EXTREMUM PROBLEMS IN THE THEORY OF ANALYTIC FUNCTIONS

By

A. J. MACINTYRE and W. W. ROGOSINSKI
of ABERDEEN of NEWCASTLE UPON TYNE

1. Introduction.

1.1 Let $p \geq 1$. We deal in the following with the class H_p of all functions $f(z)$ regular in $|z| < 1$ for which the mean values

$$M_p(f, r) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right\}^{1/p} \tag{1.1.1}$$

are bounded for $0 \leq r < 1$. If $p = \infty$, the class H_∞ is the class of all $f(z)$ regular and bounded in $|z| < 1$. Also

$$M_\infty(f, r) = \operatorname*{Max}_{|z|=r} |f(z)| \, . \tag{1.1.2}$$

By $\mathfrak{H}_p$ we denote the wider class of all functions $f(z)$, regular in $|z| < 1$ except perhaps for a finite number of poles, and such that $M_p(f, r)$ remains bounded 'eventually', i. e. for $r_0 < r < 1$ and some $r_0 < 1$.

It is well known[1] that any function $f(z)$ of H_p (or $\mathfrak{H}_p$) possesses boundary values

$$f(e^{i\theta}) = \lim_{r \to 1-0} f(re^{i\theta}) \tag{1.1.3}$$

for almost all θ, and that, if $p < \infty$,

$$\int_0^{2\pi} |f(re^{i\theta}) - f(e^{i\theta})|^p d\theta \to 0 \tag{1.1.4}$$

as $r \to 1-0$, so that $f(e^{i\theta})$ is integrable L^p. If $p = \infty$, $f(re^{i\theta}) \to f(e^{i\theta})$ boundedly for almost all θ.

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[1] Zygmund, 162–164. Compare the list of references at the end of this paper.

It follows that, if $p < \infty$,

$$(1.1.5) \qquad M_p(f, r) \to \left\{\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^p d\theta\right\}^{1/p} = M_p(f, 1) = M_p(f)$$

as $r \to 1-0$. Similarly, in the case $p = \infty$, $M_\infty(f, r) \to M_\infty(f, 1)$, the essential upper bound of the boundary moduli $|f(e^{i\theta})|$.

We define q by $p^{-1}+q^{-1} = 1$, so that to $p = 1$ corresponds $q = \infty$ and vice versa. The classes H_p and H_q, or $\mathfrak{H}_p$ and $\mathfrak{H}_q$, will be called conjugate classes. The class H_2, or $\mathfrak{H}_2$, is self-conjugate.

1.2. Consider the integral

$$(1.2.1) \qquad I = I(f) = \frac{1}{2\pi i}\int_{|\zeta|=1} f(\zeta)k(\zeta)d\zeta ,$$

where $f(z)$ is a function of H_p while the 'kernel' $k(z)$ belongs to the conjugate class $\mathfrak{H}_q$. By Hölder's inequality I exists and

$$(1.2.2) \qquad |I| \leq M_p(f)M_q(k) .$$

The problem we discuss in this paper is that of finding the *maximum of* $|I|$ *for all functions* $f(z)$ *of* H_p, *when* $M_p(f)$ *and the kernel* $k(z)$ *are given.* That the maximum is attained follows from the obvious fact that the functions are uniformly bounded in every circle $|z| \leq \varrho$, $0 < \varrho < 1$, and therefore form a normal family.

The integral $I(f)$ is the sum of the residues of $f(z)k(z)$ in $|z| < 1$. For, by Hölder's inequality, $f(z)k(z)$ belongs to $\mathfrak{H}_1$ and hence, by (1.1.4), I is the limit as $r \to 1-0$ of the corresponding integral over the circle $|z| = r$.

If, for instance, $k(z)$ has simple poles $\beta_1, \beta_2, \dots \beta_n$ with residues $c_1, c_2, \dots c_n$ in $|z| < 1$, then

$$(1.2.3) \qquad I = c_1 f(\beta_1)+c_2 f(\beta_2)+\cdots+c_n f(\beta_n) .$$

If $k(z) = r!(z-\beta)^{-r-1}$, $|\beta| < 1$, then

$$(1.2.4) \qquad I = f^{(r)}(\beta) .$$

If

$$(1.2.5) \qquad f(z) = \sum_0^\infty a_k z^k , \quad k(z) = \sum_0^n c_k z^{-(k+1)} ,$$

then

$$(1.2.6) \qquad I = c_0 a_0+c_1 a_1+\cdots+c_n a_n .$$

In all these cases the extremum problem in H_p is of special interest.

1.3. We observe that $I(f)$ does not change its value if the kernel $k(z)$ is replaced by any other kernel $\varkappa(z)$ of $\mathfrak{H}_q$ which has the same poles and principal parts in

$|z| < 1$ as $k(z)$. The inequality (1.2.2) will hold when k is replaced by $\varkappa$. Hence, if $\mathfrak{K}_q$ is the class of all such kernels $\varkappa$, we shall have

$$|I| \leq M_p(f) \cdot \underset{\varkappa \in \mathfrak{K}_q}{\mathrm{Min}}\, M_q(\varkappa) . \tag{1.3.1}$$

Again, it is easy to see that the minimum on the right hand side is attained in $\mathfrak{K}_q$. We wish to prove that (1.3.1) is the best possible estimate, i. e. that, for given $M_p(f)$ and $k(z)$,

$$\mathrm{Max}\, |I(f)| = M_p(f) \cdot \underset{\varkappa \in \mathfrak{K}_q}{\mathrm{Min}}\, M_q(\varkappa) . \tag{1.3.2}$$

This will be true if, and only if, there exists an 'extremal function' $F(z)$ of H_p and an 'extremal kernel' $K(z)$ of $\mathfrak{K}_q$ such that

$$|I(F)| = M_p(F) \cdot M_q(K) . \tag{1.3.3}$$

Formula (1.3.1) then becomes

$$|I| \leq M_p(f) M_q(K) . \tag{1.3.4}$$

Now suppose that the given kernel $k(z)$ has the poles[1] $\beta_1, \beta_2, \ldots \beta_n$ in $|z| < 1$, each pole repeated according to its multiplicity. The discussion of the case of equality in Hölder's inequality (1. 2. 2) will lead to the conclusion that (1.3.3) is true if, and only if, $K(z)$ and $F(z)$ are of the form[2]

$$K(z) = A\Pi' \frac{z-\alpha_i}{1-\bar{\alpha}_i z} \prod_1^{n-1} (1-\bar{\alpha}_i z)^{2/q} \prod_1^{n} \frac{1-\bar{\beta}_i z}{z-\beta_i} (1-\bar{\beta}_i z)^{-2/q} \tag{1.3.5}$$

and

$$F(z) = B\Pi'' \frac{z-\alpha_i}{1-\bar{\alpha}_i z} \prod_1^{n-1} (1-\bar{\alpha}_i z)^{2/p} \prod_1^{n} (1-\bar{\beta}_i z)^{-2/p} . \tag{1.3.6}$$

Here the $n-1$ parameters α_i satisfy $|\alpha_i| \leq 1$, Π' extends over all, some, or none of the α_i with $|\alpha_i| < 1$, Π'' is the complementary product with respect to all α_i, and A and B are constants.

If we take $B = 1$ and then write F_1 for F, we obtain

$$M_p^p(F_1) = |A|^{-q} M_q^q(K), \quad M_p(F_1) = |A|^{-q/p} M_q^{q-1}(K) ,$$

so that, by (1.3.3),

$$|I(F_1)| = M_q(K) M_p(F_1) = |A|^{-q/p} M_q^q(K) ,$$

and so

$$M_q(K) = |A|^{1/p} |I(F_1)|^{1/q} . \tag{1.3.7}$$

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[1] If $k(z)$ has no poles, $I = 0$; we always exclude this trivial case.

[2] The powers involved mean the principal determinations equal to unity for $z = 0$ and regular for $|z| < 1$.

This formula is sometimes useful for determining the constant $M_q(K)$.

We shall find it convenient to consider the function

$$G(z) = K(z) \prod_1^n \frac{z-\beta_i}{1-\bar{\beta}_i z} \tag{1.3.8}$$

which is regular in $|z| < 1$ and belongs to H_q.

Now $K(z)$ must have the same given principal parts as $k(z)$ at its poles β_i. Hence the values of $G(z)$ and certain of its derivatives will be prescribed at the points β_i; i. e. if β_i is a pole of order r_i, then the values

$$G(\beta_i),\ G'(\beta_i), \dots\ G^{(r_i-1)}(\beta_i) \tag{1.3.9}$$

are given.

Evidently $M_q(G) = M_q(K)$. If, therefore, $K(z)$ is an extremal kernel, then $G(z)$ will make $M_q(g)$ a minimum for all functions $g(z)$ of H_q which behave in the prescribed manner[1] for all the $z = \beta_i$.

Finally, let

$$H(z) = G(z) \prod_1^n (1-\bar{\beta}_i z)^{2/q}\,. \tag{1.3.10}$$

Then $H(z)$ belongs to H_q and takes given 'values' at the points β_i. By (1.3.5) and (1.3.8), it must be of the form

$$H(z) = A\Pi' \frac{z-\alpha_i}{1-\bar{\alpha}_i z} \prod_1^{n-1} (1-\bar{\alpha}_i z)^{2/q}\,. \tag{1.3.11}$$

This formula involves a problem of interpolation, and the desired result (1.3.2) will be true if, and only if, this interpolation function $H(z)$ exists. In fact, the solution of any of the following three problems entails that of the other two:

Problem I. *To find the maximum of $|I(f)|$ in H_p for given $M_p(f)$ and given kernel $k(z)$ of $\mathfrak{H}_q$.*

Problem II. *To find the minimum of $M_q(g)$ in the conjugate class H_q for given 'values' of g at n points β_i.*

Problem III. *To find, in H_q, a solution $H(z)$ of the interpolation formula* (1.3.11) *when the 'values' of H are given at n points β_i.*

1.4. The interpolation problem III is, logically, the simplest of the three equivalent problems. *We shall see that it always possesses a solution and that this solution is*

[1] We shall say that $g(z)$ takes 'given values' at the points β_i.

unique. It follows that the minimum problem II also possesses exactly one extremal function $G(z)$ and that the extremal kernel $K(z)$ for the maximum problem I exists uniquely.

Note first that, in the interpolation formula (1.3.11), we have the correct number n of parameters at our disposal, namely the $n-1$ parameters α_i and the constant A.

Consider, next, the case $q = 2$, when (1.3.11) becomes

$$H(z) = A\Pi'(z-\alpha_i)\Pi''(1-\bar{\alpha}_i z)\ , \tag{1.4.1}$$

so that $H(z)$ is an arbitrary polynomial of degree at most $n-1$. Hence we have the classical interpolation problem, the unique solution of which is given by the familiar interpolation formula of Lagrange.

If $q = \infty$, we have

$$H(z) = A\Pi'\frac{z-\alpha_i}{1-\bar{\alpha}_i z}. \tag{1.4.2}$$

The construction of this rational interpolation function, which again is uniquely determined, can be achieved on following a method employed by I. Schur[1] in the theory of bounded power series [class H_∞].

Next, if $q = 1$, (1.3.11) takes the form

$$H(z) = A\Pi'(z-\alpha_i)(1-\bar{\alpha}_i z)\Pi''(1-\bar{\alpha}_i z)^2\ , \tag{1.4.3}$$

and we obtain a curious interpolation problem. The degree of the polynomial on the right hand side is at most $2(n-1)$. Besides the pairs of inverse roots α_i, $\bar{\alpha}_i^{-1}$ in Π' it possesses only roots of even order not inside the unit circle. The existence and uniqueness of this interpolation polynomial has been established by S. Kakeya[2]. But his proofs does not provide a "constructive" method for actually determining the polynomial, in the manner of Lagrange's formula or Schur's algorithm. This interesting and apparently difficult problem remains open.

Finally, in the case of a general q, the unique existence of the interpolation function $H(z)$ can be established by extending the argument which Kakeya uses in the case $q = 1$. Again, no actual construction of $H(z)$ can be obtained in this way.

1.5. As we have said already, *the extremal kernel $K(z)$ in the maximum problem I is uniquely determined by the formula* (1.3.5) since this is the case for $H(z)$ in (1.3.11).

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[1] Schur, part I.

[2] Kakeya, (b).

If $p > 1$, the extremal function $F(z)$ in (1.3.6) is also unique apart from a trivial factor ε of modulus 1. For the α_i in (1.3.11), and hence in (1.3.6), are uniquely determined, and $|B|$ is determined by the given value of $M_p(F)$.

Note that, when $p = 2$,

$$F(z) = B\Pi''(z-\alpha_i)\Pi'(1-\bar{\alpha}_i z)\prod_1^n (1-\bar{\beta}_i z)^{-1}, \tag{1.5.1}$$

and that, when $p = \infty$,

$$F(z) = B\Pi'' \frac{z-\alpha_i}{1-\bar{\alpha}_i z}. \tag{1.5.2}$$

If $p = 1$, then $q = \infty$, and the product Π' in (1.4.2) may determine fewer than $n-1$ parameters α_i, say s only. In this case $n-1-s$ of the parameters α_i in the formula

$$F(z) = B\Pi''(z-\alpha_i)(1-\bar{\alpha}_i z)\Pi'(1-\bar{\alpha}_i z)^2\Pi(1-\bar{\beta}_i z)^{-2} \tag{1.5.3}$$

are arbitrary. *Hence, if $p = 1$, there may be an infinity of (genuinely different) extremal functions $F(z)$.*

It should further be noted that, whenever $H(z)$ has been determined, then $K(z)$ and thus, by (1.3.2) and (1.3.3), Max $|I|$ is known. For this it is not required to know the actual values of the α_i. This remark is of importance in some applications.

1.6. The general theory of the extremum problems, as set out in the preceding paragraphs, is, in its essential features, not new. However, what is known concerns mainly the classes H_1, H_2, and H_∞, and even this not in full generality. Also our present argument is, in many respects, simpler and more complete than that used by previous writers. Finally, there exists, as far as we know, no connected account of this theory in the otherwise very extensive literature on extremum problems of our type. All this seems to us sufficient justification for presenting the theoretical part of this paper.

The first non-trivial maximum problem in H_∞ was solved by E. Landau[1] who determined the maximum of $|a_0+a_1+\cdots+a_n|$ for the class of power series (1.2.5), for which $|f(z)| \leq 1$ in $|z| < 1$. He was the first to use the idea of "minimizing" the given kernel, and he succeeded in determining the extremal kernel $K(z)$ in a way which we shall explain, in the light of the general theory, in § 3.3 of this paper. It should be understood that Landau did not propound any general theory for extremum problems in H_∞, and that he arrived at his extremal function by a sort of "inspired

[1] Landau, (a), (c).

guess". His method, however, is applicable in many similar cases and has been followed and generalized by several subsequent writers.

The general theory takes shape when C. Carathéodory and L. Fejér[1] solve the minimum problem II in H_∞ for the case where the first coefficients

$$a_0, a_1, \dots a_n \tag{1.6.1}$$

of the power series of $g(z)$ are given. They determine, by algebraic methods, the extremal function (1.4.2)[2]. Gronwall[3] gives another and particularly simple solution based on the classical lemma of Schwarz. G. Pick[4] extends these results to the case where the values of $g(z)$ at n different points β_i are given. Finally, I. Schur[5] in his well known theory of the class H_∞, develops an algebraic algorithm, equivalent to a repeated use of Schwarz's lemma, which can be conveniently used for determining the extremal functions of minimum problems in H_∞.

The minimum problem II for the class H_1, again in the special case (1.6.1), was first discussed by F. Riesz[6] who proved, by a variational argument, the unique existence and the characteristic form of the extremal function $G(z)$. No method for constructing the solution of the interpolation problem involved is given. G. Pick[7] extends this result to the case where the values of $g(z)$ at n different points β_i are given. F. Riesz[8], in passing, also points out the relation between his minimum problem and the corresponding maximum problem in H_∞.

S. Kakeya[9] starts from the maximum problem I in H_∞, when the kernel $k(z)$ has n simple poles. *It is his argument which we follow and generalize in the present account of the general theory.* Kakeya reduces his maximum problem to the corresponding minimum problem in H_1 and this, in turn, to the interpolation problem (1.4.3). He then gives a proof of a topological character for the existence of the unique solution of the latter. Geronimus[10] gives an independent proof in the general case.

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[1] Carathéodory and Fejér.
[2] If all $\beta_i = 0$, then $G(z) = H(z)$, by (1.3.10).
[3] Gronwall.
[4] Pick, (b).
[5] Schur.
[6] Riesz. The earlier writers assume that $g(z)$ is continuous, in $|z| \leq 1$.
[7] Pick, (b)
[8] Riesz, § 7.
[9] Kakeya, (b).
[10] Geronimus, (a), (b).

L. Fejér[1], E. Egerváry[2], and others treat special maximum problems in H_1 by reducing them to the corresponding minimum problems in H_∞. G. Pick[3] discusses minimum problems in H_2.

1.7. This paper is divided in two main parts. In the first the general theory of the extremum problems for the classes H_p is developed. In cases where we follow the usual methods we content ourselves with a mere sketching of these. The second part of the paper gives systematic applications of the general theory to special problems of various types. Some of our results have been obtained previously but appear only now, that is in the light of the general theory, in their proper aspect.

Part I: The General Theory.

