ON CERTAIN EXTREMUM PROBLEMS FOR ANALYTIC FUNCTIONS.

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Introduction.

0.1. In order to state, in their simplest form, the type of problems to be discussed, we suppose, first, that

(0.1.1) $$f(z)=\sum_0^\infty a_k z^k$$

is regular for $|z|\le 1$; and that $\mathfrak{k}(z)$ is regular for $|z|\le 1$, except for a finite number of poles β_i with $|\beta_i|<1$. Then

(0.1.2) $$J(f)=\frac{1}{2\pi i}\int_{|\zeta|=1} f(\zeta)\,\mathfrak{k}(\zeta)\,d\zeta$$

is the sum of the residues of $f(z)\,\mathfrak{k}(z)$ at the points β_i. If, for instance, $\mathfrak{k}(z)=\sum_0^n c_k z^{-(k+1)}$ then $J(f)=\sum_0^n c_k a_k$; if $\mathfrak{k}(z)=n!\,(z-\beta)^{-(n+1)}$, $|\beta|<1$, then $J(f)=f^{(n)}(\beta)$.

In these and similar cases it is a natural and important problem to determine, for a given 'kernel' $\mathfrak{k}(z)$, the precise sup $|J(f)|$ when the functions $f(z)$ vary inside a suitably given class: for instance, the class of all f with $|f|\le 1$ in $|z|\le 1$.

0.2. In a previous paper [M-R][2] A. J. Macintyre and one of the present authors studied such extremum problems for the following classes H_p: Let $1\le p\le\infty$. If $p<\infty$ then H_p denotes the class of all functions $f(z)$ regular in $|z|<1$ for which the mean values

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[2] MACINTYRE and ROGOSINSKI, quoted as [M-R] throughout. Compare the list of references at the end of this paper.

$$(0.2.1) \qquad M_p(f, r) = \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p\, dt\right\}^{1/p}$$

remain bounded for $0 \leq r < 1$. If $p = \infty$, the class H_∞ is the class of all $f(z)$ regular and bounded in $|z| < 1$; that is,

$$(0.2.2) \qquad M_\infty(f, r) = \max |f(re^{it})|$$

remains bounded for $0 \leq r < 1$.

As for the given kernel $\mathfrak{k}(z)$ we assumed that it was regular in $|z| < 1$ except for a finite number of poles β_i, and that the mean values $M_q(\mathfrak{k}, r)$ remained bounded for all sufficiently large $r < 1$. Here $\frac{1}{p} + \frac{1}{q} = 1$, so that to $p = 1$ corresponds $q = \infty$ and vice versa; to $p = 2$ corresponds $q = 2$.

It is well known that under these assumptions the radial boundary values

$$(0.2.3) \qquad f(e^{it}) = \lim_{r \to 1-0} f(re^{it}), \quad \mathfrak{k}(e^{it}) = \lim_{r \to 1-0} \mathfrak{k}(re^{it})$$

exist p.p. (that is, for almost all t), and that $f(e^{it}) \in L^p$ and $\mathfrak{k}(e^{it}) \in L^q$. Here L^p, for $1 \leq p < \infty$, is the class of all complex valued functions $\varphi(t)$, measurable in $\langle 0, 2\pi \rangle$, for which $|\varphi(t)|^p$ is L-integrable; L^∞ is the class of all essentially bounded measurable $\varphi(t)$. Also $M_p(f, r) \to M_p(f, 1)$ as $r \to 1 - o$, where $M_p(f, 1)$ is the corresponding mean value for the boundary function when $p < \infty$. $M_\infty(f, 1)$ is the ess. sup $|f(e^{it})|$.

Again, the integral (0.1.2), taken with these boundary values, is the sum of the residues of $f(z)\,\mathfrak{k}(z)$ at the β_i.

In [M-R] the problem was to determine, for a given kernel of the described 'meromorphic' type, the sup $|J(f)|$ with respect to all $f \in H_p$ for which the mean value $M_p(f, 1)$ is prescribed. A complete theory of these extremum problems was obtained. It should, however, be understood that this theory was mainly obtained by joining together, and extending to general p, the various methods and results found scattered through an extensive earlier literature[1] dealing, as a rule, with the important special cases $p = 1$, $p = 2$, and $p = \infty$. Moreover, we freely adopted or quoted the heterogeneous and sometimes difficult arguments of algebraic, variational, and even topological character as we found them in this literature. The main interest in [M-R] was in the various applications of the theory, and of these we gave a systematic account.

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[1] Compare the list of references in [M-R].

0.3. In the present paper we propose both to extend the theory of extremum problems in H_p to full generality and, at the same time, to give a self-contained and uniform account of it, replacing thereby the previous heterogeneous arguments. In fact, it is by going to the natural limits of generality that also the natural approach to these problems becomes evident.

We rewrite (0.1.2) in the form

$$J(f) = \frac{1}{2\pi} \int_{|\zeta|=1} f(\zeta)\,\zeta\,\mathfrak{k}(\zeta)\,dt = \frac{1}{2\pi}\int_0^{2\pi} \varphi(t)\,\varkappa(t)\,dt, \tag{0.3.1}$$

where[1]

$$\varphi(t) \equiv f(e^{it}), \quad \varkappa(t) \equiv e^{it}\,\mathfrak{k}(e^{it}). \tag{0.3.2}$$

Next, H^p denotes the class of all (radial) boundary values $\varphi(t) \equiv f(e^{it})$ where $f(z) = H_p$. Thus H^p is a subclass of $L^p : H^p \subset L^p$. Let

$$J(f) = I(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(t)\,\varkappa(t)\,dt, \tag{0.3.3}$$

where $\varphi \in H^p$ and $\varkappa \in L^q$. *It should be noted that we have dropped the assumption, essential in [M-R], that $\varkappa(t)$ should be obtained, through* (0.3.2), *from a meromorphic kernel function $\mathfrak{k}(z)$.*

We also write, when $p < \infty$,

$$M_p(f, 1) = M_p(\varphi) = \left\{\frac{1}{2\pi}\int_0^{2\pi} |\varphi(t)|^p\,dt\right\}^{1/p}, \tag{0.3.4}$$

while $M_\infty(f, 1) = M_\infty(\varphi) = \text{ess. sup}\,|\varphi(t)|$.

Our extremum problem is then as follows:

Maximum problem in H^p: *To determine, for given $\varkappa \in L^q$, the* sup $|I(\varphi)|$ *for all $\varphi \in H^p$ with given $M_p(\varphi)$.*

In general, of course, $I(\varphi)$ will no longer have a simple interpretation as sum of residues or so. There exist, however, quite apart from the intrinsic interest of the general problem, many interesting extremum problems not covered by the 'meromorphic' case. Thus we have, when $f(z) \in H_p$,

$$\int_{-1}^{1} f(x)\,dx = -i\int_0^{\pi} f(e^{it})\,e^{it}\,dt, \tag{0.3.5}$$

[1] $\equiv$ denotes equality p. p.

by Cauchy's theorem. Here $\varkappa(t) = -2\pi i e^{it}$ for $0 \le t \le \pi$, and $\varkappa(t) = 0$ for $\pi < t < 2\pi$. The corresponding maximum problem in H_1 was discussed by Fejér and F. Riesz.[1]

0.4. Applying Hölder's inequality to $I(\varphi)$ we obtain, first,

$$|I(\varphi)| \le M_p(\varphi)\, M_q(\varkappa). \tag{0.4.1}$$

Next, we shall say that a 'kernel' $\varkappa^*(t) \in L^q$ is equivalent[2] (in L^q) to another kernel $\varkappa(t) \in L^q$, and we then write $\varkappa^* \| \varkappa$, if

$$\int_0^{2\pi} \varphi(t)\, \varkappa^*(t)\, dt = \int_0^{2\pi} \varphi(t)\, \varkappa(t)\, dt \tag{0.4.2}$$

for all $\varphi \in H^p$. We can then replace (0.4.1) by

$$|I(\varphi)| \le M_p(\varphi) \cdot \inf_{\varkappa^* \| \varkappa} M_q(\varkappa^*). \tag{0.4.3}$$

The crucial question of the whole theory is then whether this estimate is best possible; that is, whether,

$$\sup |I(\varphi)| = M_p(\varphi) \cdot \inf_{\varkappa^* \| \varkappa} M_q(\varkappa^*). \tag{0.4.4}$$

We shall see that the answer is affirmative in all cases. At the same time, this will show that our maximum problem in H^p is equivalent to the following 'conjugate' problem:

Minimum problem in L^q: *Given $\varkappa(t) \in L^q$, to determine the* $\inf M_q(\varkappa^*)$ *for all* $\varkappa^* \| \varkappa$.

If we write $\varkappa^* = \varkappa - \lambda$, then $\lambda \in L^q$ is characterized by

$$\int_0^{2\pi} \varphi(t)\, \lambda(t)\, dt = 0 \tag{0.4.5}$$

for all $\varphi \in H^p$. The minimum problem then appears as a problem of best approximation of a given $\varkappa$ by the λ.

0.5. A function $\Phi \in H^p$ with $M_p(\Phi) = 1$ is said to be an *extremal function* (for $I(\varphi)$ in H^p) if

$$I(\Phi) = \max |I(\varphi)| \tag{0.5.1}$$

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[1] Fejér and Riesz.

[2] Equivalent kernels were first used by E. Landau in his familiar determination of

$$\sup |a_0 + a_1 + \cdots + a_n|$$

for all $f(z)$ regular in $|z| < 1$ with $|f(z)| < 1$ [subclass of H_∞]; Landau.

for all $\varphi \in H^p$ with $M_p(\varphi) = 1$. Note that we have normalized Φ, first by normalizing the $M_p(\varphi)$, and secondly by requiring $I(\Phi)$ to be positive.[1] Clearly, $|I(\varepsilon \Phi)| = I(\Phi)$ if $|\varepsilon| = 1$.

Similarly, we say that $\mathrm{K}(t) \in L^q$ is an extremal kernel (for $\varkappa(t)$ in L^q) if $\mathrm{K} \| \varkappa$ and if

$$M_q(\mathrm{K}) = \min_{\varkappa^* \| \varkappa} M_q(\varkappa^*). \tag{0.5.2}$$

We then have two more conjugate problems:

Extremum problem in H^p: *Does an extremal function $\Phi \in H^p$ exist and, if so, is it unique?*

Extremum problem in L^p: *Does an extremal kernel $\mathrm{K} \| \varkappa \in L^q$ exist and, if so, is it unique?*

Our main result concerning all these extremum problems is as follows:

Theorem A. *Let* $1 \le p \le \infty$.

The identity (0.4.4) *holds in any case.*

If $1 < p \le \infty$ $(1 \le q < \infty)$, *then both the extremal function* $\Phi \in H^p$ *and the extremal kernel* $\mathrm{K} \| \varkappa \in L^q$ *exist uniquely.*[2]

If $p = 1$ $(q = \infty)$, *then at least one extremal kernel* $\mathrm{K} \| \varkappa \in L^\infty$ *exists, but there may be an infinity of such extremal kernels. An extremal function* $\Phi \in H^1$ *need not exist, and there may be an infinity of such extremal functions.*

It should be emphazised, once more, that the main difficulty in all previous treatments of extremum problems of our type lay in establishing the identity (0.4.4). It is, we think, an interesting example of the power of certain results in modern 'abstract' analysis that we shall obtain, at least when $1 \le p < \infty$, both (0.4.4) and the existence of an extremal kernel in a few lines from the familiar Hahn-Banach extension theorem for bounded linear functionals on normed vector spaces.[3] This is possible since (0.3.3) is the general form of a bounded linear functional on the Banach space L^p, if $1 \le p < \infty$. For $p = \infty$ this is no longer true. Nevertheless, the extension theorem leads to the same conclusion, though in a rather more delicate way.

