ON THE DISTRIBUTION OF VALUES OF MEROMORPHIC FUNCTIONS OF BOUNDED CHARACTERISTIC

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Introduction

1. Let $w = f(z)$ be a non-constant meromorphic function in the unit circle $|z| < 1$. Using the standard notation[1] we write $(a \neq f(0))$

$$N(r, a) = N\left(r, \frac{1}{f-a}\right) = \int_0^r \frac{n(r, a)}{r} \, dr,$$

where $n(r, a)$ denotes the number of the roots of the equation $f(z) = a$ in the disk $|z| \leq r$, multiple roots being counted with their order of multiplicity. For $a \to f(0)$ the above integral tends to the limit $+\infty$. By the customary definition of $N(r, a)$, this logarithmic singularity at $a = f(0)$ is removed, but in this paper we prefer permitting the existence of the singularity. For $\lim_{r \to 1} N(r, a)$ we write $N(1, a)$.

With the help of $N(r, a)$ the characteristic function $T(r)$ of $f(z)$ can be defined as the mean-value (Shimizu-Ahlfors's theorem)

$$T(r) = \int N(r, a) \, d\mu,$$

where the integral is extended over the whole plane and $d\mu$ denotes the spherical element of area divided by π, i.e.,

$$d\mu = \frac{|a| \, d|a| \, d \arg a}{\pi (1 + |a|^2)^2}.$$

According as $T(r)$ is bounded or not, the functions $f(z)$ meromorphic in $|z| < 1$ fall into two essentially different classes. If $f(z)$ is of bounded characteristic, then

[1] For the general theory of single-valued meromorphic functions we refer to NEVANLINNA [7].

for almost all φ in $0 \leqq \varphi < 2\pi$, $\lim_{z \to e^{i\varphi}} f(z)$ exists uniformly in the angle $|\arg(1 - z e^{-i\varphi})| \leqq \frac{\pi}{2} - \varepsilon$ for all $\varepsilon > 0$ (Fatou-Nevanlinna's theorem). If this property holds for the boundary point $z = e^{i\varphi}$ we denote the above unique limit by $f(e^{i\varphi})$ and call $f(e^{i\varphi})$ a boundary value of $f(z)$. The set of all boundary values $f(e^{i\varphi})$ in $0 \leqq \varphi < 2\pi$ is of positive capacity (Nevanlinna-Frostman's theorem).

2. In this paper we consider the following problem:

The distribution of values of a function $f(z)$, meromorphic and of bounded characteristic in $|z| < 1$, is to be investigated under the following condition: For almost all φ the boundary values $w = f(e^{i\varphi})$ belong to a given closed point set Γ which is of positive capacity and of non-void connected complement.[1]

The study of this problem is divided into two essentially different cases; for we shall show that under the above conditions only the following two alternatives are possible:

A. *$w = f(z)$ takes no value outside Γ.*

B. *$w = f(z)$ takes every value outside Γ, with the possible exception of a set of capacity zero.*

In case A the study of the distribution of values of $f(z)$ leads to a problem of *majorization*. The situation is then governed by theorems like Schwarz's lemma, Lindelöf's principle, principle of hyperbolic measure, etc. A sharp and general principle well applying to this case is the following (Littlewood [4], [5], Lehto [2]): Let the values of $f(z)$ lie in a plane domain G with at least three boundary points, and let $w = x(z)$ map the unit circle conformally onto the universal covering surface of G and satisfy the condition $x(0) = f(0)$. Then, for every $r < 1$,

$$N\left(r, \frac{1}{f-a}\right) \leqq N\left(r, \frac{1}{x-a}\right),$$

with equality if and only if $f(z) = x(e^{i\vartheta} z)$ (ϑ a real constant). This inequality contains Schwarz's lemma and Lindelöf's principle as special cases, both in a sharpened form. The case A thus being governed by known principles, we shall not deal with it in this paper.

In the above case B the study of the distribution of values of $f(z)$ gives rise to a problem of *minorization*. This problem will be treated in detail below. Our main result reads as follows: *If the point $w = f(0)$ is not in Γ, we have*

[1] A short summary of our principal results about this problem is given in [3].

(1) $$N\left(1, \frac{1}{f-a}\right) \geqq g(a, f(0), C(\Gamma)),$$

except perhaps for a set of values a of capacity zero. Here $g(a, f(0), C(\Gamma))$ denotes the Green's function of the complement of Γ with singularity at the point $f(0)$. This result is sharp in the following two senses: First, we have equality for every a if $w = f(z)$ maps the unit circle conformally onto the universal covering surface of the complement of Γ. Secondly, given an arbitrary closed set of capacity zero outside Γ, we always have a function $f(z)$, satisfying the above conditions, for which

$$N\left(1, \frac{1}{f-a}\right) < g(a, f(0), C(\Gamma))$$

at all points of this null-set.

The above results can also be stated directly in terms of the boundary values without prescribing the set Γ. We then have to impose on $f(z)$ the additional restriction that the complement of the closure of the boundary values is not void.

3. Quite symmetrical results are obtained if the points of Γ are Picard exceptional values for the function $f(z)$. In other words, suppose that all values $w = f(z)$ lie in the complement of Γ, and that for almost all φ, $w = f(e^{i\varphi})$ belongs to Γ. Let us introduce the function

$$\Phi\left(\frac{1}{f-a}\right) = \frac{1}{2\pi} \lim_{r\to 1} \int_0^{2\pi} g(f(re^{i\varphi}), a, C(\Gamma))\, d\varphi$$

which measures the convergence in the mean towards the value a in the neighbourhood of the frontier $|z| = 1$. We then have the invariant relation

(2) $$\Phi\left(\frac{1}{f-a}\right) + N\left(1, \frac{1}{f-a}\right) = g(a, f(0), C(\Gamma)).$$

Here the right-hand term, which plays the role of a characteristic function, is a *domain* function and *independent* of $f(z)$, up to the value of $f(0)$.

As for the relative magnitude of the components Φ and N in (2) it follows from the inequality (1) that

(3) $$\Phi\left(\frac{1}{f-a}\right) = 0,$$

except perhaps for a set of values of capacity zero. Therefore, it is natural to call a value a *normal* for $f(z)$ if (3) holds; otherwise a is *exceptional.* As in Nevanlinna's theory, for a more detailed study the concept of *deficiency* $\delta(a)$ *of the value* a can be introduced, the natural definition here being $(a \neq f(0))$

$$\delta(a) = \frac{\Phi\left(\frac{1}{f-a}\right)}{g(a, f(0), C(\Gamma))} = 1 - \frac{N\left(1, \frac{1}{f-a}\right)}{g(a, f(0), C(\Gamma))};$$

this definition can readily be modified for $a = f(0)$.

In the special case that the functions $f(z)$ considered satisfy the conditions $|f(z)| < 1$, $|f(e^{i\varphi})| = 1$ almost everywhere, certain analogous results have been established by Frostman [1].

§ 1. Fundamental lemma

4. This chapter is devoted to preliminary considerations, the tools necessary for the following representation being developed. The results are collected in a lemma which will be of fundamental importance below.

