THE p-ADIC GENERALISATION OF THE THUE-SIEGEL THEOREM.

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Introduction.

In 1920, Siegel[1] proved the following generalisation of the Thue-Siegel Theorem:

Let ξ be an algebraic number of degree $d \geq 2$.

1. *Let $\mathfrak{K}$ be a fixed algebraic number field, and let ξ satisfy an irreducible equation of degree $m \geq 2$ with coefficients from $\mathfrak{K}$. Let s be a natural number less than m. Then for every positive Θ the inequality*

$$|\lambda - \xi| \leq \frac{1}{A^{\frac{m}{s+1}+s+\Theta}}$$

has only a finite number of solutions in primitive numbers λ of $\mathfrak{K}$, A being the maximum of the absolute values of the coefficients of the primitive irreducible polynomial with rational integral coefficients having λ as a root.

2. *Let h and s be two natural numbers, of which s is less than d. Then for every positive Θ the inequality*

$$|\lambda - \xi| \leq \frac{1}{A^{h\left(\frac{d}{s+1}+s\right)+\Theta}}$$

has only a finite number of solutions in algebraic numbers λ of degree h.

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[1] C. SIEGEL, 'Approximation algebraischer Zahlen', Mathematische Zeitschrift, Vol. 10 (1921), pp. 173—213.

In 1932, Mahler[1] extended the Thue-Siegel Theorem, for the case when the approximating number λ is a rational number $\lambda = \frac{p}{q}$, p and q being relatively prime rational integers, to non-Archimedean as well as Archimedean valuations. He also obtained an approximation to the actual number of solutions of his inequality. His result was as follows:

Let $P_1, P_2, \ldots, P_\sigma$ be $\sigma (\geq 0)$ different natural prime numbers, and let $\xi_0, \xi_1, \xi_2, \ldots, \xi_\sigma$ be respectively real, P_1-adic, P_2-adic, . . ., P_σ-adic roots of an irreducible polynomial $f(x)$ of degree $m \geq 3$ with rational integral coefficients. Let α be the number $\min_{s=1,2,\ldots,m-1} \left(\frac{m}{s+1} + s\right)$, and β a number such that $\alpha < \beta \leq m$. Let c be a positive constant. Then the number of solutions in pairs of relatively prime rational integers p and q of the inequality

$$\min\left(1, \left|\frac{p}{q} - \xi_0\right|\right) \prod_{k=1}^{\sigma} \min\left(1, |p - \xi_k q|_{P_k}\right) \leq c \max\left(|p|, |q|\right)^{-\beta}$$

is not greater than

$$c_0 \, 2^{\frac{\beta}{\beta-\alpha}(1+\varepsilon_0)\sigma},$$

where ε_0 is a positive number and c_0 is a constant depending only on ε_0, β and $f(x)$ and not on the number and choice of the prime numbers $P_1, P_2, \ldots, P_\sigma$.

($|p - \xi_k q|_{P_k}$ denotes the P_k-adic value of $(p - \xi_k q)$).

The primary object of the present paper is to combine these two results and extend them to the case when approximation is made, by algebraic numbers λ of a fixed degree $h \geq 1$ (or of a degree dividing h) over a field $\mathfrak{R}$ of degree $n \geq 1$ over the rational number field $\mathfrak{P}$, to a number of real or complex roots and $\mathfrak{r}$-adic roots ($\mathfrak{r}$ being a finite prime ideal of $\mathfrak{R}$) of a polynomial $f(x)$ of degree $m \geq 2$ with integral coefficients from $\mathfrak{R}$. The polynomial $f(x)$ *need not now be irreducible*, the only condition imposed upon it being that it shall have a non-zero discriminant. In stating the result obtained, use is made of the term '*infinite prime ideal*'. The meaning of this term, and the definitions and notation adopted for absolute and $\mathfrak{r}$-adic valuations, are given in § 1. By the symbol $g(\mathfrak{p})$ is meant, if $\mathfrak{p}$ is a finite prime ideal of $\mathfrak{R}$, the degree of the perfect $\mathfrak{p}$-adic extension of $\mathfrak{R}$ over the perfect p-adic extension of $\mathfrak{P}$, where p is the natural prime number divisible by $\mathfrak{p}$, and if $\mathfrak{p}$ is an infinite prime ideal of $\mathfrak{R}$, the degree of the perfect

[1] K. MAHLER, 'Zur Approximation algebraischer Zahlen', Mathematische Annalen, Vol. 107 pp. 691—730 and Vol. 108, pp. 37—55 (1933).

$\mathfrak{p}$-adic extension of $\mathfrak{K}$ over the field of real numbers. $G(\mathfrak{p})$ denotes a natural number not greater than $g(\mathfrak{p})$. A denotes the maximum of the absolute values of the coefficients of that polynomial of degree hn which is a power of the primitive irreducible polynomial with rational integral coefficients having λ as a root.

The result, as stated in Theorem 1, is as follows:

Let $f(x)$ be a polynomial of degree $m \geq 2$ with integral coefficients from $\mathfrak{K}$ and a non-zero discriminant. Let $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_\varrho$, where $0 \leq \varrho \leq r_1 + r_2$, be ϱ of the $r_1 + r_2$ infinite prime ideals corresponding to the r_1 real and r_2 pairs of conjugate imaginary fields conjugate to $\mathfrak{K}$, and let $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$, where $\sigma \geq 0$, be σ different finite prime ideals of $\mathfrak{K}$. Let $h_{k\delta}$ $(k = 1, 2, \ldots, \sigma;\ \delta = 1, 2, \ldots, G(\mathfrak{r}_k))$ be a natural number not greater than h^2. Let $\xi_{j\gamma}(j = 1, 2, \ldots, \varrho;\ \gamma = 1, G(\mathfrak{q}_j))$ be a real or complex root of $f(x)$ and $\eta_{k\delta\tau}(k = 1, 2, \ldots, \sigma;\ \delta = 1, 2, \ldots, G(\mathfrak{r}_k);\ \tau = 1, 2, \ldots, h_{k\delta})$ an $\mathfrak{r}_k$-adic root of $f(x)$, and let t be the total number of these roots. Let c and ε_0 be two positive numbers and α and β two numbers such that $\alpha = \min_{s=1,2,\ldots,m-1} \left(\frac{m}{s+1} + s\right)$ and $\beta > \alpha$. Then the number of different algebraic numbers λ of degree h (or any divisor of h) over $\mathfrak{K}$, lying in the perfect $\mathfrak{r}_1$-adic, $\mathfrak{r}_2$-adic, $\ldots$, $\mathfrak{r}_\sigma$-adic extensions of $\mathfrak{K}$ and satisfying the inequality

$$\prod_{j=1}^{\varrho} \prod_{\gamma=1}^{G(\mathfrak{q}_j)} \min\left(1, |\lambda - \xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}}\right) \prod_{k=1}^{\sigma} \prod_{\delta=1}^{G(\mathfrak{r}_k)} \prod_{\tau=1}^{h_{k\delta}} \min\left(1, |\lambda - \eta_{k\delta\tau}|_{\mathfrak{r}_k}\right) \leq c\, A^{-h\beta}$$

is not greater than

$$k_0\, 2^{\frac{\beta}{\beta-\alpha}(1+\varepsilon_0)\, t}$$

where k_0 is a constant depending only on $\varepsilon_0, \beta, c, \mathfrak{K}, f(x)$ and h, and not on the number and choice of the roots to which approximation is made or on the corresponding ideals.

In the particular case of this result when $h = 1$, λ is an element ω of $\mathfrak{K}$, and A is denoted by Ω. ω may be represented as the quotient $\omega = \frac{u}{v}$ of two integers u and v of $\mathfrak{K}$ such that $N((u, v)) \leq |\sqrt{d(\mathfrak{K})}|$, where $d(\mathfrak{K})$ is the discriminant of $\mathfrak{K}$. From this fact and from Theorem 1 there follow a number of results, contained in Theorem 2 and its corollaries, on binary forms with integral coefficients from $\mathfrak{K}$. These results are simply generalisations of Mahler's results for binary forms with rational integral coefficients.

Theorem 2 states that:

If $F(x,y)$ be a binary form of degree $m \geq 2$ with integral coefficients from $\mathfrak{R}$ and a non-zero discriminant and such that the coefficient of x^m is not zero, and if $\nu_k (k = 1, 2, \ldots, \sigma)$ be the number of $\mathfrak{r}_k$-adic roots of $F(x,1)$, then the number of solutions of the inequality

$$|N(F(u,v))| \prod_{k=1}^{\sigma} |F(u,v)|_{\mathfrak{r}_k}^{G(\mathfrak{r}_k)} \leq \Omega^{m-\beta}$$

in non-associated[1] *pairs of integers u and v of $\mathfrak{R}$ such that $N((u,v)) \leq |\sqrt{d(\mathfrak{R})}|$, where $N(F(u,v))$ is the norm in $\mathfrak{R}$ over $\mathfrak{P}$ of $F(u,v)$, is not greater than*

$$k_3 2^{\frac{\beta}{\beta-\alpha}(1+\varepsilon_0)\left(\sum_{k=1}^{\sigma} G(\mathfrak{r}_k)\right)} \prod_{k=1}^{\sigma} \max(1, \nu_k),$$

where k_3 is a constant depending only on ε_0, β, $\mathfrak{R}$ and $F(x,y)$, and not on the number and choice of the finite prime ideals $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$.

From Theorem 2, Corollary 1, involving less stringent conditions for $F(x,y)$, is easily proved. It states that:

If $F(x,y)$ be a binary form of degree not less than 3 with integral coefficients from $\mathfrak{R}$ and a non-zero discriminant, then the number of solutions of the equality

$$|N(F(u,v))| \prod_{k=1}^{\sigma} |F(u,v)|_{\mathfrak{r}_k}^{g(\mathfrak{r}_k)} = 1$$

in non-associated pairs of integers u and v of $\mathfrak{R}$ such that $N((u,v)) \leq |\sqrt{d(\mathfrak{R})}|$ is not greater than

$$K^{\left(\sum_{k=1}^{\sigma} g(\mathfrak{r}_k)\right)+1},$$

where K is a constant depending only on $\mathfrak{R}$ and $F(x,y)$, and not on the number and choice of the finite prime ideals $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$.

With these conditions for $F(x,y)$, it is proved that:

The number of non-associated pairs of integers u and v of $\mathfrak{R}$, with $N((u,v)) \leq \leq |\sqrt{d(\mathfrak{R})}|$, such that $N(F(u,v))$ is divisible by no rational prime numbers other than the $\tau (\geq 0)$ given different rational prime numbers $r_1, r_2, \ldots, r_\tau$, is not greater than

[1] Two sets of $h+1$ integers $u_0, u_1, \ldots, u_h$ and $v_0, v_1, \ldots, v_h$ are said to be *associated* if $u_0/v_0 = u_1/v_1 = \cdots = u_h/v_h = \eta$ is a unit in $\mathfrak{R}$; otherwise they are *non-associated.*

$$K_0^{\tau+1},$$

where K_0 is a constant depending only on $\mathfrak{K}$ and $F(x,y)$, and not on the number and choice of $r_1, r_2, \ldots, r_\tau$.

From this result follow a number of corollaries on the representation of integers of $\mathfrak{K}$ in the form $F(u,v)$, culminating in the result:

Let U and V be any pair of integers of $\mathfrak{K}$ with $N((U,V)) \leq |\sqrt{d(\mathfrak{K})}|$, or a multiple of such a pair by an integer of $\mathfrak{K}$. Then the number of different representations of any integer I of $\mathfrak{K}$ in the form $F(U,V)$ is of order

$$|N(I)|^{\varepsilon},$$

where $N(I)$ is the norm in $\mathfrak{K}$ over $\mathfrak{P}$ of I and ε is an arbitrarily small positive constant.

Of particular interest is the result, involving even less stringent conditions for $F(x,y)$, analogous to Mahler's result on the greatest rational prime divisor of a binary form with rational integral coefficients and rational integral values for the variables. The present result is as follows:

If $F(x,y)$ be a binary form with integral coefficients from $\mathfrak{K}$, and such that $F(x,1)$ has at least three different roots, of which one may be infinite, and if u and v be any pair of integers of $\mathfrak{K}$ such that $N((u,v)) \leq |\sqrt{d(\mathfrak{K})}|$, then as

$$\max(|N(u)|, |N(v)|) \to \infty,$$

the greatest of the norms of the finite prime ideals of $\mathfrak{K}$ dividing $F(u,v)$ tends to infinity.

The proof of this result is greatly simplified by the fact that the polynomial $f(x)$ in Theorem 1, and consequently the binary form $F(x,y)$ in Theorem 2, need not be irreducible in $\mathfrak{K}$.

By using the more general form of Theorem 1, with $h \geq 1$, these results on binary forms may clearly be generalised to forms of the type:

$$\prod_{\nu=1}^{m}(x_0 + x_1\xi_\nu + x_2\xi_\nu^2 + \cdots + x_h\xi_\nu^h),$$

where $\xi_1, \xi_2, \ldots, \xi_m$ are the real or complex roots of $f(x)$ and $x_0, x_1, \ldots, x_m$ are integers of $\mathfrak{K}$. For such forms are merely products of h binary forms in which the variables take values from conjugate fields of degree h over $\mathfrak{K}$.

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§ 1. Preliminary: Definition and Properties of Finite and Infinite $\mathfrak{p}$-adic Valuations.

(a) Let $\mathfrak{K}$ be any finite algebraic field of degree $n \geq 1$ over the field $\mathfrak{P}$ of rational numbers.

By Ostrowski's Theorem, there exist only the non-equivalent valuations of the rational number field $\mathfrak{P}$ stated below. First there exists the absolute valuation $|x|$. We write this as

$$|x|_{p_\infty},$$

and refer to this valuation as '*the valuation with respect to the infinite prime number* p_∞'. The absolute valuation is thus made analogous to the second possible set of valuations, *the p-adic valuations*, p being a natural prime number. The p-adic valuations, of which there are an infinity, are defined for each p as

$$|x|_p = \begin{cases} 0 \text{ for } x = 0; \\ p^\mu \text{ for } x \neq 0, \text{ where } \mu \text{ is a rational integer and} \\ \quad \text{the reduced fraction } p^\mu x \text{ contains the factor } p \\ \quad \text{in neither numerator nor denominator.} \end{cases}$$

The remaining possible valuation, *the trivial valuation*, is

$$|x|_0 = \begin{cases} 0 \text{ for } x = 0; \\ 1 \text{ for } x \neq 0. \end{cases}$$

Thus the product of the valuations of x with respect to all natural and infinite prime numbers p satisfies the relation

$$\prod_p |x|_p = |x|_0. \tag{1}$$

[1] K. MAHLER, 'Über die Annäherung algebraischer Zahlen durch periodische Algorithmen', Acta mathematica (1937), p. 111—114.

(b) There are in general several possible continuations of $|x|_{p_\infty}$ and $|x|_p$ into the field $\mathfrak{K}$.

We consider first the continuations of $|x|_{p_\infty}$. Of the n fields $\mathfrak{K}^{(1)}, \mathfrak{K}^{(2)}, \ldots, \mathfrak{K}^{(n)}$ of real or complex numbers conjugate to $\mathfrak{K}$, let the first r_1 be the real fields

$$\mathfrak{K}^{(1)}, \mathfrak{K}^{(2)}, \ldots, \mathfrak{K}^{(r_1)},$$

and the remaining $2r_2$ ($r_1 + 2r_2 = n$) be the r_2 conjugate imaginary pairs

$$\mathfrak{K}^{(h)}, \mathfrak{K}^{(h+r_2)} \qquad (h = r_1 + 1, r_1 + 2, \ldots, r_1 + r_2).$$

We make correspond to each of the real fields $\mathfrak{K}^{(h)}$ $(h = 1, 2, \ldots, r_1)$, and to each pair of conjugate imaginary fields $\mathfrak{K}^{(h)}, \mathfrak{K}^{(h+r_2)}$ $(h = r_1 + 1, r_1 + 2, \ldots, r_1 + r_2)$, *an infinite prime ideal* $\mathfrak{p}_\infty^{(h)}$. Further, we denote by $\omega^{(h)}$ the element of $\mathfrak{K}^{(h)}$ conjugate to an element ω of $\mathfrak{K}$. Then the $r_1 + r_2$ absolute values

$$|\omega|_{\mathfrak{p}_\infty^{(h)}} = |\omega^{(h)}| \qquad (h = 1, 2, \ldots, r_1 + r_2)$$

define all the possible non-equivalent continuations of $|x|_{p_\infty}$ into $\mathfrak{K}$, and for elements x of the rational number field $\mathfrak{P}$ are identical with $|x|_{p_\infty}$, i. e.,

$$|\omega|_{\mathfrak{p}_\infty^{(h)}} = |\omega|_{p_\infty} \qquad (h = 1, 2, \ldots, r_1 + r_2)$$

if ω lies in $\mathfrak{P}$. We now write

$$g(\mathfrak{p}_\infty^{(h)}) = \begin{cases} 1 \text{ for } h = 1, 2, \ldots, r_1; \\ 2 \text{ for } h = r_1 + 1, r_1 + 2, \ldots, r_1 + r_2. \end{cases}$$

Then

$$\sum_{h=1}^{r_1+r_2} g(\mathfrak{p}_\infty^{(h)}) = n,$$

and if $N(\omega)$ be the norm in $\mathfrak{K}$ over $\mathfrak{P}$ of ω,

$$|N(\omega)|_{p_\infty} = \prod_{h=1}^{r_1+r_2} |\omega|_{\mathfrak{p}_\infty^{(h)}}^{g(\mathfrak{p}_\infty^{(h)})}. \tag{2}$$

Incidentally, $g(\mathfrak{p}_\infty^{(h)})$ is the degree of the perfect $\mathfrak{p}_\infty^{(h)}$-adic extension of $\mathfrak{K}$ over the field of real numbers, i. e., over the perfect p_∞-adic extension of $\mathfrak{P}$.

Similarly, we can continue the valuation $|x|_p$ into $\mathfrak{K}$. Let

$$(p) = \prod_{i=1}^{\pi} \mathfrak{p}^{(i)\, e(\mathfrak{p}^{(i)})}$$

be the factorisation of the natural prime number p into prime ideals of $\mathfrak{K}$, so that $e(\mathfrak{p}^{(i)})$ is the *order* of the prime ideal $\mathfrak{p}^{(i)}$. We define *the* $\mathfrak{p}^{(i)}$*-adic valuation* for $i = 1, 2, \ldots, \pi$ by

$$|\omega|_{\mathfrak{p}^{(i)}} = \begin{cases} 0 \text{ for } \omega = 0; \\ p^{\frac{\mu(\mathfrak{p}^{(i)})}{e(\mathfrak{p}^{(i)})}} \text{ for } \omega \neq 0, \text{ where } \mu(\mathfrak{p}^{(i)}) \text{ is a rational integer and} \\ \quad \text{the fractional ideal } \mathfrak{p}^{(i)\mu(\mathfrak{p}^{(i)})}(\omega) \text{ contains the factor } \mathfrak{p}^{(i)} \\ \quad \text{in neither numerator nor denominator.} \end{cases}$$

The π valuations thus defined are the only possible non-equivalent continuations of $|x|_p$ into $\mathfrak{K}$, and for elements x of the rational field $\mathfrak{P}$ are identical with $|x|_p$, i. e.,

$$|\omega|_{\mathfrak{p}^{(i)}} = |\omega|_p \qquad (i = 1, 2, \ldots, \pi)$$

if ω lies in $\mathfrak{P}$. We now write

$$g(\mathfrak{p}^{(i)}) = e(\mathfrak{p}^{(i)}) f(\mathfrak{p}^{(i)}),$$

where $f(\mathfrak{p}^{(i)})$ is the *degree* of $\mathfrak{p}^{(i)}$, i. e., the natural number such that $N(\mathfrak{p}^{(i)}) = p^{f(\mathfrak{p}^{(i)})}$. Then

$$\sum_{i=1}^{\pi} g(\mathfrak{p}^{(i)}) = n.$$

Also, since (ω) contains $\mathfrak{p}^{(i)}$ to the power $\mu(\mathfrak{p}^{(i)})$, and since

$$(p) = \prod_{i=1}^{\pi} \mathfrak{p}^{(i)e(\mathfrak{p}^{(i)})},$$

it follows that

$$N(\mathfrak{p}^{(i)})^{\mu(\mathfrak{p}^{(i)})} = p^{f(\mathfrak{p}^{(i)})\mu(\mathfrak{p}^{(i)})} = |\omega|_{\mathfrak{p}^{(i)}}^{g(\mathfrak{p}^{(i)})}, \tag{3}$$

and so

$$|N(\omega)|_p = \prod_{i=1}^{\pi} |N(\mathfrak{p}^{(i)})|_p^{\mu(\mathfrak{p}^{(i)})} = \prod_{i=1}^{\pi} |\omega|_{\mathfrak{p}^{(i)}}^{g(\mathfrak{p}^{(i)})}. \tag{4}$$

It may be noted that $g(\mathfrak{p}^{(i)})$ is the degree of the perfect $\mathfrak{p}^{(i)}$-adic extension of $\mathfrak{K}$ over the field of p-adic numbers, i. e., over the perfect p-adic extension of $\mathfrak{P}$.

Finally, $\mathfrak{K}$ has *the trivial valuation*

$$|\omega|_0 = \begin{cases} 0 \text{ for } \omega = 0; \\ 1 \text{ for } \omega \neq 0. \end{cases}$$

This is the only possible continuation of $|x|_0$ into $\mathfrak{K}$. Clearly,

$$|N(\omega)|_0 = |\omega|_0. \tag{5}$$

By Ostrowski's Theorem, every other valuation of $\mathfrak{K}$ is equivalent to one of those already defined.

From (1), (2), (4) and (5) follows the fundamental relation for $\mathfrak{K}$ corresponding to (1), viz.,

$$\prod_{\mathfrak{p}} |\omega|_{\mathfrak{p}}^{g(\mathfrak{p})} = |\omega|_0, \tag{6}$$

the product being taken over all finite and infinite prime ideals of $\mathfrak{K}$.

In particular, if ω be an integer $\omega_0 (\neq 0)$ of $\mathfrak{K}$,

$$\prod_{\mathfrak{r}} |\omega_0|_{\mathfrak{r}}^{G(\mathfrak{r})} \geq \frac{1}{|N(\omega_0)|}, \tag{6 a}$$

where the product $\prod_{\mathfrak{r}}$ is taken over any number of different finite prime ideals of $\mathfrak{K}$, and $G(\mathfrak{r})$ is a natural number not greater than $g(\mathfrak{r})$.

We shall use the relation (6) and the inequality (6 a) in proving Theorem 2 and its corollaries.

(c) From the relations (2) and (3) may be obtained an inequality which is of fundamental importance in the proof of Theorem 1. Let the $r_1 + r_2$ infinite prime ideals $\mathfrak{p}_\infty^{(h)} (h = 1, 2, \ldots, r_1 + r_2)$ be represented, in any desired order, by $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_{r_1+r_2}$, and let $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$, where $\sigma \geq 0$, be σ different finite prime ideals of $\mathfrak{K}$. Then by (2) and (3),

$$\prod_{j=1}^{r_1+r_2} |\omega|_{\mathfrak{q}_j}^{g(\mathfrak{q}_j)} \prod_{k=1}^{\sigma} |\omega|_{\mathfrak{r}_k}^{\theta(\mathfrak{r}_k)} = |N(\omega)| \prod_{k=1}^{\sigma} N(\mathfrak{r}_k)^{\mu(\mathfrak{r}_k)\frac{\theta(\mathfrak{r}_k)}{g(\mathfrak{r}_k)}},$$

where $\theta(\mathfrak{r}_k)$ $(k = 1, 2, \ldots, \sigma)$ is a positive number not greater than $g(\mathfrak{r}_k)$. Now since ω is an element of $\mathfrak{K}$, there exists a polynomial $\Upsilon(x; \omega, \mathfrak{P}, n)$ of degree n which is a power of the primitive polynomial with rational integral coefficients and irreducible in $\mathfrak{P}$ having ω as a root. If W_0 be the coefficient of the highest power of the variable x and W_n be the constant term of the polynomial $\Upsilon(x; \omega, \mathfrak{P}, n)$, then $|W_0|$ and $|W_n|$ are not less than unity, provided $\omega \neq 0$, and

$$|N(\omega)| = \frac{|W_n|}{|W_0|}.$$

But $(\omega) = \frac{\mathfrak{a}}{\mathfrak{b}}$, where $\mathfrak{a}$ and $\mathfrak{b}$ are relatively prime ideals of $\mathfrak{K}$ and $N(\mathfrak{a}) = |W_n|$, $N(\mathfrak{b}) = |W_0|$. Thus

$$\prod_{k=1}^{\sigma} N(\mathfrak{r}_k)^{\mu(\mathfrak{r}_k)} = \frac{Y_0}{Y_n},$$

where Y_0 and Y_n are relatively prime natural numbers such that $Y_0 | W_0$ and $Y_n | W_n$. It follows that the product

$$\prod_{k=1}^{\sigma} N(\mathfrak{r}_k)^{\mu(\mathfrak{r}_k)\frac{\theta(\mathfrak{r}_k)}{g(\mathfrak{r}_k)}}$$

may be written as the quotient $\frac{Z_0}{Z_n}$ of two positive numbers Z_0 and Z_n such that $1 \leq Z_0 \leq Y_0$ and $1 \leq Z_n \leq Y_n$, and so

$$(7) \qquad \prod_{j=1}^{r_1+r_2} |\omega|_{\mathfrak{q}_j}^{g(\mathfrak{q}_j)} \prod_{k=1}^{\sigma} |\omega|_{\mathfrak{r}_k}^{\theta(\mathfrak{r}_k)} = \frac{|W_n|}{|W_0|} \cdot \frac{Z_0}{Z_n} \geq \frac{1}{|W_0|} \cdot \frac{|W_n|}{Z_n} \geq \frac{1}{|W_0|}.$$

Suppose now that ω' is a number of degree n'' over $\mathfrak{K}$, where n'' is a divisor of a natural number n'. Then ω' will be an element of a field $\mathfrak{K}'$ of degree n' over $\mathfrak{K}$ and nn' over $\mathfrak{P}$. (ω' has the same meaning as ω when $n'=1$.) Let the nn' fields conjugate to $\mathfrak{K}'$ consist of r_1' real and r_2' pairs of conjugate imaginary fields, and let $\mathfrak{q}_1', \mathfrak{q}_2', \ldots, \mathfrak{q}_{r_1'+r_2'}'$ be the corresponding infinite prime ideals in any desired order. Let $\mathfrak{r}_1', \mathfrak{r}_2', \ldots, \mathfrak{r}_{\sigma'}'$ be any $\sigma'(\geq 0)$ different finite prime ideals of $\mathfrak{K}'$. Since ω' is an element of a field $\mathfrak{K}'$ of degree nn' over $\mathfrak{P}$, there exists a polynomial $\Upsilon(x; \omega', \mathfrak{P}, nn')$ of degree nn' which is a power of the primitive polynomial with rational integral coefficients and irreducible in $\mathfrak{P}$ having ω' as a root. Let W_0' be the coefficient of the highest power of the variable x in $\Upsilon(x; \omega', \mathfrak{P}, nn')$. Then, since the inequality (7) is true for the elements of any given field,

$$(8) \qquad \prod_{j=1}^{r_1'+r_2'} |\omega'|_{\mathfrak{q}_j'}^{g(\mathfrak{q}_j')} \prod_{k=1}^{\sigma'} |\omega'|_{\mathfrak{r}_k'}^{\theta(\mathfrak{r}_k')} \geq \frac{1}{|W_0'|}.$$

Here the $\mathfrak{p}'$-adic valuation, and $e(\mathfrak{p}')$, $f(\mathfrak{p}')$, $g(\mathfrak{p}')$, $\theta(\mathfrak{p}')$ and $\mu(\mathfrak{p}')$ are defined for a finite or infinite prime ideal $\mathfrak{p}'$ of $\mathfrak{K}'$, in relation to $\mathfrak{P}$, in the same way as are $\mathfrak{p}$-adic valuation, and $e(\mathfrak{p})$, $f(\mathfrak{p})$, $g(\mathfrak{p})$, $\theta(\mathfrak{p})$ and $\mu(\mathfrak{p})$ for a finite or infinite prime ideal $\mathfrak{p}$ of $\mathfrak{K}$.

Let now $\mathfrak{r}$ be any finite prime ideal of $\mathfrak{K}$. Then

$$\mathfrak{r} = \prod_{i=1}^{\pi'} \mathfrak{r}'^{(i)E(\mathfrak{r}'^{(i)})}$$

where $\mathfrak{r}'^{(1)}, \mathfrak{r}'^{(2)}, \ldots, \mathfrak{r}'^{(\pi')}$ are different finite prime ideals of $\mathfrak{K}'$, and the E's are natural numbers. Thus, if $\mathfrak{r}$ belongs to the rational prime number p, and ω' lies in the perfect $\mathfrak{r}$-adic extension of $\mathfrak{K}$ and has the $\mathfrak{r}$-adic valuation $p^{\frac{\mu(\mathfrak{r})}{e(\mathfrak{r})}}$, where $e(\mathfrak{r})$ is the order of $\mathfrak{r}$,

$$\prod_{i=1}^{\pi'} |\omega'|_{\mathfrak{r}'^{(i)}}^{g(\mathfrak{r}'^{(i)})} = p^{\sum_{i=1}^{\pi'} f(\mathfrak{r}'^{(i)})\,\mu(\mathfrak{r}'^{(i)})}$$

$$= p^{\sum_{i=1}^{\pi'} f(\mathfrak{r}'^{(i)})\,E(\mathfrak{r}'^{(i)})\,\mu(\mathfrak{r})}$$

$$= p^{\mu(\mathfrak{r})\,n'f(\mathfrak{r})} = |\omega'|_{\mathfrak{r}}^{n'g(\mathfrak{r})}.$$

Thus, if n'_k be any natural number not greater than $n'g(\mathfrak{r}_k)$ for $k=1, 2, \ldots, \sigma$, it follows that, for the appropriate σ', the appropriate prime ideals $\mathfrak{r}'_k (k=1, 2, \ldots, \sigma')$ in $\mathfrak{R}'$ and the appropriate $\theta(\mathfrak{r}'_k)$, the inequality (8) is identical with the inequality

$$\prod_{j=1}^{r'_1+r'_2} |\omega'|_{\mathfrak{q}'_j}^{g(\mathfrak{q}'_j)} \prod_{k=1}^{\sigma} |\omega'|_{\mathfrak{r}'_k}^{n'_k} \geq \frac{1}{|W'_0|}. \tag{9}$$

This is *the fundamental inequality* underlying the proof of Theorem 1.

(d) In dealing with the valuation of ω', we shall make use of the following notation:

1) *The field* $\mathfrak{R}_{j\gamma}$.

Let $\mathfrak{q}_j$ be any infinite prime ideal among the r_1+r_2 infinite prime ideals $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_{r_1+r_2}$ corresponding to the r_1 real and r_2 pairs of conjugate imaginary fields conjugate to $\mathfrak{R}$. Then if $\mathfrak{q}_j$ corresponds to a real field conjugate to $\mathfrak{R}$, let this field be called $\mathfrak{R}_{j1}$ (i. e., $\mathfrak{R}_{j\gamma}$, where $\gamma = g(\mathfrak{q}_j)$). If $\mathfrak{q}_j$ corresponds to a pair of conjugate imaginary fields conjugate to $\mathfrak{R}$, let these be called $\mathfrak{R}_{j1}$ and $\mathfrak{R}_{j2}$ (i. e., $\mathfrak{R}_{j\gamma}$, where $1 \leq \gamma \leq g(\mathfrak{q}_j)$).

2) *The* Υ *polynomials.*

By the polynomial $\Upsilon(x; \omega', \mathfrak{P}, nn')$ is meant the polynomial in x of degree nn' which is a power of the primitive polynomial with rational integral coefficients and irreducible in the rational number field $\mathfrak{P}$ having ω' as a root, ω' being as before any number of degree n'' over $\mathfrak{R}$, where n'' is a divisor of a natural number n'.

By the polynomial $\Upsilon(x; \omega', \mathfrak{R}, n')$ is meant the polynomial in x of degree n' which is a power of the primitive polynomial with integral coefficients from $\mathfrak{R}$ and irreducible in $\mathfrak{R}$ having ω' as a root.

By the polynomial $\Upsilon(x; \omega', \mathfrak{R}_{j\gamma}, n')$ is meant, for $j=1, 2, \ldots, r_1+r_2$ and $\gamma=1, g(\mathfrak{q}_j)$, the polynomial conjugate to $\Upsilon(x; \omega', \mathfrak{R}, n')$ with respect to the field $\mathfrak{R}_{j\gamma}$.

3) *The valuation* $|\omega'|_{\mathfrak{q}_j\gamma}$.

This means the absolute value of any root ω' of the polynomial $\Upsilon(x; \omega', \mathfrak{R}_{j\gamma}, n')$ $(j = 1, 2, \ldots, r_1+r_2; \gamma = 1, g(\mathfrak{q}_j))$, and reduces to the valuation $|\omega|_{\mathfrak{q}_j}$ $(j = 1, 2, \ldots, r_1+r_2)$ when ω' is an element ω of $\mathfrak{R}$.

It may also be noted that if $\mathfrak{r}$ is a finite prime ideal of $\mathfrak{R}$, $|\omega'|_{\mathfrak{r}}$ denotes the $\mathfrak{r}$-adic valuation of any root ω' of $\Upsilon(x; \omega', \mathfrak{R}, n')$ lying in the perfect $\mathfrak{r}$-adic extension of $\mathfrak{R}$. The inequality (9) holds if different roots of $\Upsilon(x; \omega', \mathfrak{R}, n')$ are evaluated with respect to the different $\mathfrak{r}_k$ $(k = 1, 2, \ldots, \sigma)$, since the $\mathfrak{r}_k$-adic roots of such a polynomial all have the same $\mathfrak{r}_k$-adic value.

In the remainder of this work, by the number ω' will be meant any root ω' of the polynomial $\Upsilon(x; \omega', \mathfrak{R}, n')$, and by 'different' ω' will be meant roots of different polynomials $\Upsilon(x; \omega', \mathfrak{R}, n')$. By 'a property satisfied by ω'' will be meant a property satisfied by some root or roots of $\Upsilon(x; \omega', \mathfrak{R}, n')$, e. g., to say that ω' lies in the perfect $\mathfrak{r}_1$-adic, $\mathfrak{r}_2$-adic, ..., $\mathfrak{r}_\sigma$-adic extensions of $\mathfrak{R}$ means that in each of these perfect extensions there lies some (not necessarily the same) root of $\Upsilon(x; \omega', \mathfrak{R}, n')$.

Notation.

(a) $\overline{|P(x, \ldots)|} = \overline{|P|}$ denotes the maximum of the absolute values of the coefficients of the polynomial P in any number of variables; in particular, this notation can also be used to represent the absolute value of a constant.

(b) If $P(x, \ldots)$ and $Q(x, \ldots)$ are two polynomials in the same variables such that the coefficients of Q are non-negative and not less than the absolute values of the corresponding coefficients of P, $Q(x, \ldots)$ is called a *majoriser* of $P(x, \ldots)$, and we write

$$P(x, \ldots) \ll Q(x, \ldots).$$

(1) $\mathfrak{R}$ is a finite algebraic field of degree $n\,(\geq 1)$ over the rational number field $\mathfrak{P}$. ζ is an algebraic integer generating $\mathfrak{R}$, and

$$\varphi(z) = z^n + \varkappa_1 z^{n-1} + \varkappa_2 z^{n-2} + \cdots + \varkappa_n$$

is the polynomial with rational integral coefficients and irreducible in $\mathfrak{P}$ having ζ as a root. $\varkappa$ is the natural number $\overline{|\varphi(z)|}$.

(2) $f(x,z)$ is a polynomial in x of degree $m\,(\geq 2)$:

$$f(x,z) = a_0(z)\,x^m + a_1(z)\,x^{m-1} + \cdots + a_m(z),$$

where $a_0(z)$ $(\not\equiv 0)$, $a_1(z)$, . . ., $a_m(z)$ are polynomials in z with rational integral coefficients and of degree not greater than $n-1$; a is the smallest natural number such that

$$a_\nu(z) \leqslant a\,(1+z)^{n-1}$$

for $\nu = 0, 1, 2, \ldots, m$; we suppose that the discriminant of $f(x,z)$ with respect to x is not divisible by $\varphi(z)$. (N. B. $f(x,\zeta)$ need not necessarily be irreducible in $\mathfrak{K}$.)

(3) $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_\varrho$, where $0 \leq \varrho \leq r_1 + r_2$, are ϱ of the r_1+r_2 infinite prime ideals corresponding to the r_1 real and r_2 pairs of conjugate imaginary fields conjugate to $\mathfrak{K}$.

$\mathfrak{q}_{\varrho+1}, \mathfrak{q}_{\varrho+2}, \ldots, \mathfrak{q}_{r_1+r_2}$ are the remaining infinite prime ideals.

(4) $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$ where $\sigma \gtrless 0$, are σ different finite prime ideals of $\mathfrak{K}$.

(5) $g(\mathfrak{p})$ where $\mathfrak{p}$ is a finite or infinite prime ideal of $\mathfrak{K}$, is the degree of the perfect $\mathfrak{p}$-adic extension of $\mathfrak{K}$ over the perfect p-adic extension of the rational number field $\mathfrak{P}$, p being the natural prime number divisible by $\mathfrak{p}$ if $\mathfrak{p}$ is finite, or the infinite prime number p_∞ if $\mathfrak{p}$ is infinite.

$G(\mathfrak{p})$ is a natural number not greater than $g(\mathfrak{p})$.

(6) h is a natural number.

$h_{k\delta}$ $(k = 1, 2, \ldots, \sigma;\ \delta = 1, 2, \ldots, G(\mathfrak{r}_k))$ is a natural number not greater than h^2.

(7) $\xi_{j\gamma}$ $(j = 1, 2, \ldots, \varrho;\ \gamma = 1,\ G(\mathfrak{q}_j))$ is a real or complex root of $f(x,\zeta)$.

$\eta_{k\delta\tau}$ $(k = 1, 2, \ldots, \sigma;\ \delta = 1, 2, \ldots, G(\mathfrak{r}_k);\ \tau = 1, 2, \ldots, h_{k\delta})$ is an $\mathfrak{r}_k$-adic root of $f(x,\zeta)$, i. e., a root of $f(x,\zeta)$ lying in the perfect $\mathfrak{r}_k$-adic extension of $\mathfrak{K}$.

(8) t is the total number of the above roots, i. e.,

$$\sum_{j=1}^{\varrho} G(\mathfrak{q}_j) + \sum_{k=1}^{\sigma} \sum_{\delta=1}^{G(\mathfrak{r}_k)} h_{k\delta}.$$

(9) λ is an algebraic number of degree h (or some divisor h' of h) over $\mathfrak{K}$ (i. e., in our notation, $\omega' = \lambda$, $n'' = h'$, $n' = h$), and lying in the perfect $\mathfrak{r}_1$-adic, $\mathfrak{r}_2$-adic, . . ., $\mathfrak{r}_\sigma$-adic extensions of $\mathfrak{K}$. Λ is the number $|\Upsilon(x; \lambda, \mathfrak{P}, hn)|$.

§ 2. Lemmas on Polynomials.

