AN EXTENSION OF THE RIEMANN MAPPING THEOREM.

By

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Introduction.

There are no limits known for the boundary conditions under which the Laplace equation

$$\Delta u = 0$$

admits a solution for a given region. The very richness of the potential theory and the great variety of its applications seem to prohibit general results in this direction. There are, however, reasons indicating that the Laplace equation would permit solution under a boundary condition

$$\Phi\left(u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial^2 u}{\partial x_1^2} \ldots\right) = 0$$

that expresses a relation between u and a given sequence of its partial derivatives of a much more general kind than those treated in the classical theory. The present paper will deal with the simplest 2-dimensional version of a problem of the indicated kind, that is directly related to the Riemann mapping theorem.

In this introduction Ω and Ω' will denote two entire complex planes and the euclidean 4-dimensional space E will be considered as the product $\Omega \times \Omega'$. A point in E will be denoted by (w, w') where

$$w = u + i\,v, \quad w' = u' + i\,v'.$$

A "surface" S in E will be defined as the boundary of some open pointset $K \subset E$. To each function $f(z)$ holomorphic in $|z| \leq 1$ we assign the curve L_f described in the space E by the point

$$(w, w') = (f(z), f'(z)) \qquad \left(f' = \frac{d f}{d z}\right),$$

when z runs through the circle $|z|=1$. In case f is only regular in the open circle $|z|<1$, L_f will be defined as the set of points (w, w') for which

$$\liminf_{|z|\uparrow 1} \| (f(z), f'(z)) - (w, w') \| = 0$$

where $\| \quad \|$ stands for the euclidean distance.

Using these concepts the Riemann mapping theorem may be expressed as follows.

Let D be a simply-connected bounded region and let K be the product set $D \times \Omega'$. Let w_0 be a given point $\in D$ and γ a given real number. Then there exists a uniquely determined univalent function $f(z)$ in $|z|<1$ such that the curve L_f lies on the boundary S of K and such that

$$f(0) = w_0, \quad \arg f'(0) = \gamma.$$

The content of this theorem can be considered as a property established for a particular kind of product sets K. It may, therefore, be of some interest to point out that the cited theorem remains true for sets $K = D \times C$, where C stands for a circle $|w'| < r$, $0 < r < \infty$. This statement, as well as more general propositions of the same kind, will not be discussed in this paper but is a simple consequence of results to be proved here concerning sets K_Φ defined by an inequality

$$|w'| < \Phi(w).$$

A brief summary of our main result runs as follows:

Let $\Phi = e^U$, where U is a real valued continuous super-harmonic function bounded from above. Then the boundary surface S of K_Φ has the property: For each complex w_0 and for each real γ there exists a unique univalent function $f(z)$ in $|z|<1$, normalized as in the Riemann theorem, and such that the curve L_f lies on S.

Preliminaries.

From now on $\Phi(w)$ will denote an arbitrary continuous positive bounded function defined for $|w| < \infty$, and M will stand for an upper bound of Φ. A function $f(z)$ holomorphic in $|z|<1$ will be called normalized if

$$f(0) = w_0, \quad f'(0) > 0, \tag{0}$$

w_0 being a given point in the w-plane. The functional equation written as

$$|f'(z)| = \Phi(f(z)) \qquad (|z|=1), \tag{1}$$

is by definition satisfied if

$$\lim_{|z|\uparrow 1} (|f'(z)| - \Phi(f(z))) = 0.$$

We will be interested in normalized univalent functions satisfying this relation, and such a function will briefly be referred to as a solution of (1).

Our problem is to decide whether or not (1) has a solution and under what conditions on Φ a unique solution exists. If for instance $w_0 = 0$ and $\Phi(w) = \Phi(r)$ is a function of $|w| = r$, then $f(z) = \varrho z$ is a solution of our problem for each $\varrho > 0$ satisfying the equation $\Phi(\varrho) - \varrho = 0$. This shows us that uniqueness requires specific additional conditions imposed on Φ. If, on the other hand, $\log \Phi$ is harmonic in the circle $|w - w_0| < M$, we easily find that the problem has a solution which is in fact unique. These two examples may serve the purpose of anticipating the result to be proved.

