SPACES WITH NON-POSITIVE CURVATURE.

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Introduction.

The theory of spaces with negative curvature began with Hadamard's famous paper [9].[1] It initiated a number of important investigations, among which we mention Cartan's generalization to higher dimensions in [7, Note III], the work on symbolic dynamics[2] for which, besides Poincaré, Hadamard's paper is the ultimate source, and the investigations of Cohn-Vossen in [8], which apply many of Hadamard's methods to more general surfaces.

For Riemann spaces the analytic requirement that the space has non-positive curvature is equivalent to the geometric condition that every point of the space has a neighborhood U such that the side bc of a geodesic triangle abc in U is at least twice as long as the (shortest) geodesic arc connecting the mid points b', c' of the other two sides:

$$(*) \qquad bc \geq 2 \cdot b'c'.$$

This condition has a meaning in any metric space in which the geodesic connection is locally unique. *It is the purpose of the present paper to show, that* (*) *allows to establish the whole theory of spaces with non-positive curvature for spaces of such a general type.* This theory proves therefore *independent of any differentiability hypothesis and,* what is perhaps more surprising, *of the Riemannian character of the metric.*

It was quite impossible to carry all the known results over without swelling the present paper beyond all reasonable limits. But an attempt was made to

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[1] The numbers refer to the References at the end of the paper.

[2] A bibliography is found in the paper [12] by M. MORSE and G. HEDLUND on this subject.

bring those facts whose proofs differ from the current ones. The following is a more detailed description of the contents.

For the convenience of the reader Section 1 compiles the definitions and results concerning *spaces with locally unique geodesics*, which were proved elsewhere but are needed here. Section 2 discusses *covering spaces and fundamental domains.* Part of it will not be used but is necessary for actually carrying over several known results not discussed here.

Then non-positive curvature is defined by (*). If the equality sign holds in (*) we say that the space has curvature 0; and if the inequality sign holds for non-degenerate triangles, the space is said to have negative curvature. Section 3 discusses the *local implications of* (*). The long proof of Theorem (3.14) is the only place where differentiability hypotheses would have permitted considerably simplifications.

If the space is straight, that is if all geodesics are congruent to euclidean straight lines, then (*) implies that for any two geodesics $x(t)$ and $y(t)$, where t is the arc length, the function $x(t)\,y(t)$ *is a convex function of* t (Section 4). This is really the central point of the theory. It permits to do without the Gauss-Bonnet Theorem, for which no analogue in general Finsler spaces has as yet been found, but which is frequently applied by Hadamard and others. The convexity of $x(t)\,y(t)$ also allows to establish a complete *theory of parallels.*

Moreover, it is basic for the other fundamental fact, that *the universal covering space of any space with non-positive curvature is straight* (Section 5). This is proved here under the additional assumption that the space has the topological property of domain invariance, which is probably always satisfied but defies present topology.

The study of general spaces with non-positive curvature is reduced to simply connected spaces by means of the covering motions of the universal covering space. These motions have no fixed points. Section 6 investigates motions of this type.

Application of these results yields among others the two facts (Section 7) on which Hadamard's work is primarily based: In a space with non-positive curvature *there is only one geodesic arc within a given homotopy class connecting two given points. In a space with negative curvature every free homotopy class contains at most one closed geodesic.*

Section 8 contains results on asymptotic geodesics which go also back to Hadamard, and points out some unsolved problems.

Then special types of spaces are discussed, first *spaces with curvature* 0 (Section 9). They are locally Minkowskian. By means of an observation by Loewner, their study can be reduced to Riemann spaces of curvature 0. They have *finite connectivity* (Cartan) and all tori with non-positive curvature have curvature 0.

There is no compact space with negative curvature and an abelian fundamental group. The fundamental group of a space with non-positive curvature has no (non-trivial) finite sub group: The simplest not simply connected spaces have therefore an infinite cyclic fundamental group. Section 10 studies these spaces, in particular cylinders.

The theory of two-dimensional manifolds, especially of *compact manifolds with negative curvature* is the subject of Section 11. The methods used by Nielsen [13, 14] for surfaces of constant negative curvature served as a guide.

Finally we return to the starting point and prove that (*) is for Riemann spaces actually equivalent to non-positive curvature (Section 12). We also show that (*) is equivalent to the »cosine inequality» (see (12.4)).

Under a minimum of differentiability hypotheses it can be seen (Section 13) that *the volume* (Hausdorff measure) *of a sphere in a Finsler space with non-positive curvature equals at least the volume of the euclidean sphere with the same radius.* At first sight it seems rather surprising that this fact, which is well known for Riemann spaces, extends to Finsler spaces. The corresponding statement for area of spheres is not correct in this simple form, but more complicated inequalities will be proved which contain the known Riemannian inequalities as special cases.

Notations.

Points are denoted by small latin letters, pointsets by latin capitals.

Small German letters indicate parametrized curves. But a curve and the pointset which carries it are not sharply distinguished when no misunderstanding is possible.

German capitals stand for groups. Motions or transformations are denoted by Greek capitals.

All spaces considered are (at least) G-spaces, whose definition is found in Section 1. The space in question is always denoted by R, its universal covering space by $\bar{R}$, and $\bar{R}$ is related to R by a definite locally isometric mapping Ω. A point $\bar{p}$ or curve $\bar{x}(t)$ of $\bar{R}$ lies over the point p or curve $x(t)$ of R if $\bar{p}\,\Omega = p$ or $\bar{x}(t)\,\Omega = x(t)$.

The fundamental group $\mathfrak{F}$ of R is thought of not as an abstract group, but as the group of motions Φ in $\bar{R}$ which lie over the identity I of R, that is $\Phi\Omega = I$. The letters Φ and Ψ (with or without subscripts) mean elements of $\mathfrak{F}$, and E is the identity of $\mathfrak{F}$.

A space in which each geodesic is congruent to a euclidean straight line is called straight. The frequently occurring hypothesis that the geodesics of the universal covering space of R are straight will therefore simply be formulated as: $\bar{R}$ is straight.

CHAPTER I.

Metric Spaces with Geodesics.

1. The Basic Properties of G-Spaces.

The conditions I to IV listed below guarantee the existence of geodesics with the geometric properties of the extremals in finite dimensional symmetric Finsler spaces, leaving aside differentiability properties. Their formulation is simplified by using the notation (xyz) to indicate that x, y, z are different points in a metric space and that their distances xy, yz, xz satisfy the relation $xy + yz = xz$. The spherical neighborhood of p which consists of the points x with $px < \varrho$ is denoted by $S(p, \varrho)$. The conditions for a G-space R are these:

I *R is metric with distance xy.*[3]

II *R is finitely compact*, or a bounded[4] sequence $x_1, x_2, \ldots$ has an accumulation point.

III *R is convex*, that is for any two different points x, z a point y with (xyz) exists.

IV *Prolongation is locally possible*, or every point p has a neighborhood $S(p, \varrho(p))$, $\varrho(p) > 0$, such that for any two different points x, y in $S(p, \varrho(p))$ a point z with (xyz) exists.

V *Prolongation is unique*, or, if (xyz_1), (xyz_2), and $yz_1 = yz_2$ then $z_1 = z_2$.

In this form V does not appear as a local requirement, but it is equivalent to a local condition (see D and Theorem (4.1) in [4, p. 215]). We recall some definitions and properties of G-spaces which were proved in [4].

[3] In contrast to [4], it is here always assumed that $xy = yx$.

[4] »Bounded» means: there is a δ such that $x_i x_k < \delta$.

A *segment* from x to y is an arc from x to y which is congruant to a euclidean segment. It has *representations* $x(t)$, $\alpha \leq t \leq \beta$, such that $x(\alpha) = x$, $x(\beta) = y$, and $x(t_1)x(t_2) = |t_1 - t_2|$ for $\alpha \leq t_i \leq \beta$. An oriented segment from x to y will be denoted by $\mathfrak{s}(x, y)$ and the point set carrying it by $T(x, y)$. Segments $\mathfrak{s}(x, y)$ exist and are shortest connections from x to y but need not be unique. However (see [4, (4.2) p. 216]).

(1.1) *If a point z with (xyz) exists, then $\mathfrak{s}(x; y)$ and $T(x, y)$ are unique.*

Segments can locally be uniformly prolonged (see [4, pp. 217, 218]): for any $\lambda \geq 2$ and any point x the numbers β, such that every segment T with endpoints in $S(x, \beta)$ is subsegment of a segment with the same center as T and with length $\lambda\beta$, have a positive least upper bound $\eta_\lambda(x)$; and $\eta_\lambda(x)$ satisfies the relations

$$|\eta_\lambda(x) - \eta_\lambda(y)| \leq xy \text{ or } \eta_\lambda(x) = \infty \text{ for all } \lambda \geq 2 \text{ and } x.$$

In the present paper another number δ_p (not mentioned in [4]) will play a rôle. δ_p is the least upper bound of those β for which the segment $\mathfrak{s}(x, y)$ is unique for any x, y in $S(p, \beta)$. By (1.1) δ_p is at least as large as $\eta_2(p)$ or the $\varrho(p)$ occurring in IV. Moreover, if $pq < \delta_p$ then

$$S(p, \delta_p) > S(q, \delta_p - pq)$$

therefore always $\delta_q \geq \delta_p - pq$ and similarly $\delta_p \geq \delta_q - qp$ hence

$$(1.2) \qquad |\delta_p - \delta_q| < pq \quad \text{or} \quad \delta_p \equiv \infty.$$

A *geodesic* $\mathfrak{g}$ is a curve which is locally a segment, or, $\mathfrak{g}$ has a parametrization $x(t)$, $-\infty < t < \infty$ such that for every t_0 a positive $\varepsilon(t_0)$ exists such that $x(t_1)x(t_2) = |t_1 - t_2|$ for $|t_0 - t_i| \leq \varepsilon(t_0)$. $x(t)$ is called a *representation* of $\mathfrak{g}$. Obviously t is the arclength. At times $\mathfrak{g}$ will have a definite orientation $\mathfrak{g}^+$. Then the word representation is to imply that t increases when $\mathfrak{g}^+$ is traversed in the positive sense.

For any segment $\mathfrak{s}(x, y)$ there is exactly one geodesic $\mathfrak{g}$ that passes through all points of the corresponding set $T(x, y)$, see [4, (8.3), p. 230]. If $x(t)$, $\alpha \leq t \leq \beta$ represents $\mathfrak{s}(x, y)$, then a representation $y(t)$ of $\mathfrak{g}$ exists such that $y(t) = x(t)$ for $\alpha \leq t \leq \beta$, [4, (5.6) p. 222].

There are two especially simple types of geodesics, the *straight lines* and the *great circles* ([4, p. 232]). $x(t)$ represents a straight line if $x(t_1)x(t_2) = |t_1 - t_2|$

for any t_i, and a great circle of length $\beta > 0$ if

$$x(t_1)\,x(t_2) = \min_{\nu=0,\,\pm 1,\,\pm 2,\,\ldots} |t_1 - t_2 + \nu\beta|. \text{ for any } t_i.$$

With one representation of a straight line or great circle, every representation has the characteristic property.

The space R is straight, or all geodesics are straight lines, if $\eta_2(p) \equiv \infty$. In that case also $\delta_p \equiv \infty$.

The following fact is not mentioned in [4] but will be needed here.

(1.3) Let $x(t)$ and $y(t)$ represent straight lines and $x(t)\,G < \alpha$ for $t \geq 0$, where G is the set that carries $y(t)$. If $y(\pi(t))$ is a foot of $x(t)$ on G, then either $\pi(t) \to \infty$ or $\pi(t) \to -\infty$.

Proof. The relation

$$|\pi(t)| = y(0)\,y(\pi(t)) \geq x(0)\,x(t) - x(0)\,y(0) - x(t)\,y(\pi(t)) > |t| - x(0)\,y(0) - \alpha$$

implies $|\pi(t)| \to \infty$. But

$$|\pi(t_1) - \pi(t_2)| = y(\pi(t_1))\,y(\pi(t_2)) \leq y(\pi(t_1))\,x(t_1) + x(t_1)\,x(t_2) + x(t_2)\,y(\pi(t_2)) < |t_1 - t_2| + 2\alpha.$$

If $|\pi(t)| > 2\alpha$ for $t > t_0$ and, say, $\pi(t_0) > 0$, then

$$|\pi(t_0) - \pi(t)| < t - t_0 + 2\alpha < 4\alpha \text{ for } 0 < t - t_0 < 2\alpha,$$

implies that $\pi(t)$ is positive for $t_0 < t < t_0 + 2\alpha$. By the same argument $\pi(t) > 0$ for $t_0 + \alpha < t < t_0 + 3\alpha$ etc., so that $\pi(t) \to \infty$.

A *ray* $\mathfrak{r}$ is a half geodesic $x(t)$, $t \geq 0$ for which $x(t_1)\,x(t_2) = |t_1 - t_2|$. If p is any point then a sequence of segments $T(p_\nu, x(t_\nu))$ with $p_\nu \to p$ and $t_\nu \to \infty$ will contain a subsequence which converges to a ray $\mathfrak{s}$ with origin p. (For this and the following see Section 11 of [4].) Any ray $\mathfrak{s}$ which is obtainable in this manner is called a co-ray from p to $\mathfrak{r}$. If $p' \neq p$ is a point of $\mathfrak{s}$ then the co-ray from p' to $\mathfrak{r}$ is unique and coincides with the sub-ray of $\mathfrak{s}$ beginning at p'.

If R is straight and $\mathfrak{g}^+$ is an oriented line represented by $x(t)$, then for any point p the line through p and $x(t)$, so oriented that $x(t)$ follows p, converges for $t \to \infty$ to an oriented line $\mathfrak{a}^+$, the so-called oriented *asymptote*[5] through p

[5] In [4] the word asymptote is also defined for not straight R. In this case the present paper avoids the word because of the difficulties mentioned in Section 8.

to $\mathfrak{g}^+$. For any point q on $\mathfrak{a}^+$ the oriented asymptote to $\mathfrak{g}^+$ coincides with $\mathfrak{a}^+$, and any positive sub-ray of $\mathfrak{a}^+$ is co-ray to any positive sub-ray of $\mathfrak{g}^+$.

It is important for the sequal to know that the concept of asymptote is in general straight spaces *neither symmetric nor transitive* (see Section III 5 of [3]). Also, if $y(t)$ represents an oriented straight line, then $y(t)\mathfrak{g}^+ < \alpha$ for $t \geq 0$ is in general neither necessary nor sufficient for $y(t)$ to be an asymptote to $\mathfrak{g}^+$ (see [4, pp. 245, 246]), even if asymptotes are symmetric and transitive.

2. Fundamental Sets.

The G-space R' is a covering space of R if a locally isometric mapping Ω' of R' on R exists (compare Section 12 of [4]). If $x'(t)$ represents a geodesic in R' then $x'(t)\,\Omega' = x(t)$ represents a geodesic in R. Conversely

(2.1) For a given representation $x(t)$ of a geodesic in R and a given point x' over $x(t_0)$ there is exactly one representation $x'(t)$ of a geodesic in R' for which $x'(t)\,\Omega' = x(t)$ and $x'(t_0) = x'$.

We say that the geodesic $\mathfrak{g}'$ in R' lies over the geodesic $\mathfrak{g}$ in R if representations $x'(t)$ of $\mathfrak{g}'$ and $x(t)$ of $\mathfrak{g}$ with $x'(t)\,\Omega' = x(t)$ exist.

(2.2) *There is exactly one geodesic $\mathfrak{g}'$ in R' over a given geodesic $\mathfrak{g}$ in R which contains a given (non-degenerate) segment $\mathfrak{s}'$ over a segment $\mathfrak{s}$ in $\mathfrak{g}$.*[6]

For the (obvious) definition of multiplicity of a point of a geodesic the reader is referred to [4, p. 231]. (2.2) implies

(2.3) The sum of the multiplicities at x' of the geodesics in R' through x' which lie over the same geodesic $\mathfrak{g}$ in R equals the multiplicity of $\mathfrak{g}$ at $x = x'\,\Omega'$.

Therefore in particular

(2.4) If x is a simple point of $\mathfrak{g}$ then only one geodesic $\mathfrak{g}'$ over $\mathfrak{g}$ through a given point x' over x exists and x' is a simple point of $\mathfrak{g}'$.