2. Reduction to a Problem of Interpolation.

2.1. We use the notation of the introduction. Let $f(z)$ be a function of H_p, $1 \leq p \leq \infty$, and let $k(z)$ be a kernel of $\mathfrak{H}_q$ with poles $\beta_1, \beta_2, \dots \beta_n$, these repeated according to multiplicity. We consider the integral

$$I = I(f) = \frac{1}{2\pi i}\int_{|\zeta|=1} f(\zeta)k(\zeta)d\zeta = \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta} f(e^{i\theta})\varkappa(e^{i\theta})d\theta \,. \tag{2.1.1}$$

Here $\varkappa(z)$ is any kernel of the class $\mathfrak{K}_q$ determined by $k(z)$.

By (1.1.3), the integrand exists and is finite for almost all θ; and, by Hölder's inequality, it is integrable.

2.2. Our first estimate is

$$|I| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})\varkappa(e^{i\theta})|d\theta \,. \tag{2.2.1}$$

Equality will hold if, and only if,

$$\arg\{\zeta f(\zeta)\varkappa(\zeta)\} \equiv \Phi \tag{2.2.2}$$

(a constant) for almost all $\zeta = e^{i\theta}$.

The function $\lambda(z) = zf(z)\varkappa(z)$ belongs to $\mathfrak{H}_1$, by Hölder's inequality, since f belongs to H_p and $\varkappa$ to $\mathfrak{H}_q$. It follows from (1.1.4), applied to $\lambda(z)$ with $p = 1$, that

[1] Fejér.
[2] Egerváry.
[3] Pick, (a), (b).

$$\int_{\theta_1}^{\theta_2} \lambda(re^{i\theta})d\theta \to \int_{\theta_1}^{\theta_2} \lambda(e^{i\theta})d\theta \tag{2.2.3}$$

as $r \to 1-0$, for every arc $< \theta_1, \theta_2 >$. Hence we may apply the usual argument for the "principle of inversion" and infer from (2.2.2) that $\lambda(z)$ is a rational function. The zeros and poles of $\lambda(z)$, if not on $|z| = 1$, appear in pairs of points inverse to each other with respect to the unit circle. Furthermore, (2.2.2) implies that any zero of $\lambda(z)$ on $|z| = 1$ must be of even order. No pole of $\lambda(z)$ can lie on $|z| = 1$ since $\lambda(z)$ belongs to $\mathfrak{H}_1$, and the only possible poles are amongst the pairs $\beta_i, \bar{\beta}_i^{-1}$. It follows that $\lambda(z)$ is of the form

$$\lambda(z) = zf(z)\varkappa(z) = Cz\frac{\prod_1^{n-1}(z-\alpha_i)(1-\bar{\alpha}_i z)}{\prod_1^{n}(z-\beta_i)(1-\bar{\beta}_i z)}, \tag{2.2.4}$$

where the $|\alpha_i| \leq 1$. For, $\lambda(z)$ must have a zero at the origin and hence at infinity if none of the β_i vanishes; it must be regular there if exactly one of the β_i vanishes; and it must have there a pole of order at most $k-1$ if k of the β_i vanish. It is easy to verify that this $\lambda(z)$ satisfies (2.2.2) for all $\zeta = e^{i\theta}$.

2.3. Next we apply Hölder's inequality and obtain

$$\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})\varkappa(e^{i\theta})|d\theta \leq M_p(f)M_q(\varkappa)\,. \tag{2.3.1}$$

Equality will hold if, and only if,

$$|f(\zeta)|^{1/q} \equiv D|\varkappa(\zeta)|^{1/p} \tag{2.3.2}$$

for almost all $\zeta = e^{i\theta}$.

Combining (2.2.2) and (2.3.1) we see that equality will hold in

$$|I| \leq M_p(f)M_q(\varkappa) \tag{2.3.3}$$

if, and only if, $f(z) = F(z)$ and $\varkappa(z) = K(z)$, where

$$F(z)K(z) = C\frac{\prod_1^{n-1}(z-\alpha_i)(1-\bar{\alpha}_i z)}{\prod_1^{n}(z-\beta_i)(1-\bar{\beta}_i z)} \tag{2.3.4}$$

and where, for almost all $\zeta = e^{i\theta}$,

$$|K(\zeta)| = |A| \left| \frac{\prod_1^{n-1} (\zeta-\alpha_i)(1-\bar{\alpha}_i\zeta)}{\prod_1^{n} (\zeta-\beta_i)(1-\bar{\beta}_i\zeta)} \right|^{1/q} \tag{2.3.5}$$

and

$$|F(\zeta)| = |B| \left| \frac{\prod_1^{n-1} (\zeta-\alpha_i)(1-\bar{\alpha}_i\zeta)}{\prod_1^{n} (\zeta-\beta_i)(1-\bar{\beta}_i\zeta)} \right|^{1/p} \tag{2.3.6}$$

It follows from (2.3.3) that, for any function $K(z)$ of $\mathfrak{K}_q$ which together with an $F(z)$ of H_p satisfies the conditions (2.3.4), (2.3.5), (2.3.6), we shall have

$$M_q(K) = \underset{\varkappa \in \mathfrak{K}_q}{\mathrm{Min}}\, M_q(\varkappa)\,. \tag{2.3.7}$$

2.4. *Let us assume, for the present, that there exists such an extremal kernel $K(z)$ and that $K(z)$ is continuous on $|z| = 1$.*

The zeros of $K(z)$ in $|z| < 1$ must be amongst the α_i in (2.3.4). The function

$$K^*(z) = K(z)\Pi' \frac{1-\bar{\alpha}_i z}{z-\alpha_i} \prod_1^n \frac{z-\beta_i}{1-\bar{\beta}_i z}\,, \tag{2.4.1}$$

where Π' is extended over the zeros of $K(z)$ in $|z| < 1$, is regular and different from zero in $|z| < 1$, continuous on $|z| = 1$, and satisfies (2.3.5) for all $\zeta = e^{i\theta}$. It follows that

$$K^*(z) = A \prod_1^{n-1} (1-\bar{\alpha}_i z)^{2/q} \prod_1^{n} (1-\bar{\beta}_i z)^{-2/q}. \tag{2.4.2}$$

This is equivalent to the formula (1.3.5) for $K(z)$ itself. The formula (1.3.6) for the associated extremal function $F(z)$ follows from (2.3.4) at once.

It should be noted that any such $K(z)$ determines all the α_i in (1.3.5) and hence those in (1.3.6), provided that $q \neq \infty$, that is $p > 1$. The associated extremal function $F(z)$ is then uniquely determined, apart from a factor ε, where $|\varepsilon| = 1$. If $p = 1$, however, some only of the α_i in (1.3.5) may be determined by $K(z)$, the remainder being quite arbitrary. There may then be genuinely different extremal functions $F(z)$.

Conversely, if $p \neq \infty$, a function $F(z)$ of the form (1.3.6) can be an associated extremal function with one extremal kernel $K(z)$ only (apart from an arbitrary constant factor), and this kernel must be of the form (1.3.5). For, $F(z)$ is continuous on $|z| = 1$, and so is $F(z)K(z)$, by (2.3.4). It follows that $K(z)$ is continuous at every point ζ, $|\zeta| = 1$, where $F(\zeta) \neq 0$. This is also true when $F(\zeta) = 0$[1]. For, $K(\zeta)$ cannot be infinite because of (2.3.5). Since $K(z)$ is continuous on $|z| = 1$, it is of the form (1.3.5), and it now follows from the prescribed principal parts at the poles β_i that its factor A in (1.3.5) is also uniquely determined.

We can now prove:

If an extremal kernel $K(z)$ exists which is continuous on $|z| = 1$ (and hence is of the form (1.3.5)) *then it is the only possible extremal kernel.*

For suppose that $K_1(z)$ be a different kernel, not necessarily continuous on $|z| = 1$, and $F_1(z)$ an extremal function associated with $K_1(z)$. Let $F(z)$ be an extremal function associated with $K(z)$ and hence of the form (1.3.6).

If $F_1(z) = e^{i\alpha}F(z)$, it would follow that $F_1(z)$ is of the form (1.3.6) and that $K_1(z)$ is of the form (1.3.5). This possibility has already been disposed of, so that we may assume that $F_1(z) \neq e^{i\alpha}F(z)$. It follows that

(2.4.3) $$|I(F_1)| = M_p(F_1)M_q(K_1); \ |I(F_1)| < M_p(F_1)M_q(K),$$

so that $M_q(K_1) < M_q(K)$ in contradiction to (2.3.7).

2.5. It remains to prove that there always exists an extremal kernel $K(z)$ in $\mathfrak{K}_q$ which is continuous on $|z| = 1$. $K(z)$ is then necessarily unique. This problem is, as we know, equivalent to that of a solution for the interpolation formula

(2.5.1) $$H(z) = A\Pi' \frac{z-\alpha_i}{1-\bar{\alpha}_i z} \prod_1^{n-1} (1-\bar{\alpha}_i z)^{2/q},$$

where the prescribed "values" of $H(z)$ at the points β_i are those of

(2.5.2) $$h(z) = k(z) \prod_1^n \frac{z-\beta_i}{1-\bar{\beta}_i z} (1-\bar{\beta}_i z)^{2/q};$$

and these, in turn, are determined by the principal parts of $k(z)$ at its poles β_i.

Note that this interpolation problem can have at most one solution. For, different solutions would lead to different extremal kernels of the form (1.3.5).

3. Extremum Problems in H_2.

3.1. If $p = 2$ then $q = 2$. The formula (2.5.1) becomes

(3.1.1) $$H(z) = A\Pi'(z-\alpha_i)\Pi''(1-\bar{\alpha}_i z),$$

and we arrive at the classical-interpolation problem the (unique) solution of which

[1] This happens only if $\zeta = \alpha_i$ and $|\alpha_i| = 1$.

is given by the familiar formula of Lagrange. Hence for the class H_2 our theory is now complete. Lagrange's interpolation formula determines explicitly $H(z)$ and thus the extremal kernel $K(z)$ which, by (1.3.8) and (1.3.10), is given by

$$K(z) = H(z) \prod_1^n (z-\beta_i)^{-1} . \tag{3.1.2}$$

This only, and not the knowledge of the roots α_i, is required for the solution of the extremum problems proper. To establish the extremal functions, note that, in view of (3.1.1), (1.5.1), and (3.1.2), we have

$$F(z) = B\bar{A}^{-1}z^{n-1}\overline{H(1/\bar{z})} \prod_1^n (1-\bar{\beta}_i z)^{-1} = Cz^{-1}\overline{K(1/\bar{z})}, \tag{3.1.3}$$

so that again the knowledge of the roots α_i of $H(z)$ is not required. Alternatively, we may say that the extremal kernel is the "natural kernel" consisting of the sum of the principal parts of the given poles. This follows from (3.1.2).

3.2. There exists another simple way of dealing with extremum problems in H_2. A power series $f(z) = \sum_0^\infty a_k z^k$ belongs to H_2 if, and only if,

$$M_2^2(f) = \sum_0^\infty |a_k|^2 < \infty . \tag{3.2.1}$$

Now, expressing the values and derivatives of $f(z)$ at the points β_i in terms of the a_k, we obtain

$$I(f) = \sum_0^\infty c_k a_k , \tag{3.2.2}$$

where the c_k are "given", the series $\sum |c_k|^2$ being convergent. More generally, consider any such sum I, not necessarily obtained in the above way. Then, by Cauchy's inequality,

$$\left| \sum_0^\infty c_k a_k \right| \leq \left\{ \sum_0^\infty |c_k|^2 \right\}^{1/2} \left\{ \sum_0^\infty |a_k|^2 \right\}^{1/2} = \left\{ \sum_0^\infty |c_k|^2 \right\}^{1/2} M_2(f) . \tag{3.2.3}$$

Equality holds if, and only if, $a_k = B\bar{c}_k$. It follows that, in the case (3.2.2), the extremal function is

$$F(z) = B\sum_0^\infty \bar{c}_k z^k , \tag{3.2.4}$$

while, for the extremal kernel, by (3.1.3),

$$K(z) = \sum_0^\infty c_k z^{-k-1} (|z| > 1); \; M_2(K) = \left\{ \sum_0^\infty |c_k|^2 \right\}^{1/2}. \tag{3.2.5}$$

It should be noted that we can replace (3.2.3) by the stricter inequality

$$\sum_0^\infty |c_k a_k| \leq \left\{ \sum_0^\infty |c_k|^2 \right\}^{1/2} M_2(f). \tag{3.2.6}$$

3.3. In certain cases the solution of an extremum problem in H_p can be reduced to one in H_2.

Let $p > 1$ and $h(z)$ be the function (2.5.2); and let there be n *different* points β_i. *Suppose, in the first place, that, for suitable determinations, the "values" $h(\beta_i)^{q/2}$ lead to a Lagrange polynomial $H^*(z)$ which has no roots in $|z| < 1$.*

Suppose, in addition, that there is a determination

$$H(z) = H^*(z)^{2/q} \tag{3.3.1}$$

which takes at the β_i the given values $h(\beta_i)$. This, for instance, will certainly be the case when $p = \infty$, $q = 1$.

Under these assumptions $H(z)$ will be a solution of (2.5.1), the product Π' being empty. On using (1.3.8) and (1.3.10), it follows that

$$K(z) = K^*(z)^{2/q} \prod_1^n \left(\frac{1 - \bar{\beta}_i z}{z - \beta_i} \right)^{1-2/q}, \tag{3.3.2}$$

$$M_q(K) = M_2(K^*)^{2/q} \tag{3.3.3}$$

and

$$F(z) = C F^*(z) \left(\frac{\prod_1^{n-1} (1 - \bar{\alpha}_i z)}{\prod_1^n (1 - \bar{\beta}_i z)} \right)^{2/p-1}, \tag{3.3.4}$$

where $K^*(z)$ and $F^*(z)$ are the extremal kernel and extremal function in H_2, corresponding to the Lagrange polynomial $H^*(z)$.

A similar method is available in the case

$$I(f) = \sum_0^n c_k a_k \quad (c_n \neq 0), \tag{3.3.5}$$

where $f(z) = \sum_0^\infty a_k z^k$ and the c_k are given. Here the given kernel is

$$k(z) = \sum_0^n c_k z^{-k-1} \tag{3.3.6}$$

and all $\beta_i = 0$ $(i = 1, 2, \ldots n+1)$. According to (1.3.8) and (1.3.10), we must have

$$(3.3.7) \qquad G(z) = H(z) = z^{n+1}K(z) = c_n + c_{n-1}z + \cdots + c_0 z^n + \cdots .$$

Now let $p > 1$. We have, for small $|z|$ and some determination,

$$(3.3.8) \qquad G(z)^{q/2} = \lambda_0 + \lambda_1 z + \cdots + \lambda_n z^n + \cdots .$$

The first coefficients $\lambda_0, \lambda_1, \ldots \lambda_n$ depend on the given $c_0, c_1, \ldots, c_n$ only. Hence, again for small $|z|$ and a certain determination,

$$(3.3.9) \qquad P_n(z)^{2/q} = (\lambda_0 + \lambda_1 z + \cdots + \lambda_n z^n)^{2/q} = c_n + c_{n-1}z + \cdots + c_0 z^n + \cdots .$$

Suppose now that the polynomial $P_n(z)$ *has no roots in* $|z| < 1$. Then $P_n(z)^{2/q}$ is of the form (2.5.1), the product Π' being empty. It follows that $G(z) = H(z) = P_n(z)^{2/q}$ and that

$$(3.3.10) \qquad P_n(z) = C \prod_1^n (1 - \bar{\alpha}_i z) .$$

The extremal function is, by (1.3.6),

$$(3.3.11) \qquad F(z) = B \prod_1^n \frac{z - \alpha_i}{1 - \bar{\alpha}_i z} \prod_1^n (1 - \bar{\alpha}_i z)^{2/p} = D \frac{z^n \overline{P_n(1/\bar{z})}}{P_n(z)^{1-2/p}},$$

and the extremal kernel is $K(z) = z^{-n-1} P_n(z)^{2/q}$, so that

$$(3.3.12) \qquad M_q(K) = M_2^{2/q}(P_n) .$$

It follows that, under our assumptions regarding $P_n(z)$,

$$(3.3.13) \qquad |c_0 a_0 + c_1 a_1 + \cdots + c_n a_n| \le M_p(f)(|\lambda_0|^2 + |\lambda_1|^2 + \cdots + |\lambda_n|^2)^{1/q} .$$

This inequality was first proved, in the case $p = \infty$, $q = 1$, by O. Szász[1]. It contains as a special case E. Landau's determination[2] of the maximum of $|a_0 + a_1 + \cdots + a_n|$ in H_∞.

It is well known that $P_n(z) \neq 0$ in $|z| < 1$ when, for instance[3],

$$(3.3.14) \qquad \lambda_0 \ge \lambda_1 \ge \ldots \ge \lambda_n \ge 0 , \quad \lambda_0 > 0 ,$$

a condition which is satisfied in Landau's case.

4. Extremum Problems in H_1.

4.1. If $p = 1$, then $q = \infty$ and the interpolation function (2.5.1) takes the form

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[1] Szász, (b).

[2] Landau, (a), (c).

[3] Kakeya, (a).

$$H(z) = A\Pi' \frac{z-\alpha_i}{1-\bar{\alpha}_i z}. \tag{4.1.1}$$

To establish its unique existence we employ an algorithm introduced into the theory of the class H_∞ by I. Schur[1].