The study of our problem will be based on the following two function classes. A_Φ consists of functions $f(z)$ holomorphic in $|z| < 1$, satisfying (0) and with the property that

$$|f'(z)| \leq \Phi(f(z)) \qquad (|z| = 1), \tag{2}$$

which relation has to be read

$$\limsup_{|z| \uparrow 1} (|f'(z)| - \Phi(f(z))) \leq 0.$$

By B_Φ we denote the class of *univalent* and holomorphic functions $g(z)$ in $|z| < 1$, normalized as above, and such that

$$|g'(z)| \geq \Phi(g(z)) \qquad (|z| = 1), \tag{3}$$

in the sense that

$$\liminf_{|z| \uparrow 1} (|g'(z)| - \Phi(g(z))) \geq 0.$$

Finally, C_Φ will denote the class of solutions of (1), and our main problem is of course to show that C_Φ is not empty. Since the function $f(z) = w_0 + \varrho z$ will belong to A_Φ for ϱ positive and sufficiently small, and to B_Φ for ϱ sufficiently large, we see that neither of these classes is vacuous.

A rather complete account of the present study runs as follows:

Theorem I. *If Φ is positive, continuous and bounded, and w is a given point, then* (1) *possesses in general two distinguished solutions f^* and g^* characterized by the following properties:*

I. *The pointset A^*, defined as the union of values w assumed in $|z| < 1$ by functions $f \in A_\Phi$, is a simply-connected region and the normalized univalent function f^* which maps $|z| < 1$ onto A^* is a solution of* (1).

II. *The pointset B^*, defined as the intersection of values w assumed in $|z|<1$ by functions $g \in B_\Phi$, is a simply-connected region and the normalized univalent function g^* which maps $|z|<1$ onto B^* is a solution of* (1).

III. *Any other solution of* (1) *maps $|z|<1$ onto a region contained in A^* and containing B^*.*

IV. *If* $\log \frac{1}{\Phi}$ *is subharmonic, A^* and B^* are identical and* (1) *has a unique solution.*

Convergence of simply-connected regions.

In the sequel we shall assume that all regions considered contain a fixed point w_0 and that they all are contained in a fixed circle $|w-w_0|<M$. The boundary sets of the simply-connected regions R, R_n shall be denoted by Γ, Γ_n respectively. The notation $[w, \Gamma_n]$ stands for the distance between the point w and the set Γ_n.

We define

$$[\Gamma, \Gamma_n] = \sup_{w \in \Gamma} [w, \Gamma_n]$$

and observe that in general $[\Gamma, \Gamma_n] \neq [\Gamma_n, \Gamma]$. By $f(z)$, $f_n(z)$ we shall mean the normalized univalent functions which map $|z|<1$ onto R, R_n respectively.

The infinite sequence $\{R_n\}_1^\infty$ is said to converge weakly to R if for any $w \in R$ there is an integer $n(w)$ such that $w \in R_n$ for $n > n(w)$, and if for any $w \in \Gamma$

$$\lim_{n=\infty} [w, \Gamma_n] = 0. \tag{4}$$

From the inequality

$$|[w_1, \Gamma_n] - [w_2, \Gamma_n]| \leq |w_1 - w_2|$$

we conclude that (4) implies that $[w, \Gamma_n]$ tends uniformly to 0 on Γ. Therefore (4) may be replaced by

$$\lim_{n=\infty} [\Gamma, \Gamma_n] = 0. \tag{5}$$

We next list a series of well-known properties that will be used later on.

α). R_n converges weakly to R if and only if $f_n(z)$ converges to $f(z)$ in $|z|<1$.

β). If

$$R_n \subset R_{n+1} \subset R \qquad (n = 1, 2, \ldots) \tag{6}$$

then R_n converges weakly to R if and only if

$$f'(0) = \lim_{n=\infty} f_n'(0). \tag{7}$$

The same conclusion is true under the condition

$$R_n \supset R_{n+1} \supset R \qquad (n=1, 2, \ldots). \tag{8}$$

γ). A region R shall be called of Schoenfliess' type if the complement of its closure is a simply-connected region D with the property that each boundary point of R is also a boundary point of D. If R is bounded by a Jordan curve it has this property but the converse is not true. The proofs of the following two propositions will be left to the reader.

The region R is of Schoenfliess' type if and only if it is the weak limit of a *strictly* shrinking sequence $\{R_n\}_1^\infty$, i.e., such that the closure of R_{n+1} is contained in R_n.