Let now $\bar{R}$ be the universal covering space of R, and Ω a definite locally isometric mapping of $\bar{R}$ on R The fundamental group $\mathfrak{F}$ of R consists of the motions of $\bar{R}$ which lie over the identity of R and $\mathfrak{F}$ is simply transitive on the points which lie over a fixed point of R. The following construction of a fundamental set in $\bar{R}$ with respect to R uses the »méthode de rayonnement» (see [7, p. 71]) and applies to any space R' which covers R regularly (compare [15, § 57]).

[6] The statement (12.4) in [4], which replaces the segments $\mathfrak{s}$ and $\mathfrak{s}'$ by points, is wrong. (The mistake lies in the assertion that $\Phi\,\Phi^{-1}(x, \alpha)$ is the identity.) The applications of (12.4) in [4] are correct, because the assumptions of the present statement (2.4) are satisfied.

If $\bar{p}_0, \bar{p}_1, \ldots$ are the different points of $\bar{R}$ which lie over the fixed point p of R then the motions of $\mathfrak{F}$ may, because of the simple transitivity of $\mathfrak{F}$ on $\{\bar{p}_i\}$ be denoted by $\Phi_0 = E, \Phi_1, \Phi_2, \ldots$ such that $\bar{p}_i = \bar{p}_0 \Phi_i$. The sets $\bar{P}_k = \Sigma_{i \neq k} \bar{p}_i$ are closed because $\mathfrak{F}$ is discrete (actually $\bar{p}_i \bar{p}_k \geq 2 \eta_2(p)$ for $i \neq k$, see [4, p. 250]). The set $H(\bar{p}_k)$ of the points $\bar{x}$ in $\bar{R}$ for which $\bar{x} \bar{P}_k > \bar{x} \bar{p}_k$ is open and

$$(2.5) \qquad H(\bar{p}_k) \Phi_k^{-1} \Phi_i = H(\bar{p}_i).$$

For if $\Phi_k^{-1} \Phi_i = \Phi$ then $\bar{p}_k \Phi = \bar{p}_i$ and $\bar{P}_k \Phi = \bar{P}_i$, hence $\bar{x} \Phi P_k \Phi > \bar{x} \Phi p_k \Phi$ for all $\bar{x} \varepsilon H(\bar{p}_k)$ so that $\bar{y} \bar{P}_i > \bar{y} \bar{p}_i$ for all $\bar{y} \varepsilon H(\bar{p}_k) \Phi$. This means that $H(\bar{p}_k) \Phi < H(\bar{p}_i)$. Similarly $H(\bar{p}_i) \Phi^{-1} < H(\bar{p}_k)$. Moreover,

$$(2.6) \qquad H(\bar{p}_k) \wedge H(\bar{p}_i) = 0 \text{ for } i \neq k$$

because $\bar{x} \varepsilon H(\bar{p}_k)$ implies $\bar{x} \bar{P}_i \leq \bar{x} \bar{p}_k < \bar{x} \bar{P}_k \leq \bar{x} \bar{p}_i$. (2.5) and (2.6) yield

$$(2.7) \qquad H(\bar{p}_k) \Phi_e \wedge H(\bar{p}_k) \Phi_m = 0 \text{ for } m \neq l.$$

The set $H(\bar{p}_k)$ is star shaped with respect to $\bar{p}_k$, that means, it contains with $\bar{x}$ every segment $T(\bar{p}_k, \bar{x})$. For if $\bar{y} \varepsilon T(\bar{p}_k, \bar{x})$ then for $i \neq k$

$$\bar{p}_i \bar{y} \geq \bar{p}_i \bar{x} - \bar{y} \bar{x} > \bar{p}_k \bar{x} - \bar{x} \bar{y} = \bar{p}_k \bar{y}.$$

Because of (2.5) $H(\bar{p}_k) \Omega$ is the same set $H(p)$ in R for all k.

(2.8) $H(p)$ contains the set $D(p)$ of those points x in R for which a point y with (pxy) exists.

The segment $T(p, x)$ is by (1.1) unique. Let $T(\bar{p}_0, \bar{x})$ be the segment over $T(p, x)$ beginning at $\bar{p}_0$. If x were not in $H(p)$, then $\bar{x}$ would not lie in $H(\bar{p}_0)$, hence $\bar{x} \bar{P}_0 \leq \bar{x} \bar{p}_0$. Therefore $\bar{p}_k \neq \bar{p}_0$ exists such that $\bar{x} \bar{p}_k \leq \bar{x} \bar{p}_0$. A segment $T(\bar{p}_k, \bar{x})$ goes under Ω into a geodesic arc from p to x which is not homotopic to $T(p, x)$ and is a segment because $\bar{x} \bar{p}_k \leq \bar{x} \bar{p}_0 = x p$. Then $T(p, x)$ would not be unique. A consequence of (2.8) is

(2.9) If $F(\bar{p}_k)$ denotes the closure of $H(\bar{p}_k)$ then $\Sigma_k F(\bar{p}_k) = \bar{R}$. Moreover, $\delta(F(\bar{p}_k)) \leq 2 \delta(R)$ and $\delta(F(\bar{p}_k)) = \infty$ if $\delta(R) = \infty$ where $\delta(A)$ denotes the diameter of the set A.

For let $\bar{x} \varepsilon \bar{R}$ and connect $x = \bar{x} \Omega$ to p by a segment T. Let $\bar{T}$ be the segment over T which begins at $\bar{x}$. It ends at a point $\bar{p}_k$ over p. Then $y \varepsilon D(p)$

for any point $y \neq x$ on T. Since $H(\bar{p}_k)\,\Omega = H(p) > D(p)$ it follows that the point $\bar{y}$ over y of $\bar{T}$ lies in $H(\bar{p}_k)$, hence $\bar{x}\,\varepsilon\,F(\bar{p}_k)$.

If $\delta(R) < \infty$ and q is the point with maximal distance from p, then the preceding construction shows, that every point of $F(\bar{p}_k)$ belongs to a segment $\bar{T}$ which lies over a segment with origin at p. Hence $F(\bar{p}_k)$ is contained in the closure of $S(\bar{p}_k, 2\,p\,q)$.

If $\delta(R) = \infty$ then p is origin of a ray (see [4, (9.5) p. 237]). By (2.8) $H(p)$ contains such rays therefore $H(\bar{p}_k)$ contains rays. This leads to

(2.10) **Theorem.** *For every point $\bar{p}_0$ of $\bar{R}$ with fundamental group $\mathfrak{F} = \{\Phi_0 = E, \Phi_1, \ldots\}$ the fundamental set $H(\bar{p}_0)$ consisting of the points $\bar{x}$ with $\bar{x}\,\bar{p}_0 < \bar{x}\,\bar{p}_0\,\Phi_i$ for $i > 0$ has the following properties:*

a) *$H(\bar{p}_0)$ is open and star shaped with respect to $\bar{p}_0$.*
b) *$H(\bar{p}_0)\,\Phi_i \wedge H(\bar{p}_0)\,\Phi_k = 0$ for $i \neq k$.*
c) *If $F(\bar{p}_0)$ is the closure of $H(\bar{p}_0)$ then $\Sigma_k F(\bar{p}_0)\,\Phi_k = \bar{R}$.*
d) *$\delta(F(\bar{p}_0)) \leq 2\,\delta(R)$ and $\delta(F(\bar{p})) = \infty$ if $\delta(R) = \infty$.*
e) *If $H(\bar{p}_k)$ and $F(\bar{p}_k)$ are correspondingly defined sets for $\bar{p}_k = \bar{p}_0\,\Phi_k$ then*

$$H(\bar{p}_k) = H(\bar{p}_0)\,\Phi_k, \quad F(\bar{p}_k) = F(\bar{p}_0)\,\Phi_k.$$

f) *$H(\bar{p}_k)\,\Omega > D(p)$, where $p = \bar{p}_k\,\Omega$ and $D(p)$ is defined in (2.8).*
g) *A sphere $S(\bar{q}, \varrho)$, $0 < \varrho < \infty$, intersects only a finite number of $F(\bar{p}_k)$.*
h) *$\mathfrak{F}$ can be generated by* (positive powers of) *those Φ_i for which points x* (depending on i) *with $\bar{p}_0\,\bar{x} = \bar{x}\,\bar{p}_0\,\Phi_i$ exist.*

Statements a) to f) follow from the preceding discussion. To see g) choose by c) the point $\bar{p}_k$ such that $\bar{q}\,\varepsilon\,F(\bar{p}_k)$. Then $\bar{q}\,\bar{p}_j \geq \bar{q}\,\bar{p}_k$ for all j. If $S(\bar{q}, \varrho) \wedge F(\bar{p}_j)$ contains a point $\bar{r}$, then $\bar{r}\,\bar{p}_k \geq \bar{r}\,\bar{p}_j$, therefore

$$\bar{p}_j\,\bar{p}_k \leq \bar{p}_j\,\bar{r} + \bar{r}\,\bar{q} + \bar{q}\,\bar{p}_k \leq \bar{p}_k\,\bar{r} + \bar{r}\,\bar{q} + \bar{q}\,\bar{p}_k < 2\,(\bar{q}\,\bar{p}_k + \varrho).$$

There is only a finite number of $\bar{p}_j$ which satisfy this inequality.

We show h) following [7, p. 76]. Let $\Phi_e \neq E$ be a given element of F. Connect $\bar{p}_0$ to $\bar{p}_1$ by a segment $\bar{\mathfrak{s}}$ and denote the last point of $F(\bar{p}_0)$ on $\bar{\mathfrak{s}}$ by $\bar{q}_1$. There is a set $F(\bar{q}_{k_1})$ that contains $\bar{q}_1$ and points of $\bar{\mathfrak{s}}$ that follow $\bar{q}_1$. For the set of points that follow $\bar{q}_1$ on $\bar{\mathfrak{s}}$ is open in $\bar{\mathfrak{s}}$ and would by g) be union of a finite number of closed sets.

Let $\bar{q}_2$ be the last point of $\bar{\mathfrak{s}}$ in $F(p_{k_1})$. Then $\bar{q}_2$ belongs to a set $F(\bar{p}_{k_2})$

that contains points of $\bar{\bar{s}}$ that follow $\bar{q}_2$. By a) and g) we arrive after a finite number m of steps at $F(\bar{p}_{k_m}) = F(p_1)$. As a consequence of the definition of $H(\bar{p}_{k_i})$

$$\bar{p}_{k_i}\bar{q}_{i+1} = \bar{q}_{i+1}\bar{p}_{k_{i+1}}, \quad i = 0, \ldots, m-1, \text{ where } \bar{p}_{k_0} = \bar{p}_0.$$

Therefore $\Phi_{k_1} = \Psi_1$ is a transformation of the type required in h) and

$$F(\bar{p}_0) = F(\bar{p}_{k_1})\Psi_1^{-1}.$$

It follows that $F(\bar{p}_0)$ has the common boundary point $\bar{q}_2\Psi_1^{-1}$ with $F(\bar{p}_{k_2})\Psi_1^{-1}$ and a Ψ_2 which satisfies h) exists that carries $F(\bar{p}_0)$ into $F(\bar{p}_{k_2})\Psi_1^{-1}$ or

$$F(\bar{p}_{k_2}) = F(p_0)\Psi_2\Psi_1.$$

Continuation of this process shows that $F(\bar{p}_1)$ has the form $F(\bar{p}_0)\Psi_m \ldots \Psi_1$ or $\Phi_e = \Psi_m \ldots \Psi_1$ where the Ψ_j satisfy h).

It is easily seen, but not needed here, that the Φ_j that satisfy h) contain with any motion its inverse, so that just half of them generate $\mathfrak{F}$ (unless $\mathfrak{F}$ consists of the identity only). We say that a G-space has finite connectivity, if its fundamental group can be generated by a finite number of elements.

If R is compact then $\alpha = \delta(F(\bar{p}_0))$ is by (2.10 d) finite. Every point $\bar{p}_i$ for which an $\bar{x}$ with $\bar{p}_0\bar{x} = \bar{x}\bar{p}_i$ exists has at most distance 2α from $\bar{p}_0$. The sphere $S(\bar{p}_0, 2\alpha)$ contains by g) only points of a finite number of $F(\bar{p}_i)$. Hence the number of Φ_i that satisfy h) is finite and we find:

(2.11) *A compact G-space has finite connectivity.*

Some additional statements are possible if $\bar{R}$ is straight:

(2.12) *If $\bar{R}$ is straight, then*

a) *A ray with origin $\bar{p}_0$ intersects the boundary of $H(\bar{p}_0)$ in at most one point* (exactly one if R is compact).

b) $H(p) = D(p)$.[7]

c) *$H(\bar{p}_0)$ is maximal*, that is, $H(\bar{p}_0)$ is not proper subset of an open set H^* with $H^*\Phi_i \wedge H^*\Phi_k = 0$ for $i \neq k$.

Proof. a) If $\bar{q}$ is on the boundary of $H(\bar{p}_0)$ then $\bar{p}_0\bar{q} \leq \bar{p}_j\bar{q}$ for all j. If $(\bar{p}_0\bar{x}\bar{q})$ then

$$\bar{p}_0\bar{x} = \bar{p}_0\bar{q} - \bar{q}\bar{x} \leq \bar{p}_j\bar{q} - \bar{q}\bar{x} \leq \bar{p}_j\bar{x}.$$

[7] The geodesics on an ellipsoid show that the straightness of R is essential for this equation.

The equality sign holds only when $\bar{p}_j\bar{q} = \bar{p}_0\bar{q}$ and $(\bar{p}_j\bar{x}\bar{q})$ but then $\bar{p}_j = \bar{p}_0$ by V. This shows $\bar{x}\,\varepsilon\,H(\bar{p}_0)$.

Next let $(\bar{p}_0\bar{q}\bar{y})$. Because $\bar{q}$ is on the boundary of $H(\bar{p}_0)$ there is a $\bar{p}_i \neq \bar{p}_0$ with $\bar{p}_i\bar{q} = \bar{q}\bar{p}$. Then

$$\bar{p}_0\bar{y} = \bar{p}_0\bar{q} + \bar{q}\bar{y} = \bar{p}_i\bar{q} + \bar{q}\bar{y} > \bar{p}_i\bar{y},$$

because the equality sign would again imply $\bar{p}_i = \bar{p}_0$. Hence $\bar{y}$ is not in $F(\bar{p}_0)$ and therefore not on the boundary of $H(\bar{p}_0)$.

b) Let $y\,\varepsilon\,H(p)$ and let $\bar{y}$ be an original of y in $H(\bar{p}_0)$. Because $H(\bar{p}_0)$ is open and $\bar{R}$ is straight $H(\bar{p}_0)$ contains by (2.10 a) a point $\bar{z}$ with $(\bar{p}_0\bar{y}\bar{z})$ and $T(\bar{p}_0, \bar{z}) < H(\bar{p}_0)$. Then $T(\bar{p}, \bar{z})\,\Omega = A$ is a geodesic arc in $H(p)$ from p to z that contains y. If A were not a segment let T be a segment from p to z and construct in $\bar{R}$ the segment over T that begins at $\bar{z}$. It ends at a point $\bar{p}_i \neq \bar{p}_0$ over p because $T \neq A$ and segments in $\bar{R}$ are unique. Since $\bar{p}\bar{z}$ is the length of A and A is not a segment it follows that $\bar{p}_i\bar{z} = pz < \bar{p}_0\bar{z}$, which contradicts the definition of $H(\bar{p}_0)$. Therefore A is a segment and (pyz) so that $y\,\varepsilon\,D(p)$ and $H(p) < D(p)$.

c) Let H^* be an open set that contains $H(\bar{p}_0)$ properly. Then a sphere $S(\bar{q}, \varrho) < H^* - H(\bar{p}_0)$ exists. By (2.10 c) $\bar{q}\,\varepsilon\,F(\bar{p}_k)$ for some $\bar{p}_k \neq \bar{p}_0$. By (2.12 a) the sphere $S(\bar{q}, \varrho)$ contains a point of $H(\bar{p}_k)$. Therefore $H^* \wedge H^*\,\Phi_k \neq 0$.

CHAPTER II.

Spaces with Non-Positive Curvature.

§ 3. Local Properties.

A center $m(x, y)$ of two points x, y is defined by the relation

$$(3.1) \qquad x\,m(x, y) = m(x, y)\,y = xy/2.$$

If x, y are both in $S(p, \delta_p)$ (see Section 1), then $m(x, y)$ is unique. $m(x, y)$ will also be called the center of $T(x, y)$.