1. Let

$$\psi(x) = b_0 x^m + b_1 x^{m-1} + \cdots + b_m, \qquad \Psi(x) = B_0 x^M + B_1 x^{M-1} + \cdots + B_M$$

be two polynomials in x with arbitrary coefficients, and let

$$d = \max(0, M - m + 1).$$

Dividing $b_0^d \Psi(x)$ by $\psi(x)$, we have

$$b_0^d \Psi(x) = \Psi^*(x)\psi(x) + \Psi^{**}(x),$$

where $\Psi^*(x)$ and $\Psi^{**}(x)$ are uniquely determined polynomials in x, $b_0, b_1, \ldots, b_m, B_0, B_1, \ldots, B_M$ with rational integral coefficients and $\Psi^{**}(x)$, which is of degree not greater than $m-1$, in x, takes the form:

$$\Psi^{**}(x) = \mathfrak{B}_0 x^{m-1} + \mathfrak{B}_1 x^{m-2} + \cdots + \mathfrak{B}_{m-1}.$$

We now show that each of the coefficients $\mathfrak{B}_0, \mathfrak{B}_1, \ldots, \mathfrak{B}_{m-1}$ is of the form:

$$\mathfrak{B}_\nu = \sum_{l=1}^{2^d} \varepsilon_{\nu l}\, b_{\nu\theta_{1l}} b_{\nu\theta_{2l}} \ldots b_{\nu\theta_{dl}} B_{\nu\mu_l} \qquad (\nu = 0, 1, \ldots, m-1), \tag{10}$$

where the $\nu\theta$ and $\nu\mu$ are certain of the suffixes $0, 1, \ldots, m$ and $0, 1, \ldots, M$ respectively, and each $\varepsilon_{\nu l}$ takes one of the three values $0, \pm 1$; in particular, for $d = 0$, the factors $b_{\nu\theta}$ are absent.

The result is obvious for $d = 0$, for then $\Psi^*(x) = 0$, $\Psi^{**}(x) = \Psi(x)$. We suppose it to be true for $d = 0, 1, \ldots, k-1$, where $k \geq 1$, and hence prove that it is true for $d = k$. By induction, the result will then hold for all non-negative d.

For convenience, we take $b_{m+1} = b_{m+2} = b_{m+3} = \cdots = 0$. Then

$$b_0^d \Psi(x) = b_0^{d-1} B_0 x^{M-m} \psi(x) + b_0^{d-1} \Psi_1(x),$$

where

$$\Psi_1(x) = c_0 x^{M-1} + c_1 x^{M-2} + \cdots + c_{M-1},$$

and

$$c_\nu = B_{\nu+1} b_0 - B_0 b_{\nu+1} \qquad (\nu = 0, 1, \ldots, M-1).$$

Now, dividing $b_0^{d-1} \Psi_1(x)$ by $\psi(x)$, we have as before the unique relation:

$$b_0^{d-1} \Psi_1(x) = \Psi_1^*(x)\psi(x) + \Psi^{**}(x),$$

and since $\Psi_1(x)$ is only of degree $M-1$, and $0 \leq d-1 \leq k-1 < k$, by our assumption the coefficients $\mathfrak{B}_\nu$ of $\Psi^{**}(x)$ are of the form:

$$\mathfrak{B}_\nu = \sum_{l=1}^{2^{d-1}} \varepsilon'_{\nu l}\, b_\nu \theta'_{1l}\, b_\nu \theta'_{2l} \ldots b_\nu \theta'_{d-1,l}\, c_{\nu\mu'_l} \quad (\nu = 0, 1, \ldots, m-1),$$

which is equivalent to the form (10).

2. The foregoing is true if we take $b_0, b_1, \ldots, b_m$ and $B_0, B_1, \ldots, B_M$ as polynomials in z with rational integral coefficients. They will satisfy the inequalities

$$b_j(z) \ll b(1+z)^{b'} \qquad (j = 0, 1, \ldots, m),$$
$$B_k(z) \ll B(1+z)^{B'} \qquad (k = 0, 1, \ldots, M),$$

where b, b', B, B' are certain non-negative rational integers. The coefficients of $\Psi^*(x)$ and $\Psi^{**}(x)$ will then be polynomials in z with rational integral coefficients, and by (10) the coefficients of $\Psi^{**}(x)$ will satisfy the inequalities

$$\mathfrak{B}_\nu(z) \ll 2^d B b^d (1+z)^{B'+db'} \qquad (\nu = 0, 1, \ldots, m-1). \tag{11}$$

Hence:

Lemma 1. *Let*

$$\psi(x,z) = b_0(z)x^m + b_1(z)x^{m-1} + \cdots + b_m(z), \quad \Psi(x,z) = B_0(z)x^M + B_1(z)x^{M-1} + \cdots + B_M(z)$$

be two polynomials in x, of which the coefficients $b_0(z), b_1(z), \ldots, b_m(z)$ and $B_0(z), B_1(z), \ldots, B_M(z)$ are polynomials in z with rational integral coefficients and satisfying the inequalities

$$b_j(z) \ll b(1+z)^{b'} \qquad (j = 0, 1, \ldots, m),$$
$$B_k(z) \ll B(1+z)^{B'} \qquad (k = 0, 1, \ldots, M),$$

where b, b', B, B' are non-negative rational integers. Let

$$d = \max(0, M-m+1).$$

Then two polynomials in x and z, $\Psi^*(x,z)$ and $\Psi^{**}(x,z)$, are uniquely determined by the relation

$$b_0(z)^d \Psi(x,z) = \Psi^*(x,z)\psi(x,z) + \Psi^{**}(x,z),$$

where

$$\Psi^{**}(x,z) = \mathfrak{B}_0(z)x^{m-1} + \mathfrak{B}_1(z)x^{m-2} + \cdots + \mathfrak{B}_{m-1}(z)$$

is of degree not greater than $m-1$ in x, and has coefficients which are polynomials in z with rational integral coefficients and which satisfy the inequalities

$$\mathfrak{B}_\nu(z) \ll 2^d B b^d (1+z)^{B'+db'} \qquad (\nu = 0, 1, \ldots, m-1). \tag{11}$$

From this lemma and its proof arise the following corollaries:

Corollary 1. *Lemma 1 remains true if d is any number not less than* $\max(0, M - m + 1)$.

Corollary 2. *If $\Psi(x,z) = \Psi_1(x,z) - \Psi_2(x,z)$, and if, for the same d, $\Psi^*(x,z)$, $\Psi_1^*(x,z)$, $\Psi_2^*(x,z)$ and $\Psi^{**}(x,z)$, $\Psi_1^{**}(x,z)$, $\Psi_2^{**}(x,z)$ are the polynomials corresponding to $\Psi(x,z)$, $\Psi_1(x,z)$, $\Psi_2(x,z)$, as in Lemma 1, then*

$$\Psi^{**}(x,z) = \Psi_1^{**}(x,z) - \psi_2^{**}(x,z).$$

Corollary 3. *If the coefficients $b_0(z), b_1(z), \ldots, b_m(z)$ and $B_0(z), B_1(z), \ldots, B_M(z)$ are independent of z, then the same is true of the coefficients $\mathfrak{B}_0(z), \mathfrak{B}_1(z), \ldots, \mathfrak{B}_{m-1}(z)$, and these will be rational integers satisfying the inequality*

$$\max_{\nu=0,1,\ldots,m-1} \overline{|\mathfrak{B}_\nu|} \leq \max_{k=0,1,\ldots,M} \overline{|B_k|} \left(2 \max_{j=0,1,\ldots,m} \overline{|b_j|}\right)^d.$$

3. In addition to $\psi(x,z)$ and $\Psi(x,z)$ we define a polynomial in z,

$$\varphi(z) = z^n + \varkappa_1 z^{n-1} + \varkappa_2 z^{n-2} + \cdots + \varkappa_n$$

with rational integral coefficients and of degree $n (\geq 1)$. We write

$$\overline{|\varphi(z)|} = \varkappa.$$

If, as in Lemma 1, $\Psi(x,z)$ is written as

$$\Psi(x,z) = \Psi^*(x,z)\,\psi(x,z) + \Psi^{**}(x,z),$$

where

$$\Psi^{**}(x,z) = \mathfrak{B}_0(z)\,x^{m-1} + \mathfrak{B}_1(z)\,x^{m-2} + \cdots + \mathfrak{B}_{m-1}(z),$$

then the $\mathfrak{B}_\nu$ satisfy the inequalities (11) and hence the inequalities

$$\mathfrak{B}_\nu(z) \ll 2^{B'+(b'+1)d}\, B b^d (1 + z + \cdots + z^{B'+b'd}) \quad (\nu = 0, 1, \ldots, m-1),$$

since

$$(1+z)^{B'+db'} = 1 + \binom{B'+db'}{1} z + \binom{B'+db'}{2} z^2 + \cdots + z^{B'+db'},$$

and

$$\binom{B'+db'}{r} \leq \sum_{\tau=0}^{B'+db'} \binom{B'+db'}{\tau} = (1+1)^{B'+db'} = 2^{B'+db'}$$

$$(r = 0, 1, \ldots, B'+db').$$

If $\delta = \max(0, B' + b'd - n + 1)$, by Lemma 1 and since $\varkappa_0^\delta = 1^\delta = 1$, $\mathfrak{B}_\nu(z)$ may be written uniquely in the form:

$$\mathfrak{B}_\nu(z) = \mathfrak{B}_\nu^*(z)\,\varphi(z) + \mathfrak{B}_\nu^{**}(z) \qquad (\nu = 0, 1, \ldots, m-1),$$

$\mathfrak{B}_\nu^*(z)$ and $\mathfrak{B}_\nu^{**}(z)$ being polynomials in z (the latter of degree not greater than $n-1$), with rational integral coefficients. Further, by Corollary 3,

$$\mathfrak{B}_\nu^{**}(z) = d_{\nu 0}\, z^{n-1} + d_{\nu 1}\, z^{n-2} + \cdots + d_{\nu, n-1} \quad (\nu = 0, 1, \ldots, m-1),$$

where the $d_{\nu\mu}$ are rational integers satisfying the inequalities

$$\overline{|d_{\nu\mu}|} \leq 2^{B'+(b'+1)d}\, B\, b^d (2\varkappa)^\delta \qquad (\nu = 0, 1, \ldots, m-1;\ \mu = 0, 1, \ldots, n-1).$$

If we write

$$\Psi^{(1)}(x,z) = \Psi^*(x,z),$$

$$\Psi^{(2)}(x,z) = \mathfrak{B}_0^*(z)\,x^{m-1} + \mathfrak{B}_1^*(z)\,x^{m-2} + \cdots + \mathfrak{B}_{m-1}^*(z),$$

$$\Psi^{(3)}(x,z) = \mathfrak{B}_0^{**}(z)\,x^{m-1} + \mathfrak{B}_1^{**}(z)\,x^{m-2} + \cdots + \mathfrak{B}_{m-1}^{**}(z),$$

then

$$b_0(z)^d\,\Psi(x,z) = \Psi^{(1)}(x,z)\,\psi(x,z) + \Psi^{(2)}(x,z)\,\varphi(z) + \Psi^{(3)}(x,z),$$

and we arrive at the following result:

Lemma 2. *Let*

$$\psi(x,z) = b_0(z)\,x^m + b_1(z)\,x^{m-1} + \cdots + b_m(z), \quad \Psi(x,z) = B_0(z)\,x^M + B_1(z)\,x^{M-1} + \cdots + B_M(z)$$

be two polynomials in x of which the coefficients $b_0(z)$, $b_1(z)$, . . ., $b_m(z)$ and $B_0(z)$, $B_1(z)$, . . ., $B_M(z)$ are polynomials in z satisfying the inequalities

$$b_j(z) \ll b\,(1+z)^{b'} \qquad (j = 0, 1, \ldots, m),$$

$$B_k(z) \ll B(1+z)^{B'} \qquad (k = 0, 1, \ldots, M),$$

where b, b', B, B' are non-negative rational integers. Let

$$\varphi(z) = z^n + \varkappa_1 z^{n-1} + \varkappa_2 z^{n-2} + \cdots + \varkappa_n$$

be a polynomial in z alone with rational integral coefficients, and let

$$d = \max(0, M - m + 1), \quad \delta = \max(0, B' + d\,b' - n + 1), \quad \varkappa = \overline{|\varphi(z)|}.$$

Then there exist three uniquely determined polynomials $\Psi^{(1)}(x,z)$, $\Psi^{(2)}(x,z)$, $\Psi^{(3)}(x,z)$ in x and z with rational integral coefficients such that

$$b_0(z)^d\,\Psi(x,z) = \Psi^{(1)}(x,z)\,\psi(x,z) + \Psi^{(2)}(x,z)\,\varphi(z) + \Psi^{(3)}(x,z),$$

where $\Psi^{(2)}(x,z)$ *is of degree not greater than* $m-1$ *in* x, *and* $\Psi^{(3)}(x,z)$ *is of degree not greater than* $m-1$ *in* x *and* $n-1$ *in* z. *Further,*

$$\overline{|\Psi^{(3)}(x,z)|} \leq 2^{B'+(b'+1)d} B b^d (2\varkappa)^\delta.$$

From this lemma and its proof the following corollaries arise:

Corollary 1. *Lemma 2 remains true if* d *and* δ *take larger values than those assigned.*

Corollary 2. *If* $\Psi(x,z) = \Psi_1(x,z) - \Psi_2(x,z)$, *and if, for the same* d *and* δ, $\Psi^{(1)}(x,z)$, $\Psi_1^{(1)}(x,z)$, $\Psi_2^{(1)}(x,z)$, $\Psi^{(2)}(x,z)$, $\Psi_1^{(2)}(x,z)$, $\Psi_2^{(2)}(x,z)$, $\Psi^{(3)}(x,z)$, $\Psi_1^{(3)}(x,z)$, $\Psi_2^{(3)}(x,z)$ *are the polynomials corresponding to* $\Psi(x,z)$, $\Psi_1(x,z)$, $\Psi_2(x,z)$, *as in Lemma 2, then*

$$\Psi^{(3)}(x,z) = \Psi_1^{(3)}(x,z) - \Psi_2^{(3)}(x,z).$$

Corollary 3. *If the coefficients of* $\psi(x,z)$ *and* $\Psi(x,z)$ *satisfy the inequalities*

$$b_j(z) \ll b(1+z)^{n-1} \qquad (j = 0, 1, \ldots, m),$$

$$B_k(z) \ll B(1+z)^{n-1} \qquad (k = 0, 1, \ldots, M),$$

then

$$\overline{|\Psi^{(3)}(x,z)|} \leq 2^{(d+1)n-1} B b^d (2\varkappa)^\delta.$$

§ 3. Construction of the *R*-Polynomial.

4. Let $\mathfrak{K}$ be a finite algebraic field of degree $n\,(\geq 1)$ over the rational number field $\mathfrak{P}$, ζ be an algebraic integer generating $\mathfrak{K}$, and

$$\varphi(z) = z^n + \varkappa_1 z^{n-1} + \varkappa_2 z^{n-2} + \cdots + \varkappa_n$$

be the irreducible polynomial with rational integral coefficients having the root ζ; as in **3** we write

$$\varkappa = \overline{|\varphi(z)|}.$$

Further, let

$$f(x,z) = a_0(z)x^m + a_1(z)x^{m-1} + \cdots + a_m(z) \qquad (a_0(z) \not\equiv 0)$$

be a polynomial in x and z with rational integral coefficients and of degree $m\,(\geq 2)$ in x and of degree not greater than $n-1$ in z; a is taken to be the smallest natural number such that

$$a_\nu(z) \ll a(1+z)^{n-1}$$

for $\nu = 0, 1, \ldots, m$. Thus the polynomial $f(x,\zeta)$ in x has integral coefficients from $\mathfrak{K}$ and is of degree exactly m in x. We impose the further condition that

its discriminant does not vanish, i. e., that *the discriminant of* $f(x, z)$ *with respect to* x *is not divisible by* $\varphi(z)$. $f(x, \zeta)$ *may be reducible in* $\mathfrak{K}$. We are concerned with the question of how closely we can approximate to a root of $f(x, \zeta)$ by numbers of a fixed degree over $\mathfrak{K}$, when we consider a finite number of valuations of $\mathfrak{K}$.

5. Let ε be a positive number, s a natural number less than m, r a natural number to be determined later, and q the rational integer determined uniquely by the inequalities

$$q \leq \left(\frac{m+\varepsilon}{s+1} - 1\right) r < q + 1.$$

To every natural number R correspond exactly

$$\mathfrak{N}_1 = (2R+1)^{n(q+r+1)(s+1)}$$

polynomials

$$R(x_1, x_2, z) = \sum_{l_1=0}^{q+r} \sum_{l_2=0}^{s} \sum_{l_3=0}^{n-1} R_{l_1 l_2 l_3} x_1^{l_1} x_2^{l_2} z^{l_3},$$

of degree $q+r$ in x_1, s in x_2, and $n-1$ in z, and with rational integral coefficients such that

$$\overline{|R(x_1, x_2, z)|} \leq R.$$

We write

$$R_{i_1 i_2}(x_1, x_2, z) = \frac{\partial^{i_1+i_2} R(x_1, x_2, z)}{i_1!\, i_2!\, \partial x_1^{i_1} \partial x_2^{i_2}}$$

$$= \sum_{l_1=0}^{q+r} \sum_{l_2=0}^{s} \sum_{l_3=0}^{n-1} R_{l_1 l_2 l_3} \binom{l_1}{i_1} \binom{l_2}{i_2} x_1^{l_1-i_1} x_2^{l_2-i_2} z^{l_3}.$$

Putting $x_1 = x_2 = x$, it follows that the polynomials $R_{i0}(x, x, z)$, for $i = 0, 1, \ldots, r-1$, are of degree not greater than $q+r+s$ in x and $n-1$ in z. Arranged in powers of x, these polynomials become

$$R_{i0}(x, x, z) = \sum_{j=0}^{q+r+s} B_{ji}(z)\, x^{q+r+s-j} \qquad (i = 0, 1, \ldots, r-1),$$

where

$$B_{ji}(z) \ll R \sum_{l_1=0}^{q+r} \binom{l_1}{i} \sum_{l_3=0}^{n-1} z^{l_3}$$

$$\ll R \sum_{l_1=0}^{q+r} 2^{l_1} \sum_{l_3=0}^{n-1} z^{l_3}$$

$$\ll 2^{q+r+1} R (1+z)^{n-1}$$

$$(j = 0, 1, \ldots, q+r+s; \quad i = 0, 1, \ldots, r-1).$$

For $i=0, 1, \ldots, r-1$, we apply Lemma 2 to the polynomials $\Psi(x,z)=R_{i0}(x,x,z)$, $\psi(x,z)=f(x,z)$, and $\varphi(z)$. Since $s \leq m-1$ and $M \leq q+r+s$,

$$\max(0, M-m+1) \leq \max(0, q+r+s-m+1) \leq q+r,$$

so that we may take $d=q+r+1$. We also take $b_\nu(z)=a_\nu(z)$ $(\nu=0, 1, \ldots, m)$, $b=a$, $B=2^{q+r+1}R$, $b'=B'=n-1$, so that

$$\max(0, B'+b'd-n+1)=(n-1)+(n-1)(q+r+1)-(n-1)=(n-1)(q+r+1).$$

Hence we may put

$$\delta=(q+r+1)(n-1).$$

Thus, by a representation corresponding to that in Lemma 2, R_{i0} can be represented uniquely thus:

$$b_0(z)^d R_{i0}(x,x,z)=R_i^{(1)}(x,z)f(x,z)+R_i^{(2)}(x,z)\varphi(z)+R_i^{(3)}(x,z) \qquad (i=0, 1, \ldots, r-1),$$

where, in particular,

$$R_i^{(3)}(x,z)=\sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n-1} P_{i\mu\nu}\, x^\mu z^\nu$$

is of degree not greater than $m-1$ in x and $n-1$ in z and has rational integral coefficients $P_{i\mu\nu}$ which satisfy the inequality

$$\overline{|R_i^{(3)}(x,z)|} \leq 2^{n-1+n(q+r+1)}\, 2^{q+r+1}\, R\, a^{q+r+1}\, (2\varkappa)^{(n-1)(q+r+1)},$$

or

$$\begin{aligned}\overline{|R_i^{(3)}(x,z)|} &\leq 2^{n-1}(2^{2n}a\varkappa^{n-1})^{q+r+1}R \\ &\leq (2^{3n-1}a\varkappa^{n-1})^{q+r+1}R \qquad (i=0, 1, \ldots, r-1).\end{aligned}$$

For given R, there are for the system of polynomials $R_i^{(3)}(x,z)$ $(i=0, 1, \ldots, r-1)$, not more than

$$\mathfrak{N}_2=\{2(2^{3n-1}a\varkappa^{n-1})^{q+r+1}R+1\}^{mnr} \leq (2^{3n-1}a\varkappa^{n-1})^{(q+r+1)mnr}(2R+1)^{mnr}$$

different possibilities. Now

$$n(q+r+1)(s+1) > n\frac{m+\varepsilon}{s+1}r(s+1)=mnr+\varepsilon nr,$$

so that

$$\mathfrak{N}_1 > (2R+1)^{mnr+\varepsilon nr}.$$

Hence

$$\mathfrak{N}_1 > \mathfrak{N}_2,$$

when

$$2R+1 \geq (2^{3n-1}a\varkappa^{n-1})^{(q+r+1)\frac{m}{\varepsilon}} > 2R-1.$$

Thus, when these inequalities hold, there must exist at least two different polynomials of the type $R(x_1, x_2, z)$, say

$$R^*(x_1, x_2, z) = \sum_{l_1=0}^{q+r} \sum_{l_2=0}^{s} \sum_{l_3=0}^{n-1} R^*_{l_1 l_2 l_3} x_1^{l_1} x_2^{l_2} z^{l_3}, \qquad R^{**}(x_1, x_2, z) = \sum_{l_1=0}^{q+r} \sum_{l_2=0}^{s} \sum_{l_3=0}^{n-1} R^{**}_{l_1 l_2 l_3} x_1^{l_1} x_2^{l_2} z^{l_3},$$

with

$$\overline{|R^*|} \leq R, \quad \overline{|R^{**}|} \leq R,$$

and such that if $R^*_{i0}(x, x, z)$ and $R^{**}_{i0}(x, x, z)$ are represented, as in Lemma 2, in the forms:

$$a_0(z)^d R^*_{i0}(x, x, z) = R_i^{*(1)}(x, z) f(x, z) + R_i^{*(2)}(x, z) \varphi(z) + R_i^{*(3)}(x, z) \qquad (i = 0, 1, \ldots, r-1),$$

$$a_0(z)^d R^{**}_{i0}(x, x, z) = R_i^{**(1)}(x, z) f(x, z) + R_i^{**(2)}(x, z) \varphi(z) + R_i^{**(3)}(x, z),$$

the r identities

$$R_i^{*(3)}(x, z) \equiv R_i^{**(3)}(x, z) \qquad (i = 0, 1, \ldots, r-1)$$

are satisfied. We write

$$R^*(x_1, x_2, z) - R^{**}(x_1, x_2, z) = R(x_1, x_2, z) = \sum_{l_1=0}^{q+r} \sum_{l_2=0}^{s} \sum_{l_3=0}^{n-1} R_{l_1 l_2 l_3} x_1^{l_1} x_2^{l_2} z^{l_3},$$

so that $R(x_1, x_2, z)$ is a polynomial with rational integral coefficients which is not identically zero and for which

$$\overline{|R(x_1, x_2, z)|} \leq 2R < (2^{3n} a \varkappa^{n-1})^{(q+r+1)\frac{m}{\varepsilon}} + 1 < (2^{3n} a \varkappa^{n-1})^{(q+r+1)\frac{m}{\varepsilon}}.$$

Further, the polynomials

$$R_{i0}(x, x, z) = \left(\frac{\partial^i R(x_1, x_2, z)}{i! \, \partial x_1^i} \right)_{x_1 = x_2 = x} \qquad (i = 0, 1, \ldots, r-1)$$

can be written in the form:

$$(12) \qquad R_{i0}(x, x, z) = a_0(z)^{-d} \{ R_i^{(1)}(x, z) f(x, z) + R_i^{(2)}(x, z) \varphi(z) \} \quad (i = 0, 1, \ldots, r-1)$$

for certain polynomials $R_i^{(1)}(x, z)$ and $R_i^{(2)}(x, z)$ with rational integral coefficients.

By Taylor's Theorem, for any fixed x,

$$R(x_1, x_2, z) = \sum_{l_1=0}^{q+r} \sum_{l_2=0}^{s} R_{l_1 l_2}(x, x, z) (x_1 - x)^{l_1} (x_2 - x)^{l_2},$$

and therefore

$$R_{i0}(x_1, x_2, z) = \sum_{l_1=0}^{q+r} \sum_{l_2=0}^{s} R_{l_1 l_2}(x, x, z) \binom{l_1}{i} (x_1 - x)^{l_1 - i} (x_2 - x)^{l_2} \quad (i = 0, 1, \ldots, r-1),$$

or, by (12),

$$R_{i0}(x_1, x_2, z) = (x_1 - x)^{r-i} F_i(x_1, x_2, x, z) + (x_2 - x)\, G_i(x_1, x_2, x, z) + f(x, z)\, H_i^{(1)}(x_1, x_2, x, z) + \varphi(z)\, H_i^{(2)}(x_1, x_2, x, z),$$

where

$$F_i(x_1, x_2, x, z) = \sum_{l_1=r}^{q+r} \sum_{l_2=0}^{s} R_{l_1 l_2}(x, x, z) \binom{l_1}{i} (x_1 - x)^{l_1 - r} (x_2 - x)^{l_2},$$

$$G_i(x_1, x_2, x, z) = \sum_{l_1=0}^{r-1} \sum_{l_2=1}^{s} R_{l_1 l_2}(x, x, z) \binom{l_1}{i} (x_1 - x)^{l_1 - i} (x_2 - x)^{l_2 - 1},$$

$$H_i^{(1)}(x_1, x_2, x, z) = a_0(z)^{-d} \sum_{l_1=0}^{r-1} R_{l_1}^{(1)}(x, z) \binom{l_1}{i} (x_1 - x)^{l_1 - i},$$

$$H_i^{(2)}(x_1, x_2, x, z) = a_0(z)^{-d} \sum_{l_1=0}^{r-1} R_{l_1}^{(2)}(x, z) \binom{l_1}{i} (x_1 - x)^{l_1 - i} \qquad (i = 0, 1, \ldots, r-1).$$

The first two of these functions are polynomials in x_1, x_2, x and z with rational integral coefficients, while the last two are polynomials in x_1, x_2, and x with coefficients which are rational functions in z with rational coefficients and which are therefore numbers of $\mathfrak{R}$ for $z = \zeta$, since $a_0(\zeta) \neq 0$. Clearly, when $z = \zeta$, the identity for $R_{i0}(x_1, x_2, z)$ takes the form:

$$R_{i0}(x_1, x_2, \zeta) = (x_1 - x)^{r-i} F_i(x_1, x_2, x, \zeta) + (x_2 - x)\, G_i(x_1, x_2, x, \zeta) + f(x, \zeta)\, H_i^{(1)}(x_1, x_2, x, \zeta) \qquad (i = 0, 1, \ldots, r-1),$$

and if we put $x = x_2$,

$$R_{i0}(x_1, x_2, \zeta) = (x_1 - x_2)^{r-i} F_i(x_1, x_2, x_2, \zeta) + f(x_2, \zeta)\, H_i^{(1)}(x_1, x_2, x_2, \zeta). \tag{13}$$

6. Since $\mathfrak{R}$ is a field, every ideal in the ring of all polynomials in one variable with coefficients from $\mathfrak{R}$ is a principal ideal[1]. There is in particular a polynomial $e(x_2)$ with coefficients from $\mathfrak{R}$ and a first coefficient unity, and of degree θ, say, which divides both $R(x_1, x_2, \zeta)$ and $f(x_2, \zeta)$, while no similar polynomial of higher degree than θ does so. Clearly,

$$0 \leq \theta \leq s \leq m - 1.$$

We put

$$f(x_2, \zeta) = e(x_2)\, \gamma(x_2), \qquad R(x_1, x_2, \zeta) = e(x_2)\, S(x_1, x_2),$$

[1] B. L. VAN DER WAERDEN, 'Moderne Algebra', 2nd edition, Vol. I (1937), Julius Springer, Berlin, p. 59.

so that $\gamma(x_2)$ is of degree exactly $m-\theta$, while $S(x_1, x_2)$ is of degree not greater than $q+r$ in x_1 and $s-\theta$ in x_2, and has coefficients from $\mathfrak{K}$. By our choice of $e(x_2)$, $S(x_1, x_2)$ and $\gamma(x_2)$ have no non-constant common factor.

We can arrange $S(x_1, x_2)$ in powers of x_2, thus:

$$S(x_1, x_2) = \sum_{l=0}^{s-\theta} w_l(x_1)\, x_2^l,$$

where the polynomials

$$w_0(x_1),\ w_1(x_1),\ \ldots,\ w_{s-\theta}(x_1) \tag{14}$$

are not all identically zero and have coefficients from $\mathfrak{K}$. Let $\chi+1$ ($\chi \geq 0$), and no more, of these polynomials be linearly independent with respect to the field $\mathfrak{K}$, say the polynomials $w_{l_0}(x_1)$, $w_{l_1}(x_1)$, $\ldots$, $w_{l\chi}(x_1)$, where $l_0 < l_1 < \cdots < l_\chi$. Then the Wronski determinant

$$\Delta(x_1) = \left| \frac{d^i w_{lj}(x_1)}{i!\, d x_1^i} \right|_{i,j=0,1,\ldots,\chi}$$

cannot be identically zero[1]. Clearly, $\Delta(x_1)$ is a polynomial in x_1 of degree not greater than $(\chi+1)(q+r)$, where $\chi \leq s-\theta$.

If we express the polynomials (14) linearly in terms of the chosen $\chi+1$ linearly independent polynomials, with coefficients from $\mathfrak{K}$, then $S(x_1, x_2)$ takes the form:

$$S(x_1, x_2) = \sum_{j=0}^{\chi} w_{lj}(x_1)\, \Omega_j(x_2),$$

and it is clear that the $\chi+2$ polynomials in x_2:

$$\Omega_0(x_2),\ \Omega_1(x_2),\ \ldots,\ \Omega_\chi(x_2),\ \gamma(x_2)$$

are none of them zero and can have no common factor. Now

$$R(x_1, x_2, \zeta) = \sum_{j=0}^{\chi} w_{lj}(x_1)\, e(x_2)\, \Omega(x_2),$$

and

$$R_{i0}(x_1, x_2, \zeta) = \sum_{j=0}^{\chi} \frac{d^i w_{lj}(x_1)}{i!\, d x_1^i}\, e(x_2)\, \Omega(x_2) \qquad (i = 0, 1, \ldots, r-1).$$

Thus, on multiplying $R_{i0}(x_1, x_2, \zeta)$ by the cofactor $\Delta_{ij}(x_1)$ of $\dfrac{d^i w_{lj}(x_1)}{i!\, d x_1^i}$ in $\Delta(x_1)$, and adding the expressions obtained for $i = 0, 1, \ldots, \chi$, we find that

[1] See note 1 on p. 1, pp. 177—8.

$$\Delta(x_1)\, e(x_2)\, \Omega_j(x_2) = \sum_{i=0}^{\chi} \Delta_{ij}(x_1)\, R_{i0}(x_1, x_2, \zeta)$$

for $j = 0, 1, \ldots, \chi$. Hence and by (13),

$$\Delta(x_1)\, e(x_2)\, \Omega_j(x_2) = (x_1 - x_2)^{r-\chi} p_j(x_1, x_2) + f(x_2, \zeta)\, q_j(x_1, x_2) \qquad (j = 0, 1, \ldots, \chi),$$

where $p_j(x_1, x_2)$ and $q_j(x_1, x_2)$ are certain polynomials with coefficients from $\mathfrak{K}$. Differentiating these identities $r - \chi - 1$ times with respect to x_1, and putting $x_1 = x_2 = x$, we see that all the polynomials

$$\Delta^{(i)}(x)\, e(x)\, \Omega_j(x) \qquad (i = 0, 1, \ldots, r - \chi - 1;\ j = 0, 1, \ldots, \chi)$$

must be divisible by $f(x, \zeta)$. Since $f(x, \zeta) = e(x)\gamma(x)$ and since $\Omega_0(x)$, $\Omega_1(x)$, ..., $\Omega_\chi(x)$, $\gamma(x)$ have no common factor, all the derivatives $\Delta^{(i)}(x)$ must be multiples of $\gamma(x)$. Thus, since $f(x, \zeta)$ and therefore $\gamma(x)$ have non-zero discriminants, $\Delta(x)$ is divisible by $\gamma(x)^{r-\chi}$, and so

$$\Delta(x) = \gamma(x)^{r-\chi} d(x), \tag{15}$$

where $d(x)$ is a polynomial with coefficients from $\mathfrak{K}$ which is not identically zero and is of degree δ, say. Since $\gamma(x)$ is of degree exactly $m - \theta$, and $\Delta(x)$ is of degree not greater than $(\chi + 1)(q + r)$, it follows that

$$\delta \leq (\chi + 1)(q + r) - (r - \chi)(m - \theta) \leq (\chi + 1)\left(\frac{m + \varepsilon}{s + 1}\right) r - (r - \chi)(m - \theta)$$

$$= \left(\frac{\chi + 1}{s + 1} m - m + \theta\right) r + \varepsilon \frac{\chi + 1}{s + 1} r + \chi(m - \theta).$$

If we write

$$\chi = s - \theta - \nu,$$

where ν is a non-negative rational integer, then δ satisfies the inequality

$$\delta \leq -\frac{(m - s - 1)\theta + m\nu}{s + 1} r + \varepsilon r + (m - 1) m.$$

If we take

$$\varepsilon \leq \tfrac{1}{2}, \qquad r \geq 2m^2,$$

it is easily verified that ν must be zero and that

$$\delta \leq \varepsilon r + (m - 1) m,$$

for otherwise δ would be negative, which is absurd. It therefore follows that all the polynomials (14) are linearly independent, and so we may take

$$\Omega_j(x_2) = x_2^j$$

for $j = 0, 1, \ldots, s - \theta$. Then the case $j = 0$ gives the identity

$$\Delta(x_1)\, e(x_2) = \sum_{i=0}^{s-\theta} \Delta_{i0}(x_1)\, R_{i0}(x_1, x_2, \zeta).$$

Differentiating successively with respect to x_1, we find that

$$\Delta^{(j)}(x_1)\, e(x_2) = \sum_{i=0}^{s-\theta+j} P_{ij}(x_1)\, R_{i0}(x_1, x_2, \zeta) \qquad (j = 0, 1, 2, \ldots), \tag{16}$$

where the P_{ij} are certain polynomials in x_1.

Now let λ_1 and λ_2 be elements of an arbitrary field over $\mathfrak{K}$ such that $f(\lambda_1, \zeta)$ and $f(\lambda_2, \zeta)$ are not zero. Then by (15) there exists a non-negative rational integer j such that

$$j \leq \varepsilon r + (m - 1)\, m$$

and

$$\Delta^{(j)}(\lambda_1) \neq 0,$$

since, firstly, $\gamma(\lambda_1) \neq 0$, and secondly, if $\Delta(x)$ is divisible by $(x - \lambda_1)^j$, but by no higher power of $(x - \lambda_1)$, $j \leq \delta$. Thus, by (16) and since $e(\lambda_2) \neq 0$, at least one of the numbers $R_{i0}(\lambda_1, \lambda_2, \zeta)$ $(i = 0, 1, \ldots, s - \theta + j)$ is not zero. We have therefore proved that *if λ_1 and λ_2 are any two elements of an arbitrary field over $\mathfrak{K}$ such that $f(\lambda_1, \zeta)$ and $f(\lambda_2, \zeta)$ are not zero, there exists a non-negative rational integer i not greater than $\varepsilon r + m^2 - 1$, where $\varepsilon \leq \frac{1}{2}$ and $r \geq 2m^2$ (thus fulfilling the conditions that $0 \leq i < r$), such that*

$$R_{i0}(\lambda_1, \lambda_2, \zeta) \neq 0.$$

7. We thus arrive at the following lemma:

Lemma 3. *Let:*

$\mathfrak{K}$ *be a finite algebraic field of degree $n\,(\geq 1)$ over the rational number field $\mathfrak{P}$;*

ζ *be an algebraic integer generating $\mathfrak{K}$;*

$\varphi(z)$ *be the polynomial*

$$\varphi(z) = z^n + \varkappa_1 z^{n-1} + \varkappa_2 z^{n-2} + \cdots + \varkappa_n$$

with rational integral coefficients and irreducible in $\mathfrak{P}$, having ζ as a root;

$\varkappa$ *be the number $\overline{|\varphi(z)|}$;*

$f(x, z)$ *be a polynomial in x of degree $m\,(\geq 2)$:*

$$f(x, z) = a_0(z)\, x^m + a_1(z)\, x^{m-1} + \cdots + a_m(z),$$

where $a_0(z)\,(\not\equiv 0)$, $a_1(z)$, $a_2(z)$, $\ldots$, $a_m(z)$ are polynomials in z with rational integral coefficients and of degree not greater than $n - 1$, and $f(x, \zeta)$ has a non-zero discriminant (N. B. *$f(x, \zeta)$ need not necessarily be irreducible in $\mathfrak{K}$*);

a *be the smallest natural number such that*

$$a_\nu(z) \ll a(1+z)^{n-1}$$

for $\nu = 0, 1, \ldots, m$;

s *be a natural number less than* m;

r *be a natural number not less than* $2m^2$;

ε *be a positive number not greater than* $\frac{1}{2}$;

q *be the rational integer determined by the inequalities*

$$q \leq \left(\frac{m+\varepsilon}{s+1} - 1\right) r < q + 1.$$

Then there exists a polynomial $R(x_1, x_2, z)$, *not identically zero, with rational integral coefficients and of degree not greater than* $q + r$ *in* x_1, s *in* x_2 *and* $n - 1$ *in* z, *with*

$$\overline{|R(x_1, x_2, z)|} < (2^{3n} a\, x^{n-1})^{(q+r+1)\frac{m}{\varepsilon}},$$

and such that the following properties are satisfied:

(a) *If*

$$F_i(x_1, x_2, x, z) = \sum_{l_1=r}^{q+r} \sum_{l_2=0}^{s} R_{l_1 l_2}(x, x, z) \binom{l_1}{i} (x_1 - x)^{l_1 - r} (x_2 - x)^{l_2},$$

$$G_i(x_1, x_2, x, z) = \sum_{l_1=0}^{r-1} \sum_{l_2=1}^{s} R_{l_1 l_2}(x, x, z) \binom{l_1}{i} (x_1 - x)^{l_1 - i} (x_2 - x)^{l_2 - 1},$$

$$H_i^{(1)}(x_1, x_2, x, z) = a_0(z)^{-(q+r+1)} \sum_{l_1=0}^{r-1} R_{l_1}^{(1)}(x, z) \binom{l_1}{i} (x_1 - x)^{l_1 - i},$$

$$H_i^{(2)}(x_1, x_2, x, z) = a_0(z)^{-(q+r+1)} \sum_{l_1=0}^{r-1} R_{l_1}^{(2)}(x, z) \binom{l_1}{i} (x_1 - x)^{l_1 - i},$$

where i *takes one of the values* $0, 1, \ldots, r-1$, $R_{l_1 l_2}(x, x, z)$ *is the function* $\left(\frac{\partial^{l_1+l_2} R(x_1, x_2, z)}{l_1!\, l_2!\, \partial x_1^{l_1} \partial x_2^{l_2}}\right)_{x_1 = x_2 = x}$, *and* $R_i^{(1)}(x, z)$ *and* $R_i^{(2)}(x, z)$ $(i = 0, 1, \ldots, r-1)$ *are certain polynomials in* x *and* z *with rational integral coefficients, then* $R_i^{(1)}(x, z)$ *and* $R_i^{(2)}(x, z)$ *can be chosen so that*

$$R_{i0}(x_1, x_2, z) = (x_1 - x)^{r-i} F_i(x_1, x_2, x, z) + (x_2 - x)\, G_i(x_1, x_2, x, z) + f(x, z)\, H_i^{(1)}(x_1, x_2, x, z) + \varphi(z)\, H_i^{(2)}(x_1, x_2, x, z).$$

(b) *If λ_1 and λ_2 are any two elements of an arbitrary field over $\mathfrak{K}$, other than roots of $f(x, \zeta)$, then there exists a non-negative rational integer i such that*

$$i \leq \varepsilon r + m^2 - 1 \leq r - 1$$

and

$$R_{i0}(\lambda_1, \lambda_2, \zeta) \neq 0.$$

§ 4 a. Inequalities Required in the Proof of Theorem 1.