Let $w = f(z)$ be a meromorphic function in the unit circle $|z| < 1$. Introducing the function

$$N(r, a) = \int_0^r \frac{n(r, a)}{r} dr$$

we first prove (Frostman [1]):

If μ *is a completely additive set function defined for all Borel measurable subsets of a closed set* S, *then*

$$\int_S N(r, a)\, d\mu(a) = N(r, \infty)\, \mu(S) + u(f(0)) - \frac{1}{2\pi} \int_0^{2\pi} u(f(r e^{i\varphi}))\, d\varphi, \tag{1.1}$$

where

$$u(w) = \int_S \log \frac{1}{|w-a|}\, d\mu(a)$$

is the logarithmic potential belonging to the set function μ.

Formula (1.1) can easily be established by means of the principle of the argument. In fact, by this principle,

$$\int_0^{2\pi} d \arg(f(r e^{i\varphi}) - a) = 2\pi (n(r, a) - n(r, \infty)).$$

Applying Cauchy-Riemann's differential equation to the left-hand side we obtain

$$r\frac{d}{dr}\int_0^{2\pi}\log|f(re^{i\varphi})-a|\,d\varphi=2\pi(n(r,a)-n(r,\infty)).$$

Multiplying both sides by $d\mu$ and integrating over S with respect to $d\mu(a)$ gives

$$-r\frac{d}{dr}\int_0^{2\pi}u(f(re^{i\varphi}))\,d\varphi=2\pi\left(\int_S n(r,a)\,d\mu-n(r,\infty)\,\mu(S)\right),$$

whence (1.1) follows by integrating with respect to r.

In (1.1) both sides may be infinite. Since we shall apply this formula to a special case where the convergence can directly be seen, we shall not study general conditions which would exclude divergence.

5. As the next step we construct a set function appropriate for our purposes. Let Γ and E be two bounded, closed, disjoint point sets in the w-plane, both of positive capacity. Let the complements of Γ and $\Gamma+E$, denoted by $C(\Gamma)$ and $C(\Gamma+E)$, respectively, be connected.

Consider the well-known Robin equilibrium potentials

$$\begin{cases} u_1(w)=\displaystyle\int_\Gamma \log\frac{1}{|\zeta-w|}\,d\mu_1(\zeta)\\ u_2(w)=\displaystyle\int_{\Gamma+E} \log\frac{1}{|\zeta-w|}\,d\mu_2(\zeta),\end{cases} \tag{1.2}$$

where μ_1 and μ_2, so-called equilibrium mass-distributions, are non-negative completely additive set functions defined for the Borel measurable subsets of Γ and $\Gamma+E$, respectively, and of total mass 1. For every subset e of capacity zero we have $\mu_1(e)=0$, $\mu_2(e)=0$. In contrast to this, $\mu_2(E)>0$, since E is of positive capacity. (For details concerning the properties of equilibrium mass-distributions we refer to Frostman [1].)

If w is a point of $C(\Gamma+E)$ we have

$$\begin{cases} u_1(w)=\gamma_1-g(w,\infty,C(\Gamma))\\ u_2(w)=\gamma_2-g(w,\infty,C(\Gamma+E)),\end{cases} \tag{1.3}$$

where γ_1 denotes the Robin's constant of $C(\Gamma)$ and $g(w, \infty, C(\Gamma))$ represents the Green's function of $C(\Gamma)$ with pole at $w=\infty$. Correspondingly, γ_2 and $g(w, \infty, C(\Gamma+E))$ denote the same quantities with respect to $C(\Gamma+E)$. If w is an internal point of Γ (if such points exist), then

$$u_1(w)=\gamma_1,$$

and similarly, if w is an inner point of $\Gamma+E$,

$$u_2(w)=\gamma_2.$$

Let us now construct the potential

$$u(w)=u_2(w)-u_1(w).$$

Because Green's function is bounded in every part of its domain of existence which does not contain a certain neighbourhood of the pole, this potential $u(w)$ is *bounded* in the whole w-plane.

By (1.2) we can write

$$u(w)=\int\limits_{\Gamma+E} \log \frac{1}{|\zeta-w|}\, d\mu(\zeta),$$

where

$$\mu(e)=\begin{cases}\mu_2(e)-\mu_1(e), & e\subset\Gamma\\ \mu_2(e), & e\subset E.\end{cases}$$

For this set function μ we have

$$\mu(\Gamma+E)=\mu_2(\Gamma+E)-\mu_1(\Gamma)=0.$$

It follows from the definition that $\mu(e)\geqq 0$, if e is an arbitrary subset of E, whereas, by the maximum principle, $\mu(e)\leqq 0$ for every measurable subset of Γ. From the properties of μ_1 and μ_2 stated above it follows that $\mu(e)=0$ if e is of capacity zero. For the sets E and Γ we have $\mu(E)>0$, $\mu(\Gamma)<0$. The former statement is immediately implied by the above remark concerning $\mu_2(E)$ and by the definition of μ the latter one being an easy consequence of the minimum principle.

In the above it was supposed that the sets Γ and E are bounded. If either of the sets does not satisfy this requirement, the above reasoning must be slightly modified. Since this can be made in an obvious manner and no essential difficulties are encountered we omit the details here. In any case we can construct a set function μ which possesses the properties stated above and for which the associated potential $u(w)$ is bounded in the whole plane.

6. Let us consider a function $w = f(z)$ satisfying the following conditions:

1. $f(z)$ is meromorphic and of bounded characteristic in the unit circle $|z| < 1$.
2. For almost all φ the boundary values $f(e^{i\varphi})$ belong to a bounded closed point set Γ of connected complement.
3. The point $w = f(0)$ is in the complement of Γ.

By the theorem of Nevanlinna and Frostman, referred to in the Introduction, the set Γ is of positive capacity.

Let E be an arbitrary closed point set of positive capacity and connected complement and lying outside Γ. We apply formula (1.1) to this $f(z)$ choosing for the set function the above μ which belongs to the sets Γ and E. Since in this case $\mu(S) = \mu(\Gamma + E) = 0$, it follows from (1.1) that

$$\int_{\Gamma + E} N(r, a)\, d\mu = u(f(0)) - \frac{1}{2\pi} \int_0^{2\pi} u(f(re^{i\varphi}))\, d\varphi. \tag{1.4}$$

Let now $r \to 1$. In the left-hand integral we can perform the passage to the limit under the integral sign since $N(r, a)$ is monotonic in r. As for the right-hand side, we first conclude from the boundedness of $u(w)$ that

$$\int_0^{2\pi} \liminf_{r \to 1} u(f(re^{i\varphi}))\, d\varphi \leqq \lim_{r \to 1} \int_0^{2\pi} u(f(re^{i\varphi}))\, d\varphi \leqq \int_0^{2\pi} \limsup_{r \to 1} u(f(re^{i\varphi}))\, d\varphi. \tag{1.5}$$

By hypothesis, the point $w = f(re^{i\varphi})$ either is in Γ or tends to Γ for almost all φ when $r \to 1$. Let us confine ourselves to studying such values φ only. Then, if $w = f(e^{i\varphi})$ is an internal point of Γ, we have, by the above,

$$\liminf_{r \to 1} u(f(re^{i\varphi})) = \limsup_{r \to 1} u(f(re^{i\varphi})) = \gamma_2 - \gamma_1. \tag{1.6}$$