The fundamental inequality (*) for spaces with non-positive curvature may then be formulated rigorously as follows:

The G-space R has non-positive curvature if every point p of R has a neighborhood $S(p, \varrho_p)$, $0 < \varrho_p \leq \delta_p$ such that any three points a, b, c in $S(p, \varrho_p)$ satisfy *the relation*

$$(3.2) \qquad 2\,m(a,\,b)\,m(a,\,c) \leq b\,c.$$

If under otherwise the same conditions

$$(3.3) \qquad 2\,m(a,\,b)\,m(a,\,c) = b\,c$$

then R is said to have curvature 0.

Three points are called collinear if one of the points lies on a segment connecting the two others. Since (3.3) holds always for collinear points, we define:

The G-space R has negative curvature if every point p of R has a neighborhood $S(p,\,\varrho_p)$, $0 < \varrho_p \leq \delta_p$ such that for any three non-collinear points $a,\,b,\,c$ in $S(p,\,\varrho_p)$.

$$(3.4) \qquad 2\,m(a,\,b)\,m(a,\,c) < b\,c.$$

Since one-dimensional G-spaces are straight lines or great circles (see [4, p. 233]), they satisfy (3.3) trivially and offer nothing interesting. It will therefore always be assumed that *the dimension of R in the sense of Menger-Urysohn is greater than one.*

In a space with non-positive curvature we introduce as auxiliary point function β_p the least upper bound of the $\varrho_p \leq \delta_p$ such that (3.2) is satisfied in $S(p,\,\varrho_p)$. As in the proof of (1.2) it is seen that

$$(3.5) \qquad |\beta_p - \beta_q| \leq p\,q \quad \text{or} \quad \beta_p = \infty.$$

It will appear soon that $\beta_p = \delta_p$.

The inequality (3.2) implies the following fundamental fact

(3.6) **Theorem**: *Let $x(t)$ and $y(s)$ represent geodesics in a space with non-positive curvature. If for suitable constants $\alpha_1 < \alpha_2$, $c \neq 0$, and d the segment $T(x(t),\,y(c\,t + d))$ is unique for $\alpha_1 \leq t \leq \alpha_2$, then $f(t) = x(t)\,y(c\,t + d)$ is in the interval $(\alpha_1,\,\alpha_2)$ a convex function of t.*

»Convex» will here always mean »weakly convex», that is

$$f((1 - \theta)\,t_1 + \theta\,t_2) \leq (1 - \theta) f(t_1) + \theta f(t_2) \quad \text{for} \quad 0 < \theta < 1.$$

If the inequality sign holds for any $t_1 \neq t_2$, we say that $f(t)$ is strictly convex. To prove that a continuous function $f(t)$ is convex it suffices to show that an $\varepsilon > 0$ exists such that

$$(3.7) \qquad 2\,f((t_1 + t_2)/2) \leq f(t_1) + f(t_2) \quad \text{for} \quad |t_1 - t_2| < \varepsilon.$$

For the proof of (3.6) put $y(ct+d)=x'(t)$. Since the set $V=\Sigma_{\alpha_1\le t\le\alpha_2} T(x(t), x'(t))$ is bounded it follows from (3.5) that $\beta=\inf_{p\varepsilon V}\beta_p$ is positive. If $x(t)=x'(t)$ for $\alpha_1\le t\le\alpha_2$ the theorem is trivial. We assume therefore that

$$k=x(t^*)x'(t^*)=\max x(t)x'(t)>0.$$

Let $z(t^*, u)$ represent $\mathfrak{S}(x(t^*), x'(t^*))$ for $0\le u\le k$ and call $z(t, u)$ the image of $z(t^*, u)$ under a linear[8] mapping of $\mathfrak{S}(x(t^*), x'(t^*))$ on $\mathfrak{S}(x(t), x'(t))$ The definition of t^* implies that

$$z(t, u_1)z(t, u_2)\le z(t^*, u_1)z(t^*, u_2)=|u_1-u_2|,\quad 0\le u_i\le k. \tag{3.8}$$

Because the segments $\mathfrak{S}(x(t), x'(t))$ are unique the point $z(t, u)$ depends continuously on t and u. Hence an $\varepsilon>0$ exists such that

$$z(t_1, u)z(t_2, u)<\beta/2\quad\text{for } |t_1-t_2|<\varepsilon. \tag{3.9}$$

Let $n>2k/\beta$ and put $u^i=ik/n$, $i=0,\ldots,n$. Then by (3.8) and (3.9)

$$z(t_1, u^i)z(t_2, u^{i\pm1})<\beta/2+|u^i-u^{i\pm1}|<\beta\text{ for all } i \text{ and } |t_1-t_2|<\varepsilon. \tag{3.10}$$

Hence the points $z(t_1, u^i)$, $z(t_2, u^i)$, $z(t_2, u^{i+1})$ lie in $S(z(t_1, u^i), \beta)$ and the points $z(t_2, u^{i+1})$, $z(t_1, u^i)$, $z(t_1, u^{i+1})$ lie in $S(z(t_2, u^{i+1}), \beta)$. Therefore (3.2) yields with the notation

$$m_{2i}=m[z(t_1, u^i), z(t_2, u^i)],\quad i=0,\ldots,n$$
$$m_{2i+1}=m[z(t_1, u^i), z(t_2, u^{i+1})],\quad i=0,\ldots,n-1$$

that

$$2m_{2i}m_{2i+1}\le z(t_2, u^i)z(t_2, u^{i+1}),\quad i=0,\ldots,n-1$$
$$2m_{2i+1}m_{2i+2}\le z(t_1, u^i)z(t_1, u^{i+1}),\quad i=0,\ldots,n-1$$

hence by addition

$$2\sum_{j=0}^{2n-1} m_j m_{j+1}\le z(t_1, 0)z(t_1, k)+z(t_2, 0)z(t_2, k). \tag{3.11}$$

Since $z(t, 0)=x(t)$, and $z(t, k)=x'(t)$ and

$$m_0=m(z(t_1, 0), z(t_2, 0))=x((t_1+t_2)/2)$$
$$m_{2n}=m(z(t_1, k), z(t_2, k))=x'((t_1+t_2)/2)$$

---

[8] The linear mapping $x\to x'$ of $\mathfrak{S}(a, b)$ on $\mathfrak{S}(a', b')$ is defined by $ax:ab=a'x':a'b'$.

it follows from the triangle inequality and (3.11) that (3.7) holds for $f(t) = x(t)\,x'(t)$.

In the case where one of the arcs $x(t)$, $y(t)$ shrinks to a point Theorem (3.6) can be strengthened. (The assumption that R has non-positive curvature is made throughout this section).

(3.12) **Theorem.** *If p, a, b are not collinear and $a, b \varepsilon S(p, \delta_p)$ then px is for $x \varepsilon T(a, b)$ a strictly convex function of ax. Consequently p has exactly one foot on $T(a, b)$.*

We prove (3.12) first under the assumption that $T(a, b) < S(p, \delta_p)$ and show later that this is always the case. Since $T(p, a)$ and $T(p, b)$ lie in $S(p, \delta_p)$ segments connecting any points of these segments are unique, so that by (3.6)

$$2\, m(p, a)\, m(b, a) \leq p\, b$$
$$2\, p\, m(a, b) \leq 2\,[p\, m(p, a) + m(p, a)\, m(a, b)] \leq p\, a + p\, b.$$

The equality sign holds only when $(p\, m(a, p)\, m(a, b))$ but then p, a, b are collinear.

(3.13) *$\beta_p = \delta_p$ and $S(p, \delta_p)$ is convex*, that is, contains $T(a, b)$ when it contains a and b.

Proof. Let δ be the least upper bound of those ϱ for which $a, b \varepsilon S(p, \varrho)$ implies $T(a, b) < S(p, \delta_p)$. For each such ϱ the sphere $S(p, \varrho)$ is convex because the special case of (3.12) already proved can be applied and yields $p\, x \leq \max\,(p\, a, p\, b) < \varrho$ when $a, b \varepsilon S(p, \varrho)$ and $(a\, x\, b)$. Therefore $S(p, \delta)$ is also convex.

If δ were smaller than δ_p, then pairs a_ν, b_ν would exist for which $p\, a_\nu < \delta + \nu^{-1}$, $p\, b_\nu < \delta + \nu^{-1}$ and $T(a_\nu, b_\nu)$ contains a point c_ν with $p\, c_\nu \geq \delta_p$. It may be assumed that $\{a_\nu\}$ and $\{b_\nu\}$ converge to points a and b respectively. Then $p\, a \leq \delta$ and $p\, b \leq \delta$. There are sequences $a'_\nu \to a$ and $b'_\nu \to a$ with $p\, a'_\nu < \delta$ and $p\, b'_\nu < \delta$. The segments $T(a_\nu, b_\nu)$ and $T(a'_\nu, b'_\nu)$ tend to the, because of $\delta < \delta_p$, unique segment $T(a, b)$. But $T(a'_\nu, b'_\nu) < S(p, \delta)$ because $S(p, \delta)$ is convex whereas $p\, c_\nu \geq \delta_p > \delta$. This proves $\delta = \delta_p$ so that $S(p, \delta_p)$ is convex, and completes the proof of (3.12).

If a, b, c are any points in $S(p, \delta_p)$ a segment connecting a point of $T(a, b)$ to a point of $T(a, c)$ is unique, hence (3.12) yields $2\, m(a, b)\, m(a, c) \leq b\, c$, which implies $\beta_p = \delta_p$.

The next theorem, whose proof is lengthy, is important for the distinction of spaces of vanishing, non-positive and negative curvature.

(3.14) **Theorem.** *Let a, b, a', b' be four non-collinear points of $S(p, \delta_p)$, $a \neq b$. If $x \to x'$ maps $\mathfrak{S}(a, b)$ linearly*[8] *on $\mathfrak{S}(a', b')$ and if the relation*

(3,15) $$x x' = \frac{x b}{a b} a a' + \frac{a x}{a b} b b' \text{ holds for one } x \text{ with } (a x b),$$

then $V = \Sigma_x T(x, x')$ is congruent to a trapezoid of a Minkowski plane.

(The segments $T(a, a')$ and $T(b, b')$ are the parallel sides of the trapezoid and the trapezoid degenerates into a triangle when $a = a'$ or $b = b'$).

Proof. By (3.6) $x x'$ is a convex function of $a x$. Therefore the equality (3.15) for one x with $(a x b)$ implies that (3.15) holds for all x on $T(a, b)$.

Put $a x = \xi$ and $a b = \alpha$. Let a'', b'' be points of $T(a, a')$ and $T(b, b')$, and let $x \to x''$ map $\mathfrak{S}(a, b)$ linearly[8] on $\mathfrak{S}(a'', b'')$. Then by (3.6)

(3.16) $$x x'' \leq \alpha^{-1}[(\alpha - \xi) a a'' + \xi b b'']$$
$$x'' x' \leq a' b'^{-1}[(a' b' - a' x') a'' a' + a' x' \cdot b'' b']$$

hence

$$x x' \leq x x'' + x'' x' \leq \alpha^{-1}[(\alpha - \xi) a a' + \xi b b']$$

therefore (3.15) and (3.16) show that the equality signs hold in (3.16) and that $x'' \varepsilon T(x, x') < V$. Putting $x x'' = \eta$ the first relation in (3.16) and the definition of the mapping $x \to x''$ yield

(3.17) $$\alpha \eta = (\alpha - \xi) a a'' + \xi b b'', \quad a'' x'' : a'' b'' = \xi : \alpha.$$

We define ξ, η as coordinates of x''. Then $x'' \to (\xi, \eta)$ maps V on the trapezoid

$\bar{V}$: $$0 \leq \xi \leq \alpha, \quad 0 \leq \eta \leq \alpha^{-1}[(\alpha - \xi) a a' + \xi b b']$$

of the Cartesian (ξ, η)-plane. In $\bar{V}$ we introduce the euclidean metric

$$\varepsilon(z_1, z_2) = [(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2]^{1/2}, \quad z_i = (\xi_i, \eta_i).$$

Let $z_i = (\xi_i, \eta_i)$ $i = 1, 2$ be points of V. If $\xi_1 = \xi_2$, then $T(z_1, z_2)$ satisfies the equation $\xi = \xi_1$. If $\xi_1 < \xi_2$ then the same argument that established (3.17) yields

(3.18) $$(\xi_2 - \xi_1)\eta = (\xi_2 - \xi)\eta_1 + (\xi - \xi_1)\eta_2, \quad z_1 z : z_1 z_2 = (\xi - \xi_1) : (\xi_2 - \xi_1)$$

for the variable point $z = (\xi, \eta)$ of $T(z_1, z_2)$. Hence V contains with any two points z_1, z_2 the segment $T(z_1, z_2)$ and the points of $T(z_1, z_2)$ lie on a straight line in $\bar{V}$. The intersection of a euclidean straight line with $\bar{V}$ appears as a segment in V, which we call a straight line in V. We call two such straight lines parallel when they correspond to pieces of parallel straight lines in $\bar{V}$ (or if their slopes are equal). (3.18) shows that

(3.19) The distances $\varepsilon(z_1, z_2)$ and $z_1 z_2$ are proportional for points on the same straight line.

To prove that the metric $z_1 z_2$ in V is Minkowskian *it must be shown that the factor of proportionality is the same for parallel lines* (so far we only know that is so for parallels to the η-axis).

It suffices to see that every interior point f of V has a neighborhood $S(q, \varrho) < V$ such that the factor of proportionality is the same for parallels which intersect $S(q, \varrho)$. The following considerations are restricted to a suitable $S(q, \varrho)$.

The line H is perpendicular to L at f (and L transversal to H at f) if all points of H have the intersection f of H and L as foot on L. The transversals of H at f are, because of the convexity of the circles in V (see (3.13)), the supporting lines of any circle with center $c \neq f$ on H through f.

(3.20) If H is perpendicular to L, then H is perpendicular to all parallels L' to L.

For if $H \wedge L' = f'$ and r is a point of H different from f and f' let $a' \varepsilon L'$ and let the straight line through r and a' intersect L at a (compare figure). Then by (3.19)

$$ar : a'r = \varepsilon(a, r) : \varepsilon(a' r) = \varepsilon(f, r) : \varepsilon(f' r) = fr : f'r$$

and $a'r > f'r$ because $ar > fr$.

Under adequate differentiability hypotheses this means that the parallels to L are transversal curves to the perpendiculars to L in the sense of the calculus of variations. Hence two parallels cut out equal pieces from all perpendiculars, or the parallels to L are equidistant to L (see [2, p. 339]).

To prove this without differentiability hypotheses observe first that the equidistant curves to L are convex curves which turn their concavity towards L. This is contained in the following general fact:

(3.21) *If in a space with non-positive curvature a, b, c, d are points of $S(p, \delta_p)$ and f_x denotes the foot of the point $x \varepsilon T(a, b)$ on $T(c, d)$ then xf_x is a convex function of ax.*

For if $x, y \varepsilon T(a, b)$ then by (3.6)

$$2\,m(x, y) f_{m(x, y)} \leq 2\,m(x, y)\, m(f_x, f_y) \leq x f_x + y f_y. \tag{3.22}$$

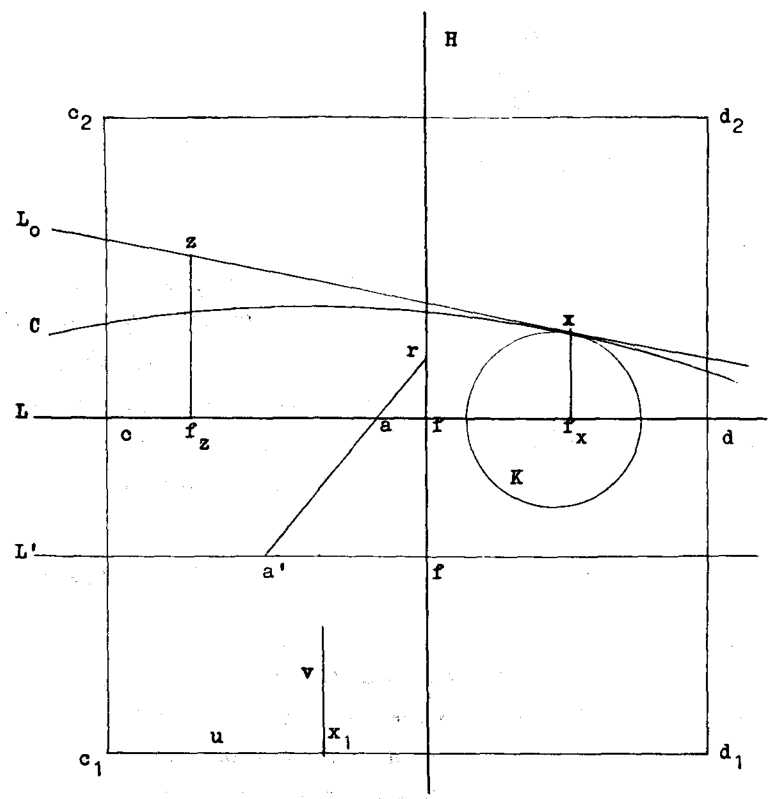

Fig. 1.

