Then the particular integral equation which has a formal Laplace transform equal to (65) must necessarily be of the form

$$r\,t\,V_{j\,\omega}(t) = \Lambda_{j\,\omega}(t) + \sum_{\nu=1}^{\infty} C_\nu\, t^\nu/\nu\,! \tag{81}$$

where the integral

$$\begin{aligned}\Lambda_{j\,\omega}(t) = \int_0^t [J_j\, V_{j\,\omega}(\tau) - V_{j\,\omega}(\tau)\, J_j - \omega\, V_{j\,w}(\tau) + \\ + B_{j\,1}(t-\tau)^{\delta r}\, V_{j,\,\omega-1+\delta r}(\tau)/(\delta\, r)\,! + \\ + \sum_{i=1}^{m} \sum_{k=1}^{\omega-2+\delta r} \sum_{\eta=\delta}^{\infty} B_{i\,k\,\eta}(t-\tau)^\eta\, V_{i\,k}(\tau)/\eta\,! + \\ + \sum_{i=1}^{m} \sum_{k=\omega-1+r\delta}^{r} \sum_{\eta=1+\delta}^{\infty} B_{i\,k\,\eta}(t-\tau)^\eta\, V_{i\,k}(\tau)/\eta\,!]\,d\,\tau\end{aligned}$$

and $\omega = 1, \ldots, r$.

Similarly the integral equation corresponding to (66) is

$$\begin{aligned}(r\,t - r_{\beta_i})\, V_{i\,\omega}(t) = \Lambda_{i\,\omega}(t) + \sum_{\nu=1}^{\infty} C_\nu\, t^{\nu-1}/(\nu-1)\,! + \\ + \sum_{k=1}^{\omega} B_{j,\,\omega-k}\, V_{j\,k}(t) + \sum_{k=1}^{w-1} B_{i,\,\omega-k}\, V_{i\,k}(t)\end{aligned} \tag{82}$$

where the integral

$$\begin{aligned}\Lambda_{i\,\omega}(t) = \int_0^t \Bigg[\sum_{k=\omega}^{r} B_{j,\,\omega+r-k}\, V_{j\,k}(\tau) - \omega\, V_{i\,\omega}(\tau) + \\ + \sum_{k=\omega}^{r} B_{i,\,\omega+r-k}\, V_{i\,k}(\tau) + \\ + \sum_{k=1}^{m} \sum_{\mu=1}^{r} \sum_{\nu=1}^{\infty} B_{k\,\mu\,r}(t-\tau)^{\nu-1}\, V_{k\,\mu}(\tau)/(\nu-1)\,!\Bigg]\,d\,\tau\end{aligned}$$

and $\omega = 1, \ldots, r$; and $\beta_i = 0$.

Likewise the integral equation corresponding to (67) is

$$t\,V_{i\,1}(t) = \Lambda_i(t) + \sum_{\nu=1}^{\infty} C_\nu\, t^\nu/\nu\,! \tag{83}$$

where the integral

$$\Lambda_i(t) = \int_0^t [(r_{h-1,i} - 2) V_{i1}(\tau) + J_i V_{i1}(\tau) - V_{i1}(\tau) J_j + B_{j0} V_{j1}(\tau) +$$

$$+ \sum_{k=1}^{m} \sum_{\nu=1}^{\infty} B_{k\nu} (t-\tau)^\nu V_{k1}(\tau)/\nu!] \, d\tau$$

and $i \neq j$ and $\beta_i = 1$.

Equations (81–83) make up the desired system of *related integral equations.*

Since the absolute values of the elements in the B matrices and C vectors are bounded as described in section § 6 all the infinite series appearing in this system of related integral equations represent entire functions convergent for all values of t and τ because of the presence of the factorials in the denominators.

This system of integral equations may then be considered in its own right, regardless of the particular way it has been derived. Formally this system is satisfied by the series

$$V_{i\,\omega}(t) = \sum_{\eta=1}^{\infty} T_{i\,\omega\eta}\, t^{\eta-1}/(\eta-1)! \tag{84}$$

see (61), (64), and (80). But the absolute value of each of the elements in the matrix $T_{i\,\omega\eta}$ is less than $(n-1)!\,c\,e^{q\,n}$, and therefore *the series in* (84) *converge if* $|t| < e^{-q}$ *for* $i = 1, \ldots, m$ *and* $\omega = 1, \ldots, r$ *and thus define in a rigorous fashion functions* $V_{i\,\omega}(t)$ *which are solutions of the system of related integral equations.*