In case $w = f(e^{i\varphi})$ is a boundary point of Γ it follows from the definition of $u(w)$ that

$$\liminf_{r \to 1} u(f(re^{i\varphi})) \geqq \gamma_2 - \gamma_1 \tag{1.7}$$

and

$$\limsup_{r \to 1} u(f(re^{i\varphi})) \leqq \gamma_2 - \gamma_1 + \sup g(f(e^{i\varphi}), \infty, C(\Gamma)). \tag{1.8}$$

By a theorem of Frostman [1], if w is a boundary point of Γ, then

$$\sup g(w, \infty, C(\Gamma)) = 0,$$

except at most in a set of capacity zero. On the other hand, by Nevanlinna-Frostman's theorem, the set of boundary values which $f(z)$ takes on $|z|=1$ in a set of positive linear measure always is of positive capacity. Hence, by (1.6), (1.7), and (1.8), for almost all φ

$$\lim_{r\to 1} u(f(r e^{i\varphi})) = \gamma_2 - \gamma_1,$$

and thus, by (1.5),

$$\lim_{r\to 1} \int_0^{2\pi} u(f(r e^{i\varphi}))\, d\varphi = 2\pi(\gamma_2 - \gamma_1).$$

Consequently, the equation (1.4) yields the following relation for $r\to 1$

$$\int_{\Gamma+E} N(1, a)\, d\mu = v(f(0)), \tag{1.9}$$

where

$$v(f(0)) = g(f(0), \infty, C(\Gamma)) - g(f(0), \infty, C(\Gamma+E))$$

or

$$v(f(0)) = g(f(0), \infty, C(\Gamma)),$$

according as $f(0)$ belongs to $C(E)$ or is an inner point of E. In both cases $v(f(0))$ is *positive*. In the latter case this is evident, and in the former case it follows from the minimum principle. For if $v(f(0))=0$, i.e., if

$$g(f(0), \infty, C(\Gamma)) = g(f(0), \infty, C(\Gamma+E)),$$

we conclude from the minimum principle that in the whole domain $C(\Gamma+E)$

$$g(w, \infty, C(\Gamma)) \equiv g(w, \infty, C(\Gamma+E)).$$

This, however, is impossible since E is of positive capacity. Finally, it is readily seen that $v(f(0))$ is positive also if $w=f(0)$ is a boundary point of E.

If the sets Γ or E are not bounded we also obtain a formula (1.9) where the right-hand term $v(f(0))$ is positive and depends on $f(0)$ and on the sets Γ and E only.

We summarize the results needed below in the following

Lemma. *Let Γ and E be two closed disjoint point sets in the w-plane, both of positive capacity and connected complement. We then have a completely additive set function μ, defined for all Borel measurable subsets of $\Gamma+E$, which possesses the following properties:*

1. *$\mu(e)\leqq 0$ for all measurable sets of Γ, $\mu(e)\geqq 0$ for all measurable sets of E.*
2. *$\mu(e)=0$ for all sets e of capacity zero, whereas $\mu(E)>0$.*

3. *If* $w = f(z)$ *is a meromorphic function of bounded characteristic in the unit circle* $|z| < 1$ *such that for almost all* φ *its boundary values* $f(e^{i\varphi})$ *belong to* Γ *and* $w = f(0)$ *is not in* Γ, *then*

$$\int_{\Gamma+E} N(1, a)\, d\mu = v(f(0)),$$

where $v(f(0))$ *is positive and only depends on the sets* Γ *and* E *and on the value* $w = f(0)$.

§ 2. Minorant theorems

7. In the w-plane let Γ be a closed point set of positive capacity and non-void connected complement. Let $w = x(z)$ be a function which maps the unit circle $|z| < 1$ onto the universal covering surface of the complement of Γ. By this property, $x(z)$ is uniquely determined e.g. up to the values of $x(0)$ and $\arg x'(0)$.

Since Γ is of positive capacity $x(z)$ has, by a well-known theorem ([7]), a bounded characteristic function. Hence, the boundary values $w = x(e^{i\varphi})$ exist almost everywhere. They necessarily belong to the set Γ. For if w_0 is an inner point of $C(\Gamma)$ we can draw a circular disk K around w_0 such that K entirely lies in $C(\Gamma)$. The maps of this disk by the function $z = x^{-1}(w)$ (x^{-1} denotes the inverse function of x) lie disjoint in the unit circle $|z| < 1$. Hence, when the point $z = re^{i\varphi}$ continuously approaches the circumference of the unit circle the image point $w = x(z)$ again and again leaves the disk K and thus cannot converge towards the point w_0.

Accordingly, the boundary values $x(e^{i\varphi})$ are in Γ, and we can apply the Lemma to $w = x(z)$. Since $x(z)$ omits all values belonging to the set Γ, we have for these values a, $N(1, a) = 0$, and so

$$\int_{\Gamma} N(1, a)\, d\mu = 0.$$

Hence, for $x(z)$ the Lemma yields

$$\int_{E} N(1, a)\, d\mu = v(x(0)). \tag{2.1}$$

Let now $w = f(z)$ be an arbitrary function of bounded characteristic which satisfies the conditions of the Lemma. In other words, let $w = f(e^{i\varphi})$ belong to Γ for almost all φ and let $w = f(0)$ be outside Γ. By the Lemma, we have

$$\int_{E} N(1, a)\, d\mu = v(f(0)) - \int_{\Gamma} N(1, a)\, d\mu.$$

Since $d\mu \leqq 0$ in Γ, this implies that

$$\int_E N(1, a)\, d\mu \geqq v(f(0)). \tag{2.2}$$

We now choose $x(z)$ so that $x(0) = f(0)$. Then, combining (2.1) and (2.2),

$$\int_E N\left(1, \frac{1}{f-a}\right) d\mu \geqq \int_E N\left(1, \frac{1}{x-a}\right) d\mu. \tag{2.3}$$

From this relation we conclude that $N\left(1, \frac{1}{f-a}\right)$ cannot be less than $N\left(1, \frac{1}{x-a}\right)$ at all points of E. For if so, i.e. if

$$N\left(1, \frac{1}{x-a}\right) - N\left(1, \frac{1}{f-a}\right) = \delta(a) > 0$$

in E, it follows from (2.3), because $d\mu \geqq 0$ in E, that

$$\int_E \delta(a)\, d\mu = 0.$$

This, however, is impossible. For consider a monotonically decreasing sequence of positive numbers $p_1 > p_2 > \cdots$, where $p_\nu \to 0$ for $\nu \to \infty$. Define E_0 as the subset of E where $\delta(a) > p_1$, and let E_ν $(\nu = 1, 2, \ldots)$ be the set where $p_{\nu+1} < \delta(a) \leqq p_\nu$; these sets E_ν $(\nu = 0, 1, \ldots)$ are measurable. Since $d\mu \geqq 0$ in E we have for each ν

$$p_{\nu+1} \int_{E_\nu} d\mu \leqq \int_{E_\nu} \delta\, d\mu \leqq \int_E \delta\, d\mu = 0,$$

and hence

$$\int_{E_\nu} d\mu = 0.$$

Because $E = \sum E_\nu$ and μ is additive this implies that

$$\sum_\nu \int_{E_\nu} d\mu = \int_E d\mu = \mu(E) = 0,$$

in contradiction to the Lemma according to which $\mu(E) > 0$.