Returning to the present special case let x be a point of an equidistant curve C to L and, generally, f_x the foot of the point x on L. Then for $x \varepsilon C$ the circle $K = K(f_x, xf_x)$ (the locus of the points z in V with $zf_x = xf_x$) has the point x in common with C. A supporting line L_0 to C at x is also a supporting line to K at x.

By (3.13) K is a convex curve and has therefore a unique supporting line or tangent at all but a countable number of points. We assume first that circles have tangents everywhere. Since then the transversal to $T(x, f_x)$ at x is unique, L_0 must by (3.20) be parallel to L.

Let z be any point of L_0 different from x. Then $zf_z \geq xf_x$ because C turns its concavity toward L. The equidistant curve C' through z to L has, for the same reason as above, L_0 as supporting line at z. Therefore C' lies between L_0 and C, it must contain x, hence $C' = C$. Because C is convex $C = L_0$.

It now follows that $x \rightarrow f_x$ maps L_0 linearly on L. For if $x, y \,\varepsilon\, L_0$ and

$z = m(x, y)$ then as in (3.22)

$$2zf_z \leq 2zm(f_x, f_y) \leq xf_x + yf_y = 2zf_z.$$

Because the foot f_z of z is unique $m(f_x, f_y) = z$.

Take a subsegment $T(c, d)$ of L with center f and segments $T(c_1, c_2)$, $T(d_1, d_2)$ of the same length with centers c, d and perpendicular to L at c and d respectively. Then $T(c_i, d_i)$ lie on parallels to L. For the points z in the quadrangle W with vertices c_1, c_2, d_2, d_1 we introduce coordinates u, v as follows: if the perpendicular to L through z intersects $T(c_1, d_1)$ at x_1, then $u = c_1 x_1$ and $v = x_1 z$. As in the proof of (3.18) it follows from the linearity of the mapping $x \to f_x$ that the straight lines in W have linear equations in u and v. The coordinates (ξ, η) and (u, v) of the same point in W satisfy therefore a relation of the form

$$(3.23) \qquad \xi : \eta : 1 = (a_1 u + b_1 v + c_1) : (a_2 u + b_2 v + c_2) : (a_3 u + b_3 v + c_3).$$

By (3.19) the distance $z_1 z_2$, $z_i = (u_i, v_i)$, is on a fixed line in W proportional to $[(u_1 - u_2)^2 + (v_1 - v_2)^2]^{1/2}$. Therefore this distance and $[(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2]^{1/2}$ are proportional on a fixed line. This means that (3.23) leaves the line at infinity fixed and is an affinity. The lines $u = \text{const}$ which are parallel for (u, v) are therefore also parallel in (ξ, η). Because the curves $v = \text{const}$ are equidistant and parallel, the factor of proportionality is the same for the lines $u = \text{const}$, or the parallels to H. Since H was arbitrary, *the theorem is proved in the case of differentiable circles.*

If *one* circle with center on a line H' intersects H' in a point where the circle has a unique tangent, then (3.20) implies that *all* circles with center on H' will intersect H' in such points. For the sake of brevity we call a line H' with this property smooth. All lines through a fixed point except an at most countable number are smooth.

If the line L through f varies, the family of perpendiculars to L (and to the parallels to L) changes continuously. A simple measure theoretical consideration yields the following: For a sufficiently small positive β and a given line H through f a line L' through f exists such that the perpendicular H' to L' at f is as close to H as desired and L' contains points s_1, s_2 with $(s_1 f s_2)$ and $s_i f > \beta$ at which the perpendiculars P_1, P_2 to L' are smooth.

The proof for differentiable circles used only that $T(x, f_x)$ and $T(z, f_z)$ are smooth and shows therefore, that the equidistant curves L' coincide between P_1

and P_2 with the parallels to L'. By the previous arguments the factor of proportionality is the same for all parallels to H' between P_1 and P_2, Since the factor depends continuously on the line and $s_l f > \beta$ it follows that the factor is the same for all parallels to H sufficiently close to H. This completes the proof of (3.14).

Notice the corollary

(3.24) *If, under the assumptions of* (3.6), *the space has negative curvature and the two geodesics are different then $x(t)y(ct+d)$ is a strictly convex function of t.*

4. The Theory of Parallel Lines in Straight Spaces.

If all geodesics are straight lines, then $\eta_\lambda(p) = \delta_p = \infty$ for all λ and p and the facts of the preceding section hold in the large. Because of their frequent occurrence we formulate the implications of (3.6) (3.21) and (3.24) explicitely:

(4.1) *If R is a straight space with non-positive curvature then $px(t)$ is for any geodesic $x(t)$ and any point p not on $x(t)$ a strictly convex function of t. The spheres of R are convex.*

(4.2) *If $x(t)$ and $y(t)$ represent different geodesics $\mathfrak{g}$ and $\mathfrak{h}$ in a straight space R with non-positive (negative) curvature then for any constants $c \neq 0$, $c' \neq 0$, d, d' the function $x(ct+d)y(c't+d')$ and $x(ct+d)\mathfrak{h}$, and $y(c't+d')\mathfrak{g}$ are (strictly) convex.*[9]

This slightly more general formulation follows from (3.6) because a linear transformation of the independent variable does not influence convexity. (4.2) permits to show that asymptotes have all the usual properties. It was pointed out at the end of Section 1 that this is not true in general straight spaces.

(4.3) **Theorem.** *If the line $\mathfrak{h}^+$ is an asymptote to $\mathfrak{g}^+$ then $\mathfrak{g}^+$ is an asymptote to $\mathfrak{h}^+$. If $\mathfrak{h}^+$ is an asymptote to $\mathfrak{g}^+$, and $\mathfrak{k}^+$ to $\mathfrak{h}^+$, then $\mathfrak{k}^+$ is an asymptote to $\mathfrak{g}^+$.*

Therefore we may simply say that two oriented lines are asymptotic to each other. We prove at the same time

(4.4) **Theorem.** *Let $x(t)$ and $y(t)$ represent $\mathfrak{g}^+$ and $\mathfrak{h}^+$. Each of the following conditions is necessary and sufficient for $\mathfrak{g}^+$ and $\mathfrak{h}^+$ to be asymptotes to each other*

a) *$x(t)y(t)$ is bounded for $t > 0$*

b) *$x(t)\mathfrak{h}_1$ (or $y(t)\mathfrak{g}_1$) is bounded for $t \geq 0$, where $\mathfrak{h}_1$ and $\mathfrak{g}_1$ are positive subrays of $\mathfrak{h}^+$ and $\mathfrak{g}^+$.*

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[9] If $x(t)$ and $y(t)$ represent the same geodesic, then $x(ct+d)y(c't+d')$ is either a linear function of t or there is a value t_0 such that the function is linear both for $t \geq t_0$ and for $t \leq t_0$.

It is clear that (4.4 a) implies the symmetry of the asymptote relation. It also implies transitivity because boundedness of $x(t)y(t)$ and of $y(t)z(t)$, where $z(t)$ represents $\mathfrak{k}^+$, implies boundedness of $x(t)z(t)$.

For the proof of (4.4 a) denote generally by $\mathfrak{g}^+(a, b)$ the straight line through a and b with the orientation in which b follows a. The line $\mathfrak{g}^+(a, x(t))$ tends to the asymptote $\mathfrak{a}^+$ to $\mathfrak{g}^+$ through a. Let $a(s)$ and $a_t(s)$ represent $\mathfrak{a}^+$ to $\mathfrak{g}^+(a, x(t))$ with $a(0) = a_t(0) = a$. Then $a_t(s) \to a(s)$ for $t \to \infty$. Put

$$(4.5) \qquad c_t = t / a x(t). \quad c_t \to 1 \text{ for } t \to \infty$$

because $t - a x(0) \leq a x(t) \leq t + a x(0)$.

By (4.2) $a_t(s) x(c_t s)$ is a convex function of s which vanishes for $s = a x(t)$ and decreases therefore for $-\infty < s \leq a x(t)$. By (4.5)

$$\lim a_t(s) x(c_t s) = a(s) x(s)$$

and $a(s) x(s)$ is a non-increasing convex function of s. Moreover, $a(s) x(s)$ is bounded for $t \geq 0$ which proves the necessity of a) and also of b) because $a(s) \mathfrak{g}_1 \leq a(s) x(s)$ for large s.

To see the sufficiency of a) we show: if the oriented line represented by $x(t)$ is not an asymptote to the oriented line $\mathfrak{h}^+$ represented by $y(t)$, then $x(t) y(t) \to \infty$ for $t \to \infty$. Let $a(t)$ represent the asymptote to $\mathfrak{h}^+$ through $x(0)$ with $a(0) = x(0)$. Then $x(t) \neq a(t)$ for $t \neq 0$, hence $x(t) a(t)$ is convex, vanishes at $t = 0$ and is positive otherwise. It follows that $x(t) a(t) \to \infty$ for $t \to \infty$. Because the necessity of a) was already proved, $a(t) y(t)$ is bounded for $t \geq 0$, hence $x(t) y(t) \to \infty$.

Finally we prove the sufficiency of b). Let $y(\pi(t))$ be the foot of $x(t)$ on H^+ and $x(t) y(\pi(t)) < \alpha$. Because of b) and (1.3) $\pi(t) \to \infty$. If $x(t)$ were not an asymptote to $\mathfrak{h}^+$ let $a(t)$ with $a(0) = x(0)$ be the asymptote $\mathfrak{a}^+$ to $\mathfrak{h}^+$ through $x(0)$. Then $y(\pi(t)) a(\pi(t)) < \alpha_1$ by (4.4 a) hence

$$x(t) \mathfrak{a}^+ \leq x(t) a(\pi(t)) \leq x(t) y(\pi(t)) + y(\pi(t)) a(\pi(t)) < \alpha + \alpha_1.$$

But this is impossible because $x(t) \mathfrak{a}^+$ is by (4.2) a convex function of t, which vanishes for $t = 0$ and tends therefore to ∞.

(4.6) If $p x_\nu \to \infty$ and $\mathfrak{g}^+(p, x_\nu) \to \mathfrak{l}^+$, then $\mathfrak{g}^+(q, x_\nu)$ tends for any point q to an asymptote to $\mathfrak{l}^+$.

It suffices to see that this is true for every subsequence i of $\{\nu\}$ for which $\mathfrak{g}^+(q, x_i)$ converges to a line $\mathfrak{a}^+$. Let $x_i(t)$ and $y_i(t)$ represent $\mathfrak{g}^+(p, x_i)$ and $\mathfrak{g}^+(q, x_i)$

with $x_i(0) = p$, $y_i(0) = q$. Then $x_i(t)\, y_i(t\, q x_i / p x_i)$ decreases for $0 \leq t \leq p x_i$. Since $x_i(t)$ and $y_i(t)$ tend to representations $x(t)$ and $y(t)$ of $\mathfrak{l}^+$ and $\mathfrak{a}^+$ and $q x_i / p x_i \to 1$ it follows as before that $x(t)\, y(t)$ decreases for $t \geq 0$, hence $\mathfrak{a}^+$ and $\mathfrak{l}^+$ are by (4.4 a) asymptotes to each other.

If the lines $\mathfrak{p}$ and $\mathfrak{g}$ can be so oriented that $\mathfrak{p}^+$ is an asymptote to $\mathfrak{g}^+$ and $\mathfrak{p}^-$ to $\mathfrak{g}^-$, then we call $\mathfrak{p}$ and $\mathfrak{g}$ *parallels to each other* (see 4.3). If $p(t)$ and $x(t)$ represent $\mathfrak{p}^+$ and $\mathfrak{g}^+$, then both $p(t)\, x(t)$ and $p(-t)\, q(-t)$ are non-increasing functions of t, therefore $p(t)\, x(t)$ is constant. Conversely, if $p(t)\, x(t)$ is constant or only bounded, it follows from (4.4 a) that $\mathfrak{p}$ and $\mathfrak{g}$ are parallels.[10] In the same way it follows from (4.4 b) that the boundedness of $p(t)\mathfrak{g}$ or of $x(t)\mathfrak{p}$ is necessary and sufficient for $\mathfrak{g}$ and $\mathfrak{p}$ to be parallels.

(4.7) *The lines $\mathfrak{p}$ and $\mathfrak{g}$ are parallels to each other if and only if they have representations $p(t)$ and $x(t)$ which have one of the following properties*

a) *$p(t)\, x(t)$ is constant or bounded*
b) *$x(t)\mathfrak{p}$ (or $p(t)\mathfrak{g}$) is constant or bounded*
c) *$\Sigma_t\, T(p(t), x(t))$ is congruent to a strip of a Minkowski plane bounded by parallel lines.*

Part c) follows immediately from part a) and Theorem (3.14). Notice the following corollaries

(4.8) *In a straight space with negative curvature the asymptotes to the two orientations of a line $\mathfrak{h}$ through a point not on $\mathfrak{h}$ are carried by different straight lines.*

(4.9) If in a straight space of non-positive curvature the asymptotes to the two orientations of any straight line through any point lie on the same straight line (that is the euclidean parallel axiom holds) then the space is Minkowskian.

This fact is a special case of the more general theorems IV 6.2 and IV 7.4 in [3].

We conclude this section with a theorem which rests on the following lemma

(4.10) For any points $p_1, \ldots, p_n$ in a straight space with non-positive curvature and any $\alpha > 1$ there is exactly one point q for which $\Sigma_\nu\, x p_\nu^\alpha$ reaches its minimum.

Let $x(t)$ represent any straight line Then $p_\nu x(t)$ is a no where constant convex function of t (see (4.1)), hence $p_\nu x(t)^\alpha$ is strictly convex. Therefore $\Sigma p_\nu x(t)^\alpha$ is a strictly convex function of t which tends to ∞ when $|t| \to \infty$, and reaches therefore its minimum at exactly one t.

[10] We admit the case where $p(t) = x(t + d)$ so that every line is parallel to itself.

The minimum of $\Sigma x p_\nu^\alpha$ as x varies over R is reached at at least one point q. If it were reached at another point r and $x(t)$ represents $\mathfrak{g}(q, r)$ then $\Sigma p_\nu x(t)^\alpha$ would have two minima as a function of t.

Cartan observed in [7, pp. 266, 267] that (4.10) contains the theorem

(4.11) *If $\mathfrak{G}$ is a finite group of motions in a straight space with non-positive curvature then a point q exists which remains fixed under all motions of $\mathfrak{G}$.*

For let p_1 be any point and $p_2, \ldots, p_n$ its images under the motions of $\mathfrak{G}$. The set $\{p_i\}$ goes into itself under all motions of $\mathfrak{G}$. Therefore the by (4.10) unique point q where $\Sigma x p_\nu^\alpha$, $\alpha > 1$, reaches its minimum goes into itself under the motions of $\mathfrak{G}$.

5. The Universal Covering Space of a Space with Domain Invariance.

The importance of straight spaces lies in the fact that essentially all simply connected spaces with non-positive curvature are straight. The term »essentially all» refers to the assumption made in the proof presented here that the space has the property of *domain invariance:*

(5.1) *If X and X' are homeomorphic subsets of R and X is open in R then X' is open in R.*

Probably every G-space has this property, so that assuming (5.1) means no restriction. But so far this has been proved only for two-dimensional G-spaces (see [3, p. 29]). The only known fact for general G-spaces which goes in this direction is (4.12) in [4, p. 219]. All Finsler spaces in the usual sense are by their very definition topological manifolds and satisfy therefore (5.1). Property (5.1) is essential for Sections 7, 8, 10 of the present paper, automatically satisfied by the spaces considered in Sections 9, 11, 12, 13 and not necessary for Section 6.