0.6. In any concrete extremum problem of our type it will be desirable to determine either an extremal function Φ or an extremal kernel K. This, of course, is

[1] We exclude, throughout, the trivial case when $I(\varphi) = 0$ for all $\varphi \in H^p$.

[2] The unique existence of K was first proved by Doob who studied the Minimum problem in L^q.

[3] Banach, p. 55. We require this theorem for complex valued functionals (Bohnenblust and Sobczyk); compare Hille, p. 20.

usually impossible. One would then wish to have at least some indication as regards the possible forms of Φ or K. In certain cases such indication can be obtained from the following remark.

By (0.4.4), and since $M_p(\Phi)=1$, we shall have $I(\Phi)=M_q(\mathrm{K})$. On the other hand

$$I(\Phi)=\frac{1}{2\pi}\int_0^{2\pi}\Phi\,\mathrm{K}\,dt\le\frac{1}{2\pi}\int_0^{2\pi}|\Phi\,\mathrm{K}|\,dt\le M_q(\mathrm{K}). \tag{0.6.1}$$

A simple discussion of the two signs of equality here shows that we must have

$$\arg(\Phi(t)\,\mathrm{K}(t))\equiv 0,\quad |\Phi(t)|^{1/q}\equiv A\,|\mathrm{K}(t)|^{1/p}. \tag{0.6.2}$$

Now $\Phi(t)$ is the boundary function of a certain associated function $F(z)\in H_p$. If also $\varkappa(t)$ is the boundary function of an associated function $k(z)$, regular and with bounded $M_q(k,r)$ in some annulus $\varrho<|z|<1$, then $\mathrm{K}(t)$ will have corresponding properties. In this case it is possible to employ the Schwarz principle of inversion and to continue the function $F(z)\,K(z)$ across $|z|=1$. This method was used in the meromorphic case treated in [M-R]. It followed then that $F(z)\,K(z)$ was a rational function. From this we were able to determine the possible forms of $F(z)$ and $K(z)$ themselves; Φ and K always exist in this case even when $p=1$. However, we had again to use heterogeneous results and arguments to obtain these forms. In the present paper we shall regain all this by a self-contained uniform method.

1. The classes H_p and H^p.

1.1. Throughout this paper we have $1\le p\le\infty$. The classes L^p and H_p, and the mean values therein, have been defined in the introduction. We require the following known properties of the functions $f(z)\in H_p$:

(i) The radial boundary values

$$\varphi(t)\equiv f(e^{it})=\lim_{r\to 1-0} f(re^{it}) \tag{1.1.1}$$

exist p. p.[1]

(ii)[1] If $r\to 1-0$, then $f(re^{it})\underset{p}{\to}\varphi(t)$. By this we mean

$$\int_0^{2\pi}|f(re^{it})-\varphi(t)|^p\,dt\to 0 \tag{1.1.2}$$

[1] Compare Zygmund, p. 162—164. If $p<\infty$, then p-convergence is 'strong' convergence of order p. If $p=\infty$, this is not the case since strong convergence then means uniform convergence p. p.

when $p < \infty$. If $p = \infty$, we mean bounded convergence p.p. It follows, in both cases, that $\varphi(t) \in L^p$ and that

$$M_p(f, r) \to M_p(\varphi) \tag{1.1.3}$$

as $r \to 1 - 0$.

(iii) *If* $f(z) \in H_p$ *and if* $\varphi(t) = 0$ *in a set of positive measure, then* $f(z) \equiv 0$. This theorem is classical[1] in the case $p = \infty$, and it also holds[2] for functions $f(z)$, regular in $|z| < 1$ and with bounded characteristic

$$T(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(r e^{it})| \, dt. \tag{1.1.4}$$

Now, if $f(z) \in H_p$, then

$$T(r) \leq \log M_1(f^*, r) \leq \log(1 + M_1(f, r)) \leq \log(1 + M_p(f, r)), \tag{1.1.5}$$

where $f^* = \max(|f|, 1)$, since the geometrical mean of $|f^*|$ is not greater than its arithmetical one.

1.2. If $\varphi(t) \in L^p$ we write its formal Laurent expansion

$$\varphi(t) \sim \sum_{-\infty}^{\infty} c_k e^{ikt} \tag{1.2.1}$$

where

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) e^{-ikt} \, dt. \tag{1.2.2}$$

Let H^p denote the class of boundary functions $\varphi(t) \equiv f(e^{it})$ where $f(z) \in H_p$. By § 1.1 (ii), $H^p \subset L^p$.

The following theorem is certainly not new. We are, however, unable to quote it and hence add its simple proof.

Theorem 1. *The class* H^p *is the class of all functions* $\varphi(t) \in L^p$ *with formal Laurent-Taylor expansion*

$$\varphi(t) \sim \sum_{0}^{\infty} c_k e^{ikt}. \tag{1.2.3}$$

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[1] F. and M. Riesz; see also Zygmund, p. 145.

[2] R. Nevanlinna, p. 197—198.

Proof. (i) Let $f(z)=\sum_0^\infty c_k z^k \in H_p$. Then, for $r<1$,

(a) $$\frac{1}{2\pi}\int_0^{2\pi} f(re^{it})\, e^{-ikt}\, dt = \begin{cases} c_k & \text{for } k\geq 0 \\ 0 & \text{for } k<0 \end{cases}.$$

Hence (1.2.3) follows from $f(re^{it}) \underset{p}{\to} \varphi(t)$. Also $\varphi(t)\in L^p$ (§ 1.1).

(ii) Let $\varphi(t)\sim\sum_0^\infty c_k e^{ikt}\in L^p$, so that

(b) $$\frac{1}{2\pi}\int_0^{2\pi} \varphi(t)\, e^{-ikt}\, dt = \begin{cases} c_k & \text{for } k\geq 0 \\ 0 & \text{for } k<0 \end{cases}.$$

Hence, for $k\geq 1$,

(c) $$c_k=\frac{1}{\pi}\int_0^{2\pi}\varphi(t)\cos kt\, dt = -\frac{i}{\pi}\int_0^{2\pi}\varphi(t)\sin kt\, dt.$$

Writing $\varphi=u-iv$, $c_k=a_k-ib_k$, we see that $u\in L^p$, $v\in L^p$; that u has Fourier coefficients a_k, b_k and that v is conjugate to u. Next, the function $f(z)=\sum_0^\infty c_k z^k$ is regular in $|z|<1$, since $c_k\to 0$. Also, with obvious notations, $u(re^{it})\to u(t)$, $v(re^{it})\to v(t)$ p.p., and[1] $u(re^{it})\underset{p}{\to} u(t)$, $v(re^{it})\underset{p}{\to} v(t)$. Hence $f(re^{it})\to\varphi(t)$ p.p. and $f(re^{it})\underset{p}{\to}\varphi(t)$. It follows that $M_p(f,r)\uparrow M_p(\varphi)$ if $r\uparrow 1$, so that $f(z)\in H_p$.

We note that $f(z)$ and $\varphi(t)$ determine each other uniquely p.p.

2. Equivalent kernels.

2.1. Let $\varkappa(t)\in L^q$ where $\frac{1}{p}+\frac{1}{q}=1$. A kernel $\varkappa^*(t)\in L^q$ is said to be *equivalent* to $\varkappa(t)$ (in L^q), and we then write $\varkappa^*\,\|\,\varkappa$, if

(2.1.1) $$\int_0^{2\pi}\varphi(t)\,\varkappa^*(t)\,dt=\int_0^{2\pi}\varphi(t)\,\varkappa(t)\,dt$$

for all $\varphi\in H^p$. The integrals exist, by Hölder's inequality.

Theorem 2. *$\varkappa^*(t)\,\|\,\varkappa(t)$ (in L^q) if, and only if,*

(2.1.2) $$\varkappa^*(t)=\varkappa(t)+e^{it}\gamma(t)$$

where $\gamma(t)\in H^q$.

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[1] Compare Zygmund, p. 87.

Proof. (i) If $\varkappa^* \| \varkappa$, then $\varkappa^* - \varkappa \in L^q$ and hence $\gamma(t) = (\varkappa^*(t) - \varkappa(t))\, e^{-it} \in L^q$. Let $e^{it}\gamma(t) \sim \sum_{-\infty}^{\infty} c_k e^{ikt}$. Putting $\varphi(t) = e^{-ikt}$, with $k \le 0$, into (2.1.1), we see that $c_k = 0$ for $k \le 0$. Hence $\gamma \in H^q$, by Theorem 1.

(ii) Suppose that $\varkappa^*$ is of the form (2.1:2). Then, clearly, $\varkappa^* \in L^q$. Now, let $f(z) \in H_p$ and $g(z) \in H_q$ be the functions associated, by Theorem 1, with $\varphi(t) \in H^p$ and $\gamma(t) \in H^q$, respectively. By § 1.1 (ii), it is possible to apply Cauchy's theorem to the function $f(z)\, g(z)$ *on* the unit circle $|\zeta| = 1$. This gives

$$\text{(a)} \qquad \int_0^{2\pi} \varphi(t)\, \gamma(t)\, e^{it}\, dt = -i \int_{|\zeta|=1} f(\zeta)\, g(\zeta)\, d\zeta = 0,$$

which is equivalent to (2.1.1). Hence $\varkappa^* \| \varkappa$.

2.2. Let $\varkappa(t) \sim \sum_{-\infty}^{\infty} c_k e^{ikt} \in L^q$. For some applications it is of interest to know whether $\varkappa_1(t) \in L^q$, where $\varkappa_1 \sim \sum_{-\infty}^{0} c_k e^{ikt}$.

(i) $\varkappa_1(t) \| \varkappa(t)$ *if, and only if*, $\varkappa_1(t) \in L^q$. This is obvious.

(ii)[1] *If* $1 < q < \infty$ *then* $\varkappa_1(t) \| \varkappa(t)$.

We sketch the proof the formal details of which are easily verified. If $\varkappa = u - iv$, then $u \in L^q$, $v \in L^q$. If $\tilde{u}$ and $\tilde{v}$ are the conjugate functions of u and v, respectively, then it is known[2] that $\tilde{u} \in L^q$, $\tilde{v} \in L^q$. Now, putting $c_0 = \alpha_0 - i\beta_0$, one verifies that $\varkappa_1 = U - iV$ where

$$\text{(b)} \qquad U = \tfrac{1}{2}(\alpha_0 + u + \tilde{v}), \quad V = \tfrac{1}{2}(\beta_0 + v - \tilde{u}).$$

Here $U \in L^q$, $V \in L^q$, so that $\varkappa_1 \in L^q$.