Since E was an arbitrary set of positive capacity we thus conclude that *the inequality*

$$N\left(1, \frac{1}{f-a}\right) < N\left(1, \frac{1}{x-a}\right) \tag{2.4}$$

can hold at most in a set of capacity zero.

This result is sharp in the following sense: *Given an arbitrary closed null-set B outside Γ we can always find a function $f(z)$, satisfying the above conditions, such that* (2.4) *is true at all points of B.* To prove this consider a function $x^*(z)$ which maps the unit circle onto the universal covering surface of the complement of $\Gamma+B$. As shown above, the boundary values $w=x^*(e^{i\varphi})$ belong to the set $\Gamma+B$. But since B is of capacity zero it follows from Nevanlinna-Frostman's theorem that for almost all φ the boundary values $x^*(e^{i\varphi})$ must belong to Γ. Hence, $x^*(z)$ fulfills the conditions imposed on $f(z)$ above. On the other hand, all points of B are Picard exceptional values for $x^*(z)$ and so for these values a we really have the inequality

$$N\left(1, \frac{1}{x^*-a}\right) < N\left(1, \frac{1}{x-a}\right).$$

8. In order to express the above results in a somewhat different form we make use of the lemma in [2] according to which the function $N(r, a)$ is subharmonic in a. Hence,

$$N(r, a) \leqq g(a, f(0), C(\Gamma)),$$

and thus, putting $f(z)=x(z)$ and letting $r\to 1$, also

$$N\left(1, \frac{1}{x-a}\right) \leqq g(a, x(0), C(\Gamma)). \tag{2.5}$$

By the mean-value argument used in [2] to establish the subharmonicity of $N(r, a)$ and applying Harnack's principle, it is readily seen that $N\left(1, \frac{1}{x-a}\right)$ is not only subharmonic but even regular harmonic in $C(\Gamma)$, except for the logarithmic singularity at $a=x(0)$. Now it immediately follows from the maximum principle that the Green's function $g(a, x(0), C(\Gamma))$ is smaller than any other function harmonic and non-negative in $C(\Gamma)$ and with a similar singularity at the point $a=x(0)$. Consequently, we obtain from (2.5) the following equality (Poincaré [8], Myrberg [6])

$$N\left(1, \frac{1}{x-a}\right) = g(a, x(0), C(\Gamma)). \tag{2.6}$$

Considering this relation we can summarize the above results as follows.

Theorem 1. *Let $f(z)$ be meromorphic and of bounded characteristic in the unit circle. For almost all φ let its boundary values $w=f(e^{i\varphi})$ belong to a closed point set Γ of non-void connected complement, and let $w=f(0)$ be outside Γ. We then have in the complement of Γ,*

$$N(1, a) \geqq g(a, f(0), C(\Gamma)) \tag{2.7}$$

with the possible exception of a set of values a of capacity zero. Such an exceptional null-set can exist. Equality holds in (2.7) *for every a e.g. if* $w = f(z)$ *maps the unit circle onto the universal covering surface of the complement of* Γ.

9. This theorem, in character somewhat similar to the classical principle of Phragmén and Lindelöf, can be expressed in an integrated form which is convenient from the point of view of various applications. For this purpose, let us cover the w-plane with a non-negative mass μ, i.e., μ is a completely additive, non-negative set function defined for all Borel measurable point sets of the plane.

By the definition of $N(1, a)$ we can write

$$\int N(1, a)\, d\mu = \int_0^1 \frac{dr}{r} \int n(r, a)\, d\mu = \int_0^1 \frac{\Omega(r)}{r}\, dr, \tag{2.8}$$

where the integrals may also be divergent. Here

$$\Omega(r) = \int n(r, a)\, d\mu$$

obviously denotes the total mass lying on the Riemann surface onto which the function $f(z)$ in question maps the disk $|z| \leq r$. Each surface element e of this image surface is then furnished with the mass $\mu(e)$.

Incidentally, we make some remarks about the convergence of the integrals (2.8). We do not go into details, in particular since the divergence case too may not be without interest. If the total mass M of the plane is finite, an easy estimate based on Nevanlinna's first fundamental theorem in the exact spherical form ([7]) yields

$$\int_0^1 \frac{\Omega(r)}{r}\, dr \leq M\, T(1) + u(f(0)),$$

where T is the characteristic function of $f(z)$ and

$$u(w) = \int \log \frac{\sqrt{1+|w|^2}\,\sqrt{1+|a|^2}}{|w-a|}\, d\mu$$

denotes the "spherical-logarithmic" potential belonging to the mass distribution μ. Since $f(z)$ is of bounded characteristic, i.e., $T(1) < \infty$, it follows that the integrals (2.8) are convergent if the potential u has a finite value at the point $w = f(0)$. If, however, the total mass M is infinite the domain $C(\Gamma)$ must generally satisfy certain additional restrictions in order that the integrals (2.8) were convergent.

Let us now impose the following condition on the mass μ: *Let* $\mu(e) = 0$ *for each measurable set e of capacity zero.* It follows from the metrical properties of harmonic null-sets (see [7]) that this condition is fulfilled e.g. if $d\mu$ equals the Euclidean or spherical element of area. We then obtain, by (2.8), the following integrated form of Theorem 1.

Theorem 1'. *Let $f(z)$ be a meromorphic function of bounded characteristic whose boundary values $w=f(e^{i\varphi})$ belong for almost all φ to a closed point set Γ of connected complement, and for which $w=f(0)$ is not in Γ. Further, let $w=x(z)$ map the unit circle onto the universal covering surface of the complement of Γ and satisfy the condition $x(0)=f(0)$. Then, if the w-plane is covered with a non-negative mass μ for which $\mu(e)=0$ for every set e of capacity zero, we have*

$$\int_0^1 \frac{\Omega(r;f)}{r}\,dr \geqq \int_0^1 \frac{\Omega(r;x)}{r}\,dr,$$

where $\Omega(r;f)$ and $\Omega(r;x)$ denote the total mass on the image of $|z|\leqq r$ by $f(z)$ and $x(z)$, respectively. If the right-hand integral diverges the inequality is interpreted to mean that the left-hand integral also is infinite.