It is clear that by successive substitutions the system (81–83) can be rewritten in the more convenient form

$$t\, V_{j\,\omega}(t) = \mathfrak{J}_{j\,\omega}(t) + \sum_{\nu=1}^{\infty} C_\nu t^\nu/\nu! \tag{85}$$

$$t\,(r\,t - r_{\beta_i})^\omega\, V_{i\,\omega}(t) = \sum_{\nu=0}^{\omega} t^\nu \mathfrak{J}_{i\,\omega\nu}(t) + \sum_{\nu=1}^{\infty} C_\nu t^\nu/\nu! \tag{86}$$

where $i \neq j$ and $\beta_i = 0$; and

$$t\, V_{i1}(t) = \mathfrak{J}_{i1}(t) + \sum_{\nu=1}^{\infty} C_\nu t^\nu/\nu! \tag{87}$$

where $i \neq j$ and $\beta_i = 1$. Here each of the $\mathfrak{J}_{i\,\omega}(t)$, $(i = 1, \ldots, m)$, and $\mathfrak{J}_{i\,\omega\nu}$'s are integrals of the form

$$\mathfrak{J}(t) = \int_0^t \sum_{k=1,}^{m} \sum_{\eta=1}^{r} \sum_{\nu=0}^{\infty} B_{k\eta\nu} V_{k\eta}(\tau) [(t-\tau)^\nu/\nu!] \, d\tau$$

where each series

$$\sum_{\nu=0}^{\infty} B_{k\eta\nu}(t-\tau)^{\nu}/\nu!$$

is convergent and dominated by the series

$$\sum_{\nu=0}^{\infty} \zeta^{\nu}\,\Theta_{k\eta}\,|t-\tau|^{\nu}/\nu! = \Theta_{k\eta}\,e^{\zeta|t-\tau|}. \tag{88}$$

Likewise each series $\sum_{\nu=0}^{\infty} C_{\nu} t^{\nu}/\nu!$ is convergent and dominated by the series

$$\sum_{\nu=0}^{\infty} \zeta^{\nu}\,\Theta\,|t|^{\nu}/\nu! = \Theta\,e^{\zeta|t|}. \tag{89}$$

Admittedly the positive constants ζ and θ used here may be larger than the ζ and θ used in section § 6.

§ 8. Rate of Growth of the $V_{i\omega}(t)$ as $t\to\infty$

In order to estimate the rate of growth of the elements of the matrix $V_{i\omega}(t)$ as $t\to\infty$ begin by marking the points $t = r_{\beta_i}/r$ for $i = 1, \ldots, m$ in the complex t-plane and then draw the rays which radiate from the origin $t = 0$ and pass through these marked points. Cover each of the distinct rays by a sector of very small angular opening. Between any two of these successive covering sectors there will be a relatively larger sector, say $\mathfrak{S}$. Each of these $\mathfrak{S}$-sectors is to be considered closed in the sense that points falling on the two rays which form the edges of $\mathfrak{S}$ are to be counted as part of $\mathfrak{S}$.

Let the variable t be restricted temporarily to some particular one of these sectors, say to $\mathfrak{S}$; and let it be understood that the path of integration for each of the integrals in (85–87) runs out radially from the origin to the point t in $\mathfrak{S}$. The various $V_{i\omega}(t)$ can then be analytically continued indefinitely out along the rays in $\mathfrak{S}$ as is evident from the structure of the integral equations. Let the norm $\| V_{i\omega}(t) \|$ of the matrix $V_{i\omega}(t)$ be the largest of the absolute values of the various elements in the matrix for the given value of t.

Lemma 1. *The matrices $V_{i\omega}(t)$ satisfying the integral equations* (85–87) *and defined in the neighborhood of the origin by the series* (84) *satisfy the inequalities*

$$\| V_{i\omega}(t) \| < c\,e^{pt}, \qquad (i = 1, \ldots, m;\ \omega = 1, \ldots, r),$$

along every ray in each $\mathfrak{S}$-sector if the positive constants c and p are chosen sufficiently large.

To show the existence of two such constants c and p, select a positive constant $t_0 < e^{-q}$ and then there will obviously exist a constant c such that

$$\| V_{i\omega}(t) \| < c \quad \text{and} \quad \| V_{i\omega}(t) \| < c\, e^{q|t|}, \qquad (i = 1, \ldots, m;\ \omega = 1, \ldots, r),$$

for all $|t| \leq t_0$ and all $q \geq 0$.