(iii) *Suppose that*

$$\text{(2.2.1)} \qquad k(z) = \sum_{-\infty}^{\infty} c_n z^n$$

is regular in some annulus $\varrho < |z| < 1$. *Then* $\varkappa_1(t) \| \varkappa(t)$.

For, $k_1(z) = \sum_{-\infty}^{0} c_n z^n$ is regular for $|z| > \varrho$ and hence $\varkappa_1(t) = k(e^{it})$ is bounded, so that $\varkappa_1 \in L^\infty \subset L^q$.

[1] Compare Carleson.

[2] See Zygmund, p. 147.

2.3. It is convenient to use geometrical language. We consider the equivalence classes of functions $\chi \in L^p$ as 'points' in a 'space' L^p: if $\chi_1 \equiv \chi_2$, then χ_1 and χ_2 determine the same point in L^p. This space becomes a Banach space on introducing as metric the norm

$$\|\chi\| = \|\chi\|_p = M_p(\chi). \tag{2.3.1}$$

The 'triangle inequality', that is $\|\chi_1 + \chi_2\| \leq \|\chi_1\| + \|\chi_2\|$, is here Minkowski's inequality

$$M_p(\chi_1 + \chi_2) \leq M_p(\chi_1) + M_p(\chi_2). \tag{2.3.2}$$

Convergence in the space L^p is 'strong convergence' according to the norm: $\chi_n \to \chi$ is $\|\chi_n - \chi\|_p \to 0$. If $1 \leq p < \infty$, this is also $\chi_n \underset{p}{\to} \chi$. If $p = \infty$, however, it is $\chi_n(t) \to \chi(t)$ uniformly p.p. and not just boundedly p.p., as $\chi_n \underset{\infty}{\to} \chi$ denotes.

Theorem 3. *The set $S_\varkappa$ of all $\varkappa^* \| \varkappa$ is closed and convex in L^q.*

Proof. From the definition (2.1.1) of equivalence it is clear that $\varkappa_n \| \varkappa$ and $\varkappa_n \to \varkappa^*$ imply $\varkappa^* \| \varkappa$. Hence $S_\varkappa$ is closed. Again, by (2.1.1), if $\varkappa_1 \| \varkappa$, $\varkappa_2 \| \varkappa$, then $A\varkappa_1 + (1-A)\varkappa_2 \| \varkappa$ for any complex A, and, in particular, for $0 \leq A \leq 1$. Hence $S_\varkappa$ is convex.

2.4. Let $\varkappa^* \| \varkappa$. By (2.1.2), $\varkappa^* = \varkappa - \lambda$ where $\lambda \in H^q$. The λ form a linear subspace $\Lambda \subset H^q$, which is characterised by the property

$$\int_0^{2\pi} \varphi(t)\,\lambda(t)\,dt = 0 \quad \text{for all} \quad \varphi \in H^p. \tag{2.4.1}$$

We call, therefore, Λ the *annihilator subspace* of H^p (in H^q).

3. Extremal kernels and extremal functions.

3.1. Given $\varkappa \in L^q$, our integral

$$I(\varphi) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(t)\,\varkappa(t)\,dt \tag{3.1.1}$$

is a bounded linear functional on the subspace H^p of L^p. Its norm, on H^p, is defined as

$$\|I\|_{H^p} = \|I\| = \sup_{\|\varphi\|_p = 1} |I(\varphi)|, \quad \varphi \in H^p. \tag{3.1.2}$$

Its determination is (after an obvious normalization to $M_p(\varphi) = \|\varphi\| = 1$) exactly our maximum problem in H^p.

We also write, with the notation in § 2.4,

(3.1.3) $$\delta_\varkappa = \inf_{\varkappa^* \| \varkappa} M_q(\varkappa^*) = \inf_{\varkappa^* \| \varkappa} \| \varkappa^* \|_q = \inf_{\lambda \in \Lambda} \| \varkappa - \lambda \|_q,$$

so that $\delta_\varkappa$ is the distance of the given kernel $\varkappa$ from the annihilator space Λ.

By Hölder's inequality

(3.1.4) $$\| I \| \leq \delta_\varkappa.$$

Our main aim will be to establish equality here.

We shall, usually, exclude the 'trivial' case $\| I \| = 0$; that is, $I(\varphi) = 0$ for all $\varphi \in H^p$. In this case $e^{-it} \varkappa \in H^q$, as we see on putting $\varphi = e^{-ikt}$ for $k \leq 0$. Hence $\varkappa \| 0$ and $\delta_\varkappa = 0$. Conversely, $\delta_\varkappa = 0$ implies $\| I \| = 0$, by (3.1.4). Thus, in the trivial case, we have equality in (3.1.4).

3.2. A kernel $\mathrm{K} \| \varkappa$ is said to be an *extremal kernel* (associated with $\varkappa$) if

(3.2.1) $$\| \mathrm{K} \|_q = \delta_\varkappa,$$

so that the inferior in (3.1.3) is a minimum attained at K. In the 'trivial' case $\delta_\varkappa = 0$, $\mathrm{K} \equiv 0$ is an extremal kernel.

Theorem 4. *The set* $S_\varkappa^*$ *of all extremal kernels* $\mathrm{K} \| \varkappa$ *is either empty, or convex and closed in* L^q.

Proof. The sphere $\| \chi \|_q \leq \delta_\varkappa$, with centre 0, is convex (by (2.3.2)) and closed in L^q. Hence $S_\varkappa^*$, the intersection of the convex and closed set $S_\varkappa$ with this sphere, is either empty, or convex and closed in L^q.

We shall later see that, actually, $S_\varkappa^*$ is not empty.

3.3. A function $\Phi \in H^p$ is said to be an *extremal function* (for I) if $\| \Phi \| = 1$ and

(3.3.1) $$I(\Phi) = \| I \|.$$

Theorem 5. *If* $\| I \| > 0$, *then the set* E *of all extremal functions* Φ *is either empty, or closed and convex in* H^p.

Proof. (i) If $\Phi_n \in E$ and $\Phi_n \to \Phi$ (in L^p), then, clearly, $\Phi \in H^p$, $\| \Phi_n \| \to \| \Phi \|$ and $I(\Phi_n) \to I(\Phi)$. Thus $\| \Phi \| = 1$, $I(\Phi) = \| I \|$, and $\Phi \in E$: E is closed.

(ii) Let $\Phi_1 \in E$, $\Phi_2 \in E$, and

(a) $$\Phi = A \Phi_1 + (1 - A) \Phi_2, \quad 0 \leq A \leq 1.$$

Then $\Phi \in H^p$ and

(b) $$I(\Phi) = A\, I(\Phi_1) + (1-A)\, I(\Phi_2) = \|I\|,$$

while

(c) $$\|\Phi\| \le A\,\|\Phi_1\| + (1-A)\,\|\Phi_2\| = 1.$$

Also $\|\Phi\| > 0$, because otherwise $\Phi = 0$, $I(\Phi) = \|I\| = 0$. Hence $\|\Phi\| = \vartheta$ where $0 < \vartheta \le 1$. It follows that $\vartheta^{-1}\Phi \in H^p$, $\|\vartheta^{-1}\Phi\| = 1$, and

(d) $$\|I\| \ge |I(\vartheta^{-1}\Phi)| = \vartheta^{-1} I(\Phi) = \vartheta^{-1}\|I\| \geq \|I\|,$$

so that $\vartheta = 1$ and $\Phi \in E$: E is convex.

We note that, in the trivial case $\|I\| = 0$, $I(\varphi) = 0$ for all $\varphi \in H^p$. E is the sphere $\|\Phi\| = 1$ which is closed but not convex.

4. Extremum problems in normed linear spaces.

4.1. The situation we are dealing with becomes clearer when we consider it as an instance of a more general one.

Let X be a normed linear space over the field of complex numbers. The norm for any $\chi \in \mathrm{X}$ is denoted by $\|\chi\|$; it satisfies, in particular, the triangle inequality $\|\chi_1 + \chi_2\| \le \|\chi_1\| + \|\chi_2\|$. If B is a (bounded) linear functional on X, its norm is defined as

(4.1.1) $$\|B\| = \sup_{\|\chi\|=1} |B(\chi)|, \quad \chi \in \mathrm{X}.$$

With this norm the B form the '*conjugate*' normed linear vector space X^* of X. In particular, $\|B\|$ satisfies the triangle inequality.

Now let Φ be a linear subspace of X, and let I be a linear functional on Φ. Its norm, on Φ, is

(4.1.2) $$\|I\|_\Phi = \sup_{\|\varphi\|=1} |I(\varphi)|, \quad \varphi \in \Phi.$$

In particular, a B on X becomes an I on Φ by 'restricting' it to the $\varphi \in \Phi$. Clearly,

(4.1.3) $$\|B\|_\Phi \le \|B\|.$$

A functional $B^* \in \mathrm{X}^*$ is said to be equivalent to I if $B^*(\varphi) = I(\varphi)$ for all $\varphi \in \Phi$; we then write $B^* \| I$. Clearly, $\|B^*\|_\Phi = \|I\|_\Phi$ so that, by (4.1.3),

(4.1.4) $$\|I\|_\Phi \le \inf_{B^* \| I} \|B^*\|.$$

Theorem I. *The set S_I of all $B^* \| I$ is closed and convex in* X^*.

The proof is like that of Theorem 3.

4.2. A functional $B^{*\!*} \in X^*$ is called an *extremal functional*, associated with I, if $B^{*\!*} \| I$ and

$$\|B^{*\!*}\| = \min_{B^* \| I} \|B^*\|. \tag{4.2.1}$$

Theorem II. *The set $S_I^{*\!*}$ of all extremal functionals' $B^{*\!*} \| I$ is convex and closed in* X^*.

The proof is like that of Theorem 4, (2.3.2) being replaced by the triangle inequality in X^*. That $S_I^{*\!*}$ (and hence S_I in Theorem I) is not empty, is part of the following theorem, which expresses a general '*principle of duality*' connecting a maximum problem in X with a minimum problem in the conjugate X^*.

Theorem III. *There exists at least one extremal functional $B^{*\!*} \| I$. Moreover*

$$\|I\|_\Phi = \sup_{\|\varphi\| \le 1} |I(\varphi)| = \min_{B^* \| I} \|B^*\| = \|B^{*\!*}\|. \tag{4.2.2}$$

Proof. By the Hahn-Banach extension theorem for a normed linear space over the complex field[1], every I on Φ is the restriction of some $B^* \in X^* : B^* \| I$. Moreover, there is at least one such $B^{*\!*} \| I$, for which $\|B^{*\!*}\| = \|I\|_\Phi$.

We note that, if B_0^* is a particular linear functional equivalent to I, then the $B^* \| I$ are exactly the linear functionals $B^* = B_0^* - L$, where $L(\varphi) = 0$ for all $\varphi \in \Phi$. The L form a linear subspace Λ of X^*, the *annihilator space of* Φ. We thus have

$$\|B^{*\!*}\| = \min_{L \in \Lambda} \|B_0^* - L\| = \delta_I, \tag{4.2.3}$$

so that δ_I is the distance of any B_0^* from Λ. Again, S_I is a 'parallel' at distance δ_I, to Λ.