The purpose of the present section is the proof of the following fact which is well known for Riemann spaces, see [7, p. 261].

(5.2) **Theorem.** *The universal covering space of a space with non-positive curvature and domain invariance is straight* (and has, of course, also non-positive curvature).

Proof. δ_p may be assumed finite since otherwise nothing is to be proved. Fix a point p of the given space R and call V the locus $px = \delta_p/2$. For $u \varepsilon V$ let $x(u, t)$, $t \geq 0$, represent the half geodesic which coincides with $\mathfrak{s}(p, u)$ for $0 \leq t \leq \delta_p/2$. For every point q of the space there is at least one pair u, t such that $q = x(u, t)$. The mapping $(u, t) \rightarrow x(u, t)$ is one-to-one for $0 \leq t < \delta_p$.

$$(5.3) \qquad \beta_\alpha = \inf \delta_x/2 \ \text{ for } \ x \,\varepsilon\, S(p, 2\alpha)$$

is by (1.2) positive, and since the function $x(u, t)$ is uniformly continuous for $u \,\varepsilon\, V$ and $0 \leq t \leq k$, compare [4, p. 224], an $\varepsilon_\alpha > 0$ exists such that

$$(5.4) \qquad u, u' \,\varepsilon\, V, \ u, u' < \varepsilon_\alpha, \ \text{and} \ 0 \leq t \leq 2\alpha, \ |t - t'| < 2\varepsilon_\alpha \ \text{imply} \\ x(u, t)\, x(u', t') < \beta_\alpha.$$

Let $\alpha \geq \delta_p$ and define $W(v, \alpha)$ as the set of $x(u, t)$ for which $u \,\varepsilon\, V \wedge S(v, \varepsilon_\alpha) = V_{\alpha v}$ and $|t - \delta_p/2| < \varepsilon_\alpha \cdot M^{-1}$ where $M = 2\alpha/\delta_p$. Then $W(v, \alpha)$ is open and

$$(5.5) \qquad x(u, t) \to x(u, Mt), \ u \,\varepsilon\, V_{\alpha v}, \ |t - \delta_p/2| < \varepsilon_\alpha M^{-1}$$

maps $W(v, \alpha)$ continuously on the set $W'(v, \alpha)$ consisting of those $x(u, t)$ for which $u \,\varepsilon\, V_{\alpha v}$ and $|t - \alpha| < \varepsilon_\alpha$. If $u_i \,\varepsilon\, V_{\alpha v}$ and $|t_i - \delta_p/2| < \varepsilon_\alpha M^{-1}$ then

$$x(u_1, s t_1) \to x(u_2, s t_2), \quad 0 \leq s \leq 1$$

maps, by (5.3) and (5.4) the geodesic arc $0 \leq t' \leq t_1$ of $x(u_1, t)$ linearly on the arc $0 \leq t' \leq t_2$ of $x(u_2, t)$ in such a way that the segment connecting corresponding points is unique. It follows from (3.6) that $x(u_1, s t_1)\, x(u_2, s t_2)$ is a convex function of s. Since it vanishes for $s = 0$ it increases (unless $u_1 = u_2$ and $t_1 = t_2$) and has for $s = M$ a greater value than for $s = 1$, or

$$x(u_1, M t_1)\, x(u_2, M t_2) \geq x(u_1, t_1)\, x(u_2, t_2).$$

This shows that the inverse of the mapping (5.5) is single valued and continuous. *Because of the invariance of the domain* (5.1) *the set* $W'(v, \alpha)$ *is open.*

We observe also that (5.5) furnishes a one-to-one and continuous mapping of the set $W^*(v, \alpha)$ of pairs (u, t) (with the metric $(u_1, t_1)(u_2, t_2) = u_1 u_2 + |t_1 - t_2|$) on $W'(v, \alpha)$ (because the correspondence $(u, t) \to x(u, t)$ is topological for $W(v, \alpha)$). Let $U(v, \alpha)$ be the maximal sphere $S(x(v, \alpha), \varrho)$ in $W(v, \alpha)$ and $U^*(v, \alpha)$ the corresponding set of (u, t) in $W^*(v, \alpha)$. Because of (3.13), (5.3) and (5.4) the set $U(v, \alpha)$ is convex.

The universal covering space will be the set $\bar{R}$ of all pairs (u, t), $u \,\varepsilon\, v$, $t \geq 0$ locally metrized as follows. We put

$$(u_1, t_1)(u_2, t_2) = x(u_1, t_1)\, x(u_2, t_2) \begin{cases} \text{for } 0 \leq t_i < \delta_p \\ (u_i, t_i) \,\varepsilon\, U(v, \alpha). \end{cases}$$

Therefore the curves in $\bar{R}$ get definite lengths. We metrize $\bar{R}$ in the large by defining as distance of two points the greatest lower bound of the lengths of all curves that connect two points. Because of the convexity of the $U(v, \alpha)$ and of $S(p, \delta_p)$ distances already defined do not change. Therefore $(u, t) \to x(u, t)$ is a locally isometric mapping $\bar{R}$ on R.

If $\bar{R}$ is finitely compact then $\bar{R}$ is automatically convex because of the way distances were defined in $\bar{R}$ (see Conditions II and III in Section 1 and [4, p. 219 and p. 248]). Finite compactness will be obvious when it has been shown that for any fixed u the curve (u, t), $t \geq 0$ in $\bar{R}$ is a ray, or, since t is from the definition of distances in $\bar{R}$ the arc length on (u, t), that the arc (u, t), $0 \leq t \leq s$ is for every s a shortest connection of $(u, 0) = \bar{p}$ and (u, s).

Let ϱ be the least upper bound of those s (called admissible) such that for every $u \varepsilon V$ the arc $0 \leq t \leq s$ is a shortest connection of $\bar{p}$ and (u, s). Then $\varrho \geq \delta_p/2$, but we have to prove $\varrho = \infty$.

If $\varrho_{v\alpha}$ is the radius of the sphere $U(v, \alpha)$ in $W(v, \alpha)$ then $\varrho_s = \inf \varrho_{v\alpha}$ for $\alpha \leq s$ and $v \varepsilon V$ is positive. If all $s < s_0$ are admissible then s_0 is admissible. Therefore it suffices to see that with any s the number $s' = s + \varrho_s/8$ is also admissible. For a given $v \varepsilon V$ let $\bar{q} = (v, s')$ and $p_\nu(s)$, $0 \leq \sigma < \sigma_\nu$ a sequence of curves from $\bar{p}$ to $\bar{q}$ referred to the arclength σ as parameter whose length σ_ν tends to the distance $\bar{p}\bar{q}$ of the points $\bar{p}$ and $\bar{q}$ in R. Since the arc (v, t), $0 \leq t \leq s'$ has length s' it may be assumed that $\sigma_\nu \leq s'$.

The curve $p_\nu(\sigma)$ contains a point $p_\nu(\sigma_\nu^0)$ of the form (u_ν, s). If the arc $0 < \sigma < \sigma_\nu^0$ of $p_\nu(\sigma)$ is replaced by the arc (u_ν, t), $0 \leq t \leq s$ the new curve is, because of the admissibility of s, not longer than $p_\nu(\sigma)$. Therefore it may be assumed that $p_\nu(\sigma)$ represents for $0 \leq \sigma \leq s$ an arc (u_ν, t), $0 \leq t \leq s$.

No point of $p_\nu(\sigma)$ with $\sigma \geq s$ can be outside of the sphere $S((v, s), \varrho_s/4)$ in R, since this sphere is congruent to the sphere $S(x(v, s), \varrho_s/4)$ in R and the length of the arc $s \leq \sigma \leq \sigma_\nu$ of $p_\nu(\sigma)$ would be at least $\varrho_s/4$. Consequently, the arc $s \leq \sigma \leq \sigma_\nu$ of $p_\nu(\sigma)$ may be replaced by the segment from $p_\nu(s)$ to q without increasing the length of $p_\nu(\sigma)$. Then a subsequence of the new $p_\nu(\sigma)$ will tend to a curve $\bar{p}(\sigma)$ from $\bar{p}$ to $\bar{q}$ of length $\bar{p}\bar{q} \leq s'$ which consists of an arc (u_ν, t), $0 \leq t \leq s$ and a segment from (u_ν, s) to $\bar{q}$. The minimizing property of $\bar{p}(\sigma)$ (it has length $\bar{p}\bar{q}$) shows that the segment from (u_ν, s) to $\bar{q}$ must be a continuation of the arc (u_ν, t). By construction (u_ν, t) and (v, t) have common points different

The preceding discussion shows that this definition is consistent in the sense that pairs that belong to two neighborhoods have the same distance in both.

from $\bar{p}$ only for $u_v = v$. This shows that (v, t), $0 \leq t \leq s'$ is shortest connection of $\bar{p}$ and (v, s').

Since every half geodesic issuing from p is a ray, every set in $\bar{R}$ can be contracted to $\bar{p}$, hence $\bar{R}$ is simply connected and therefore the (unique, see [4, p. 255]) universal covering space of R. By the motions of $\mathfrak{F}$ the point $\bar{p}$ can be moved into any point over p. Since p was arbitrary in R, the half geodesics issuing from any point of $\bar{R}$ are rays, which implies that $\bar{R}$ is straight. (5.2) yields

(5.6) *A simply connected space with domain invariance and non-positive curvature is straight.*

Since a straight space is not compact we obtain from (5.6) and (2.11)

(5.7) *A compact space with non-positive curvature (and (5.1)) is not simply connected, but has finite connectivity.*

This implies, for instance, that spheres of dimensions $n \geq 2$ and topological products of such spheres (see [15, § 43]) cannot be metrized such that they become G-spaces with non-positive curvature.

6. Motions without Fixed Points in Straight Spaces.

The covering motions of $\bar{R}$ over R have no fixed points· Properties of such motions will therefore be important for the study of R.

The motion Φ without fixed points of a straight space R (not necessarily with non-positivite curvature) is called *axial* if Φ transforms a straight line $\mathfrak{g}$ into itself: $\mathfrak{g}\Phi = \mathfrak{g}$. If $z \varepsilon \mathfrak{g}$ then $(z\, z\Phi\, z\Phi^2)$ otherwise $z = z\Phi^2$ because $z z\Phi = z\Phi z\Phi^2$, but then $m(z, z\Phi) = m(z\Phi, z\Phi^2)$ would be a fixed point of Φ. A line $\mathfrak{g}$ with $\mathfrak{g}\Phi = \mathfrak{g}$ is called an axis of Φ, the orientation $\mathfrak{g}^+$ of $\mathfrak{g}$ for which $z\Phi$ follows z in an oriented axis of Φ. The above argument shows that also $\mathfrak{g}^+\Phi = \mathfrak{g}^+$. Clearly $z z\Phi = z' z'\Phi$ for any two points z, z' of $\mathfrak{g}$.

A characterization of axial motions is contained in

(6.1) **Theorem**. *Let Φ be a motion without fixed points of a straight space R. Then $z z\Phi = \inf_{x \varepsilon R} x x\Phi$, if and only if the points $z\Phi^i$ lie on a straight line.*

Proof. Let $z_i = z\Phi^i$, $z_0 = z$ lie on a straight line $\mathfrak{g}$. Then $\mathfrak{g}$ is an axis of Φ and it follows from the preceding remarks that $z z_n = n z z_1$. Since $x\Phi^i x\Phi^{i+1} = x x\Phi$ for any i

$$n z z_1 = z z_n \leq z x + x x \Phi + x \Phi x \Phi_2 + \cdots + x \Phi^{n-1} x \Phi^n + x \Phi^n z_n = 2 z x + n x x \Phi.$$

For $n \to \infty$ we obtain $z z_1 \leq x x \Phi$.

Let $z z \Phi = \inf x x \Phi$. It suffices to show that $(z z_1 z_2)$. If this were not true let $(z x z_1)$. Then $(z_1 x \Phi z_2)$, but not $(x z_1 x \Phi)$, so that

$$x x \Phi < x z_1 + z_1 x \Phi = z x + x z_1 = z z_1$$

which contradicts the hypothesis.

(6.2) Corollary. If for the motion Φ of a straight space R a point z with $z z \Phi = \inf\limits_{x \varepsilon R} x x \Phi > 0$ exists, then no power of Φ (except the identity) has fixed points.

For $z z \Phi > 0$ implies that Φ has no fixed points. If Φ^i, $i \neq 0$, left the point p fixed then by (6.1) for any j

$$p z = p \Phi^{ij} z \Phi^{ij} = p z \Phi^{ij} \geq z z \Phi^{ij} - p z = |ij| z z \Phi - p z$$

which is impossible.

We put $\inf\limits_{x \varepsilon R} x x \Phi = \lambda(\Phi)$ if this number is positive. By (6.1) points z_1, z_2 on different oriented axes $\mathfrak{g}_1^+, \mathfrak{g}_2^+$ of Φ satisfy the relation $z_1 z_1 \Phi = \lambda(\Phi) = \alpha = z_2 z_2 \Phi$. If $z_i(t)$ represents $\mathfrak{g}_i^+$ then

$$z_i(k\alpha) = z_i(0) \Phi^k, \text{ hence } z_1(0) z_2(0) = z_1(k\alpha) z_2(k\alpha)$$

and for any t, if $k\alpha \leq t < (k+1)\alpha$

$$\begin{aligned} z_1(t) z_2(t) &< z_1(t) z_1(k\alpha) + z_1(k\alpha) z_2(k\alpha) + z_2(k\alpha) z_2(t) \\ &= z_1(0) z_2(0) + 2(t - k\alpha) < z_1(0) z_2(0) + 2\alpha. \end{aligned}$$

Therefore (4.7) yields

(6.3) *If R is straight and has non-positive curvature then two axes of the same axial motion of R are parallel.*

The following simple fact is often useful

(6.4) If Φ is a motion with axis $\mathfrak{g}$ of the straight space R and Ψ is any motion of R, then $\Psi^{-1} \Phi \Psi$ is a motion with axis $\mathfrak{g}\Psi$ and $\lambda(\Psi^{-1} \Phi \Psi) = \lambda(\Phi)$.

For $\Psi^{-1} \Phi \Psi$ has no fixed point and $\mathfrak{g}\Psi(\Psi^{-1}\Phi\Psi) = \mathfrak{g}\Phi\Psi = \mathfrak{g}\Psi$. If $x \varepsilon \mathfrak{g}\Psi$ then $x = z\Psi$ for a suitable $z \varepsilon \mathfrak{g}$ and

$$x x \Psi^{-1} \Phi \Psi = z \Psi z \Psi \Psi^{-1} \Phi \Psi = z \Psi z \Phi \Psi = z z \Phi.$$

Now we come to facts where non-positive curvature is essential. We dispose of the simplest case first

(6.5) If $\Phi \neq E$ is a motion of a straight space with non-positive curvature for which $xx\Phi$ is bounded, then $xx\Phi$ is constant. The points $p\Phi^i$ lie for any p on a straight line L_p. A point q not on L_p determines with L_p a Minkowski plane P and Φ^i is a translation of P along L_p.

Let $x(t)$ represent a straight line, then $x(t)x(t)\Phi$ is bounded and therefore by (4.7) constant. Moreover $x(t)$ is parallel to $x(t)\Phi$. Since any two points can be connected by a line it follows that $xx\Phi$ is constant. This constant is not zero because Φ is not the identity. By (6.1) the points $p\Phi^i$ lie for any p on a straight line L_p. If q is not on L_p then $g(p, q)$ is parallel to $g(p, q)\Phi = \mathfrak{g}(p\Phi, q\Phi)$ and the two lines bound by (4.7) a strip S of a Minkowski plane. (6.3) shows that $\Sigma_i S\Phi_i$ is a Minkowski plane.

Corollary. If, under the assumption of (6.5), the space is two-dimensional, then it is a Minkowski plane.

But the corresponding statement for higher dimensions is false, see the example on pp. 140, 141 in [3].