8. We shall require the following slight variation of a lemma proved by Siegel[1]:

Lemma 4. *Let $\mu_1, \mu_2, \ldots, \mu_l$ be any l numbers, where $l \geq 1$, and let L be the maximum of the absolute values of the coefficients of the polynomial in z:*

$$\prod_{\nu=1}^{l} (z - \mu_\nu).$$

Then

$$\prod_{\nu=1}^{l} \max(1, |\mu_\nu|) \leq 4^l L.$$

Proof. Suppose $l_1 (0 \leq l_1 \leq l)$ of the numbers $\mu_1, \mu_2, \ldots, \mu_l$ have absolute values not greater than 2. Without loss of generality, these may be taken as $\mu_1, \mu_2, \ldots, \mu_{l_1}$. Then

$$\prod_{\nu=1}^{l_1} \max(1, |\mu_\nu|) \leq 2^{l_1}.$$

Put $f(z) = \prod_{\nu=1}^{l_1} (z - \mu_\nu)$. Then for at least one of the (l_1+1)th roots of unity $\varepsilon_0 = 1, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{l_1}$, say for $z_0 = \varepsilon_j \; (0 \leq j \leq l_1)$, $|f(z_0)|$ is not less than 1, since

$$\sum_{\nu=0}^{l_1} \varepsilon_\nu f(\varepsilon_\nu) = l_1 + 1.$$

Thus

$$\prod_{\nu=1}^{l_1} \max(1, |\mu_\nu|) \leq 2^{l_1} \left| \prod_{\nu=1}^{l_1} (z_0 - \mu_\nu) \right|.$$

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[1] See note 1 on p. 1, p. 175.

Since $\mu_{l_1+1}, \mu_{l_1+2}, \ldots, \mu_l$ all have absolute values greater than 2,

$$\prod_{\nu=l_1+1}^{l}\left\{\frac{\max(1,|\mu_\nu|)}{|z_0-\mu_\nu|}\right\} \leq \prod_{\nu=l_1+1}^{l}\frac{|\mu_\nu|}{|\mu_\nu|-1} = \prod_{\nu=l_1+1}^{l}\left(1+\frac{1}{|\mu_\nu|-1}\right)$$

$$< \prod_{\nu=l_1+1}^{l}(1+1) = 2^{l-l_1}.$$

Thus

$$\prod_{\nu=1}^{l}\max(1,|\mu_\nu|) \leq 2^l\prod_{\nu=1}^{l}|z_0-\mu_\nu| \leq 2^l L(|z_0|^l+|z_0|^{l-1}+\cdots+1)$$

$$= 2^l(l+1)L \leq 4^l L,$$

and the lemma is proved, since $l+1 \leq 2^l$ for $l \geq 1$.

From Lemma 4 follows immediately:

Corollary. *Let*

$\mathfrak{K}$ *be a finite algebraic field of degree* $n(\geq 1)$ *over the rational number field* $\mathfrak{P}$;

h *be a natural number;*

λ *be an algebraic number of degree* h *(or some divisor of* h*) over* $\mathfrak{K}$, *and so an element of a field* $\mathfrak{K}'$ *of degree* h *over* $\mathfrak{K}$ *and* hn *over* $\mathfrak{P}$;

L_0 *be the coefficient of the highest power of* x *in the polynomial* $\Upsilon(x;\lambda,\mathfrak{P},hn)$;

A *be the number* $|\overline{\Upsilon(x;\lambda,\mathfrak{P},hn)}|$;

$\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(w)}$ *be any* $w\,(0 \leq w \leq hn)$ *conjugates to* λ, *with respect to the field* $\mathfrak{K}'$.

Then

$$\text{(a)}\quad \prod_{\nu=1}^{w}\max(1,|\lambda^{(\nu)}|) \leq 4^{hn}\frac{A}{|L_0|},$$

and

$$\text{(b)}\quad \prod_{\nu=1}^{w}(1+|\lambda^{(\nu)}|) \leq 8^{hn}\frac{A}{|L_0|}.$$

9. We shall also require *bounds for the valuations of any root of* $f(x,\zeta)$, or for a product of valuations of such roots. (The symbols $\varphi(z)$, $\varkappa$, ζ, $\mathfrak{K}$, n, $f(x,z)$, a, m have the same meaning as in Lemma 3.)

We write $f(x,z)$ as a polynomial in z, thus:

$$f(x,z) = a_0(x)z^{n-1}+a_1(x)z^{n-2}+\cdots+a_{n-1}(x),$$

where the $a_r(x)$ $(r=0,1,\ldots,n-1)$ are polynomials in x with rational integral coefficients and of degree not greater than m. They are not all identically zero.

Let $D(x)$ be the resultant of $f(x,z)$ and $\varphi(z)$ with respect to z, so that

$$\varphi(z)\,\Phi_D(x,z) + f(x,z)\,F_D(x,z) = D(x), \tag{17}$$

where $\Phi_D(x,z)$ and $F_D(x,z)$ are certain polynomials in z of degrees not greater than $n-2$ and $n-1$ respectively, and with coefficients which are polynomials in x with rational integral coefficients. The resultant $D(x)$ is a polynomial in x with rational integral coefficients. Also,

$$D(x) \not\equiv 0.$$

For otherwise the equations $\varphi(z)=0$ and $f(x,z)=0$ would have a common solution in z independent of the value of x. Thus $\varphi(z)$ and $f(x,z)$ would have as a common divisor a polynomial in z with coefficients not all zero, and not involving x. By the irreducibility of $\varphi(z)$, this would be possible only if $\varphi(z)$ were a divisor of the coefficients $a_0(z), a_1(z), \ldots, a_m(z)$ of $f(x,z)$, considered as a polynomial in x. But this is impossible, since $a_0(z), a_1(z), \ldots, a_m(z)$ are of degree not greater than $n-1$, while $\varphi(z)$ is of degree n.

Thus we may write

$$D(x) = D_0 x^M + D_1 x^{M-1} + \cdots + D_M,$$

where M is a non-negative rational integer, $D_0, D_1, \ldots, D_M$ are rational integers, and $D_0 \neq 0$.

By (17), any root of $f(x,\zeta)$ is also a root of $D(x)$, so that the problem is reduced to that of finding bounds for the roots of $D(x)$.

10. (a) *Let ξ be a real or complex root of $f(x,\zeta)$.* Then

$$D_0 \xi^M = -(D_1 \xi^{M-1} + D_2 \xi^{M-2} + \cdots + D_M),$$

so that, provided $\xi \neq 0$,

$$D_0 \xi = -\left(D_1 + \frac{D_2}{\xi} + \cdots + \frac{D_M}{\xi^{M-1}}\right),$$

and

$$|D_0|\,|\xi| \leq |\overline{D}|(1 + |\xi|^{-1} + |\xi|^{-2} + \cdots + |\xi|^{-(M-1)}).$$

Suppose now that $|\xi| > 1$. Then

$$|\xi| < \frac{|\overline{D}|}{|D_0|} \cdot \frac{1}{1-|\xi|^{-1}}.$$

Hence

$$|\xi| < \frac{|\overline{D}|}{|D_0|} + 1.$$

This inequality is obviously true also for $|\xi| \leq 1$; hence, since D_0 is a non-zero rational integer, so that $|D_0| \geq 1$, it follows that

$$|\xi| < |\overline{D}| + 1$$

for all real or complex roots ξ of $f(x, \zeta)$.

(b) *Let η be a root of $f(x, \zeta)$ in the perfect* $\mathfrak{r}$*-adic extension of* $\mathfrak{R}$, i. e., an $\mathfrak{r}$-adic root of $f(x, \zeta)$, where $\mathfrak{r}$ is a finite prime ideal of $\mathfrak{R}$. As before, provided $\eta \neq 0$,

$$D_0 \eta = -\left(D_1 + \frac{D_2}{\eta} + \cdots + \frac{D_M}{\eta^{M-1}}\right).$$

Hence

$$|D_0|_{\mathfrak{r}} |\eta|_{\mathfrak{r}} \leq \max\left(1, |\eta|_{\mathfrak{r}}^{-1}, |\eta|_{\mathfrak{r}}^{-2}, \ldots, |\eta|_{\mathfrak{r}}^{-(M-1)}\right),$$

since $D_1, D_2, \ldots, D_M$ are rational integers. Thus, if $|\eta|_{\mathfrak{r}} > 1$,

$$|\eta|_{\mathfrak{r}} \leq \frac{1}{|D_0|_{\mathfrak{r}}};$$

and this inequality also holds if $|\eta|_{\mathfrak{r}} \leq 1$, for D_0 is a rational integer, so that $|D_0|_{\mathfrak{r}} \leq 1$.

Let now $\eta_{1\mathfrak{r}}, \eta_{2\mathfrak{r}}, \ldots, \eta_{g(\mathfrak{r})\mathfrak{r}}$ be any $g(\mathfrak{r})$ roots of $f(x, \zeta)$ in the perfect $\mathfrak{r}$-adic extension of $\mathfrak{R}$. Then

$$\prod_{\mathfrak{r}} \left(\prod_{\nu=1}^{g(\mathfrak{r})} \max\left(1, |\eta_{\nu\mathfrak{r}}|_{\mathfrak{r}}\right) \right) \leq \prod_{\mathfrak{r}} \frac{1}{|D_0|_{\mathfrak{r}}^{g(\mathfrak{r})}},$$

the product being taken over any number of different finite prime ideals $\mathfrak{r}$ of $\mathfrak{R}$. But it was stated in § 1 that if $\mathfrak{p}^{(1)}, \mathfrak{p}^{(2)}, \ldots, \mathfrak{p}^{(\pi)}$ are the different prime ideal factors in $\mathfrak{R}$ of a natural prime number p, then

$$\sum_{i=1}^{\pi} g(\mathfrak{p}^{(i)}) = n.$$

Thus, since D_0 is a non-zero rational integer,

$$\prod_{\mathfrak{r}} \left(\prod_{\nu=1}^{g(\mathfrak{r})} \max\left(1, |\eta_{\nu\mathfrak{r}}|_{\mathfrak{r}}\right) \right) \leq |D_0|^n,$$

and so

$$\prod_{\mathfrak{r}} \left(\prod_{\nu=1}^{g(\mathfrak{r})} \max\left(1, |\eta_{\nu\mathfrak{r}}|_{\mathfrak{r}}\right) \right) \leq (|\overline{D}|)^n. \tag{18}$$

11. It now remains to find an upper bound for $|D|$ in terms of the coefficients and degrees of $\varphi(z)$ and $f(x, z)$.

The resultant $D(x)$ of $\varphi(z)$ and $f(x, z)$ is the determinant of $2n-1$ rows and columns:

$$\left.\begin{array}{|ccccccccccccc|}
1, & \varkappa_1, & \varkappa_2, & \ldots, & \varkappa_{n-1}, & \varkappa_n, & . & . & . & . & . & & \\
0, & 1, & \varkappa_1, & \ldots, & \varkappa_{n-2}, & \varkappa_{n-1}, & \varkappa_n, & . & . & . & . & & \\
. & . & & & & & & & . & . & . & & \\
. & . & . & & & & & & & . & . & & \\
. & . & . & . & & & & & & & . & & \\
. & . & . & . & . & 1, & \varkappa_1, & \varkappa_2, & \ldots, & \varkappa_{n-1}, & \varkappa_n & & \\
\alpha_0, & \alpha_1, & \alpha_2, & \ldots, & \alpha_{n-1}, & . & . & . & . & . & . & & \\
0, & \alpha_0, & \alpha_1, & \ldots, & \alpha_{n-2}, & \alpha_{n-1}, & . & . & . & . & . & & \\
. & . & & & & & & . & . & . & . & & \\
. & . & . & & & & & & . & . & . & & \\
. & . & . & . & & & & & & . & . & & \\
. & . & . & . & . & & & & & & . & & \\
. & . & . & . & . & . & \alpha_0, & \alpha_1, & \ldots, & \alpha_{n-2}, & \alpha_{n-1} & &
\end{array}\right\} \begin{array}{l} n-1 \text{ rows} \\ \\ \\ \\ \\ \\ n \text{ rows} \\ \\ \\ \\ \\ \\ \end{array}$$

From this determinant, $\Phi_D(x, z)$ is obtained by replacing the final column by $z^{n-2}, z^{n-3}, \ldots, 1, 0, \ldots, 0$, and $F_D(x, z)$ by replacing the final column by $0, \ldots, 0, z^{n-1}, z^{n-2}, \ldots, 1$.

Now

$$a_\mu(z) \ll a(1+z)^{n-1} \ll 2^{n-1} a(1 + z + z^2 + \cdots + z^{n-1}) \quad (\mu = 0, 1, \ldots, m),$$

since

$$\binom{n-1}{\mu} \leq \sum_{\nu=0}^{n-1} \binom{n-1}{\nu} = 2^{n-1}.$$

Hence

$$\alpha_\nu(x) \ll 2^{n-1} a(1 + x + x^2 + \cdots + x^m) \ll 2^{n-1} a(1+x)^m \quad (\nu = 0, 1, \ldots, n-1).$$

Let $\pi(x)$ be the polynomial obtained on taking any one complete product of elements of the determinant $D(x)$. Then since $|\varkappa_w| \leq \varkappa$ for $w = 0, 1, \ldots, n$,

$$\begin{aligned} \pi(x) &\ll \varkappa^{n-1} (2^{n-1} a)^n (1+x)^{mn} \\ &\ll \varkappa^{n-1} 2^{(n-1)n} a^n 2^{mn} (1 + x + x^2 + \cdots + x^{mn}). \end{aligned}$$

There are at most $(2n-1)!$ such products $\pi(x)$. Hence

$$D(x) \ll (2n-1)!\, 2^{(n-1)n} \varkappa^{n-1} a^n 2^{mn} (1 + x + x^2 + \cdots + x^{mn}).$$

Thus

$$\begin{aligned}
\overline{|D|} + 1 &\leq (2n-1)!\, 2^{(n-1)n+mn} a^n \varkappa^{n-1} + 1 \\
&< 2^{2n-1} n^{2n-1} 2^{(n-1)n+mn} a^n \varkappa^{n-1} \\
&= 2^{n^2+n+mn-1} n^{2n-1} a^n \varkappa^{n-1} \\
&< 2^{n^2+n+mn+4n\log n} a^n \varkappa^{n-1} \qquad \text{(since } e < 2^2\text{)} \\
&< 2^{n(2n+2m+4\log n)} a^n \varkappa^{n-1} \\
&= 4^{n(n+m+2\log n)} a^n \varkappa^{n-1} \\
&< 4^{n(2n+m)} a^n \varkappa^{n-1} \qquad \text{(since } 2\log n < n \text{ for } n \geq 1\text{).}
\end{aligned} \tag{19}$$

By (17), (18) and (19) we arrive at:

Lemma 5. *Let* $\mathfrak{K}$, $\varkappa$, n, a, m, $f(x,\zeta)$ *be defined as in Lemma 3.*

(a) *If* ξ *be any real or complex root of* $f(x,\zeta)$, *then*

$$|\xi| < 4^{n(2n+m)} a^n \varkappa^{n-1}.$$

(b) *If* $\eta_{1\mathfrak{r}}, \eta_{2\mathfrak{r}}, \ldots, \eta_{g(\mathfrak{r})\mathfrak{r}}$ *be any* $g(\mathfrak{r})$ *roots of* $f(x,\zeta)$ *in the perfect* $\mathfrak{r}$*-adic extension of* $\mathfrak{K}$, *where* $\mathfrak{r}$ *is any finite prime ideal of* $\mathfrak{K}$, *then*

$$\prod_{\mathfrak{r}} \left(\prod_{\nu=1}^{g(\mathfrak{r})} \max\left(1, |\eta_{\nu\mathfrak{r}}|_{\mathfrak{r}}\right) \right) < 4^{n^2(2n+m)} a^{n^2} \varkappa^{n(n-1)},$$

the product being taken over any number of different finite prime ideals $\mathfrak{r}$ *of* $\mathfrak{K}$.

§ 4. Proof of the Approximation Theorem.

12. Let λ_1 and λ_2 be any two algebraic numbers of degrees h_1 and h_2 (where h_1 and h_2 are divisors of a fixed natural number h) over the field $\mathfrak{K}$, which is of degree n (≥ 1) over the rational number field $\mathfrak{P}$ (i. e., in our notation, $\omega' = \lambda_1$ or λ_2 and $n'' = h_1$ or h_2); let L_{10} and L_{20} be the coefficients of the highest powers of x in the polynomials $\Upsilon(x; \lambda_1, \mathfrak{P}, hn)$ and $\Upsilon(x; \lambda_2, \mathfrak{P}, hn)$, and let Λ_1 and Λ_2 be the numbers $\overline{|\Upsilon(x; \lambda_1, \mathfrak{P}, hn)|}$ and $\overline{|\Upsilon(x; \lambda_2, \mathfrak{P}, hn)|}$.

By Lemma 3, provided neither λ_1 nor λ_2 is a root of $f(x,\zeta)$, for some non-negative rational integer i not greater than $\varepsilon r + m^2 - 1$,

$$\mathfrak{R}_i = R_{i0}(\lambda_1, \lambda_2, \zeta) \neq 0,$$

$R_{i0}(x_1, x_2, z)$ being defined as in Lemma 3. Now $R_{i0}(x_1, x_2, z)$ is a polynomial in x_1, x_2 and z with rational integral coefficients and is of degree not greater

than $q+r-i$ in x_1, s in x_2 and $n-1$ in z. Thus, since ζ is an integer of $\mathfrak{K}$ and λ_1 and λ_2 are of degrees h_1 and h_2 dividing h over $\mathfrak{K}$, $\mathfrak{R}_i$ is a non-zero element of a field $\mathfrak{K}'$ of degree h^2 over $\mathfrak{K}$. Further, the coefficient of the highest power of x in the polynomial $\Upsilon(x; \mathfrak{R}_i, \mathfrak{P}, h^2 n)$ is a divisor of $(L_{10}^{q+r-i} L_{20}^{s})^h$, since λ_1 and λ_2 lie in fields $\mathfrak{K}_1$ and $\mathfrak{K}_2$, of degree hn over $\mathfrak{P}$, which are subfields of $\mathfrak{K}'$, which is of degree $h^2 n$ over $\mathfrak{P}$ and therefore of degree h over $\mathfrak{K}_1$ and $\mathfrak{K}_2$. Thus, if $\mathfrak{R}_i^{(1)}, \mathfrak{R}_i^{(2)}, \ldots, \mathfrak{R}_i^{(h^2 n)}$ be the real or complex values conjugate to $\mathfrak{R}_i$, with respect to $\mathfrak{K}'$, and if $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$ be any $\sigma (\geq 0)$ different finite prime ideals of $\mathfrak{K}$, and if, for $k = 1, 2, \ldots, \sigma$, h_k be any natural number not greater than $g(\mathfrak{r}_k) h^2$, the inequality (9), with $\omega' = \mathfrak{R}_i$ and $n' = h^2$, takes the form:

$$\prod_{\nu=1}^{h^2 n} |\mathfrak{R}_i^{(\nu)}| \prod_{k=1}^{\sigma} |\mathfrak{R}_i|_{\mathfrak{r}_k}^{h_k} \geq |L_{10}^{q+r-1} L_{20}^{s}|^{-h},$$

provided λ_1 and λ_2 lie in the perfect $\mathfrak{r}_1$-adic, $\mathfrak{r}_2$-adic, ..., $\mathfrak{r}_\sigma$-adic extensions of $\mathfrak{K}$

As before, let there be r_1 real and r_2 pairs of conjugate imaginary fields conjugate to $\mathfrak{K}$, and let $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_{r_1+r_2}$ be the corresponding infinite prime ideals in any desired order. Then $|\mathfrak{R}_i|_{\mathfrak{q}_j \gamma}$ $(j = 1, 2, \ldots, r_1+r_2;\ \gamma = 1, g(\mathfrak{q}_j))$ represents the absolute value of any of the $\mathfrak{R}_i^{(\nu)}$ $(\nu = 1, 2, \ldots, h^2 n)$ which are roots of the polynomial $\Upsilon(x; \mathfrak{R}_i, \mathfrak{K}_{j\gamma}, h^2)$. Thus, if ϱ be any non-negative rational integer not greater than $r_1 + r_2$, and if $G(\mathfrak{q}_j)$ be a natural number not greater than $g(\mathfrak{q}_j)$ $(j = 1, 2, \ldots, \varrho)$,

$$\prod_{j=1}^{\varrho} \prod_{\gamma=1}^{G(\mathfrak{q}_j)} |\mathfrak{R}_i|_{\mathfrak{q}_j \gamma} \prod_{k=1}^{\sigma} |\mathfrak{R}_i|_{\mathfrak{r}_k}^{h_k} \geq \left\{ |L_{10}^{q+r-i} L_{20}^{s}|^h \prod_{\nu}' |\mathfrak{R}_i^{(\nu)}| \right\}^{-1}, \tag{20}$$

where

$$\prod_{\nu}' |\mathfrak{R}_i^{(\nu)}| = \prod_{\nu=1}^{h^2 n} |\mathfrak{R}_i^{(\nu)}| \left(\prod_{j=1}^{\varrho} \prod_{\gamma=1}^{G(\mathfrak{q}_j)} |\mathfrak{R}_i|_{\mathfrak{q}_j \gamma} \right)^{-1}.$$

Now, as defined in Lemma 3,

$$\overline{|R(x_1, x_2, z)|} < (2^{3n} a x^{n-1})^{(q+r+1)\frac{m}{\varepsilon}}.$$

Hence

$$R(x_1, x_2, z) \ll (2^{3n} a x^{n-1})^{(q+r+1)\frac{m}{\varepsilon}} \sum_{l_1=0}^{q+r} \sum_{l_2=0}^{s} \sum_{l_3=0}^{n-1} x_1^{l_1} x_2^{l_2} z^{l_3},$$

and

$$R_{i0}(x_1, x_2, z) \ll (2^{3n} a x^{n-1})^{(q+r+1)\frac{m}{\varepsilon}} \sum_{l_1=0}^{q+r} \sum_{l_2=0}^{s} \sum_{l_3=0}^{n-1} \binom{l_1}{i} x_1^{l_1-i} x_2^{l_2} z^{l_3}.$$

But

$$\binom{l_1}{i} \leq \sum_{\nu=0}^{l_1} \binom{l_1}{\nu} = 2^{l_1} \leq 2^{q+r},$$

and so

$$\sum_{l_1=0}^{q+r} \binom{l_1}{i} x_1^{l_1-i} \ll 2^{q+r} \sum_{l_1=0}^{q+r-i} x_1^{l_1} \ll 2^{q+r}(1+x_1)^{q+r-i}.$$

Also

$$\sum_{l_2=0}^{s} x_2^{l_2} \ll (1+x_2)^s, \quad \sum_{l_3=0}^{n-1} z^{l_3} \ll (1+z)^{n-1}.$$

Hence

$$R_{i0}(x_1, x_2, z) \ll \left(2^{3n} a x^{n-1}\right)^{(q+r+1)\frac{m}{\varepsilon}} 2^{q+r}(1+x_1)^{q+r-i}(1+x_2)^5(1+z)^{n-1},$$

and so

$$|\mathfrak{R}_i|_{q_j\gamma} \leq \left(2^{3n} a x^{n-1}\right)^{(q+r+1)\frac{m}{\varepsilon}} 2^{q+r}(1+|\lambda_1|_{q_j\gamma})^{q+r-i}(1+|\lambda_2|_{q_j\gamma})^s(1+|\zeta|_{q_j})^{n-1}$$
$$(j=1, 2, \ldots, r_1+r_2;\ \gamma=1, g(q_j)),$$

where $|\lambda_1|_{q_j\gamma}$ and $|\lambda_2|_{q_j\gamma}$ represent the absolute values of any roots $\lambda_{1j\gamma}$ and $\lambda_{2j\gamma}$ of the polynomials $\Upsilon(x; \lambda_1, \mathfrak{K}_{j\gamma}, h)$ and $\Upsilon(x; \lambda_2, \mathfrak{K}_{j\gamma}, h)$, and $|\mathfrak{R}_i|_{q_j\gamma}$ represents the absolute value of that $\mathfrak{R}_i^{(\nu)}$ corresponding to the pair of such roots chosen. It follows that

$$\prod_{\nu}{}' |\mathfrak{R}_i^{(\nu)}| \leq \left(2^{3n} a x^{n-1}\right)^{h^2 n(q+r+1)\frac{m}{\varepsilon}} 2^{(q+r)h^2 n} \prod_{j=1}^{r_1+r_2} \prod_{\gamma=1}^{g(q_j)} \prod_{\nu_1=1}^{h} (1+|\lambda_1^{(\nu_1)}|_{q_j\gamma})^{(q+r-i)h} \cdot$$
$$\cdot \prod_{j=1}^{r_1+r_2} \prod_{\gamma=1}^{g(q_j)} \prod_{\nu_2=1}^{h} (1+|\lambda_2^{(\nu_2)}|_{q_j\gamma})^{sh} \prod_{j=1}^{r_1+r_2} (1+|\zeta|_{q_j})^{g(q_j)(n-1)h^2},$$

$\lambda_1^{(1)}, \lambda_1^{(2)}, \ldots, \lambda_1^{(h)}$ and $\lambda_2^{(1)}, \lambda_2^{(2)}, \ldots, \lambda_2^{(h)}$ being the roots of the polynomials $\Upsilon(x; \lambda_1, \mathfrak{K}, h)$ and $\Upsilon(x; \lambda_2, \mathfrak{K}, h)$. Thus, by Lemma 4, Corollary (b),

$$\prod_{\nu}{}' |\mathfrak{R}_i^{(\nu)}| \leq \left(2^{3n} a^n x^{n-1}\right)^{h^2 n(q+r+1)\frac{m}{\varepsilon}} 2^{h^2 n(q+r)} 8^{h^2 n(q+r-i)} \left(\frac{\Lambda_1}{|L_{10}|}\right)^{(q+r-i)h} 8^{h^2 ns} \cdot$$
$$\cdot \left(\frac{\Lambda_2}{|L_{20}|}\right)^{sh} 8^{h^2 n(n-1)} x^{h^2(n-1)}.$$

Hence and by (20),

$$(21) \quad \prod_{j=1}^{\varrho} \prod_{\gamma=1}^{G(q_j)} |\mathfrak{R}_i|_{q_j\gamma} \prod_{k=1}^{\sigma} |\mathfrak{R}_i|_{r_k}^{h_k}$$
$$\geq (\Lambda_1^{q+r-i} \Lambda_2^s)^{-h} \left\{\left(2^{3n} a x^{n-1}\right)^{n(q+r+1)\frac{m}{\varepsilon}} 2^{n(q+r)} 8^{n(q+r+m+n-2)} x^{n-1}\right\}^{-h^2}.$$

13. We write:

$$\mathfrak{F}_{ij\gamma} = F_i(\lambda_1, \lambda_2, \xi_{j\gamma}, \zeta), \quad \mathfrak{G}_{ij\gamma} = G_i(\lambda_1, \lambda_2, \xi_{j\gamma}, \zeta) \qquad (j = 1, 2, \ldots, \varrho;\ \gamma = 1, G(\mathfrak{q}_j)),$$

$$\mathfrak{F}_{ik} = F_i(\lambda_1, \lambda_2, \eta_k, \zeta), \quad \mathfrak{G}_{ik} = G_i(\lambda_1, \lambda_2, \eta_k, \zeta) \qquad (k = 1, 2, \ldots, \sigma),$$

where $F_i(x_1, x_2, x, z)$ and $G_i(x_1, x_2, x, z)$ are defined as in Lemma 3, and $\xi_{j\gamma}$ is a real or complex root and η_k an $\mathfrak{r}_k$-adic root of $f(x, \zeta)$.

We seek first for upper bounds for $\mathfrak{F}_{ij\gamma}$ and $\mathfrak{G}_{ij\gamma}$, where $j = 1, 2, \ldots, \varrho$ and $\gamma = 1, \ G(\mathfrak{q}_j)$. It should be noted that by $|A(\lambda_1, \lambda_2, \xi_{j\gamma}, \zeta)|_{\mathfrak{q}_{j\gamma}}$, where $A(x_1, x_2, x, z)$ is any function of x_1, x_2, x and z, is meant the absolute value of A when $x_1 = \lambda_{1j\gamma}$, $x_2 = \lambda_{2j\gamma}$, $x = \xi_{j\gamma}$ and $z = \zeta_{j\gamma}$, where $\lambda_{1j\gamma}$ and $\lambda_{2j\gamma}$ are any roots of the polynomials $\Upsilon(x; \lambda_1, \mathfrak{K}_{j\gamma}, h)$ and $\Upsilon(x; \lambda_2, \mathfrak{K}_{j\gamma}, h)$, and $\zeta_{j\gamma}$ is the root in the field $\mathfrak{K}_{j\gamma}$ of the polynomial $\varphi(z)$.

By the definitions of $F_i(x_1, x_2, x, z)$ and $G_i(x_1, x_2, x, z)$,

$$\max\left(|\mathfrak{F}_{ij\gamma}|_{\mathfrak{q}_{j\gamma}}, \ |\mathfrak{G}_{ij\gamma}|_{\mathfrak{q}_{j\gamma}}\right) \leq \sum_{l_1=0}^{q+r} \sum_{l_2=0}^{s} \binom{l_1}{i} |R_{l_1 l_2}(\xi_{j\gamma}, \xi_{j\gamma}, \zeta)|_{\mathfrak{q}_{j\gamma}}$$
$$(j = 1, 2, \ldots, \varrho;\ \gamma = 1, \ G(\mathfrak{q}_j)),$$

provided

$$|\lambda_1 - \xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}} \leq 1, \quad |\lambda_2 - \xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}} \leq 1.$$

But

$$|R_{l_1 l_2}(\xi_{j\gamma}, \xi_{j\gamma}, \zeta)|_{\mathfrak{q}_{j\gamma}} \leq \left(2^{3n} a \varkappa^{n-1}\right)^{(q+r+1)\frac{m}{\varepsilon}} \sum_{k_1=0}^{q+r} \sum_{k_2=0}^{s} \sum_{l_3=0}^{n-1} \binom{k_1}{l_1} \binom{k_2}{l_2} |\xi_{j\gamma}|^{k_1+k_2-l_1-l_2} |\zeta|^{l_3}_{\mathfrak{q}_{j\gamma}}$$

and by Lemma 5 (a),

$$|\xi_{j\gamma}| < 4^{n(2n+m)} a^n \varkappa^{n-1},$$

and, further, it is easily verified, by a method identical with that of 10 (a), but with $\varphi(z)$ in place of $D(x)$, that

$$|\zeta|_{\mathfrak{q}_j} \leq \varkappa + 1.$$

Hence

$$\max\left(|\mathfrak{F}_{ij\gamma}|_{\mathfrak{q}_{j\gamma}}, \ |\mathfrak{G}_{ij\gamma}|_{\mathfrak{q}_{j\gamma}}\right) \leq \left(2^{3n} a \varkappa^{n-1}\right)^{(q+r+1)\frac{m}{\varepsilon}} \cdot$$
$$\cdot \sum_{k_1=0}^{q+r} \sum_{k_2=0}^{s} \sum_{l_3=0}^{n-1} \left\{ \sum_{l_1=0}^{q+r} \sum_{l_2=0}^{s} \binom{l_1}{i} \binom{k_1}{l_1} \binom{k_2}{l_2} \left(4^{n(2n+m)} a^n \varkappa^{n-1}\right)^{k_1+k_2-l_1-l_2} (\varkappa + 1)^{l_3} \right\}$$
$$= \left(2^{3n} a \varkappa^{n-1}\right)^{(q+r+1)\frac{m}{\varepsilon}} \sum_{k_1=0}^{q+r} \sum_{k_2=0}^{s} \sum_{l_3=0}^{n-1} \left\{ \binom{k_1}{i} \left(4^{n(2n+m)} a^n \varkappa^{n-1} + 1\right)^{k_1+k_2-i} (\varkappa + 1)^{l_3} \right\}$$
$$(j = 1, 2, \ldots, \varrho;\ \gamma = 1, \ G(\mathfrak{q}_j)).$$

Now

$$\binom{k_1}{i} \leq \sum_{\nu=0}^{k_1} \binom{k_1}{\nu} = 2^{k_1} \leq 2^{q+r}.$$

Hence

$$\sum_{k_1=0}^{q+r} \binom{k_1}{i} (4^{n(2n+m)} a^n x^{n-1} + 1)^{k_1-i} \leq 2^{q+r} \sum_{k_1=0}^{q+r} (4^{n(2n+m)} a^n x^{n-1} + 1)^{k_1-i}$$

$$\leq 2^{q+r} \sum_{k_1=0}^{q+r} \binom{q+r}{k_1} (4^{n(2n+m)} a^n x^{n-1} + 1)^{k_1}$$

$$= 2^{q+r} (4^{n(2n+m)} a^n x^{n-1} + 2)^{q+r}$$

$$< 3^{q+r} (4^{n(2n+m)} a^n x^{n-1})^{q+r}.$$

Also

$$\sum_{k_2=0}^{s} (4^{n(2n+m)} a^n x^{n-1} + 1)^{k_2} \leq \sum_{k_2=0}^{s} \binom{s}{k_2} (4^{n(2n+m)} a^n x^{n-1} + 1)^{k_2}$$

$$= (4^{n(2n+m)} a^n x^{n-1} + 2)^s$$

$$< \left(\tfrac{3}{2}\right)^{m-1} (4^{n(2n+m)} a^n x^{n-1})^{m-1}.$$

Also

$$\sum_{l_3=0}^{n-1} (x+1)^{l_3} = \left\{ \frac{(x+1)^n - 1}{x} \right\} < 2^n x^{n-1}.$$

Thus

$$\max (|\mathfrak{F}_{ij\gamma}|_{q_{j\gamma}}, |\mathfrak{G}_{ij\gamma}|_{q_{j\gamma}})$$

$$< (2^{3n} a x^{n-1})^{(q+r+1)\frac{m}{\varepsilon}} 2^{n-m+1} 3^{q+r+m-1} x^{n-1} (4^{n(2n+m)} a^n x^{n-1})^{q+r+m-1}$$

$$(j = 1, 2, \ldots, \varrho;\ \gamma = 1, G(q_j)).$$

Hence, since

$$\sum_{j=1}^{\varrho} G(q_j) \leq n,$$

$$(22) \quad \prod_{j=1}^{\varrho} \prod_{\gamma=1}^{G(q_j)} \max (|\mathfrak{F}_{ij\gamma}|_{q_{j\gamma}}, |\mathfrak{G}_{ij\gamma}|_{q_{j\gamma}})$$

$$< (2^{3n} a x^{n-1})^{(q+r+1)\frac{mn}{\varepsilon}} 2^{n(n-m+1)} 3^{n(q+r+m-1)} x^{n(n-1)} (4^{n(2n+m)} a^n x^{n-1})^{n(q+r+m-1)},$$

provided

$$(23) \qquad |\lambda_1 - \xi_{j\gamma}|_{q_{j\gamma}} \leq 1, \quad |\lambda_2 - \xi_{j\gamma}|_{q_{j\gamma}} \leq 1$$

for $j = 1, 2, \ldots, \varrho$ and $\gamma = 1, G(q_j)$.

We seek next for upper bounds for the r_k-adic valuations of $\mathfrak{F}_{ik}$ and $\mathfrak{G}_{ik}$. It should be noted that by $|A(\lambda_1, \lambda_2, \eta_k, \zeta)|_{r_k}$, where $A(x_1, x_2, x, z)$ is any function

of x_1, x_2, x and z, is meant the $\mathfrak{r}_k$-adic valuation of $A(x_1, x_2, x, z)$ when $x_1 = \lambda_1$, $x_2 = \lambda_2$, $x = \eta_k$ and $z = \zeta$, where λ_1 and λ_2 are any roots of the polynomials $\Upsilon(x; \lambda_1, \mathfrak{K}, h)$ and $\Upsilon(x; \lambda_2, \mathfrak{K}, h)$ lying in the perfect $\mathfrak{r}_k$-adic extension of $\mathfrak{K}$.

By the definitions of $F_i(x_1, x_2, x, z)$ and $G_i(x_1, x_2, x, z)$,

$$\max(|\mathfrak{F}_{ik}|_{\mathfrak{r}_k}, |\mathfrak{G}_{ik}|_{\mathfrak{r}_k}) \leq \max_{\substack{l_1=i, i+1, \ldots, q+r \\ l_2=0, 1, 2, \ldots, s}} |R_{l_1 l_2}(\eta_k, \eta_k, \zeta)|_{\mathfrak{r}_k} \qquad (k = 1, 2, \ldots, \sigma),$$

provided

$$|\lambda_1 - \eta_k|_{\mathfrak{r}_k} \leq 1, \qquad |\lambda_2 - \eta_k|_{\mathfrak{r}_k} \leq 1.$$

But $R_{l_1 l_2}(x, x, z)$ is a polynomial in x and z with rational integral coefficients and of degree not greater than $q+r+s$ in x. Thus, since ζ is an integer of the field $\mathfrak{K}$,

$$\max(|\mathfrak{F}_{ik}|_{\mathfrak{r}_k}, |\mathfrak{G}_{ik}|_{\mathfrak{r}_k}) \leq \max(1, |\eta_k|_{\mathfrak{r}_k})^{q+r+s} \qquad (k = 1, 2, \ldots, \sigma), \tag{24}$$

provided

$$|\lambda_1 - \eta_k|_{\mathfrak{r}_k} \leq 1, \qquad |\lambda_2 - \eta_k|_{\mathfrak{r}_k} \leq 1. \tag{25}$$

14. By Lemma 3,

$$R_{i0}(\lambda_1, \lambda_2, \zeta) = (\lambda_1 - x)^{r-i} F_i(\lambda_1, \lambda_2, x, \zeta) + (\lambda_2 - x) G_i(\lambda_1, \lambda_2, x, \zeta) + f(x, \zeta) H_i^{(1)}(\lambda_1, \lambda_2, x, \zeta).$$

Hence, putting $x = \xi_{j\gamma}$, it follows that

$$|\mathfrak{R}_i|_{\mathfrak{q}_{j\gamma}} = |(\lambda_1 - \xi_{j\gamma})^{r-i} \mathfrak{F}_{ij\gamma} + (\lambda_2 - \xi_{j\gamma}) \mathfrak{G}_{ij\gamma}|_{\mathfrak{q}_{j\gamma}} \qquad (j = 1, 2, \ldots, \varrho;\ \gamma = 1, \ldots G(\mathfrak{q}_j)),$$

and putting $x = \eta_k$, it follows that

$$|\mathfrak{R}_i|_{\mathfrak{r}_k} = |(\lambda_1 - \eta_k)^{r-i} \mathfrak{F}_{ik} + (\lambda_2 - \eta_k) \mathfrak{G}_{ik}|_{\mathfrak{r}_k} \qquad (k = 1, 2, \ldots, \sigma).$$

Thus, firstly,

$$|\mathfrak{R}_i|_{\mathfrak{q}_{j\gamma}} \leq 2 \max(|\mathfrak{F}_{ij\gamma}|_{\mathfrak{q}_{j\gamma}}, |\mathfrak{G}_{ij\gamma}|_{\mathfrak{q}_{j\gamma}}) \max(|\lambda_1 - \xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}}^{r-i}, |\lambda_2 - \xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}})$$
$$(j = 1, 2, \ldots, \varrho;\ \gamma = 1, \ldots G(\mathfrak{q}_j)),$$

so that by (22) and (23),

$$\prod_{j=1}^{\varrho} \prod_{\gamma=1}^{G(\mathfrak{q}_j)} |\mathfrak{R}_i|_{\mathfrak{q}_{j\gamma}} \leq 2^n (2^{3n} a \varkappa^{n-1})^{(q+r+1)\frac{mn}{\varepsilon}} 2^{n(n-m+1)} 3^{n(q+r+m-1)} \varkappa^{n(n-1)} \cdot$$
$$\cdot (4^{n(2n+m)} a^n \varkappa^{n-1})^{n(q+r+m-1)} \prod_{j=1}^{\varrho} \prod_{\gamma=1}^{G(\mathfrak{q}_j)} \max(|\lambda_1 - \xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}}^{r-i}, |\lambda_2 - \xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}}), \tag{26}$$

provided

$$|\lambda_1 - \xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}} \leq 1, \qquad |\lambda_2 - \xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}} \leq 1 \tag{27}$$

for $j = 1, 2, \ldots, \varrho$ and $\gamma = 1, \ldots G(\mathfrak{q}_j)$.