4.3. The space X is said to be '*strictly convex*' if

$$\|\chi_1 + \chi_2\| = \|\chi_1\| + \|\chi_2\| \tag{4.3.1}$$

holds if, and only if, $a_1 \chi_1 = a_2 \chi_2$ where $a_1 \ge 0$, $a_2 \ge 0$, $a_1 + a_2 > 0$.

Theorem IV. *If X^* is strictly convex, then there exists exactly one extremal functional $B^{*\!*} \| I$.*

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[1] Compare Hille, p. 20. For 'separable' spaces, like L^p, the proof of the extension theorem does not require transfinite induction.

Proof. Let B_1^* and B_2^* be extremal functionals. Then, by Theorem II, $B_1^* = \frac{1}{2}(B_1^* + B_2^*)$ is also extremal. Hence

$$\|\tfrac{1}{2}(B_1^* + B_2^*)\| = \|I\|_\Phi = \tfrac{1}{2}(\|B_1^*\| + \|B_2^*\|).$$

Since X^* is strictly convex, it follows that, say, $B_2^* = \lambda B_1^*$ where $\lambda \geq 0$. But $\|B_2^*\| = \|B_1^*\| = \|I\|_\Phi$ so that $\lambda = 1$.

4.4. A point $\varphi^* \in \Phi$ is said to be an *extremal point*, for I, if $\|\varphi^*\| = 1$ and if

$$I(\varphi^*) = \max_{\varphi \in \Phi} |I(\varphi)|. \tag{4.4.1}$$

We note that, if $\|I\|_\Phi > 0$, and if $\varphi \in \Phi$, $\|\varphi\| \leq 1$, and $I(\varphi) = \|I\|_\Phi$, then φ is extremal. For, certainly, $\varphi \neq \Theta$ where Θ is the zero point of X. Now, if $\|\varphi\| = \lambda$, $0 < \lambda \leq 1$, then $\|\lambda^{-1}\varphi\| = 1$ and thus $|I(\lambda^{-1}\varphi)| \leq \|I\|_\Phi$. On the other hand, $I(\lambda^{-1}\varphi) = \lambda^{-1} I(\varphi) = \lambda^{-1}\|I\|_\Phi \geq \|I\|_\Phi$. Hence $\lambda = 1$ and $\|\varphi\| = 1$.

Theorem V. *If* $\|I\|_\Phi > 0$, *then the set* E_I^* *of all extremal points* φ^* *for* I *is either empty or convex in* X.

The proof is like that of part (ii) of the proof for Theorem 5.

We also note that, if Φ is a closed subspace of X, then E_I^* is also closed.

Theorem VI. *If* X *is strictly convex and if* $\|I\|_\Phi > 0$, *then there exists at most one extremal point* φ^* *for* I.

The proof is like that of Theorem IV.

4.5. The space X is said to be (locally) weakly compact[1], if, given any sequence $\{\chi_n\}$ in X with $\|\chi_n\| \leq 1$, there exists a subsequence $\{\chi_{n_k}\}$ and a $\chi \in X$ such that

$$B(\chi_{n_k}) \to B(\chi) \tag{4.5.1}$$

for every $B \in X^*$.

It follows that $\|\chi\| \leq 1$. For, $|B(\chi_{n_k})| \leq \|B\| \|\chi_{n_k}\| \leq \|B\|$, and hence, by (4.5.1), $|B(\chi)| \leq \|B\|$ for all $B \in X^*$. Now, if $\chi \neq \Theta$, there exists[2] a B such that $B(\chi) = 1$ and $\|B\| = \|\chi\|^{-1}$. Hence $\|\chi\| \leq 1$.

Theorem VII. *If* X *is weakly compact, and if* Φ *is a closed subspace of* X, *then there exists at least one extremal point* φ^* *for* I.

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[1] Such a space is reflexive, and vice versa; compare Hille, Theorem 2.11.2; Bourbaki, Eberlein.

[2] Compare Banach, p. 55 and 57.

Proof. We may assume that $\|I\|_\Phi > 0$. Then there exists a sequence of $\varphi_n \in \Phi$ such that $\|\varphi_n\| = 1$ and $I(\varphi_n) = B^*(\varphi_n) \to \|I\|_\Phi$ for all $B^* \| I$. Since X is weakly compact, there exists a subsequence $\{\varphi_{n_k}\}$ and a $\chi \in X$ with $\|\chi\| \leq 1$ such that $B(\varphi_{n_k}) \to B(\chi)$ for all $B \in X^*$ and, in particular, for all $B^* \| I$. Hence $B^*(\chi) = \|I\|_\Phi$ for all $B^* \| I$.

On writing $B^* = B_0^* - L$, we see that $L(\chi) = 0$ for all $L \in \Lambda$, where Λ is the annihilator space of Φ. Hence $\chi \in \Phi$. For, otherwise and since Φ is closed, χ would have a positive distance from Φ, and hence there would exist[1] an $L \in \Lambda$ such that $L(\chi) \neq 0$. It now follows, from our remark after (4.4.1), that χ is extremal for I.

5. Existence theorems in H^p.

5.1. The Banach spaces $X = L^p$ have the following properties:

(i) If $1 \leq p < \infty$, then the general linear functional B on L^p has a unique representation[2]

$$B(\chi) = \frac{1}{2\pi} \int_0^{2\pi} \chi(t)\, \varkappa(t)\, dt, \tag{5.1.1}$$

where $\varkappa \in L^q$ and $\|B\| = \|\varkappa\|_q$. Hence, if $1 \leq p < \infty$, the conjugate space X^* of L^p is the space L^q, in the sense that X^* and L^q are isomorphic and isometric under the mapping $B \leftrightarrow \varkappa$.

(ii) If $p = \infty$, the formula (5.1.1) does not represent the most general linear functional on L^∞; the form of the latter is rather complicated.

If C is the subspace of the continuous functions $c(t)$, then the general functional B on C is of the form[2]

$$B(c) = \frac{1}{2\pi} \int_0^{2\pi} c(t)\, d\mu(t), \tag{5.1.2}$$

where the complex-valued function $\mu(t)$ is of bounded variation in $\langle 0, 2\pi \rangle$, and

$$\|B\|_C = \frac{1}{2\pi} \int_0^{2\pi} |d\mu(t)|. \tag{5.1.3}$$

This representation through μ is unique apart from an additive constant to μ.

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[1] Compare BANACH, p. 55 and 57.

[2] Compare BANACH, p. 61—65, for real L^p. The generalization to complex L^p is easy.

(iii) If $1<p<\infty$, then L^p is (locally) weakly compact[1]; that is, given any sequence $\{\chi_n\}$ in L^p with $\|\chi_n\|\leq 1$, there exists a subsequence $\{\chi_{n_k}\}$ and a $\chi\in L^p$ with $\|\chi\|\leq 1$ such that

$$\int_0^{2\pi} \chi_{n_k}\varkappa\,dt \to \int_0^{2\pi}\chi\varkappa\,dt \tag{5.1.4}$$

for every $\varkappa\in L^q$.

If $p=\infty$, (5.1.4) still holds, but is no longer equivalent to weak compactness since (5.1.1) is not the form of the general linear functional on L^∞.

If $p=1$, (5.1.4) does not hold.

5.2. The class H^p is a closed linear subspace of L^p.

For, let $\varphi_n\in H^p$ and let $\varphi_n\to\chi$ in X; that is, $\varphi_n \underset{p}{\to}\chi$ if $1\leq p<\infty$, and $\varphi_n\to\chi$ uniformly p.p. when $p=\infty$. Then

$$\int_0^{2\pi}\varphi_n(t)e^{-irt}\,dt\to\int_0^{2\pi}\chi(t)e^{-irt}\,dt \tag{5.2.1}$$

for all r. On taking $r=-1,-2,\ldots$, we see that $\chi\in H^p$.

The integral

$$I(\varphi)=\frac{1}{2\pi}\int_0^{2\pi}\varphi\varkappa\,dt,\quad \varphi\in H^p,\ \varkappa\in L^q, \tag{5.2.2}$$

we are discussing here, is a restriction to H^p of (5.1.1). If $1\leq p<\infty$, then, conversely, the restriction of every B on L^p yields an I.

Our maximum problem is the determination of $\|I\|_{H^p}$.

5.3. Theorem 6. *If $1<p\leq\infty$, then there exists at least one extremal function Φ.*

Proof. If $1<p<\infty$, this follows from Theorem VII.

The following similar argument holds also for $p=\infty$:

There exists a sequence $\{\varphi_n\}$ in H^p with $\|\varphi_n\|=1$ such that

$$I(\varphi_n)=\frac{1}{2\pi}\int_0^{2\pi}\varphi_n\varkappa\,dt\to\|I\|_{H^p}. \tag{a}$$

Hence, by § 5.1. (iii), there exists a subsequence $\{\varphi_{n_k}\}$ and a $\chi\in L^p$ with $\|\chi\|\leq 1$ such that (5.1.4) holds. In particular, (5.2.1) holds for the φ_{n_k}. It follows that $\chi\in H^p$ and hence that χ is extremal, since $I(\chi)=\|I\|_{H^p}$ by (a) and (5.1.4).

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[1] Compare Banach, p. 130—131.

5.4. Theorem 7. *If* $1 \leq p < \infty$, *then there exists at least one extremal kernel* $\mathrm{K} \| \varkappa$. Moreover,

$$\| I \|_{H^p} = \sup_{\|\varphi\| \leq 1} | I(\varphi) | = \min_{\varkappa^* \| \varkappa} \| \varkappa^* \|_q = \| \mathrm{K} \|_q. \tag{5.4.1}$$

This is an immediate consequence of Theorem III[1], since I is the restriction to H^p of a general linear functional B on L^p.

5.5. In the case $p = \infty$, Theorem III is not immediately available. For, it is not obvious that the extremal functional B^* of that theorem should be of the special form (5.1.1).

We shall require the following result due to F. and M. Riesz[2]:

Suppose that $\mu(t)$ is of bounded variation in $\langle 0, 2\pi \rangle$. If

$$\int_0^{2\pi} e^{ikt} \, d\mu(t) = 0 \qquad (k = 0, 1, 2 \ldots), \tag{5.5.1}$$

then $\mu(t)$ is absolutely continuous.

Theorem 8. *If* $p = \infty$, *then there exists at least one extremal kernel* $\mathrm{K} \| \varkappa$, *satisfying* (5.4.1).

Proof. (i) Let C be the space of continuous functions $c(t)$, and Γ the subspace of continuous $\gamma(t) \in H^\infty$. By Theorem III, there exists an extremal B^* on C, such that $\|B^*\| = \|I\|_\Gamma$ and $B^*(\gamma) = I(\gamma)$ for all $\gamma \in \Gamma$. By (5.1.2), B^* is of the form

$$B^*(c) = \frac{1}{2\pi} \int_0^{2\pi} c(t) \, d\mu(t). \tag{5.5.2}$$

Now, $e^{ikt} \in \Gamma$ for all $k = 0, 1, 2 \ldots$. Hence

$$\int_0^{2\pi} e^{ikt} \, d\mu(t) = \int_0^{2\pi} e^{ikt} \varkappa(t) \, dt, \tag{a}$$

or

$$\int_0^{2\pi} e^{ikt} \, d(\{\mu - \mu_1\}(t)) = 0, \tag{b}$$

where $\mu_1(t) = \int_0^t \varkappa(\tau) \, d\tau$. It follows, from (5.5.1), that $\mu - \mu_1$ is absolutely continuous. Since μ_1 is absolutely continuous, μ itself is so. On putting $\mu'(t) \equiv \mathrm{K}(t)$, we have $\mathrm{K}(t) \in L^1$ and

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[1] Compare footnote on p. 299.