Let R be of Schoenfliess' type and let $\{R_n\}_1^\infty$ satisfy (8). Then the additional condition

$$\lim_{n=\infty} [\Gamma_n, \Gamma] = 0 \tag{9}$$

implies the weak convergence of R_n to R.

δ). Assume that $U(w)$ is a continuous function in the circle $|w-w_0|<M$, and let u, u_n denote the solution of the Dirichlet problem for R, R_n respectively with boundary values in both cases $=U(w)$. Then the weak convergence of R_n to R implies that

$$\lim_{n=\infty} u_n(w) = u(w) \qquad (w \in R). \tag{10}$$

Compactness.

Any $f \in A_\Phi$ is continuous in the closed unit circle and satisfies the inequality

$$|f(z_1) - f(z_2)| \leq M\,|z_1 - z_2| \qquad (|z_1|, |z_2| \leq 1). \tag{11}$$

This equicontinuity would permit us to apply to A_Φ the uniform topology on $|z| \leq 1$, and to establish all statements concerning this class in a rather simple way. Since, however, this topology does not apply to B_Φ it will not on the whole offer us any great advantage in the proof of Theorem I.

For reasons that will be clear later on we can restrict our study of B_Φ to functions $f(z)$ subject to the condition

$$|f(z) - f(0)| \leq M \qquad (|z| < 1), \tag{12}$$

and the weak topology considered in the previous section will be the adequate concept.

Lemma I. *Let* $\{f_n\}_1^\infty$ *be a sequence of functions univalent in* $|z|<1$ *and satisfying* (12). *Let* $\{f_n\}_1^\infty$ *belong to one of the classes* A_Φ, B_Φ *or* C_Φ. *Then* $\{f_n\}_1^\infty$ *contains a subsequence that converges in* $|z|<1$ *to a univalent* f *which belongs to the same class as does the original sequence.*

Throughout this paper $U(w)$ will be defined by the relation

$$\Phi(w)=e^{U(w)}$$

and $U_R(w)$ will denote the function which outside R coincides with $U(w)$ and in R is defined as the solution of the Dirichlet problem for R with boundary values $=U(w)$. Since a simply-connected region is regular for the Dirichlet problem and $U(w)$ is continuous, it follows that $U_R(w)$ is a continuous function.

Consider now a normalized univalent $f(z)$ which maps $|z|<1$ onto a region R. If $f\in A_\Phi$ we have by definition

$$\limsup_{|z|\uparrow 1}\,(\log|f'(z)|-U(f(z)))\leq 0. \tag{13}$$

Consequently,

$$\limsup_{|z|\uparrow 1}\,(\log|f'(z)|-U_R(f(z)))\leq 0. \tag{14}$$

In the last formula the function considered is harmonic in $|z|<1$, and it follows by the maximum principle for harmonic functions that

$$\log|f'(z)|\leq U_R(f(z)) \qquad (|z|<1). \tag{15}$$

Since (15) implies (13), the truth of (15) is both necessary and sufficient for a univalent normalized f to belong to A_Φ.

Using the minimum principle for harmonic functions we find by an analogous argument that f will belong to B_Φ if and only if

$$\log|f'(z)|\geq U_R(f(z)) \qquad (|z|<1). \tag{16}$$

Finally, the relation

$$\log|f'(z)|=U_R(f(z)) \qquad (|z|<1) \tag{17}$$

is the criterion for f being a solution of (1).

In the proof of Lemma I we may of course assume that the original sequence converges in $|z|<1$ to some univalent f. Consider first the case $\{f_n\}_1^\infty\subset A_\Phi$. Since f_n converges to f, R_n will converge weakly to R, and it follows that $U_{R_n}(w)$ converges to $U_R(w)$ for $w\in R$. We also recall that this implies uniform convergence in each closed subset of R. Therefore, for any fixed z, $|z|<1$,

$$\log |f'(z)| = \lim_{n=\infty} \log |f_n'(z)| \leq \lim_{n=\infty} U_{R_n}(f_n(z)) = U_R(f(z)) \tag{18}$$

and the conclusion $f \in A_\Phi$ follows.

For the class B_Φ the proof is analogous, and for C_Φ obvious, this class being the intersection of A_Φ and B_Φ.

Unions and intersections of maps.