Secondly,

$$|\Re_i|_{\mathfrak{r}_k} \leq \max\left(|\mathfrak{F}_{ik}|_{\mathfrak{r}_k}, |\mathfrak{G}_{ik}|_{\mathfrak{r}_k}\right) \max\left(|\lambda_1 - \eta_k|_{\mathfrak{r}_k}^{r-i}, |\lambda_2 - \eta_k|_{\mathfrak{r}_k}\right) \qquad (k = 1, 2, \ldots, \sigma),$$

so that by (24) and (25),

$$|\Re_i|_{\mathfrak{r}_k} \leq \max\left(1, |\eta_k|_{\mathfrak{r}_k}\right)^{q+r+s} \max\left(|\lambda_1 - \eta_k|_{\mathfrak{r}_k}^{r-i}, |\lambda_2 - \eta_k|_{\mathfrak{r}_k}\right);$$

and if $\eta_{k\delta\tau}$ $(k = 1, 2, \ldots, \sigma;\ \delta = 1, 2, \ldots, G(\mathfrak{r}_k);\ \tau = 1, 2, \ldots, h_{k\delta})$ be an $\mathfrak{r}_k$-adic root of $f(x, \zeta)$,

$$\prod_{k=1}^{\sigma} |\Re_i|_{\mathfrak{r}_k}^{h_k} \leq \prod_{k=1}^{\sigma} \prod_{\delta=1}^{G(\mathfrak{r}_k)} \prod_{\tau=1}^{h_{k\delta}} \left\{\max\left(1, |\eta_{k\delta\tau}|_{\mathfrak{r}_k}^{q+r+s}\right) \max\left(|\lambda_1 - \eta_{k\delta\tau}|_{\mathfrak{r}_k}^{r-i}, |\lambda_2 - \eta_{k\delta\tau}|_{\mathfrak{r}_k}\right)\right\},$$

where $h_{k\delta}$ $(k = 1, 2, \ldots, \sigma;\ \delta = 1, 2, \ldots, G(\mathfrak{r}_k))$ is a natural number not greater than h^2 and such that

$$\sum_{\delta=1}^{G(\mathfrak{r}_k)} h_{k\delta} = h_k.$$

Then by Lemma 5 (b),

$$(28) \qquad \prod_{k=1}^{\sigma} |\Re_i|_{\mathfrak{r}_k}^{h_k} \leq \left(4^{n^2(2n+m)} a^{n^2} \varkappa^{n(n-1)}\right)^{(q+r+s)h} \cdot \cdot \prod_{k=1}^{\sigma} \prod_{\delta=1}^{G(\mathfrak{r}_k)} \prod_{\tau=1}^{h_{k\delta}} \max\left(|\lambda_1 - \eta_{k\delta\tau}|_{\mathfrak{r}_k}^{r-i}, |\lambda_2 - \eta_{k\delta\tau}|_{\mathfrak{r}_k}\right),$$

provided

$$(29) \qquad |\lambda_1 - \eta_{k\delta\tau}|_{\mathfrak{r}_k} \leq 1, \qquad |\lambda_2 - \eta_{k\delta\tau}|_{\mathfrak{r}_k} \leq 1$$

for $k = 1, 2, \ldots, \sigma;\ \delta = 1, 2, \ldots, G(\mathfrak{r}_k);\ \tau = 1, 2, \ldots, h_{k\delta}$.

From (21), (26), (27), (28) and (29), it follows that

$$\prod_{=1}^{\varrho} \prod_{\gamma=1}^{G(\mathfrak{q}_j)} \max\left(|\lambda_1 - \xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}}^{r-i}, |\lambda_2 - \xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}}\right) \prod_{k=1}^{\sigma} \prod_{\delta=1}^{G(\mathfrak{r}_k)} \prod_{\tau=1}^{h_{k\delta}} \max\left(|\lambda_1 - \eta_{k\delta\tau}|_{\mathfrak{r}_k}^{r-i}, |\lambda_2 - \eta_{k\delta\tau}|_{\mathfrak{r}_k}\right)$$

$$> \left(A_1^{q+r-i} A_2^{s}\right)^{-h} \left\{\left(2^{3n} a \varkappa^{n-1}\right)^{n(q+r+1)\frac{m}{\varepsilon}} 2^{n(q+r)} 8^{n(q+r+m+n-2)} \varkappa^{n-1}\right\}^{-h^2} \cdot$$

$$\cdot \left\{2^n \left(2^{3n} a \varkappa^{n-1}\right)^{n(q+r+1)\frac{m}{\varepsilon}} 2^{n(n-m+1)} 3^{n(q+r+m-1)} \varkappa^{n(n-1)} \left(4^{n(2n+m)} a^n \varkappa^{n-1}\right)^{n(q+r+m-1)}\right\}^{-h^2} \cdot$$

$$\cdot \left(4^{n(2n+m)} a^n \varkappa^{n-1}\right)^{-n(q+r+m-1)h^2}$$

$$= \left(A_1^{q+r-i} A_2^{s}\right)^{-h} T^{-h^2},$$

say, provided

$$(27) \qquad |\lambda_1 - \xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}} \leq 1, \qquad |\lambda_2 - \xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}} \leq 1$$

for $j = 1, 2, \ldots, \varrho$ and $\gamma = 1$, $G(q_j)$, and

$$|\lambda_1 - \eta_{k\delta\tau}|_{r_k} \leq 1, \qquad |\lambda_2 - \eta_{k\delta\tau}|_{r_k} \leq 1 \tag{29}$$

for $k = 1, 2, \ldots, \sigma$; $\delta = 1, 2, \ldots, G(r_k)$; $\tau = 1, 2, \ldots, h_{k\delta}$.

Now

$$T \leq a^{\frac{2mn}{\varepsilon}(q+r+1)+2n^2(q+r+m-1)} \varkappa^{\frac{2mn(n-1)}{\varepsilon}(q+r+1)+(n-1)(n+1)+2n(n-1)(q+r+m-1)} \cdot$$

$$\cdot 2^{(6n^2m/\varepsilon)(q+r+1)+(q+r)n+(n-m+2)n+4n^2(2n+m)(q+r+m-1)+3n(q+r+m+n-2)+2n(q+r+m-1)}$$

$$= a^{I_1} \varkappa^{I_2} 2^{I_3},$$

say. Then, since $m \geq 2$, $n \geq 1$ and $\varepsilon \leq \frac{1}{2}$,

$$I_1 < 2(q+r+m)\left(\frac{mn}{\varepsilon} + n^2\right) < 2n^2(q+r+m)\left(2n + \frac{m}{\varepsilon}\right),$$

$$I_2 < 2(q+r+m)\left(\frac{mn(n-1)}{\varepsilon} + n(n-1) + n(n-1)\right)$$

$$< 2n^2(q+r+m)\left(2n + \frac{m}{\varepsilon}\right),$$

$$I_3 < (q+r+m)\left(\frac{6n^2m}{\varepsilon} + n + n^2 + 4n^2(2n+m) + 3n + 3n(n-2) + 2n\right)$$

$$= 2(q+r+m)\left(\frac{3n^2m}{\varepsilon} + 2n^2(2n+m+1)\right)$$

$$= 2(q+r+m)n^2\left(\frac{3m}{\varepsilon} + 4n + 2m + 2\right)$$

$$< 2(q+r+m)n^2\left(\frac{4m}{\varepsilon} + 8n\right)$$

$$= 2n^2(q+r+m)\left(2n + \frac{m}{\varepsilon}\right)4.$$

Hence

$$T < (16a\varkappa)^{2n^2(q+r+m)\left(2n+\frac{m}{\varepsilon}\right)},$$

and so, if

$$E(\varrho, \sigma) = (16a\varkappa)^{2n^2(q+r+m)\left(2n+\frac{m}{\varepsilon}\right)h^2} (A_1^{q+r-i} A_2^s)^h \cdot$$

$$\prod_{j=1}^{\varrho} \prod_{\gamma=1}^{G(q_j)} \max\left(|\lambda_1 - \xi_{j\gamma}|_{q_{j\gamma}}^{r-i}, |\lambda_2 - \xi_{j\gamma}|_{q_{j\gamma}}\right) \prod_{k=1}^{\sigma} \prod_{\delta=1}^{G(r_k)} \prod_{\tau=1}^{h_{k\delta}} \max\left(|\lambda_1 - \eta_{k\delta\tau}|_{r_k}^{r-i}, |\lambda_2 - \eta_{k\delta\tau}|_{r_k}\right),$$

then

$$E(\varrho, \sigma) > 1, \tag{30}$$

provided

$$|\lambda_1 - \xi_{j\gamma}|_{\mathfrak{q}_j} \leq 1, \qquad |\lambda_2 - \xi_{j\gamma}|_{\mathfrak{q}_j} \leq 1 \tag{27}$$

for $j = 1, 2, \ldots, \varrho$ and $\gamma = 1, G(\mathfrak{q}_j)$, and

$$|\lambda_1 - \eta_{k\delta\tau}|_{\mathfrak{r}_k} \leq 1, \qquad |\lambda_2 - \eta_{k\delta\tau}|_{\mathfrak{r}_k} \leq 1 \tag{29}$$

for $k = 1, 2, \ldots, \sigma$; $\delta = 1, 2, \ldots, G(\mathfrak{r}_k)$; $\tau = 1, 2, \ldots, h_{k\delta}$.

15. Let c and Θ be positive numbers such that

$$c \geq 1, \quad \beta = \frac{m}{s+1} + s + \Theta < m + 1,$$

and let $\Gamma_{j\gamma}$ $(j = 1, 2, \ldots, \varrho;\ \gamma = 1,\ G(\mathfrak{q}_j))$ and $\Gamma_{k\delta\tau}$ $(k = 1, 2, \ldots, \sigma;\ \delta = 1, 2, \ldots, G(\mathfrak{r}_k);\ \tau = 1, 2, \ldots, h_{k\delta})$ form a system of $t = \sum_{j=1}^{\varrho} G(\mathfrak{q}_j) + \sum_{k=1}^{\sigma} \sum_{\delta=1}^{G(\mathfrak{r}_k)} h_{k\delta}$ positive numbers with sum unity.

We now impose the following conditions on ε, r, λ_1 and λ_2:

1) $$0 < \varepsilon \leq \tfrac{1}{2}, \qquad \varepsilon < \frac{\Theta}{\beta};$$

2) $$r \geq \frac{2m^3}{\varepsilon} \geq \frac{s+1}{s}\,\frac{m^3}{\varepsilon};$$

3) $$A_1 \geq (16\,a\,z)^{2n^2 \left(\frac{m}{s+1} + \varepsilon\right)\left(2n + \frac{m}{\varepsilon}\right) h \Big/ (\Theta - \beta\varepsilon)}\, c^{\left(1 - \frac{s}{\beta} + \varepsilon\right) \Big/ (\Theta - \beta\varepsilon)\, h} = C, \text{ say};$$

4) $$(c^{-\frac{1}{h\beta}} A_1)^r \leq A_2 < (c^{-\frac{1}{h\beta}} A_1)^{r+1};$$

5) $$|\lambda_1 - \xi_{j\gamma}|_{\mathfrak{q}_j} \leq (c\,A_1^{-h\beta})^{\Gamma_{j\gamma}}, \qquad |\lambda_2 - \xi_{j\gamma}|_{\mathfrak{q}_j} \leq (c\,A_2^{-h\beta})^{\Gamma_{j\gamma}}$$
for $j = 1, 2, \ldots, \varrho$ and $\gamma = 1, G(\mathfrak{q}_j)$;

6) $$|\lambda_1 - \eta_{k\delta\tau}|_{\mathfrak{r}_k} \leq (c\,A_1^{-h\beta})^{\Gamma_{k\delta\tau}}, \qquad |\lambda_2 - \eta_{k\delta\tau}|_{\mathfrak{r}_k} \leq (c\,A_2^{-h\beta})^{\Gamma_{k\delta\tau}}$$
for $k = 1, 2, \ldots, \sigma$; $\delta = 1, 2, \ldots, G(\mathfrak{r}_k)$; $\tau = 1, 2, \ldots, h_{k\delta}$;

7) λ_1 and λ_2 are not roots of $f(x, \zeta)$.

The conditions already imposed are:

A) $$0 < \varepsilon \leq \tfrac{1}{2}, \qquad r \geq 2m^2;$$

B) $$|\lambda_1 - \xi_{j\gamma}|_{q_j\gamma} \leq 1, \qquad |\lambda_2 - \xi_{j\gamma}|_{q_j\gamma} \leq 1$$
for $j = 1, 2, \ldots, \varrho$ and $\gamma = 1, G(\mathfrak{q}_j)$;

C) $$|\lambda_1 - \eta_{k\delta\tau}|_{\mathfrak{r}_k} \leq 1, \qquad |\lambda_2 - \eta_{k\delta\tau}|_{\mathfrak{r}_k} \leq 1$$
for $k = 1, 2, \ldots, \sigma$; $\delta = 1, 2, \ldots, G(\mathfrak{r}_k)$; $\tau = 1, 2, \ldots, h_{k\delta}$;

D) λ_1 and λ_2 are not roots of $f(x, \zeta)$.

These conditions are contained in the new ones. A) follows immediately from 1) and 2), and D) is identical with 7). B) and C) follow from 1), 2), 3), 4), 5) and 6). For by 1), $\Theta - \beta\varepsilon > 0$, and so, by 3) and since $c \geq 1$,

$$\Lambda_1 > c^{\left(1-\frac{s}{\beta}+\varepsilon\right)/(\Theta-\beta\varepsilon)h} \geq c^{\frac{\left(1-\frac{s}{\beta}\right)}{h\Theta}}.$$

But

$$\frac{1-\frac{s}{\beta}}{\Theta} = \frac{\beta(s+1)-s(s+1)}{\Theta\beta(s+1)} = \frac{m+\Theta(s+1)}{\Theta\beta(s+1)} = \frac{m}{\Theta\beta(s+1)} + \frac{1}{\beta},$$

and so

$$\Lambda_1 > c^{\frac{1}{h}\left(\frac{m}{\Theta\beta(s+1)}+\frac{1}{\beta}\right)} \geq c^{\frac{1}{h\beta}},$$

i. e.,

$$c\Lambda_1^{-h\beta} < 1.$$

Also, by 4),

$$\Lambda_2 \geq c^{-\frac{r}{h\beta}}\Lambda_1^r > c^{\left\{-\frac{r}{\beta}+r\left(\frac{m}{\Theta\beta(s+1)}+\frac{1}{\beta}\right)\right\}\frac{1}{h}} = c^{r\left(\frac{m}{\Theta\beta(s+1)}\right)\frac{1}{h}}$$
$$\geq c^{\frac{2m^3}{\Theta\beta\varepsilon h}} \qquad \text{(by 2))}$$
$$\geq c^{\frac{1}{h\beta}},$$

since $\Theta < m$, as $\beta < m + 1$. Thus

$$c\Lambda_1^{-h\beta} < 1, \qquad c\Lambda_2^{-h\beta} < 1,$$

and by 5) and 6), B) and C) are satisfied.

It therefore follows, by the result obtained in **14**, that the inequality

(30) $$E(\varrho, \sigma) > 1$$

holds when the conditions 1) to 7) are satisfied.

16. When these conditions hold, by 5) and 6) it is clear that

$$E(\varrho, \sigma) \leq \max(E_1, E_2),$$

where

$$E_1 = (16\,a\varkappa)^{2n^2(q+r+m)\left(2n+\frac{m}{\varepsilon}\right)h^2} (\Lambda_1^{q+r-i-\beta(r-i)}\,\Lambda_2^{s})^h\, c^{r-i},$$

$$E_2 = (16\,a\varkappa)^{2n^2(q+r+m)\left(2n+\frac{m}{\varepsilon}\right)h^2} (\Lambda_1^{q+r-i}\,\Lambda_2^{s-\beta})^h\, c,$$

since

$$\prod_{j=1}^{\varrho}\prod_{\gamma=1}^{G(q_j)} \max\left(|\lambda_1-\xi_{j\gamma}|_{q_j\gamma}^{r-i},\ |\lambda_2-\xi_{j\gamma}|_{q_j\gamma}\right) \prod_{k=1}^{\sigma}\prod_{\delta=1}^{G(r_k)}\prod_{\tau=1}^{h_{k\delta}} \max\left(|\lambda_1-\eta_{k\delta\tau}|_{r_k}^{r-i},\ |\lambda_2-\eta_{k\delta\tau}|_{r_k}\right)$$

$$\leq \prod_{j=1}^{\varrho}\prod_{\gamma=1}^{G(q_j)} \max\left(c^{r-i}\,\Lambda_1^{-h\beta(r-i)},\ c\,\Lambda_2^{-h\beta}\right)^{\Gamma_{j\gamma}} \prod_{k=1}^{\sigma}\prod_{\delta=1}^{G(r_k)}\prod_{\tau=1}^{h_{k\delta}} \max\left(c^{r-i}\,\Lambda_1^{-h\beta(r-i)},\ c\,\Lambda_2^{-h\beta}\right)^{\Gamma_{k\delta\tau}}$$

$$= \max\left(c^{r-i}\,\Lambda_1^{-h\beta(r-i)},\ c\,\Lambda_2^{-h\beta}\right),$$

the sum of the Γs being 1.

Now

$$E_1 \leq (16\,a\varkappa)^{2n^2(q+r+m)\left(2n+\frac{m}{\varepsilon}\right)h^2}\,\Lambda_1^{he_1}\,c^{hf_1}, \qquad E_2 \leq (16\,a\varkappa)^{2n^2(q+r+m)\left(2n+\frac{m}{\varepsilon}\right)h^2}\,\Lambda_1^{he_2}\,c^{hf_2},$$

where, by 4),

$$e_1 = q + r - i - \beta(r-i) + s(r+1) = q + r - i - (\beta - s)\,r + i\beta + s,$$

$$e_2 = q + r - i + (s-\beta)\,r = q + r - i - (\beta - s)\,r < e_1,$$

$$f_1 = r - i - \frac{s}{\beta}(r+1) < \left(1 - \frac{s}{\beta}\right)r + 1,$$

$$f_2 = 1 - \left(\frac{s-\beta}{\beta}\right)r = \left(1 - \frac{s}{\beta}\right)r + 1.$$

By the inequalities of Lemma 3,

$$q + r \leq \left(\frac{m+\varepsilon}{s+1}\right)r, \qquad i \leq \varepsilon r + m^2 - 1,$$

and so

$$\max(e_1, e_2) \leq \left(\frac{m+\varepsilon}{s+1}\right)r - (\beta - s)\,r + (\varepsilon r + m^2 - 1)(\beta - 1) + s.$$

But

$$\beta < m + 1, \qquad s \leq m - 1, \qquad r \geq \frac{s+1}{s}\,\frac{m^3}{\varepsilon},$$

so that

$$(m^2-1)(\beta-1) + s < (m^2-1)\,m + (m-1) < m^3 \leq \frac{\varepsilon s r}{s+1}.$$

Also, by the definition of β,

$$\left(\frac{m+\varepsilon}{s+1}\right) r - (\beta - s) r + \varepsilon r(\beta - 1) \leq -\left(\Theta - \left(\beta - \frac{s}{s+1}\right)\varepsilon\right) r.$$

Hence

$$\max(e_1, e_2) < -(\Theta - \beta\varepsilon) r.$$

Further, since $\varepsilon r \geq 2m^3 > 1$,

$$\max(f_1, f_2) < \left(1 - \frac{s}{\beta} + \varepsilon\right) r.$$

Thus

$$\max(E_1, E_2) \leq (16 a \varkappa)^{2n^2(q+r+m)\left(2n+\frac{m}{\varepsilon}\right)h^2} A_1^{-(\Theta-\beta\varepsilon) r h} c^{\left(1-\frac{s}{\beta}+\varepsilon\right) r},$$

and by 3), and since $(q + r) \leq \left(\frac{m+\varepsilon}{s+1}\right) r$ and $m < m^3 \leq \frac{s}{s+1}\varepsilon r$,

$$\max(E_1, E_2) < (16 a \varkappa)^{2n^2\left(\frac{m}{s+1}+\varepsilon\right) r\left(2n+\frac{m}{\varepsilon}\right)h^2} \cdot (16 a \varkappa)^{-2n^2\left(\frac{m}{s+1}+\varepsilon\right)\left(2n+\frac{m}{\varepsilon}\right)h^2 r} c^{-\left(1-\frac{s}{\beta}+\varepsilon\right) r} \cdot c^{\left(1-\frac{s}{\beta}+\varepsilon\right) r}$$

$$= 1,$$

i. e.,

$$E(\varrho, \sigma) < 1,$$

which contradicts (30).

17. It therefore follows that *the only algebraic numbers λ of degree h (or some divisor of h) over $\mathfrak{K}$, lying in the perfect $\mathfrak{r}_1$-adic, $\mathfrak{r}_2$-adic, . . ., $\mathfrak{r}_\sigma$-adic extensions of $\mathfrak{K}$, which can possibly satisfy all the inequalities*

$$|\lambda - \xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}} \leq (c A^{-h\beta})^{\Gamma_{j\gamma}} \tag{31}$$

for $j = 1, 2, \ldots, \varrho$ and $\gamma = 1, G(\mathfrak{q}_j)$, and

$$|\lambda - \eta_{k\delta\tau}|_{\mathfrak{r}_k} \leq (c A^{-h\beta})^{\Gamma_{k\delta\tau}} \tag{32}$$

for $k = 1, 2, \ldots, \sigma$; $\delta = 1, 2, \ldots, G(\mathfrak{r}_k)$; $\tau = 1, 2, \ldots, h_{k\delta}$, A being the number $|\Upsilon(x; \lambda, \mathfrak{P}, hn)|$, are such that:

either (a) $A < C = (16 a \varkappa)^{2n^2\left(\frac{m}{s+1}+\varepsilon\right)\left(2n+\frac{m}{\varepsilon}\right)h/(\Theta-\beta\varepsilon)} c^{\left(1-\frac{s}{\beta}+\varepsilon\right)/(\Theta-\beta\varepsilon)h}$;

or (b) *if A_1 be the minimum value of A not less than C for which a corresponding λ is a solution of the inequalities, then any other values of A not less than C for which a corresponding λ is a solution satisfy the inequalities*

$$A_1 \leq A < (c^{-\frac{1}{h\beta}} A_1)^{\left[\frac{2m^3}{\varepsilon}\right]+1};$$

or (c) *λ is a root of $f(x, \zeta)$.*

Clearly, the total number of solutions of the inequalities (31) and (32) is finite. We seek now for an upper bound for the number of solutions in terms of $\varkappa$, n, a, m, h, c and β.

18. Let now λ_1 and λ_2 be two different algebraic numbers of degrees h_1 and h_2 dividing h over $\mathfrak{R}$, such that $\Delta_2 \geq \Delta_1$, and let both be solutions of the inequalities (31) and (32). Then, since the sum of the Γ's is 1,

$$(33)\quad X = \prod_{j=1}^{\varrho} \prod_{\gamma=1}^{G(\mathfrak{q}_j)} \max\left(|\lambda_1 - \xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}},\ |\lambda_2 - \xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}}\right) \cdot \prod_{k=1}^{\sigma} \prod_{\delta=1}^{G(\mathfrak{r}_k)} \prod_{\tau=1}^{h_{k\delta}} \max\left(|\lambda_1 - \eta_{k\delta\tau}|_{\mathfrak{r}_k},\ |\lambda_2 - \eta_{k\delta\tau}|_{\mathfrak{r}_k}\right) \leq c\,\Delta_1^{-h\beta}.$$

Also

$$2 \max\left(|\lambda_1 - \xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}},\ |\lambda_2 - \xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}}\right) \geq |(\lambda_1 - \xi_{j\gamma}) - (\lambda_2 - \xi_{j\gamma})|_{\mathfrak{q}_{j\gamma}} = |\lambda_1 - \lambda_2|_{\mathfrak{q}_{j\gamma}}$$

$$(j = 1, 2, \ldots, \varrho;\ \gamma = 1, G(\mathfrak{q}_j)),$$

and

$$\max\left(|\lambda_1 - \eta_{k\delta\tau}|_{\mathfrak{r}_k},\ |\lambda_2 - \eta_{k\delta\tau}|_{\mathfrak{r}_k}\right) \geq |(\lambda_1 - \eta_{k\delta\tau}) - (\lambda_2 - \eta_{k\delta\tau})|_{\mathfrak{q}_{j\gamma}} = |\lambda_1 - \lambda_2|_{\mathfrak{r}_k}$$

$$(k = 1, 2, \ldots, \sigma;\ \delta = 1, 2, \ldots, G(\mathfrak{r}_k);\ \tau = 1, 2, \ldots, h_{k\delta}),$$

so that

$$X \geq \frac{1}{2^{\sum_{j=1}^{\varrho} G(\mathfrak{q}_j)}} \prod_{j=1}^{\varrho} \prod_{\gamma=1}^{G(\mathfrak{q}_j)} |\lambda_1 - \lambda_2|_{\mathfrak{q}_{j\gamma}} \prod_{k=1}^{\sigma} |\lambda_1 - \lambda_2|_{\mathfrak{r}_k}^{h_k}$$

$$(34)\qquad \geq \frac{1}{2^{\sum_{j=1}^{\varrho} G(\mathfrak{q}_j)}} \{|L_{10}\, L_{20}|^h \prod_{\nu}{}' |(\lambda_1 - \lambda_2)^{(\nu)}|\}^{-1},$$

by the fundamental inequality (9) (taking $\omega' = \lambda_1 - \lambda_2$, $n' = h^2$), since $(L_{10}\, L_{20})^h$ is a multiple of the coefficient of the highest power of x in the polynomial $\Upsilon(x; \lambda_1 - \lambda_2, \mathfrak{P}, h^2 n)$. $\left(\prod_{\nu}{}' |(\lambda_1 - \lambda_2)^{(\nu)}|\right.$ is the product

$$\prod_{\nu=1}^{h^2 n} |(\lambda_1 - \lambda_2)^{(\nu)}| \left(\prod_{j=1}^{\varrho} \prod_{\gamma=1}^{G(\mathfrak{q}_j)} |\lambda_1 - \lambda_2|_{\mathfrak{q}_{j\gamma}}\right)^{-1},$$

where the $(\lambda_1 - \lambda_2)^{(\nu)}$ $(\nu = 1, 2, \ldots, h^2 n)$ are the conjugate values of $\lambda_1 - \lambda_2$ in the polynomial $\Upsilon(x; \lambda_1 - \lambda_2, \mathfrak{P}, h^2 n)$.)

Now

$$\prod_{\nu}{}' |(\lambda_1 - \lambda_2)^{(\nu)}| \leq \prod_{\nu}{}' (2 \max (|\lambda_1^{(\nu)}|, |\lambda_2^{(\nu)}|))$$

$$\leq 2^{h^2 n - \sum_{j=1}^{\varrho} G(q_j)} \left\{ \prod_{\nu_1=1}^{hn} \max (1, |\lambda_1^{(\nu_1)}|) \prod_{\nu_2=1}^{hn} \max (1, |\lambda_2^{(\nu_2)}|) \right\}^h,$$

where $\lambda_1^{(\nu_1)} (\nu_1 = 1, 2, \ldots, hn)$ and $\lambda_2^{(\nu_2)} (\nu_2 = 1, 2, \ldots, hn)$ are the conjugate values of λ_1 and λ_2 in the polynomials $\Upsilon(x; \lambda_1, \mathfrak{P}, hn)$ and $\Upsilon(x; \lambda_2, \mathfrak{P}, hn)$, and $\lambda_1^{(\nu)}$ and $\lambda_2^{(\nu)}$ denote the values of λ_1 and λ_2 corresponding to $(\lambda_1 - \lambda_2)^{(\nu)}$. Thus, by Lemma 4 (a),

$$\prod_{\nu}{}' |(\lambda_1 - \lambda_2)^{(\nu)}| \leq 2^{h^2 n - \sum_{j=1}^{\varrho} G(q_j)} 16^{h^2 n} \left(\frac{A_1}{|L_{10}|} \frac{A_2}{|L_{20}|} \right)^h.$$

Hence and from (34),

$$X \geq (2^{h^2 n} 16^{h^2 n} A_1^h A_2^h)^{-1},$$

and so by (33),

$$c A_1^{-h\beta} \geq (32^{h^2 n} A_1^h A_2^h)^{-1},$$

i. e.,

$$A_2 \geq (32^{hn} c^{\frac{1}{h}})^{-1} A_1^{\beta - 1}. \tag{35}$$

19. We divide the solutions of (31) and (32) into three groups, J_1, J_2 and J_3, as follows:

J_1: If A_1 be the minimum value of A not less than C for which a corresponding λ is a solution, J_1 contains those solutions for which

$$C \leq A_1 \leq A < (c^{-\frac{1}{h\beta}} A_1)^{\left[\frac{2m^2}{\varepsilon}\right]+1};$$

J_2: contains those solutions for which

$$(64^{hn} c^{\frac{1}{h}})^{\frac{1}{\beta-2}} \leq A < C;$$

J_3: contains those solutions for which either

(a) $$A < (64^{hn} c^{\frac{1}{h}})^{\frac{1}{\beta-2}},$$

or (b) $$f(\lambda, \zeta) = 0$$

(except that any roots of $f(x, \zeta)$ included in J_1, J_2 and J_3 (a) are excluded from J_3 (b)).

In J_1, since

$$C > (16^{2hn} c^{\frac{1}{h}})^{\frac{1-\frac{s}{\beta}+\varepsilon}{\Theta-\beta\varepsilon}},$$

it follows that

$$\Lambda_1 > (32^{hn} c^{\frac{1}{h}})^{\frac{1-\frac{s}{\beta}+\varepsilon}{\Theta-\beta\varepsilon}}.$$

Put

$$\Theta - \beta\varepsilon \leq \tfrac{1}{2}\left(1 - \frac{s}{\beta} + \varepsilon\right)(\beta - 2). \tag{36}$$

Then

$$\Lambda_1 > (32^{hn} c^{\frac{1}{h}})^{\frac{2}{\beta-2}}. \tag{37}$$

Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_\mu$ be the values of Λ not less than C giving the μ different solutions in J_1, and let $\Lambda_1 \leq \Lambda_2 \leq \cdots \leq \Lambda_\mu$. Then

$$(32^{hn} c^{\frac{1}{h}})^{\frac{2}{\beta-2}} < \Lambda_1 \leq \Lambda_2 \leq \cdots \leq \Lambda_\mu < (c^{-\frac{1}{h\beta}} \Lambda_1)^{\left[\frac{2m^3}{\varepsilon}\right]+1} \leq \Lambda_1^{\left[\frac{2m^3}{\varepsilon}\right]+1}.$$

But by (35),

$$\begin{aligned}
\Lambda_\mu &\geq (32^{hn} c^{\frac{1}{h}})^{-1} \Lambda_{\mu-1}^{\beta-1} \\
&\geq (32^{hn} c^{\frac{1}{h}})^{-(1+(\beta-1))} \Lambda_{\mu-2}^{(\beta-1)^2} \\
&\geq \cdots \geq (32^{hn} c^{\frac{1}{h}})^{-(1+(\beta-1)+(\beta-1)^2+\cdots+(\beta-1)^{\mu-2})} \Lambda_1^{(\beta-1)^{\mu-1}} \\
&> \{(32^{hn} c^{\frac{1}{h}})^{-\frac{1}{\beta-2}} \Lambda_1\}^{(\beta-1)^{\mu-1}}.
\end{aligned}$$

Thus

$$\Lambda_1^{\left[\frac{2m^3}{\varepsilon}\right]+1} > (32^{hn} c^{\frac{1}{h}})^{-\frac{(\beta-1)^{\mu-1}}{\beta-2}} \Lambda^{(\beta-1)^{\mu-1}},$$

and by (37),

$$\Lambda_1^{\left[\frac{2m^3}{\varepsilon}\right]+1} > \Lambda_1^{-\frac{1}{2}(\beta-1)^{\mu-1}} \Lambda_1^{(\beta-1)^{\mu-1}} = \Lambda_1^{\frac{1}{2}(\beta-1)^{\mu-1}}.$$

Hence

$$\log\left(2\left[\frac{2m^3}{\varepsilon}\right]+2\right) > (\mu-1)\log(\beta-1),$$

and so

$$\mu < 1 + \frac{\log \frac{5m^3}{e}}{\log(\beta-1)}. \tag{38}$$

Let $A_{\mu+1}, A_{\mu+2}, \ldots, A_{\mu+\nu}$ be the As corresponding to the different solutions of (31) and (32) in J_2, and let $A_{\mu+1} \leq A_{\mu+2} \leq \cdots \leq A_{\mu+\nu}$. Then

$$(64^{hn} c^{h})^{\frac{1}{\beta-2}} \leq A_{\mu+1} \leq A_{\mu+2} \leq \cdots \leq A_{\mu+\nu} < C.$$

As before,

$$A_{\mu+\nu} > \{(32^{hn} c^{h})^{-\frac{1}{\beta-2}} A_{\mu+1}\}^{(\beta-1)^{\nu-1}},$$

and so

$$C > 2^{\frac{hn}{\beta-2}(\beta-1)^{\nu-1}}.$$

Hence

$$\log \log C > \log \left(\frac{hn}{\beta-2} \log 2\right) + (\nu - 1) \log (\beta - 1),$$

so that

$$\nu < 1 + \frac{\left(\log \frac{\log C}{\frac{hn}{\beta-2} \log 2}\right)}{\log (\beta - 1)}. \tag{39}$$

20. From (38) and (39) we now obtain bounds for μ and ν involving only $\varkappa$, n, a, m, h, c and β.

Siegel[1] has shown that if

$$\alpha = \min_{s=1, 2, \ldots, m-1} \left(\frac{m}{s+1} + s\right),$$

then

$$2\sqrt{m} - 1 \leq \alpha \leq \sqrt{4m+1} - 1, \tag{40}$$

and the value s' of s giving α is

$$s' = \left[\frac{\sqrt{4m+1} - 1}{2}\right]. \tag{41}$$

We choose β, Θ and ε so that

$$\beta = \alpha + \Theta, \quad 0 < \Theta \leq \frac{1}{4m}, \quad \Theta - \beta\varepsilon = \tfrac{1}{4}\Theta > 0. \tag{42}$$

Then

$$\varepsilon = \frac{\frac{3}{4}\Theta}{\alpha + \Theta} \leq \frac{\frac{3}{16m}}{\alpha} < \tfrac{1}{2},$$

[1] See note 1, p. 1, p. 191.

and since α is not greater than m,

$$\beta < m + 1,$$

which is consistent with the original definitions of ε and β. Further, the inequality (36) is satisfied for $s = s'$. For, if $m \geq 2$,

$$\tfrac{1}{2}\left(1 - \frac{s'}{\beta} + \varepsilon\right)(\beta - 2) > \tfrac{1}{2}\left(1 - \frac{s'}{\beta}\right)(\beta - 2) = \frac{(\beta - s')(\beta - 2)}{2\beta} \geq \frac{(\beta - 2)(1 + \Theta)}{2(\alpha + \Theta)}.$$

Thus, if $m = 2$, so that $\alpha = 2$, it follows that

$$\tfrac{1}{2}\left(1 - \frac{s'}{\beta} + \varepsilon\right)(\beta - 2) > \frac{\Theta(1 + \Theta)}{2(2 + \Theta)} > \frac{\Theta}{4} = \Theta - \beta\varepsilon,$$

by (42); and if $m > 2$, so that by (40) and since, if $m = 3$, $\alpha = 2\tfrac{1}{2}$,

$$\beta - 2 > \tfrac{1}{2},$$

it follows, by (40) and (41), that

$$\tfrac{1}{2}\left(1 - \frac{s'}{\beta} + \varepsilon\right)(\beta - 2) > \frac{\tfrac{1}{2}(1 + \Theta)}{2(\alpha + \Theta)} > \frac{1}{4\sqrt{4m + 1}} > \frac{1}{16m} \geq \tfrac{1}{4}\Theta = \Theta - \beta\varepsilon.$$

Since (36) is therefore satisfied for $s = s'$ when $m \geq 2$, the inequalities (38) and (39) for μ and ν are true when the conditions (42) hold.

Now from (42),

$$\log(\beta - 1) \geq \log(1 + \Theta) > 0. \tag{43}$$

Also

$$\varepsilon = \frac{\tfrac{3}{4}\Theta}{\alpha + \Theta} > \frac{\tfrac{3}{4}\Theta}{\sqrt{4m + 1}} > \frac{\tfrac{3}{4}\Theta}{3\sqrt{m}} = \frac{\Theta}{4\sqrt{m}} \geq \frac{\Theta^{3/2}}{2},$$

so that

$$\log\frac{5m^3}{\varepsilon} < \log(5m^3 \cdot 2\Theta^{-3/2}) < \log\left\{\frac{(4m)^3}{\Theta^{3/2}}\right\} \leq \log\frac{1}{\Theta^{3/2}} < 5\log\frac{1}{\Theta}.$$

Hence and from (43) and (38),

$$\mu < 1 + \frac{5\log\dfrac{1}{\Theta}}{\log(1 + \Theta)}. \tag{44}$$

Further, by (43),

$$\frac{\log\left(\dfrac{\log C}{\dfrac{hn}{\beta-2}\log 2}\right)}{\log(\beta-1)} \leq \frac{\log\log C-\log hn}{\log(1+\Theta)}+\frac{\log(\beta-2)}{\log(\beta-1)}+\frac{\log\dfrac{1}{\log 2}}{\log(1+\Theta)},$$

since $\log C > hn$, i. e., $\log\log C-\log hn>0$, and $\dfrac{1}{\log 2}>1$, i. e., $\log\dfrac{1}{\log 2}>0$. But

$$\frac{\log(\beta-2)}{\log(\beta-1)}<1.$$

Also

$$\frac{\log\dfrac{1}{\log 2}}{\log(1+\Theta)}<\frac{\log\frac{3}{2}}{\log(1+\Theta)},$$

since $\log 2>0.693>\frac{2}{3}$. Thus, by (39),

$$v<2+\frac{\log\frac{3}{2}}{\log(1+\Theta)}+\frac{\log\log C-\log hn}{\log(1+\Theta)}. \tag{45}$$

Now

$$C=(16\,a\,\varkappa)^{2n^2\left(\frac{m}{s+1}+\varepsilon\right)\left(2n+\frac{m}{\varepsilon}\right)h/(\Theta-\beta\varepsilon)}\,c^{\frac{1-\frac{s}{\beta}+\varepsilon}{h(\Theta-\beta\varepsilon)}}.$$

But, since $m\leq\dfrac{1}{4\Theta}$, $\varepsilon\geq\dfrac{\Theta^{5/2}}{2}$ and $\Theta-\beta\varepsilon=\frac{1}{4}\Theta$, it follows that

$$2n^2\left(\frac{m}{s+1}+\varepsilon\right)\left(2n+\frac{m}{\varepsilon}\right)h\Big/(\Theta-\beta\varepsilon)<hn^2(m+1)\left(2n+\frac{1}{2\Theta^{5/2}}\right)\Big/\tfrac{1}{4}\Theta$$

$$<hn^2\frac{3}{8\Theta}\cdot\frac{2n}{2\Theta^{5/2}}\cdot\frac{4}{\Theta}\ \text{(since } m\geq 2 \text{ and } 2\Theta^{5/2}<\tfrac{1}{2})$$

$$=hn^3\tfrac{3}{2}\Theta^{-9/2}.$$

Also,

$$\frac{1-\frac{s}{\beta}+\varepsilon}{h(\Theta-\beta\varepsilon)}<\frac{\frac{3}{2}}{\frac{1}{4}h\Theta}=\frac{6}{h\Theta}.$$

Thus

$$C<(16\,a\,\varkappa)^{3/2\,h\,n^3\Theta^{-9/2}}\,c^{\frac{6}{h\Theta}},$$

and so either

$$\text{(a)}\ C<(16\,a\,\varkappa)^{3\,h\,n^3\Theta^{-9/2}},\quad\text{or}\quad\text{(b)}\ C<c^{\frac{12}{h\Theta}}.$$

In case (a),

$$\log\log C < \log\log(16\,a\,\varkappa) + 3\log n + \log h + \tfrac{9}{2}\log\frac{1}{\Theta} + \log 3;$$

and in case (b),

$$\log\log C < \log\log c + \log\frac{1}{\Theta} + \log 12 - \log h.$$

Hence and from (45),

$$\nu < 2 + \max\left\{\frac{\log\log(16\,a\,\varkappa) + 2\log n + \tfrac{9}{2}\log\frac{1}{\Theta} + \log\tfrac{9}{2}}{\log(1+\Theta)}, \frac{\log\log c + \log\frac{1}{\Theta} + \log 18}{\log(1+\Theta)}\right\}.$$

Hence and from (44), and since $\log\frac{9}{2} < 2$ and $\log 18 < 3$,

$$(46)\quad \mu + \nu < 3 + \frac{1}{\log(1+\Theta)}\max\left\{\log\log(16\,a\,\varkappa) + \right.$$
$$\left. + 2\log n + 10\log\frac{1}{\Theta} + 2, \log\log c + 6\log\frac{1}{\Theta} + 3\right\}.$$

The number of different solutions in J_3 is clearly finite and is bounded by a number depending only on n, m, h, c and β, and not on a and $\varkappa$, and it is of interest to note that the total number of different solutions of the inequalities (31) and (32) is therefore of order $\log\log(16\,a\,\varkappa)$.