4.2. Consider the class B of functions $w = \varphi(z)$ regular and satisfying $|w| \leq 1$ in $|z| < 1$.

Let n points β_i in $|z| < 1$ be given, taken multiply if desired and arranged in some order. We wish to discuss the possible "values" w_i of $\varphi(z)$ at these points. Put

$$\varphi_1(z) = \varphi(z)\,, \qquad \gamma_1 = \varphi_1(\beta_1) = w_1\,. \tag{4.2.1}$$

Then $|\gamma_1| \leq 1$; and $|\gamma_1| = 1$ if, and only if, $\varphi_1(z) \equiv \gamma_1$. If $|\gamma_1| < 1$, put[2]

$$\varphi_2(z) = \frac{1-\bar{\beta}_1 z}{z-\beta_1}\,\frac{\varphi_1(z)-\gamma_1}{1-\bar{\gamma}_1\varphi_1(z)}\,, \qquad \gamma_2 = \varphi_2(\beta_2)\,, \tag{4.2.2}$$

so that

$$\gamma_2 = \frac{1-\bar{\beta}_1\beta_2}{\beta_2-\beta_1}\,\frac{w_2-w_1}{1-\bar{w}_1 w_2}\,, \qquad w_2 = \varphi(\beta_2)\,, \tag{4.2.3}$$

if $\beta_2 \neq \beta_1$, and

$$\gamma_2 = \frac{1-|\beta_1|^2}{1-|w_1|^2}\,w_2\,, \qquad w_2 = \varphi_1'(\beta_1) = \varphi'(\beta_1)\,, \tag{4.2.4}$$

if $\beta_2 = \beta_1$. $\varphi_2(z)$ belongs to the class B, so that $|\gamma_2| \leq 1$.

Similarly, if $|\gamma_k| < 1$ $(k < n)$ we put

$$\varphi_{k+1}(z) = \frac{1-\bar{\beta}_k z}{z-\beta_k}\,\frac{\varphi_k(z)-\gamma_k}{1-\bar{\gamma}_k\varphi_k(z)}\,, \qquad \gamma_{k+1} = \varphi_{k+1}(\beta_{k+1})\,. \tag{4.2.5}$$

The numbers γ_k are certain rational functions of the $\beta_i, \bar{\beta}_i$ and $w_i, \bar{w}_i$ with $1 \leq i \leq k$, and $|\gamma_{k+1}| \leq 1$. A necessary restriction for possible values w_i is, therefore, that either (i) $|\gamma_i| < 1$ for all i $(1 \leq i \leq n)$ or (ii) that there exists an s $(1 \leq s \leq n)$ such that $|\gamma_s| = 1$, while $|\gamma_i| < 1$ for $1 \leq i \leq s$.

In the second case, $\varphi_s(z) \equiv \gamma_s$, *and all the values* w_i *with* $i > s$ *are determined by the preceding ones.*

These restrictions on possible values w_i are also sufficient in order that there should exist some function $\varphi(z)$ in B which takes these values at the points β.

In case (i) there exists an infinity of such functions. We may start with *any*

[1] Schur.

[2] Compare Schur: Schur considers the case $\beta_i \equiv 0$ only.

function $\varphi_n(z)$ of B such that $\varphi_n(\beta_n) = \gamma_n$. Resolving (4.2.5) backwards step by step we arrive at a function $\varphi(z)$ of the desired kind.

In case (*ii*) there exists exactly one such function. For necessarily $\varphi_s(z) \equiv \gamma_s$. Hence, resolving (4.2.5) backwards as before, we arrive after $s-1$ steps at a uniquely determined rational function $\varphi(z)$ in B, of degree at most $s-1$,[1] which takes the values w_i at the points β_i.

It is easy to see[2] that this function is of the form

$$\varphi(z) = \varepsilon \prod_{1}^{s-1} \frac{z-\alpha_i}{1-\bar{\alpha}_i z} \qquad (|\alpha_i| < 1,\ |\varepsilon| = 1)\,. \tag{4.2.6}$$

For, since $|\varphi_s(z)| \equiv |\gamma_s| = 1$, we conclude from (4.2.5) that $|\varphi_{s-1}(z)| = 1$ when $|z| = 1$, and, finally, that $|\varphi(z)| = 1$ when $|z| = 1$. Since $\varphi(z)$ is a rational function of degree at most $s-1$ in B, it must be of the form (4.2.6), with s possibly replaced by some $p < s$. But $p < s$ is impossible, since then $|\gamma_p| = 1$, as is easily verified.

4.3. Consider now our interpolation problem where n arbitrary "values" w_i are prescribed at the points β_i.

Let $\varrho > 0$. If ϱ is large, the values $\varrho^{-1}w_i$ will determine numbers $\gamma_i(\varrho)$ which satisfy the conditions (*i*) of 4.2. For it follows, by induction, from the definition (4.2.5) that $\gamma_i(\varrho) \to 0$ (of order ϱ^{-1}) as $\varrho \to \infty$. Hence, for large ϱ, there will exist a function $\varphi_\varrho(z)$ in B which takes the values $w_i\varrho^{-1}$ at the points β_i.

On the other hand, if $\varrho \to 0$, some of the $w_i\varrho^{-1}$ will tend to infinity[3], so that no such function $\varphi_\varrho(z)$ can exist when ϱ is small enough.

Now let $P > 0$ be the greatest lower bound of all values $\varrho > 0$ for which such a $\varphi_\varrho(z)$ in B exists. Since the functions of B form a normal family, there exists also a function $\varphi_P(z)$ in B. Not all the corresponding $|\gamma_i(P)|$ can be less than 1. For then, by reason of continuity, the same would hold for some $\varrho < P$, and a $\varphi_\varrho(z)$ would exist contrary to the definition of P. Hence the case (*ii*) of (4.2) must hold when $\varrho = P$, and $\varphi_P(z)$ is of the form (4.2.6) with some $s \leq n$. It follows that the function

$$H(z) = P\varphi_P(z) = \varepsilon P \prod_{1}^{s-1} \frac{z-\alpha_i}{1-\bar{\alpha}_i z}\,, \qquad |\varepsilon| = 1\,, \tag{4.3.1}$$

is a solution of our interpolation problem.

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[1] A quotient of two polynomials each of degree at most $s-1$.

[2] Schur, in the case $\beta_i \equiv 0$, uses an algebraic argument to prove (4.2.6).

[3] We exclude the trivial case that all $w_i = 0$ when $H(z) \equiv 0$ is of the form (4.1.1).

We know already that this solution is unique. It follows, by the above arguments, that $\varphi_\varrho(z)$ exists for *all* $\varrho > P$ and that all $|\gamma_i(\varrho)| < 1$ for these ϱ. The latter property ceases first, as ϱ decreases, for $\varrho = P$.

4.4. The value of P is of main interest in the applications. Since the $\gamma_i(\varrho)$ can be calculated for large ϱ and are rational functions of ϱ, the actual determination of P becomes an algebraic problem. In the "classical" case where all $\beta_i = 0$, i. e. where the first $n+1$ coefficients $c_0, c_1, \dots c_n$ of

$$H(z) = A\Pi' \frac{z-\alpha_i}{1-\bar{\alpha}_i z} = c_0 + c_1 z + \cdots + c_n z^n + \cdots \tag{4.4.1}$$

are prescribed, it has been proved by C. Carathéodory and L. Fejér[1] that $|A| = P$, where P is the greatest root of the equation

$$\begin{vmatrix} \varrho^2 - h_{00} & -h_{01} & \cdot & \cdot & \cdot & -h_{0n} \\ -h_{10} & \varrho^2 - h_{11} & \cdot & \cdot & \cdot & -h_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -h_{n0} & -h_{n1} & \cdot & \cdot & \cdot & \varrho^2 - h_{nn} \end{vmatrix} = 0 \tag{4.4.2}$$

and

$$h_{\varkappa\lambda} = \sum_{\nu=0}^{\varkappa} \bar{c}_{\varkappa-\nu} c_{\lambda-\nu} \quad (0 \le \varkappa \le \lambda \le n), \quad h_{\lambda\varkappa} = \bar{h}_{\varkappa\lambda}. \tag{4.4.3}$$

The case $\beta_i \equiv \beta \neq 0$ is easily reduced to the above by a linear transformation.

4.5. Let $f(z) = \sum_0^\infty a_k z^k$, when $I(f)$ is of the form

$$I(f) = \sum_0^\infty c_k a_k . \tag{4.5.1}$$

Now any function $f(z)$ of H_1 can be represented in the form[2]

$$f(z) = \mathfrak{B}(z)\mathfrak{F}(z) , \tag{4.5.2}$$

where $\mathfrak{B}(z)$ is a "Blaschke product", satisfying $|\mathfrak{B}(z)| \le 1$ in $|z| < 1$ and $|\mathfrak{B}(\zeta)| = 1$ for almost all $\zeta = e^{i\theta}$; where $\mathfrak{F}(z) \neq 0$ in $|z| < 1$; and where $\mathfrak{F}(z)$ belongs to H_1. Also $M_1(\mathfrak{F}) = M_1(f)$. We write, with some determination of the root,

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[1] Carathéodory and Fejér.

[2] See Zygmund, p. 161: A corresponding representation holds for $f(z)$ of H_p, when $\mathfrak{F}(z)$ will belong to H_p.

$$f(z) = \mathfrak{B}(z)\mathfrak{F}(z)^{1/2}\mathfrak{F}(z)^{1/2} = \mathfrak{F}_1(z)\mathfrak{F}_2(z)\,. \tag{4.5.3}$$

Both $\mathfrak{F}_1(z)$ and $\mathfrak{F}_2(z)$ belong to H_2, and

$$M_2^2(\mathfrak{F}_1) = M_2^2(\mathfrak{F}_2) = M_1(\mathfrak{F}) = M_1(f)\,. \tag{4.5.4}$$

Let $\mathfrak{F}_1(z) = \sum\limits_0^\infty A_k z^k$, $\mathfrak{F}_2(z) = \sum\limits_0^\infty B_k z^k$ so that

$$a_k = \sum_{l=0}^{k} A_l B_{k-l}\,. \tag{4.5.5}$$

Next consider

$$f^*(z) = \mathfrak{F}_1^*(z)\mathfrak{F}_2^*(z) = \sum_0^\infty a_k^* z^k\,, \tag{4.5.6}$$

where $\mathfrak{F}_1^*(z) = \sum\limits_0^\infty |A_k| z^k$, $\mathfrak{F}_2^*(z) = \sum\limits_0^\infty |B_k| z^k$. Both $\mathfrak{F}_1^*(z)$ and $\mathfrak{F}_2^*(z)$ belong to H_2. Hence, by Schwarz's inequality, $f^*(z)$ belongs to H_1, and

$$M_1(f^*) \leq M_2(\mathfrak{F}_1^*)M_2(\mathfrak{F}_2^*) = M_2(\mathfrak{F}_1)M_2(\mathfrak{F}_2) = M_1(f)\,. \tag{4.5.7}$$

Also

$$|a_k| \leq \sum_{l=0}^{k} |A_l||B_{k-l}| = a_k^*\,. \tag{4.5.8}$$

It follows that, if $c_k \geq 0$,

$$\sum_0^\infty c_k |a_k| \leq \sum_0^\infty c_k a_k^* \leq M_1(f^*)M_\infty(K) \leq M_1(f)M_\infty(K)\,. \tag{4.5.9}$$

This is an interesting improvement on $|I(f)| \leq M_1(f)M_\infty(K)$ and shows that the coefficients of an extremal function $F(z)$ must have constant signs if the c_k have this property[1].

5. Extremum Problems in H_∞.

5.1. If $p = \infty$ then $q = 1$, and the interpolation function (2.5.1) becomes

$$H(z) = A\Pi'(z-\alpha_i)(1-\bar{\alpha}_i z)\Pi''(1-\bar{\alpha}_i z)^2\,. \tag{5.1.1}$$

As we have pointed out in the introduction to this paper, the existence of a (unique) solution of the interpolation problem involved has been indirectly established by F. Riesz[2] and G. Pick[3] who show that the corresponding minimum function $G(z)$ of H_1 exists uniquely and is of the appropriate form. A direct proof, based on a

[1] Compare Egerváry; Landau (b).
[2] Riesz.
[3] Pick, (a), (b).

topological argument, has been given by S. Kakeya[1]. None of these proofs is constructive. It would be of considerable interest to find a method for actually determining the interpolation function.

It should be noted that the above proofs, as published, deal only with the two cases where either all $\beta_i = 0$, or where all β_i are different from one another. It is, however, evident that the most general case can be obtained from the latter one by a limiting process, the uniqueness of the result following from our general theory[2].

5.2. The topological proof of Kakeya extends to the general interpolation problem (2.5.1)[3]; if $p > 1$, $q < \infty$. We refrain therefore from giving any further account of it.

5.3. We proceed to *construct* the interpolation polynomial (5.1.1) in the simple case of two different points β_1 and β_2. This is of importance for applications in H_∞.

Given w_1 and w_2 we have to show that there exists a function $H(z)$ either of the form

(5.3.1) $$H(z) = A(z-\alpha)(1-\bar{\alpha}z) \qquad (|\alpha| < 1)$$

or

(5.3.2) $$H(z) = A(1-\bar{\alpha}z)^2 \qquad (|\alpha| \leq 1)\,,$$

such that $H(\beta_1) = w_1$ and $H(\beta_2) = w_2$.

We shall assume that w_1 and w_2 do not vanish simultaneously, in which case $H(z) \equiv 0$. To avoid minor complications we shall also assume that both β_1 and β_2 are different from zero. The excluded case can easily be treated by a limiting process.

A function $H(z)$ of the form (5.3.1) satisfies the equation

(5.3.3) $$z^2\overline{H(1/\bar{z})} = \bar{\varepsilon}H(z) \qquad (\varepsilon = A/\bar{A},\ |\varepsilon| = 1)\,.$$

In particular, $\beta_i^2\overline{H(1/\bar{\beta}_i)} = \bar{\varepsilon}w_i$ or

(5.3.4) $$w_i^* = H(1/\bar{\beta}_i) = \varepsilon\bar{\beta}_i^{-2}\bar{w}_i \qquad (i = 1, 2)\,.$$

It follows that, in the case (5.3.1), $H(z)$ *must be the Lagrange polynomial that takes the given values* w_i *at the points* β_i *and the values* w_i^* *at the points* $1/\bar{\beta}_i$.

Given an arbitrary ε with $|\varepsilon| = 1$, let us start, therefore, with the Lagrange polynomial $L(z)$ that takes the values w_i and w_i^* at the points β_i and $1/\bar{\beta}_i$, respectively.

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[1] Kakeya, (b).

[2] See also Geronimus.

[3] Following the notation of Kakeya (b) one has to put $E(z, t) = (1-\bar{t}z)^{2/q}$ where t is a point on the first sheet of his double unit circle S, and $E(z, t) = (z-t)(1-\bar{t}z)^{2/q-1}$ when t is on the second sheet.

$L(z)$ is of degree at most three. An elementary calculation shows that the coefficient of z^3 is $\bar{\beta}_1\bar{\beta}_2(v-\varepsilon\bar{v})$, where

$$v = \frac{w_1}{(\beta_1-\beta_2)(1-\beta_1\bar{\beta}_2)(1-|\beta_1|^2)} + \frac{w_2}{(\beta_2-\beta_1)(1-\beta_2\bar{\beta}_1)(1-|\beta_2|^2)}. \tag{5.3.5}$$

Hence $L(z)$ will be of degree at most two, if either $v = 0$ and ε is arbitrary, or if $v \neq 0$ and $\varepsilon = v/\bar{v}$. In both cases, by definition,

$$z^2\overline{L(1/\bar{z})} = \bar{\varepsilon}L(z) \tag{5.3.6}$$

is true for the four points β_i, $1/\bar{\beta}_i$. Since (5.3.6) is a quadratic equation in z, it must hold identically. It follows that, if a root α, with $|\alpha| \neq 1$ of $L(z)$ exists, then $1/\bar{\alpha}$ is also a root. $L(z)$ is of the form (5.3.1), and $L(z)$ is $H(z)$.

5.4. There remains the case when

$$L(z) = A(z-\eta_1)(z-\eta_2) \qquad (|\eta_1| = |\eta_2| = 1)\,. \tag{5.4.1}$$

We then have

$$\frac{w_1}{w_2} = \frac{(\beta_1-\eta_1)(\beta_1-\eta_2)}{(\beta_2-\eta_1)(\beta_2-\eta_2)}. \tag{5.4.2}$$

Now $t = \dfrac{\beta_1-z}{\beta_2-z}$ maps the circle $|z| < 1$ on the exterior of a certain circle C. Since $t = 0$ if $z = \beta_1$, and $t = \infty$ if $z = \beta_2$, C does not contain the origin $t = 0$. To $|z| > 1$ corresponds the interior of C, and to $|z| = 1$ the circumference of C. *It remains to show that* (5.4.2) *implies that w_1/w_2 is inside or on the boundary of the curve C^2.*[1] For then

$$\frac{w_1}{w_2} = \left(\frac{\beta_1-\zeta}{\beta_2-\zeta}\right)^2 \tag{5.4.3}$$

with $|\zeta| \geq 1$, and writing $\zeta = 1/\bar{\alpha}$,

$$\frac{w_1}{w_2} = \left(\frac{1-\beta_1\bar{\alpha}}{1-\beta_2\bar{\alpha}}\right)^2 \qquad (|\alpha| \leq 1)\,. \tag{5.4.4}$$

In this case an $H(z) = A(1-\bar{\alpha}z)^2$ with $|\alpha| \leq 1$ clearly exists.