[2] Compare ZYGMUND, p. 158.

(5.5.3) $$B^{*}(c)=\frac{1}{2\pi}\int_0^{2\pi} c(t)\,\mathrm{K}(t)\,dt,$$

so that $\mathrm{K}\,\|\,\varkappa$ in Γ. Also $\|\mathrm{K}\|_1=\|I\|_\Gamma$, and hence K is extremal in Γ.

(ii) If we put $\mathrm{K}=\varkappa+\lambda$, then $\lambda\in L^1$ and it follows, from (a), that

(c) $$\int_0^{2\pi} e^{ikt}\lambda(t)\,dt=0 \qquad (k=0,\,1,\,2\ldots).$$

Hence, by Theorem 2, $\mathrm{K}\,\|\,\varkappa$ in H^∞ itself. It follows that $\|I\|_{H^\infty}\le\|\mathrm{K}\|_1=\|I\|_\Gamma$. On the other hand, clearly, $\|I\|_\Gamma\le\|I\|_{H^\infty}$. Hence $\|I\|_{H^\infty}=\|\mathrm{K}\|_1$, and K is the required extremal kernel in H^∞.

6. Uniqueness theorems.

6.1. *We assume, from now on, that* $\|I\|>0$.

If $1<p<\infty$, then L^p is 'strictly convex'. This expresses a familiar fact concerning the sign of equality in Minkowski's inequality (2.3.2).

Theorem 9. *If* $1<p<\infty$, *then there exist exactly one extremal function* Φ *for* I, *and exactly one extremal kernel* $\mathrm{K}\,\|\,\varkappa$. *Moreover*

(6.1.1) $$I(\Phi)=\|\mathrm{K}\|_q.$$

This follows from Theorems 6 and 7, and from Theorems IV and VI.

Theorem 10. *If* $p=2$, *and* $\varkappa(t)\sim\sum_{-\infty}^{\infty}c_k e^{ikt}$, *then* $\|I\|^2=\sum_{-\infty}^{0}|c_k|^2$ *and*

(6.1.2) $$\mathrm{K}(t)\sim\sum_{-\infty}^{0}c_k e^{ikt};\quad \Phi(t)=\{\sum_{-\infty}^{0}|c_k|^2\}^{-1/2}\sum_{0}^{\infty}\bar{c}_{-k}e^{ikt}.$$

Proof. First, by § 2.2. (ii), $\mathrm{K}\,\|\,\varkappa$. Also any $\varkappa^*\,\|\,\varkappa$ is of the form $\varkappa^*=\mathrm{K}+\lambda^*$, where $\lambda^*\sim\sum_{1}^{\infty}c_k^* e^{ikt}\in H^2$. Hence

(a) $$\|\varkappa^*\|^2=\sum_{-\infty}^{0}|c_k|^2+\sum_{1}^{\infty}|c_k^*|^2\ge\sum_{-\infty}^{0}|c_k|^2=\|K\|^2,$$

so that K is extremal. Clearly, $\|\Phi\|=1$ and

(b) $$I(\Phi)=\frac{1}{2\pi}\int_0^{2\pi}\Phi\,\mathrm{K}\,dt=\{\sum_{-\infty}^{0}|c_k|^2\}^{1/2}=\|\mathrm{K}\|,$$

so that Φ is the extremal function.

This theorem shows that all extremum problems in H^2 are of elementary nature.[1]

[1] Compare [M-R], p. 297.

6.2. Theorem 11. *If* $1 \leq p \leq \infty$, *let* $\Phi \in H^p$ *with* $\|\Phi\|_p = 1$ *and let* $\mathrm{K} \| \varkappa$ *in* L_q. *Then* Φ *is an extremal function for* I, *and* K *is an extremal kernel if, and only if, for almost all* t

(6.2.1) $$\Phi(t)\,\mathrm{K}(t) \geq 0$$

and

(6.2.2) $$|\mathrm{K}(t)|^{1/p} \equiv A\,|\Phi(t)|^{1/q},$$

where

(6.2.3) $$A = \|I\|^{1/p}.$$

If $p = 1$, *then* (6.2.2) *means*

$(6.2.2)_1$ $$|\mathrm{K}(t)| \equiv \|I\|;$$

if $p = \infty$, *then*

$(6.2.2)_\infty$ $$|\Phi(t)| \equiv 1$$

for almost all those t *for which* $\mathrm{K}(t) \neq 0$.

Proof. Extremal functions Φ and extremal kernels K are characterized by their satisfying (6.1.1), and this equation can be written as

(a) $$\frac{1}{2\pi}\int_0^{2\pi} \Phi\,\mathrm{K}\,dt = \frac{1}{2\pi}\int_0^{2\pi} |\Phi\,\mathrm{K}|\,dt = \|\Phi\|_p\,\|\mathrm{K}\|_q = \|\mathrm{K}\|_q.$$

The first equality here is equivalent to (6.2.1), and the second to (6.2.2).

If $1 < p < \infty$, we have $|\mathrm{K}(t)|^q \equiv A^{pq}\,|\Phi(t)|^p$ and hence

(a) $$\|I\|^q = \|\mathrm{K}\|^q = A^{pq}\,\|\Phi\|^p = A^{pq}.$$

This gives (6.2.3). Similarly, with $p = 1$, $q = \infty$, we obtain $(6.2.2)_1$; one should observe that, by § 1.1. (iii), $\Phi(t) \neq 0$ p.p. If $p = \infty$, $q = 1$, we obtain $(6.2.2)_\infty$.

Theorem 12. *If* $1 \leq p < \infty$, *and if an extremal function* Φ *exists, then the extremal kernel* K *is unique.*

Proof. The theorem is new only for $p = 1$ [see Theorem 9]. We have $\Phi(t) \neq 0$ p. p., by § 1.1. (iii). Hence (6.2.1) determines $\arg \mathrm{K}(t)$, and (6.2.2) [or $(6.2.2)_1$] determines $|\mathrm{K}(t)|$ p.p.

Theorem 13. *If* $1 < p \leq \infty$, *then the extremal function* Φ *is unique.*

Proof. Φ and K both exist, and the theorem is new only for $p=\infty$ [Theorem 9].

Now $\|\mathrm{K}\|_q=\|I\|>0$. Hence $\mathrm{K}(t)\neq 0$ in a set E of positive measure. It follows that (6.2.1) determines $\arg \Phi(t)$, and (6.2.2) [or $(6.2.2)_\infty$] determines $|\Phi(t)|$ p.p. in E, when *some* K has been chosen. By § 1.1. (iii), $\Phi(t)$ is then determined almost everywhere.

6.3. Theorem 14. *If* $1\leq p\leq\infty$, *and if an extremal function* Φ *exists, then the extremal kernel* K *is unique. In particular, if* $1<p\leq\infty$, *then the extremal kernel is unique.*

Proof. The theorem is new only for $p=\infty$ [Theorem 12]. Let $\mathrm{K}(t)\equiv\varkappa(t)+e^{it}\Lambda(t)$ where $\Lambda\in H^q$. Then

(a) $$\Phi(t)\,\mathrm{K}(t)\equiv\Phi(t)\,\varkappa(t)+e^{it}\,\Phi(t)\,\Lambda(t),$$

where $\Phi\Lambda\in H^1$ since $\Phi\in H^p$. Hence $e^{it}\,\Phi(t)\,\Lambda(t)$ are the radial boundary values $G(e^{it})$ of a function $G(z)\in H_1$ for which $G(0)=0$. If we put $G(z)=U(z)+i\,V(z)$, then, by (6.2.1),

(b) $$V(e^{it})\equiv-\Im(\Phi(t)\,\varkappa(t)).$$

Now, since $G(0)=0$, a classical formula by Schwarz gives, for $0<r<1$,

(c) $$G(rz)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}V(r\zeta)\frac{z+\zeta}{z-\zeta}d\zeta.$$

Again, by (1.1.2), $V(r\zeta)\underset{1}{\to}V(\zeta)$ as $r\to 1$. Hence, by (b) and (c),

(6.3.1) $$G(z)=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\Im(\Phi(t)\,\varkappa(t))\frac{\zeta+z}{\zeta-z}d\zeta,\qquad \zeta=e^{it},$$

so that $G(z)$ is determined, and hence $G(e^{it})\equiv e^{it}\,\Phi(t)\,\Lambda(t)$ is determined p.p. Since $\Phi(t)\neq 0$ p.p., $\Lambda(t)$ and thus $\mathrm{K}(t)$ are also determined p.p. by Φ.

7. The case $p=1$.

7.1. We have now proved all the clauses of Theorem A (of the Introduction) except those relating to the non-existence of extremal functions Φ and the non-uniqueness of the extremal kernels K, in the case $p=1$.

Before we turn to these clauses, it is of importance for applications to state a case in which at least one Φ exists. By Theorem 12, the extremal kernel K is then unique.

Given a kernel $\varkappa(t) \sim \sum_{-\infty}^{\infty} c_k e^{ikt}$, we associate with it, formally, the Laurent-function

(7.1.1) $$k(z) \sim \sum_{-\infty}^{\infty} c_k z^k.$$

Theorem 15. *If $p=1$, $\varkappa \in L^\infty$, and if the associated function $k(z)$ is regular in some annulus $\varrho < |z| < 1$, then there exists at least one extremal function Φ. Also the extremal kernel* $\mathrm{K} \| \varkappa$ *is then unique.*

Proof. The function $k_1(z) = \sum_{-\infty}^{-1} c_k z^k$ is regular for $|z| > \varrho$, and $k_2(z) = \sum_{0}^{\infty} c_k z^k$ is regular for $|z| < 1$. Also, as $r \to 1$, $k_1(re^{it}) \to \varkappa_1(t) = \sum_{-\infty}^{-1} c_k e^{ikt}$ uniformly so that $\varkappa_1(t)$ is continuous. Hence $\varkappa_2(t) \sim \sum_{0}^{\infty} c_k e^{ikt} \in H^\infty$, $k_2(z) \in H_\infty$, and $k_2(re^{it}) \underset{\infty}{\rightarrow} \varkappa_2(t)$. Finally,

(a) $$k(re^{it}) \underset{\infty}{\rightarrow} \varkappa(t).$$

Now let $\varphi \in H^1$ and let $f(z)\,(\in H_1)$ be associated with φ; that is, $\varphi(t) \equiv f(e^{it})$. Then

(b) $$I(\varphi) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(t)\, \varkappa(t)\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(re^{it})\, k(re^{it})\, dt$$

holds for $\varrho < r < 1$, since $f(z)\, k(z)$ is regular in $\varrho < |z| < 1$ and because of (a) and $f(re^{it}) \underset{1}{\rightarrow} \varphi(t)$.