10. If the boundary values of $f(z)$ belong to Γ for almost all φ, Theorem 1 gives us answer to the following question: *How large sets of values can $f(z)$ omit outside Γ?*

Of course, $f(z)$ can omit all values outside Γ. Disregarding this trivial case we suppose that there is a point z_0 in the unit circle such that $w=f(z_0)$ belongs to the complement of Γ. We shall first show that it does not mean any restriction for our study to suppose that $w=f(0)$ is not in Γ. For consider the function

$$f^*(z)=f\left(\frac{z+z_0}{1+\bar{z}_0 z}\right),$$

for which $f^*(0)$ $(=f(z_0))$ belongs to the complement of Γ. Now

$$z^*=\frac{z+z_0}{1+\bar{z}_0 z}$$

transforms the unit circle into itself in a one-one manner. Hence, the boundary values of $f^*(z)$ also belong to Γ for almost all φ. Moreover, the functions $f(z)$ and $f^*(z)$ omit exactly the same values in the unit circle. Consequently, $f(z)$ can be replaced by $f^*(z)$, i.e., we can assume that $w=f(0)$ belongs to the complement of Γ.

By Theorem 1 we have, up to a null-set,

$$N(1,a)\geqq g(a,f(0),C(\Gamma))>0.$$

Hence, if $f(z)$ takes one value outside Γ it must take all values in the complement of Γ, except perhaps a set of capacity zero. Such a null-set is possible as is shown by the function $x^*(z)$ considered above in Section 7. It takes values outside Γ but omits all values which belong to an arbitrarily given closed set of capacity zero in the complement of Γ. We thus obtain

Theorem 2. *Let $w = f(z)$ be a meromorphic function of bounded characteristic whose boundary values $f(e^{i\varphi})$ for almost all φ belong to a closed point set Γ of non-void connected complement. Then only the following two alternatives are possible:*

A. *The function $w = f(z)$ takes no value belonging to the complement of Γ.*

B. *The function $w = f(z)$ takes every value belonging to the complement of Γ, with the possible exception of a set of capacity zero. Such an exceptional null-set can actually exist.*

11. The above Theorems 1 and 2 can also be expressed in a more direct form without prescribing the set Γ. In fact, let $f(z)$ be of bounded characteristic in the unit circle, and let the complement of the closure of its boundary values be not void. This complement then consists of a finite or enumerable set of domains; let D be an arbitrary domain of this set. We apply Theorem 2 by choosing Γ equal to the complement of D and obtain

Theorem 3. *Let $f(z)$ be of bounded characteristic in $|z| < 1$, and let the complement of the closure of its boundary values be not void. Then, if D is an arbitrary connected portion of this complement, only two alternatives are possible: Either $w = f(z)$ takes no value belonging to D, or $w = f(z)$ takes every value in D, except perhaps a set of capacity zero.*

As a simple special case let us e.g. note the following

Corollary 1. *Let $f(z)$ be of bounded characteristic in the unit circle, and let the boundary values $w = f(e^{i\varphi})$ for almost all φ belong to a closed Jordan curve which divides the w-plane into two open parts A and B. Then, up to perhaps a set of capacity zero, either the set of values of $f(z)$ coincides with A or with B, or $f(z)$ takes all values in $A + B$.*

12. In order to express conveniently the result of Theorem 1 in similar terms, we first slightly generalize the definition of the function

$$N(r, a) = \int_0^r \frac{n(r, a)}{r} \, dr.$$

Let $n_t(r, a)$ denote the number of the roots of the equation $f(z) = a$ in the circular disk

$$\left|\frac{z-t}{1-\bar{t}z}\right| \leqq r \qquad (|t| < 1),$$

multiple roots being counted with their multiplicity. For $t = 0$, $n_t(r, a)$ coincides with $n(r, a)$. By means of $n_t(r, a)$ we now define the function

$$N_t(r, a) = \int_0^r \frac{n_t(r, a)}{r} \, dr.$$

From the standpoint of the theory of the distribution of values this function can be used equally well as $N(r, a)$ for characterizing the density of the roots of the equation $f(z) = a$. While $N(r, a)$ has a logarithmic singularity at $a = f(0)$, this function possesses a singularity at $a = f(t)$.

It follows from the definition that

$$N_t\left(r, \frac{1}{f-a}\right) = N\left(r, \frac{1}{f^* - a}\right),$$

where

$$f^*(z) = f\left(\frac{z+t}{1+\bar{t}z}\right).$$

Hence, we immediately get from Theorem 1

Theorem 4. *Let $f(z)$ be of bounded characteristic in the unit circle, and let the complement of the closure of its boundary values be not void. Let D be a connected part of this complement such that there is a point t in the unit circle $|z| < 1$ at which $w = f(z)$ takes a value belonging to D. Then, for all values a belonging to D,*

$$N_t(1, a) \geqq g(a, f(t), D),$$

with the possible exception of a set of capacity zero.

13. Let us study when a proper inequality holds in (2.7), i.e., when

$$N\left(1, \frac{1}{f-a}\right) > g(a, f(0), C(\Gamma)), \tag{2.9}$$

except perhaps for a null-set. A necessary condition is readily obtained: If all values of $w = f(z)$ lie in the complement of Γ, then ([2])

$$N\left(r, \frac{1}{f-a}\right) \leq N\left(r, \frac{1}{x-a}\right),$$

and hence also

$$N\left(1, \frac{1}{f-a}\right) \leqq N\left(1, \frac{1}{x-a}\right) = g\,(a, f\,(0), C\,(\Gamma)).$$

Thus, in order that (2.9) is valid $f(z)$ must take values which belong to the set Γ.

For further study we again make use of the Lemma, which gives

$$\int_E \left(N\left(1, \frac{1}{f-a}\right) - N\left(1, \frac{1}{x-a}\right)\right) d\mu = \int_\Gamma N\left(1, \frac{1}{f-a}\right) |d\mu|. \tag{2.10}$$

Suppose first that $N\left(1, \frac{1}{f-a}\right)$ is positive in Γ only in a set e of capacity zero. By the Lemma, then $\mu(e) = 0$ and it follows that

$$\int_E \left(N\left(1, \frac{1}{f-a}\right) - N\left(1, \frac{1}{x-a}\right)\right) d\mu = 0.$$

By a reasoning exactly similar to the one used in Section 7 we conclude from this that (2.9) cannot be true in a set of positive capacity.

Suppose hereafter that $N\left(1, \frac{1}{f-a}\right) > 0$ in Γ in a set e of positive capacity. As is easily seen, we then have $\mu(e) < 0$, whence it follows that

$$\int_\Gamma N\left(1, \frac{1}{f-a}\right) |d\mu| > 0,$$

and hence, by (2.10),

$$\int_E \left(N\left(1, \frac{1}{f-a}\right) - N\left(1, \frac{1}{x-a}\right)\right) d\mu > 0.$$

Therefore, the inequality

$$N\left(1, \frac{1}{f-a}\right) \leqq N\left(1, \frac{1}{x-a}\right)$$

cannot hold at all points of E. Since E was an arbitrary set of positive capacity we thus obtain:

Theorem 5. *Let the function $f(z)$ and the set Γ have the same meaning as in Theorem 1. Regarding the inequality*

$$N\,(1, a) \geqq g\,(a, f\,(0), C\,(\Gamma))$$

only the following two possibilities can occur: Except perhaps a set of capacity zero we either have the equality

$$N(1, a) = g(a, f(0), C(\Gamma))$$

or the proper inequality

$$N(1, a) > g(a, f(0), C(\Gamma)).$$

The latter alternative is valid if and only if $f(z)$ takes a set of values of positive capacity which belongs to Γ.