Conversely, if such an $H(z)$ exists, then (5.4.4) holds, and w_1/w_2 is inside or on the boundary of C^2. *It follows that the case* (5.3.1) *corresponds to w_1/w_2 being outside C^2*; for we know that only one interpolation function can exist, and the two cases are exclusive. In particular, in the very special case mentioned in § 5.3, namely $v = 0$, ε arbitrary, we have

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[1] C^2 is the image of C by the transform $u = t^2$.

$$\frac{w_1}{w_2} = \frac{(1-\beta_1\bar{\beta}_2)(1-|\beta_1|^2)}{(1-\beta_2\bar{\beta}_1)(1-|\beta_2|^2)}, \tag{5.4.5}$$

and it is easy to verify that this value is inside C^2 (see also § 5.6.). Hence $L(z)$ is in this case of the form (5.4.1).

5.5. To prove that w_1/w_2 is inside or on the boundary of C^2 when (5.4.2) holds, we consider the two points

$$\tau_1 = \frac{\beta_1-\eta_1}{\beta_2-\eta_1}, \qquad \tau_2 = \frac{\beta_1-\eta_2}{\beta_2-\eta_2}, \tag{5.5.1}$$

which are both on the circumference of C. Let $\tau_i = \varrho_i e^{i\theta_i}$. By (5.4.2)

$$\frac{w_1}{w_2} = \tau_1\tau_2 = \varrho_1\varrho_2 e^{i(\theta_1+\theta_2)} = \varrho e^{i\theta}, \tag{5.5.2}$$

say. The line joining $t = 0$ to the point $\varrho e^{i\theta/2}$ bisects the angle $(\tau_1, 0, \tau_2)$, and this point is contained in the angle bounded by the two tangents from 0 to the circle C. Hence w_1/w_2 is contained in the corresponding angle with respect to C^2.

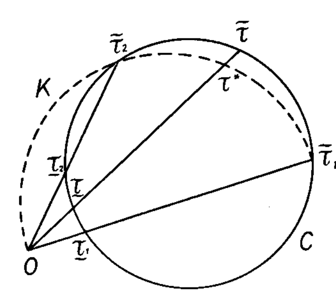
Next consider the points

$$\underset{\sim}{\tau}_i = \underset{\sim}{\varrho}_i e^{i\theta_i}, \quad \tilde{\tau}_i = \tilde{\varrho}_i e^{i\theta_i} \quad (\underset{\sim}{\varrho}_i \le \tilde{\varrho}_i, \quad i = 1, 2) \tag{5.5.3}$$

and

$$\underset{\sim}{\tau} = \underset{\sim}{\varrho} e^{i\theta/2}, \quad \tilde{\tau} = \tilde{\varrho} e^{i\theta/2} \qquad (\underset{\sim}{\varrho} \le \tilde{\varrho}) \tag{5.5.4}$$

which are the (possibly coinciding) points on the circumference of C^2 corresponding to these arguments (Compare figure). The points τ_i are amongst the points (5.5.3). The point w_1/w_2 will clearly be inside or on the boundary of C^2 if



$$\underset{\sim}{\varrho}^2 \le \underset{\sim}{\varrho}_1 \underset{\sim}{\varrho}_2 \le \tilde{\varrho}_1\tilde{\varrho}_2 \le \tilde{\varrho}^2 . \tag{5.5.5}$$

To prove this we may assume that $\theta_1 < \theta_2$ say, the case $\theta_1 = \theta_2 (= \frac{1}{2}\theta)$ being trivial[1]. Draw the circle K through 0, $\tilde{\tau}_1$ and $\tilde{\tau}_2$. Since 0 is outside C, that arc $(\tilde{\tau}_1, \tilde{\tau}_2)$ of K which does not contain 0 will be inside C.

[1] We owe the following simple geometrical argument to Mr. O. F. T. Roberts.

Consider the point $\tau^* = \varrho^* e^{i\theta/2}$ on this arc. If d is the diameter of K and 2φ is the angle that the chord $(0, \tau^*)$ subtends at the centre of K, then $\varrho^* = d \sin \varphi$. Since the line joining 0 to τ^* bisects the angle $(\tilde{\tau}_1, 0, \tilde{\tau}_2)$, we have, similarly, $\tilde{\varrho}_i = d \sin \varphi_i$ where $\varphi_i = \varphi \pm \alpha$, say.

Hence

$$\tilde{\varrho}_1 \tilde{\varrho}_2 = d^2 \sin(\varphi+\alpha) \sin(\varphi-\alpha) = \tfrac{1}{2} d^2 (\cos 2\alpha - \cos 2\varphi)$$

$$\leq \tfrac{1}{2} d^2 (1 - \cos 2\varphi) = d^2 \sin^2 \varphi = \varrho^{*2} \leq \tilde{\varrho}^2 . \tag{5.5.6}$$

On the other hand, $\underset{\sim}{\varrho}_1 \tilde{\varrho}_1 = \underset{\sim}{\varrho}_2 \tilde{\varrho}_2 = \underset{\sim}{\varrho} \tilde{\varrho}$. Hence, by (5.5.6),

$$(\underset{\sim}{\varrho} \tilde{\varrho})^2 = (\underset{\sim}{\varrho}_1 \underset{\sim}{\varrho}_2)(\tilde{\varrho}_1 \tilde{\varrho}_2) \leq \underset{\sim}{\varrho}_1 \underset{\sim}{\varrho}_2 \tilde{\varrho}^2 , \tag{5.5.7}$$

or $\underset{\sim}{\varrho}^2 \leq \underset{\sim}{\varrho}_1 \underset{\sim}{\varrho}_2$. This completes our proof.

5.6. To sum up, the interpolation function $H(z)$ can be found in the case of two different points β_1 and β_2 in the following simple way. If w_1/w_2 is outside C^2, then $H(z)$ is the Lagrange polynomial $L(z)$, uniquely[1] defined in § 5.3., and $H(z)$ is of the form (5.3.1). If w_1/w_2 is inside or on the boundary of C^2, then $H(z) = \Lambda^2(z)$ where $\Lambda(z)$ is the (linear) Lagrange polynomial for which $\Lambda(\beta_i) = w_i^{1/2}$, the square roots being chosen so that $\Lambda(z) \neq 0$ in $|z| < 1$. This is possible according to our discussion. $H(z)$ is then of the form (5.3.2).

The uniqueness of $H(z)$ could be proved directly. This proof which we omit here leads also to the following result which in itself is of some interest:

We have seen that, whenever τ_1 and τ_2 are two points on the circumference of C, then the point $u = \tau_1 \tau_2$ is not outside C^2. Conversely, it can be shown that any point u, not outside C^2, can be represented in the form $u = \tau_1 \tau_2$ and that the two factors τ_1 and τ_2 are in general, apart from their order, uniquely determined. The only exception is that point[2] U on the line joining 0 to the "centre"[3] u_0 of C^2, whose distance from 0 equals the length of the tangents from 0 to C^2. In fact, it is geometrically clear that $\tau_1 \tau_2 = U$ whenever, with our above notations, $\tau_1 = \tilde{\tau}_1$ say, and $\tau_2 = \underset{\sim}{\tau}_2$.

---

[1] The case $v = 0$, ε arbitrary, does not arise here as appears below.

[2] U is the point (5.4.5) corresponding to the case $v = 0$, ε arbitrary, disclosed in § 5.3.

[3] $u_0 = z_0^2$ where z_0 is the centre of C.

Part II. Applications.

6. Inequalities in H_2.

6.1. Extremum problems in H_2 are simple: the 'natural' kernel is the extremal kernel, and the associated extremal functions are then determined by (3.1.3).

Thus the pair

$$K(z) = (z-\beta)^{-(n+1)}, \qquad F(z) = Bz^n(1-\bar{\beta}z)^{-(n+1)} \tag{6.1.1}$$

are extremal kernel and associated extremal function for the inequality

$$|f^{(n)}(\beta)| \leq n!M_2\{(z-\beta)^{-(n+1)}\}M_2(f) . \tag{6.1.2}$$

To calculate $M_2(K)$, we put $\zeta = (\beta+w)/(1+\bar{\beta}w)$ and obtain

$$\begin{aligned} M_2^2(K) &= \frac{1}{2\pi}\int_{|\zeta|=1}\frac{|d\zeta|}{|\zeta-\beta|^{2(n+1)}} = \frac{1}{2\pi}\int_{|w|=1}\left|\frac{1+\bar{\beta}w}{w(1-|\beta|^2)}\right|^{2(n+1)}\frac{1-|\beta|^2}{|1+\bar{\beta}w|^2}|dw| \\ &= \frac{1}{2\pi(1-|\beta|^2)^{2n+1}}\int_{|w|=1}|(1+\bar{\beta}w)^n|^2|dw| . \end{aligned} \tag{6.1.3}$$

Hence[1]

$$|f^{(n)}(\beta)| \leq \frac{n!}{(1-|\beta|^2)^{n+1/2}}\left\{1+\binom{n}{1}^2|\beta|^2+\binom{n}{2}^2|\beta|^4+\cdots+\binom{n}{n}^2|\beta|^{2n}\right\}^{1/2}M_2(f) . \tag{6.1.4$_2$}$$

Alternatively, by (1.3.7),

$$M_2(K) = |I(F_1)|^{1/2} = |F_1^{(n)}(\beta)n!|^{1/2} , \tag{6.1.5}$$

so that (6.1.4) can also be written as

$$|f^{(n)}(\beta)| \leq |n!F_1^{(n)}(\beta)|^{1/2}M_2(f) , \tag{6.1.6$_2$}$$

where $F_1(z) = z^n(1-\bar{\beta}z)^{-(n+1)}$.

6.2. Let $\beta_1 \neq \beta_2$. By (3.1.3), the pair

$$K(z) = [(z-\beta_1)(z-\beta_2)]^{-1} , \quad F(z) = Bz[(1-\bar{\beta}_1z)(1-\bar{\beta}_2z)]^{-1} \tag{6.2.1}$$

are (natural) extremal kernel and associated extremal functions for the inequality

$$\left|\frac{f(\beta_2)-f(\beta_1)}{\beta_2-\beta_1}\right| \leq M_2\{[(z-\beta_1)(z-\beta_2)]^{-1}\}M_2(f) . \tag{6.2.2}$$

[1] Suffixes p, such as in $(6.1.4)_2$, indicate the class H_p (here $p = 2$) for which the formula holds. This will help the reader to find the main results for each class.

Here

$$M_2^2(K) = \frac{1}{2\pi}\int_{|\zeta|=1}\frac{|d\zeta|}{|\zeta-\beta_1|^2|\zeta-\beta_2|^2} = \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{\zeta d\zeta}{(\zeta-\beta_1)(\zeta-\beta_2)(1-\bar\beta_1\zeta)(1-\bar\beta_2\zeta)}$$

$$(6.2.3)\qquad = \frac{\beta_1}{(1-|\beta_1|^2)(\beta_1-\beta_2)(1-\beta_1\bar\beta_2)}+\frac{\beta_2}{(1-|\beta_2|^2)(\beta_2-\beta_1)(1-\beta_2\bar\beta_1)}$$

$$= \frac{1-|\beta_1\beta_2|^2}{|1-\beta_1\bar\beta_2|^2(1-|\beta_1|^2)(1-|\beta_2|^2)},$$

and we obtain

$$(6.2.4)_2\qquad \left|\frac{f(\beta_2)-f(\beta_1)}{\beta_2-\beta_1}\right| \le \left[\frac{1-|\beta_1\beta_2|^2}{|1-\beta_1\bar\beta_2|^2(1-|\beta_1|^2)(1-|\beta_2|^2)}\right]^{1/2} M_2(f)\,.$$

6.3. There is an interesting application of (6.2.4). Let $0 \le a \le b < 1$. If $f(z) = \sum_0^\infty a_n z^n$ belongs to H_2, then its 'majorant' $f^*(z) = \sum_0^\infty |a_n| z^n$ also belongs to H_2, and $M_2(f^*) = M_2(f)$. Clearly $|f'(x)| \le f^{*\prime}(x)$ when $a \le x \le b$, so that

$$\int_a^b |f'(x)|dx \le \int_a^b f^{*\prime}(x)dx = f^*(b)-f^*(a)\,.$$

Hence, by (6.2.4),

$$(6.3.1)_2\qquad \int_a^b |f'(x)|dx \le (b-a)\left[\frac{1+ab}{(1-ab)(1-a^2)(1-b^2)}\right]^{1/2} M_2(f)\,,$$

when $0 \le a < b < 1$. In particular,

$$(6.3.2)_2\qquad \int_0^b |f'(x)|dx \le \frac{b}{(1-b^2)^{1/2}} M_2(f) \qquad (0 \le b < 1)\,.$$

The extremal functions are $F(z) = Bz[(1-az)(1-bz)]^{-1}$.

The integral in (6.3.1) is the length of the map of the interval $a \le x \le b$ by the transformation $w = f(x)$.

6.4. We have seen, in § 3.2, that Cauchy's inequality provides another simple way of dealing with extremum problems in H_2. Thus we have, for all $r \ge 0$,

$$(6.4.1)_2\qquad |a_0|+|a_1|r+\cdots+|a_n|r^n \le (1+r^2+\cdots+r^{2n})^{1/2} M_2(f)$$

with the extremal functions $F(z) = B(1+rz+r^2z^2+\cdots+r^nz^n)$.

The right hand side is of the order $(1-r)^{-1/2}$ as $n \to \infty$, uniformly for all f with given $M_2(f)$. For fixed f this order can be reduced to[1]

[1] Hardy.

$$(6.4.2)_2 \qquad f^*(r) = \sum_0^\infty |a_n| r^n = o(1-r)^{1/2}$$

as $r \to 1-0$. For, when $m \geq 0$ is given, then

$$f^*(r) \leq \sum_0^m |a_n| r^n + (1-r^2)^{-1/2}\Big(\sum_{m+1}^\infty |a_n|^2\Big)^{1/2}$$

and so

$$\overline{\lim_{r\to 1-0}}\, (1-r^2)^{1/2} f^*(r) \leq \Big(\sum_{m+1}^\infty |a_n|^2\Big)^{1/2}.$$

On letting $m \to \infty$, we obtain (6.4.2).

Cauchy's inequality is also available for extremum problems involving certain transcendental kernels not covered by our general theory. Thus

$$(6.4.3) \qquad \int_{\beta_1}^{\beta_2} f(z)dz = \frac{1}{2\pi i}\int_{|\zeta|=1} f(\zeta) \log \frac{\zeta-\beta_1}{\zeta-\beta_2}\, d\zeta$$

has the logarithmic kernel $k(z) = \log\{(z-\beta_1)/(z-\beta_2)\}$. Here

$$(6.4.4)_2 \qquad \left|\int_{\beta_1}^{\beta_2} f(z)dz\right| = \left|\sum_0^\infty a_n \frac{\beta_2^{n+1}-\beta_1^{n+1}}{n+1}\right| \leq \left\{\sum_0^\infty \left|\frac{\beta_2^{n+1}-\beta_1^{n+1}}{n+1}\right|^2\right\}^{1/2} M_2(f).$$

Equality holds when $a_n = B\dfrac{\bar\beta_2^{n+1}-\bar\beta_1^{n+1}}{n+1}$, so that

$$K(z) = k(z) = \log\frac{z-\beta_1}{z-\beta_2}, \quad F(z) = \frac{B}{z}\log\frac{1-\bar\beta_1 z}{1-\bar\beta_2 z}$$

may be regarded as extremal kernel and associated extremal functions. We note that (6.4.4) implies

$$(6.4.5)_2 \qquad \left|\int_{\beta_1}^{\beta_2} f(z)dz\right| \leq 2\left\{\sum_0^\infty \frac{1}{(n+1)^2}\right\}^{1/2} M_2(f) = \frac{2\pi}{\sqrt{6}} M_2(f)$$

for all $|\beta_1| < 1$, $|\beta_2| < 1$.

If $B = 1$, then, clearly, $M_2(F) = M_2(K)$ and hence $|I(F)| = M_2^2(K)$. The 'constant' in (6.4.4) is therefore

$$(6.4.6) \qquad M_2(K) = \left|\int_{\beta_1}^{\beta_2} \log\frac{1-\bar\beta_1 z}{1-\bar\beta_2 z}\frac{dz}{z}\right|^{1/2} = \left\{\int_0^1 \log\frac{(1-\beta_1\bar\beta_2 x)(1-\beta_2\bar\beta_1 x)}{(1-|\beta_1|^2 x)(1-|\beta_2|^2 x)}\frac{dx}{x}\right\}^{1/2}.$$

If $\beta_1 = a$, $\beta_2 = b$ and $0 \leq a < b < 1$, then, integrating over the interval $a \leq x \leq b$, we can replace $f(x)$ in (6.4.4) by $|f(x)|$. For, (6.4.4) holds also for the majorant $f^*(x)$. In particular,

$$\int_0^b |f(x)|dx \le \left\{\int_0^1 \log\frac{1}{1-b^2x}\frac{dx}{x}\right\}^{1/2} M_2(f) \qquad (0 < b < 1)\,, \tag{6.4.7$_2$}$$

and, on letting $b \to 1$,

$$\int_0^1 |f(x)|dx \le \left\{\sum_0^\infty \frac{1}{(n+1)^2}\right\}^{1/2} M_2(f) = \frac{\pi}{\sqrt{6}} M_2(f)\,. \tag{6.4.8$_2$}$$

7. Other Inequalities for which the 'Natural' Kernel is Extremal.

7.1. While the natural kernel is always the extremal kernel in H_2, it is, so to speak, accidentally the extremal kernel in certain special cases for other classes H_p.