Suppose that the lemma is false. Then there will exist a positive constant $t_1 = t_1(p) > t_0$ such that

$$\| V_{i\omega}(t) \| < c\, e^{p|t|}, \qquad (i = 1, \ldots, m;\ \omega = 1, \ldots, r), \tag{90}$$

for all $|t| < t_1$ along every ray in $\mathfrak{S}$, while for some point $t' = t_1 e^{i\Phi}$ in $\mathfrak{S}$

$$\| V_{j\omega}(t') \| = c\, e^{p t_1}, \qquad (|t'| = t_1),$$

for at least one choice of values for i and ω, say $i = i'$, $\omega = \omega'$. Let the integration be along the ray running from the origin out to and through t'.

If by chance i' is equal to a value of i such that $i \neq j$ and $\beta_i = 0$, then from (86), (90), and (89)

$$t_1 |r t' - r_{\beta_i}|^{\omega'} \| V_{i'\omega'}(t') \| = t_1 |r t' - r_{\beta_i}|^{\omega'} c\, e^{p t_1} < \theta\, e^{\zeta |t_1|} + \tag{91}$$
$$+ \sum_{\nu=0}^{\omega'} t_1^{\nu} \sum_{k=1}^{m} \sum_{\eta=1}^{r} \int_0^{t_1} \lambda \left\| \sum_{\mu=0}^{\infty} B_{\nu k \eta \mu} (t-\tau)^{\mu}/\mu! \right\| c\, e^{p|\tau|}\, d|\tau|$$

where λ is the maximum number of elements to be found in the columns of the various matrices $V_{i\omega}(t)$ under consideration.

But the

$$\sum_{k=1}^{m} \sum_{\eta=1}^{r} \int_0^{t_0} \lambda \left\| \sum_{\mu=0}^{\infty} B_{\nu k \eta \mu} (t-\tau)^{\mu}/\mu! \right\| c\, e^{p|\tau|}\, d|\tau| < m r \lambda \sigma c\, e^{p t_0} \tag{92}$$

where σ is the maximum value of all the norms

$$\left\| \sum_{\mu=0}^{\infty} B_{\nu k \eta \mu} (t-\tau)^{\mu}/\mu! \right\|$$

in the region $|t - \tau| \leq t_0$ for $\nu = 1, \ldots, \omega'$; $k = 1, \ldots, m$; $\eta = 1, \ldots, r$. Hence dividing (91) by $t_1 |r t' - r_{\beta_i}|^{\omega'} c\, e^{p t_1}$ and utilizing (88) and inequalities similar to (92)

$$1 < \sum_{\nu=0}^{\omega'} \frac{t_1^{\nu-1}\, m\, r\, \lambda\, \sigma\, e^{p(t_0-t_1)}}{t_1^{\omega'}\, |r - r_{\beta_i}/t'|^{\omega'}} +$$

$$+ \sum_{\nu=0}^{\omega'} t_1^{\nu-1} \sum_{\kappa=1}^{m} \sum_{\eta=1}^{r} \frac{\lambda\, \theta \int_{t_0}^{t_1} e^{(t_1-|\tau|)(\zeta-p)}\, d\,|\tau|}{t_1^{\omega'}\, |r - r_{\beta_i}/t'|^{\omega'}} +$$

$$+ \frac{\theta\, e^{|t_1|(\zeta-p)}}{c\, t_1^{\omega'+1}\, |r - r_{\beta_i}/t'|^{\omega'}}.$$

Taking $p > \zeta$ and observing that

$$\int_{t_0}^{t_1} e^{(t_1-|\tau|)(\zeta-p)}\, d\,|\tau| = \int_0^{t_1-t_0} e^{x(\zeta-p)}\, d\,x < \int_0^{\infty} e^{x(\zeta-p)}\, d\,x = 1/(p-\zeta),$$

it follows that

$$1 < \sum_{\nu=0}^{\omega'} \frac{t_1^{\nu-1}\, m\, r\, \lambda\, \sigma\, e^{p(t_0-t_1)}}{t_1^{\omega'}\, |r - r_{\beta_i}/t_1'|^{\omega'}} + \sum_{\nu=0}^{\omega'} \frac{t_1^{\nu-1}\, \lambda\, \theta\, m\, r}{t_1^{\omega'}\, |r - r_{\beta_i}/t_1'|^{\omega'}\, (p-\zeta)} + \tag{93}$$

$$+ \frac{\theta\, e^{|t_1|(\zeta-p)}}{c\, t_1^{\omega'+1}\, |r - r_{\beta_i}/t'|^{\omega'}}.$$

Noting that all the $|r - r_{\beta_i}/t'|$ for $t_1 > t_0$ are uniformly bounded away from zero in a $\mathfrak{S}$-sector, it is clear from (93) that if p is chosen large enough the inequality is an absurdity for the right member will be less than 1.