Consider a finite sequence $\{f_\nu\}_1^n \subset A_\Phi$, $(1 \leq n < \infty)$, and let S_ν denote the Riemann surface onto which f_ν maps $|z|<1$. The notation $\bar{S}_\nu$ stands for the projection of S_ν on the w-plane, i.e. the set of values taken by f_ν in $|z|<1$. The union R_0 of $\{\bar{S}_\nu\}_1^n$ is a connected open pointset. The *extended union* of $\{\bar{S}_\nu\}_1^n$,

$$R = \mathrm{E\,U}\,\{\bar{S}_\nu\}_1^n$$

is defined as follows: R is the collection of complex numbers w which may be surrounded by a Jordan curve γ each point of which belongs to the ordinary union R_0. The extended union R is a simply-connected region containing w_0. The normalized univalent function which maps $|z|<1$ onto R will be denoted by

$$f = \mathrm{E\,U}\,\{f_\nu\}_1^n.$$

We next assume that $\{f_\nu\}_1^n \subset B_\Phi$. Each S_ν is now a simply-connected plane region containing w_0 and the *reduced intersection*

$$D = \mathrm{R\,I}\,\{S_\nu\}_1^n$$

is defined as the set of those points w which can be joined with w_0 by a Jordan arc γ such that each $w_1 \in \gamma$ is assumed by each f_ν, $(\gamma = 1, 2, \ldots n)$, in $|z|<1$. D is a simply-connected region, and the normalized f which maps $|z|<1$ onto D will be denoted by

$$f = \mathrm{R\,I}\,\{f_\nu\}_1^n.$$

The proof of the statements $f^* \in A_\Phi$, $g^* \in B_\Phi$ depends essentially on the following lemma which is itself a consequence of the maximum-minimum principle.

Lemma II. *If* $\{f_\nu\}_1^n \in A_\Phi$ *so does*

$$f = \mathrm{E\,U}\,\{f_\nu\}_1^n.$$

Similarly, if $\{g_\nu\}_1^n \in B_\Phi$ *so does*

$$g = \mathrm{R\,I}\,\{g_\nu\}_1^n.$$

Consider first the case that the f_ν are holomorphic in the closed unit circle. The curves described by the points

$$w = f_\nu(e^{i\theta}) \qquad (0 \le \theta \le 2\pi,\ \nu = 1, 2, \ldots n)$$

are then analytic, and the ordinary union R_0 of $\{\bar{S}_\nu\}_1^n$ is a finitely-connected region bounded by a finite number of analytic arcs. The extended union R is bounded by the outer boundary component Γ of R_0. Γ itself is composed of a finite number of arcs Γ_k, each of which is an arc of the boundary of some S_ν, say for $\nu = \nu(k)$. Let F, F_ν denote the inverse functions of f, f_ν respectively. F_ν is defined on S_ν and

$$G_\nu = \log \left| \frac{1}{F_\nu} \right|$$

is the Green's function of S_ν, singular at the distinguished point $w_0 = f_\nu(0)$. Similarly,

$$G = \log \left| \frac{1}{F} \right|$$

is the Green's function of R. Since the boundary of S_ν lies either under R or under the boundary of R, we conclude by the minimum principle that the inequality

$$G - G_\nu \ge 0 \tag{19}$$

holds throughout S_ν, $(\nu = 1, 2, \ldots n)$. At a point w which is an inner point of some arc Γ_k we will therefore have

$$\frac{\partial G}{\partial n} \ge \frac{\partial G_\nu}{\partial n} \qquad (\nu = \nu(k)), \tag{20}$$

where the derivative is taken in the direction of the inner normal. Since F and F_ν are regular at the point considered, (20) implies

$$|F'(w)| \ge |F'_\nu(w)| \ge \frac{1}{\Phi(w)}. \tag{21}$$

This yields

$$|f'(z)| \le \Phi(f(z)) \qquad (|z| = 1), \tag{22}$$

except perhaps at a finite number of points which correspond to intersections of the Γ_k. However, the regularity of the boundary of R permits us to use the following Poisson representation

$$\log |f'(z)| = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|z - e^{i\theta}|^2} \log |f'(e^{i\theta})| \, d\theta \qquad (|z| < 1), \tag{23}$$

which combined with (22) yields

$$\log |f'(z)| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1-|z|^2}{|z-e^{i\theta}|^2} U(f(e^{i\theta}))\, d\theta = U_R(f(z)).$$

Therefore $f \in A_\Phi$.