6.6) For a motion $\Phi \neq E$ of a straight space R with non-positive curvature let a point z and a sequence $\{x_\nu\}$ exist such that $x_\nu x_\nu \Phi$ is bounded, $zx_\nu \to \infty$ and $\mathfrak{g}^+(z, x_\nu)$ converges to a line $\mathfrak{g}^+$. Then Φ transforms any asymptote to g^+ (in particular $\mathfrak{g}^+$ itself) into an asymptote to $\mathfrak{g}^+$.

Note. Φ satisfies the hypothesis if no points z with $zz\Phi = \inf_{x \in R} xx\Phi$ exists.

The proof is simple: Let $x_\nu(t)$ represent $g^+(z, x_\nu)$ with $x_\nu(0) = z$. Then $x_\nu(t)$ tends to a representation $x(t)$ of $\mathfrak{g}^+$ and $y_\nu(t) = x_\nu(t)\Phi \to x(t)\Phi = y(t)$. If $\sup(x_\nu x_\nu \Phi, zz\Phi) = \beta$ then

$$x_\nu(t)y_\nu(t) \leq \beta \quad \text{for} \quad 0 \leq t \leq zx_\nu$$

because $x_\nu(t)y_\nu(t)$ is convex, hence

$$x(t)y(t) \leq \beta \quad \text{for} \quad 0 \leq t < \infty$$

which shows according to (4.4) that $\mathfrak{g}^+\Phi$ is an asymptote to $\mathfrak{g}^+$. Because of the transitivity $\mathfrak{g}^+\Phi^i$ is also an asymptote to $\mathfrak{g}^+$.

If $\mathfrak{h}^+$ is any asymptote to $\mathfrak{g}^+$ then $\mathfrak{h}^+\Phi^i$ is an asymptote to $\mathfrak{g}^+\Phi^i$ and therefore to $\mathfrak{g}^+$.

(6.7) If in addition to the assumptions (6.6) z and $\mathfrak{g}^+$ have the property that $zz\Phi = \inf_{x \varepsilon R} xx\Phi$ and $\mathfrak{g} \neq \mathfrak{g}(z, z\Phi)$ then $\mathfrak{g}(z, z\Phi)$ bounds a Minkowski half plane (imbedded in R).

For, on the one hand $x(t)y(t)$ reaches (with the previous notations) a minimum for $t=0$, on the other hand $x(t)y(t)$ is non increasing because $x(t)$ and $y(t)$ are asymptotes. Hence $x(t)y(t)$ is constant for $t \geq 0$. By (3.14) the rays $x(t), y(t)$, $t \geq 0$ bound together with $T(z, z\Phi)$ a piece V of a Minkowski plane and $\Sigma_i V\Phi^i$ is a Minkowski halfplane.

An application of (6.7) is

(6.8) *An axial motion Φ of a straight space with negative curvature has exactly one axis $\mathfrak{g}$ and $x_\nu x_\nu \Phi \to \infty$ when $x_\nu \mathfrak{g} \to \infty$.*

The uniqueness of the axis follows from (6.3) and (4.7). Assume for an indirect proof that a sequence x'_ν with $x'_\nu \mathfrak{g} \to \infty$ and $x'_\nu x'_\nu \Phi < \alpha$ exists. Let f'_ν be the foot of x'_ν on $\mathfrak{g}$ and b any point of $\mathfrak{g}$. Choose i_ν such that $f_\nu = f'_\nu \Phi^{i_\nu} \varepsilon T(b, b\Phi)$. Then $x_\nu = x'_\nu \Phi^{i_\nu}$ has f_ν as foot on $\mathfrak{g}$ and $x'_\nu f'_\nu = x_\nu f_\nu = x_\nu \mathfrak{g} \to \infty$, moreover $x_\nu x_\nu \Phi = x'_\nu \Phi^{i_\nu} x'_\nu \Phi^{i_\nu+1} = x'_\nu x'_\nu \Phi < \alpha$.

If $\{k\}$ is a subsequence of $\{\nu\}$ for which $f_k \to z$, then $\mathfrak{g}(f_k, x_k)$ and therefore also $\mathfrak{g}(z, x_k)$ tend to the perpendicular L to $\mathfrak{g}$ at z. The assumption of (6.7) is satisfied for L, z and x_k so that R would not have negative curvature.

If R is a plane, the discussion can be carried much farther. Let $\Phi \neq E$ be an axial orientation preserving motion of the plane R with non-positive curvature.

If $\mathfrak{g}$ is an axis and $\mathfrak{h}$ is parallel to $\mathfrak{g}$, then $\mathfrak{h}\Phi = \mathfrak{h}$. For $\mathfrak{h}$ lies on the same side of $\mathfrak{g}$ as $\mathfrak{h}$. If $x \varepsilon \mathfrak{h}$ then $x\Phi^i \mathfrak{g}$ is constant. On the other hand $x\mathfrak{g}$ is constant for $x \varepsilon \mathfrak{h}$, so that $x\Phi^i \varepsilon \mathfrak{h}$.

Let $z(t)$ represent the oriented axis $\mathfrak{g}$ and $y(s)$ the perpendicular to $\mathfrak{g}$ at $z(t)$ oriented to the »right» of $\mathfrak{g}$ with $y_t(0) = z(t)$. Then every point in the plane has two coordinates s, t and Φ is the transformation

$$s' = s, \quad t' = t + \lambda(\Phi).$$

The parallels to $\mathfrak{g}$ have equations $s = \text{const}$. If α is a non-positive and β a non-negative number, then the set W of those values of s for which $s = \text{const}$ is a parallel to $\mathfrak{g}$ has one of the following forms: $\alpha \leq s \leq \beta$, $\alpha \leq s < \infty$, $-\infty \leq s \leq \beta$, $-\infty < s < \infty$. If R has negative curvature, then $\alpha = \beta = 0$ by (4.7) and (6.3). In the other cases the set of points (s, t) with $s \varepsilon W$ is a piece of a Minkowski plane.

(6.9) An asymptote to $s=\alpha\,(s=\beta)$ through a point (s', t') with $s'<\alpha\,(s'>\beta)$ has distance 0 from $s=\alpha\,(s=\beta)$.

Proof. Let $\mathfrak{g}_\alpha^+$ denote the oriented axis $s=\alpha$ of Φ, and let $\mathfrak{h}$ be asymptote to $\mathfrak{g}_\alpha^+$ through (s', t'). $\mathfrak{h}\Phi^i$ lies for $i<j$ between $\mathfrak{h}\Phi^j$ and $\mathfrak{g}_\alpha^+$. For $i\to-\infty$ the line $\mathfrak{h}\Phi^i$ tends therefore to an asymptote $\mathfrak{h}'$ to $\mathfrak{g}_\alpha^+$. But $\mathfrak{h}'$ is invariant under all Φ^i and therefore a parallel to $\mathfrak{g}_\gamma$. The definition of α shows that $\mathfrak{h}'=\mathfrak{g}_\alpha$. Since $\mathfrak{h}\Phi^i\mathfrak{g}_\alpha=\mathfrak{h}\mathfrak{g}_\alpha$ it follows that $\mathfrak{h}\mathfrak{g}_\iota=0$.

Let Ψ be an axial motion that reverses the orientation. Then $\Psi^2=\Phi$ preserves it. The preceding considerations apply to Φ but it is easily seen that W must be symmetric to the (0, 0), that is it has either the form $-\beta\leq s\leq\beta$ or, $-\infty<s<\infty$. The analytic expression for Ψ is

$$s'=-s,\quad t'=t+\lambda(\Psi).$$

Following Nielsen [14, pp. 198—199] we prove:

(6.10) If Φ and Ψ are orientations preserving axial transformations of a plane with negative curvature whose orientated axes $\mathfrak{g}^+$ and $\mathfrak{h}^+$ are asymptotes, then the commutator $\Psi^{-1}\Phi^{-1}\Psi\Phi$ is non-axial.

By (4.4) $\Phi^*=\Psi^{-1}\Phi^{-1}\Psi$ has axis $\mathfrak{g}^-\Psi$ with $\lambda(\Phi^*)=\lambda(\Phi)$. Now $x\Phi^*x\Phi^*\Phi\geq$ $\geq\lambda(\Phi)$, but $x\Phi^*x\Phi^*\Phi\to\lambda(\Phi)$ when $x\Phi^*\mathfrak{g}\to 0$. By (6.9) all asymptotes to $\mathfrak{g}$ and $\mathfrak{h}$ have distance 0 from each other. Therefore, as x traverses $\mathfrak{g}^+$ in the positive direction $x\mathfrak{g}\Psi\Phi\to 0$ and since $x\Phi^*$ lies on $\mathfrak{g}\Psi\Phi$ it follows that $x\Phi^*x\Phi^*\Phi\to 0$. Taking orientation into account we see that $xx\Phi^*\Phi\to 0$, which shows that $\inf xx\Phi^{-1}\Phi^{-1}\Psi\Phi=0$ so that the commutator cannot be axial.

7. Geodesic Connections and Closed Geodesics.

In a G-space let $p(t)$ and $q(t)$ be two continuous curves defined for the same connected set M_t of values t. Denote generally by $p(t_1, t_2)$ the subarc $t_1\leq t\leq t_2$ of $p(t)$ and by $p(t_2, t_1)$ the same arc with the opposite orientation. Two continuous curves $\mathfrak{c}_i$, $i=1, 2$, connecting $p(t_i)$ to $g(t_i)$ are said to be *homotopic along* (p, q) if

$$\mathfrak{c}_1\,p(t_1\,t_2)\,\mathfrak{c}_2^{-1}\,q(t_2, t_1)\sim 0. \tag{7.1}$$

Since then also

$$\mathfrak{c}_2\,p(t_2, t_1)\,\mathfrak{c}_1^{-1}\,q(t_1, t_2)\sim 0$$

the concept is symmetric and transitive (and, of course, reflexive).

(7.2) Let $\bar{R}$ be straight (see Notations). If $p(t)$, $q(t)$, $t \varepsilon M_t$, are two continuous curves in R and $\mathfrak{c}$ a curve connecting $p(t_0)$ to $q(t_0)$, then for every $t \varepsilon M_t$ exactly one geodesic arc $\mathfrak{g}_t \sim \mathfrak{c}$ along (p, q) exists and $\mathfrak{g}_t$ depends continuously on t.

For let an arbitrary continuous curve $\bar{\mathfrak{c}}$ over $\mathfrak{c}$ begin at $\bar{p}$ and end at $\bar{q}$. Choose $\bar{p}(t)$ with $\bar{p}(t)\Omega = p(t)$ and $\bar{p}(t_0) = \bar{p}$, similarly $\bar{q}(t)$ with $\bar{q}(t)\Omega = q(t)$ and $\bar{q}(t_0) = q$. Because $\bar{R}$ is simply connected the segment $\bar{\mathfrak{s}}_t = \mathfrak{s}(\bar{p}(t), \bar{q}(t))$ satisfies the relation $\bar{\mathfrak{s}}_{t_0} \sim \bar{\mathfrak{c}}$ and for t_1, t_2

$$\bar{\mathfrak{s}}_t \bar{p}(t_1, t_2) \bar{\mathfrak{s}}_t^{-1} \bar{q}(t_2, t_1) \sim 0.$$

Therefore $\bar{\mathfrak{s}}_t \Omega = \mathfrak{g}_t$ is a geodesic arc in R with $\mathfrak{g}_{t_0} \sim \mathfrak{c}$ and $\mathfrak{g}_t \sim \mathfrak{c}$ along (p, q). For a given $t_1 \varepsilon M_t$ let $\mathfrak{g}_1$ be any geodesic arc connecting $p(t_1)$ to $q(t_1)$ which is homotopic to $\mathfrak{c}$ along (p, q). Then $\mathfrak{g}_1 \sim \mathfrak{g}_{t_1}$. Therefore the arc $\bar{\mathfrak{g}}_1$ over $\mathfrak{g}_1$ which begins at $\bar{p}(t_1)$ ends at $\bar{q}(t_1)$. Moreover, $\bar{\mathfrak{g}}_1$ is a geodesic arc and must coincide with $\bar{\mathfrak{s}}_{t_1}$ because $\bar{R}$ is straight. Therefore $\mathfrak{g}_1 = \mathfrak{g}_{t_1}$.

The last consideration, or the special case where $p(t)$ and $g(t)$ are constant, yield

(7.3) *If $\bar{R}$ is straight then for arbitrary p, q in R every class of homotopic curves from p to q contains exactly one geodesic arc.*

In the remainder of this section and in the next we assume that, whenever a space R of non-positive curvature is considered, R has the property of domain invariance. Then $\bar{R}$ is by (5.2) straight and (4.1), (4.2) and (7.2) yield.

(7.4) Let $x(t)$ represent a geodesic in a space of non-positive curvature, and let $\mathfrak{c}$ be a curve from the (arbitrary) point p to $x(t_0)$. Then exactly one geodesic arc $\mathfrak{g}_t \sim \mathfrak{c}$ along (p, x) from p to $x(t)$ exists and the length of $\mathfrak{g}_t$ is a strictly convex function of t.

(7.5) If $x(t)$ and $y(t)$ represent geodesics in a space with non-positive (negative) curvature and $\mathfrak{c}$ is a curve from $x(t_0)$ to $y(t_0)$ then exactly one geodesic arc $\mathfrak{g}_t \sim \mathfrak{c}$ along (x, y) exists and the length of $\mathfrak{g}_t$ is a (strictly) convex function of t.

A geodesic one-gon of length $|\alpha| > 0$ is a geodesic arc $x(t, t + \alpha)$ with $x(t) = x(t + \alpha)$. A closed geodesic of length $|\alpha|$ is a geodesic for which $x(t + \alpha) \equiv x(t)$ (that is $x(t + \alpha) = x(t)$ for all t. If this is true for one representation, it is true for all). If also $\beta \neq 0$, α and $x(t + \beta) \equiv x(t)$ then we consider the corresponding closed geodesic as different from the first. For $\beta = i\alpha$ we say that the second geodesic is i times the first. (7.3) yields

(7.6) *If $\bar{R}$ is straight, then R contains no geodesic one-gons or closed geodesics which are homotopic to* 0.

There is a well known one-to-one correspondence between the classes of freely homotopic curves (free homotopy classes) in R and classes of conjugate elements in the fundamental group $\mathfrak{F}$ of R (compare [15, § 49]. It may be briefly described as follows: Let $c(s)$ be a closed curve $0 \leq s \leq \alpha$, $c(0) = c(\alpha)$. If $\bar{c}_0$ lies over $c(0)$ let $\bar{c}(s)\Omega = c(s)$ with $\bar{c}(0) = \bar{c}_0$. Then $\bar{c}(\alpha)$ lies over $c(\alpha) = c(0)$, hence a motion $\Phi(c, \bar{c}_0)$ in $\mathfrak{F}$ exists with $\bar{c}(\alpha) = \bar{c}_0 \Phi(c, \bar{c}_0)$. If $\bar{c}_0'$ is another point over $c(0)$, then $\bar{c}_0' = \bar{c}_0 \Psi$ for a suitable Ψ in $\mathfrak{F}$ and $\Phi(c, \bar{c}_0') = \Psi^{-1}\Phi(c, c_0)\Psi$. When $\bar{c}_0'$ traverses the points over $c(0)$, then Ψ traverses $\mathfrak{F}$.

Conversely, if $\Phi \varepsilon \mathfrak{F}$ and $\bar{c}_0 \Phi = \bar{c}_1$ and $\bar{c}(s)$ is a curve from $\bar{c}_0$ to $\bar{c}_1$, then $\bar{c}(s)\Omega = c(s)$ is a closed curve in R and the class of conjugate elements in $\mathfrak{F}$ determined by $c(s)$ contains Φ. The identity of $\mathfrak{F}$ belongs to all curves that can be contracted to a point.

Standard arguments furnish the following facts

(7.7) If (in a G-space) a free homotopy class $(\neq 0)$ contains a shortest curve $\mathfrak{c}$, then $\mathfrak{c}$ is a closed geodesic.

(7.8) If the free homotopy class K either does not contain curves outside $S(p, \nu)$ for large ν, or the length of curves in K that contain points outside of $S(p, \nu)$ tends with ν to ∞, then K contains a closed geodesic.