It follows from the principle of harmonic measure that the latter alternative occurs if there is a point set of positive capacity in the unit circle $|z| < 1$ such that at every point of this set the value of $f(z)$ belongs to Γ.

Theorem 5 can be expressed in a slightly different form which often appears in applications. Let Γ^* be a closed subset of Γ such that Γ^* is of connected complement and that $\Gamma - \Gamma^*$ is of positive capacity. Let $w = f^*(z)$ be a function of bounded characteristic, with boundary values in Γ^* for almost all φ, and having $w = f^*(0)$ outside Γ. Then

$$N\left(1, \frac{1}{f^* - a}\right) > g(a, f^*(0), C(\Gamma)),$$

except at most for a set of capacity zero. To prove this we first conclude from Theorem 2 that $f^*(z)$ takes all values outside Γ^* up to a set of capacity zero. In particular, $f^*(z)$ takes a set of values of positive capacity which belongs to $\Gamma - \Gamma^*$, and the inequality follows from Theorem 5.

In case of a proper inequality we have an r_0 such that

$$N\left(r, \frac{1}{f - a}\right) \geqq N\left(r, \frac{1}{x - a}\right)$$

for every $r \geqq r_0$. In general, this does not hold uniformly with respect to a.

§ 3. Functions omitting a set of values of positive capacity

14. In this chapter we suppose that $w = f(z)$ is a meromorphic function in $|z| < 1$ such that its values lie in a plane domain G whose boundary Γ is of positive capacity. As is well known, $f(z)$ is then of bounded characteristic.

Let $w = x(z)$ map the unit circle onto the universal covering surface G^∞ of G and satisfy the condition $x(0) = f(0)$. Denoting by x^{-1} the inverse function of x we form the function $\psi(z) = x^{-1}(f(z))$ and choose the branch of x^{-1} so that $\psi(0) = 0$. Since $\psi(z)$ can be continued in the whole unit circle it follows from the monodromy theorem that $\psi(z)$ is single-valued. Obviously, $|\psi(z)| < 1$ in $|z| < 1$.

Let a $(\neq f(0))$ be an arbitrary point in G and let $z_1, z_2, \ldots$ denote all roots of the equation $x(z) = a$ in the unit circle. Since $|\psi(z)| < 1$, the product

$$\pi(\psi(z)) = \prod_{|z_\nu|<1} \frac{\psi(z) - z_\nu}{1 - \bar{z}_\nu \psi(z)}$$

converges and represents a regular analytic function in the unit circle. It follows from the principle of the argument that

$$\int_0^{2\pi} d \arg \pi(\psi(r e^{i\varphi})) = r \frac{d}{d r} \int_0^{2\pi} \log |\pi(\psi(r e^{i\varphi}))| \, d\varphi = 2\pi n(r, 0), \tag{3.1}$$

where $n(r, 0)$ denotes the number of the zeros of $\pi(\psi(z))$ in $|z| \leqq r$. Now $\pi(\psi(z)) = 0$ if and only if $\psi(z) = z_\nu$, i.e., if and only if $f(z) = x(z_\nu) = a$. Hence, $n(r, 0)$ for $\pi(\psi(z))$ equals $n(r, a)$ for $f(z)$, and we obtain by integrating (3.1) with respect to r,

$$\frac{1}{2\pi} \int_0^{2\pi} \log |\pi(\psi(r e^{i\varphi}))| \, d\varphi - \log |\pi(\psi(0))| = N\left(r, \frac{1}{f-a}\right). \tag{3.2}$$

Here

$$\log |\pi(\psi(0))| = \log |\pi(0)| = \sum_\nu \log |z_\nu| = -N\left(1, \frac{1}{x-a}\right),$$

and thus, by (2.6),

$$\log |\pi(\psi(0))| = -g(a, f(0), G).$$

Similarly,

$$\log |\pi(\psi(z))| = -g(a, f(z), G) = -g(f(z), a, G).$$

Hence, (3.2) yields the following relation

$$N\left(r, \frac{1}{f-a}\right) + \frac{1}{2\pi} \int_0^{2\pi} g(f(r e^{i\varphi}), a, G) \, d\varphi = g(a, f(0), G). \tag{3.3}$$

15. We shall now analyze the equation (3.3) in more detail in the case that the boundary values $f(e^{i\varphi})$ of $f(z)$ belong to Γ for almost all φ.

Denoting

$$\Phi(a) = \Phi\left(\frac{1}{f-a}\right) = \frac{1}{2\pi} \lim_{r \to 1} \int_0^{2\pi} g(f(r e^{i\varphi}), a, G) \, d\varphi,$$

we obtain a non-negative function $\Phi(a)$ which measures the mean convergence of $f(z)$ towards the value a in the neighbourhood of the boundary $|z| = 1$. By (3.3) we have

$$N(1, a) + \Phi(a) = g(a, f(0), G), \tag{3.4}$$

where the right-hand term is a *domain* function and *independent* of the particular function $f(z)$ up to the value of $f(0)$. Hence, if some $f(z)$ takes a given value a relatively seldom, i.e., if $N\left(1, \frac{1}{f-a}\right)$ is small, this is compensated by a strong approximation of the value a in the neighbourhood of $|z|=1$ so that the sum $N+\Phi$ attains its invariant value, characteristic of the domain G. The situation shows an obvious similarity to Nevanlinna's first fundamental theorem for meromorphic functions. However, here the sum $N+\Phi$ is essentially independent of the function $f(z)$ but depends on the value a considered, whereas the corresponding sum $N+m$ in Nevanlinna's theory depends on the function but not on the value a.

A further analogue of Nevanlinna's theory is found by investigating the relative magnitude of the components N and Φ in (3.4). Since $\Phi(a)$ is non-negative we have for all a

$$N(1, a) \leqq g(a, f(0), G).$$

On the other hand, it follows from Theorem 1 that

$$N(1, a) \geqq g(a, f(0), G),$$

except perhaps for a set of values a of capacity zero. Hence, up to such a null-set,

$$N(1, a) = g(a, f(0), G)$$

or, which is the same,

$$\Phi(a) = 0.$$

This result can be considered as an analogue of Nevanlinna's second fundamental theorem.

By an example similar to the one used above in Section 7 it can be shown that given an arbitrary closed set in G of capacity zero we always have a function $f(z)$, satisfying the above conditions, such that for this $f(z)$ the function $\Phi(a)$ is positive at all points of the given null-set.

We summarize the above results in

Theorem 6. *Let $f(z)$ be a meromorphic function in the unit circle $|z|<1$. Let the values of $f(z)$ lie in a plane domain G, whose boundary Γ is of positive capacity, and let the boundary values $w=f(e^{i\varphi})$ belong to Γ for almost all φ. Introducing the function*

$$\Phi\left(\frac{1}{f-a}\right) = \frac{1}{2\pi} \lim_{r\to 1} \int_0^{2\pi} g(f(re^{i\varphi}), a, G)\, d\varphi,$$

which characterizes the mean convergence towards the value a in the neighbourhood of the frontier, we have the invariant relation

$$N\left(1, \frac{1}{f-a}\right) + \Phi\left(\frac{1}{f-a}\right) = g\,(a, f\,(0), G).$$

Here $N\left(1, \frac{1}{f-a}\right)$ *is the main term for we have*

$$\Phi\left(\frac{1}{f-a}\right) = 0$$

except perhaps for a set of values a of capacity zero. Such an exceptional null-set can exist.