For example, if $p = \infty, q = 1$, the kernel $(z-\beta)^{-(n+1)}$ is extremal provided that $n+1$ is even. For, if $n+1 = 2(m+1)$ say, we may take, in (1.3.5), $A = 1, \Pi'$ to be empty, $m+1$ of the α_i equal to β and the remaining m of the α_i to be zero. The extremal function (1.3.6) then becomes

$$F(z) = Bz^m \left(\frac{z-\beta}{1-\bar{\beta}z}\right)^{m+1}, \tag{7.1.1}$$

and we obtain[1]

$$\begin{aligned} |f^{(2m+1)}(\beta)| &\le (2m+1)!\,M_1\{(z-\beta)^{-2(m+1)}\}\ \mathrm{Max}\,|f| \\ &= (2m+1)!\,M_2^2\{(z-\beta)^{-(m+1)}\}\ \mathrm{Max}\,|f|\,. \end{aligned} \tag{7.1.2}$$

Hence, by (6.1.3),[2]

$$\begin{aligned} |f^{(2m+1)}(\beta)| \le \frac{(2m+1)!}{(1-|\beta|^2)^{2m+1}} \Bigg\{ 1^2 + \binom{m}{1}^2 |\beta|^2 + \binom{m}{2}^2 |\beta|^4 + \cdots \\ + \binom{m}{m}^2 |\beta|^{2m} \Bigg\} \mathrm{Max}\,|f|\,. \end{aligned} \tag{7.1.3$_\infty$}$$

7.2. The extremal kernel (1.3.5) will take the form $\Pi(z-\beta_i)^{-1}$, if Π' is empty and $\Pi(1-\bar{\alpha}_i z)^{2/q} = \Pi(1-\bar{\beta}_i z)^{2/q-1}$. This will occur whenever $\Pi(1-\bar{\beta}_i z)^{1-q/2}$ is a polynomial.

In particular, if $q \neq 2$ and if all the β_i equal β and are $n+1$ in number, then the degree $(n+1)\,(1-q/2)$ of the polynomial must be a positive integer, so that q must be rational and less than 2. If $q/2 = h/k$ in its lowest terms, then $n+1$ must be a multiple of k. Since $q = p/(p-1)$, this case will arise, in particular, when p is an integer greater than 2. If then p is even, $n+1$ must be a multiple of $p-1$; if p is odd,

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[1] We write Max $|f|$ for the least upper bound of $|f(z)|$ in $|z| < 1$.

[2] (7.1.3) was first proved, in a different way, by Szász (b).

then $n+1$ must be a multiple of $2(p-1)$. If $p=\infty$, $q=1$, then $n+1$ must be even [§ 7.1]; if $p=2$, $q=2$, then n may be arbitrary [§ 6.1].

The 'constant' of the corresponding inequality is

$$M_q[(z-\beta)^{-(n+1)}] = M_2^{2/q}[(z-\beta)^{-(n+1)q/2}] .$$

Hence, by (6.1.3), we obtain

$$(7.2.1)_p \qquad |f^{(n)}(\beta)| \leq \frac{n!}{(1-|\beta|^2)^{n+1/p}}\left[1+\binom{\mu}{1}^2|\beta|^2+\binom{\mu}{2}^2|\beta|^4+\cdots +\binom{\mu}{\mu}^2|\beta|^{2\mu}\right]^{1/q} M_p(f) ,$$

where $\mu=(n+1)q/2-1$. *This inequality holds for all integral* $p\geq 2$: *if* p *is even, then* $n+1$ *must be a multiple of* $p-1$; *if* p *is odd, then* n *must be a multiple of* $2(p-1)$.

Since

$$\Pi(1-\bar{\beta}_i z)^{1-q/2} = (1-\bar{\beta}z)^{(n+1)(1-q/2)} = (1-\bar{\beta}z)^{n-\mu},$$

we see that $n-\mu$ of the n roots α_i of the extremal functions (1.3.6) must equal β, while the remaining μ roots vanish. Hence these extremal functions are

$$(7.2.2) \qquad F(z) = Bz^\mu(z-\beta)^{n-\mu}(1-\bar{\beta}z)^{(n-\mu)(2/p-1)-(n+1)2/p} = Bz^\mu(z-\beta)^{n-\mu}(1-\bar{\beta}z)^{-(\mu+1)}.$$

8. Extremal Kernels without Zeros in $|z|<1$.

8.1. It is sometimes possible, if Π' in (1.3.5) is empty, to determine the extremal kernel, even when it is not the natural kernel. This can be done by Landau's method which we discussed in § 3.3.

Let $\alpha>-1$. If $\sigma_n^{(\alpha)}(z) = s_n^{(\alpha)}(z)\Big/\binom{\alpha+n}{n}$, where

$$(8.1.1) \qquad s_n^{(\alpha)}(z) = \binom{\alpha+n}{n}a_0+\binom{\alpha+n-1}{n-1}a_1z+\cdots+a_nz^n ,$$

then the $\sigma_n^{(\alpha)}(z)$ are the (C,α)-transformations of the partial sums $s_n(z)=s_n^{(0)}(z)$ of $f(z)$. We write $\sigma_n^{(\alpha)}=\sigma_n^{(\alpha)}(1)$, $s_n^{(\alpha)}=s_n^{(\alpha)}(1)$.

Let $p>1$. The function $G(z)$ of (3.3.7) corresponding to $I(f)=s_n^{(\alpha)}$ is, for small $|z|$,

$$(8.1.2) \qquad G(z) = 1+\binom{\alpha+1}{1}z+\cdots+\binom{\alpha+n}{n}z^n+\cdots = (1-z)^{-(\alpha+1)}+O(|z|^{n+1}) ,$$

so that

$$(8.1.3) \qquad G(z)^{q/2} = (1-z)^{-(\alpha+1)q/2}+O(|z|^{n+1}) = 1+\binom{\lambda+1}{1}z+\cdots+\binom{\lambda+n}{n}z^n+O(|z|^{n+1}),$$

where $\lambda = (\alpha+1)q/2-1$. Hence, by (3.3.13), we shall have

$$(8.1.4)_{p>1} \quad |\sigma_n^{(\alpha)}| \leq \frac{1}{\binom{\alpha+n}{n}} \left[1+\binom{\lambda+1}{1}^2+\binom{\lambda+2}{2}^2+\cdots+\binom{\lambda+n}{n}^2\right]^{1/q} M_p(f),$$

provided that the polynomial

$$(8.1.5) \qquad P_n(z) = 1+\binom{\lambda+1}{1}z+\binom{\lambda+2}{2}z^2+\cdots+\binom{\lambda+n}{n}z^n$$

has no roots in $|z| < 1$. By Kakeya's Lemma (3.3.14), this will certainly be the case when $\lambda \leq 0$, that is when $q \leq 2/(\alpha+1)$. In particular, when $\alpha = 0$, we obtain for all $q \leq 2$, that is for all $p \geq 2$,

$$(8.1.6)_{p\geq 2} \quad \begin{aligned} |s_n| = |a_0+a_1+\cdots+a_n| \leq \Big[1+\left(\frac{q}{2}\right)^2+\left(\frac{q(q+2)}{2\cdot 4}\right)^2+\cdots \\ +\left(\frac{q(q+2)\cdots(q+2n-2)}{2\cdot 4\cdots 2n}\right)^2\Big]^{1/q} M_p(f)\,. \end{aligned}$$

For $q = 1, p = \infty$ this is Landau's inequality.

If $\lambda = 0, q = 2/(\alpha+1)$, then we have, for all $-1 < \alpha \leq 1$,

$$(8.1.7) \qquad |\sigma_n^{(\alpha)}| \leq \frac{(n+1)^{\frac{\alpha+1}{2}}}{\binom{\alpha+n}{n}} M_{\frac{2}{1-\alpha}}(f) \sim n^{\frac{1-\alpha}{2}} M_{\frac{2}{1-\alpha}}(f)\,.$$

If $\alpha = 1$, this is the familiar inequality $|\sigma_n^{(1)}| \leq \text{Max}\,(f)$. In all these cases the extremal functions are, by (3.3.11), of the form $F(z) = Bz^n\overline{P_n(1/\bar{z})}/P_n(z)^{1-2/p}$.

More generally,

$$(8.1.8)_p \quad |\sigma_n^{(\alpha)}(z)| \leq \frac{r^n}{\binom{\alpha+n}{n}} \left[1+\binom{\lambda+1}{1}^2 r^{-2}+\binom{\lambda+2}{2}^2 r^{-4}+\cdots+\binom{\lambda+n}{n}^2 r^{-2n}\right]^{1/q} M_p(f),$$

where $|z| = r$, *provided that the polynomial*

$$(8.1.9) \qquad Q_n(\zeta) = r^n+\binom{\lambda+1}{1}r^{n-1}\zeta+\cdots+\binom{\lambda+n}{n}\zeta^n$$

has no roots in $|\zeta| < 1$. By Kakeya's Lemma, this will certainly be the case when either $\lambda \leq 0$ and $r \geq 1-\frac{|\lambda|}{n}$, or when $\lambda \geq 0$ and $r \geq 1+\lambda$.

8.2. Let $p > 1$. It is often convenient to shift a pole β of a kernel $k(z)$ to $w = 0$ by the linear transformation

$$z = \frac{w+\beta}{1+\bar{\beta}w}, \quad dz = \frac{1-|\beta|^2}{(1+\bar{\beta}w)^2}\,dw \tag{8.2.1}$$

[see § 6.1]. Conversely, we then have

$$w = \frac{z-\beta}{1-\bar{\beta}z}, \quad 1+\bar{\beta}w = \frac{1-|\beta|^2}{1-\bar{\beta}z}, \tag{8.2.2}$$

$$w-\alpha = (1+\alpha\bar{\beta})\frac{z-\gamma}{1-\bar{\beta}z}, \quad 1-\bar{\alpha}w = (1+\bar{\alpha}\beta)\frac{1-\bar{\gamma}z}{1-\bar{\beta}z},$$

where $\gamma = (\alpha+\beta)/(1+\alpha\bar{\beta})$.

On transformation $I(f)$ becomes

$$\begin{aligned} I(f) &= \frac{1}{2\pi i}\int_{|w|=1} f\left(\frac{w+\beta}{1+\bar{\beta}w}\right) k\left(\frac{w+\beta}{1+\bar{\beta}w}\right)\frac{1-|\beta|^2}{(1+\bar{\beta}w)^2}\,dw \\ &= \frac{1}{2\pi i}\int_{|w|=1} \varphi(w)\varkappa(w)dw\,, \end{aligned} \tag{8.2.3}$$

where

$$\varphi(w) = f\left(\frac{w+\beta}{1+\bar{\beta}w}\right)\left[\frac{1-|\beta|^2}{(1+\bar{\beta}w)^2}\right]^{1/p}, \quad \varkappa(w) = k\left(\frac{w+\beta}{1+\bar{\beta}w}\right)\left[\frac{1-|\beta|^2}{(1+\bar{\beta}w)^2}\right]^{1/q}. \tag{8.2.4}$$

Also

$$M_q(k) = \left\{\frac{1}{2\pi}\int_{|w|=1}\left|k\left(\frac{w+\beta}{1+\bar{\beta}w}\right)\right|^q \frac{1-|\beta|^2}{|1+\bar{\beta}w|^2}\,|dw|\right\}^{1/q} = M_q(\varkappa)\,. \tag{8.2.5}$$

8.3. Consider, as an example, the kernel $k(z) = (z-\beta)^{-(n+1)}$, when

$$\varkappa(w) = (1-|\beta|^2)^{-(n+1/p)}w^{-(n+1)}(1+\bar{\beta}w)^{n+1-2/q}. \tag{8.3.1}$$

We can now employ the same method as in § 3.3 and § 8.1. For any equivalent kernel $K(w)$ the function $G(w)$ of (3.3.7) satisfies, for small $|w|$,

$$\begin{aligned} (1-|\beta|^2)^{n+1/p}G(w) &= (1-|\beta|^2)^{n+1/p}w^{n+1}K(w) = (1+\bar{\beta}w)^{n+1-2/q}+O(|w|^{n+1})\,, \\ [(1-|\beta|^2)^{n+1/p}G(w)]^{q/2} &= (1+\bar{\beta}w)^{(n+1)q/2-1}+O(|w|^{n+1}) \\ &= 1+\binom{\mu}{1}\bar{\beta}w+\binom{\mu}{2}\bar{\beta}^2w^2+\cdots+\binom{\mu}{n}\bar{\beta}^nw^n+O(|w|^{n+1})\,, \end{aligned} \tag{8.3.2}$$

where $\mu = (n+1)q/2-1$. Suppose now that the polynomial

$$P_n(w) = 1+\binom{\mu}{1}\bar{\beta}w+\binom{\mu}{2}\bar{\beta}^2w^2+\cdots+\binom{\mu}{n}\bar{\beta}^nw^n \tag{8.3.3}$$

has no roots in $|w| < 1$ and hence is of the form $C\prod_1^n(1-\bar{\alpha}_i w)$. Also

$$(8.3.4)\qquad K(w) = (1-|\beta|^2)^{-(n+1/p)}w^{-(n+1)}P_n^{2/q}(w)$$

is then of the form (1.3.5) [with $\Pi' = 0$, $\beta_i = 0$ for $0 \le i \le n+1$], and hence is the extremal kernel which makes $M_q(\varkappa) = M_q(k)$ a minimum. We thus obtain

$$(8.3.5)_p\qquad |f^{(n)}(\beta)| \le \frac{n!}{(1-|\beta|^2)^{n+1/p}}\left[1+\binom{\mu}{1}^2|\beta|^2+\binom{\mu}{2}^2|\beta|^4+\cdots+\binom{\mu}{n}^2|\beta|^{2n}\right]^{1/q} M_p(f)\,,$$

provided that $P_n(w) \neq 0$ *in* $|w| < 1$. By (3.3.1) and (8.2.4) the corresponding extremal functions are of the form

$$(8.3.6)\qquad F(z) = Bw^n\overline{P_n(1/\bar{w})}/P_n(w)^{1-2/p}.\ (1+\bar{\beta}w)^{2/p} \qquad \left(w = \frac{z-\beta}{1-\bar{\beta}z}\right).$$

The condition that $P_n(w) \neq 0$ in $|w| < 1$ is certainly satisfied when $\mu \le n$ and is an integer. Then $P_n(w) = (1+\bar{\beta}w)^\mu$ and we obtain again (7. 2. 1).

When $n = 0$ then $P_n \equiv 1$, so that for all $p \ge 1$

$$(8.3.7)_p\qquad |f(\beta)| \le \frac{1}{(1-|\beta|^2)^{1/p}} M_p(f).$$

The extremal functions are $F(z) = B(1-\bar{\beta}z)^{-2/p}$.

It is easy to verify (8.3.7) directly. By (1.3.5), the extremal kernel, equivalent to $(z-\beta)^{-1}$, is clearly

$$K(z) = \left(\frac{1-\bar{\beta}z}{1-|\beta|^2}\right)^{1-2/q}(z-\beta)^{-1}\,.$$

Hence, using (6.1.3), we have

$$\begin{aligned} M_q(K) &= (1-|\beta|^2)^{2/q-1}M_q\{(1-\bar{\beta}z)^{-2/q}\} = (1-|\beta|^2)^{2/q-1}M_2^{2/q}\{(1-\bar{\beta}z)^{-1}\} \\ &= (1-|\beta|^2)^{2/q-1}(1-|\beta|^2)^{-1/q} = (1-|\beta|^2)^{-1/p}, \end{aligned}$$

which proves (8.3.7) anew.

If $n = 1$, then $\mu = q-1 = (p-1)^{-1}$, $P_1(w) = 1+(q-1)\bar{\beta}w$, and we obtain

$$(8.3.8)_p\qquad |f'(\beta)| \le \frac{1}{(1-|\beta|^2)^{1+1/p}}\left(1+\frac{|\beta|^2}{(p-1)^2}\right)^{1/q} M_p(f)\,,$$

valid for $p > 1$ *and all* $|\beta| \le p-1$. In particular, when $p \ge 2$, (8.3.8) holds for all $|\beta| < 1$.

If $n = 2$, then $\mu = \frac{3}{2}q-1$. Also

$$P_2(w) = 1+\mu\bar{\beta}w+\frac{\mu(\mu-1)}{2}\bar{\beta}^2w^2$$

has the roots

$$(\bar{\beta}w)^{-1} = \tfrac{1}{2}[-\mu\pm\sqrt{2\mu-\mu^2}]\,.$$

If the modulus of the right hand side does not exceed one, then $P_2(w)$ will have no roots in $|w| < 1$ for any $|\beta| < 1$. An elementary argument shows that this is the case when either $\mu = 2$ or when $\mu \leq 1$. Hence we find that[1]

$$(8.3.9)_p \qquad |f''(\beta)| \leq \frac{2}{(1-|\beta|^2)^{2+1/p}}\left(1+\mu^2|\beta|^2+\frac{\mu^2(\mu-1)^2}{4}|\beta|^4\right)^{1/q} M_p(f)$$

holds for all $|\beta| < 1$, *when either* $p = 2$ *or* $p \geq 4$.