In particular, if the space is compact, then every class K contains closed geodesics. The following criterion is of importance for general spaces:

(7.9) *Let $\bar{R}$ be straight. The free homotopy class K belonging to a given element $\Phi \neq E$ of $\mathfrak{F}$ contains a closed geodesic if and only if Φ is an axial motion. The closed geodesics in K are the images of the axes of Φ and have length $\lambda(\Phi)$.*

Proof. Let K contain the closed geodesic $\mathfrak{g}$: $x(t)$, $x(t+\alpha) \equiv x(t)$. For a suitable point $\bar{x}_0$ over $x(0)$ the motion Φ will be the motion $\Phi(x, \bar{x}_0)$ defined above. Choose $\bar{x}(t)$ such that $\bar{x}(t)\Omega = x(t)$ and $\bar{x}(0) = \bar{x}_0$. Call L a segment with center $x(0)$ that is represented by $x(t)$, for $|t| \leq \beta$, where $\beta = \min(\eta_2(x(0)), 1)$, and $\bar{L}$ the segment over L with center $\bar{x}_0$. Since Φ lies over the identity of R (i.e. $\Phi\Omega = I$) it carries $\bar{L}$ into a segment $\bar{L}'$ over L through $x(0)$.

The straight line $\bar{\mathfrak{g}}(=\bar{x}(t))$ that contains $\bar{L}$ goes under Φ into the line $\bar{\mathfrak{g}}'$ through $\bar{L}'$. Since $\bar{\mathfrak{g}}$ lies over $\bar{\mathfrak{g}}$ it contains the segment $\bar{L}''$ through $\bar{x}(0)\Phi = \bar{x}(\alpha)$ that lies over the segment L'' represented by $x(t+\alpha)$ for $|t| \leq \beta$. But $L'' = L$ because $\mathfrak{g}$ is closed, hence $\bar{L}' = \bar{L}''$ or $\bar{\mathfrak{g}} = \bar{\mathfrak{g}}\Phi$, which shows that Φ is axial and that $\lambda(\Phi)$ is the length of $\mathfrak{g}$.

The sufficiency of the condition that Φ is axial can be proved by retracing our steps: Let $\bar{x}, \bar{x}\Phi, \bar{x}\Phi^2$ be on a straight line $\mathfrak{g}$ represented by $\bar{x}(t)$ with $\bar{x}(0)=\bar{x}, \bar{x}(\alpha)=\bar{x}\Phi, \alpha=\lambda(\Phi)$. Let $\mathfrak{g}$ be the geodesic $x(t)=\bar{x}(t)\Omega$ and L the segment represented by $x(t)$ for $|t|\leq\beta$. As t goes from 0 to α the point $x(t)$ and L go into $x(\alpha)$ and the segment L'' represented by $x(t+\alpha)$ for $|t|\leq\beta$. If $\bar{L}''$ is the segment over L'' with center at $\bar{x}\Phi$ and $\bar{L}'$ is the segment of length 2β on $\mathfrak{g}$ with center $\bar{x}\Phi$, then $\bar{\mathfrak{g}}\Phi=\bar{\mathfrak{g}}$ implies $\bar{L}''=\bar{L}'$, hence $L=L''$ so that $\mathfrak{g}$ is closed ($x(t+\alpha)=x(t)$ for $|t|\leq\beta$ implies $x(t+\alpha)\equiv x(t)$, see [4, (5.6)]).

This proof shows also that Ω maps the axes of Φ into the closed geodesics freely homotopic to $\mathfrak{g}$.

(7.10) If $\bar{R}$ is straight then the shortest curves in a free homotopy class K of R coincide with its closed geodesics.

By (7.5) every shortest curve in K is a closed geodesic without the assumption that $\bar{R}$ is straight. This assumption is essential for the converse, as any ellipsoid of revolution $a^{-2}(x^2+y^2)+c^{-2}z^2=1$ shows, where the meridians $y=mx$ are closed geodesics but homotopic to 0.

Let $\mathfrak{g}$ be any closed geodesic and $c(s)$ any closed curve freely homotopic to $\mathfrak{g}$. Choose a point $\bar{c}_0$ over $c(0)$ and define $\Phi(c, \bar{c}_0)$ as above. Since the class contains $\mathfrak{g}$ the motion $\Phi(c,\bar{c}_0)$ leaves a line $\bar{\mathfrak{g}}$ over $\mathfrak{g}$ invariant. Then for $\bar{z}\,\varepsilon\,\bar{\mathfrak{g}}$ the length of $\mathfrak{g}$ is $\lambda(\Phi)=\bar{z}\bar{z}\Phi$ and c has by (6.1) at least length $\bar{c}_0\bar{c}_0\Phi\geq\bar{z}\bar{z}\Phi$.

(6.8) and (7.9) yield the following basic result for spaces of negative curvature.

(7.11) *In a space of negative curvature every free homotopy class K contains at most one closed geodesic $\mathfrak{g}$. The length of a geodesic one-gon in K tends to ∞ when the distance of its vertex from $\mathfrak{g}$ tends to ∞.*

8. **Asymptotic Geodesics.**

The oriented geodesics $\mathfrak{g}^+$ and $\mathfrak{h}^+$ in a space R with non-positive curvature and domain invariance are called *asymptotes to each other* if straight lines $\bar{\mathfrak{g}}^+$ and $\bar{\mathfrak{h}}^+$ in $\bar{R}$ over $\mathfrak{g}^+$ and $\mathfrak{h}^+$ exist which are asymptotes to each other. Following Hadamard [9] this may be formulated without using $\bar{R}$ as follows:

Let $x(t)$ represent $\mathfrak{g}^+$ and connect a given point p to $x(0)$ by a curve $\mathfrak{c}$. Let $\mathfrak{h}_s$ be the (unique, see (7.3)) geodesic arc from p to $x(s)$ which is homotopic to $\mathfrak{c}x(0,s)$. If $y_s(t)$ represents the geodesic which contains $\mathfrak{h}_s$ and for which $y_s(0)=p$, y_s (length $\mathfrak{h}_s$) $=x(s)$, then $y_s(t)$ tends for $s\to\infty$ to an asymptote $y(t)$

to $\mathfrak{g}^+$, more precisely the *asymptote* $\mathfrak{h}^+$ *to* $\mathfrak{g}^+$ *through* p *of type* $\mathfrak{c}$. Whereas $\mathfrak{c}$ determines $\mathfrak{h}^+$ uniquely, $\mathfrak{h}^+$ may not determine the type $\mathfrak{c}$ uniquely. For instance, parallel generators on a cylinder are asymptotes to each other of infinitely many types.

Let $x(t), y(t)$ represent geodesics in R and let $\mathfrak{c}$ connect $x(0)$ to $x(0)$. If $\bar{\mathfrak{c}}\,\Omega = c$ and $\bar{\mathfrak{c}}$ begins at $\bar{p}$ and ends at $\bar{x}$, let $\bar{x}(t)\,\Omega = x(t)$ with $\bar{x}(0) = x$. If $\bar{x}(\pi(t))$ is the foot of $\bar{y}(t)$ on $\bar{x}(t)$, then $T_t = \overline{T}[\bar{y}(t), \bar{x}(\pi(t))]\,\Omega$ is a geodesic arc in R with the property that $y(0, t)\,T_t \sim \mathfrak{c}\,x(0, \pi(t))$ and that T_t is perpendicular to $x(t)$ at $x(\pi(t))$. The arc T_t is uniquely determined by this property. The length $y(t)\,x(\pi(t))$ of T_t is called the distance $\delta(\mathfrak{c}; y(t), \mathfrak{g}^+)$ of type c from $y(t)$ to the geodesic $\mathfrak{g}^+$ represented by $x(t)$. Then (4.2) and (4.4) yield

(8.1) Let $x(t)$ and $y(t)$ represent oriented geodesics $\mathfrak{g}^+$ and $\mathfrak{h}^+$ in R and let c connect $y(0)$ to $x(0)$. Then $\delta(\mathfrak{c}; y(t), \mathfrak{g}^+)$ is a convex function of t and $\mathfrak{h}^+$ is an asymptote to $\mathfrak{g}^+$ of type $\mathfrak{c}$ if and only if $\delta(\mathfrak{c}; y(t), \mathfrak{g}^+)$ is bounded for $t \geq 0$.

If the curvature is negative, the following can be added

(8.2) *If* $y(t)$ *represents an asymptote of type* $\mathfrak{c}$ *to* $\mathfrak{g}^+$ *in a space with negative curvature and* $y(0)\,y(t) < \alpha < \infty$, *then* $\delta(\mathfrak{c}; y(t), \mathfrak{g}^+) \to 0$.

Since $\delta(\mathfrak{c}; y(t), \mathfrak{g}^+)$ decreases it tends for $t \to \infty$ to a limit δ. For a fixed $\beta > 0$ consider the geodesic arcs $T_{t-\beta}$, T_t, $T_{t+\beta}$ defined above. Their length tends to δ. For a suitable sequence $t_\nu \to \infty$ the arcs $y(t_\nu - \beta, t_\nu + \beta)$ tend because of $y(0)\,y(t) < \alpha$ to an arc of the form $z(t_0 - \beta, t_0 + \beta)$ of a geodesic $z(t)$, and $x(\pi(t_\nu - \beta), \pi(t_\nu + \beta))$ tends to an arc of a geodesic $\mathfrak{K}^+$. The limits of $T_{t_\nu - \beta}$, T_{t_ν}, $T_{t_\nu + \beta}$ are geodesic arcs of length β perpendicular to $\mathfrak{K}^+$ and of the same type $\mathfrak{c}'$. The function $\delta(\mathfrak{c}'; z(t), \mathfrak{K}^+)$ would be linear in the interval $t_0 - \beta \leq t \leq t_0 + \beta$. By (3.14) and (3.22) the space could not have negative curvature.

The following theorem was proved by Hadamard (see [9, pp. 42, 65, 66]) by using strongly the Riemannian character of his metric:

(8.3) *In a space of negative curvature let* $y(t)$ *represent an asymptote to* $x(t)$ *both of type* $\mathfrak{c}_1$ *and of type* $\mathfrak{c}_2$, *where* $\mathfrak{c}_1$ *and* $\mathfrak{c}_2$ *connect* $x(0)$ *to* $y(0)$ *and* $\mathfrak{c}_1 \neq \mathfrak{c}_2$.

If $y(0)\,y(t) < \alpha$ *and the free homotopy class of* $\mathfrak{c}_1\mathfrak{c}_2^{-1}$ *contains a closed geodesic* $\mathfrak{g}$, *then* $x(t)$ *and* $y(t)$ *are asymptotes to a suitable orientation of* $\mathfrak{g}$.

Proof. Let T_{it} be perpendiculars of type $\mathfrak{c}_i$ from $y(t)$ to $x(t)$ and $x(\pi_i(t))$ the endpoint of T_{it}. Then

$$T_{1t}\,x[\pi_1(t), \pi_2(t)]\,T_{2t}^{-1} \sim \mathfrak{c}_1\,\mathfrak{c}_2^{-1} \neq 0.$$

The curve on the left side has by (7.10) at least the length 2ϱ of $\mathfrak{g}$. By (8.2) the length $\delta(c_i;\, y(t), x)$ of T_{it} tends to zero when $t \to \infty$, hence

$$x[\pi_1(t)]\, x[\pi_2(t)] \to 0 \text{ for } t \to \infty \text{ and } |\pi(t_1) - \pi(t_2)| > \varrho \text{ for large } t.$$

The inequality $y(0)y(t) < \alpha$ implies $y(0)x(t) < 2\alpha$ for large t. Therefore a sequence $t_\nu \to \infty$ exists such that $x(\pi_1(t_\nu) + s)$ tends for all s to a representation $z(s)$ of a geodesic (see [3, pp. 22, 23]). The argument converse to the reasoning that leads to (7.11) yields that $\pi_2(t_\nu)$ converges to a finite value s_2 for which the arc $z(0, s_2)$ is a geodesic one-gon freely homotopic to a suitable orientation $\mathfrak{g}^+$ of $\mathfrak{g}$. We may assume that $s_2 > 0$. Let

$$\pi_1(t'_\nu) = (1 - \theta)\, \pi_1(t_\nu) + \theta\, \pi_2(t_\nu).$$

Then $\pi_1(t'_\nu) \to \theta s_2$ and $\pi_2(t'_\nu)$ tends to a value s_3 such that $z(\theta s_2, s_3)$ is another geodesic one-gon freely homotopic to $\mathfrak{g}^+$.

Hence $\mathfrak{g}^+$ has a multiple point at θs_2 unless the line elements of $\mathfrak{g}^+$ at θs_3 coincide. But $\mathfrak{g}^+$ has only a countable number of multiple points (see [4, p. 231]). Therefore there are θ for which the line elements coincide, and $z(s)$ is a closed geodesic. By (7.11) it must represent $\mathfrak{g}^+$.

It follows easily that $x(t)$ and $y(t)$ are asymptotes to $\mathfrak{g}^+$.

The connection between co-rays and asymptotic geodesics, which is entirely clear for simply connected spaces, *is obscure for general spaces* and it seems difficult to find a general theorem. The following questions suggest themselves (compare the end of Section 1).

Let $x(t)$ and $y(t)$ represent oriented asymptotic geodesics which are rays for $t \geq 0$. Are these then co-rays to each other?

The converse is certainly in general not true, that is, rays which are co-rays to each other in a space of non-positive curvature need not belong to asymptotic geodesics. The following is an example:

In the Cartesian (x, y, z)-space let D_n denote the disk in the (x, y)-plane punctured at the center defined by

$$D_n : (x - 2n + 1)^2 + y^2 < 5^{-1} n^{-2}, \quad (x, y) \neq (2n - 1, 0).$$

Define $f(x, y)$ by

$$f(x, y) = \begin{cases} 0 \text{ if } (x, y) \text{ is not in } \Sigma_n D_n \text{ or } (x, y) \neq (2n - 1, 0) \\ \tan\, [(5^{-1} n^{-2} - (x - 2n + 1)^2 - y^2)^2 \cdot \pi\, 50^{-1} n^{-2}] \text{ if } (x, y) \in D_n. \end{cases}$$

Since $\tan[2^{-1}\pi(x-a)^4(b-a)^{-4}]$ is convex for $a \leq x < b$ and tends to ∞ for $x \to b-$ the surface $S: z = f(x, y)$ consists of the (x, y)-plane except at the disks D_n, where infinite tubes of negative curvature are erected. The surface is of class C^3 because the first three derivatives of $\tan x^4$ vanish at 0. With the ordinary geodesic distance s becomes a surface of non-positive curvature.

The numbers have been chosen such that the straight lines through $p = (0, 1, 0)$ and $(2n, 0, 0)$ has no common point with D_n and lies therefore in S. The limit of these lines is $q(t) = (t, 1, 0)$ and the ray $t \geq 0$ of this line is a co-ray to every ray $q_\alpha(t) = (t, \alpha, 0)$ with $\alpha \leq -1$, $t \geq 0$. But the oriented geodesics represented by $q(t)$ and $q_\alpha(t)$ are not asymptotes to each other because no two segments from p through $(2n, 0, 0)$ to the point $(2n(1-\alpha), \alpha, 0)$ of $q_\alpha(t)$ are of the same type.

It seems probable that the infinite connectivity of S is essential for such examples, but the question is open and worth investigating.

9. Spaces with Curvature 0.

Any attempt to enumerate the different types of spaces with non-positive curvature is futile because the n-dimensional manifolds have not even been classified topologically for $n \geq 3$. Therefore the attention must be restricted to some special types of spaces. The following general fact follows from (4.11).

(9.1) *The fundamental group of a space with non-positive curvature has no finite subgroup* (except the group consisting of the identity).

In a space with curvature 0 it follows from (3.3), (3.13) and (3.14) that the metric in any triangle with vertices in $S(p, \delta_p)$ is Minkowskian.[11] Therefore the geometry is locally Minkowskian. The space is a manifold and domains are invariant. Therefore

(9.2) *If R has curvature 0, then $\bar{R}$ is a Minkowski space.*

The study of spaces with curvature 0 can be reduced to the study of Riemann spaces with curvature 0 by means of the following fact:

(9.3) *For a given Minkowski space M there is an associated*[12] *euclidean space S for which all Minkowskian motions are euclidean motions.*

Let K be the Minkowskian unitsphere in a definite associated euclidean

[11] The long proof of (3.14) is not necessary in this case, because much more is known than the hypotheses of (3.14).