21. We have now proved the following lemma:

Lemma 6. *Let:*

$\mathfrak{K}$ — *be a finite algebraic field of degree* n (≥ 1) *over the rational number field* $\mathfrak{P}$;

ζ — *be an algebraic integer generating* $\mathfrak{K}$;

$\varphi(z)$ — *be the polynomial*

$$\varphi(z) = z^n + \varkappa_1 z^{n-1} + \varkappa_2 z^{n-2} + \cdots + \varkappa_n,$$

with rational integral coefficients and irreducible in $\mathfrak{P}$ *having* ζ *as a root*;

$\varkappa$ — *be the number* $|\varphi(z)|$;

$f(x, z)$ — *be a polynominal in* x *of degree* m (≥ 2);

$$f(x,z) = a_0(z)\,x^m + a_1(z)\,x^{m-1} + \cdots + a_m(z),$$

where $a_0(z)$ $(\not\equiv 0)$, $a_1(z), \ldots, a_m(z)$ *are polynomials in* z *with rational integral coefficients and of degree not greater than* $n-1$, *and* $f(x,\zeta)$ *has a non-zero discriminant* (N. B. $f(x,\zeta)$ *need not necessarily be irreducible in* $\mathfrak{K}$);

a *be the smallest natural number such that*

$$a_\nu(z) \ll a(1+z)^{n-1}$$

for $\nu = 0, 1, \ldots, m$;

$\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_\varrho$, *where* $0 \leq \varrho \leq r_1 + r_2$, *be* ϱ *of the* $r_1 + r_2$ *infinite prime ideals corresponding to the* r_1 *real and* r_2 *pairs of conjugate imaginary fields conjugate to* $\mathfrak{K}$;

$\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$, *where* $\sigma \geq 0$, *be* σ *different finite prime ideals of* $\mathfrak{K}$;

$G(\mathfrak{q}_j)$ $(j = 1, 2, \ldots, \varrho)$ *be a natural number not greater than* $g(\mathfrak{q}_j)$;

$G(\mathfrak{r}_k)$ $(k = 1, 2, \ldots, \sigma)$ *be a natural number not greater than* $g(\mathfrak{r}_k)$;

h *be a natural number*;

$h_{k\delta}$ $(k = 1, 2, \ldots, \sigma;\ \delta = 1, 2, \ldots, G(\mathfrak{r}_k))$ *be a natural number not greater than* h^2;

$\xi_{j\gamma}$ $(j = 1, 2, \ldots, \varrho;\ \gamma = 1, G(\mathfrak{q}_j))$ *be a real or complex root of* $f(x,\zeta)$;

$\eta_{k\delta\tau}$ $(k = 1, 2, \ldots, \sigma;\ \delta = 1, 2, \ldots, G(\mathfrak{r}_k);\ \tau = 1, 2, \ldots, h_{k\delta})$ *be an* $\mathfrak{r}_k$*-adic root of* $f(x,\zeta)$;

c *be a number not less than* 1;

α *be the number* $\min\limits_{s=1,2,\ldots,m-1}\left(\dfrac{m}{s+1}+s\right)$;

Θ *be a positive number not greater than* $\dfrac{1}{4m}$;

β *be the number* $\alpha + \Theta$;

$\Gamma_{j\gamma}, \Gamma_{k\delta\tau}$ $(j = 1, 2, \ldots, \varrho;\ \gamma = 1, G(\mathfrak{q}_j);\ k = 1, 2, \ldots, \sigma;\ \delta = 1, \ldots, G(\mathfrak{r}_k);$ $\tau = 1, 2, \ldots, h_{k\delta})$ *form a system of* $t = \sum\limits_{j=1}^{\varrho} G(\mathfrak{q}_j) + \sum\limits_{k=1}^{\sigma}\sum\limits_{\delta=1}^{G(\mathfrak{r}_k)} h_{k\delta}$ *positive numbers with sum* 1.

Then if λ *be any algebraic number of degree* h (*or any divisor of* h) *over* $\mathfrak{K}$; *lying in the perfect* $\mathfrak{r}_1$*-adic,* $\mathfrak{r}_2$*-adic,* $\ldots$, $\mathfrak{r}_\sigma$*-adic extensions of* $\mathfrak{K}$, *and if* Λ *be the number* $\overline{|\Upsilon(x;\ \lambda, \mathfrak{P}, hn)|}$, *the number of different numbers* λ, *which are neither roots of* $f(x,\zeta)$ *nor such that*

$$\Lambda < (64^{hn} c^{\frac{1}{h}})^{\frac{1}{\beta-2}},$$

and which satisfy the inequalities

$$|\lambda - \xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}} \leq (c\,\Lambda^{-h\beta})^{\Gamma_{j\gamma}} \tag{31}$$

for $j = 1, 2, \ldots, \varrho$ *and* $\gamma = 1, G(\mathfrak{q}_j)$, *and*

$$|\lambda - \eta_{k\delta\tau}|_{\mathfrak{r}_k} \leq (c\,\Lambda^{-h\beta})^{\Gamma_{k\delta\tau}} \tag{32}$$

for $k = 1, 2, \ldots, \sigma$; $\delta = 1, 2, \ldots, G(\mathfrak{r}_k)$; $\tau = 1, 2, \ldots, h_{k\delta}$, *is less than*

$$3 + \frac{1}{\log(1+\Theta)} \max\left(\log\log(16\,a\,\varkappa) + \right. \\ \left. + 2\log n + 10\log\frac{1}{\Theta} + 2, \log\log c + 6\log\frac{1}{\Theta} + 3\right). \tag{46}$$

22. We now require to find a bound for the number of different algebraic numbers λ of degree h (or any divisor of h) over $\mathfrak{R}$, lying in the perfect $\mathfrak{r}_1$-adic, $\mathfrak{r}_2$-adic, ..., $\mathfrak{r}_\sigma$-adic extensions of $\mathfrak{R}$, which satisfy the inequality

$$\prod_{j=1}^{\varrho}\prod_{\gamma=1}^{G(\mathfrak{q}_j)} \min\left(1, |\lambda - \xi_{j\gamma}|_{\mathfrak{q}_j\gamma}\right) \prod_{k=1}^{\sigma}\prod_{\delta=1}^{G(\mathfrak{r}_k)}\prod_{\tau=1}^{h_{k\delta}} \min\left(1, |\lambda - \eta_{k\delta\tau}|_{\mathfrak{r}_k}\right) \leq c\,\Lambda^{-h\beta^*}, \tag{47}$$

where β^* is any number greater than α.

Clearly, we can choose Θ so that $\beta < \beta^*$. Then

$$\beta^* = \beta(1+\theta),$$

where θ is a positive number. Thus, since $c \geq 1$,

$$c\,\Lambda^{-h\beta^*} = c^{-\frac{\beta^*-\beta}{\beta}}(c\,\Lambda^{-h\beta})^{\frac{\beta^*}{\beta}} \leq (c\,\Lambda^{-h\beta})^{1+\theta}. \tag{48}$$

Let now λ be any one of the solutions of (47). We exclude for the present solutions which are roots of $f(x, \zeta)$ and those for which $\Lambda < (64^{hn}\,c^{\frac{1}{h}})^{\frac{1}{\beta-2}}$. Thus, for the λ we consider,

$$\Lambda \geq (64^{hn}\,c^{\frac{1}{h}})^{\frac{1}{\beta-2}} > c^{\frac{1}{h\beta}},$$

and so

$$c\,\Lambda^{-h\beta^*} < c\,\Lambda^{-h\beta} < 1.$$

Hence and from (47) and from (48), there exists a system of

$$t = \sum_{j=1}^{\varrho} G(\mathfrak{q}_j) + \sum_{k=1}^{\sigma}\sum_{\delta=1}^{G(\mathfrak{r}_k)} h_{k\delta}$$

non-negative numbers $q_{j\gamma}$ $(j = 1, 2, \ldots, \varrho;\ \gamma = 1, G(\mathfrak{q}_j))$ and $q_{k\delta\tau}$ $(k = 1, 2, \ldots, \sigma;\ \delta = 1, 2, \ldots, G(\mathfrak{r}_k);\ \tau = 1, 2, \ldots, h_{k\delta})$, with a sum not less than 1, such that

$$\min\left(1, |\lambda - \xi_{j\gamma}|_{\mathfrak{q}_j}\right) = (c\,\Lambda^{-h\beta^*})^{q_{j\gamma}} \leq (c\,\Lambda^{-h\beta})^{(1+\theta)\,q_{j\gamma}} \tag{49}$$

for $j = 1, 2, \ldots, \varrho$ and $\gamma = 1, G(\mathfrak{q}_j)$, and

$$\min\left(1, |\lambda - \eta_{k\delta\tau}|_{\mathfrak{r}_k}\right) = (c\,\Lambda^{-h\beta^*})^{q_{k\delta\tau}} \leq (c\,\Lambda^{-h\beta})^{(1+\theta)\,q_{k\delta\tau}} \tag{50}$$

for $k = 1, 2, \ldots, \sigma;\ \delta = 1, 2, \ldots, G(\mathfrak{r}_k);\ \tau = 1, 2, \ldots, h_{k\delta}$.

But there exists a natural number T such that

$$\theta\,T \geq t,$$

so that if the system of q's coincides with a system of numbers $q_1, q_2, \ldots, q_t$, then for $l = 1, 2, \ldots, l$,

$$(1 + \theta)\,q_l = \frac{H_l}{T} + \chi_l, \tag{51}$$

where

$$H_l = [T(1 + \theta)\,q_l],$$

and the residue,

$$\chi_l = (1 + \theta)\,q_l - \frac{H_l}{T},$$

satisfies the inequalities

$$0 \leq \chi_l < \frac{1}{T},$$

so that

$$\sum_{l=1}^{t} \chi_l < \frac{t}{T} \leq \theta.$$

Since

$$\sum_{l=1}^{t} q_l \geq 1,$$

it follows that

$$\frac{\sum_{l=1}^{t} H_l}{T} = (1 + \theta) \sum_{l=1}^{t} q_l - \sum_{l=1}^{t} \chi_l \geq (1 + \theta) - \theta = 1,$$

i. e.,

$$\sum_{l=1}^{t} H_l \geq T. \tag{52}$$

Let the system of inequalities (49) and (50) be represented in the form:

$$\min\left(1, |\lambda - \mu_l|_{\mathfrak{p}_l}\right) = (c\,\Lambda^{-h\beta^*})^{q_l} \leq (c\,\Lambda^{-h\beta})^{(1+\theta)\,q_l} \qquad (l = 1, 2, \ldots, t).$$

Then by (51), for $l = 1, 2, \ldots, t$,

$$\min (1, |\lambda - \mu_l|_{p_l}) \leq (c A^{-h\beta})^{(1+\theta)\eta_l} \leq (c A^{-h\beta})^{\frac{H_l}{T}},$$

since $c A^{-h\beta} < 1$. Thus, by (52), we can choose t non-negative rational integers $\nu_1, \nu_2, \ldots, \nu_t$ such that $\nu_1 \leq H_1, \nu_2 \leq H_2, \ldots, \nu_t \leq H_t$ and with sum T, so that, for $l = 1, 2, \ldots, t$,

$$\min (1, |\lambda - \mu_l|_{p_l}) \leq (c A^{-h\beta})^{\frac{\nu_l}{T}}. \tag{53}$$

It follows that to every solution λ of (47) (except the excluded cases) there corresponds at least one system of non-negative rational integers $\nu_1, \nu_2, \ldots, \nu_t$ with sum T such that all the t inequalities (53) are satisfied. Consider any one such system, S, of ν's. The ν's of S cannot all be zero if t is positive, since their sum is T. Further, for any non-zero ν,

$$(c A^{-h\beta})^{\frac{\nu}{T}} < 1,$$

and so all of the t inequalities

$$|\lambda - \mu_l|_{p_l} \leq (c A^{-h\beta})^{\frac{\nu_l}{T}} \qquad (l = 1, 2, \ldots, t) \tag{54}$$

for which the corresponding ν's are not zero will be satisfied. Let these be t' in number. Now Lemma 6 holds if t is replaced by t', and the Γ's by the corresponding non-zero ν's, and so, by this lemma, the above t' inequalities have less than

$$3 + \frac{1}{\log (1+\Theta)} \max\left(\log\log (16 a z) + 2 \log n + 10 \log \frac{1}{\Theta} + 2, \log\log c + 6 \log \frac{1}{\Theta} + 3\right)$$

solutions for which λ is neither a root of $f(x, \zeta)$ nor such that $A < (64^{hn} c^{h})^{\frac{1}{\beta-2}}$. Clearly, the t inequalities (53) cannot have more than this number of solutions satisfying the conditions stated, for the system S of ν's, since every solution of the t inequalities (53) is a solution of the t' inequalities taken from (54). Further, there are at most

$$\binom{T + t - 1}{t - 1}$$

systems of ν's, this being the number of different solutions of the equality

$$\sum_{l=1}^{t} \nu_l = T$$

in non-negative rational integers $\nu_1, \nu_2, \ldots, \nu_t$. Also, every solution of the inequality (47) for which λ is neither a root of $f(x, \zeta)$ nor such that $A < (64^{hn} c^{\frac{1}{h}})^{\frac{1}{\beta-2}}$ is also a solution of the t inequalities (53) for at least one of these systems.

It therefore follows that the number $\mathfrak{N}$ of different solutions of the inequality (47) for which λ is neither a root of $f(x, \zeta)$ nor such that $A < (64^{hn} c^{\frac{1}{h}})^{\frac{1}{\beta-2}}$ is less than

$$\binom{T+t-1}{t-1}\left\{3 + \frac{1}{\log(1+\Theta)} \max\left(\log\log(16\,a\,\varkappa) + \right.\right.$$
$$\left.\left. + 2\log n + 10\log\frac{1}{\Theta} + 2, \log\log c + 6\log\frac{1}{\Theta} + 3\right)\right\}.$$

23. We can select T, if $t > 0$, so that

$$\frac{\beta}{\beta^* - \beta} t \leq T < \frac{\beta}{\beta^* - \beta} t + 1,$$

since we have only imposed on T the condition that $T \geq \frac{t}{\theta} = t\frac{\beta}{\beta^* - \beta}$. Then

$$\frac{\beta}{\beta^* - \beta} t \leq T + t - 1 < \frac{\beta^*}{\beta^* - \beta} t,$$

and since

$$\binom{T+t-1}{t-1} < \sum_{\nu=0}^{T+t-1} \binom{T+t-1}{\nu} = 2^{T+t-1} < 2^{\frac{\beta^*}{\beta^*-\beta} t},$$

it follows that

$$\mathfrak{N} < 2^{\frac{\beta^*}{\beta^*-\beta} t}\left\{3 + \frac{1}{\log(1+\Theta)} \max\left(\log\log(16\,a\,\varkappa) + \right.\right.$$
$$\left.\left. + 2\log n + 10\log\frac{1}{\Theta} + 2, \log\log c + 6\log\frac{1}{\Theta} + 3\right)\right\}.$$

For any sufficiently small number ε_0, we can choose $\Theta\left(\leq \frac{1}{4m}\right)$ so that

$$\frac{\beta^*}{\beta^* - \beta} = \frac{\beta^*}{\beta^* - \alpha}(1 + \varepsilon_0).$$

Then

$$\mathfrak{N} \leq k_1 2^{\frac{\beta^*}{\beta^*-\alpha}(1+\varepsilon_0) t},$$

where k_1 is a constant depending only on ε_0, β^*, c, $\varkappa$, n, a and m, and not on the number and choice of roots of $f(x, \zeta)$ to which approximation is made, or on the corresponding ideals.

Further, the number of different solutions of the inequality (47) such that $A < (64^{hn} c^{\bar{h}})^{\frac{1}{\beta-2}}$ clearly depends only on Θ (i. e., ε_0, m and β^*), n and h, and the number of different solutions which are roots of $f(x, \zeta)$ is at most m. Thus the number of solutions of the inequality (47) is not greater than

$$k_0 2^{\frac{\beta^*}{\beta^*-\alpha}(1+\varepsilon_0)t}, \tag{55}$$

where k_0 is a constant depending only on ε_0, β^*, c, $\varkappa$, n, a, m and h, i. e., on ε_0, β^*, c, $\mathfrak{K}$, $f(x, \zeta)$ and h. Now the inequality (47) cannot have more solutions when $c < 1$ than when $c \geq 1$. Again, any increase in ε_0 can only increase the exponent in the bound (55). Thus, if k_0 remains constant for these changes in value of c and ε_0, the bound (55) holds for any positive c and ε_0, and k_0 still does not depend on the number and choice of the roots to which approximation is made, or on the corresponding ideals.

We have therefore proved the following theorem (in which we replace β^* by β and $f(x, \zeta)$ by $f(x)$, $a_0(\zeta)$, $a_1(\zeta)$, . . ., $a_m(\zeta)$ being now any integers $a_0 (\neq 0)$, a_1, . . ., a_m of $\mathfrak{K}$):

Theorem 1. *Let:*

$\mathfrak{K}$	*be a finite algebraic field of degree* n (≥ 1) *over the rational number field* $\mathfrak{P}$;
$f(x)$	*be a polynomial of degree* m (≥ 2) *with integral coefficients from* $\mathfrak{K}$ *and a non-zero discriminant*;
$\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_\varrho$,	*where* $0 \leq \varrho \leq r_1 + r_2$, *be* ϱ *of the* $r_1 + r_2$ *infinite prime ideals corresponding to the* r_1 *real and* r_2 *pairs of conjugate imaginary fields conjugate to* $\mathfrak{K}$;
$\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$,	*where* $\sigma \geq 0$, *be* σ *different finite prime ideals of* $\mathfrak{K}$;
$G(\mathfrak{q}_j)$	$(j = 1, 2, \ldots, \varrho)$ *be a natural number not greater than* $g(\mathfrak{q}_j)$;
$G(\mathfrak{r}_k)$	$(k = 1, 2, \ldots, \sigma)$ *be a natural number not greater than* $g(\mathfrak{r}_k)$;
h	*be a natural number*;
$h_{k\delta}$	$(k = 1, 2, \ldots, \sigma;\ \delta = 1, 2, \ldots, G(\mathfrak{r}_k))$ *be a natural number not greater than* h^2;
$\xi_{j\gamma}$	$(j = 1, 2, \ldots, \varrho;\ \gamma = 1, G(\mathfrak{q}_j))$ *be a real or complex root of* $f(x)$;
$\eta_{k\delta\tau}$	$(k = 1, 2, \ldots, \sigma;\ \delta = 1, 2, \ldots, G(\mathfrak{r}_k);\ \tau = 1, 2, \ldots, h_{k\delta})$ *be an* $\mathfrak{r}_k$*-adic root of* $f(x)$;

t *be the total number of roots considered, i.e.,* $\sum_{j=1}^{\varrho} G(\mathfrak{q}_j) + \sum_{k=1}^{\sigma} \sum_{\delta=1}^{G(\mathfrak{r}_k)} h_{k\delta}$;

c, ε_0 *be two positive numbers;*

α, β *be two numbers such that*

$$\alpha = \min_{s=1,2,\ldots,m-1} \left(\frac{m}{s+1} + s \right), \quad \beta > \alpha.$$

Then the number of different algebraic numbers λ of degree h (or any divisor of h) over $\mathfrak{K}$, lying in the perfect $\mathfrak{r}_1$-adic, $\mathfrak{r}_2$-adic, . . ., $\mathfrak{r}_\sigma$-adic extensions of $\mathfrak{K}$, and satisfying the inequality

$$\prod_{j=1}^{\varrho} \prod_{\gamma=1}^{G(\mathfrak{q}_j)} \min\left(1, |\lambda - \xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}}\right) \prod_{k=1}^{\sigma} \prod_{\delta=1}^{G(\mathfrak{r}_k)} \prod_{\tau=1}^{h_{k\delta}} \min\left(1, \lambda - \eta_{k\delta\tau}|_{\mathfrak{r}_k}\right) \leq c\, \Lambda^{-h\beta},$$

where Λ is the number $|\Upsilon(x; \lambda, \mathfrak{P}, hn)|$, is not greater than

$$k_0\, 2^{\frac{\alpha}{\beta-\alpha}(1+\varepsilon_0)t},$$

where k_0 is a constant depending only on $\varepsilon_0, \beta, c, \mathfrak{K}, f(x)$ and h, and not on the number and choice of the roots to which approximation is made, or on the corresponding ideals.

24. Remarks.

(a) In the particular case when λ is an element ω of the field $\mathfrak{K}$, $h = 1$ and $\tau = 1$, and we may write $\eta_{k\delta 1} = \eta_{k\delta}$. Then Theorem 1 takes the form:

The number of different numbers ω of $\mathfrak{K}$ satisfying the inequality

$$\prod_{j=1}^{\varrho} \prod_{\gamma=1}^{G(\mathfrak{q}_j)} \min\left(1, |\omega - \xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}}\right) \prod_{k=1}^{\sigma} \prod_{\delta=1}^{G(\mathfrak{r}_k)} \min\left(1, |\omega - \eta_{k\delta}|_{\mathfrak{r}_k}\right) \leq c\, \Omega^{-\beta},$$

where Ω is the number $|\Upsilon(x, \omega, \mathfrak{P}, n)|$, is not greater than

$$k_0\, 2^{\frac{\beta}{\beta-\alpha}(1+\varepsilon_0)t},$$

where k_0 is a constant depending only on $\varepsilon_0, \beta, c, \mathfrak{K}$ and $f(x)$, and not on the number and choice of roots to which approximation is made, nor on the corresponding ideals.

(b) *Let $\theta_1, \theta_2, \ldots, \theta_n$ be a fixed basis of $\mathfrak{K}$, and let $z_1, z_2, \ldots, z_n$ and $z'_1, z'_2, \ldots, z'_n$ denote any $2n$ rational integers such that the maximum of their absolute values is a natural number z. Then the inequality*

$$Y = \prod_{j=1}^{\varrho} \prod_{\gamma=1}^{G(\mathfrak{q}j)} \min\left(1, \left|\frac{z_1\theta_1 + z_2\theta_2 + \cdots + z_n\theta_n}{z'_1\theta_1 + z'_2\theta_2 + \cdots + z'_n\theta_n} - \xi_{j\gamma}\right|_{\mathfrak{q}_{j\gamma}}\right) \cdot$$

$$\cdot \prod_{k=1}^{\sigma} \prod_{\gamma=1}^{G(\mathfrak{r}_k)} \min\left(1, \left|\frac{z_1\theta_1 + z_2\theta_2 + \cdots + z_n\theta_n}{z'_1\theta_1 + z'_2\theta_2 + \cdots + z'_n\theta_n} - \eta_{k\vartheta}\right|_{\mathfrak{r}_k}\right) \leq c' z^{-n\beta},$$

where c' is a positive number, has not more than

$$k_\theta 2^{\frac{\beta}{\beta-\alpha}(1+\varepsilon_0)t}$$

solutions in selections of $z_1, z_2, \ldots, z_n, z'_1, z'_2, \ldots, z'_n$ giving different numbers $\frac{z_1\theta_1 + z_2\theta_2 + \cdots + z_n\theta_n}{z'_1\theta_1 + z'_2\theta_2 + \cdots + z'_n\theta_n}$ of the field $\mathfrak{K}$, k_θ being a constant depending only on the given basis and on $\varepsilon_0, \beta, c', \mathfrak{K}$ and $f(x)$, and not on the number and choice of roots to which approximation is made, or on the corresponding ideals.

This statement can be proved as follows:

Denote by ω the number $\frac{z_1\theta_1 + z_2\theta_2 + \cdots + z_n\theta_n}{z'_1\theta_1 + z'_2\theta_2 + \cdots + z'_n\theta_n}$, which is a number of $\mathfrak{K}$. Then ω is a root of the polynomial

$$P(x) = \prod_{\nu=1}^{n} \{(z'_1\theta_1^{(\nu)} + z'_2\theta_2^{(\nu)} + \cdots + z'_n\theta_n^{(\nu)})x - (z_1\theta_1^{(\nu)} + z_2\theta_2^{(\nu)} + \cdots + z_n\theta_n^{(\nu)})\},$$

the product being taken over all the conjugate values of $\theta_1, \theta_2, \ldots, \theta_n$. $P(x)$ is of the form:

$$P(x) = b\,\Upsilon(x;\, \omega,\, \mathfrak{P},\, n),$$

where b is a rational integer. Let Ω_d be the absolute value of the coefficient of x^{n-d} in $\Upsilon(x;\, \omega,\, \mathfrak{P},\, n)$, let Ω be the number $|\overline{\Upsilon(x;\, \omega,\, \mathfrak{P},\, n)}|$, and let θ be the greatest of the n^2 numbers $|\theta_1^{(1)}|, \ldots, |\theta_n^{(n)}|$. Then for $d = 0, 1, 2, \ldots, n$,

$$\Omega_d \leq \binom{n}{d} (n\theta z)^n,$$

and so

$$\Omega \leq n!\,(n\theta)^n z^n = \theta_0 z^n,$$

say, so that θ_0 is a positive constant depending only on the given basis and on $\mathfrak{K}$. Thus, by (a), the inequality

$$Y \leq c\,\theta_0^{-\beta}\,z^{-n\beta}$$

has not more than

$$k_0\,2^{\frac{\beta}{\beta-\alpha}(1+\varepsilon_0)t}$$

solutions in different numbers ω of $\mathfrak{K}$, i.e., in selections of $z_1, z_2, \ldots, z_n$ $z_1', z_2', \ldots, z_n'$ giving different numbers $\dfrac{z_1\theta_1 + z_2\theta_2 + \cdots + z_n\theta_n}{z_1'\theta_1 + z_2'\theta_2 + \cdots + z_n'\theta_n}$ of $\mathfrak{K}$. Thus, on writing $c\,\theta_0^{-\beta} = c'$, the result follows, k_0 being replaced by a constant k_θ depending on the given basis as well as on ε_0, β, c', $\mathfrak{K}$ and $f(x)$.

§ 5. Properties of Binary Forms.

25. Let $F(x, y)$ denote the binary form of degree m (≥ 2):

$$y^m f\left(\frac{x}{y}\right) = a_0 x^m + a_1 x^{m-1} y + \cdots + a_m y^m,$$

so that, firstly, the coefficients $a_0 (\neq 0), a_1, \ldots, a_m$ are integers of the field $\mathfrak{K}$, and, secondly, $F(x, y)$ has a non-zero discriminant.

Let now $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(m)}$ be the roots of $F(x, 1)$ in the complex field. Then

$$F(x, y) = a_0 (x - \xi^{(1)} y)(x - \xi^{(2)} y) \cdots (x - \xi^{(m)} y). \tag{56}$$

Also, let $\eta_k^{(1)}, \eta_k^{(2)}, \ldots, \eta_k^{(\nu_k)}$ $(k = 1, 2, \ldots, \sigma;\ 0 \leq \nu_k \leq m)$ be the $\mathfrak{r}_k$-adic roots of $F(x, 1)$, $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$ being, as before, $\sigma (\geq 0)$ different finite prime ideals of $\mathfrak{K}$. Then, for $k = 1, 2, \ldots, \sigma$,

$$F(x, y) = (x - \eta_k^{(1)} y)(x - \eta_k^{(2)} y) \cdots (x - \eta_k^{(\nu_k)} y)\, G_k(x, y), \tag{57}$$

where $G_k(x, y)$ is not reducible to linear factors in the perfect $\mathfrak{r}_k$-adic extension of $\mathfrak{K}$.

26. Differentiating (56) logarithmically with respect to x,

$$\frac{F'(x, y)}{F(x, y)} = \frac{1}{x - \xi^{(1)} y} + \frac{1}{x - \xi^{(2)} y} + \cdots + \frac{1}{x - \xi^{(m)} y}$$

(where the dash denotes partial differentiation with respect to x). Thus, if $\xi_{j\gamma}^{(1)}, \xi_{j\gamma}^{(2)}, \ldots, \xi_{j\gamma}^{(m)}$ $(j = 1, 2, \ldots, r_1 + r_2;\ \gamma = 1, g(\mathfrak{q}_j))$ be the real or complex roots of the polynomial conjugate to $F(x, 1)$ with respect to the field $\mathfrak{K}_{j\gamma}$,

$$(58) \qquad |F(x,y)|_{\mathfrak{q}_{j\gamma}} \geq \frac{|F'(x,y)|_{\mathfrak{q}_{j\gamma}}}{m} \min\left(|x - \xi_{j\gamma}^{(1)} y|, |x - \xi_{j\gamma}^{(2)} y|, \ldots, |x - \xi_{j\gamma}^{(m)} y|\right).$$

($|F(x,y)|_{\mathfrak{q}_{j\gamma}}$ denotes the absolute value, for any real or complex values of the variables x and y, of the binary form, $F_{j\gamma}(x,y)$, conjugate to $F(x,y)$ with respect to the field $\mathfrak{K}_{j\gamma}$.)

Now there exist binary forms $H_1(x,y)$ and $K_1(x,y)$ of degree $m-2$, and $H_2(x,y)$ and $K_2(x,y)$ of degree $m-1$, with coefficients from $\mathfrak{K}$, such that

$$(59) \qquad F(x,y)\,H_1(x,y) + F'(x,y)\,H_2(x,y) = x^{2m-2},$$

$$(60) \qquad F(x,y)\,K_1(x,y) + F'(x,y)\,K_2(x,y) = y^{2m-2}.$$

But for all x and y, and for $j = 1, 2, \ldots, r_1 + r_2$ and $\gamma = 1, g(\mathfrak{q}_j)$, there exist constants c^{I} and c^{II} such that:

$$|H_1(x,y)|_{\mathfrak{q}_{j\gamma}} \leq c^{\mathrm{I}} \max(|x|, |y|)^{m-2},$$
$$|H_2(x,y)|_{\mathfrak{q}_{j\gamma}} \leq c^{\mathrm{II}} \max(|x|, |y|)^{m-1},$$
$$|K_1(x,y)|_{\mathfrak{q}_{j\gamma}} \leq c^{\mathrm{I}} \max(|x|, |y|)^{m-2},$$
$$|K_2(x,y)|_{\mathfrak{q}_{j\gamma}} \leq c^{\mathrm{II}} \max(|x|, |y|)^{m-1}.$$

It therefore follows, from considering that one of the identities (59) and (60) the right-hand side of which has the greater absolute value (or either identity if $|x| = |y|$), that for each pair of real or complex numbers x and y either

$$|F(x,y)|_{\mathfrak{q}_{j\gamma}} \geq \frac{1}{2\,c^{\mathrm{I}}} \max(|x|, |y|)^m,$$

or

$$|F'(x,y)|_{\mathfrak{q}_{j\gamma}} \geq \frac{1}{2\,c^{\mathrm{II}}} \max(|x|, |y|)^{m-1},$$

for $j = 1, 2, \ldots, r_1 + r_2$ and $\gamma = 1, g(\mathfrak{q}_j)$.

In the second case, it follows from (58) that

$$|F(x,y)|_{\mathfrak{q}_{j\gamma}} \geq \frac{1}{2m\,c^{\mathrm{II}}} \max(|x|, |y|)^{m-1} \min\left(|x - \xi_{j\gamma}^{(1)} y|, |x - \xi_{j\gamma}^{(2)} y|, \ldots, |x - \xi_{j\gamma}^{(m)} y|\right).$$

Let $c^{\mathrm{III}} = \max_{\substack{j=1,2,\ldots,r_1=r_2 \\ \gamma=1,\ g(\mathfrak{q}_j)}} (|\xi_{j\gamma}^{(1)}|, |\xi_{j\gamma}^{(2)}|, \ldots, |\xi_{j\gamma}^{(m)}|)$ Then, for $\nu = 1, 2, \ldots, m$, if $|x| \geq (c^{\mathrm{III}} + 1)|y|$,

$$|x - \xi_{j\gamma}^{(\nu)} y| \geq \max\left(\frac{|x|}{c^{\mathrm{III}} + 1}, |y|\right) = \frac{1}{c^{\mathrm{III}} + 1} \max(|x|, |y|,$$

and if $|x| \leq (c^{\mathrm{III}} + 1)|y|$,

$$|x - \xi_{j\gamma}^{(\nu)} y| = \left|\frac{x}{y} - \xi_{j\gamma}^{(\nu)}\right| |y| \geq \frac{1}{c^{\mathrm{III}} + 1} \max(|x|, |y|) \left|\frac{x}{y} - \xi_{j\gamma}^{(\nu)}\right|.$$

Hence

$$|F(x,y)|_{\mathfrak{q}_j\gamma} \geq c^{\mathrm{IV}} \max(|x|, |y|)^m \min\left(\left|\frac{x}{y} - \xi_{j\gamma}^{(1)}\right|, \left|\frac{x}{y} - \xi_{j\gamma}^{(2)}\right|, \ldots, \left|\frac{x}{y} - \xi_{j\gamma}^{(m)}\right|, 1\right),$$

where c^{IV} is a positive constant. Thus, in both cases,

$$|F(x,y)|_{\mathfrak{q}_j\gamma} \geq c^{\mathrm{V}} \max(|x|, |y|)^m \min\left(\left|\frac{x}{y} - \xi_{j\gamma}^{(1)}\right|, \left|\frac{x}{y} - \xi_{j\gamma}^{(2)}\right|, \ldots, \left|\frac{x}{y} - \xi_{j\gamma}^{(m)}\right|, 1\right),$$

where

$$c^{\mathrm{V}} = \min\left(\frac{1}{2\,c^{\mathrm{I}}}, c^{\mathrm{IV}}\right).$$

If $\omega = \frac{u}{v}$ be any element of $\mathfrak{K}$, it is well-known that it may be represented as a quotient $\omega = \frac{u}{v}$ of integers u and v of $\mathfrak{K}$ the greatest common ideal divisor (u, v) of which has a norm $N((u, v))$ in $\mathfrak{K}$ over $\mathfrak{P}$ not greater than $|\sqrt{d(\mathfrak{K})}|$, where $d(\mathfrak{K})$ is the discriminant of $\mathfrak{K}$.[1] For such integers u and v of $\mathfrak{K}$,

$$|F(u,v)|_{\mathfrak{q}_j\gamma} \geq c^{\mathrm{V}} \max(|u|_{\mathfrak{q}_j\gamma}, |v|_{\mathfrak{q}_j\gamma})^m \min(|\omega - \xi_{j\gamma}^{(1)}|_{\mathfrak{q}_j\gamma}, |\omega - \xi_{j\gamma}^{(2)}|_{\mathfrak{q}_j\gamma}, \ldots, |\omega - \xi_{j\gamma}^{(m)}|_{\mathfrak{q}_j\gamma}, 1),$$

for $j = 1, 2, \ldots, r_1 + r_2$ and $\gamma = 1, g(\mathfrak{q}_j)$. ($|F(u,v)|_{\mathfrak{q}_j\gamma}$ denotes the absolute value of the binary form $F_{j\gamma}(x,y)$ when x and y take the conjugate values in $\mathfrak{K}_{j\gamma}$ to u and v.) Hence the norm $N(F(u,v))$ in $\mathfrak{K}$ over $\mathfrak{P}$ of $F(u,v)$ satisfies the inequality

$$|N(F(u,v))| \geq (c^{\mathrm{V}})^m \prod_{j=1}^{r_1+r_2} \prod_{\gamma=1}^{g(\mathfrak{q}_j)} \max(|u|_{\mathfrak{q}_j\gamma}, |v|_{\mathfrak{q}_j\gamma}) \cdot \prod_{j=1}^{r_1+r_2} \prod_{\gamma=1}^{g(\mathfrak{q}_j)} \min(|\omega - \xi_{j\gamma}^{(1)}|_{\mathfrak{q}_j\gamma}, |\omega - \xi_{j\gamma}^{(2)}|_{\mathfrak{q}_j\gamma}, \ldots, |\omega - \xi_{j\gamma}^{(m)}|_{\mathfrak{q}_j\gamma}, 1).$$

[1] Follows from: E. HECKE, 'Theorie der algebraischen Zahlen', Akademische Verlagsgesellschaft, Leipzig (1923), p. 120.