16. Let us give certain immediate conclusions of this theorem. Since the union of two sets of capacity zero also is of capacity zero the latter statement of Theorem 6 can be expressed in the following form which stresses the invariance in the distribution of values under the above circumstances.

Corollary 2. *Let* $f_1(z)$ and $f_2(z)$ *be two functions meromorphic in the unit circle and satisfying the following conditions:*

1. *The values of* $f_1(z)$ *and* $f_2(z)$ *lie in a domain whose boundary* Γ *is of positive capacity.*
2. *The boundary values* $f_1(e^{i\varphi})$ *and* $f_2(e^{i\varphi})$ *belong to* Γ *for almost all* φ.
3. $f_1(0) = f_2(0)$.

Under these conditions

$$N\left(1, \frac{1}{f_1-a}\right) = N\left(1, \frac{1}{f_2-a}\right),$$

with the possible exception of a set of values of capacity zero.

This result can also be stated in terms of the integral

$$\int_0^1 \frac{\Omega(r)}{r}\,dr$$

which was introduced in Section 9, as follows:

Corollary 3. *Let* $w = f_1(z)$ *and* $w = f_2(z)$ *be two meromorphic functions which fulfill the requirements 1—3 of Corollary 2. Let the w-plane be covered with a non-negative* *mass μ, which satisfies the condition $\mu(e)=0$ for every set e of capacity zero, and let $\Omega(r)$ denote the total mass on the image of $|z|\leq r$ by the considered function. Then*

$$\int_0^1 \frac{\Omega(r)}{r}\,dr$$

has the same value for $f_1(z)$ and $f_2(z)$.

According to Shimizu and Ahlfors, Nevanlinna's first fundamental theorem can be written in the form ([7])

$$m(r,a)+N(r,a)=T(r), \tag{3.5}$$

where

$$m(r,a)=\frac{1}{2\pi}\int_0^{2\pi}\log\frac{\sqrt{1+|f(re^{i\varphi})|^2}\,\sqrt{1+|a|^2}}{|f(re^{i\varphi})-a|}\,d\varphi-\log\frac{\sqrt{1+|f(0)|^2}\,\sqrt{1+|a|^2}}{|f(0)-a|}$$

and

$$T(r)=\frac{1}{\pi}\int_0^r\frac{A(t)}{t}\,dt, \tag{3.6}$$

$A(t)$ denoting the area of the image of $|z|\leq t$ by $f(z)$ in the mapping onto the Riemann sphere.

As remarked above, μ satisfies the condition of Corollary 3 if $d\mu$ is put equal to the spherical element of area. Hence, by (3.6), in this special case Corollary 3 yields

$$T(1,f_1)=T(1,f_2).$$

Further, considering this result it follows from (3.5), by Corollary 2, that

$$m\left(1,\frac{1}{f_1-a}\right)=m\left(1,\frac{1}{f_2-a}\right),$$

except at most for a set of values a of capacity zero. Choosing especially $f_1(z)=f(z)$ arbitrarily and setting $f_2(z)=x(z)$, we have

$$m\left(1,\frac{1}{f-a}\right)\geqq m\left(1,\frac{1}{x-a}\right),$$

where inequality can hold only in a set of capacity zero.

Considering the representation of $N(1,a)$ by means of the Green's function of the image of $|z|<1$ by $f(z)$, the latter part of Theorem 6 can also be expressed in the following form which yields a geometric interpretation.

Corollary 4. *Let $f(z)$ be a meromorphic function in the unit circle and let it satisfy the conditions of Theorem 6. Then, with the possible exception of a set of values a of capacity zero,*

$$\sum_{\nu} g(P_\nu, f(0), F) = g(a, f(0), G),$$

where F is the image of $|z|<1$ by $w=f(z)$, and P_ν $(\nu=1, 2, \ldots)$ denote all points of F above the point $w=a$.

17. Let us return to Theorem 6, and consider a function $f(z)$ which satisfies the conditions stated there. We call a value a *normal* for $f(z)$ if $\Phi(a)=0$; in case $\Phi(a)>0$ the value a is called *exceptional*. By Theorem 6, the set of exceptional values always is of capacity zero.

For a more detailed study we introduce, following the example of Nevanlinna's theory, the concept of *deficiency* $\delta(a)$ by defining for $a \neq f(0)$,

$$\delta(a) = \frac{\Phi(a)}{g(a, f(0), G)} = 1 - \frac{N(1, a)}{g(a, f(0), G)}.$$

For $a=f(0)$ we put

$$\delta(a) = 1 - \frac{N^*\left(1, \dfrac{1}{f-a}\right)}{N^*\left(1, \dfrac{1}{x-a}\right)},$$

where

$$N^*(1, a) = \int_0^1 \frac{n(r, a) - n(0, a)}{r} \, dr + n(0, a) \log r.$$

By this definition, $0 \leq \delta(a) \leq 1$. For normal values $\delta(a)=0$, whereas the maximal deficiency 1 is obtained if and only if $N(1, a)=0$, i.e., if and only if a is a Picard exceptional value for $f(z)$. As mentioned above, given an arbitrary closed null-set B in G we can find a function $f(z)$ such that all points of B are Picard exceptional values for $f(z)$, i.e. at all points of B the deficiency attains its maximum value 1.

Except the values 0 and 1, $\delta(a)$ can take any value k in the interval (0,1). To show this consider the function

$$f(z) = \frac{ez-1}{e-z} e^{\lambda \frac{z+1}{z-1}}, \tag{3.7}$$

where $\lambda>0$. It is easily seen that $|f(z)|<1$ in $|z|<1$, and that $|f(e^{i\varphi})|=1$ except for $\varphi=0$. Hence, in this case G is the unit circle $|w|<1$, Γ being the circumference $|w|=1$.

This function $f(z)$ has its single zero at the point $z=\frac{1}{e}$, so that

$$N(1,0)=\log e=1.$$

Since now

$$g(w,f(0),G)=\log\left|\frac{1-\overline{f(0)}\,w}{w-f(0)}\right|,$$

we have

$$g(0,f(0),G)=\log\frac{1}{|f(0)|}=1+\lambda.$$

Thus

$$\delta(0)=1-\frac{1}{1+\lambda},$$

and we see that $a=0$ is an exceptional value for this $f(z)$.

As a function of λ, $\delta(0)$ is continuous and monotonically increasing. If $\lambda\to 0$, $\delta(0)\to 0$, and if $\lambda\to\infty$, $\delta(0)\to 1$. Hence, given any k, $0<k<1$, we always have a $\lambda>0$ such that $\delta(0)=k$ for the function (3.7).