Similarly, if either $i' = j$, or if simultaneously $i' \neq j$, $\beta_i = 1$, and $h = 2$, then an absurdity can again be reached by a chain of inequalities quite like those just given. Thus in every case it is evident Lemma 1 must be correct in order to avoid these absurdities.

The analysis at this stage is paralleling Trjitzinsky's work [2] so closely that the details from this point forward can be omitted and the results of the analysis merely stated.

The formal Laplace operator indicated in (80) can now be put on a rigorous basis. Select some ray in sector $\mathfrak{S}$ where the arg $t = \Phi$ and then in evaluating all Laplace integrals, such as

$$T(s) = \int_0^{\infty} e^{-st}\, V(t)\, dt$$

integrate from $t = 0$ to $t = \infty$ along this Φ-ray. Limit the complex variable $s = |s|\, e^{i\sigma}$ to the half-plane $H(\Phi)$ defined by the inequality

$$R(e^{i\Phi} s) = |s| \cos(\Phi + \alpha) > p'$$

where $p' = p + \varepsilon$, $\varepsilon > 0$ and arbitrary. With this agreement the integrals

$$\int_0^\infty e^{-st} V_{i\omega}(t)\, dt$$

all converge absolutely in $H(\Phi)$ and there define analytic functions

$$T_{i\omega} = T_{i\omega}(s^{1/r}) = \int_0^\infty e^{-st} V_{i\omega}(t)\, dt. \tag{94}$$

Moreover in $H(\Phi)$

$$\int_0^\infty e^{-st}\, t\, V_{i\omega}(t)\, dt = -\frac{dT_{i\omega}}{ds};$$

$$\int_0^\infty e^{-st} \sum_{\nu=1}^\infty [C_\nu t^\nu/\nu!]\, dt = \sum_{\nu=1}^\infty C_\nu/s^{\nu+1};$$

and for all of our B series

$$\int_0^\infty e^{-st} \int_0^t \sum_{\nu=0}^\infty [B_\nu (t-\tau)^\nu V_{i\omega}(\tau)/\nu!]\, d\tau\, dt = \sum_{\nu=0}^\infty B_\nu T_{i\omega}/s^{\nu+1}$$

where both the integration $\int_0^\infty$ and $\int_0^t$ are taken along the Φ-ray.

In short the analytic function T_{iw} *of* s, *defined by the Laplace integrals* (94) *satisfy the decomposed related non-homogeneous system* (65–67). Moreover a few obvious transformations make it evident Nörlund's theory applies and it follows that the analytic functions which are solutions of (65–67). can be represented in the half-plane $H(\Phi)$ by the convergent factorial series

$$T_{i\omega}(s^{-1/r}) = \sum_{\nu=0}^\infty \frac{K_{i\omega\nu}(\Phi, \gamma)}{s(s+\gamma e^{-i\Phi})(s+2\gamma e^{-i\Phi}) \dots (s+\nu\gamma e^{-i\Phi})}$$

$(i = 1, \dots, m;\ \omega = 1, \dots, r)$, where the positive constant γ is sufficiently large and the constant matrices $K_{i\omega\nu}$, as indicated, depend upon the choice of Φ and γ. Any constant $\gamma > 1$ is suitable provided it is large enough so that, when the interior Ψ of the circle $|\xi - 1| = 1$ is mapped into the complex t-plane by the transformation

$$t = [e^{-i\Phi} \log \xi]/\gamma,$$

the map of Ψ' is contained completely within a region which is the union of the sector $\mathfrak{S}$ under consideration and the circle $|t| < e^{-q}$. The function $T_{i\,\omega}(s^{-1/r})$ also possesses an asymptotic expansion

$$T_{i\,\omega}(s^{-1/r}) \approx \sum_{\eta=1}^{\infty} \mathfrak{U}_{i,\,\eta r+\omega}/s^{\eta}$$

in the sector

$$-\frac{\pi}{2} - \Phi + \varepsilon \leq \arg s \leq \frac{\pi}{2} - \Phi - \varepsilon,$$

where $\varepsilon > 0$ and arbitrary, for all sufficiently large $|s|$, see Theorem 1 in Doetsch's text [14], p. 231.