But the conjugates of u and v can be formed into products of n members in not more than $(2n)!$ ways (one conjugate, to either u or v, being taken from each field $\mathfrak{K}_{j\gamma}$ conjugate to $\mathfrak{K}$). Thus, since each coefficient of the polynomial $\Upsilon(x; \omega, \mathfrak{P}, n)$ is a divisor of a sum of not more than $(2n)!$ such products, it follows that

$$(2n)! \prod_{j=1}^{r_1+r_2} \prod_{\gamma=1}^{g(\mathfrak{q}j)} \max\left(|u|_{\mathfrak{q}j\gamma}, |v|_{\mathfrak{q}j\gamma}\right) \geq \Omega,$$

where Ω is the number $\overline{|\Upsilon(x; \omega, \mathfrak{P}, n)|}$. Hence, taking $c(\mathfrak{q}) = \{c^{\nu}/(2n)!\}^m$, so that $c(\mathfrak{q})$ is a positive constant depending only on $\mathfrak{K}$ and the binary form $F(x, y)$,

$$(61)\quad |N(F(u, v))| \geq c(\mathfrak{q})\, \Omega^m \prod_{j=1}^{r_1+r_2} \prod_{\gamma=1}^{g\mathfrak{q}(j)} \min\left(|\omega - \xi_{j\gamma}^{(1)}|_{\mathfrak{q}j\gamma}, |\omega - \xi_{j\gamma}^{(2)}|_{\mathfrak{q}j\gamma}, \ldots, |\omega - \xi_{j\gamma}^{(m)}|_{\mathfrak{q}j\gamma}, 1\right).$$

27. We can obtain from (57) corresponding inequalities for the $\mathfrak{r}_1$-adic, $\mathfrak{r}_2$-adic, . . ., $\mathfrak{r}_\sigma$-adic valuations, but we must first prove the following lemma:

Lemma 7. *The* $\mathfrak{r}$ *adic value of a polynomial* $r(x)$ *with* $\mathfrak{r}$*-adic coefficients, where* $\mathfrak{r}$ *is a finite prime ideal of an algebraic number field* $\mathfrak{K}$*, and with a non-zero discriminant and no* $\mathfrak{r}$*-adic roots, has a positive lower bound for all values of* x *which are* $\mathfrak{r}$*-adic numbers. In particular, if all the coefficients of* $r(x)$ *are* $\mathfrak{r}$*-adic integers, the first coefficient unity, and the discriminant an* $\mathfrak{r}$*-adic unit, then for all* $\mathfrak{r}$*-adic numbers* x*,*

$$|r(x)|_{\mathfrak{r}} \geq 1.$$

Proof. We may suppose without loss of generality that the first coefficient of $r(x)$ is 1. Then $r(x)$ may be written as

$$r(x) = x^M + b_1 x^{M-1} + b_2 x^{M-2} + \cdots + b_M,$$

and we may write

$$b = \max\left(1, |b_1|_{\mathfrak{r}}, |b_2|_{\mathfrak{r}}, \ldots, |b_M|_{\mathfrak{r}}\right).$$

If $|x|_{\mathfrak{r}} > b$, it is clear that

$$|r(x)|_{\mathfrak{r}} = |x^M|_{\mathfrak{r}} > b^M \geq 1.$$

Thus we need only consider values of x for which

$$|x|_{\mathfrak{r}} \leq b.$$

The discriminant of $r(x)$ is:

$$D_r = \left|\begin{array}{ccccccccccc}
1, & b_1, & b_2, & \ldots\ldots, & b_{M-1}, & b_M, & . & . & . & . & . \\
0, & 1, & b_1, & \ldots\ldots, & b_{M-2}, & b_{M-1}, & b_M, & . & . & . & . \\
. & . & & & & & & & & & \\
. & . & . & & & & & & & & \\
. & . & . & . & & & & & & & \\
. & . & . & . & 1, & b_1, & b_2, & \ldots\ldots\ldots, & b_{M-1}, & b_M & \\
M, & (M-1)\,b_1, & (M-2)\,b_2, & \ldots, & b_{M-1}, & . & . & . & . & . & . \\
0, & M, & (M-1)\,b_1, & \ldots, & 2\,b_{M-2}, & b_{M-1}, & . & . & . & . & . \\
. & . & & & & & & . & . & . & . \\
. & . & . & & & & & & . & . & . \\
. & . & . & . & & & & & & . & . \\
. & . & . & . & . & & & & & & . \\
. & . & . & . & . & M, & (M-1)\,b_1, & \ldots\ldots, & 2\,b_{M-2}, & b_{M-1} &
\end{array}\right|,$$

($M-1$ rows; M rows)

and is assumed non-zero in the enunciation of the lemma. Let $p(x)$ be the polynomial obtained from this determinant by replacing the final column by $x^{M-2}, x^{M-3}, \ldots, 1, 0, 0, \ldots, 0$, and $q(x)$ the polynomial obtained be replacing the final column by $0, 0, \ldots, 0, x^{M-1}, x^{M-2}, \ldots, 1$. Then

$$r(x)\,p(x) + r'(x)\,q(x) = D_r. \tag{62}$$

Now $p(x)$ is a polynomial of degree $M-2$ and the r-adic value of each of its coefficients is not greater than b^{2M-2}. Thus

$$|p(x)|_r \leq b^{2M-2}\,b^{M-2} \leq b^{3(M-1)}. \tag{63}$$

Also, $q(x)$ is a polynomial of degree $M-1$ and the r-adic value of each of its coefficients is not greater than b^{2M-2}. Thus

$$|q(x)|_r \leq b^{2M-2}\,b^{M-1} = b^{3(M-1)}. \tag{64}$$

Then if

$$|r(x)|_r < d\,b^{-3(M-1)}, \tag{65}$$

where $|D_r|_r = d$, it follows from (63) that

$$|r(x)\,p(x)|_r < d.$$

Hence and by (62),

$$|r'(x)|_r = \frac{\max(d, |r(x)p(x)|_r)}{|q(x)|_r} = \frac{d}{|q(x)|_r},$$

and so, by (64),

$$|r'(x)|_r \geq d\,b^{-3(M-1)}. \tag{66}$$

We now show that $|r(x)|_r$ can be arbitrarily small for r-adic numbers x only if $r(x)$ has an r-adic root. We suppose that there exists an r-adic number χ such that

$$|\chi|_r \leq b, \quad |r(\chi)|_r < \min\left(\frac{d^2}{b^{7(M-1)}}, \frac{d}{b^{3(M-1)}}\right),$$

and that θ is the r-adic number $-\dfrac{r(\chi)}{r'(\chi)}$. Then by (65) and (66),

$$|\theta|_r \leq \frac{b^{3(M-1)}}{d}|r(\chi)|_r < 1. \tag{67}$$

Further, since $|\chi|_r \leq b$,

$$\left|\frac{r^{(l)}(\chi)}{l!}\right|_r \leq b^{M-l} \leq b^{M-1}$$

for $l = 1, 2, \ldots, M$. Hence

$$\left|\sum_{l=2}^{M} \frac{r^{(l)}(\chi)}{l!}\theta^l\right|_r \leq b^{M-1}|\theta|_r^2 = b^{M-1}\left|\frac{r(\chi)}{r'(\chi)^2}\right|_r |r(\chi)|_r <$$

$$< b^{M-1}\frac{\frac{d^2}{b^{7(M-1)}}}{\frac{d^2}{b^{6(M-1)}}}|r(\chi)|_r = |r(\chi)|_r.$$

Thus

$$|r(\chi+\theta)|_r < |r(\chi)|_r, \tag{68}$$

for

$$r(\chi+\theta) = r(\chi) + r'(\chi)\theta + \sum_{l=2}^{M}\frac{r^{(l)}(\chi)}{l!}\theta^l = \sum_{l=2}^{M}\frac{r^{(l)}(\chi)}{l!}\theta^l.$$

Hence also,

$$|r(\chi+\theta)|_r < d\,b^{-3(M-1)},$$

and so, by (66),

$$|r'(\chi+\theta)|_r \geq d\,b^{-3(M-1)}. \tag{69}$$

Now there exist sequences of numbers χ_ν, θ_ν such that

$$\theta = -\frac{r(\chi)}{r'(\chi)}, \qquad \chi_1 = \chi + \theta,$$

$$\theta_1 = -\frac{r(\chi_1)}{r'(\chi_1)}, \qquad \chi_2 = \chi_1 + \theta_1,$$

$$\theta_2 = -\frac{r(\chi_2)}{r'(\chi_2)}, \qquad \chi_3 = \chi_2 + \theta_2,$$

$$\theta_3 = -\frac{r(\chi_3)}{r'(\chi_3)}, \qquad \chi_4 = \chi_3 + \theta_3,$$

and so on. By (67), (68) and (69),

$$|\chi_1|_\mathfrak{r} \leq b, \ |r(\chi_1)|_\mathfrak{r} < |r(\chi)|_\mathfrak{r} < \min\left(\frac{d^2}{b^{7(M-1)}}, \frac{d}{b^{3(M-1)}}\right), \ |\theta_1|_\mathfrak{r} \leq \frac{b^{3(M-1)}}{d}|r(\chi_1)|_\mathfrak{r},$$

$$|\chi_2|_\mathfrak{r} \leq b, \ |r(\chi_2)|_\mathfrak{r} < |r(\chi_1)|_\mathfrak{r} < \min\left(\frac{d^2}{b^{7(M-1)}}, \frac{d}{b^{3(M-1)}}\right), \ |\theta_2|_\mathfrak{r} \leq \frac{b^{3(M-1)}}{d}|r(\chi_2)|_\mathfrak{r},$$

$$|\chi_3|_\mathfrak{r} \leq b, \ |r(\chi_3)|_\mathfrak{r} < |r(\chi_3)|_\mathfrak{r} < \min\left(\frac{d^2}{b^{7(M-1)}}, \frac{d}{b^{3(M-1)}}\right), \ |\theta_3|_\mathfrak{r} \leq \frac{b^{3(M-1)}}{d}|r(\chi_3)|_\mathfrak{r},$$

and so on. Thus

$$\lim_{\nu\to\infty}|r(\chi_\nu)|_\mathfrak{r} = 0, \quad \lim_{\nu\to\infty}|\chi_{\nu+1} - \chi_\nu|_\mathfrak{r} = \lim_{\nu\to\infty}|\theta_\nu|_\mathfrak{r} = 0.$$

Thus $\{\chi_\nu\}$ is a fundamental $\mathfrak{r}$-adic sequence having a limit χ^* which is a root of $r(x)$. We have therefore proved that $|r(x)|_\mathfrak{r}$ can be less than $\min\left(\frac{d^2}{b^{7(M-1)}}, \frac{d}{b^{3(M-1)}}\right)$ for an $\mathfrak{r}$-adic number x such that $|x|_\mathfrak{r} \leq b$ only if the polynomial $r(x)$ has an $\mathfrak{r}$-adic root.

Thus, if $r(x)$ has no $\mathfrak{r}$-adic root, for all $\mathfrak{r}$-adic numbers x,

$$|r(x)|_\mathfrak{r} \geq \min\left(b^M, \frac{d^2}{b^{7(M-1)}}, \frac{d}{b^{3(M-1)}}\right),$$

and the first part of the lemma is proved. The conditions stated in the second part of the lemma involve the conditions $b = 1$ and $d = 1$, so that immediately

$$|r(x)|_\mathfrak{r} \geq 1.$$

28. Differentiating (57) logarithmically with respect to x,

$$\frac{F'(x,y)}{F(x,y)} = \frac{1}{x-\eta_k^{(1)}y} + \frac{1}{x-\eta_k^{(2)}y} + \cdots + \frac{1}{x-\eta_k^{(\nu_k)}y} + \frac{G_k'(x,y)}{G_k(x,y)}$$
$$(k = 1, 2, \ldots, \sigma);\ 0 \leq \nu_k \leq m).$$

Thus

$$(70)\quad |F(x,y)|_{\mathfrak{r}_k} \geq |F'(x,y)|_{\mathfrak{r}_k} \min\left(|x-\eta_k^{(1)}y|_{\mathfrak{r}_k}, |x-\eta_k^{(2)}y|_{\mathfrak{r}_k}, \ldots, |x-\eta_k^{(\nu_k)}y|_{\mathfrak{r}_k}, \left|\frac{G_k(x,y)}{G_k'(x,y)}\right|_{\mathfrak{r}_k}\right) \qquad (k = 1, 2, \ldots, \sigma).$$

If the maximum of the $\mathfrak{r}_k$-adic valuations of the coefficients of $G_k(x,y)$ is b_{0k}, and if u and v be, as before, any integers of $\mathfrak{K}$ such that $N((u,v)) \leq |\sqrt{d(\mathfrak{K})}|$, then

$$(71)\qquad |G_k'(u,v)| \leq b_{0k} \qquad (k = 1, 2, \ldots, \sigma).$$

Further, neither $G_k(x, 1)$ nor $G_k(1, y)$ has an $\mathfrak{r}_k$-adic root, and both have non-zero discriminants, since $F(x,y)$ has a non-zero discriminant. Thus, by Lemma 7, for every $\mathfrak{r}_k$-adic number x,

$$|G_k(x,1)|_{\mathfrak{r}_k} \geq b_{1k}, \quad |G_k(1,y)|_{\mathfrak{r}_k} \geq b_{1k} \qquad (k = 1, 2, \ldots, \sigma),$$

where b_{1k} is a positive constant. Hence, since $G_k(x,y)$ is homogeneous,

$$(72)\quad |G_k(u,v)|_{\mathfrak{r}_k} = \left|G_k\left(\frac{u}{v}, 1\right)v^{m-\nu_k}\right|_{\mathfrak{r}_k} \geq b_{1k}|v|_{\mathfrak{r}_k}^{m-\nu_k}, \quad |G_k(u,v)|_{\mathfrak{r}_k} = \left|G_k\left(1, \frac{v}{u}\right)u^{m-\nu_k}\right|_{\mathfrak{r}_k} \geq b_{1k}|u|_{\mathfrak{r}_k}^{m-\nu_k} \qquad (k = 1, 2, \ldots, \sigma).$$

Now by definition u and v have a greatest common ideal divisor $(u,v) = \mathfrak{c}$ such that $N(\mathfrak{c}) \leq |\sqrt{d(\mathfrak{K})}|$. There are at most a finite number of ideals in $\mathfrak{K}$ with norms not greater than $|\sqrt{d(\mathfrak{K})}|$, and so

$$|\mathfrak{c}|_{\mathfrak{r}_k} \geq b_{2k} \qquad (k = 1, 2, \ldots, \sigma),$$

where b_{2k} is a positive constant not greater than 1. (Here $|\mathfrak{c}|_{\mathfrak{r}}$ means simply $r^{-\frac{\mu}{e}}$, where r is the rational prime number to which the prime ideal $\mathfrak{r}$ of $\mathfrak{K}$ belongs, e is the order of $\mathfrak{r}$, and μ is the power to which $\mathfrak{r}$ divides $\mathfrak{c}$.) Hence and by (72),

$$|G_k(u,v)|_{\mathfrak{r}_k} \geq b_{1k}\, b_{2k}^m \qquad (k = 1, 2, \ldots, \sigma),$$

since $b_{2k}^m \leq b_{2k}^{m-\nu_k}$, and since at least one of the ideals $\frac{(u)}{\mathfrak{c}}$ and $\frac{(v)}{\mathfrak{c}}$ is prime to $\mathfrak{r}_k$. Thus, by (71),

$$\left|\frac{G_k(u,v)}{G_k'(u,v)}\right|_{\mathfrak{r}_k} \geq \frac{b_{1k}\, b_{2k}^m}{b_{0k}} \qquad (k = 1, 2, \ldots, \sigma). \tag{73}$$

The discriminant Δ of $F(x,y)$ is a non-zero integer of $\mathfrak{K}$, and so we may choose binary forms $H_1(x,y)$ and $H_2(x,y)$ of degree $m-2$, and $K_1(x,y)$ and $K_2(x,y)$ of degree $m-1$, with integral coefficients from $\mathfrak{K}$, such that

$$F(x,y)\,H_1(x,y) + F'(x,y)\,K_1(x,y) = \Delta\, x^{2m-2},$$
$$F(x,y)\,H_2(x,y) + F'(x,y)\,K_2(x,y) = \Delta\, y^{2m-2}.$$

Thus, taking $x = u$ and $y = v$,

$$\begin{aligned}\max\left(|F(u,v)|_{\mathfrak{r}_k},\ |F'(u,v)|_{\mathfrak{r}_k}\right) &\geq |\Delta|_{\mathfrak{r}_k} \max\left(|u|_{\mathfrak{r}_k}^{2m-2},\ |v|_{\mathfrak{r}_k}^{2m-2}\right)\\ &\geq |\Delta|_{\mathfrak{r}_k}\, b_{2k}^{2m-2} \qquad (k = 1, 2, \ldots, \sigma).\end{aligned}$$

Hence, if for any k from $k = 1, 2, \ldots, \sigma$ and any u and v,

$$|F(u,v)|_{\mathfrak{r}_k} < |\Delta|_{\mathfrak{r}_k}\, b_{2k}^{2m-2},$$

then

$$|F'(u,v)|_{\mathfrak{r}_k} \geq |\Delta|_{\mathfrak{r}_k}\, b_{2k}^{2m-2},$$

and so, by (70) and (73),

$$|F(u,v)|_{\mathfrak{r}_k} \geq |\Delta|_{\mathfrak{r}_k}\, b_{2k}^{2m-2} \min\left(|u - \eta_k^{(1)} v|_{\mathfrak{r}_k},\ |u - \eta_k^{(2)} v|_{\mathfrak{r}_k},\ \ldots,\ |u - \eta_k^{(\nu_k)} v|_{\mathfrak{r}_k},\ \frac{b_{1k}\, b_{2k}^m}{b_{0k}}\right).$$

Thus, for $k = 1, 2, \ldots, \sigma$ and all u and v, if $c_k = |\Delta|_{\mathfrak{r}_k}\, b_{2k}^{2m-2} \min\left(1, \frac{b_{1k}\, b_{2k}^m}{b_{0k}}\right)$, it follows that

$$|F(u,v)|_{\mathfrak{r}_k} \geq c_k \min\left(|u - \eta_k^{(1)} v|_{\mathfrak{r}_k},\ |u - \eta_k^{(2)} v|_{\mathfrak{r}_k},\ \ldots,\ |u - \eta_k^{(\nu_k)} v|_{\mathfrak{r}_k},\ 1\right). \tag{74}$$

For all those $\mathfrak{r}_k$ the norms of which exceed a certain value depending only on $\mathfrak{K}$ and $F(x,y)$, the greatest common ideal divisor of u and v, and the discriminant Δ and the first coefficient a_0 of $F(x,1)$ are prime to $\mathfrak{r}_k$. It then follows for these $\mathfrak{r}_k$ that

$$c_k = 1.$$

For, firstly,

$$|\mathfrak{c}|_{\mathfrak{r}_k} = b_{2k} = 1.$$

Secondly,

$$|\Delta|_{\mathfrak{r}_k} = 1.$$

Thirdly, the coefficients of the polynomial

$$\frac{1}{a_0} F(x, 1) = x^m + \frac{a_1}{a_0} x^{m-1} + \cdots + \frac{a_m}{a_0}$$

are $\mathfrak{r}_k$-adic integers, so that the $\mathfrak{r}_k$-adic roots $\eta_k^{(1)}, \eta_k^{(2)}, \ldots, \eta_k^{(r_k)}$ of $F(x, 1)$ are $\mathfrak{r}_k$-adic integers. Thus, the form

$$\frac{1}{a_0} G(x, y) = \frac{\frac{1}{a_0} F(x, y)}{(x - \eta_k^{(1)} y)(x - \eta_k^{(2)} y) \ldots (x - \eta_k^{(r_k)} y)}$$

has $\mathfrak{r}_k$-adic integral coefficients and a first coefficient unity, so that $b_{0k} = 1$. Fourthly, the discriminant of $\frac{1}{a_0} G(x, y)$ is an $\mathfrak{r}_k$-adic integer and divides Δ, and is thus an $\mathfrak{r}_k$-adic unit. It therefore follows from Lemma 7 that we can take b_{1k} as unity. Thus we have proved that

$$c_k = |\Delta|_{\mathfrak{r}_k} b_{2k}^{2m-2} \min\left(1, \frac{b_{1k} b_{2k}^m}{b_{0k}}\right) = 1$$

for all those $\mathfrak{r}_k$ with norms exceeding a certain value depending only on $\mathfrak{R}$ and $F(x, y)$.

Now by (74),

$$\prod_{k=1}^{\sigma} |F(u, v)|_{\mathfrak{r}_k}^{G(\mathfrak{r}_k)} \geq c_1^{G(\mathfrak{r}_1)} c_2^{G(\mathfrak{r}_2)} \ldots c_\sigma^{G(\mathfrak{r}_\sigma)} \cdot \prod_{k=1}^{\sigma} \min\left(|u - \eta_k^{(1)} v|_{\mathfrak{r}_k}, |u - \eta_k^{(2)} v|_{\mathfrak{r}_k}, \ldots, |u - \eta_k^{(r_k)} v|_{\mathfrak{r}_k}, 1\right)^{G(\mathfrak{r}_k)},$$

where, for $k = 1, 2, \ldots, \sigma$, $G(\mathfrak{r}_k)$ is a natural number not greater than $g(\mathfrak{r}_k)$. But for all those $\mathfrak{r}_k$ with norms exceeding a certain value depending only on $\mathfrak{R}$ and $F(x, y)$, $c_k = 1$, and for other $\mathfrak{r}_k$, which can be only finite in number, c_k can assume only a finite number of values. Thus, the product $c_1^{G(\mathfrak{r}_1)} c_2^{G(\mathfrak{r}_2)} \ldots c_\sigma^{G(\mathfrak{r}_\sigma)}$, however large the number and whatever the choice of prime ideals $\mathfrak{r}_k$, can assume only a finite number of values. The minimum of these values will be a positive constant c_0 depending only on $\mathfrak{R}$ and $F(x, y)$, and not on the number or choice of the prime ideals $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$. Also,

$$(75)\quad \prod_{k=1}^{\sigma} |F(u,v)|_{\mathfrak{r}_k}^{G(\mathfrak{r}_k)} \geq c_0 \prod_{k=1}^{\sigma} \min\left(|u-\eta_k^{(1)}v|_{\mathfrak{r}_k}, |u-\eta_k^{(2)}v|_{\mathfrak{r}_k}, \ldots, |u-\eta_k^{(\nu_k)}v|_{\mathfrak{r}_k}, 1\right)^{G(\mathfrak{r}_k)}.$$

29. As before, let ω be any element of $\mathfrak{K}$ and u and v any pair of integers of $\mathfrak{K}$ such that $\omega = \frac{u}{v}$ and $(u,v) = \mathfrak{c}$ has a norm not greater than $|\sqrt{d(\mathfrak{K})}|$. Then it remains to prove that

$$(76)\quad \prod_{k=1}^{\sigma} |F(u,v)|_{\mathfrak{r}_k}^{G(\mathfrak{r}_k)} \geq c(\mathfrak{r}) \prod_{k=1}^{\sigma} \min\left(|\omega-\eta_k^{(1)}|_{\mathfrak{r}_k}, |\omega-\eta_k^{(2)}|_{\mathfrak{r}_k}, \ldots, |\omega-\eta_k^{(\nu_k)}|_{\mathfrak{r}_k}, 1\right)^{G(\mathfrak{r}_k)},$$

where $c(\mathfrak{r})$ is a positive constant depending only on $\mathfrak{K}$ and $F(x,y)$.

(a) Suppose that for $\sigma_1\,(0 \leq \sigma_1 \leq \sigma)$ of the $\mathfrak{r}_k$, which without loss of generality may be taken as $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_{\sigma_1}$,

$$|v|_{\mathfrak{r}_k} \geq |\mathfrak{c}|_{\mathfrak{r}_k} |a_0|_{\mathfrak{r}_k}.$$

Then for $\theta = 1, 2, \ldots, \nu_k$,

$$|\omega - \eta_k^{(\theta)}|_{\mathfrak{r}_k} = \frac{|u - \eta_k^{(\theta)} v|_{\mathfrak{r}_k}}{|v|_{\mathfrak{r}_k}} \leq \frac{|u - \eta_k^{(\theta)} v|_{\mathfrak{r}_k}}{|\mathfrak{c}|_{\mathfrak{r}_k} |a_0|_{\mathfrak{r}_k}},$$

and by (75), replacing σ by σ_1,

$$\prod_{k=1}^{\sigma_1} |F(u,v)|_{\mathfrak{r}_k}^{G(\mathfrak{r}_k)} \geq$$

$$\geq c_0 \prod_{k=1}^{\sigma_1} (|\mathfrak{c}|_{\mathfrak{r}_k} |a_0|_{\mathfrak{r}_k})^{G(\mathfrak{r}_k)} \prod_{k=1}^{\sigma_1} \min\left(|\omega - \eta_k^{(1)}|_{\mathfrak{r}_k}, |\omega - \eta_k^{(2)}|_{\mathfrak{r}_k}, \ldots, |\omega - \eta_k^{(\nu_k)}|_{\mathfrak{r}_k}, 1\right)^{G(\mathfrak{r}_k)},$$

since $|v|_{\mathfrak{r}_k} \leq 1$, and so $|\mathfrak{c}|_{\mathfrak{r}_k} |a_0|_{\mathfrak{r}_k} \leq 1$, for $k = 1, 2, \ldots, \sigma_1$. But by the inequality (6 a), which clearly still holds if ω_0 is replaced by the ideal $\mathfrak{c}$,

$$\prod_{k=1}^{\sigma_1} (|\mathfrak{c}|_{\mathfrak{r}_k} |a_0|_{\mathfrak{r}_k})^{G(\mathfrak{r}_k)} \geq \frac{1}{N(\mathfrak{c})\,|N(a_0)|} \geq \frac{1}{|\sqrt{d(\mathfrak{K})}|\,|N(a_0)|}.$$

Hence

$$\prod_{k=1}^{\sigma_1} |F(u,v)|_{\mathfrak{r}_k}^{G(\mathfrak{r}_k)} \geq c_{01} \prod_{k=1}^{\sigma_1} \min\left(|\omega - \eta_k^{(1)}|_{\mathfrak{r}_k}, |\omega - \eta_k^{(2)}|_{\mathfrak{r}_k}, \ldots, |\omega - \eta_k^{(\nu_k)}|_{\mathfrak{r}_k}, 1\right)^{G(\mathfrak{r}_k)},$$

where c_{01} is a positive constant depending only on $\mathfrak{K}$ and $F(x,y)$.

(b) For the remaining $\mathfrak{r}_k$, i. e., $\mathfrak{r}_{\sigma_1+1}, \mathfrak{r}_{\sigma_1+2}, \ldots, \mathfrak{r}_\sigma$,

$$|v|_{\mathfrak{r}_k} < |c|_{\mathfrak{r}_k} |a_0|_{\mathfrak{r}_k},$$

and so

$$|u|_{\mathfrak{r}_k} = |c|_{\mathfrak{r}_k}.$$

Hence

$$|F(u,v)|_{\mathfrak{r}_k} = \max(|a_0 u^m|_{\mathfrak{r}_k}, |a_1 u^{m-1} v|_{\mathfrak{r}_k}, \ldots, |a_m v^m|_{\mathfrak{r}_k})$$
$$= |a_0 u^m|_{\mathfrak{r}_k} = |a_0|_{\mathfrak{r}_k} |c|^m_{\mathfrak{r}_k}.$$

Thus

$$\prod_{k=\sigma_1+1}^{\sigma} |F(u,v)|^{G(\mathfrak{r}_k)}_{\mathfrak{r}_k} \geq$$
$$\geq \prod_{k=\sigma_1+1}^{g} (|a_0|_{\mathfrak{r}_k} |c|^m_{\mathfrak{r}_k})^{G(\mathfrak{r}_k)} \prod_{k=\sigma_1+1}^{\sigma} \min(|\omega - \eta_k^{(1)}|_{\mathfrak{r}_k}, |\omega - \eta_k^{(2)}|_{\mathfrak{r}_k}, \ldots, |\omega - \eta_k^{(r_k)}|_{\mathfrak{r}_k}, 1)^{G(\mathfrak{r}_k)},$$

and by the inequality (6 a),

$$\prod_{k=\sigma_1+1}^{\sigma} |F(u,v)|^{G(\mathfrak{r}_k)}_{\mathfrak{r}_k} \geq \frac{1}{|N(a_0)|\,|\sqrt{d(\mathfrak{K})}|^m} = c_{02},$$

say. Hence

$$\prod_{k=\sigma_1+1}^{\sigma} |F(u,v)|^{G(\mathfrak{r}_k)}_{\mathfrak{r}_k} \geq c_{02} \prod_{k=\sigma_1+1}^{\sigma} \min(|\omega - \eta_k^{(1)}|_{\mathfrak{r}_k}, |\omega - \eta_k^{(2)}|_{\mathfrak{r}_k}, \ldots, |\omega - \eta_k^{(r_k)}|_{\mathfrak{r}_k}, 1)^{G(\mathfrak{r}_k)},$$

where c_{02} is a positive constant depending only on $\mathfrak{K}$ and $F(x,y)$.

Thus (76) follows, with $c(\mathfrak{r}) = c_{01} c_{02}$.

30. Combining the inequalities (61) and (76) and writing $c(\mathfrak{q})\, c(\mathfrak{r}) = C_0$, we arrive at the following lemma:

Lemma 8. *Let:*

$\mathfrak{K}$	*be a finite algebraic field of degree* $n (\geq 1)$ *over the rational number field* $\mathfrak{P}$;
$F(x,y)$	*be a binary form of degree* $m (\geq 2)$, *with a non-zero discriminant and integral coefficients from the field* $\mathfrak{K}$ *of which the coefficient of* x^m *is not zero*;
$\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_{r_1+r_2}$	*be the* $r_1 + r_2$ *infinite prime ideals corresponding to the* r_1 *real and* r_2 *pairs of conjugate imaginary fields conjugate to* $\mathfrak{K}$;
$\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$,	*where* $\sigma \geq 0$, *be* σ *different finite prime ideals of* $\mathfrak{K}$;
$G(\mathfrak{r}_k)$	$(k = 1, 2, \ldots, \sigma)$ *be a natural number not greater than* $g(\mathfrak{r}_k)$;

$\xi_{j\gamma}^{(1)}, \xi_{j\gamma}^{(2)}, \ldots, \xi_{j\gamma}^{(m)}$ $(j = 1, 2, \ldots, r_1 + r_2;\ \gamma = 1, g(\mathfrak{q}_j))$ *be the real or complex roots of the polynomial conjugate to* $F(x, 1)$ *with respect to the field* $\mathfrak{K}_{j\gamma}$;

$\eta_k^{(1)}, \eta_k^{(2)}, \ldots, \eta_k^{(\nu_k)}$ $(k = 1, 2, \ldots, \sigma)$ *be the* ν_k $\mathfrak{r}_k$*-adic roots of* $F(x, 1)$;

ω *be any non-zero element of* $\mathfrak{K}$;

u, v *be any two integers of* $\mathfrak{K}$ *such that* $\omega = \frac{u}{v}$ *and* $N((u, v)) \leq |\sqrt{d(\mathfrak{K})}|$, *where* $d(\mathfrak{K})$ *is the discriminant of* $\mathfrak{K}$;

Ω *be the number* $|\Upsilon(x; \omega, \mathfrak{P}, n)|$;

$N(F(u, v))$ *be the norm in* $\mathfrak{K}$ *over* $\mathfrak{P}$ *of* $F(u, v)$.

Then there exists a positive constant C_0, *depending only on* $\mathfrak{K}$ *and* $F(x, y)$, *and not on the number and choice of* $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$, *such that*

$$|N(F(u,v))| \prod_{k=1}^{\sigma} |F(u,v)|_{\mathfrak{r}_k}^{G(\mathfrak{r}_k)} \geq C_0\, \Omega^m \cdot$$

$$\cdot \prod_{j=1}^{r_1+r_2} \prod_{\gamma=1}^{g(\mathfrak{q}_j)} \min\left(|\omega - \xi_{j\gamma}^{(1)}|_{\mathfrak{q}_{j\gamma}}, |\omega - \xi_{j\gamma}^{(2)}|_{\mathfrak{q}_{j\gamma}}, \ldots, |\omega - \xi_{j\gamma}^{(m)}|_{\mathfrak{q}_{j\gamma}}, 1\right) \cdot$$

$$\cdot \prod_{k=1}^{\sigma} \min\left(|\omega - \eta_k^{(1)}|_{\mathfrak{r}_k}, |\omega - \eta_k^{(2)}|_{\mathfrak{r}_k}, \ldots, |\omega - \eta_k^{(\nu_k)}|_{\mathfrak{r}_k}, 1\right)^{G(\mathfrak{r}_k)}$$

for all ω *and all* u *and* v.

(This lemma is, of course, also true for $m = 1$.)

31. For a certain number $\mu\,(0 \leq \mu \leq \sigma)$ of the finite prime ideals $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$,

$$\nu_k > 0.$$

Without loss of generality, we can suppose these to be $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\mu$, so that $\nu_k = 0$ for $k = \mu + 1, \mu + 2, \ldots, \sigma$. Then the inequality of Lemma 8 becomes

$$(77)\quad |N(F(u,v))| \prod_{k=1}^{\sigma} |F(u,v)|_{\mathfrak{r}_k}^{G(\mathfrak{r}_k)} \geq C_0\, \Omega^m \cdot$$

$$\prod_{j=1}^{r_1+r_2} \prod_{\gamma=1}^{g(\mathfrak{q}_j)} \min\left(|\omega - \xi_{j\gamma}^{(1)}|_{\mathfrak{q}_{j\gamma}}, |\omega - \xi_{j\gamma}^{(2)}|_{\mathfrak{q}_{j\gamma}}, \ldots, |\omega - \xi_{j\gamma}^{(m)}|_{\mathfrak{q}_{j\gamma}}, 1\right) \cdot$$

$$\cdot \prod_{k=1}^{\mu} \min\left(|\omega - \eta_k^{(1)}|_{\mathfrak{r}_k}, |\omega - \eta_k^{(2)}|_{\mathfrak{r}_k}, \ldots, |\omega - \eta_k^{(\nu_k)}|_{\mathfrak{r}_k}, 1\right)^{G(\mathfrak{r}_k)}.$$

Now each value of ω satisfying the inequality

$$(78)\quad \prod_{j=1}^{r_1+r_2}\prod_{\gamma=1}^{g(\mathfrak{q}_j)} \min\left(|\omega-\xi_{j\gamma}^{(1)}|_{\mathfrak{q}_{j\gamma}}, |\omega-\xi_{j\gamma}^{(2)}|_{\mathfrak{q}_{j\gamma}}, \ldots, |\omega-\xi_{j\gamma}^{(m)}|_{\mathfrak{q}_{j\gamma}}, 1\right)\cdot$$
$$\cdot \prod_{k=1}^{\mu} \min\left(|\omega-\eta_k^{(1)}|_{\mathfrak{r}_k}, |\omega-\eta_k^{(2)}|_{\mathfrak{r}_k}, \ldots, |\omega-\eta_k^{(\nu_k)}|_{\mathfrak{r}_k}, 1\right)^{G(\mathfrak{r}_k)} \leq \frac{1}{C_0}\Omega^{-\beta}$$

also satisfies at least one of the inequalities

$$(79)\quad \prod_{j=1}^{r_1+r_2}\prod_{\gamma=1}^{g(\mathfrak{q}_j)} \min\left(|\omega-\xi_{j\gamma}^{(\alpha_{j\gamma})}|_{\mathfrak{q}_{j\gamma}}, 1\right) \prod_{k=1}^{\mu} \min\left(|\omega-\eta_k^{(\alpha_k)}|_{\mathfrak{r}_k}, 1\right)^{G(\mathfrak{r}_k)} \leq \frac{1}{C_0}\Omega^{-\beta},$$

where $\alpha_{j\gamma}(j=1, 2, \ldots, r_1+r_2;\ \gamma=1, g(\mathfrak{q}_j))$ takes each of the values $1, 2, \ldots, m$ and $\alpha_k(k=1, 2, \ldots, \mu)$ each of the values $1, 2, \ldots, \nu_k$. But from Theorem 1, with $h=1$, $\lambda=\omega$, $\Lambda=\Omega$, $c=\frac{1}{C_0}$, $\varrho=r_1+r_2$, $\sigma=\mu$, $G(\mathfrak{q}_j)=g(\mathfrak{q}_j)$ and $\xi_{j\gamma}=\xi_{j\gamma}^{(\alpha_{j\gamma})}$ for $j=1, 2, \ldots, r_1+r_2$ and $\gamma=1, g(\mathfrak{q}_j)$, and $\eta_{k\delta\tau}=\eta_k^{(\alpha_k)}$ for $k=1, 2, \ldots, \mu$, $\delta=1, 2, \ldots, G(\mathfrak{r}_k)$ and $\tau=h_{k\delta}=1$, it follows that each of the inequalities (79) is satisfied by not more than

$$k_0\, 2^{\frac{\beta}{\beta-\alpha}(1+\varepsilon_0)t}$$

different numbers ω of $\mathfrak{K}$, where $t=\sum_{j=1}^{r_1+r_2} g(\mathfrak{q}_j) + \sum_{k=1}^{\mu} G(\mathfrak{r}_k) = n + \sum_{k=1}^{\mu} G(\mathfrak{r}_k)$, ε_0 is any positive number, and k_0 is a constant depending only on ε_0, β, $\mathfrak{K}$ and $F(x, y)$, and not on the number and choice of $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$. Also, there are not more than $m^n \nu_1 \nu_2 \ldots \nu_\mu$ different inequalities of the form (79), so that the number $\mathfrak{N}_0$ of different numbers ω of $\mathfrak{K}$ satisfying (78) is not greater than

$$k_0\, 2^{\frac{\beta}{\beta-\alpha}(1+\varepsilon_0)\left(n+\sum_{k=1}^{\mu} G(\mathfrak{r}_k)\right)} m^n \nu_1 \nu_2 \ldots \nu_\mu \leq k_1\, 2^{\frac{\beta}{\beta-\alpha}(1+\varepsilon_0)\left(\sum_{k=1}^{\sigma} G(\mathfrak{r}_k)\right)} \prod_{k=1}^{\sigma} \max(1, \nu_k),$$

where k_1 is a constant depending only on ε_0, β, $\mathfrak{K}$ and $F(x, y)$.

Now to each ω satisfying (78) correspond not more than k_2 pairs of integers u and v of $\mathfrak{K}$ such that $\omega=\frac{u}{v}$ and $N((u, v)) \leq \left|\sqrt{d(\mathfrak{K})}\right|$, where k_2 is a constant depending only on $\mathfrak{K}$. This follows from the well-known fact that the number of ideals with norms not greater than a number x is of order x[1], and because to

[1] See p. 61, note 1: HECKE, p. 160.

each such ideal $\mathfrak{c}$ corresponds not more than one pair of integers u and v of $\mathfrak{K}$ (or two pairs if sign be taken into account) such that $\frac{u}{v} = \omega$ and $(u, v) = \mathfrak{c}$. Thus the number of non-associated pairs of integers u and v of $\mathfrak{K}$ $\left(\text{writing } \omega \text{ as } \frac{u}{v}\right)$ with $N((u, v)) \leq \left|\sqrt{d(\mathfrak{K})}\right|$ and satisfying (78) is not greater than

$$k_1 k_2 2^{\frac{\beta}{\beta-\alpha}(1+\varepsilon_0)\left(\sum_{k=1}^{\sigma} G(\mathfrak{r}_k)\right)} \prod_{k=1}^{\sigma} \max(1, \nu_k),$$

and by (77) this expression is also an upper bound for the number of solutions of the inequality

$$|N(F(u, v))| \prod_{k=1}^{\sigma} |F(u, v)|_{\mathfrak{r}_k}^{G(\mathfrak{r}_k)} \leq \Omega^{m-\beta},$$

in non-associated pairs of integers u and v of $\mathfrak{K}$ such that $N((u, v)) \leq \left|\sqrt{d(\mathfrak{K})}\right|$.

Thus, writing $k_1 k_2 = k_3$, we have proved the following theorem:

Theorem 2. *Let*:

$\mathfrak{K}$ — *be a finite algebraic field of degree* $n (\geq 1)$ *over the rational number field* $\mathfrak{P}$;

$F(x, y)$ — *be a binary form of degree* m (≥ 2) *with integral coefficients from* $\mathfrak{K}$ *and a non-zero discriminant and such that the coefficient of* x^m *is not zero*;

$\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$, *where* $\sigma \geq 0$, *be* σ *different finite prime ideals of* $\mathfrak{K}$;

$G(\mathfrak{r}_k)$ — $(k = 1, 2, \ldots, \sigma)$ *be a natural number not greater than* $g(\mathfrak{r}_k)$;

ν_k — $(k = 1, 2, \ldots, \sigma)$ *be the number of* $\mathfrak{r}_k$*-adic roots of* $F(x, 1)$;

α, β — *be two numbers such that*

$$\alpha = \min_{s=1, 2, \ldots, m-1} \left(\frac{m}{s+1} + s\right), \quad \alpha < \beta;$$

ε_0 — *be a positive number.*

Then the number of solutions of the inequality

$$|N(F(u, v))| \prod_{k=1}^{\sigma} |F(u, v)|_{\mathfrak{r}_k}^{G(\mathfrak{r}_k)} \leq \Omega^{m-\beta},$$

in non-associated pairs of integers u and v of $\mathfrak{K}$ such that $N((u,v)) \leq |\sqrt{d(\mathfrak{K})}|$, where $N(F(u,v))$ is the norm in $\mathfrak{K}$ over $\mathfrak{P}$ of $F(u,v)$, Ω is the number $\left|\Upsilon\left(x; \frac{u}{v}, \mathfrak{P}, n\right)\right|$ and $d(\mathfrak{K})$ is the discriminant of $\mathfrak{K}$, is not greater than

$$k_3 2^{\frac{\beta}{\beta-\alpha}(1+\varepsilon_0)\left(\sum_{k=1}^{\sigma} G(\mathfrak{r}_k)\right)} \prod_{k=1}^{\sigma} \max(1, \nu_k),$$

where k_3 is a constant depending only on $\varepsilon_0, \beta, \mathfrak{K}$ and $F(x,y)$, and not on the number and choice of the finite prime ideals $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$.