Next, there exists a sequence of functions $\varphi_m \in H^1$, with $\|\varphi_m\| = 1$, such that

(c) $$I(\varphi_m) \to \|I\|.$$

The associated functions $f_m(z)$, plainly, are uniformly bounded in every fixed circle $|z| \le r\ (<1)$, and hence are 'normal' in $|z| < 1$. There exists, therefore, a subsequence $\{m_n\}$ and a function $F(z)$, regular in $|z| < 1$, such that $f_{m_k}(z) \to F(z)$ uniformly in every fixed circle $|z| \le r\ (<1)$. It follows that

(d) $$M_1(F, r) = \lim_{k \to \infty} M_1(f_{m_k}, r) \le 1,$$

so that $F(z) \in H_1$ and $\|\Phi\| \le 1$, where $\Phi\ (\in H^1)$ is the boundary function of $F(z)$.

Now, by (b),

(e) $$I(\varphi_{m_k}) = \int_0^{2\pi} f_{m_k}(re^{it})\, k(re^{it})\, dt \to \frac{1}{2\pi} \int_0^{2\pi} F(re^{it})\, k(re^{it})\, dt = I(\Phi),$$

so that, by (c), $I(\Phi) = \|I\|$. Hence Φ is extremal.

7.2. In the following example, which was first discussed by L. Fejér and F. Riesz[1], there exists *no extremal function* Φ.

Let $f(z) \in H_1$ with $\|\varphi\| = 1$, where φ $(\in H^1)$ is the boundary function of f. We consider the integral

$$(7.2.1) \qquad J(f) = \int_{-1}^{1} f(x)\,dx.$$

Since $f(re^{it}) \underset{1}{\to} \varphi(t)$, we can apply Cauchy's theorem and obtain

$$(7.2.2) \qquad J(f) = -i\int_0^{\pi} \varphi(t)\,e^{it}\,dt = I(\varphi).$$

Here the given kernel is $\varkappa(t) = \langle -2\pi i e^{it}, 0\rangle$; that is $\varkappa(t) = -2\pi i e^{it}$ for $0 \le t \le \pi$, and $\varkappa(t) = 0$ for $\pi < t \le 2\pi$. An equivalent kernel is $\mathrm{K}(t) = \varkappa(t) + \pi i e^{it}$; that is $\mathrm{K} = \langle -\pi i e^{it}, \pi i e^{it}\rangle$. It follows that

$$(7.2.3) \qquad |J(f)| \le \pi.$$

It was shown by Fejér and Riesz that this estimate is best possible, so that $\|I\| = \pi$ and K is an extremal kernel.

We shall now show that equality in (7.2.3) is impossible. Suppose that Φ were an extremal function, so that $I(\Phi) = \pi$. By (6.2.1), we would have

$$\arg \Phi(t) \equiv \left\langle \frac{\pi}{2} - t,\ -\frac{\pi}{2} - t \right\rangle.$$

Since $\Phi \in H^1$, we have for all $k \ge 0$,

$$\text{(a)} \qquad 0 = -i\int_0^{2\pi} \Phi(t)\,e^{i(k+1)t}\,dt = \int_0^{\pi} |\Phi(t)|\,e^{ikt}\,dt - \int_{\pi}^{2\pi} |\Phi(t)|\,e^{ikt}\,dt.$$

On taking conjugate complex values, we see that this holds for *all* k. Thus the two functions of L^1, $\psi_1 = \langle |\Phi|, 0\rangle$ and $\psi_2 = \langle 0, \Phi\rangle$ would have the same Laurent coefficients. It follows that $\psi_1 \equiv \psi_2$ and $\Phi \equiv 0$, which is impossible.

7.3. In our second example *the extremal kernel is not unique* By Theorem 12, again no extremal function can exist.

We consider

$$(7.3.1) \qquad I(\varphi) = \frac{1}{2\pi}\int_0^{\pi/2} \varphi(t)\,dt - \frac{1}{2\pi}\int_{\pi/2}^{\pi} \varphi(t)\,dt$$

with

[1] L. Fejér and F. Riesz; see references.

$$\varkappa(t) = \begin{cases} 1 & 0 \le t \le \frac{\pi}{2} \\ -1 \text{ for } & \frac{\pi}{2} < t \le \pi \\ 0 & \pi < t \le 2\pi. \end{cases} \tag{7.3.2}$$

Clearly, $\|I\| \le 1$. We shall show that $\|I\| = 1$, so that $\varkappa(t)$, for which $\|\varkappa\| = 1$, is an extremal kernel.

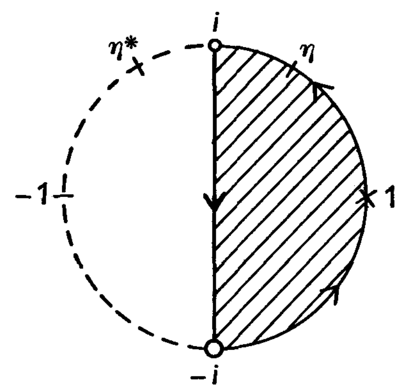

Fig. 1.

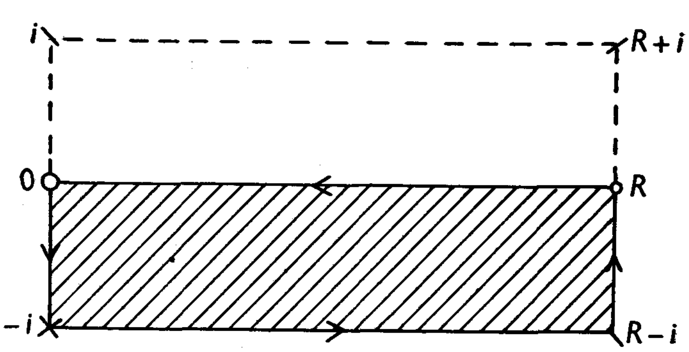

Fig. 2.

Let $R > 0$, and let $f(z)$ be the function that maps the semi-circle $|z| \le 1$, $\Re z \ge 0$ schlicht onto the rectangle with vertices $-i$, $R-i$, R and 0 in such a way that $1 \leftrightarrow -i$, $i \leftrightarrow R$, and $-i \leftrightarrow 0$ [Figures 1 and 2]. The whole circle $|z| \le 1$ is then mapped onto the rectangle with vertices $-i$, $R-i$, $R+i$, i, and $-1 \leftrightarrow i$. Also there is an $\eta = e^{i\vartheta}$, $0 < \vartheta < \frac{\pi}{2}$, such that $\eta \leftrightarrow R-i$; and there is an $\eta^* = e^{i\vartheta^*}$, $\vartheta^* = \pi - \vartheta$, such that $\eta^* \leftrightarrow R+i$.

Let $\varphi(t) \equiv i\zeta f'(\zeta)$ where $\zeta = e^{it}$. Then $|\varphi(t)|\,dt = |f'(\zeta)|\,dt$ is the element of arc along the rectangular map of $|\zeta|=1$, while $\arg\varphi(t)$ determines the direction of the tangential vector to this map. Hence $\varphi(t) \geq 0$ for $0<t<\vartheta$, $\varphi(t)\leq 0$ for $\pi-\vartheta<t<\pi$, while $\varphi(t)$ is purely imaginary on the remaining arcs of $|\zeta|=1$. It follows that

(a) $$I(\varphi) = \frac{1}{2\pi}\left\{\int_0^{\vartheta} + \int_{\pi-\vartheta}^{\pi} |f'(\zeta)|\,dt\right\} + \frac{i}{2\pi}\left\{\int_{\vartheta}^{\pi/2} - \int_{\pi/2}^{\pi-\vartheta} |f'(\zeta)|\,dt\right\},$$

(b) $$|I(\varphi)| \geq \frac{1}{2\pi}\left\{\int_0^{\vartheta} + \int_{\pi-\vartheta}^{\pi} |f'(\zeta)|\,dt\right\} = \frac{R}{\pi}\cdot$$

On the other hand,

(c) $$2\pi\,\|\varphi\| = \int_0^{2\pi} |f'(\zeta)|\,dt = 2R+4,$$

the total length of the rectangle. On replacing φ by $\varphi^* = \varphi\cdot\|\varphi\|^{-1}$, we have $\|\varphi^*\|=1$ and $|I(\varphi^*)| \geq \dfrac{R}{R+2}\cdot$ Hence, on letting $R\to\infty$, we find $\|I\|\geq 1$; and thus $\|I\|=1$.

Next, it is easy to verify that the function

(7.3.3) $$w = g(z) = -\left(\frac{1-\varepsilon\sqrt{\dfrac{1-z}{1+z}}}{1+\varepsilon\sqrt{\dfrac{1-z}{1+z}}}\right)^2, \qquad \varepsilon = e^{i\pi/4},$$

where the root has the principal value, maps $|z|<1$ schlicht onto the circle $|w|<1$ slit along the real interval $-1\leq w\leq 0$, in such a way that $1\leftrightarrow -1$, $i\leftrightarrow 0$, $-1\leftrightarrow -1$.

Now, we may add to our extremal kernel $\varkappa$ in (7.3.2) the boundary values $\zeta g(\zeta)$, $\zeta = e^{it}$, to obtain an equivalent kernel $\varkappa^*$. If we can prove that $\|\varkappa^*\| = \operatorname{Max}|\varkappa^*(t)| = 1$, then $\varkappa^*$ is also an extremal kernel; we have then an infinity of extremal kernels, by Theorem 4. First, $|\varkappa^*(t)|\equiv 1$ for $\pi<t<2\pi$, since $|g(\zeta)|\equiv 1$ and $\varkappa\equiv 0$ there. So we have to prove that $|\varkappa^*(t)|\leq 1$ for $0\leq t\leq \pi$. For reasons of symmetry, we may assume that $0\leq t\leq \dfrac{\pi}{2}\cdot$

Now, for these t,

(d) $$\frac{1-\zeta}{1+\zeta} = -i\tan t/2, \qquad \varepsilon\sqrt{\frac{1-\zeta}{1+\zeta}} = \sqrt{\tan\frac{t}{2}},$$

so that

(e) $$g(\zeta) = -\left(\frac{1-\sqrt{\tan t/2}}{1+\sqrt{\tan t/2}}\right)^2 = -h^2(t),$$

say; and $\varkappa^*(t) = 1 - e^{it} h^2(t)$. Hence

(f) $$|\varkappa^*(t)|^2 = 1 - 2\cos t\, h^2(t) + h^4(t) = 1 + h^2(t)\,[h^2(t) - 2\cos t].$$

If we put $\lambda = \sqrt{\tan t/2}$, then $0 \le \lambda \le 1$, $\cos t = \dfrac{1-\lambda^4}{1+\lambda^4}$, and

(g) $$h^2(t) - 2\cos t = \left(\frac{1-\lambda}{1+\lambda}\right)^2 - 2\,\frac{1-\lambda^4}{1+\lambda^4} = \frac{P(\lambda)}{(1+\lambda)^2(1+\lambda^4)},$$

where

(h) $$\begin{aligned} P(\lambda) &= (1-\lambda)^2(1+\lambda^4) - 2(1+\lambda)^2(1-\lambda^4) \\ &= -[(1-\lambda^3)(3\lambda+\lambda^2) + (1-\lambda^5)(1+3\lambda)] \le 0. \end{aligned}$$

Hence $|\varkappa^*(t)| \le 1$, by (*f*).