Transforming these results back to the t-plane by the transformation $t = s^{-1/r}$ we summarize our conclusions in

Theorem III. *Let a differential equation of canonical form* (34) *be given where* Y *is a vector and consider the* j*-th column of blocks*

$$(95) \qquad Y_j(t) = \left\| \begin{array}{l} t^{h-\beta_{ij}}(\Gamma_{1j} + U_{1j}(t)) \\ \vdots \\ t^{h-\beta_{j-1,j}}(\Gamma_{j-1,j} + U_{j-1,j}(t)) \\ I_j + U_{jj}(t) \\ t^{h-\beta_{j+1,j}}(\Gamma_{j+1,j} + U_{j+1,j}(t)) \\ \vdots \\ t^{h-\beta_{mj}}(\Gamma_{mj} + U_{mj}(t)) \end{array} \right\| \exp\{f_j(t)\, I_j + J_j \log t\}$$

in the formal series solution (46). *If either* $h = 2$ *with no restriction on the nature of the characteristic roots, or if* $h > 2$ *and the characteristic root* ϱ_{j0} *differs from all the other characteristic roots* ϱ_{i0}, $i \neq j$, *then the* $U_{ij}(t)$, $(i = 1, \ldots, m)$ *in* (95) *can be considered as known analytic functions which can be represented in the form*

$$(96) \qquad U_{ij}(t) = \sum_{k=1}^{r} \{t^k U_{ijk} + t^k \mathfrak{T}_{ijk}(t)\}, \qquad (i = 1, \ldots, m),$$

where all the

$$\mathfrak{T}_{ijk}(t) = \sum_{\nu=0}^{\infty} \frac{K_{ijk\nu}(\Phi, \gamma)}{t^{-r}(t^{-r} + \gamma e^{-i\Phi})(t^{-r} + 2\gamma e^{-i\Phi}) \ldots (t^{-r} + \nu\gamma e^{-i\Phi})}$$

are convergent factorial series provided

(i) *angle* $\Phi \neq \arg(\varrho_{i0} - \varrho_{j0})$ *for* $i = 1, \ldots, j-1, j+1, \ldots, m$;

(ii) *positive constant* γ *is sufficiently large; and*

(iii) *the point* t *is located inside any of the* r *loop-shaped regions which map into the half-plane* $H(\Phi)$ *under the transformation* $s = t^{-r}$.

Furthermore each column of matrix $Y_j(t)$ is an independent analytic vector solution of equation (34) *when Y is treated as a vector. The analytic functions $U_{ij}(t)$ can also be represented asymptotically by the formal series*

$$U_{ij}(t) \approx \sum_{k=1}^{\infty} U_{ijk}\, t^k$$

provided the $|t|$ is sufficiently small and t is located in one of the sectors

$$(2\Phi - \pi + 2\varepsilon + 4\pi k)/2r \leq \arg t \leq (2\Phi + \pi - 2\varepsilon + 4\pi k)/2r$$

where $\varepsilon > 0$ and is arbitrary and $k = 0, 1, \ldots, r-1$. The U_{ijk} in (96) *and the $K_{ijk\nu}(\Phi, \gamma)$ are appropriate known constant matrices.*

§ 9. Summary and Critique

When this paper was first undertaken it was hoped that all the formal series solutions of a vector equation of type (34) could be summed in every case. This objective has not been attained. We have succeeded completely only when $h = 2$ or when $m = 2$. If $h > 2$ and $m = 3$ the method presented in this paper will be applicable and provide at least one analytic vector solution expressed in terms of convergent factorial series, even though a full independent set of such convergent vector solutions may not have been obtained.

The simplest case which can not be fully treated is a certain equation of the third order, but not the equation

$$\frac{d^3 y}{d x^3} + \frac{a}{x}\frac{d y}{d x} + \frac{b\, y}{x^3} = 0, \quad a \neq 0, \quad b \neq 0, \tag{97}$$

given by Trjitzinsky [2] to show that his work was of the greatest possible completeness. Curiously enough what this example does show is that Trjitzinsky has not really pointed out the full power of his method, for the substitution $x = s^2$ will transform (97) into the equation

$$\frac{d^3 y}{d s^3} - \frac{3}{s}\frac{d^2 y}{d s^2} + \left(4a + \frac{3}{s^2}\right)\frac{d y}{d s} + \frac{8 b y}{s^3} = 0$$

which has three distinct characteristic roots and either Trjitzinsky's analysis [2] or that of the present paper will give a full independent set of solutions expressed in terms of convergent generalized factorial series. It is believed that the analysis presented here brings out more completely the scope and power of Trjitzinsky's method.

To summarize the progress made:

(1) A step-by-step procedure for computing formal solutions is given.

(2) The canonical form has been refined.

(3) No distinction need be made between normal and anormal solutions.

(4) At least one formal solution, although not a fundamental set, has been summed if $h > 2$ and $m = 3$.

(5) If in the canonical form of an equation $h = 2$ or $m = 2$, a fundamental set of convergent solutions has been obtained regardless of whether or not the formal solutions are normal or anormal or whether or not there is in the sense of Trjitzinsky, one or more logarithmic groups associated with each characteristic root.

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