32. If $m \geq 3$, we can take $\beta = m$, and the inequality of Theorem 2 then becomes

$$|N(F(u,v))| \prod_{k=1}^{\sigma} |F(u,v)|_{\mathfrak{r}_k}^{G(\mathfrak{r}_k)} \leq 1,$$

which, by the inequality (6 a), is equivalent to the equality

$$|N(F(u,v))| \prod_{k=1}^{\sigma} |F(u,v)|_{\mathfrak{r}_k}^{g(\mathfrak{r}_k)} = 1,$$

since $F(u,v)$ is an integer of $\mathfrak{K}$. Let $\varepsilon_0 = 1$. Then, since $0 \leq \nu_k \leq m$ for $k = 1, 2, \ldots, \sigma$, the number of solutions of this equality in non-associated pairs of integers u und v of $\mathfrak{K}$ such that $N((u,v)) \leq |\sqrt{d(\mathfrak{K})}|$ is not greater than

$$k_3 4^{\frac{m}{m-\alpha}\left(\sum_{k=1}^{\sigma} g(\mathfrak{r}_k)\right)} m^{\sigma},$$

and therefore not greater than

$$K^{\left(\sum_{k=1}^{\sigma} g(\mathfrak{r}_k)\right)+1},$$

where K is a constant depending only on $\mathfrak{K}$ and $F(x,y)$, and not on the number and choice of the finite prime ideals $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$.

Further, we may replace the conditions that the coefficient of $x^m (m \geq 3)$ in $F(x,y)$ is not zero and that $F(x,y)$ has a non-zero discriminant by the conditions that $F(x,y)$ is of degree not less than 3 and has a non-zero discriminant. For if $F_1(x,y)$ is a binary form satisfying the latter but not the former conditions, it may be transformed by a linear transformation of determinant 1 with rational

integral coefficients into a binary form $F_2(x, y)$ which satisfies the former conditions. Thus the number of solutions of the equality

$$|N(F_1(u,v))| \prod_{k=1}^{\sigma} |F_1(u,v)|_{\mathfrak{r}_k}^{g(\mathfrak{r}_k)} = 1$$

will be exactly the same as the number of solutions of the equality

$$|N(F_2(u,v))| \prod_{k=1}^{\sigma} |F_2(u,v)|_{\mathfrak{r}_k}^{g(\mathfrak{r}_k)} = 1.$$

We have therefore proved the following corollary:

Corollary 1. *Let*:

$\mathfrak{K}$ *be a finite algebraic field over the rational number field* $\mathfrak{P}$;

$F(x, y)$ *be a binary form of degree not less than* 3 *with integral coefficients from* $\mathfrak{K}$ *and a non-zero discriminant*;

$\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$, *where* $\sigma \geq 0$, *be* σ *different finite prime ideals of* $\mathfrak{K}$.

Then the number of solutions of the equality

$$|N(F(u,v))| \prod_{k=1}^{\sigma} |F(u,v)|_{\mathfrak{r}_k}^{g(\mathfrak{r}_k)} = 1,$$

in non-associated pairs of integers u *and* v *of* $\mathfrak{K}$ *such that* $N((u,v)) \leq |\sqrt{d(\mathfrak{K})}|$, *where* $N(F(u,v))$ *is the norm in* $\mathfrak{K}$ *over* $\mathfrak{P}$ *of* $F(u,v)$ *and* $d(\mathfrak{K})$ *is the discriminant of* $\mathfrak{K}$, *is not greater than*

$$K^{\left(\sum_{k=1}^{\sigma} g(\mathfrak{r}_k)\right)+1},$$

where K *is a constant depending only on* $\mathfrak{K}$ *and* $F(x, y)$, *and not on the number and choice of the finite prime ideals* $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$.

33. From Corollary 1 follow a number of other corollaries. The first is as follows:

Corollary 2. *If* $\mathfrak{K}$ *be a finite algebraic field over the rational number field* $\mathfrak{P}$, *if* $F(x, y)$ *be a binary form with integral coefficients from* $\mathfrak{K}$ *and such that* $F(x, 1)$ *has at least three different roots, of which one may be infinite, and if* u *and* v *be any pair of integers of* $\mathfrak{K}$ *such that* $N((u,v)) \leq |\sqrt{d(\mathfrak{K})}|$, *where* $d(\mathfrak{K})$ *is the discriminant of* $\mathfrak{K}$, *then as*

$$\max(|N(u)|, |N(v)|) \to \infty,$$

the greatest of the norms of the finite prime ideals dividing $F(u, v)$ *tends to infinity.*

Proof. Suppose first that $F(x, y)$ satisfies the conditions imposed in Corollary 1. Now if Corollary 2 were false for such $F(x, y)$, $F(u, v)$ could have only a finite number of prime ideal divisors, say $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$, for all members of some infinite sequence S of pairs of integers u and v of $\mathfrak{R}$ such that $N((u, v)) \leq |\sqrt{d(\mathfrak{R})}|$ and $\max(|N(u)|, |N(v)|) \to \infty$. It would then follow from the relation (6) that for all pairs u and v of the infinite sequence S,

$$|N(F(u, v))| \prod_{k=1}^{\sigma_0} |F(u, v)|_{\mathfrak{r}_k}^{g(\mathfrak{r}_k)} = 1,$$

which would contradict Corollary 1 (with $\sigma = \sigma_0$). We have thus proved the corollary true if $F(x, y)$ is of degree not less than 3 and has a non-zero discriminant.

It is also true if $F(x, 1)$ has at least three different roots, whether or not $F(x, y)$ has a non-zero discriminant, for then $F(x, y)$ decomposes into two binary forms $F_1(x, y)$ and $F_2(x, y)$, with integral coefficients from $\mathfrak{R}$, such that $F_1(x, 1)$ has at least three different roots but no coincident roots. Thus $F_1(x, y)$ is of degree not less than 3 and has a non-zero discriminant. The corollary is therefore true for $F_1(x, y)$, and consequently for $F(x, y)$.

34. Let $F(x, y)$ be defined as in Corollary 1. The norm of $F(u, v)$ in $\mathfrak{R}$ over $\mathfrak{P}$, i. e., $N(F(u, v))$, will be a rational integer for any integers u and v of $\mathfrak{R}$. Suppose that u and v are such that $N(F(u, v))$ is divisible by no rational prime numbers other than the $\tau (\geq 0)$ different prime numbers $r_1, r_2, \ldots, r_\tau$. Let the σ prime ideals $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$ consist of all the different prime ideal divisors in $\mathfrak{R}$ of $r_1, r_2, \ldots, r_\tau$. Then the valuations of $F(u, v)$ with respect to all other finite prime ideals of $\mathfrak{R}$ are unity, for otherwise, by the relation (6), $N(F(u, v))$ would be divisible by other rational prime numbers besides $r_1, r_2, \ldots, r_\tau$. From the same relation,

$$|N(F(u, v))| \prod_{k=1}^{\sigma} |F(u, v)|_{\mathfrak{r}_k}^{g(\mathfrak{r}_k)} = 1.$$

But by Corollary 1, the number of solutions of this equality in non-associated pairs of integers u and v of $\mathfrak{R}$ such that $N((u, v)) \leq |\sqrt{d(\mathfrak{R})}|$ is not greater than

$$K^{\left(\sum_{k=1}^{\sigma} g(\mathfrak{r}_k)\right)+1},$$

where K is a constant depending only on $\mathfrak{K}$ and $F(x, y)$. Now

$$\sum_{k=1}^{\sigma} g(\mathfrak{r}_k) = n,$$

since the sum of the g's corresponding to the prime ideals of $\mathfrak{K}$ dividing a rational prime number is n. Hence

$$K^{\left(\sum_{k=1}^{\sigma} g(\mathfrak{r}_k)\right)+1} = K^{\tau n+1} \leq K_0^{\tau+1},$$

say, where K_0 is a constant depending only on $\mathfrak{K}$ and $F(x, y)$.

Thus *the number of non-associated pairs of integers* u *and* v *of* $\mathfrak{K}$, *with* $N((u, v)) \leq |\sqrt{d(\mathfrak{K})}|$, *such that* $N(F(u, v))$ *is divisible by no rational prime numbers other than* $r_1, r_2, \ldots, r_\tau$, *is not greater than*

$$K_0^{\tau+1},$$

where K_0 *is a constant depending only on* $\mathfrak{K}$ *and* $F(x, y)$, *and not on the number and choice of the prime numbers* $r_1, r_2, \ldots, r_\tau$.

This result may be given either of the interpretations contained in the following corollary:

Corollary 3. *Let* $\mathfrak{K}$ *be a finite algebraic field over the rational number field* $\mathfrak{P}$, *let* $F(x, y)$ *be a binary form of degree not less than* 3 *with integral coefficients from* $\mathfrak{K}$ *and a non-zero discriminant, let* $r_1, r_2, \ldots, r_\tau$ *be* τ (≥ 0) *different rational prime numbers, and let* u *and* v *be any integers of* $\mathfrak{K}$ *such that* $N((u, v)) \leq |\sqrt{d(\mathfrak{K})}|$, *where* $d(\mathfrak{K})$ *is the discriminant of* $\mathfrak{K}$. *Then*:

(a) *the number of different integers* I *of* $\mathfrak{K}$ *with norms in* $\mathfrak{K}$ *over* $\mathfrak{P}$ *which are divisible by no rational prime numbers other than* $r_1, r_1, \ldots, r_\tau$, *and which are expressible in the form* $F(u, v)$, *is not greater than*

$$K_1^{\tau+1},$$

where K_1 *is a constant depending only on* $\mathfrak{K}$ *and* $F(x, y)$, *and not on the number and choice of* $r_1, r_2, \ldots, r_\tau$;

(b) *the number of different representations of a non-zero integer I of $\mathfrak{K}$, the norm in $\mathfrak{K}$ over $\mathfrak{P}$ of which is divisible by no rational prime numbers other than $r_1, r_2, \ldots, r_\tau$, in the form $F(u,v)$ is not greater than*

$$K_2^{\tau+1},$$

where K_2 is a constant depending only on $\mathfrak{K}$ and $F(x,y)$;

(c) *in particular, if the norm in $\mathfrak{K}$ over $\mathfrak{P}$ of I is a rational prime number, the number of different representations of I in the form $F(U, V)$, where U and V are any pair of integers of $\mathfrak{K}$ which are multiples of a pair u and v by an integer of $\mathfrak{K}$, is bounded by a number depending only on $\mathfrak{K}$ and $F(x,y)$.*

(c) follows since, if a pair U and V are equal to a pair ku and kv, where k is an integer of $\mathfrak{K}$,

$$I = F(U, V) = k^m F(u, v),$$

where m is the degree of $F(x,y)$, and so

$$N(I) = N(F(U, V)) = (N(k))^m N(F(u, v)),$$

which, by the definition of I, is impossible unless $N(k) = 1$. But then

$$N((U, V)) \leq \left|\sqrt{d(\mathfrak{K})}\right|,$$

so that U and V are identical with a pair u and v. (c) then follows from (b) by taking $\tau = 1$.

From this corollary may be obtained a number of interesting results concerning the representations of systems of integers by homogeneous forms in more than two variables.

Example. Consider the cubic binary form in x and y

$$F(x,y) = a_0 x^3 + a_1 x^2 y + a_2 x y^2 + a_3 y^3,$$

with integral coefficients from the quadratic field $\mathfrak{K} = \mathfrak{P}(\sqrt{C})$, where C is a non-zero rational integer not equal to 1 and square-free. Then by Corollary 3 (b), provided $F(x,y)$ has a non-zero discriminant, the number of different representations of any integer I of the field $\mathfrak{P}(\sqrt{C})$ by $F(u,v)$, where u and v are integers of $\mathfrak{P}(\sqrt{C})$ such that $N((u,v)) \leq \left|\sqrt{d(\mathfrak{P}(\sqrt{C}))}\right|$, is not greater than

$$K_2^{\tau+1},$$

where K_2 is a constant independent of I and τ is the number of rational prime divisors of the norm $N(I)$ in $\mathfrak{P}(\sqrt{C})$ over $\mathfrak{P}$ of I.

Now we may write

$$\alpha_0 = a_0 + \sqrt{C}\,b_0,\ \alpha_1 = a_1 + \sqrt{C}\,b_1,\ \alpha_2 = a_2 + \sqrt{C}\,b_2,\ \alpha_3 = a_3 + \sqrt{C}\,b_3,$$

$$I = q + \sqrt{C}\,r,$$

$$u = u_1 + \sqrt{C}\,u_2,\ v = v_1 + \sqrt{C}\,v_2,$$

where $a_0, b_0, a_1, b_1, a_2, b_2, a_3, b_3, q, r, u_1, u_2, v_1, v_2$ are rational integers. It is easily verified that if

$$F(u, v) = q + \sqrt{C}\,r,$$

then

$$\begin{aligned}
&a_0 u_1^3 - C b_0 u_2^3 + a_3 v_1^3 - C b_3 v_2^3 + 3 C b_0 u_1^2 u_2 + a_0 u_1^2 v_1 + C b_1 u_1^2 v_2 \\
&- 3 a_0 u_2^2 u_1 - a_1 u_2^2 v_1 - C b_1 u_2^2 v_2 + a_2 v_1^2 u_1 + C b_2 v_1^2 u_2 + 3 C b_3 v_1^2 v_2 \\
&- a_2 v_2^2 u_1 - C b_2 v_2^2 u_2 - 3 a_3 v_2^2 v_1 + 2 C b_1 u_1 u_2 v_1 - 2 a_2 u_2 v_1 v_2 + 2 C b_2 v_1 v_2 u_1 - 2 a_1 v_2 u_1 u_2 \\
&= q,
\end{aligned}$$

$$\begin{aligned}
&b_0 u_1^3 - a_0 u_2^3 + b_3 v_1^3 - a_3 v_2^3 + 3 a_0 u_1^2 u_2 + b_1 u_1^2 v_1 + a_1 u_1^2 v_2 \\
&- 3 b_0 u_2^2 u_1 - b_1 u_2^2 v_1 - a_1 u_2^2 v_2 + b_2 v_1^2 u_1 + a_2 v_1^2 u_2 + 3 a_3 v_1^2 v_2 \\
&- b_2 v_2^2 u_1 - a_2 v_2^2 u_2 - 3 b_3 v_2^2 v_1 + 2 a_1 u_1 u_2 v_1 - 2 b_2 u_2 v_1 v_2 + 2 a_2 v_1 v_2 u_1 - 2 b_1 v_2 u_1 u_2 \\
&= r.
\end{aligned}$$

Thus the number of different sets of rational integers u_1, u_2, v_1, v_2 such that $N((u_1 + \sqrt{C}\,u_2,\ v_1 + \sqrt{C}\,v_2)) \leq \left|\sqrt{d(\mathfrak{P}(\sqrt{C}))}\right|$ and satisfying the above pair of equations is not greater than

$$K_2^{\tau+1}.$$

In particular, if $C = -1$, i.e., $\mathfrak{K} = \mathfrak{P}(i)$, then $N(I) = q^2 + r^2$ and $\left|\sqrt{d(\mathfrak{P}(\sqrt{C}))}\right| = 2$. An interesting case arises when

$$a_0 = 1,\ b_0 = 0,\ a_1 = 0,\ b_1 = 0,\ a_2 = 0,\ b_2 = 0,\ a_3 = 0,\ b_3 = 1.$$

For these values of the coefficients,

$$F(x, y) = x^3 + i\,y^3,$$

so that $F(x, y)$ has a non-zero discriminant. It therefore follows, by taking these values for the coefficients, that the pair of equations

$$u_1^3 - 3u_2^2 u_1 - 3v_1^2 v_2 + v_2^3 = q,$$
$$3u_1^2 u_2 - u_2^3 - 3v_1 v_2^2 + v_1^3 = r$$

are satisfied by not more than

$$K_2^{\tau+1}$$

different sets of rational integers u_1, u_2, v_1, v_2 such that $N((u_1 + iu_2, v_1 + iv_2)) \leq 2$, where K_2 is a constant independent of q and r, and τ is the number of rational prime divisors of $q^2 + r^2$.

35. A result similar to that of Corollary 3(c) can be obtained for any non-zero integer I of the field $\mathfrak{K}$. Let $B(I)$ be the number of different representations of such an integer I in the form $F(U, V)$, where U and V are a pair of integers of $\mathfrak{K}$ with $N((U, V)) \leq |\sqrt{d(\mathfrak{K})}|$, or a multiple of such a pair by an integer of $\mathfrak{K}$, and let $b(I)$ be the number of different representations of I by $F(u, v)$, where u and v are a pair of integers of $\mathfrak{K}$ with $N((u, v)) \leq |\sqrt{d(\mathfrak{K})}|$. Then

$$B(I) \leq \sum_{\substack{\delta \\ \delta^m \mid I}} b\left(\frac{I}{\delta^m}\right),$$

where the sum is taken over all positive integers δ of $\mathfrak{K}$ such that δ^m divides I.

Now if the norm $N(I)$ in $\mathfrak{K}$ over $\mathfrak{P}$ of I has τ rational prime divisors, the norm of $\frac{I}{\delta^m}$ has the same or fewer rational prime divisors, and so, by the result preceding Corollary 3, for each set of δ's having the same norm Δ, say,

$$\sum_{\substack{\delta \\ N(\delta)=\Delta}} b\left(\frac{I}{\delta^m}\right) \leq K_2^{\tau+1}.$$

But there are at most $T(I)$ δ's having different norms, where $T(I)$ is the number of rational integer divisors of $N(I)$. Thus

$$B(I) \leq T(I) K_2^{\tau+1}.$$

Further, it is well-known that the logarithm of the number of rational integer divisors of the rational integer $N(I)$ is $O(\log |N(I)|/\log\log |N(I)|)$, and that the

number of rational prime divisors of $N(I)$ is also $O(\log|N(I)|/\log\log|N(I)|)$.[1] Thus

$$\log T(I) = O\left(\frac{\log|N(I)|}{\log\log|N(I)|}\right), \quad \tau = O\left(\frac{\log|N(I)|}{\log\log|N(I)|}\right),$$

and so

$$\log B(I) = O\left(\frac{\log|N(I)|}{\log\log|N(I)|}\right).$$

We have therefore proved the following corollary:

Corollary 4. *Let $\mathfrak{K}$ be a finite algebraic field over the rational number field $\mathfrak{P}$, let $F(x,y)$ be a binary form of degree not less than 3 with integral coefficients from $\mathfrak{K}$ and a non-zero discriminant, and let U and V be any pair of integers of $\mathfrak{K}$ with $N((U,V)) \leq |\sqrt{d(\mathfrak{K})}|$, where $d(\mathfrak{K})$ is the discriminant of $\mathfrak{K}$, or a multiple of such a pair by an integer of $\mathfrak{K}$. Then the number $B(I)$ of different representations of an integer I of $\mathfrak{K}$ (with a sufficiently large norm $N(I)$ in $\mathfrak{K}$ over $\mathfrak{P}$) in the form $F(U,V)$ is not greater than*

$$K_4^{\frac{\log|N(I)|}{\log\log|N(I)|}},$$

where K_4 is a constant depending only on $\mathfrak{K}$ and $F(x,y)$. Also,

$$B(I) = O(|N(I)|^{\varepsilon}),$$

where ε is an arbitrarily small positive constant.

36. From the argument preceding Corollary 3 follows a further result on the greatest rational prime divisor of the norm of the product of a number of integers of $\mathfrak{K}$, provided, firstly, that the number of integers is sufficiently great, and, secondly, that all the integers may be represented by means of a binary form $F(x,y)$ with integral coefficients from $\mathfrak{K}$ and such that $F(x,1)$ has at least three different roots. $F(x,y)$ need not now have a non-zero discriminant. The result is as follows:

Corollary 5. *Let $\mathfrak{K}$ be a finite algebraic field over the rational number field $\mathfrak{P}$, and let $F(x,y)$ be a binary form with integral coefficients from $\mathfrak{K}$ and be such that $F(x,1)$ has at least three different roots, of which one may be infinite. Then if M is a sufficiently large natural number and if*

$$u_1, v_1;\ u_2, v_2;\ \ldots;\ u_M, v_M$$

[1] E. LANDAU, 'Handbuch der Lehre von der Verteilung der Primzahlen', Vol. 1 (1909), B. G. Teubner, Leipzig, pp. 220—222.

are M non-associated pairs of integers u and v of $\mathfrak{K}$ such that $N((u,v)) \leq |\sqrt{d(\mathfrak{K})}|$, where $d(\mathfrak{K})$ is the discriminant of $\mathfrak{K}$, the greatest rational prime divisor of the norm in $\mathfrak{K}$ over $\mathfrak{P}$ of

$$F(u_1, v_1) F(u_2, v_2) \ldots F(u_M, v_M)$$

is greater than

$$K_5 \log M \operatorname{loglog} M,$$

where K_5 is a positive constant depending only on $\mathfrak{K}$ and $F(x,y)$.

Proof. If $F(x,y)$ has a zero discriminant, it may be decomposed, as in the proof of Corollary 2, into binary forms $F_1(x,y)$ and $F_2(x,y)$, with integral coefficients from $\mathfrak{K}$, such that $F_1(x,y)$ is of degree not less than 3 and has a non-zero discriminant. But if the corollary is true for $F_1(x,y)$, it is clearly also true for $F(x,y)$. We may therefore assume, without loss of generality, that $F(x,y)$ is of degree not less than 3 and has a non-zero discriminant. Then by the argument preceding Corollary 3 (with the same τ and K_0) we can choose τ so that

$$K_0^{\tau} + 1 \leq M < K_0^{\tau+1} + 1,$$

provided that M is sufficiently large and that K_0 is chosen greater than unity. By the result of the argument preceding Corollary 3 the numbers

$$N(F(u_1, v_1)),\ N(F(u_2, v_2)), \ldots, N(F(u_M, v_M))$$

cannot all be divisible only by the same $\tau - 1$ rational prime numbers, and so their product must be divisible by a rational prime number not less than the τth rational prime number. But it is well-known that τth rational prime number is greater than $\frac{1}{2}\tau \log \tau$, provided that τ is sufficiently large.[1] Further, we have defined τ so that

$$\tau \geq \left[\frac{\log (M-1)}{\log K_0}\right],$$

whence follows the result.

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[1] See p. 81, note 1: LANDAU, p. 214.

A Further Application of the $\mathfrak{p}$-adic Generalisation of the Thue-Siegel Theorem.

1. The object of this paper is to extend the results on binary forms contained in my previous paper, 'The $\mathfrak{p}$-adic generalisation of the Thue-Siegel Theorem', to forms of the type

$$\prod_{\nu=1}^{m}(x_0\xi^{(\nu)h} + x_1\xi^{(\nu)h-1} + \cdots + x_h),$$

where h is a natural number and $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(m)}$ are the m roots of a polynomial $f(x)$ of degree $m(\geq h)$ with coefficients from a field $\mathfrak{K}$ of degree $n(\geq 1)$ over the rational number field $\mathfrak{P}$, and with a non-zero discriminant. The method of proof will differ from that given for binary forms, and will be shorter, but the results obtained will not be quite so refined.

The notation used will, as far as practicable, be the same as in my previous paper. As there, $g(\mathfrak{p})$, where $\mathfrak{p}$ is a finite or infinite prime ideal of $\mathfrak{K}$, will represent the degree of the perfect $\mathfrak{p}$-adic extension of $\mathfrak{K}$, over the field of real numbers if $\mathfrak{p}$ is infinite, and over the field of p-adic numbers if $\mathfrak{p}$ is finite, p being the rational prime number divisible by $\mathfrak{p}$. As before also, $\Upsilon(x; \omega'; \mathfrak{P}, nn')$ will represent that polynomial of degree nn' which is a power of the primitive polynomial with rational integral coefficients having as a root the number ω', which is an element of a field $\mathfrak{K}'$ of degree n' over $\mathfrak{K}$. The symbol $\mathfrak{q}_{j\gamma}$ also retains the same meaning. It will now be convenient to refer to the symbol $\mathfrak{q}_{j\gamma}$, as well as the symbol $\mathfrak{q}_j$, as an 'infinite prime ideal'.

The main theorem will be as follows:

Theorem 2 a. *Let:*

$\mathfrak{K}$ — *be a finite algebraic field of degree* $n(\geq 1)$ *over the rational number field* $\mathfrak{P}$;

h — *be a natural number*;

$F(x_0, x_1, \ldots, x_h)$ *be the form*

$$\prod_{r=1}^{m} (x_0 \xi^{(r)h} + x_1 \xi^{(r)h-1} + \cdots + x_h),$$

where $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(m)}$ *are the roots of a polynomial* $f(x)$ *of degree* $m (\geq h)$ *with coefficients from* $\mathfrak{K}$ *and a non-zero discriminant;*

$\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$, *where* $\sigma \geq 0$, *be* σ *different finite prime ideals of* $\mathfrak{K}$;

$G(\mathfrak{r}_k)$ $(k = 1, 2, \ldots, \sigma)$ *be a natural number not greater than* $g(\mathfrak{r}_k)$;

α, β *be two numbers such that*

$$\alpha = \min_{s=1, 2, \ldots, m-1} \left(\frac{m}{s+1} + s\right), \quad \beta > \alpha;$$

ε_0 *be a positive number;*

$u_0, u_1, \ldots, u_h$ *be any system of integers of* $\mathfrak{K}$ *such that* $N((u_0, u_1, \ldots, u_h)) \leq \left|\sqrt{d(\mathfrak{K})}\right|$, *where* $d(\mathfrak{K})$ *is the discriminant of* $\mathfrak{K}$;

Λ *be the maximum of the absolute values of the coefficients of the polynomial in* x

$$\prod_{\mu=1}^{n} (u_0^{(\mu)} x^h + u_1^{(\mu)} x^{h-1} + \cdots + u_h^{(\mu)}),$$

where $u_\theta^{(1)}, u_\theta^{(2)}, \ldots, u_\theta^{(n)}$ $(\theta = 0, 1, 2, \ldots, h)$ *are the conjugate values to* u_θ.

Then the number of solutions in non-associated sets of integers $u_0, u_1, \ldots, u_h$ *of the inequality*

$$|N(F(u_0, u_1, \ldots, u_h))| \prod_{k=1}^{\sigma} |F(u_0, u_1, \ldots, u_h)|_{\mathfrak{r}_k}^{G(\mathfrak{r}_k)} \leq \Lambda^{m-h^2\beta-h},$$

where $N(F(u_0, u_1, \ldots, u_h))$ *is the norm in* $\mathfrak{K}$ *over* $\mathfrak{P}$ *of* $F(u_0, u_1, \ldots, u_h)$, *is not greater than*

$$k_6 (2^{\frac{\beta}{\beta-\alpha}(1+\varepsilon_0)} k_7)^{\left(\sum_{k=1}^{\sigma} G(\mathfrak{r}_k)\right)},$$

where k_6 *is a constant depending only on* $\varepsilon_0, \beta, h, \mathfrak{K}$ *and* $F(x_0, x_1, \ldots, x_h)$, *and not on the number and choice of the finite prime ideals* $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$, *and* k_7 *is a constant depending only on* m *and* h.

2. To prove this theorem, it will first be necessary to extend Theorem 1 of my previous paper to include:

(a) approximation by numbers λ of degree h over $\mathfrak{K}$ which do not lie in the perfect $\mathfrak{r}_1$-adic, $\mathfrak{r}_2$-adic, ..., $\mathfrak{r}_\sigma$-adic extensions of $\mathfrak{K}$;

(b) approximation by the roots $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(h)}$ of a polynomial

$$u_0 x^h + u_1 x^{h-1} + \cdots + u_h,$$

where $u_0, u_1, \ldots, u_h$ are integers of $\mathfrak{K}$.

(a) To include the first set of numbers, we must *extend our definition of valuation with respect to a finite prime ideal.* If ω' be a non-zero element of a field $\mathfrak{K}'$ of degree n' over $\mathfrak{K}$, lying in the perfect $\mathfrak{r}$-adic extension of $\mathfrak{K}$, where $\mathfrak{r}$ is any finite prime ideal of $\mathfrak{K}$, the $\mathfrak{r}$-adic valuation of ω' was defined as

$$|\omega'|_{\mathfrak{r}} = p^{\frac{\mu(\mathfrak{r})}{e(\mathfrak{r})}},$$

where p is the rational prime number divisible by $\mathfrak{r}$, $e(\mathfrak{r})$ is the order of $\mathfrak{r}$, and $\mu(\mathfrak{r})$ is a rational integer such that the fractional ideal $\mathfrak{r}^{\mu(\mathfrak{r})}(\omega')$ contains the factor $\mathfrak{r}$ in neither numerator nor denominator. The valuation of ω' with respect to any of the π' finite prime ideals $\mathfrak{r}'^{(1)}, \mathfrak{r}'^{(2)}, \ldots, \mathfrak{r}'^{(\pi')}$ of $\mathfrak{K}'$ dividing $\mathfrak{r}$ is, of course, $p^{\frac{\mu(\mathfrak{r}'^{(i)})}{e(\mathfrak{r}'^{(i)})}}$ $(i = 1, 2, \ldots, \pi')$, where $\mu(\mathfrak{r}'^{(i)})$ and $e(\mathfrak{r}'^{(i)})$ are defined in the same way as $\mu(\mathfrak{r})$ and $e(\mathfrak{r})$. It is clear that *each of these valuations will be equal to* $|\omega'|_{\mathfrak{r}}$, for if $\mathfrak{r} = \prod_{i=1}^{\pi'} \mathfrak{r}'^{(i)\,E(\mathfrak{r}'^{(i)})}$, $\mu(\mathfrak{r}'^{(i)}) = \mu(\mathfrak{r})\,E(\mathfrak{r}'^{(i)})$ and $e(\mathfrak{r}'^{(i)}) = e(\mathfrak{r})\,E(\mathfrak{r}'^{(i)})$ $(i = 1, 2, \ldots, \pi')$.

If ω' does not lie in the perfect $\mathfrak{r}$-adic extension of $\mathfrak{K}$, the valuation $|\omega'|_{\mathfrak{r}}$ no longer exists according to the above definition, but we now define it to be *any one of the valuations* $|\omega'|_{\mathfrak{r}'^{(1)}}, |\omega'|_{\mathfrak{r}'^{(2)}}, \ldots, |\omega'|_{\mathfrak{r}'^{(\pi')}}$, *which may now be different.*

Now it was stated in the inequality (8) of my previous paper, and proved, that

$$\prod_{j=1}^{r'_1+r'_2} |\omega|_{\mathfrak{q}'_j}^{g(\mathfrak{q}'_j)} \prod_{k=1}^{\sigma'} |\omega'|_{\mathfrak{r}'_k}^{\theta(\mathfrak{r}'_k)} \geq \frac{1}{|W'_0|},$$

where $\mathfrak{q}'_1, \mathfrak{q}'_2, \ldots, \mathfrak{q}'_{r'_1+r'_2}$ are the $r'_1 + r'_2$ infinite prime ideals corresponding to the r'_1 real and r'_2 pairs of conjugate imaginary fields conjugate to $\mathfrak{K}'$, $\mathfrak{r}'_1, \mathfrak{r}'_2, \ldots, \mathfrak{r}'_{\sigma'}$ are σ' (≥ 0) different finite prime ideals of $\mathfrak{K}'$, $\theta(\mathfrak{r}'_k)$ $(k = 1, 2, \ldots, \sigma')$ is a positive number not greater than $g(\mathfrak{r}'_k)$, and W'_0 is the coefficient of the highest power of x in the polynomial $\Upsilon(x; \omega', \mathfrak{P}, nn')$. $(g(\mathfrak{p}'),$ where $\mathfrak{p}'$ is a finite or infinite prime ideal of $\mathfrak{K}'$, is defined in the same way in relation to $\mathfrak{K}'$ as is $g(\mathfrak{p})$ in relation to $\mathfrak{K}$.) Now from the definition of $g(\mathfrak{p})$ given at the beginning of the present paper $g(\mathfrak{r}'^{(i)}) \geq g(\mathfrak{r})$ $(i = 1, 2, \ldots, \pi')$. It therefore follows that inequality (9) of my previous paper, i. e.,

$$\prod_{j=1}^{r'_1+r'_2} |\omega'|_{\mathfrak{q}'_j}^{g(\mathfrak{q}'_j)} \prod_{k=1}^{\sigma} |\omega'|_{\mathfrak{r}_k}^{n'_k} \geq \frac{1}{|W'_0|},$$

remains true, provided $n'_k \leq g(\mathfrak{r}_k)$ $(k = 1, 2, \ldots, \sigma)$. The whole of the proof of Theorem 1 is then valid without alteration, except that $h_{k\delta} = 1$ $(k = 1, 2, \ldots, \sigma;$ $\delta = 1, 2, \ldots, G(\mathfrak{r}_k))$, it being understood that at each step valuations are taken in fields of sufficiently high degree over $\mathfrak{K}$ to give the argument meaning. It may also be noted that no restriction need now be placed on the $\mathfrak{r}$-adic roots of $f(x)$, which can now be any roots of that polynomial.

Theorem 1 therefore states, in its revised form, that if $f(x)$ be a polynomial of degree $m (\geq 2)$ with coefficients from $\mathfrak{K}$ and a non-zero discriminant, if $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_\varrho$, where $0 \leq \varrho \leq r_1 + r_2$, be ϱ of the $r_1 + r_2$ infinite prime ideals corresponding to the r_1 real and r_2 pairs of conjugate imaginary fields conjugate to $\mathfrak{K}$, if $G(\mathfrak{q}_j)$ $(j = 1, 2, \ldots, \varrho)$ be a natural number not greater than $g(\mathfrak{q}_j)$ and $G(\mathfrak{r}_k)$ $(k = 1, 2, \ldots, \sigma)$ a natural number not greater than $g(\mathfrak{r}_k)$, if $\xi_{j\gamma}$ $(j = 1, 2, \ldots, \varrho;$ $\gamma = 1, G(\mathfrak{q}_j))$ and $\eta_{k\delta} (k = 1, 2, \ldots, \sigma;$ $\delta = 1, 2, \ldots, G(\mathfrak{r}_k))$ be roots of $f(x)$, if c, ε_0 be two positive numbers, and if λ be any algebraic number of degree h (or any divisor of h) over $\mathfrak{K}$ and A be the number $|\Upsilon(x; \lambda; \mathfrak{P}; hn)|$, then *the inequality*

$$\prod_{j=1}^{\varrho} \prod_{\gamma=1}^{G(\mathfrak{q}_j)} \min(1, |\lambda - \xi_{j\gamma}|_{\mathfrak{q}_j\gamma}) \prod_{k=1}^{\sigma} \prod_{\delta=1}^{G(\mathfrak{r}_k)} \min(1, |\lambda - \eta_{k\delta}|_{\mathfrak{r}_k}) \leq c A^{-h\beta}$$

is satisfied by not more than

$$k_0 2^{\frac{\beta}{\beta-a}(1+\varepsilon_0)\left(\sum_{j=1}^{\varrho} G(\mathfrak{q}_j) + \sum_{k=1}^{\sigma} G(\mathfrak{r}_k)\right)}$$

different numbers λ, where k_0 is a constant depending only on $\varepsilon_0, \beta, \mathfrak{K}, f(x)$ and h, and not on the number and choice of roots to which approximation is made, nor on the corresponding ideals.

(b) On replacing h by h', the revised Theorem 1 is clearly true for numbers λ of degree h' over $\mathfrak{K}$, where h' is a natural number not greater than h, and A is the number $|\Upsilon(x; \lambda, \mathfrak{P}, h'n)|$.

Now the roots of the polynomial $u_0 x^h + u_1 x^{h-1} + \cdots + u_h$, for any system of integers $u_0, u_1, \ldots, u_h$ of $\mathfrak{K}$, can be of degree $1, 2, \ldots, h-1$ or h over $\mathfrak{K}$. Let the class of numbers $\lambda^{(\nu)}$ contain all possible roots of degree $\nu (\nu = 1, 2, \ldots, h)$ over $\mathfrak{K}$. Then we may select from these h classes those such that the roots of some polynomial $u_0 x^h + u_1 x^{h-1} + \cdots + u_h$ are contained in the selected classes,

at least one in each class. Consider any one such selection S: $\lambda'^{(1)}, \lambda'^{(2)}, \ldots, \lambda'^{(h_0)}$ $(1 \leq h_0 \leq h)$, of degrees $h^{(1)}, h^{(2)}, \ldots, h^{(h_0)}$ over $\mathfrak{K}$. Consider also the inequality

$$(1) \quad \prod_{j=1}^{\varrho} \prod_{\gamma=1}^{G(\mathfrak{q}_j)} \min\left(1, |\lambda - \xi_{j\gamma}|_{\mathfrak{q}_{j\gamma}}\right) \prod_{k=1}^{\sigma} \prod_{\delta=1}^{G(\mathfrak{r}_k)} \min\left(1, |\lambda - \eta_{k\delta}|_{\mathfrak{r}_k}\right) \leq c \prod_{\nu=1}^{h_0} A'^{(\nu)-h^{(\nu)}\beta},$$

where the λ in each valuation is selected from any of the classes $\lambda'^{(1)}, \lambda'^{(2)}, \ldots, \lambda'^{(h_0)}$, and $A'^{(\nu)}$ $(\nu = 1, 2, \ldots, h_0)$ denotes the number $|\overline{\Upsilon(x; \lambda'^{(\nu)}, \mathfrak{P}, h^{(\nu)}n)}|$. This inequality can be split up into h_0 inequalities in each of which the left-hand side includes only valuations involving $\lambda'^{(\nu)}$ $(\nu = 1, 2, \ldots, h_0)$ and the right-hand side is

$$\left|\sqrt[h_0]{c}\right| A'^{(\nu)-h^{(\nu)}\beta}.$$

Now if the inequality (1) is satisfied, so is at least one of the h_0 subsidiary inequalities. But by the revised Theorem 1, each such inequality has not more than

$$k'^{(\nu)} 2^{\frac{\beta}{\beta-\alpha}(1+\varepsilon_0)\left(\sum_{j=1}^{\varrho} G(\mathfrak{q}_j) + \sum_{k=1}^{\sigma} G(\mathfrak{r}_k)\right)}$$

solutions in numbers from the class $\lambda'^{(\nu)}$, where $k'^{(\nu)}$ is a constant depending only on $\varepsilon_0, \beta, c, \mathfrak{K}, f(x), h^{(\nu)}$ and h_0. Also, there are not more than $h^{\left(\sum_{j=1}^{\varrho} G(\mathfrak{q}_j) + \sum_{k=1}^{\sigma} G(\mathfrak{r}_k)\right)}$ ways of selecting the h_0 inequalities. Thus the number of solutions of the inequality (1) in numbers from the classes $\lambda'^{(1)}, \lambda'^{(2)}, \ldots, \lambda'^{(h_0)}$ is not greater than

$$(k'^{(1)} + k'^{(2)} + \cdots + k'^{(h_0)}) \left(2^{\frac{\beta}{\beta-\alpha}(1+\varepsilon_0)} h\right)^{\left(\sum_{j=1}^{\varrho} G(\mathfrak{q}_j) + \sum_{k=1}^{\sigma} G(\mathfrak{r}_k)\right)},$$

or

$$k' \left(2^{\frac{\beta}{\beta-\alpha}(1+\varepsilon_0)} h\right)^{\left(\sum_{j=1}^{\varrho} G(\mathfrak{q}_j) + \sum_{k=1}^{\sigma} G(\mathfrak{r}_k)\right)},$$

where k' is the number $(k'^{(1)} + k'^{(2)} + \cdots + k'^{(h_0)})$ and therefore depends only on $\varepsilon_0, \beta, c, \mathfrak{K}, f(x)$ and h, and not on the number and choice of roots to which approximation is made, nor on the corresponding ideals. It therefore follows, since the total number of selections of the type S from the classes $\lambda^{(\nu)}$ $(\nu = 1, 2, \ldots, h)$ is not greater than h^h, that the number of solutions of the inequality (1) for all possible selections S of classes from these classes is not greater than

$$k'^{(0)}\left(2^{\frac{\beta}{\beta-\alpha}(1+\varepsilon_0)}h\right)^{\left(\sum_{j=1}^{\varrho} G(\mathfrak{q}_j)+\sum_{k=1}^{\sigma} G(\mathfrak{r}_k)\right)},$$

where $k'^{(0)}$ depends only on ε_0, β, c, $\mathfrak{K}$, $f(x)$ and h.