It should be noted that, quite generally, (8.3.5) holds, for fixed p, when $|\beta|$ is small enough. For, $P_n(w)$ will then have no roots in $|w| < 1$.

9. The Inequality for $f'(\beta)$.

9.1. In general we are unable to complete the analysis of an inequality whose extremal kernel has zeros in $|z| < 1$. But the case of $f'(\beta)$ is sufficiently simple for us to do so.

We have proved (8.3.8) for all $|\beta| \leq p-1$; it holds, in particular, for all $|\beta| < 1$ when $p \geq 2$. Let now $p < 2$ and $|\beta| > p-1$. The kernel (8.3.1) is here

$$(9.1.1) \qquad \varkappa(w) = (1-|\beta|^2)^{-(1+1/p)}w^{-2}(1+\bar{\beta}w)^{2/p}.$$

The equivalent extremal kernel $K(w)$ must have a root γ in $|w| < 1$, so that, by (1.3.5), it is of the form

$$(9.1.2) \qquad K(w) = A(w-\gamma)(1-\bar{\gamma}w)^{2/q-1}w^{-2}\,.$$

The constants A and γ must satisfy

$$-\gamma A = (1-|\beta|^2)^{-(1+1/p)},$$

$$(9.1.3) \qquad A\left\{1+\left(\frac{2}{q}-1\right)|\gamma|^2\right\} = (1-|\beta|^2)^{-(1+1/p)}\frac{2}{p}\bar{\beta} = -\frac{2}{p}\gamma\bar{\beta}A\,.$$

Hence $\gamma\bar{\beta}$ is negative, that is $\gamma\bar{\beta} = -|\gamma|\,|\beta|$, so that

$$1-\frac{2}{p}|\gamma|\,|\beta|+\left(1-\frac{2}{p}\right)|\gamma|^2 = 0$$

[1] Szász (b), in the case $p = \infty$.

or

$$(9.1.4)\qquad |\gamma| = \frac{[|\beta|^2+p(2-p)]^{1/2}-|\beta|}{2-p} = \frac{p}{[|\beta|^2+p(2-p)]^{1/2}+|\beta|}.$$

Clearly, $|\gamma| < 1$ when $p \leq 2$ and $|\beta| > p-1$. Also

$$M_q(K) = |A|M_q\{(1-\bar{\gamma}w)^{2/q}\} = |A|M_2^{2/q}(1-\bar{\gamma}w) = |A|(1+|\gamma|^2)^{1/q}.$$

Hence

$$(9.1.5)_p\qquad |f'(\beta)| \leq \frac{(1+|\gamma|^2)^{1/q}}{|\gamma|(1-|\beta|^2)^{1+1/p}} M_p(f)$$

is valid for $p < 2$ *and* $|\beta| > p-1$; $|\gamma|$ is given by (9.1.4). In particular, when $p = 1$, $q = \infty$, we have[1]

$$(9.1.6)_1\qquad |f'(\beta)| \leq \frac{[|\beta|+(1+|\beta|^2)^{1/2}]}{(1-|\beta|^2)^2} M_1(f).$$

By (1.3.6) and (8.2.4), the extremal functions for (9.1.5), are of the form

$$(9.1.7)\qquad \begin{aligned} F(z) &= B[(1-\bar{\gamma}w)(1+\bar{\beta}w)]^{2/p} && \left(w = \frac{z-\beta}{1-\bar{\beta}z}\right) \\ &= C(1-\bar{\alpha}z)^{2/p}(1-\bar{\beta}z)^{-4/p} && \left(\alpha = \frac{\beta+\gamma}{1+\gamma\bar{\beta}}\right). \end{aligned}$$

The inequality (9.1.6) holds also for $\beta = 0$. But then $|\gamma| = 1$, and there are an infinity of extremal functions of the form

$$(9.1.8)\qquad F(z) = B(z-\alpha)(1-\bar{\alpha}z),$$

where α is an arbitrary parameter with $|\alpha| < 1$.

10. Inequalities in H_1.

10.1. Inequalities in H_1 have special features as we explained in § 4. In particular, there will be genuinely different extremal functions if the extremal kernel has fewer than the maximum number of zeros. This case, however, must be considered as exceptional, since the number of parameters appearing in the kernel is then less than the order of the problem.

Consider, for instance, the kernel $[(1-\bar{\beta}z)/(z-\beta)]^{n+1}$ which is of extremal form. It yields the inequality

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[1] Macintyre and Rogosinski.

$$\left|\left(\frac{d}{dz}\right)^n \{f(z)(1-\bar{\beta}z)^{n+1}\}_{z=\beta}\right| \leq n!M_1(f) \tag{10.1.1$_1$}$$

with extremal functions

$$F(z) = B\prod_1^n (z-\alpha_i)(1-\bar{\alpha}_i z)(1-\bar{\beta}z)^{-2(n+1)}, \tag{10.1.2}$$

the parameters α_i, $|\alpha_i| < 1$, being arbitrary. In particular, when $\beta = 0$, the elementary inequality

$$|f^{(n)}(0)| \leq n!M_1(f), \quad |a_n| \leq M_1(f) \tag{10.1.3}$$

has this variety of extremal functions.

It is also interesting to note that, in the inequality (9.1.6) for $f'(\beta)$, the extremal function is of unique type when $\beta \neq 0$. In the next paragraph we shall find the same with regard to $f''(\beta)$. We also note that (9.1.6) may easily be obtained once more from (10.1.1) when $n = 1$.[1]

10.2. Consider the kernel

$$k(z) = \frac{A}{z-\beta_1}+\frac{B}{z-\beta_2}$$

where $\beta_1 \neq \beta_2$. The corresponding extremal kernel must be, apart from a constant factor, one of the two kernels

$$\frac{1-\bar{\beta}_1 z}{z-\beta_1}\frac{1-\bar{\beta}_2 z}{z-\beta_2}, \quad \frac{z-\alpha}{1-\bar{\alpha}z}\frac{1-\bar{\beta}_1 z}{z-\beta_1}\frac{1-\bar{\beta}_2 z}{z-\beta_2} \quad (|\alpha| < 1). \tag{10.2.1}$$

To the first case corresponds the inequality

$$|(1-|\beta_1|^2)(1-\bar{\beta}_2\beta_1)f(\beta_1)-(1-|\beta_2|^2)(1-\bar{\beta}_1\beta_2)f(\beta_2)| \leq |\beta_2-\beta_1|M_1(f), \tag{10.2.2$_1$}$$

and the extremal functions depend on an arbitrary parameter α with $|\alpha| < 1$. We find also, on considering the residues at β_1 and β_2, that

$$\frac{A}{B} = -\frac{1-|\beta_1|^2}{1-|\beta_2|^2}\frac{1-\bar{\beta}_2\beta_1}{1-\bar{\beta}_1\beta_2}, \tag{10.2.3}$$

so that this case is an exceptional one.

In the general case, we have, for every given α with $|\alpha| < 1$,

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[1] Macintyre and Rogosinski.

$$(10.2.4)_1 \qquad \left|(1-|\beta_1|^2)(1-\bar\beta_2\beta_1)\frac{\beta_1-\alpha}{1-\bar\alpha\beta_1}f(\beta_1)-(1-|\beta_2|^2)(1-\bar\beta_1\beta_2)\frac{\beta_2-\alpha}{1-\bar\alpha\beta_2}f(\beta_2)\right| \le |\beta_2-\beta_1| M_1(f)\,,$$

and the extremal functions are essentially unique.

10.3. If we take, in (10.2.4), $\beta_2 = 0$ and $\beta_1 = r (0 < r < 1)$ and if we determine α, with $|\alpha| < 1$, from

$$\alpha = (1-r^2)\frac{\alpha-r}{1-\bar\alpha r},$$

then an elementary calculation gives

$$(10.3.1) \qquad \alpha = \tfrac{1}{2}(r-\sqrt{4-3r^2})\,, \qquad |\alpha|^{-1} = \frac{r+\sqrt{4-3r^2}}{2(1-r^2)}.$$

Hence, by (10.2.4),

$$\left|\int_0^r f'(x)dx\right| = |f(r)-f(0)| \le \frac{r}{|\alpha|}M_1(f)\,.$$

Here, by what we have proved in § 4.5, we may replace f' by $|f'|$, and obtain

$$(10.3.2)_1 \qquad \int_0^r |f'(x)|dx \le \frac{r(r+\sqrt{4-3r^2})}{2(1-r^2)}M_1(f)\,.$$

The extremal functions are, by (1.3.6),

$$(10.3.3) \qquad F(z) = B\left(\frac{1-\alpha z}{1-rz}\right)^2$$

with the above α.

11. The Inequality for $f''(\beta)$ in H_1.

11.1. The inequalities for $f^{(n)}(\beta)$ in H_1 can be found by means of the algebraic equation (4.4.2), after having shifted the pole β to 0 by a linear transformation. In the case of $f''(\beta)$ this method would lead to a cubic equation. We prefer here a direct argument which yields a parametric expression for the desired upper bound.

To the kernel $k(z) = (z-\beta)^{-3}$ corresponds, by (8.2.4), the kernel

$$(11.1.1) \qquad \varkappa(w) = \left\{\frac{1+\bar\beta w}{w(1-|\beta|^2)}\right\}^3$$

If $K(w)$ is the equivalent extremal kernel, then

(11.1.2) $$K(w) = \{w(1-|\beta|^2)\}^{-3}\chi(w)$$

where

(11.1.3) $$\chi(w) = 1+3\bar{\beta}w+3(\bar{\beta}w)^2+O(|w|^3)$$

is that function, regular in $|w| < 1$, for which $M = \text{Max}\,|\chi|$ is a minimum. Clearly $M \geq 1$.

If $M = 1$, then $\beta = 0$, $\chi \equiv 1$, and the extremal kernel $K(w)$ has no root in $|w| < 1$.

If $\beta \neq 0$, then $M > 1$ and

(11.1.4) $$\psi(w) = \frac{M(\chi-1)}{w(M^2-\chi)} = \frac{3\bar{\beta}M}{M^2-1}\left\{1+\frac{M^2+2}{M^2-1}\bar{\beta}w+O(|w|^2)\right\}$$

will be regular, and $|\psi| < 1$, in $|w| < 1$. Hence[1]

(11.1.5) $$\frac{3|\beta|^2M(M^2+2)}{(M^2-1)^2} \leq 1-\left(\frac{3|\beta|M}{M^2-1}\right)^2.$$

The minimum condition on M requires equality here, that is

(11.1.6) $$|\beta|^2 = \frac{(M-1)^2(M+1)}{3M(M+2)};$$

and we must also have

(11.1.7) $$\psi(w) = \frac{w-\alpha}{1-\bar{\alpha}w}, \quad \alpha = -\frac{3\bar{\beta}M}{M^2-1}.$$

The right hand side of (11.1.6) increases from 0 to 1 when M increases from 1 to M_0, where M_0 is the root, greater than 1, of

(11.1.8) $$(M-1)^2(M+1)-3M(M+2) \equiv M^3-4M^2-7M+1 = 0.$$

This root is slightly greater than 5. Hence

$(11.1.9)_1$ $$|f''(\beta)| \leq \frac{2M}{(1-|\beta|^2)^3}M_1(f),$$

where $|\beta|$ is given in the parametric form (11.1.6), the parameter M running from 1 to M_0.

If $\beta \neq 0$, $M > 1$, then it follows from (11.1.4) and (11.1.7) that χ is a rational function of degree 2, with $|\chi| = M$ on $|w| = 1$. Hence it has two zeros in $|w| < 1$, and therefore $K(w)$ has also two zeros in $|w| < 1$, and $K(z)$ has two zeros in $|z| < 1$. It follows that the extremal functions $F(z)$ are essentially unique. On the other hand, if $\beta = 0$, then the extremal functions depend on two arbitrary parameters α_1 and α_2, with $|\alpha_i| < 1$.

[1] For, $|\varphi'(0)| \leq 1-|\varphi(0)|^2$, by (4.2.4) with $\beta_1 = 0$.

12. Egerváry's Inequalities in H_1.

12.1. Let $f(z) = \sum_0^\infty a_n z^n$ be of class H_1, and let $r > 0$. We wish to find inequalities for the expressions

$$S_{n-1}(r) = |a_0|+|a_1|r+|a_2|r^2+\cdots+|a_{n-1}|r^{n-1} . \tag{12.1.1}$$

As we have seen in § 4.5, we may here replace the $|a_k|$ by the a_k, so that the corresponding kernel is

$$k(z) = z^{-n}[r^{n-1}+r^{n-2}z+\cdots+z^{n-1}] . \tag{12.1.2}$$

By (1.3.6), the equivalent extremal kernel is of the form

$$K(z) = Az^{-n}\Pi' \frac{z-\alpha_i}{1-\bar{\alpha}_i z} = Az^{-n}[r^{n-1}+r^{n-2}z+\cdots+z^{n-1}+O(|z|^n)] . \tag{12.1.3}$$

The problem of determining this kernel has been solved, for $n/(n+1) < r \leq 1$, by Egerváry[1]. We verify his results and extend them for all $r > 0$. First, consider the polynomial

$$P(z) = \sin\theta+z\sin 2\theta+\cdots+z^{n-1}\sin n\theta = \frac{\sin\theta-z^n\sin(n+1)\theta+z^{n+1}\sin n\theta}{1-2z\cos\theta+z^2} . \tag{12.1.4}$$

Let θ be a root of $\sin n\theta = r\sin(n+1)\theta$. Then, for small $|z|$,

$$\begin{aligned}\frac{P(z)}{z^{n-1}P(1/z)} &= \frac{\sin\theta-z^n\sin(n+1)\theta+z^{n+1}\sin n\theta}{\sin n\theta-z\sin(n+1)\theta+z^{n+1}\sin\theta}\\ &= \frac{\sin\theta}{\sin n\theta}\,\frac{1+O(|z|^n)}{1-z/r+O(|z|^{n+1})}\\ &= \frac{\sin\theta}{\sin n\theta}\left[1+\frac{z}{r}+\cdots+\left(\frac{z}{r}\right)^{n-1}+O(|z|^n)\right] .\end{aligned} \tag{12.1.5}$$

If now $P(z)$ has all its roots in $|z| < 1$, then

$$K(z) = \frac{\sin n\theta}{\sin\theta}\,\frac{r^{n-1}}{z^{2n-1}}\,\frac{P(z)}{P(1/z)} \tag{12.1.6}$$

will be of the form (12.1.3), and hence will be the extremal kernel. We shall prove that $P(z)$ has all its roots in $|z| < 1$, when $0 < \theta < \pi/(n+1)$, that is for

$$r = \sin n\theta/\sin(n+1)\theta > n/(n+1) .$$

Now this is true when $0 < \theta \leq \pi/2n$, by Kakeya's Lemma (3.3.14) applied to $P(1/z)$.

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[1] Egerváry; see also Landau (b).

It suffices, therefore, to show that, for $0 < \theta < \pi/(n+1)$ no root of $P(z)$ can be on the unit circle. Now $P(e^{i\varphi}) = 0$ implies

$$\begin{aligned} &\cos(n+1)\varphi \sin n\theta - \cos n\varphi \sin(n+1)\theta = -\sin\varphi\,, \\ &\sin(n+1)\varphi \sin n\theta - \sin n\varphi \sin(n+1)\theta = 0 \end{aligned} \tag{12.1.7}$$

and thus

$$\sin^2 n\theta + \sin^2(n+1)\theta - 2\sin n\theta \sin(n+1)\theta\cos\varphi = \sin^2\theta\,. \tag{12.1.8}$$

This equation determines $\cos\varphi$ when $0 < \theta < \pi/(n+1)$; and (12.1.7) then gives $\varphi = \pm\theta$. But, clearly, $e^{\pm i\theta}$ is not a root of $P(z)$. We find, therefore, by (12.1.6), that, *for* $r > n/(n+1)$,

$$|a_0| + |a_1|r + \cdots + |a_{n-1}|r^{n-1} \leq r^{n-1}\frac{\sin n\theta}{\sin\theta}M_1(f)\,, \tag{12.1.9$_1$}$$

where $\theta\big(0 < \theta < \pi/(n+1)\big)$ *is the root of* $\sin n\theta = r\sin(n+1)\theta$.