7.4. The following simple example shows[1] that there may be *an infinity of extremal functions* Φ; by Theorem 12, K will be unique in this case. Consider

(7.4.1) $$I(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(t)\, e^{-it}\, dt.$$

Clearly, $\|I\| = 1$ and $\Phi(t) = e^{it}$ is an extremal function. Also $\varkappa(t) = e^{-it}$ is the extremal kernel.

However, any of the functions

(7.4.2) $$\Phi_\alpha(t) = \{1 + |\alpha|^2\}^{-1} e^{it}\, |e^{it} - \alpha|^2, \qquad |\alpha| < 1,$$

is also extremal. For, $\Phi_\alpha(t) \in H^1$, as boundary function of

$$F_\alpha(z) = \{1 + |\alpha|^2\}^{-1}(z-\alpha)(1-\bar{\alpha} z).$$

Also

(a) $$\frac{1}{2\pi}\int_0^{2\pi} |e^{it} - \alpha|^2\, dt = 1 + |\alpha|^2,$$

so that, clearly, $\|\Phi_\alpha\| = 1$ and $I(\Phi_\alpha) = 1$.

8. Rational kernels.

8.1. In [M-R] integrals of the form

(8.1.1) $$J(f) = \frac{1}{2\pi i}\int_{|\zeta|=1} f(\zeta)\, \mathfrak{k}(\zeta)\, d\zeta = \frac{1}{2\pi}\int_0^{2\pi} \varphi(t)\, \varkappa(t)\, dt = I(\varphi)$$

[1] Compare [M-R], p. 307.

were discussed, where $f \in H_p$ with boundary function $\varphi \in H^p$. The kernel function $\mathfrak{k}(z)$ was supposed to be meromorphic in $|z|<1$ with a finite number of poles β_i there, each pole counted according to multiplicity. Moreover, $M_q(\mathfrak{k}, r)$ was to remain bounded for sufficiently large $r<1$, so that the boundary kernel

$$(8.1.2) \qquad \varkappa(t) \equiv e^{it}\,\mathfrak{k}(e^{it})$$

belonged to L^q.

The function $\mathfrak{k}(z)$ can be replaced by a rational kernel function $\mathfrak{k}^*(z)$, which is the sum of the principal parts of $\mathfrak{k}$ at the β_i. For, if $\varkappa^*$ is associated with $\mathfrak{k}^*$ as in (8.1.2), then, clearly, $\varkappa^* \| \varkappa$.

Our general theory shows that, in this rational case, the following holds:

I. Φ *always exists.* For, in the only problematic case $p=1$, we may apply Theorem 15.

However, if $p=1$, Φ need not be unique as the example in § 7.4 shows.

II. K *always exists uniquely.* For, in the case $p=1$, we may apply I and Theorem 12.

8.2. If Φ $(\in H^p)$ is extremal, we write $F(z)$ for the associated extremal function of H_p. Similarly, if $\mathrm{K} \| \varkappa$ $(\in L^q)$ is extremal, so that the associated function $\mathrm{K}(z)$ is meromorphic in $|z|<1$, we put, in view of (8.1.2),

$$(8.2.1) \qquad \mathfrak{K}(z) = z^{-1}\,\mathrm{K}(z)$$

and speak of the extremal kernel function $\mathfrak{K}(z)$.

The main theoretical result in [M-R] was[1]:

Theorem 16. *Let* $1 \le p \le \infty$, *and let* $\mathfrak{k}(z)$ *have* n *poles* β_i *in* $|z|<1$, *each counted according to multiplicity. Then there exist* $n-1$ *numbers* α_i *with* $|\alpha_i| \le 1$, *such that*

(i) $\mathfrak{K}(z)$ *has a unique representation*

$$(8.2.2) \qquad \mathfrak{K}(z) = A \prod{}' \frac{z-\alpha_i}{1-\bar{\alpha}_i z} \prod_1^{n-1} (1-\bar{\alpha}_i z)^{2/q} \prod_1^{n} \frac{1-\bar{\beta}_i z}{z-\beta_i} (1-\bar{\beta}_i z)^{-2/q},$$

where $\prod'$ *is extended over all, some, or none of the* α_i *with* $|\alpha_i|<1$.

For no other function of the form (8.2.2) *is* $\mathrm{K} \| \varkappa$.

(ii) $F(z)$ *is extremal if, and only if,* $M_p(F, 1) = \|\Phi\| = 1$, $J(F) = I(\Phi) > 0$, *and if it is of the form*

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[1] [M-R], formulae (1.3.5) and (1.3.6), p. 277.

$$F(z) = B \prod{}'' \frac{z-\alpha_i}{1-\bar{\alpha}_i z} \prod_1^{n-1} (1-\bar{\alpha}_i z)^{2/p} \prod_1^{n} (1-\bar{\beta}_i z)^{-2/p}. \tag{8.2.3}$$

Here $\prod''$ *is complementary to* $\prod'$ *with respect to the* α_i *with* $|\alpha_i|<1$.

The powers, occurring in both formulae, are principal determinations, having value 1 *at* $z=0$.

We note that, when $p>1$ $(q<\infty)$, then *all* the α_i are determined, through (8.2.2), by the unique function $\mathfrak{K}(z)$. Hence $F(z)$, in (8.2.3), is then also fully determined, in accordance with our general theory. If, however, $p=1$ $(q=\infty)$, then only the α_i in $\prod'$ are determined by $\mathfrak{K}(z)$. If their number m is less than $n-1$, then $n-1-m$ of the parameters α_i in (8.2.3) are arbitrary, and we obtain an infinity of extremal functions $F(z)$ (and Φ).

8.3. If we write

$$H(z) = \prod_1^n \frac{z-\beta_i}{1-\bar{\beta}_i z} (1-\bar{\beta}_i z)^{2/q} \mathfrak{K}(z), \tag{8.3.1}$$

then $H(z)$ is regular in $|z|<1$ and belongs to H_q. Moreover, if $\mathfrak{K}(z)$ has a pole of order p_i at β_i, then the values of $H(z)$ and of its first p_i-1 derivatives at β_i are prescribed. For, $\mathfrak{K}(z)$ has the same principal parts at the β_i as has the given kernel $\mathfrak{k}(z)$. Formula (8.2.2) now appears as a *unique interpolation formula*[1]

$$H(z) = A \prod{}' \frac{z-\alpha_i}{1-\bar{\alpha}_i z} \prod_1^{n-1} (1-\bar{\alpha}_i z)^{2/q} \tag{8.3.2}$$

for a function $H(z) \in H_q$ with n prescribed 'values' at the β_i. For $p=2$, $q=2$, this is exactly the classical interpolation formula of Lagrange. For $q=\infty$ $(p=1)$, we have

$$H(z) = A \prod{}' \frac{z-\alpha_i}{1-\bar{\alpha}_i z}, \tag{8.3.2$_1$}$$

and for $q=1$ $(p=\infty)$ the polynomial interpolation formula

$$H(z) = A \prod{}' (z-\alpha_i)(1-\bar{\alpha}_i z) \prod{}^* (1-\bar{\alpha}_i z)^2, \tag{8.3.2$_\infty$}$$

where $\prod^*$ is complementary to $\prod'$ with respect to *all* the $n-1$ parameters α_i, $|\alpha_i| \le 1$.

The interpolation formula $(8.3.2)_1$ was first obtained by I. Schur[1]; and $(8.3.2)_\infty$ by Kakeya[1].

By Theorem 16, the solution of the interpolation problem (8.3.2) is equivalent to the determination of $\mathfrak{K}(z)$ and $F(z)$, that is, to our original extremal problems.

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[1] Compare [M-R], p. 278–279.

8.4. In [M-R], Theorem 16 was obtained, partly by quoting results of previous writers, partly by generalising their methods of proof, and often only in a sketchy way.[1] We now propose to give an independent and complete proof for Theorem 16.

We first require the following

Lemma. *If* $F(z)$ *and* $\mathfrak{K}(z)$ *are some extremal function and the extremal kernel function, respectively, then there exist* $n-1$ *numbers* α_i, *with* $|\alpha_i| \leq 1$, *such that*

$$L(z) = F(z)\, K(z) = z\, F(z)\, \mathfrak{K}(z) = C\, z\, \frac{\prod_1^{n-1} (z-\alpha_i)(1-\bar{\alpha}_i z)}{\prod_1^{n} (z-\beta_i)(1-\bar{\beta}_i z)}. \tag{8.4.1}$$

Proof. As we remarked in § 8.1., we may assume that the given kernel $\mathfrak{k}(z)$ is rational. Also, by Theorem 2 and (8.2.1),

$$K(z) = z(\mathfrak{k}(z) + G(z)), \tag{a}$$

where $G \in H_q$. It follows that $K(re^{it}) \underset{q}{\to} \mathrm{K}(t)$ as $r \to 1-0$. Also $F(re^{it}) \underset{p}{\to} \Phi(t)$, since $F \in H_p$. Hence

$$L(re^{it}) \underset{1}{\to} \Lambda(t) \equiv \Phi(t)\, \mathrm{K}(t). \tag{b}$$

Again, by (6.2.1), for almost all t,

$$\Lambda(t) \geq 0. \tag{c}$$

We can now apply the Schwarz principle of continuation. For, because of (*b*), the usual proof for this principle is here available. We conclude that $L(z)$ is a rational function, satisfying $L(z) = \bar{L}(1/\bar{z})$.

The poles of $L(z)$ in $|z|<1$ must be amongst the β_i, and with each such pole there is also the pole $\bar{\beta}_i^{-1}$, of the same order. Similarly, with every zero α_i in $|z|<1$ there occurs also the zero $\bar{\alpha}_i^{-1}$, of the same order. Moreover, the zeros α_i on $|z|=1$ must be of even order, because of (c). From this follows that $L(z)$ is of the form (8.4.1).

Suppose, for example, that all $\beta_i \neq 0$. Then $L(z)$ has a zero at $z=0$, and hence has one, of the same order, at $z=\infty$. If this order is k, then exactly $k-1$ of the α_i are nought; and, in order that there be a zero of order k at $z=\infty$, we see that $L(z)$ must be of the form (8.4.1). The discussion in the general case is similar.

[1] Compare [M-R], footnote 3 on p. 293.

8.5. To prove Theorem 16, we proceed in several steps.

(i) Given $F(z)$ and $\mathfrak{K}(z)$, the function $L(z) = z F(z)\mathfrak{K}(z) = F(z) K(z)$, and hence C and all the α_i in (8.4.1), are given. By § 8.4. (c),

$$|\Lambda(t)| \equiv \Lambda(t) = \Phi(t)\,\mathrm{K}(t) \equiv |C| \frac{\prod_1^{n-1} |e^{it} - \alpha_i|^2}{\prod_1^{n} |e^{it} - \beta_i|^2}. \tag{8.5.1}$$

Also, by (6.2.2) and (6.2.3),

$$|\mathrm{K}(t)| \equiv \|I\|^{1/p} \Lambda(t)^{1/q}, \qquad |\Phi(t)| \equiv \|I\|^{-1/p} \Lambda(t)^{1/p}. \tag{8.5.2}$$

(ii) The α_i in (8.4.1), for which $|\alpha_i| < 1$, divide themselves into the zeros of $F(z)$ and $\mathfrak{K}(z)$ in $|z| < 1$. Those of $\mathfrak{K}(z)$ are, because of the uniqueness of this function, fully determined, even in the case $p = 1$, where there may be several $F(z)$. Also they do not coincide with any of the β_i, since $\mathfrak{K}(z)$ must have the same poles as $\mathfrak{k}(z)$.