Let now Λ be defined as in the Theorem 2 a already enunciated. Then by a result due to Siegel[1], for every class $\lambda'^{(\nu)}$, and appropriate $u_0, u_1, \ldots, u_h$,

$$\Lambda \geq \frac{\Lambda'^{(\nu)}}{h!}.$$

Further, $h^{(1)} + h^{(2)} + \cdots + h^{(h_0)} = h$. It therefore follows that the inequality

$$\prod_{j=1}^{\varrho}\prod_{\gamma=1}^{G(\mathfrak{q}_j)} \min\left(1, |\lambda - \xi_{j\gamma}|_{\mathfrak{q}_j\gamma}\right) \prod_{k=1}^{\sigma}\prod_{\delta=1}^{G(\mathfrak{r}_k)} \min\left(1, |\lambda - \eta_{k\delta}|_{\mathfrak{r}_k}\right) \leq c\,(h!)^{-h\beta}\,\Lambda^{-h\beta}$$

cannot have more solutions in numbers λ which are roots of a polynomial $u_0 x^h + u_1 x^{h-1} + \cdots + u_h$ than has the inequality (1) in numbers λ chosen from all possible selections S of classes $\lambda'^{(\nu)}$ ($\nu = 1, 2, \ldots, h_0$; $1 \leq h_0 \leq h$). Also, c is arbitrary, and the number of sets of integers $u_0, u_1, \ldots, u_h$ of $\mathfrak{K}$ such that $N((u_0, u_1, \ldots, u_h)) \leq \left|V\overline{d(\mathfrak{K})}\right|$, and such that the polynomials $u_0 x^h + u_1 x^{h-1} + \cdots + u_h$ have the same roots, is not greater than a constant depending only on $\mathfrak{K}$. We have therefore proved the following extension of Theorem 1:

Theorem 1 a. *Let:*

$\mathfrak{K}$	*be a finite algebraic field of degree* $n (\geq 1)$ *over the rational number field* $\mathfrak{P}$;
$f(x)$	*be a polynomial of degree* $m (\geq 2)$ *with coefficients from* $\mathfrak{K}$ *and a non-zero discriminant*;
$\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_\varrho$,	*where* $0 \leq \varrho \leq r_1 + r_2$, *be* ϱ *of the* $r_1 + r_2$ *infinite prime ideals corresponding to the* r_1 *real and* r_2 *pairs of conjugate imaginary fields conjugate to* $\mathfrak{K}$;
$\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$,	*where* $\sigma \geq 0$, *be* σ *different finite prime ideals of* $\mathfrak{K}$;
$G(\mathfrak{q}_j)$	$(j = 1, 2, \ldots, \varrho)$ *be a natural number not greater than* $g(\mathfrak{q}_j)$;
$G(\mathfrak{r}_k)$	$(k = 1, 2, \ldots, \sigma)$ *be a natural number not greater than* $g(\mathfrak{r}_k)$;
h	*be a natural number*;

[1] C. SIEGEL: 'Approximation algebraischer Zahlen', Mathematische Zeitschrift, Vol. 10 (1921), p. 176, Hilfsatz III.

$\xi_{j\gamma}, \eta_{k\delta}$ ($j = 1, 2, \ldots, \varrho$; $\gamma = 1, G(\mathfrak{q}_j)$; $k = 1, 2, \ldots, \sigma$; $\delta = 1, 2, \ldots, G(\mathfrak{r}_k)$) *be roots of* $f(x)$;

c, ε_0 *be two positive numbers*;

α, β *be two numbers such that*

$$\alpha = \min_{s=1,2,\ldots,m-1} \left(\frac{m}{s+1} + s\right), \quad \beta > \alpha;$$

$u_0, u_1, \ldots, u_h$ *be any system of integers of* $\mathfrak{K}$ *such that* $N((u_0, u_1, \ldots, u_h)) \leq \left|\sqrt{d(\mathfrak{K})}\right|$, *where* $d(\mathfrak{K})$ *is the discriminant of* $\mathfrak{K}$, *and* $(u_0, u_1, \ldots, u_h)$ *is the ideal generated by* $u_0, u_1, \ldots, u_h$;

λ *be any root of the polynomial* $u_0 x^h + u_1 x^{h-1} + \cdots + u_h$;

Λ *be the maximum of the absolute values of the coefficients of the polynomial in* x, $\prod_{\mu=1}^{n}(u_0^{(\mu)} x^h + u_1^{(\mu)} x^{h-1} + \cdots + u_h^{(\mu)})$, *where* $u_\theta^{(1)}, u_\theta^{(2)}, \ldots, u_\theta^{(n)}$ ($\theta = 0, 1, 2, \ldots, h$) *are the conjugate values to* u_θ.

Then the number of non-associated systems of integers $u_0, u_1, \ldots, u_h$ *such that roots* λ *of the polynomial* $u_0 x^h + u_1 x^{h-1} + \cdots + u_h$ *satisfy the inequality*

$$\prod_{j=1}^{\varrho} \prod_{\gamma=1}^{G(\mathfrak{q}_j)} \min(1, |\lambda - \xi_{j\gamma}|_{\mathfrak{q}_j}) \prod_{k=1}^{\sigma} \prod_{\delta=1}^{G(\mathfrak{r}_k)} \min(1, |\lambda - \eta_{k\delta}|_{\mathfrak{r}_k}) \leq c\, \Lambda^{-h\beta}$$

is not greater than

$$k^{(0)}\left(2^{\frac{\beta}{\beta-\alpha}(1+\varepsilon_0)} h\right)^{\left(\sum_{j=1}^{\varrho} G(\mathfrak{q}_j) + \sum_{k=1}^{\sigma} G(\mathfrak{r}_k)\right)}$$

where $k^{(0)}$ *is a constant depending only on* $\varepsilon_0, \beta, c, \mathfrak{K}, f(x)$ *and* h, *and not on the number and choice of roots to which approximation is made, nor on the corresponding deals.*

3. We shall require the following extension of a lemma due to Siegel[1]:

Lemma. *Let* $\mathfrak{K}$, $f(x)$, h *and* $\mathfrak{q}_{j\gamma}$ ($j = 1, 2, \ldots, r_1 + r_2$; $\gamma = 1, g(\mathfrak{q}_j)$) *be defined as in Theorem 1 a. Let* $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(m)}$ *be the roots of the polynomial* $f(x)$. *Let*

$$y_\nu = u_0 \xi^{(\nu)h} + u_1 \xi^{(\nu)h-1} + \cdots + u_h \qquad (\nu = 1, 2, \ldots, m),$$

where $u_0, u_1, \ldots, u_h$ *are integers of* $\mathfrak{K}$ *such that* $N((u_0, u_1, \ldots, u_h)) \leq \left|\sqrt{d(\mathfrak{K})}\right|$, *where* $d(\mathfrak{K})$ *is the discriminant of* $\mathfrak{K}$. *Then:*

[1] See p. 88, note 1: SIEGEL, p. 196.

1) *if*

$$\max\left(|u_0|_{\mathfrak{q}_{j\gamma}}, |u_1|_{\mathfrak{q}_{j\gamma}}, \ldots, |u_h|_{\mathfrak{q}_{j\gamma}}\right) = u_{j\gamma} \qquad (j = 1, 2, \ldots, r_1 + r_2;\ \gamma = 1, g(\mathfrak{q}_{j\gamma})),$$

there exists a positive constant c_0, depending only on $f(x)$, such that of the m linear forms y_ν not more than h are such that

$$|y_\nu|_{\mathfrak{q}_{j\gamma}} = |u_0 \xi_{j\gamma}^{(\nu)h} + u_1 \xi_{j\gamma}^{(\nu)h-1} + \cdots + u_h|_{\mathfrak{q}_{j\gamma}} < c_0 u_{j\gamma},$$

for valuation with respect to each infinite prime ideal $\mathfrak{q}_{j\gamma}$ $(j = 1, 2, \ldots, r_1 + r_2;$ $\gamma = 1, g(\mathfrak{q}_j))$ $(\xi_{j\gamma}^{(1)}, \xi_{j\gamma}^{(2)}, \ldots, \xi_{j\gamma}^{(m)}$ being the roots of the polynomial conjugate to $f(x)$ with respect to the field $\mathfrak{K}_{j\gamma})$;

2) *if $\mathfrak{r}$ be any finite prime ideal of $\mathfrak{K}$, there exists a positive constant $c_\mathfrak{r}$, depending only on $f(x)$ and $\mathfrak{r}$, such that of the m linear forms y_ν not more than h are such that*

$$|y_\nu|_\mathfrak{r} = |u_0 \xi^{(\nu)h} + \cdots + u_h|_\mathfrak{r} < c_\mathfrak{r},$$

and $c_\mathfrak{r} = 1$ for all $\mathfrak{r}$ with norms exceeding a certain value depending only on $\mathfrak{K}$ and $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(m)}$.

(N. B. The symbol $|u_0 \xi_{j\gamma}^{(\nu)h} + u_1 \xi_{j\gamma}^{(\nu)h-1} + \cdots + u_h|_{\mathfrak{q}_{j\gamma}}$ represents the absolute value of the expression $u_0^{(j\gamma)} \xi_{j\gamma}^{(\nu)h} + u_1^{(j\gamma)} \xi_{j\gamma}^{(\nu)h-1} + \cdots + u_h^{(j\gamma)}$, where $u_0^{(j\gamma)}, u_1^{(j\gamma)}, \ldots, u_h^{(j\gamma)}$ are the conjugate values to $u_0, u_1, \ldots, u_h$ with respect to the field $\mathfrak{K}_{j\gamma}$.)

Proof. The lemma is trivial for $h \geq m$. We may therefore suppose that $h \leq m - 1$.

We select any $h + 1$ of the linear forms y_ν, which without loss of generality may be written as

$$y_\nu = \sum_{\theta=0}^{h} \xi^{(\nu)h-\theta} \mu_\theta \qquad (\nu = 1, 2, \ldots, h + 1).$$

Since $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(m)}$ are all different, $u_0, u_1, \ldots, u_h$ may be written as linear functions of $y_1, y_2, \ldots, y_{h+1}$, in which the coefficients $l_1^{(\nu)}, l_2^{(\nu)}, \ldots, l_{h+1}^{(\nu)}$ $(\nu = 0, 1, \ldots, h)$ depend only on $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(m)}$.

1) By the above

$$\max\left(|l_1^{(\nu)}|_{\mathfrak{q}_{j\gamma}}, |l_2^{(\nu)}|_{\mathfrak{q}_{j\gamma}}, \ldots, |l_{h+1}^{(\nu)}|_{\mathfrak{q}_{j\gamma}}\right) \leq c_1$$

$$(j = 1, 2, \ldots, r_1 + r_2;\ \gamma = 1, g(\mathfrak{q}_j);\ \nu = 0, 1, \ldots, h),$$

where c_1 is a constant depending only on $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(m)}$. Hence

$$u_{j\gamma} \leq (h+1)\, c_1\, y_{j\gamma} \qquad (j = 1, 2, \ldots, r_1 + r_2;\ \gamma = 1, g(\mathfrak{q}_j)),$$

where

$$y_{j\gamma} = \max\,(|y_1|_{\mathfrak{q}_{j\gamma}}, |y_2|_{\mathfrak{q}_{j\gamma}}, \ldots, |y_{h+1}|_{\mathfrak{q}_{j\gamma}}).$$

Thus

$$y_{j\gamma} \geq \frac{u_{j\gamma}}{(h+1)\, c_1} \geq \frac{u_{j\gamma}}{m\, c_1} \qquad (j = 1, 2, \ldots, r_1 + r_2;\ \gamma = 1, g(\mathfrak{q}_j)).$$

We can determine such a c_1 for every possible combination of $h+1$ linear forms y_ν. Then if $c_0 \leq \frac{1}{m\, c_1}$ for each c_1, c_0 is the constant required.

2) We have, further,

$$\max\,(|l_1^{(\nu)}|_{\mathfrak{r}}, |l_2^{(\nu)}|_{\mathfrak{r}}, \ldots, |l_{h+1}^{(\nu)}|_{\mathfrak{r}}) \leq c'_{\mathfrak{r}} \qquad (\nu = 0, 1, \ldots, h),$$

where $c'_{\mathfrak{r}}$ is a constant depending only on $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(m)}$ and on $\mathfrak{r}$. Then if

$$\max\,(|u_0|_{\mathfrak{r}}, |u_1|_{\mathfrak{r}}, \ldots, |u_h|_{\mathfrak{r}}) = u_{\mathfrak{r}},\ \max\,(|y_1|_{\mathfrak{r}}, |y_2|_{\mathfrak{r}}, \ldots, |y_{h+1}|_{\mathfrak{r}}) = y_{\mathfrak{r}},$$

it follows as before that

$$u_{\mathfrak{r}} \leq c'_{\mathfrak{r}}\, y_{\mathfrak{r}},$$

so that

$$y_{\mathfrak{r}} \geq \frac{u_{\mathfrak{r}}}{c'_{\mathfrak{r}}}.$$

Now since $N((u_0, u_1, \ldots, u_h)) \leq |\sqrt{d(\mathfrak{K})}|$,

$$u_{\mathfrak{r}} \geq c''_{\mathfrak{r}},$$

where $c''_{\mathfrak{r}}$ is a constant depending only on $\mathfrak{K}$ and $\mathfrak{r}$. Hence

$$y_{\mathfrak{r}} \geq \frac{c''_{\mathfrak{r}}}{c'_{\mathfrak{r}}}.$$

We can determine a constant $c'_{\mathfrak{r}}$ for every possible combination of $h+1$ linear forms y_ν. Then if $c_{\mathfrak{r}} \leq \frac{c''_{\mathfrak{r}}}{c'_{\mathfrak{r}}}$ for each such $c'_{\mathfrak{r}}$, $c_{\mathfrak{r}}$ is the constant required.

Further, for all $\mathfrak{r}$ with norms greater than a certain value depending only on $\mathfrak{K}$ and $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(m)}$, we may take $c_{\mathfrak{r}} = c'_{\mathfrak{r}} = c''_{\mathfrak{r}} = 1$.

4. We can now prove the main theorem by a method analogous to that used by Siegel[1] in dealing with the same problem. We write

[1] See p. 88, note 1: SIEGEL, pp. 197—8.

$$F(u_0, u_1, \ldots, u_h) = \prod_{\nu=1}^{m} y_\nu = \prod_{\nu=1}^{m} (u_0 \xi^{(\nu)h} + u_1 \xi^{(\nu)h-1} + \cdots + u_h),$$

where $u_0, u_1, \ldots, u_h$ are integers of $\mathfrak{K}$ such that $N((u_0, u_1, \ldots, u_h)) \leq |\sqrt{d(\mathfrak{K})}|$.

We may assume $u_0 \neq 0$, as we may prove the theorem separately, by taking appropriate values in place of h, for the cases $u_0 = 0$, $u_1 \neq 0$; $u_0 = 0$, $u_1 = 0$, $u_2 \neq 0$; $\ldots$; $u_0 = 0$, $u_1 = 0, \ldots, u_{h-1} = 0$, $u_h \neq 0$, and sum the numbers of solutions.

Since $u_0 \neq 0$, we may write

$$y_\nu = u_0 \xi^{(\nu)h} + u_1 \xi^{(\nu)h-1} + \cdots + u_h = u_0 \prod_{\theta=1}^{h} (\xi^{(\nu)} - \lambda^{(\theta)}) \qquad (\nu = 1, 2, \ldots, m).$$

We choose combinations of h different forms from the m linear forms y_ν. (We assume $h \leq m$.) Let now $y_1^{(j\gamma)}, y_2^{(j\gamma)}, \ldots, y_h^{(j\gamma)}$ $(1 \leq j \leq r_1 + r_2;\ 1 \leq \gamma \leq g(\mathfrak{q}_j))$ be such a combination. Then

$$|y_1^{(j\gamma)} y_2^{(j\gamma)} \ldots y_h^{(j\gamma)}|_{\mathfrak{q}_{j\gamma}} = |u_0|_{\mathfrak{q}_{j\gamma}}^{h} \prod_{\nu=1}^{h} \prod_{\theta=1}^{h} |\xi_{j\gamma}^{(\nu)} - \lambda^{(\theta)}|_{\mathfrak{q}_{j\gamma}}.$$

Let $y_1^{(k)}, y_2^{(k)}, \ldots, y_h^{(k)}$ $(1 \leq k \leq \sigma)$ be another, not necessarily different, such combination. Then

$$|y_1^{(k)}, y_2^{(k)} \ldots y_h^{(k)}|_{\mathfrak{r}_k} = |u_0|_{\mathfrak{r}_k}^{h} \prod_{\nu=1}^{h} \prod_{\theta=1}^{h} |\xi_k^{(\nu)} - \lambda^{(\theta)}|_{\mathfrak{r}_k}.$$

In each case the ξ's are the roots of $f(x)$ or the appropriate conjugate polynomial corresponding to the linear forms chosen and their valuations. Now there exist positive constants $b_{j\gamma}$ $(j = 1, 2, \ldots, r_1 + r_2;\ \gamma = 1, g(\mathfrak{q}_j))$ and b_k $(k = 1, 2, \ldots, \sigma)$, depending only on $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(m)}$ and the prime ideals $\mathfrak{q}_{j\gamma}$ and $\mathfrak{r}_k$, such that the minima of the valuations of the differences of the numbers $\xi_{j\gamma}^{(1)}, \xi_{j\gamma}^{(2)}, \ldots, \xi_{j\gamma}^{(m)}$ with respect to $\mathfrak{q}_{j\gamma}$ and $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(m)}$ with respect to $\mathfrak{r}_k$ are respectively not less than $b_{j\gamma}$ and b_k. Thus in each double product there are not more than h factors with valuations less than $\frac{1}{2} b_{j\gamma}$ or b_k, as the case may be. For if $|\xi_{j\gamma}^{(\nu)} - \lambda^{(\theta)}|_{\mathfrak{q}_{j\gamma}} < \frac{1}{2} b_{j\gamma}$ and $|\xi_{j\gamma}^{(\nu')} - \lambda^{(\theta)}|_{\mathfrak{q}_{j\gamma}} < \frac{1}{2} b_{j\gamma}$, where $\nu \neq \nu'$, then $|\xi_{j\gamma}^{(\nu)} - \xi_{j\gamma}^{(\nu')}|_{\mathfrak{q}_{j\gamma}} < b_{j\gamma}$. Similarly, if $|\xi_k^{(\nu)} - \lambda^{(\theta)}|_{\mathfrak{r}_k} < b_k$ and $|\xi_k^{(\nu')} - \lambda^{(\theta)}|_{\mathfrak{r}_k} < b_k$, then $|\xi_k^{(\nu)} - \xi_k^{(\nu')}|_{\mathfrak{r}_k} < b_k$. Both these results are impossible.

It therefore follows, since, by the inequality (6 a) of my previous paper,

$$\prod_{j=1}^{r_1+r_2}\prod_{\gamma=1}^{g(\mathfrak{q}_j)} |u|_{\mathfrak{q}_{j\gamma}} \prod_{k=1}^{\sigma} |u_0|_{\mathfrak{r}_k}^{G(\mathfrak{r}_k)} \geq 1,$$

that the number of solutions of the inequality

$$(2) \qquad \prod_{j=1}^{r_1+r_2}\prod_{\gamma=1}^{g(\mathfrak{q}_j)} |y_1^{(j\gamma)} y_2^{(j\gamma)} \dots y_h^{(j\gamma)}|_{\mathfrak{q}_{j\gamma}} \prod_{k=1}^{\sigma} |y_1^{(k)} y_2^{(k)} \dots y_h^{(k)}|_{\mathfrak{r}_k}^{G(\mathfrak{r}_k)} \leq C\Delta^{-h^2\beta},$$

Δ being defined as in Theorems 1 a and 2 a and C being any positive constant, is not greater than the total number of solutions in roots λ of polynomials $u_0 x^h + u_1 x^{h-1} + \cdots + u_h$ of all the possible inequalities

$$(3) \qquad \prod_{j=1}^{r_1+r_2}\prod_{\gamma=1}^{g(\mathfrak{q}_j)}\prod_{\nu=1}^{h} |\xi_{j\gamma}^{\prime(\nu)} - \lambda|_{\mathfrak{q}_{j\gamma}} \prod_{k=1}^{\sigma}\prod_{\nu=1}^{h} |\xi_k^{\prime(\nu)} - \lambda|_{\mathfrak{r}_k}^{G(\mathfrak{r}_k)} \geq$$

$$\geq C \prod_{j=1}^{r_1+r_2}\prod_{\gamma=1}^{g(\mathfrak{q}_j)} (\tfrac{1}{2} b_{j\gamma})^{-h(h-1)} \prod_{k=1}^{\sigma} b_k^{-h(h-1)G(\mathfrak{r}_k)} \cdot \Delta^{-h^2\beta},$$

where the selection of roots $\xi_{j\gamma}^{\prime(1)}, \xi_{j\gamma}^{\prime(2)}, \dots, \xi_{j\gamma}^{\prime(h)}$ passes through all possible combinations, with duplications, of $\xi_{j\gamma}^{(1)}, \xi_{j\gamma}^{(2)}, \dots, \xi_{j\gamma}^{(h)}$, and the selection $\xi_k^{\prime(1)}, \xi_k^{\prime(2)}, \dots, \xi_k^{\prime(h)}$ passes similarly through all combinations of $\xi_k^{(1)}, \xi_k^{(2)}, \dots, \xi_k^{(h)}$.

When the inequality (3) is satisfied, so also is at least one of the inequalities

$$(4) \qquad \prod_{j=1}^{r_1+r_2}\prod_{\gamma=1}^{g(\mathfrak{q}_j)} |\xi_{j\gamma}^{(\nu_{j\gamma})} - \lambda|_{\mathfrak{q}_{j\gamma}} \prod_{k=1}^{\sigma} |\xi_k^{(\nu_k)} - \lambda|_{\mathfrak{r}_k}^{G(\mathfrak{r}_k)} \leq C \prod_{j=1}^{r_1+r_2}\prod_{\gamma=1}^{g(\mathfrak{q}_j)} (\tfrac{1}{2} b_{j\gamma})^{-(h-1)} \prod_{k=1}^{\sigma} b_k^{-(h-1)G(\mathfrak{r}_k)} \cdot \Delta^{-h\beta}$$

$$(1 \leq \nu_{j\gamma} \leq h;\ 1 \leq \nu_k \leq h).$$

Now $b_k = 1$ for all except a finite number of finite prime ideals $\mathfrak{r}_k$, since b_k depends only on $\xi^{(1)}, \xi^{(2)}, \dots, \xi^{(m)}$ in addition to $\mathfrak{r}_k$. Hence

$$\prod_{j=1}^{r_1+r_2}\prod_{\gamma=1}^{g(\mathfrak{q}_j)} (\tfrac{1}{2} b_{j\gamma})^{h-1} \prod_{k=1}^{\sigma} b_k^{(h-1)G(\mathfrak{r}_k)} \geq b_0,$$

where b_0 is a positive constant depending only on $\xi^{(1)}, \xi^{(2)}, \dots, \xi^{(m)}$. Thus the number of solutions of (4) is not greater than the number of solutions of the inequality

$$(5)\qquad \prod_{j=1}^{r_1+r_2}\prod_{\gamma=1}^{g(\mathfrak{q}_j)} |\xi^{(\nu_{j\gamma})}_{j\gamma}-\lambda|_{\mathfrak{q}_{j\gamma}} \prod_{k=1}^{\sigma} |\xi^{(\nu_k)}_{k}-\lambda|^{G(\nu_k)}_{\nu_k} \leq C\, b_0^{-1}\, A^{-h\beta}$$

$$(1 \leq \nu_{j\gamma} \leq h;\ 1 \leq \nu_k \leq h).$$

By Theorem 1 a, to this inequality there correspond not more than

$$k^{(0)}\left(2^{\frac{\beta}{\beta-\alpha}(1+\varepsilon_0)}h\right)^{\left(n+\sum_{k=1}^{\sigma} G(\nu_k)\right)}$$

non-associated systems of integers $u_0, u_1, \ldots, u_h$ of $\mathfrak{K}$ with $N((u_0, u_1, \ldots, u_h)) \leq$ $\leq \left|V\overline{d(\mathfrak{K})}\right|$ and such that roots λ of the polynomial $u_0 x^h + u_1 x^{h-1} + \cdots + u_h$ satisfy the inequality, where $k^{(0)}$ depends only on $\varepsilon_0, \beta, C, \mathfrak{K}, f(x)$ and h. (Such systems of integers will be called 'solutions' of the corresponding inequality.)

Corresponding to each innequality (3) there are not more than $h^{n+\sigma}$ inequalities (5). There are, further, not more than $h^{(n+\sigma)h}$ possible inequalities (3). It therefore follows that the number of solutions of (2) in non-associated systems of integers $u_0, u_1, \ldots, u_h$ is not greater than

$$k^{(0)}h^{(h+1)(n+\sigma)}\left(2^{\frac{\beta}{\beta-\alpha}(1+\varepsilon_0)}h\right)^{\left(n+\sum_{k=1}^{\sigma} G(\nu_k)\right)}.$$

Now the $n+\sigma$ sets of h linear forms contained in the inequality (2) can be chosen from the m forms $y_1, y_2, \ldots, y_m$ in $\left(\frac{m!}{h!(m-h)!}\right)^{n+\sigma}$ ways. Thus the total number of solutions of all the possible inequalities (2) in non-associated systems of integers $u_0, u_1, \ldots, u_h$, with $u_0 \neq 0$, is not greater than

$$k^{(0)}\left(2^{\frac{\beta}{\beta-\alpha}(1+\varepsilon_0)}h\right)^{\left(n+\sum_{k=1}^{\sigma} G(\nu_k)\right)}\left(\frac{h^{h+1}\cdot m!}{h!(m-h)!}\right)^{n+\sigma}.$$

It follows that the number of solutions with $u \neq 0$ is not greater than

$$k_4\left(2^{\frac{\beta}{\beta-\alpha}(1+\varepsilon_0)}k_5\right)^{\left(\sum_{k=1}^{\sigma} G(\nu_k)\right)},$$

where k_4 is a constant depending only on $\varepsilon_0, \beta, C, \mathfrak{K}, h$ and $f(x)$, and k_5 a constant depending only on m and h. Replacing h by $h-1, h-2, \ldots, 1$ successively, and summing the number of sulutions in each case, it follows that the number of solutions of all possible inequalities (2) in non-associated systems of

integers $u_0, u_1, \ldots, u_h$ of $\mathfrak{K}$ such that $N((u_0, u_1, \ldots, u_h)) \leq |\sqrt{d(\mathfrak{K})}|$ is not greater than

$$k_6 (2^{\frac{\beta}{\beta-\alpha}(1+\varepsilon_0)} k_7)^{\left(\sum_{k=1}^{\sigma} G(\mathfrak{r}_k)\right)},$$

where k_6 is a constant depending only on ε_0, β, C, $\mathfrak{K}$, h and $f(x)$, and k_7 a constant depending only on m and h.

But by the lemma, for each system of integers $u_0, u_1, \ldots, u_h$,

$$\prod_{j=1}^{r_1+r_2} \prod_{\gamma=1}^{g(\mathfrak{q}_j)} \prod_{\nu=1}^{m} |u_0 \xi_{j\gamma}^{(\nu)h} + u_1 \xi_{j\gamma}^{(\nu)h-1} + \cdots + u_h|_{\mathfrak{q}_{j\gamma}} \prod_{k=1}^{\sigma} \prod_{\nu=1}^{m} |u_0 \xi_k^{(\nu)h} + u_1 \xi_k^{(\nu)h-1} + \cdots + u_h|_{\mathfrak{r}_k}^{G(\mathfrak{r}_k)}$$

$$\geq c_0^{(m-h)n} \prod_{j=1}^{r_1+r_2} \prod_{\gamma=1}^{g(\mathfrak{q}_j)} u_{j\gamma}^{m-h} \prod_{k=1}^{\sigma} c_k^{(m-h)G(\mathfrak{r}_k)} \prod_{j=1}^{r_1+r_2} \prod_{\gamma=1}^{g(\mathfrak{q}_j)} |y_1^{(j\gamma)} y_2^{(j\gamma)} \ldots y_h^{(j\gamma)}|_{\mathfrak{q}_{j\gamma}} \prod_{k=1}^{\sigma} |y_1^{(k)} y_2^{(k)} \ldots y_h^{(k)}|_{\mathfrak{r}_k}^{G(\mathfrak{r}_k)},$$

for some $n + \sigma$ systems of h linear forms $y_1^{(j\gamma)}, y_2^{(j\gamma)}, \ldots, y_h^{(j\gamma)}$ $(1 \leq j \leq r_1 + r_2;$ $1 \leq \gamma \leq g(\mathfrak{q}_j))$ and $y_1^{(k)}, y_2^{(k)}, \ldots, y_h^{(k)}$ $(1 \leq k \leq \sigma)$, and some constants c_k $(k = 1, 2, \ldots, \sigma)$ depending only on $\mathfrak{K}$, $f(x)$ and $\mathfrak{r}_k$, and equal to 1 for all $\mathfrak{r}_k$ with norms greater than a certain constant depending only on $\mathfrak{K}$ and $f(x)$. Thus, since

$$\prod_{j=1}^{r_1+r_2} \prod_{\gamma=1}^{g(\mathfrak{q}_j)} u_{j\gamma} \geq \Lambda,$$

and

$$c_0^{(m-h)n} \prod_{k=1}^{\sigma} c_k^{(m-h)G(\mathfrak{r}_k)} \geq C',$$

where C' is a constant depending only on h, $\mathfrak{K}$ and $f(x)$, it follows, since $h \leq m$, that

$$|N(F(u_0, u_1, \ldots, u_h))| \prod_{k=1}^{\sigma} |F(u_0, u_1, \ldots, u_h)|_{\mathfrak{r}_k}^{G(\mathfrak{r}_k)} \geq$$

$$\geq C' \prod_{j=1}^{r_1+r_2} \prod_{\gamma=1}^{g(\mathfrak{q}_j)} |y_1^{(j\gamma)} y_2^{(j\gamma)} \ldots y_h^{(j\gamma)}|_{\mathfrak{q}_{j\gamma}} \prod_{k=1}^{\sigma} |y_1^{(k)} y_2^{(k)} \ldots y_h^{(k)}|_{\mathfrak{r}_k} \Lambda^{m-h}.$$

On taking $C = \frac{1}{C'}$, Theorem 2 a follows.

5. Provided $m > h^2 \alpha + h$, we can choose β so that $h^2 \beta + h = m$. Then the inequality of Theorem 2 a becomes

$$|N(F(u_0, u_1, \ldots, u_h))| \prod_{k=1}^{\sigma} |F(u_0, u_1, \ldots, u_h)|_{\mathfrak{r}_k}^{G(\mathfrak{r}_k)} \leq 1.$$

If $\varepsilon_0 = 1$, the number of solutions of this inequality in non-associated systems of integers $u_0, u_1, \ldots, u_h$ of $\mathfrak{K}$ such that $N((u_0, u_1, \ldots, u_h)) \leq |\sqrt{d(\mathfrak{K})}$ is not greater than

$$k_6\left(4^{\frac{m-h}{h^2\left(\frac{m-h}{h^2}-\alpha\right)}} k_7\right)^{\left(\sum_{k=1}^{\sigma} G(\mathfrak{r}_k)\right)}$$

and therefore not greater than

$$K_6^{\left(\sum_{k=1}^{\sigma} G(\mathfrak{r}_k)\right)+1},$$

where K_6 is a constant depending only on $\mathfrak{K}$ and $F(x_0, x_1, \ldots, x_h)$, and not on the number and choice of the finite prime ideals $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$.

Further, the result is still true if one of $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(m)}$ is infinite, i. e., if one factor of $F(x_0, x_1, \ldots, x_h)$ is x_0, for any form $F(x_0, x_1, \ldots, x_h)$ of this type such that the corresponding $f(x)$ has a non-zero discriminant may be transformed by a linear transformation of determinant 1 with rational integral coefficients into a form of the type already dealt with.

We have therefore proved the following corollary:

Corollary 1. *Let:*

$\mathfrak{K}$ *be a finite algebraic field of degree* $n (\geq 1)$ *over the rational number field* $\mathfrak{P}$;

$F(x_0, x_1, \ldots, x_h)$ *be the form*

$$\prod_{\nu=1}^{m} (x_0 \xi^{(\nu)h} + x_1 \xi^{(\nu)h-1} + \cdots + x_h),$$

where $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(m)}$ (*of which one may be infinite*) *are the roots of a polynomial* $f(x)$ *with coefficients from* $\mathfrak{K}$ *and a non-zero discriminant, and of degree* $m > h^2\alpha + h$, *where*

$$\alpha = \min_{s=1, 2, \ldots, m-1} \left(\frac{m}{s+1} + s\right);$$

$\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$, *where* $\sigma \geq 0$, *be* σ *different finite prime ideals of* $\mathfrak{K}$;

$G(\mathfrak{r}_k)$ $(k = 1, 2, \ldots, \sigma)$ *be a natural number not greater than* $g(\mathfrak{r}_k)$.

Then the number of solutions of the inequality

$$|N(F(u_0, u_1, \ldots, u_h))| \prod_{k=1}^{\sigma} |F(u_0, u_1, \ldots, u_h)|_{\mathfrak{r}_k}^{G(\mathfrak{r}_k)} \leq 1$$

in non-associated systems of integers $u_0, u_1, \ldots, u_h$ *of* $\mathfrak{K}$ *such that* $N((u_0, u_1, \ldots, u_h)) \leq$ $\leq |\sqrt{d(\mathfrak{K})}|$, *where* $N(F(u_0, u_1, \ldots, u_h))$ *is the norm in* $\mathfrak{K}$ *over* $\mathfrak{P}$ *of* $F(u_0, u_1, \ldots, u_h)$ *and* $d(\mathfrak{K})$ *is the discriminant of* $\mathfrak{K}$, *is not greater than*

$$K_6^{\sum_{k=1}^{\sigma} G(\mathfrak{r}_k)+1},$$

where K_6 *is a constant depending only on* $\mathfrak{K}$ *and* $F(x_0, x_1, \ldots, x_h)$, *and not on the number and choice of the finite prime ideals* $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\sigma$.

6. From this corollary arises, as before, the following corollary, the proof of which follows exactly the same lines as in the case of binary forms:

Corollary 2. *If* $\mathfrak{K}$ *be a finite algebraic field over the rational number field* $\mathfrak{P}$, *if* $F(x_0, x_1, \ldots, x_h)$ *be the form*

$$\prod_{\nu=1}^{m} (x_0 \xi^{(\nu)h} + x_1 \xi^{(\nu)h-1} + \cdots + x_h),$$

where $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(m)}$ (*of which one may be infinite*) *are the conjugate roots of a polynomial* $f(x)$ *with coefficients from* $\mathfrak{K}$ *and at least three different roots, and of degree* $m > h^2\alpha + h$, *where* $\alpha = \min_{s=1,2,\ldots,m-1} \left(\frac{m}{s+1} + s\right)$, *and if* $u_0, u_1, \ldots, u_h$ *be any system of integers of* $\mathfrak{K}$ *such that* $N((u_0, u_1, \ldots, u_h)) \leq |\sqrt{d(\mathfrak{K})}|$, *where* $d(\mathfrak{K})$ *is the discriminant of* $\mathfrak{K}$, *then as*

$$\max(|N(u_0)|, |N(u_1)|, \ldots, |N(u_h)|) \to \infty,$$

the greatest of the norms of the finite prime ideals of $\mathfrak{K}$ *dividing the numerator of* $F(u_0, u_1, \ldots, u_h)$ *in its reduced form tends to infinity.*

(We are justified in writing 'numerator' because $F(u_0, u_1, \ldots, u_h)$ becomes an integer of $\mathfrak{K}$ on multiplying by a_0^h, where a_0 is the coefficient of x^m in the polynomial $f(x)$ (taken now to have integral coefficients), except when one of $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(m)}$ is infinite. Then $a_0 = 0$, and we must multiply by a_1^h, where a_1 is the coefficient of x^{m-1} in the polynomial $f(x)$.)

We can obtain a lower bound for m in terms of h, such that $m > h^2\alpha + h$, by using a method due to Siegel. Siegel proved[1] that $\alpha \leq \sqrt{4m+1} - 1$. It therefore follows, if we write $m \geq 4h^4 - 2h^2 + m^*$, where m^* is a rational integer ≥ 0, that

$$\begin{aligned} m - h^2\alpha &\geq 4h^4 - 2h^2 + m^* - h^2(\sqrt{16h^4 - 8h^2 + 1 + 4m^*} - 1) \\ &\geq 4h^4 - 2h^2 + m^* - h^2\left(4h^2 - 1 + \frac{2m^*}{4h^2-1} - 1\right) \\ &= m^* - \frac{2h^2 m^*}{4h^2-1} = m^*\frac{2h^2-1}{4h^2-1}. \end{aligned}$$

Now

$$m^*\frac{2h^2-1}{4h^2-1} > h,$$

provided $m^* > h\frac{4h^2-1}{2h^2-1} = 2h + \frac{h}{2h^2-1}$, i. e., provided

$$m^* \geq 2h + \frac{h}{2h^2-1} + 1 = 4 \text{ if } h = 1,$$

or

$$m^* \geq 2h + 1 \text{ if } h > 1.$$

Thus $m > h^2\alpha + h$, provided

$$m \geq 6 \text{ if } h = 1$$
$$m \geq 4h^4 - 2h^2 + 2h + 1 \text{ if } h > 1.$$

Thus, for example,

$$m \geq 61 \quad \text{for } h = 2,$$
$$m \geq 313 \quad \text{for } h = 3,$$
$$m \geq 1001 \quad \text{for } h = 4.$$

(N. B. It has already been shown in the previous paper that m can be any integer greater than 3 if $h = 1$.)

7. The remaining corollaries on binary forms, concerning the representation of integers of $\mathfrak{R}$ by binary forms and the greatest rational prime divisor of the norm of a number of such representations, also have their exact counterparts in

[1] See p. 88, note 1: SIEGEL, pp. 191—2.

the present case, and these too depend on the basic corollary. It is necessary however, in making this generalisation, that $f(x)$ should have integral coefficients and be of degree $m > h^2\alpha + h$, and to consider forms of the type $a_0^h F(u_0, u_1, \ldots, u_k)$, where a_0 is the coefficient of x^m in $f(x)$, instead of forms of the type $F(u_0, u_1, \ldots, u_h)$, since forms of the latter type are not necessarily integers. In cases where one of the roots of $f(x)$ is infinite (as is now possible), $a_0 = 0$, and we must consider forms of the type $a_1^h F(u_0, u_1, \ldots, u_h)$, where a_1 is the coefficient of x^{m-1} in $f(x)$. All these forms are denoted for convenience by $F^*(u_0, u_1, , \ldots, u_h)$. With these modifications, and retaining otherwise the conditions of Theorems 1 a and 2 a, the following result is true:

The number of non-associated systems of integers $u_0, u_1, \ldots, u_h$ of $\mathfrak{R}$ such that $N(F^*(u_0, u_1, \ldots, u_h))$ is divisible by no rational prime numbers other than the $\tau (\geq 0)$ given different rational prime numbers $r_1, r_2, \ldots, r_\tau$ is not greater than

$$K_0^{\tau+1},$$

where K_0 is a number depending only on $\mathfrak{R}$ and $F^*(x_0, x_1, \ldots, x_h)$, and not on the number and choice of $r_1, r_2, \ldots, r_\tau$.

From this result follow various others on the representation of integers of $\mathfrak{R}$ in the form $F^*(u_0, u_1, \ldots, u_h)$, corresponding to those obtained for binary forms.

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