Since

$$\lambda = \frac{1}{r}\frac{\sin n\theta}{\sin\theta} = \frac{\cos n\theta}{1-r\cos\theta}\,, \quad \lambda^2(1-r\cos\theta)^2 = 1-\sin^2 n\theta = 1-r^2\lambda^2\sin^2\theta,$$

$$\lambda^2(1-2r\cos\theta+r^2) = 1$$

we can restate this result as

$$|a_0| + |a_1|r + \cdots + |a_{n-1}|r^{n-1} \leq \frac{r^n}{(1-2r\cos\theta+r^2)^{1/2}}M_1(f) \quad \big(r > n/(n+1)\big)\,. \tag{12.1.10$_1$}$$

In particular, when $r = 1$, then $\theta = \pi/(2n+1)$, and we obtain

$$|a_0| + |a_1| + \cdots + |a_{n-1}| \leq \frac{1}{2\sin\pi/(4n+2)}M_1(f)\,. \tag{12.1.11$_1$}$$

The extremal functions for (12.1.10) are, by (1.3.6),

$$F(z) = Bz^{2(n-1)}P^2(1/z)\,. \tag{12.1.12}$$

Next, if $r = n/(n+1)$, $\theta = 0$, we can take

$$Q(z) = \lim_{\theta\to 0}\frac{P(z)}{\sin\theta} = 1+2z+3z^2+\cdots+nz^{n-1} \tag{12.1.13}$$

which has all its roots in $|z| < 1$. We thus have

$$|a_0| + |a_1|\frac{n}{n+1} + \cdots + |a_{n-1}|\left(\frac{n}{n+1}\right)^{n-1} \leq n\left(\frac{n}{n+1}\right)^{n-1}M_1(f) \tag{12.1.14$_1$}$$

with

$$K(z) = \frac{n\left(\frac{n}{n+1}\right)^{n-1}}{z^{2n-1}} \frac{Q(z)}{Q(1/z)}, \quad F(z) = Bz^{2(n-1)}Q^2(1/z) \tag{12.1.15}$$

as extremal kernel and extremal function.

Finally, when $0 < r < n/(n+1)$, we consider

$$R(z) = \sinh\theta + z\sinh 2\theta + \cdots + z^{n-1}\sinh n\theta\,. \tag{12.1.16}$$

Defining θ by $\sinh n\theta = r\sinh(n+1)\theta$, we obtain, similarly to (12.1.5),

$$\frac{R(z)}{z^{n-1}R(1/z)} = \frac{\sinh\theta}{\sinh n\theta}\left\{1+\frac{z}{r}+\cdots+\left(\frac{z}{r}\right)^{n-1}+O(|z|^n)\right\}. \tag{12.1.17}$$

Again, by Kakeya's Lemma, $R(z)$ has all its roots in $|z| < 1$. Hence

$$\begin{aligned} |a_0|+|a_1|r+\cdots+|a_{n-1}|r^{n-1} &\le r^{n-1}\frac{\sinh n\theta}{\sinh\theta}M_1(f) \\ &= \frac{r^n}{(1-2r\cosh\theta+r^2)^{1/2}}M_1(f) \end{aligned} \qquad \big(0 < r < n/(n+1)\big) \tag{12.1.18$_1$}$$

with

$$K(z) = \frac{\sinh n\theta}{\sinh\theta}\frac{r^{n-1}}{z^{2n-1}}\frac{R(z)}{R(1/z)}, \quad F(z) = Bz^{2(n-1)}R^2(1/z) \tag{12.1.19}$$

as extremal kernel and extremal function.

We note that, if $r < 1$, then $r < n/(n+1)$ for large n. The positive root θ of $\sinh n\theta = r\sinh(n+1)\theta$ tends to $\log 1/r$ when $n \to \infty$, and it easy to see that the constant $r^{n-1}\sinh n\theta/\sinh\theta$, in (12.1.18), tends to $(1-r^2)^{-1}$, in agreement with (8.3.7).

13. Radial Mean Values in H_1 (a Logarithmic Kernel).

13.1. If f is of class H_1, and $0 < r < 1$, then any inequality for $\int_0^r f(x)dx$ will also hold for $\int_0^r |f(x)|dx$, as we saw in § 4.5. Also $|f(0)| \le M_1(f)$. Hence we shall have an inequality of the type

$$\int_0^r |f(x)|dx \le \lambda(r)M_1(f)\,. \tag{13.1.1}$$

If $\lambda(r)$ is the best possible constant', then $\lambda(r)/r \to 1$ as $r \to 0$. Also $\lambda(r)/r$ increases with r, as is readily seen on considering $f(\varrho z)$, where $0 < \varrho < 1$. It is well known[1] that $\lambda(r)/r$ increases to π as r increases to 1. We wish to determine this $\lambda(r)$.

By (6.4.3), the kernel of the problem is $k(z) = \log\bigl(z/(z-r)\bigr)$, where the log has its principal value and is regular and one-valued outside the segment $0 \leq z \leq r$. Any equivalent kernel is of the form

$$\varkappa(z) = \log\frac{z}{z-r} + \varkappa^*(z) \tag{13.1.2}$$

where $\varkappa^*(z)$ is regular in $|z| < 1$.

We can repeat our argument of § 2 with obvious modifications. If an extremal kernel $K(z)$ exists, and if $F(z)$ is an associated extremal function, then it follows, as in (2.2.2), that the function $zF(z)K(z)$ has constant argument for almost all z on $|z| = 1$. By the principle of inversion we then conclude that

$$zF(z)K(z) = C\log\frac{z}{(z-r)(1-rz)}. \tag{13.1.3}$$

Also, as in (2.3.2) ($q = \infty$), $|K(z)|$ must be constant for almost all z on $|z| = 1$. As for the unique existence of $K(z)$, it is again sufficient to show that a kernel $K(z)$, of the form (13.1.2), exists which is continuous on $|z| = 1$ and for which $|K(z)| = \lambda$ on $|z| = 1$. This $\lambda = \lambda(r)$ is then the desired constant.

Consider the function.

$$\tau = \frac{re^s}{e^s-1}, \qquad e^s = \frac{\tau}{\tau - r} \tag{13.1.4}$$

which, apart from a pole at $s = 0$, is regular and schlicht for $|\Im(s)| < \pi$, and which omits the values 0 and r. To the circle $|s| = \lambda$, where $\lambda < \pi$, corresponds a certain 'dumbell' like curve Γ_λ enclosing the points 0 and r. For small λ this curve is approximately a circle of centre 0 and radius r/λ. Γ_λ shrinks as λ increases, and, when $\lambda = \pi$, it touches the segment $0 \leq \tau \leq r$ at $\tau = \frac{1}{2}r$ from both sides.

Next, let $z = z(\tau)$ be the function which maps the interior of Γ_λ on a circle $|z| < \varrho$ in such a way that $z(0) = 0$ and $z(r) = r$. This function, and the corresponding $\varrho = \varrho(\lambda)$, is uniquely determined. As λ decreases from π to 0, ϱ will increase from r to infinity; and there will be exactly one value $\lambda = \lambda(r)$ for which $\varrho = 1$. For this value of λ, the inverse function $\tau = \tau(z)$ will be regular in $|z| < 1$ and continuous on $|z| = 1$. Solving (13.1.4), we find that the function

[1] Fejér and Riesz.

$$s = K(z) = \log \frac{\tau(z)}{\tau(z)-r} \tag{13.1.5}$$

is of the form (13.1.2), that $K(z)$ is continuous and $|K(z)| = \lambda(r)$ on $|z| = 1$. Hence $K(z)$ is the extremal kernel of our problem, and $\lambda(r)$ is the desired best possible constant in (13.1.1). The extremal functions $F(z)$ are determined by (13.1.3).

There is an alternative way to define $K(z)$. It follows from our discussion that the function $s = K(z)$ maps the circle $|z| < 1$, cut along the segment $0 \leq z \leq r$, on the interior of the strip $|\Im(s)| < \pi$ less the circle $|s| \leq \lambda$. It is clear that this mapping property defines $K(z)$ and $\lambda(r)$ uniquely, since the 'moduli' of the two twice-connected domains must be the same.

We can use this property to obtain estimates for $\lambda(r)$. For instance, the transformation $\zeta = (z+\alpha)/(1+\alpha z)$ will map the unit circle on itself and the segment $0 \leq z \leq r$ onto the segment $-\alpha \leq \zeta \leq \alpha$, provided that

$$\frac{r+\alpha}{1+\alpha r} = -\alpha \, , \; \alpha = \frac{-r}{1+\sqrt{1-r^2}}. \tag{13.1.6}$$

The transformation $s = i\lambda/\zeta$ will map the circle $|z| < 1$, slit along the segment $0 \leq z \leq r$, onto the exterior of the circle $|s| = \lambda$ slit along the two parts $|\Im(s)| \geq \lambda/\alpha$ of the imaginary axis. Since this domain must have the same modulus as the strip $|\Im(s)| < \pi$ less the circle $|s| \leq \lambda$, we conclude that these two slits must enter the strip. Hence $\lambda/\alpha < \pi$, or

$$\lambda(r) < \frac{r\pi}{1+\sqrt{1-r^2}}. \tag{13.1.7}$$

This estimate for $\lambda(r)/r$ ranges from $\frac{1}{2}\pi$ to π as r increases from 0 to 1, while the true range is from 1 to π. However, for r near to 1, it is a useful estimate. For small r, the crude estimate

$$\lambda(r) < \tfrac{1}{2} \log \frac{1+r}{1-r}, \tag{13.1.8}$$

obtained on integrating (8.3.7), is better.

According to the footnote to (4.5.2), the inequality (13.1.1) can be applied to $|f|^p$, when f belongs to H_p. We then obtain

$$\int_0^r |f(x)|^p dx \leq \lambda(r) M_p^p(f) \, . \tag{13.1.9}$$

Equality, however, for the $\lambda(r)$ defined above, is only possible when $p = 1$. If $p = 2$ and $f(z) = \sum_0^\infty a_n z^n$, we find

$$(13.1.10) \qquad \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{|a_m||a_n|}{m+n+1} r^{m+n} \leq \frac{\lambda(r)}{r} \sum_0^\infty |a_n|^2 < \frac{\pi}{1+(1-r)^{1/2}} \sum_0^\infty |a_n|^2 .$$

This is a generalisation of Hilbert's well known inequality[1], which itself is obtained, with best possible constant π, on letting $r \to 1$.

14. Linear Restrictions.

14.1. We have been concerned so far with two conjugate extremum problems, I and II of § 1.3. It is possible to combine the two and propose the question, what are the possible values of $I(f)$ when $M_p(f)$ and the values of f at a finite number of points in $|z| < 1$ are prescribed.

This problem has attracted considerable interest in the case $p = \infty$. When $I(f) = f(\beta)$, then the problem is effectively solved by Schur's algorithm: unless the class of functions is empty, the region of possible values of $f(\beta)$ is a certain circle[2]. For other forms of $I(f)$ the discussion is much more complicated, but a number of special cases have been investigated by Dieudonné[3] and Rogosinski[4].

For other values of p the problem is usually difficult, but we are able to treat a few special cases.

14.2. Let us consider the subclass $\underline{H}_p$ of functions f of H_p, for which $f(0) = 0$. We then have $f(z) = zg(z)$ where g also belongs to H_p. In fact, $M_p(f) = M_p(g)$. If $I(f)$ in volves the kernel $k(z)$, then, clearly, $I(f) = I^*(g)$ where $I^*(g)$ involves the kernel $k^*(z) = zk(z)$. Extremum problems for the class $\underline{H}_p$ are, therefore, amenable to our general theory. We obtain, for instance, from (8.3.7) at once the inequality[5]

$$(14.2.1)_{\underline{p}} \qquad |f(\beta)| \leq \frac{|\beta|}{(1-|\beta|^2)^{1/p}} M_p(f)$$

with extremal functions $F(z) = Bz(1-\bar{\beta}z)^{-2/p}$.

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[1] Hardy, Littlewood and Pólya, Chapter IX.
[2] Pick (a), (b).
[3] Dieudonné.
[4] Rogosinski.
[5] A suffix $\underline{p}$ relates to the class $\underline{H}_p$.

14.3. To find inequalities for $f'(\beta)$ in $\underline{H}_p$, we have to consider the kernel $k^*(z) = z/(z-\beta)^2$. On using the transformation (8.2.1) we are led, by (8.2.4), to the kernel

$$\varkappa^*(w) = (1-|\beta|^2)^{-(1+1/p)} w^{-2}(w+\beta)(1+\bar{\beta}w)^{1-2/q}. \tag{14.3.1}$$

The corresponding extremal kernel is, by (1.3.5), of one of the two forms

$$(i)\ A(1-\bar{\alpha}w)^{2/q}w^{-2}, \quad (ii)\ A(w-\alpha)(1-\bar{\alpha}w)^{2/q-1}w^{-2}, \tag{14.3.2}$$

where $A = \beta(1-|\beta|^2)^{-(1+1/p)}$ or $-\alpha A = \beta(1-|\beta|^2)^{-(1+1/p)}$, respectively. In both cases

$$M_q(K^*) = |A| M_2^{2/q}\{(1-\bar{\alpha}w)^2\} = |A|(1+|\alpha|^2)^{1/q}. \tag{14.3.3}$$

The corresponding extremal functions are, by (1.3.6),

$$(i)\ \Phi(w) = B(w-\alpha)(1-\bar{\alpha}w)^{2/p-1}, \quad (ii)\ \Phi(w) = B(1-\bar{\alpha}w)^{2/p}; \tag{14.3.4}$$

or, by (8.2.4), $F(z) = zG(z) = Cz(1+\bar{\beta}w)^{2/p}\Phi(w)$, that is

$$(i)\ F(z) = Cz\frac{(z-\gamma)(1-\bar{\gamma}z)^{2/p-1}}{(1-\bar{\beta}z)^{1/p}}, \quad (ii)\ F(z) = Cz\frac{(1-\bar{\gamma}z)^{2/p}}{(1-\bar{\beta}z)^{1/p}}, \tag{14.3.5}$$

where $\gamma = (\alpha+\beta)/(1+\alpha\bar{\beta})$.

First, consider the case (i). Equating the coefficients of w in $\varkappa^*(w)$ and $K^*(w)$ we find for α the equation

$$-\frac{2\beta}{q}\bar{\alpha} = 1+(1-2/q)|\beta|^2\,. \tag{14.3.6}$$

Hence $\beta\bar{\alpha}$ must be negative. We also require $|\alpha| \leq 1$. This implies

$$\frac{q}{2}+\left(\frac{q}{2}-1\right)|\beta|^2 \leq |\beta|\,,$$

or

$$1 \leq q \leq \frac{2|\beta|(1+|\beta|)}{1+|\beta|^2}, \qquad \frac{q}{1+\sqrt{1+2q-q^2}} \leq |\beta| < 1\,. \tag{14.3.7}$$

This case is only possible when $q < 2$, $p > 2$ [as is also seen from (14.3.6)]. By (14.3.2), (i), we obtain for these $|\beta|$

$$|f'(\beta)| \leq \frac{|\beta|}{(1-|\beta|^2)^{1+1/p}}(1+|\alpha|^2)^{1/q} M_p(f). \tag{14.3.8$_{\underline{p}}$}$$

The extremal functions are those of (14.3.5), (i).

The case (ii) must cover the remaining range of $|\beta|$; in particular, all $|\beta|$ when $q \geq 2$. We find, similarly, that

$$-\frac{\beta}{\alpha}[1+(2/q-1)|\alpha|^2] = 1+(1-2/q)|\beta|^2\,. \tag{14.3.9}$$

Again $\beta\bar{\alpha}$ is negative, and we must have $|\alpha| < 1$. Solving (14.3.9) we find

$$(14.3.10) \qquad \alpha = \frac{-2\beta}{(1-\varrho|\beta|^2)+\sqrt{\varrho^2|\beta|^4-6\varrho|\beta|^2+1}} \qquad \left(\varrho = \frac{2}{q}-1\right).$$

We then obtain, in this case,

$$(14.3.11)_{\underline{p}} \qquad |f'(\beta)| \le \frac{|\beta|}{(1-|\beta|^2)^{1+1/p}} \frac{(1+|\alpha|^2)^{1/q}}{|\alpha|} M_p(f)$$

with the extremal functions (14.3.5), (*ii*).

When $|\alpha| \to 1$, then (14.3.9) becomes (14.3.6): we are in the extreme case $q = 2|\beta|(1+|\beta|)(1+|\beta|^2)^{-1}$, and the two estimates (14.3.8) and (14.3.11) become the same.

We remark that a single formula holds for all β when $p \le 2$, while two are required when $p > 2$. Without the condition $f(0) = 0$ this situation is reversed.

If $p = 1$, then $\varrho = -1$, and (14.3.11) becomes

$$(14.3.12)_{\underline{1}} \qquad |f'(\beta)| \le \frac{(1+|\beta|^2)+\sqrt{|\beta|^4+6|\beta|^2+1}}{2(1-|\beta|^2)^2} M_1(f) .$$

If $p = 2$, then $\varrho = 0$, $\alpha = -\beta$, and so

$$(14.3.13)_{\underline{2}} \qquad |f'(\beta)| \le \frac{(1+|\beta|^2)^{1/2}}{(1-|\beta|^2)^{3/2}} M_2(f) .$$

If $p = \infty$, then, by (14.3.8), the case (*i*) occurs when $|\beta| \ge \sqrt{2}-1$. We then find, by (14.3.6), that

$$-\bar{\alpha} = \frac{1-|\beta|^2}{2\beta}, \quad 1+|\alpha|^2 = \frac{(1+|\beta|^2)^2}{4|\beta|^2}.$$

Hence, by (14.3.8),

$$(14.3.14)_{\underline{\infty}} \qquad |f'(\beta)| \le \frac{(1+|\beta|^2)^2}{4|\beta|(1-|\beta|^2)} \operatorname{Max}|f| \qquad (\sqrt{2}-1 \le |\beta| < 1) .$$

In the second case, we have, by (14.3.9),

$$\frac{1+|\alpha|^2}{\alpha} = -\frac{1-|\beta|^2}{|\beta|},$$

so that (14.3.11) gives

$$(14.3.15)_{\underline{\infty}} \qquad |f'(\beta)| \le \operatorname{Max}|f| \qquad (|\beta| < \sqrt{2}-1) .$$

By (14.3.5), the extremal functions are

$$(14.3.16) \qquad (i)\ F(z) = Cz\frac{z-\gamma}{1-\bar{\gamma}z}, \qquad (ii)\ F(z) = Cz,$$

respectively. The formulae (14.3.14) and (14.3.15) were first proved by Dieudonné[1].