§ 4. Applications

18. Throughout this chapter we suppose that $f(z)$ is a meromorphic function of bounded characteristic in $|z|<1$ whose boundary values $w=f(e^{i\varphi})$ belong to $|w|\leq 1$ for almost all φ. By Theorem 2, the function $f(z)$ either is bounded: $|f(z)|\leq 1$, or $f(z)$ takes every value a, $|a|>1$, except perhaps a set of capacity zero.

Let us first consider the case $|f(z)|\leq 1$, and normalize $f(z)$ by the requirement $f(0)=0$. Since the Green's function $g(w,a)$ of the unit circle $|w|<1$ is

$$g(w,a)=\log\left|\frac{1-\bar{a}\,w}{w-a}\right|,$$

the relation (3.3) becomes $(a\neq 0)$

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{1-\bar{a}f(re^{i\varphi})}{f(re^{i\varphi})-a}\right|d\varphi+N(r,a)=\log\frac{1}{|a|}. \tag{4.1}$$

If $z_1, z_2, \ldots$ denote the a-points of $f(z)$ we have

$$N(r,a)=\sum_{|z_\nu|<r}\log\frac{r}{|z_\nu|}.$$

Hence, by (4.1),

$$|a|\leq\prod_{|z_\nu|<1}|z_\nu|, \tag{4.2}$$

where every z_ν appears according to its multiplicity. If the product does not contain any factor it is interpreted, here as well as below, to mean unity.

It follows from (4.1) that if equality holds in (4.2), $f(z)$ must necessarily have boundary values of modulus 1 almost everywhere. On the other hand, we conclude from Theorem 6 that if this condition is fulfilled equality is true up to a set of capacity zero. We thus get

Corollary 5. *Let $f(z)$ be regular in the unit circle and satisfy the conditions $|f(z)| \leqq 1$, $f(0) = 0$. Let $z_1, z_2, \ldots$ denote all points in $|z| < 1$, where $f(z)$ takes a given value a $(\neq 0)$. Then*

$$|a| \leqq \prod_{|z_\nu| < 1} |z_\nu|.$$

Equality can hold only if $|f(e^{i\varphi})| = 1$ for almost all φ. If this condition is fulfilled equality is valid except perhaps for a set of values a of capacity zero.

Hence, for a bounded function $f(z)$ for which $f(0) = 0$ and $|f(e^{i\varphi})| = 1$ for almost all φ, a value a $(\neq 0)$ is normal, in the sense of the definition given in Section 17, if and only if

$$\prod_{|z_\nu| < 1} |z_\nu| = |a|.$$

In case $a = 0$ the corresponding result reads as follows: If

$$f(z) = c_n z^n + c_{n+1} z^{n+1} + \cdots \qquad (c_n \neq 0)$$

the value $a = 0$ is normal if and only if

$$\prod_{0 < |z_\nu| < 1} |z_\nu| = |c_n|, \tag{4.3}$$

where $z_1, z_2, \ldots$ denote the zeros of $f(z)$. If $a = 0$ is an exceptional value we have

$$\prod_{0 < |z_\nu| < 1} |z_\nu| > |c_n|.$$

19. Let us consider these relations from a somewhat different point of view. We construct the function

$$\omega(z, r) = \frac{r(f(z) - a)}{r^2 - \bar{a} f(z)} : \prod_{|z_\nu| \leqq r} \frac{r(z - z_\nu)}{r^2 - \bar{z}_\nu z},$$

where $z_1, z_2, \ldots$ again denote the a-points of $f(z)$. By Schwarz's lemma, $|\omega(z, r)| \leqq 1$ in $|z| < 1$. Letting $r \to 1$ we thus obtain the inequality

$$\left|\frac{f(z)-a}{1-\bar{a}f(z)}\right| \leq \prod_{|z_\nu|<1}\left|\frac{z-z_\nu}{1-\bar{z}_\nu z}\right|. \tag{4.4}$$

From the maximum principle we conclude that if equality is true in (4.4) for one point z, it identically holds in z. Now, for $z=0$ the relation (4.4) coincides with (4.2). By considering Corollary 5 we thus find again the following result of Frostman [1]:

Let $f(z)$ be a regular function in the unit circle such that $|f(z)|\leq 1$ and $|f(e^{i\varphi})|=1$ for almost all φ. If a is a complex number, $|a|<1$, and $z_1, z_2, \ldots$ denote the a-points of $f(z)$, we have the representation formula

$$\frac{f(z)-a}{1-\bar{a}f(z)} = e^{i\lambda}\prod_{|z_\nu|<1}\frac{z-z_\nu}{1-\bar{z}_\nu z}e^{-i\alpha_\nu} \qquad (\alpha_\nu=\pi+\arg z_\nu) \tag{4.5}$$

if and only if a is a normal value for $f(z)$. Hence, (4.5) *is valid except perhaps for a set of values a of capacity zero.*

As is well known, every convergent Blaschke-product $\pi(z)$ has boundary values $\pi(e^{i\varphi})$ of modulus 1 for almost all φ. From the above we conclude the following interrelation between the class of bounded functions with boundary values of modulus 1 almost everywhere and its subclass of convergent Blaschke-products: *If $f(z)$ belongs to the former class then for "almost all" a the linear transforms*

$$\frac{f(z)-a}{1-\bar{a}f(z)}$$

belong to the latter class. In particular, if $a=0$ is a normal value for $f(z)$, i.e., if (4.3) holds, then $f(z)$ can be expanded as a Blaschke-product by means of its zeros.

Heins has recently proved[1] that an infinite convergent Blaschke-product takes every value $e^{i\vartheta}$ $(0\leq\vartheta<2\pi)$ infinitely often on the boundary. Hence, by the above result, if $f(z)$ is non-rational and a is a normal value for $f(z)$, the function

$$\frac{f(z)-a}{1-\bar{a}f(z)}$$

also has this property. This yields us

Theorem 7. *Let $f(z)$ be regular and non-rational in the unit circle, $|f(z)|\leq 1$ and $|f(e^{i\varphi})|=1$ almost everywhere. Then $f(z)$ takes every value $e^{i\vartheta}$ $(0\leq\vartheta<2\pi)$ infinitely often on the frontier $|z|=1$.*

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[1] In a lecture given in Zürich in February 1953.

20. Let us now suppose that $w = f(z)$ also takes values outside $|w| \leqq 1$, and normalize $f(z)$ by the requirement $f(0) = \infty$. If $f(z) = a$ $(|a| > 1,\ a \neq \infty)$ at the points $z_1, z_2, \ldots$, we have by Theorem 1, up to a set of capacity zero,

$$N(1, a) = \sum \log \frac{1}{|z_\nu|} \geqq g(a, \infty, (|w| > 1)) = \log |a|.$$

We take the antilogarithm and express the result in

Corollary 6. *Let $f(z)$ be meromorphic and of bounded characteristic in the unit circle, $f(0) = \infty$, and let $|f(e^{i\varphi})| \leqq 1$ for almost all φ. If a is a complex number of modulus > 1, and $z_1, z_2, \ldots$ denote all a-points of $f(z)$, then*

$$\prod_{|z_\nu| < 1} |z_\nu| \leqq \frac{1}{|a|},$$

except perhaps for a set of values a of capacity zero.

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