In the general case where the f_ν are not holomorphic in the closed circle, we set

$$f_{\nu,\varepsilon}(z) = f_\nu((1-\varepsilon)z), \qquad \varepsilon > 0.$$

These functions are analytic in $|z| \leq 1$ and certainly belong to the class $A_{t\Phi}$ where t stands for a number >1 which tends to 1 for $\varepsilon \to 0$. Therefore

$$f_\varepsilon = \mathrm{E\,U}\{f_{\nu,\varepsilon}\}_1^n$$

will belong to $A_{t\Phi}$ and map $|z|<1$ onto

$$R_\varepsilon = \mathrm{E\,U}\{\bar{S}_{\nu,\varepsilon}\}_1^n,$$

$S_{\nu,\varepsilon}$ being the map of $|z|<1-\varepsilon$. For $\varepsilon \to 0$, R_ε will converge weakly to

$$R = \mathrm{E\,U}\{\bar{S}_\nu\}_1^n$$

and f_ε will converge to f in $|z|<1$. By Lemma I we finally conclude that f will belong to the class $A_{t\Phi}$ for any $t>1$. This proves our statement that $f \in A_\Phi$.

For the class B_Φ the proof is similar. If $\{f_\nu\}_1^n \subset B_\Phi$, we compare the Green's function for S_ν and for

$$D = \mathrm{R\,I}\{S_\nu\}_1^n$$

and obtain by the maximum principle

$$G - G_\nu \leq 0 \qquad (w \in D).$$

For the rest the proof is the same as in the previous case and will not be repeated.

The maximal region A^*.

From the equicontinuity expressed by (11) we derive the existence of an enumerable subset $\{h_n\}_1^\infty$ of A_Φ having the property that for any $f \in A_\Phi$ and for any $\varepsilon > 0$ there will exist an integer n such that

$$\max_{|z|\leq 1} |f(z) - h_n(z)| < \varepsilon.$$

The pointset A^* is therefore identical with the union of values taken in $|z|<1$ by $\{h_n\}_1^\infty$. Let S_n denote the Riemann surface onto which h_n maps the unit circle. By Lemma II,

$$f_n = \mathrm{E\,U}\,\{h_\nu\}_1^n \in A_\Phi.$$

Consequently,

$$R_n = \mathrm{E\,U}\,\{\bar{S}_\nu\}_1^n \subset A^*.$$

Hence, $\{R_n\}_1^\infty$ is an increasing sequence of simply-connected regions which exhaust A^* for $n\to\infty$. Therefore A^* is simply-connected and f_n converges to f^* for $n\to\infty$. Finally, by Lemma I, $f^*\in A_\Phi$.

Next to be proved is that f^* is a solution. To this purpose we will profit from the following remarkable relation between the given function Φ and the maximal region $A^*=A_\Phi^*$. Let $\psi(w)$ be a positive continuous function which coincides with $\Phi(w)$, outside A^*, whereas in A^*, ψ is not subject to any additional condition. Since $\Phi=\psi$ on the boundary of A^*, f^* will also belong to the class A_ψ. Therefore $A_\psi^*\supset A_\Phi^*$, and it follows that $\Phi=\psi$ on the boundary of A_ψ^*. This implies that the mapping function h^* for A_ψ^* belongs to A_Φ, and it follows that $A_\Phi^*=A_\psi^*$.

In particular, A^* is also maximal for the class A_ψ with

$$\psi(w)=e^{U_{A^*}(w)}.$$

Since

$$\log|f^{*\prime}(z)|-U_{A^*}(f^*(z)) \tag{24}$$

is a non-positive harmonic function in $|z|<1$, it is either $\equiv 0$, or <0 throughout the circle. Under the last alternative

$$|F^{*\prime}(w)|>e^{-U_{A^*}(w)} \qquad (w\in A^*), \tag{25}$$

F^* being the inverse of f^*. Let V be the conjugate harmonic function of U_{A^*} in A^* and let V be normalized by the condition $V(w_0)=0$. Define

$$H(w)=\int_{w_0}^{w} e^{-(U_{A^*}+iV)}\,dw \qquad (w\in A^*).$$

By (25)

$$|H'(w)|<|F^{*\prime}(w)| \qquad (w\in A^*),$$

and it follows that

$$r_0=\inf_{w\in A^*}\left|\frac{H}{F^*}\right|<1.$$

The region $R_0=\{w,\ |H(w)|<r_0\}$ is therefore contained in A^* and has at least one boundary point, say w_1, located on the boundary of A^*. By $H(w)$ the region R_0