14.4. In the case of the second derivative $f''(\beta)$ in H_p, the kernel $k^*(z)=2z(z-\beta)^{-3}$ is transformed into

$$(14.4.1) \qquad K^*(w) = 2(1-|\beta|^2)^{-(2+1/p)}w^{-3}(w+\beta)(1+\bar{\beta}w)^{2/p}$$

The equivalent extremal kernel $K^*(w)$ is of one of the three forms

$$(14.4.2) \qquad \begin{gathered} (i)\ A[(1-\bar{\alpha}_1 w)(1-\bar{\alpha}_2 w)]^{2/q}w^{-3}, \quad (ii)\ A(w-\alpha_1)(1-\bar{\alpha}_1 w)^{2/q-1}(1-\bar{\alpha}_2 w)^{2/q}w^{-3}, \\ (iii)\ A(w-\alpha_1)(w-\alpha_2)[(1-\bar{\alpha}_1 w)(1-\bar{\alpha}_2 w)]^{2/q-1}w^{-3}, \end{gathered}$$

where A, or $-\alpha_1 A$, or $\alpha_1\alpha_2 A$ equals $2\beta(1-|\beta|^2)^{-(2+1/p)}$, respectively. For a general p the actual determination of this extremal kernel is bound to be very involved. We shall, therefore, have to confine ourselves to a few remarks.

Consider the case when $K^*(w)$ is of form (i). We may assume that $0 < \beta < 1$ and must then have

$$\beta[(1-\bar{\alpha}_1 w)(1-\bar{\alpha}_2 w)]^{2/q} = (w+\beta)(1+\beta w)^{2/p}+O(|w|^3).$$

Clearly, $p = 1$, $q = \infty$ is impossible. If $p > 1$, we must have

$$(14.4.3) \qquad \begin{aligned} &(1-\bar{\alpha}_1 w)(1-\bar{\alpha}_2 w) \\ &= 1+\left[\frac{q}{2\beta}+(q-1)\beta\right]w+\left[\frac{q(q-2)}{8\beta^2}+\frac{q(q-1)}{2}+\frac{(q-1)(q-2)}{2}\beta^2\right]w^2; \end{aligned}$$

that is, this polynomial $P(w) = 1+Uw+Vw^2$, say, must have no roots in $|w| < 1$. It follows, first, that $U = |\bar{\alpha}_1+\bar{\alpha}_2| < 2$, and it is readily seen that this implies $q < 2$. Hence the case (i) is certainly impossible when $p \leq 2$. If $q < 2$, then an elementary discussion shows that $P(w)$ has two real roots, and our condition on these becomes $U+\sqrt{U^2-4V} \leq 2$, or $U-V \leq 1$. For such q and β we shall then have

$$(14.4.4)_p \qquad \begin{aligned} |f''(\beta)| \leq \frac{2|\beta|}{(1-|\beta|^2)^{2+1/p}}\Bigg[1&+\left(\frac{q}{2|\beta|}+(q-1)|\beta|\right)^2 \\ &+\left(\frac{q(q-2)}{8|\beta|^2}+\frac{q(q-1)}{2}+\frac{(q-1)(q-2)}{2}|\beta|^2\right)^2\Bigg]^{1/q} M_p(f). \end{aligned}$$

[1] Dieudonné.

When $p = 2$, then the natural kernel (14.4.1) is the extremal kernel: it has the root $w = -\beta$ in $|w| < 1$ and is of the form (ii). We obtain for all $|\beta| < 1$

$$(14.4.5)_{\underline{2}} \qquad |f''(\beta)| \leq \frac{2}{(1-|\beta|^2)^{5/2}} [1+4|\beta|^2+|\beta|^4]^{1/2} M_2(f) .$$

In the case $p = 1$ the extremal kernel is of the form (iii) when $\beta \neq 0$. For, we know that (i) is impossible, and (ii) would imply

$$(1-w/\alpha_1)(1-\bar{\alpha}_1 w)^{-1} = (1+w/\beta)(1+\beta w)^2+O(|w|^3) \qquad (0 < \beta < 1) .$$

Equating the coefficients of w^2, we would have $\bar{\alpha}_1^2-\bar{\alpha}_1/\alpha_1 = \beta^2+2$ which is impossible when $\beta > 0$. By (4.4.2) the resulting inequality is

$$(14.4.6)_{\underline{1}} \qquad |f''(\beta)| \leq \frac{2C}{(1-|\beta|^2)^3} M_1(f) ,$$

where C is the greatest root of the cubic equation

$$(14.4.7) \qquad \begin{vmatrix} -\varrho^2-|\beta|^2 & -(|\beta|+2|\beta|^3) & -(2|\beta|^2+|\beta|^4) \\ -(|\beta|+2|\beta|^3) & \varrho^2-(1+5|\beta|^2+4|\beta|^4) & -(3|\beta|+7|\beta|^3+2|\beta|^5) \\ -(2|\beta|^2+|\beta|^4) & -(3|\beta|+7|\beta|^3+2|\beta|^5) & \varrho^2-(1+9|\beta|^2+8|\beta|^4+ \\ & & +|\beta|^6) \end{vmatrix} = 0 .$$

14.5. Inequalities for $f''(\beta)$ in $\underline{H}_\infty$ can be obtained in elementary form. We may assume again that $0 \leq \beta < 1$. The form (14.4.2), (i), of the extremal kernel will be required whenever

$$P(w) = 1+\frac{w}{2\beta}-\frac{w^2}{8\beta^2} \qquad (\beta > 0)$$

has no roots in $|w| < 1$. Since these roots are $2(1\pm\sqrt{3})\beta$, we obtain, by (14.4.4),

$$(14.5.1)_{\underline{\infty}} \qquad \begin{aligned} |f''(\beta)| &\leq \frac{2|\beta|}{(1-|\beta|^2)^2}\left[1+\frac{1}{4|\beta|^2}+\frac{1}{64|\beta|^4}\right] \mathrm{Max}|f| \\ &= \frac{(1+8|\beta|^2)^2}{32|\beta|^3(1-|\beta|^2)^2} \mathrm{Max}\,|f| \end{aligned}$$

for $|\beta| \geq \dfrac{1}{2(\sqrt{3}-1)} = \dfrac{1+\sqrt{3}}{4}$.[1]

[1] Rogosinski. Part of the inequality (26) there is wrong, owing to an arithmetical slip on p. 104, and should be replaced by the present formula (14.5.3).

For smaller $|\beta|$ the form (14.4.2), (*ii*), of the extremal kernel becomes available; that is

$$K^*(w) = A(w-\alpha_1)(1-\bar{\alpha}_1 w)(1-\bar{\alpha}_2 w)^2 w^{-3}$$

where $-A\alpha_1 = 2\beta(1-|\beta|^2)^{-2}$. We require

$$\left(1-\frac{w}{\alpha_1}\right)(1-\bar{\alpha}_1 w)(1-\alpha_2 w)^2 = 1+\frac{w}{\beta}+O(|w|^3)\,.$$

If $0<\beta<1$, the α will be real, because of the uniqueness of the extremal kernel. Comparing coefficients we find

$$\alpha_1+1/\alpha_1+2\alpha_2 = -1/\beta\,, \qquad 1+2\alpha_2(\alpha_1+1/\alpha_1)+\alpha_2^2 = 0\,.$$

Writing $\gamma = \alpha_1+1/\alpha_1$ we obtain $\gamma = -[2\alpha_2+1/\beta]$ and, finally,

$$\alpha_2 = \frac{\beta}{1+\sqrt{1+3\beta^2}}\,, \quad \gamma = -\frac{1+2\sqrt{1+3\beta^2}}{3\beta}. \tag{14.5.2}$$

Clearly, $0<\alpha_2<1$. Also γ increases from $-\infty$ to -2 when β increases from 0 to $\frac{1}{4}(1+\sqrt{3})$. Hence α_1 is negative and $|\alpha_1|<1$ for $0<\beta<\frac{1}{4}(1+\sqrt{3})$.

Next,

$$\begin{aligned} M_1(K^*) &= \frac{|A|}{2\pi}\int_{|w|=1} |(w-\alpha_1)(1-\alpha_1 w)(1-\alpha_2 w)^2|\,|dw| \\ &= \frac{|A|}{2\pi}\int_{|w|=1} |(1-\alpha_1 w)(1-\alpha_2 w)|^2|dw| = \frac{2\beta}{|\alpha_1|(1-\beta^2)^2}[1+(\alpha_1+\alpha_2)^2+(\alpha_1\alpha_2)^2] \\ &= -\frac{2\beta}{(1-\beta^2)^2}[\gamma+2\alpha_2+\alpha_2^2\gamma] = \frac{2\beta}{(1-\beta^2)^2}[1/\beta-\alpha_2^2\gamma] \\ &= \frac{2}{(1-\beta^2)^2}\left[1+\frac{\beta^2(1+2\sqrt{1+3\beta^2})}{3(1+\sqrt{1+3\beta^2})^2}\right], \end{aligned}$$

or, after an elementary calculation,

$$M_1(K^*) = \frac{4}{3(1-\beta^2)^2}\left[1+\frac{1+3\beta^2+3\beta^4}{1+(1+3\beta^2)^{3/2}}\right].$$

We obtain, therefore,

$$|f''(\beta)| \le \frac{4}{3(1-|\beta|^2)^2}\left[1+\frac{1+3|\beta|^2+3|\beta|^4}{1+(1+3|\beta|^2)^{3/2}}\right]\operatorname{Max}|f| \tag{14.5.3$_\infty$}$$

for $0<|\beta|<\frac{1}{4}(1+\sqrt{3})$.

If $|\beta| = \frac{1}{4}(1+\sqrt{3})$, both (14.5.1) and (14.5.3) become identical, the common 'constant' being $128(2+\sqrt{3})/(29+19\sqrt{3})$. The extremal functions can be obtained in each case in the usual way.

14.6. In the preceding paragraphs it was assumed that $f(0) = 0$. We discuss now the case where a value $f(0)$ different from zero and $T = M_p(f)$ are given. If $f(z) = \sum_0^\infty a_n z^n$ belongs to H_2, then, for $|\beta| < 1$,

$$|f(\beta)-f(0)|^2 = \left|\sum_1^\infty \alpha_n \beta^n\right|^2 \leq \sum_1^\infty |\alpha_n|^2 \sum_1^\infty |\beta|^{2n} = (T^2-|f(0)|^2)|\beta|^2(1-|\beta|^2)^{-1},$$

that is

$$(14.6.1)_2 \qquad |f(\beta)-f(0)| \leq \frac{|\beta|}{(1-|\beta|^2)^{1/2}}(T^2-|f(0)|^2)^{1/2} \qquad \big(T = M_2(f)\big).$$

Equality will be attained if, and only if, $a_n = \lambda \bar{\beta}^n$ for $n \geq 1$; that is when $f(z)$ is of the form

$$(14.6.2) \qquad F(z) = f(0) + \lambda \frac{\bar{\beta} z}{1-\bar{\beta} z},$$

where $|\lambda|$ is determined by $T = M_2(F)$. Any point on the circumference of the circle (14.6.1) is attained for suitable λ. For smaller $|\lambda|$ interior points of the circle are obtained, but, in order to retain T, we have to add to $F(z)$ a suitable term $cz(z-\beta)$, say. The closed circle (14.6.1) is thus the exact region for possible values of $f(\beta)$.

14.7. The corresponding result for H_∞ is well known. A simple use of Schwarz's Lemma shows that the exact region of variability for $f(\beta)$ is the closed circle C that corresponds to the circle $|t| \leq |\beta|$ by the transformation

$$w = T\frac{tT+f(0)}{T+\overline{f(0)}t} \qquad (T = \operatorname{Max}|f|).$$

14.8. For the class H_1 the problem is a little more difficult. We may assume that $0 < \beta < 1$ and that $0 < f(0) \leq T = M_1(f)$.

The two inequalities (10.2.2) and (10.2.4) are at our disposal. We use first the latter, that is

$$(14.8.1)_1 \qquad \left|(1-\beta^2)\frac{\beta-\alpha}{1-\bar{\alpha}\beta}f(\beta)+\alpha f(0)\right| \leq \beta T \qquad (|\alpha| < 1).$$

Equality is attained for

(14.8.2) $$F_\alpha(z) = f(0)(1-\bar{\alpha}z)^2(1-\beta z)^{-2},$$

where

(14.8.3) $$T = M_1(F_\alpha) = f(0)\left[1+\frac{|\beta-\alpha|^2}{1-\beta^2}\right].$$

Hence α is restricted to the circle

(14.8.4) $$|\alpha-\beta| = \left[(1-\beta^2)\left(\frac{T}{f(0)}-1\right)\right]^{1/2}.$$

Here the points

(14.8.5) $$\alpha_{\pm} = \beta \pm \left[(1-\beta^2)\left(\frac{T}{f(0)}-1\right)\right]^{1/2}$$

are endpoints of a diameter. We must also have $|\alpha| < 1$ if (14.8.1) is to be available.

First suppose that $f(0) > \frac{1}{2}(1+\beta)T$. Then $\alpha_+ < 1$ and the circle (14.8.4) belongs to $|\alpha| < 1$. When α describes (14.8.4) then the value of $F_\alpha^{1/2}(\beta)$ describes the circle

(K) $$s = f^{1/2}(0)\frac{1-\bar{\alpha}\beta}{1-\beta^2}$$

in the half plane $\Re s > 0$. For suitable α_1, α_2 on (14.8.4), and a suitable t with $0 \leq t \leq 1$, the function

$$G(z) = tF_{\alpha_1}^{1/2}(z)+(1-t)F_{\alpha_2}^{1/2}(z)$$

takes any prescribed value $G(\beta)$ inside or on K. Also $G(0) = f^{1/2}(0)$.

Hence $w = H(z) = G^2(z)$ takes any prescribed value inside or on the curve $\Gamma = K^2$, the transform of K by $w = s^2$. Also $H(0) = f(0)$ and $M_1(H) = M_2^2(G) \leq T$. Hence, adding to $H(z)$ a suitable term $cz(z-\beta)$, *we see that, for given T and given $f(0)(> \frac{1}{2}(1+\beta)T)$, any prescribed value of $f(\beta)$ inside or on Γ can be attained.*

Next, let $f(0) \leq \frac{1}{2}(1-\beta)T$, in which case $\alpha_- \leq -1$ and the circle (14.8.4) belongs to $|\alpha| \geq 1$. Here formula (10.2.2), that is

$(14.8.6)_1$ $$|(1-\beta^2)f(\beta)-f(0)| \leq \beta T$$

becomes available. Equality is attained for

(14.8.7) $$F_\alpha^*(z) = -\frac{f(0)}{\alpha}(z-\alpha)(1-\bar{\alpha}z)(1-\beta z)^{-2},$$

where α is restricted to the curve

(14.8.8) $$T = M_1(F_\alpha^*) = \frac{f(0)}{|\alpha|}\left[1+\frac{|\beta-\alpha|^2}{1-\beta^2}\right].$$

Now the circle (14.8.4) encloses $|\alpha| \leq 1$. For fixed arg α and $|\alpha| = 1$ the right hand side of (14.8.8) equals that of (14.8.3), and hence is not greater than T. It follows, that, for every given arg α, there exists an α with $0 < |\alpha| \leq 1$ satisfying (14.8.8): the curve of these α encloses $\alpha = 0$. On the other hand, if α satisfies (14.8.8), then an elementary calculation shows that

$$(14.8.9) \qquad (1-\beta^2)F^*_\alpha(\beta)-f(0) = -\beta\frac{|\alpha|}{\alpha}T\,,$$

so that the values $F^*_\alpha(\beta)$ describe the whole circumference of the circle (14.8.6). An argument, similar to that used above, shows *that every point of the closed circle* (14.8.6) *is a possible value for* $f(\beta)$, *provided that* $f(0) \leq \frac{1}{2}(1-\beta)T$.

Lastly, if $\frac{1}{2}(1-\beta)T < f(0) \leq \frac{1}{2}(1+\beta)T$, then the circle (14.8.4) meets the circle $|\alpha| = 1$ at two points (or, when $f(0) = \frac{1}{2}(1+\beta)T$, touches it at $\alpha_+ = 1$). Apart from these points the values of arg α are divided into two categories. Either the equation (14.8.4), or the equation (14.8.8) has a root α with $|\alpha| < 1$. All these possible values of α form a simple closed curve consisting of that part of the circle (14.8.4) for which $|\alpha| < 1$, and, for the rest, of an arc of the curve (14.8.8). The corresponding values $F(\beta)$ of the extremal functions lie on a simple closed curve Γ^* consisting of an arc of Γ and an arc of the circle (14.8.6). *This curve* Γ^* *and its interior constitute the exact region of possible values* $f(\beta)$, *when* $\frac{1}{2}(1-\beta)T < f(0) \leq \frac{1}{2}(1+\beta)T$.

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