Consider the functions

$$\mathfrak{K}^*(z) = \mathfrak{K}(z) \prod{}' \frac{1 - \bar{\alpha}_i z}{z - \alpha_i} \prod_1^n \frac{z - \beta_i}{1 - \bar{\beta}_i z}, \qquad F^*(z) = F(z) \prod{}'' \frac{1 - \bar{\alpha}_i z}{z - \alpha_i}, \tag{a}$$

where $\prod{}'$ is extended over the zeros of $\mathfrak{K}(z)$, and $\prod{}''$ over those of $F(z)$, in $|z| < 1$.

These functions have the following properties:

(α). $\mathfrak{K}^*(z)$ and $F^*(z)$ are regular, and $\neq 0$, in $|z| < 1$.

(β). $\mathfrak{K}^*(z) \in H_q$, $F^*(z) \in H_p$.

(γ). $L^*(z) = F^*(z)\mathfrak{K}^*(z)$ is regular in $|z| \leq 1$.

(δ). $|\mathrm{K}^*(t)| \equiv \|I\|^{1/p} \Lambda(t)^{1/q}$, $|\Phi^*(t)| \equiv \|I\|^{-1/p} \Lambda(t)^{1/p}$, where K^* and Φ^* are the boundary functions of $\mathfrak{K}^*$ and F^*, respectively.

(iii) If we can show that any pair of functions $\mathfrak{K}^*(z)$ and $F^*(z)$, satisfying (α)—(δ), is determined, apart from constant factors of modulus 1, then our theorem is proved. For, the functions

$$\mathfrak{K}^*(z) = A \prod_1^{n-1} (1 - \bar{\alpha}_i z)^{2/q} \prod_1^n (1 - \bar{\beta}_i z)^{-2/q} \tag{8.5.3}$$

and

$$F^*(z) = B \prod_1^{n-1} (1 - \bar{\alpha}_i z)^{2/p} \prod_1^n (1 - \bar{\beta}_i z)^{-2/p}, \tag{8.5.4}$$

where

$$|A| = \|I\|^{1/p} |C|^{1/q}, \qquad |B| = \|I\|^{-1/p} |C|^{1/p}, \tag{8.5.5}$$

satisfy these conditions. This is evident for (α)—(γ). As regards (δ), we have, by (8.5.1),

$$\text{(b)}\qquad |K^*(t)| = |A| \left\{ \frac{\prod_1^{n-1} |e^{it} - \alpha_i|^2}{\prod_1^{n} |e^{it} - \beta_i|^2} \right\}^{1/q} = \|I\|^{1/p} |C|^{1/q} |C|^{-1/q} \Lambda(t)^{1/q}$$

and

$$\text{(c)}\qquad |\Phi^*(t)| = |B| \left\{ \frac{\prod_1^{n-1} |e^{it} - \alpha_i|^2}{\prod_1^{n} |e^{it} - \beta_i|^2} \right\}^{1/p} = \|I\|^{-1/p} |C|^{1/p} |C|^{-1/p} \Lambda(t)^{1/p}.$$

It will then follow that $\mathfrak{K}^*(z)$ and $F^*(z)$ are of the forms (8.5.3) and (8.5.4); and these are equivalent to (8.2.2) and (8.2.3), respectively.

(iv) It suffices to consider $\mathfrak{K}^*(z)$, the proof for the determination of $F^*(z)$ being analogous. Alternatively, once the form (8.2.2) for $\mathfrak{K}(z)$ has been established, that for $F(z)$ follows from (8.4.1). Now, by (α), the principal determination of $\log \mathfrak{K}^*(z)$ is regular in $|z|<1$. Hence, by the formula of Schwarz,

$$\text{(8.5.6)}\qquad \log \mathfrak{K}^*(rz) = \frac{1}{2\pi} \int_0^{2\pi} \log |\mathfrak{K}^*(re^{it})| \frac{e^{it}+z}{e^{it}-z} \, dt + i \arg \mathfrak{K}^*(0),$$

for all $0<r<1$. If we can prove that this formula still holds for $r=1$, then the function $\mathfrak{K}^*(z)$ will be determined, apart from a constant factor of modulus 1, since $|\mathfrak{K}^*(e^{it})| \equiv |K^*(t)|$ is determined by (δ). It, clearly, suffices to show that

$$\text{(8.5.7)}\qquad \log |\mathfrak{K}^*(re^{it})| \xrightarrow[1]{} \log |K^*(t)|.$$

(v) We, first, note that

$$\text{(d)}\qquad \int_0^{2\pi} |\log |K^*(t)|| \, dt = \int_0^{2\pi} |\log \{\|I\|^{1/p} \Lambda(t)^{1/q}\}| \, dt < \infty,$$

since, by (8.5.1) and (δ), the integrand involves at most logarithmic infinities. Hence $\log |K^*(t)| \in L^1$.

Next, by (α) and (γ), the function $L^*(z) = F^*(z)\, \mathfrak{K}^*(z)$ is regular in $|z| \le 1$, $\neq 0$ in $|z|<1$, and has at most a finite number m, say, of roots γ_l on $|z|=1$. Each of these roots is, by the way, of even order; anyhow, we count them according to multiplicity. Now the function $L^{**}(z) = L^*(z) \prod_1^m (z-\gamma_l)^{-1}$ is regular and $\neq 0$ in $|z| \le 1$. Hence $|L^{**}(z)| \ge d > 0$ in $|z| \le 1$. It follows that, for $|z|<1$,

$$\text{(e)} \qquad \left|\frac{1}{\mathfrak{K}^*(z)}\right| = \left|\frac{F^*(z)\prod_1^m (z-\gamma_i)^{-1}}{L^{**}(z)}\right| \le \frac{1}{d}\left|F^*(z)\prod_1^m |z-\gamma_i|^{-1}\right|,$$

and hence, for $r<1$,

$$\text{(f)} \qquad \log^+\frac{1}{|\mathfrak{K}^*(re^{it})|} \le \log^+\frac{1}{d} + \log^+|F^*(re^{it})| + \log^+\{r^{-m}\prod_1^m |e^{it}-\gamma_i|^{-1}\}$$

since $|re^{it}-\gamma_i| = r|e^{it}-\gamma_i r^{-1}| \ge r|e^{it}-\gamma_i|$.

We now make the points t_i, where $\gamma_i = e^{it_i}$, centres of a set I_δ of intervals of equal length and total length $\delta < 2\pi$. By the argument, used in (1.1.5).

$$\text{(g)} \qquad \frac{1}{2\pi}\int_{I_\delta} \log^+|\mathfrak{K}^*(re^{it})|\,dt \le \log\left\{1+\frac{1}{2\pi}\int_{I_\delta}|\mathfrak{K}^*(re^{it})|\,dt\right\}\cdot$$

Similarly, and on account of (f),

$$\text{(h)} \qquad \frac{1}{2\pi}\int_{I_\delta}\log^+\left|\frac{1}{\mathfrak{K}^*(re^{it})}\right|dt \le \frac{\delta}{2\pi}\log^+\frac{1}{d} + \log\left\{1+\frac{1}{2\pi}\int_{I_\delta}|F^*(re^{it})|\,dt\right\} +$$

$$+\frac{1}{2\pi}\int_{I_\delta}\log^+\left\{r^{-m}\prod_1^m |e^{it}-\gamma_i|^{-1}\right\}dt.$$

By (β), $F^*(re^{it}) \underset{p}{\to} \Phi^*(t)$ and $\mathfrak{K}^*(re^{it}) \underset{q}{\to} e^{-it}\mathrm{K}^*(t)$. Hence the right hand sides of (g) and (h) tend with $r\to 1-0$ to the corresponding expressions for $r=1$. But then these expressions, and hence their sum, are small for small δ, since Φ^* and K^* are integrable. Since, by (d), $\log|\mathrm{K}^*|$ is also integrable, we conclude that, given $\varepsilon>0$,

$$\text{(i)} \qquad \overline{\lim_{r\to 1-0}}\,\frac{1}{2\pi}\int_{I_\delta}\big|\log|\mathfrak{K}^*(re^{it})| - \log|\mathrm{K}^*(t)|\big|\,dt < \varepsilon,$$

provided that $\delta = \delta(\varepsilon)$ has been fixed sufficiently small.

Finally, we consider the complement $\bar{I}_\delta$ of I_δ. Here we use the elementary inequality

$$\text{(j)} \qquad |\log a - \log b| \le \frac{|a-b|}{\operatorname{Min}(a,b)},$$

and note that $|L^*(re^{it})| \ge d^* > 0$ on $\bar{I}_\delta$ for all $0 \le r < 1$. Hence, if $|\mathfrak{K}^*(re^{it})| \ge |\mathrm{K}^*(t)|$, then

$$\text{(k)} \qquad \big|\log|\mathfrak{K}^* re^{it})| - \log|K^*(t)|\big| \le \frac{|\mathfrak{K}^*(re^{it})| - |\mathrm{K}^*(t)|}{|\mathrm{K}^*(t)|} \le \frac{|\mathfrak{K}^*(re^{it}) - \mathfrak{K}^* e^{it})|\,|\Phi^*(t)|}{d^*};$$

and if $|K^*(t)| \geq |\mathfrak{K}^*(re^{it}|$, then, similarly,

$$\text{(l)} \qquad \big|\log|\mathfrak{K}^*(re^{it})| - \log|\mathrm{K}^*(t)|\big| \leq \frac{|\mathfrak{K}^*(re^{it}) - \mathfrak{K}^*(e^{it})|\,|F^*(re^{it})|}{d^*}.$$

It follows that

$$\frac{1}{2\pi}\int_{I_\delta} \big|\log|\mathfrak{K}^*(re^{it})| - \log|\mathrm{K}^*(t)|\big|\,dt \leq$$

$$\text{(m)} \qquad \leq \frac{1}{2\pi d^*}\int_0^{2\pi} |\mathfrak{K}^*(re^{it}) - \mathfrak{K}^*(e^{it})|\,\{|F^*(re^{it})| + |\Phi^*(t)|\}\,dt$$

$$\leq \frac{1}{d^*}\|\mathfrak{K}^*(re^{it}) - \mathfrak{K}^*(e^{it})\|_q\,\{M_p(F^*, r) + \|\Phi^*\|_p\}$$

$$\leq \frac{2}{d^*}\|\Phi^*\|_p\,\|\mathfrak{K}^*(re^{it}) - \mathfrak{K}^*(e^{it})\|_q.$$

Here the right hand side tends to zero as $r \to 1-0$, since $\mathfrak{K}^*(re^{it}) \underset{q}{\to} \mathfrak{K}^*(e^{it})$.[1] The interval set I_δ, in (i), can, therefore, be replaced by the whole interval $\langle 0, 2\pi\rangle$. Since $\varepsilon > 0$ was arbitrary, (8.5.7) follows, and Theorem 16 is proved.

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[1] If $q = \infty$, we use (m) directly.