is mapped onto the circle $|z| < r_0$. If h is the inverse of H, the function $f(z) = h(r_0 z)$ will satisfy the equation

$$|f'(z)| = r_0 \psi(f(z)) \qquad (|z| = 1).$$

Without loss of generality we may assume that $w_0 = 0$. Since r_0 is < 1 and ψ is continuous there will exist a positive number ε such that for any complex number $a = |a| e^{i\gamma}$ in the circle $|a - 1| < \varepsilon$, the function $a f(z e^{-i\gamma})$ will belong to the class A_ψ. The union of values taken by this family in $|z| < 1$ will contain a neighborhood of w_1, which is a contradiction since w_1 is on the boundary of A^* and A^* is maximal for the class A_ψ. Therefore (24) vanishes identically in $|z| < 1$ and it is proved that f^* is a solution

The minimal region B^*.

The proof of the statement $g^* \in B_\Phi$ follows the same line as in the previous case and is quite easy. The difficult part of the theorem is, however, to show that g^* is a solution.

We first observe that $f^* \in B_\Phi$. This implies that

$$m \equiv \inf_{g \in B_\Phi} g'(0) \geq \operatorname{Min}_{w \in A^*} \Phi(w) > 0. \tag{26}$$

In fact, on defining

$$g_1 = \mathrm{R\,I}(g, f^*) \qquad (g \in B_\Phi).$$

we obtain a function $g_1 \in B_\Phi$ which assumes only values in A^* and is such that $g_1'(0) \leq g'(0)$. This proves the truth of (26).

The sequence $\{h_n\}_1^\infty$ considered in the previous section will now be chosen in B_Φ and such that

$$m = \lim_{n=\infty} h_n'(0).$$

By Lemma II,

$$g_n = \mathrm{R\,I}\{h_\nu\}_1^n \in B_\Phi.$$

Consequently, if h_ν maps $|z| < 1$ onto S_ν, the regions

$$R_n = \mathrm{R\,I}\{S_\nu\}_1^n$$

will have the property

$$B^* \subset R_{n+1} \subset R_n \qquad (n = 1, 2, \ldots).$$

By definition of the operator R I it follows that

$$g_n'(0) \leq \operatorname{Min}_{\nu=1,2,\ldots n} h_\nu'(0).$$

For $n \to \infty$, R_n will thus converge weakly to some region $R \supset B^*$, and g_n will converge in $|z| < 1$ to g which maps the unit circle onto R. This R is identical with B^*, since otherwise there would exist an $h \in B_\Phi$ which in $|z| < 1$ omits some value $w \in R$, and the function

$$k = \mathrm{R\,I}(g, h) \in B_\Phi$$

would have the property $k'(0) < g'(0) = m$, which is a contradiction. Therefore $g = g^* \in B_\Phi$.

The proof that g^* is a solution of (1) is rather complicated and will first be carried through under the assumption that B^* is a region of Schoenfliess' type. To this purpose we construct a sequence of continuous functions

$$\Phi_n = e^{U_n}$$

with the following properties: each Φ_n coincides with Φ in B^*, whereas outside the closure of B^*,

(27) $$\Phi_{n+1} \leq \Phi_n \leq \Phi,$$

(28) $$\lim_{n=\infty} \Phi_n = 0.$$

Let A_n^* and B_n^* be the maximal and the minimal region that corresponds to Φ_n. Obviously, $A_{\Phi_{n+1}} \subset A_{\Phi_n} \subset A_\Phi$, $B_{\Phi_{n+1}} \supset B_{\Phi_n} \supset B_\Phi$, and it follows that

(29) $$B_n^* \subset B^* \subset A_{n+1}^* \subset A_n^* \subset A^*.$$

Since $\Phi_n = \Phi$ on B^*, the minimal function of the class B_{Φ_n} takes only values in B^*. Therefore

$$B_n^* = B^* \qquad (n = 1, 2, \ldots).$$

The maximal function f_n^* of A_{Φ_n} is a solution of equation (1) with Φ replaced by Φ_n. Consequently,

(30) $$\log f_n^{*\prime}(0) = U_{n, A_n^*}(w_0) \leq U_{A_n^*}(w_0).$$

If g^* were not a solution of (1) we would have

(31) $$\log g^{*\prime}(0) = U_{B^*}(w_0) + \varepsilon, \qquad (\varepsilon > 0).$$

Since $B^* \subset A_n^*$,

$$g^{*\prime}(0) \leq f_n^{*\prime}(0).$$

On combining these relations we obtain

(32) $$U_{A_n^*}(w_0) \geq U_{B^*}(w_0) + \varepsilon \qquad (n = 1, 2, \ldots).$$

For the inverse F_n^* of f_n^* we have

$$2\pi = \int |F_n^{*\prime}|\,|d\,w| = \int \frac{|d\,w|}{\Phi_n(w)} \tag{33}$$

where the integral is extended over the boundary of A_n^*. As $n \to \infty$, Φ_n converges uniformly to 0 on each closed set contained in the complement of the closure of B^*. Relation (33) therefore implies that the boundary Γ_n^* of A_n^* converges to the boundary Γ^* of B^* in the sense that

$$\lim_{n=\infty} [\Gamma_n^*, \Gamma^*] = 0. \tag{34}$$

For a region B^* of Schoenfliess' type (34) implies that the shrinking sequence $\{A_n^*\}_1^\infty$ converges weakly to B^*. Hence

$$\lim_{n=\infty} U_{A_n^*}(w_0) = U_{B^*}(w^*) \tag{35}$$

which contradicts (32). Our assumption (31) is therefore false and consequently g^* is a solution of (1).

In case B^* is not of Schoenfliess' type we proceed as follows. Instead of the class B_Φ we consider the family of classes $B_t \equiv B_{t\Phi}$ where t is a positive parameter, and we define

$$m(t) = \inf_{g \in B_t} g'(0).$$

For $t_1 < t_2$, $B_{t_1} \supset B_{t_2}$ and it follows that $m(t_1) \le m(t_2)$ and $B_{t_1}^* \subset B_{t_2}^*$. As a monotonic function $m(t)$ is continuous except at an enumerable set at most. At first sight it seems absurd that $m(t)$ could be discontinuous, but this actually can occur and is one of the reasons why the minimal region is hard to deal with.

We first prove that B_t^* is a *strictly* monotonic family of regions, i.e. that the closure of $B_{t_1}^*$ is contained in $B_{t_2}^*$. Assume $w_0 = 0$ and let $g_{t_2}^*(z)$ be the minimal function of the class B_{t_2}. From the continuity of Φ it follows that $a\,g_{t_2}^*(z\,e^{-i\gamma})$ will belong to B_{t_1}, provided $a = |a|\,e^{i\gamma}$ is a complex number in the circle $|a-1| < \varepsilon$, ε being a positive number depending on t_1, t_2 and Φ. This proves our statement.

Let now t be a number such that

$$m(t) = \lim_{s \downarrow t} m(s).$$

This implies that for $s \downarrow t$, B_s^* will converge weakly to B_t^*, and this region is therefore of Schoenfliess' type. Accordingly, g_t^* is a solution of equation (1) with Φ replaced by $t\,\Phi$. Let $\{t_n\}_1^\infty$ be an infinite sequence of numbers tending *increasingly* to 1 and let m be continuous at each t_n. The previous result then applies to each

$g^*_{t_n}$. For $n \to \infty$, $B^*_{t_n}$ will converge weakly to some region $B \subset B^*$, and $g^*_{t_n}$ will converge in $|z| < 1$ to some function g, which by Lemma I must belong to B_Φ. Thus $g = g^*$, $B = B^*$ and a second application of Lemma I proves that g^* is a solution. This ends the proof of the two first parts of the theorem.

Since the third part is a direct consequence of the definition of the sets A^* and B^* there only remains the uniqueness problem to be discussed. Our assumption $\log \frac{1}{\Phi}$ subharmonic, i.e. U super-harmonic, is obviously a sufficient condition for uniqueness. In fact, since $B^* \subset A^*$ and U is super-harmonic it follows that

$$U_{B^*}(w_0) \geq U_{A^*}(w_0), \tag{36}$$

$$g^{*\prime}(0) \leq f^{*\prime}(0), \tag{37}$$

where equality in relation (37) implies $B^* = A^*$. On the other hand we have

$$\log g^{*\prime}(0) = U_{B^*}(w_0), \qquad \log f^{*\prime}(0) = U_{A^*}(w_0).$$

Consequently

$$g^{*\prime}(0) \geq f^{*\prime}(0)$$

and the uniqueness is proved.

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