[12] Compare [5, Section 2].

space S'. According to an unpublished result of Loewner there exists exactly one ellipsoid E with the origin as center of smallest (Minkowskian [12] or euclidean) volume that contains K.[13]

Any motion of M carries Minkowskian, and therefore euclidean straight lines into straight lines and is therefore an affine mapping of the euclidean space. This mapping preserves the euclidean volume, since it preserves the Minkowskian volume.

Any motion Φ of M can be composed of a motion Φ' that leaves the origin o fixed and a translation Ψ (if Ψ is a translation that carries o into oΦ, then $\Phi = (\Phi\Psi^{-1})\Psi$ and $\Phi' = \Phi\Psi^{-1}$). Because Φ' is an affine mapping that preserves volume and carries K into itself, E goes into itself. If S is the euclidean space with E as unitsphere, then any motion of M which leaves o fixed is also a motion of S. Since the translations of M are also translations for S, the theorem is proved. Applying (9.3) to the motions of the fundamental group of a space with curvature o yields (compare [4, (13.8)]):

(9.4) *Every G-space R of curvature* o *can be metrized as a Riemann space of curvature* o *such that* S *has the same geodesics as R and every motion of R is a motion of S.*

But it is important to notice that *not every locally euclidean space can be realized by a given Minkowki metric.* For instance, the two-dimensional locally euclidean spaces belong to the following five topological types (see [10], or [7, Chapter II, Section VII]):

The plane, the cylinder, the torus, the Moebius strip, and the one-sided torus or Klein bottle.

The covering transformations of the plane over a cylinder or a torus consist of translations and can therefore be realized with any Minkowski metric. But the covering transformations for *the Moebius strip and the one-sided torus* contain products of translations and reflections in a line, and *can therefore be realized by a given Minkowski metric, only if this metric admits a reflection in some straight line.*

More generally, products of n circles and straight lines can be provided with any Minkowski metric, but other n-dimensional types cannot be realized by arbitrary Minkowski metrics.

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[13] While this paper was in print a proof was given by M. M. DAY in: Some characterizations of inner-product spaces, Trans. Am. Math. Soc. vol. 62, 1947, pp. 320—337.

The geodesics in a Moebius strip R have properties which will illustrate certain statements of the next section. The fundamental group of the Minkowskian covering space $\bar{R}$ is cyclic and is with a suitable associated euclidean rectangular coordinate system generated by the motion

$$\Psi: \qquad \bar{x}' = -\bar{x},\ \bar{y}' = \bar{y} + a,$$

where the curves $\bar{y} = \text{const}$ are perpendicular to $\bar{x} = 0$ (and $a = \lambda(\Psi)$, (compare section 6). The line $x = 0$ is the only axis. The interval $0 \leq \bar{y} \leq a$, $\bar{x} = 0$ goes into a great circle $\mathfrak{g}$ in R. The intervals $0 \leq \bar{y} \leq 2a$, $\bar{x} = k \neq 0$ go into closed geodesics of length $2a$ homotopic to $2\mathfrak{g}$.

Rays $\bar{y} = k$ and $\bar{x} \geq 0$, or $\bar{y} = k$ and $\bar{x} \leq 0$ go into rays in R, but the whole line $\bar{y} = k$ does not go into a straight line in R, because points of the form $(\bar{x}_0, \bar{y}_0)$, $(\bar{x}_0, \bar{y}_0 + a)$ can for large $\bar{x}_0$ be connected by curves which are shorter than the interval from $(-\bar{x}_0, \bar{y}_0 + a)$ to $(\bar{x}_0, \bar{y}_0 + a)$.

Spaces with non-positive curvature which are not compact need not have finite connectivity as the example in Section 8 shows; but

(9.5) *Spaces of curvature* 0 *have finite connectivity.*

A proof for three-dimensional Riemann spaces of curvature 0, which extends to n dimensions is found in Cartan [7, pp. 75, 76]. (9.4) shows then that (9.5) holds also for locally Minkowskian spaces.

For certain spaces non-positive curvature implies vanishing curvature:

(9.6) *A torus of non-positive curvature has curvature* 0.

The proof rests on the following fact, which will be used again later on:

(9.7) If the fundamental group of a compact space with non-positive curvature is abelian, then $\bar{y}\bar{y}\Phi$ is bounded for a fixed $\Phi \varepsilon \mathfrak{F}$ and all $\bar{y} \varepsilon \bar{R}$.

Because R is compact any fundamental set $F(\bar{p})$ (see (2.10 d)) is compact, hence $\bar{x}\bar{x}\Phi$ is bounded for $\bar{x} \varepsilon F(\bar{p})$. If $\bar{y}$ is any point in $\bar{R}$ then a point $\bar{x}$ in $F(\bar{p})$ and an element Ψ of $\mathfrak{F}$ exists with $\bar{x}\Psi = \bar{y}$. Because $\mathfrak{F}$ is abelian

$$\bar{y}\bar{y}\Phi = \bar{x}\Psi\,\bar{x}\Psi\Phi = \bar{x}\Psi\,\bar{x}\Phi\Psi = \bar{x}\bar{x}\Phi,$$

so that $\bar{y}\bar{y}\Phi$ is bounded.

If $\Phi \neq E$, then $\bar{y}\bar{y}\Phi$ is by (6.5) constant, every point of $\bar{R}$ is on an axis of Φ and these axes form a family of parallel lines.

If R is a torus we may represent R as a product of n circles and (because $\mathfrak{F}$ is abelian and conjugate elements are equal) these circles may be chosen as

n closed geodesics, $G_1, \ldots, G_n$. The sub product $G_1 \times G_2$ is a torus with a Minkowski metric (see (6.5)), applying (6.5) again it follows that $(G_1 \times G_2) \times G_3$ is a Minkowski space, etc.

10. Spaces with Cyclic Fundamental Groups.

Statements (9.7), (6.5) and (5.7) yield

(10.1) *There is no compact space of negative curvature with domain invariance and an abelian fundamental group.*

Since group spaces have abelian fundamental groups (10.1) implies the following two facts:

A compact group space cannot carry a metric with negative curvature (whether invariant under the group or not).

There is no compact space with negative curvature and a simply transitive group of motions.

The hyperboloid of one sheet shows that the assumption of compactness in (10.1) is essential. However, the hyperboloid is in a certain sense typical.

(10.2) *If R contains a closed geodesic $\mathfrak{g}$, has negative curvature and an abelian fundamental group, then $\mathfrak{F}$ is cyclic. All closed geodesics in R are multiples of a great circle.*

Proof. Let Φ be a motion in $\mathfrak{F}$ with an axis $\bar{\mathfrak{g}}$ over $\mathfrak{g}$ (compare (7.9)) and $\Psi \neq E$ any motion in $\mathfrak{F}$. Then $\Psi^{-1}\Phi\Psi$ is by (4.4) an axial motion and has $\bar{\mathfrak{g}}\Psi$ as axis. But $\Psi^{-1}\Phi\Psi = \Phi$ and Φ has only one axis (see (4.8)), hence $\bar{\mathfrak{g}}\Psi = \bar{\mathfrak{g}}$; that is all elements of $\mathfrak{F}$ have $\bar{\mathfrak{g}}$ as axis. Because $\mathfrak{F}$ is discrete there is an element $\Phi_0 \neq E$ in $\mathfrak{F}$ for which $\lambda(\Phi_0)$ is minimal. If $\Psi \neq E$ and $\bar{z} \varepsilon \bar{\mathfrak{g}}$ then $\lambda(\Psi) = \bar{z}\bar{z}\Psi$ is an integral multiple of $\lambda(\Phi_0)$. Therefore i exists such that $\bar{z}\Phi_0^i = \bar{z}\Psi$ or $\bar{z}\Phi_0^i\Psi^{-1} = \bar{z}$. Since E is the only motion in $\mathfrak{F}$ with fixed points it follows that $\Phi_0^i = \Psi$. This shows that $\mathfrak{F}$ is cyclic and that all closed geodesics are multiples of the geodesic $\mathfrak{g}_0$ that corresponds to Φ_0. The following discussion of a more general case will show that $\mathfrak{g}_0$ is a great circle.

Let Φ be an axial motion of a simply connected space $\bar{R}$ of non-positive curvature. Since no power $(\neq E)$ of Φ has fixed points (see (6.2)) the cyclic group $\{\Phi^i\}$ is the fundamental group of a space R with $\bar{R}$ as universal covering space.

(10.3) If $\bar{\mathfrak{g}}$ is an axis of Φ, then $\mathfrak{g} = \bar{\mathfrak{g}}\Omega$ is a great circle. The image $\bar{\mathfrak{h}}\Omega$ of a perpendicular $\bar{\mathfrak{h}}$ to $\bar{\mathfrak{g}}$ at a point $\bar{y}$ is a geodesic $\mathfrak{h}$ whose halfextremals $\mathfrak{h}^+, \mathfrak{h}^-$

with origin $y = \bar{y}\,\Omega$ are rays. $\mathfrak{h}$ then is locus of all points that have y as foot on $\mathfrak{g}$.

Proof. We prove a little more than that $\mathfrak{g}$ is a great circle, namely that for any two different points $x, z \,\varepsilon\, \mathfrak{g}$ *every* segment $T(x, y)$ lies on $\mathfrak{g}$. ($\mathfrak{g}$ is a great circle if a *suitable* segment $T(x, y)$ lies on $\mathfrak{g}$, see [4, p. 232].) Let $\bar{x}\,\Omega = x$. The segment over $T(x, y)$ that begins at $\bar{x}$ ends at a point $\bar{y}$ over y, and $\bar{y}\,\varepsilon\,\bar{\mathfrak{g}}$. For $\bar{\mathfrak{g}}$ contains at least one point $\bar{y}_1$ over y, hence all points $\bar{y}_1 \Phi^i$, but $F = \{\Phi^i\}$. Therefore $\bar{T}(\bar{x}, \bar{y}) < \bar{\mathfrak{g}}$ and $T(x, y) < \mathfrak{g}$. So far only the fact that $\bar{R}$ is straight has been used.

Let $x \,\varepsilon\, \mathfrak{h}$. Let A be a subarc from x to y of $\mathfrak{h}$ and f a foot of x on $\mathfrak{g}$. Then the length $\lambda(A)$ of A satisfies the inequality $\lambda(A) \geq xy \geq xf$. Let $\bar{x}\,\Omega = x$ and let $\bar{A}$ be the arc over A that begins at $\bar{x}$, and $\bar{T}$ a segment beginning at $\bar{x}$ over a segment $T(x, f)$. $\bar{A}$ and $\bar{T}$ end at points $\bar{y}$ and $\bar{f}$ of $\bar{\mathfrak{g}}$ because, as was shown before, $\bar{\mathfrak{g}}$ contains all points over $\mathfrak{g}$. A and T are locally perpendicular to $\mathfrak{g}$. Therefore $\bar{A}$ and $\bar{T}$ are perpendicular to $\bar{\mathfrak{g}}$. Since they have the common point $\bar{x}$, it follows that $\bar{A} = \bar{T}$ and $A = T(x, f)$.

The example of the Minkowskian Moebius strip in Section 9 shows that $\mathfrak{h}$ need not be a straight line.

Specializing further assume that $\bar{R}$ is two dimensional and that Φ preserves the orientation. If we use the notations of Section 6 with dashes to distinguish between $\bar{R}$ and R and introduce in $\bar{R}$ the coordinates s, t, then the representation

$$\Phi: \qquad s' = s, \quad t' = t + \lambda(\Phi)$$

shows that R *is a cylinder*, and that $0 \leq t \leq \lambda(\Phi)$ may serve as (closed) fundamental set. The closed geodesics of R are images of the axes of the motions $\Phi \neq E$ in $\mathfrak{F}$. Therefore they are the multiples of the great circles $s = s_0 \,\varepsilon\, W$, $0 \leq t \leq \lambda(\Phi)$. If R has negative curvature there is only the one great circle $s = 0$.

The images $y_t(s) = \bar{y}_t(s)\,\Omega$ of the perpendiculars to $\bar{\mathfrak{g}}\,(s = 0)$ are straight lines (and not only union of two rays as in the preceding theorem). For if $s_1 < 0$ and $s_2 > 0$, then a segment from $\bar{y}_t(s_1)$ to $\bar{y}_{t+i\varepsilon}(s_2)$, $i \neq 0$, $\varepsilon = \lambda(\Phi)$, intersects $\bar{\mathfrak{g}}$ at some point $\bar{z}(t_0)$ with $t < t_0 < t + i\varepsilon$. Because $\bar{y}_t(s)$ is perpendicular to $\bar{\mathfrak{g}}$

$$\bar{y}_t(s_1)\,\bar{z}(t_0) > \bar{y}_t(s_1)\,\bar{y}_t(0) = |s_1| \text{ and } \bar{y}_{t+i\varepsilon}\,\bar{z}(t_0) > s_2$$

hence $y_t(s_1)\,y_{t+i\varepsilon}(s_2) > s_2 - s_1$, so that the subarc $y_t(s_1 - s_2)$ which has length $s_2 - s_1$ is the shortest connection of $y_t(s_1)$ and $y_t(s_2)$ in R.

In general there are lines in R between $y_t(s)$ and $y_{t+\varepsilon}(s)$ whose images in R are also straight lines.

The *behavior of the other geodesics* in R can easily be discussed. Since we know the behavior on a Minkowskian cylinder, assume that W has either the form $\alpha \leq s \leq \beta$ or $\alpha \leq s < \infty$. and take an arbitrary point $\bar{p}_0 = (s_0, t_0)$ in R with $s_0 < \alpha$. Let $\bar{\mathfrak{g}}_\alpha$ be the line $s = \alpha$. A line through $\bar{p}_0$ either intersects $\bar{\mathfrak{g}}_\alpha$ or is a non-parallel asymptote to $\bar{\mathfrak{g}}_\alpha$, or is neither asymptote to $\bar{\mathfrak{g}}_\alpha$ nor intersects $\bar{\mathfrak{g}}_\alpha$.

The first type intersects, also $\bar{\mathfrak{g}}$. If it is so represented by $x(t)$ that $x(0)$ is on $\bar{\mathfrak{g}}$, then $\bar{x}(t)\bar{\mathfrak{g}}$ tends monotonically to ∞ when $t \to \infty$ or $t \to -\infty$. If $x(t) = \bar{x}(t)\Omega$ then the preceding discussion shows that $x(t)\mathfrak{g} = \bar{x}(t)\bar{\mathfrak{g}}$. Thus $x(t)$ is a Jordan curve with $x(t)\mathfrak{g} \to \infty$ for $|t| \to \infty$.

If $\bar{x}(t)$ represents the oriented asymptote through $\bar{p}_0$ to $\bar{\mathfrak{g}}_\alpha^+$, then $\bar{x}(t)\bar{\mathfrak{g}}_\alpha$ decreases monotonically and tends to 0 for $t \to \infty$. Therefore $x(t) = \bar{x}(t)\Omega$ is a Jordan curve for which $x(t)\mathfrak{g}_\alpha$ varies monotonically from ∞ to 0.

If finally $\bar{x}(t)$ represents a geodesic through $\bar{p}_0$ of the third type then $\bar{x}(t)\bar{\mathfrak{g}}$ reaches a minimum for some t_0 and is monotone for $t \geq t_0$ and for $t \leq t_0$, moreover $\bar{x}(t)\bar{\mathfrak{g}} \to \infty$ for $|t| \to \infty$. Therefore each of the half extremals $t \geq t_0$ and $t \geq t_0$ of $x(t)$ is a Jordan curve on which $x(t)\mathfrak{g}_\alpha$ (or $x(t)\mathfrak{g}$) is monotone but the two half extremals intersect each other.

If R has negative curvature, we find thus exactly the same behavior of the geodesics as on the hyperboloid of one sheet.

If Ψ is an axial motion of $\bar{R}$ which does not preserve the orientation, then the representation (compare Section 6)

$$\Psi: \qquad s' = -s, \quad t' = t + \lambda(\Psi)$$

shows that R *is a Moebius strip*. Because $\Phi = \Psi^2$ is the transformation

$$s' = s, \quad t' = t + 2\lambda(\Psi) = t' + \lambda(\Phi),$$

R has a cylinder of the previously discussed type as two sheeted covering space, but W must be symmetric to the origin. As on the Moebius strip $s = 0$, $0 \leq t \leq \lambda(\Psi)$ is the only great circle on R, the lines $s = s_0$, $0 \leq t \leq 2\lambda(\Psi)$, $s_0 \varepsilon W$, are closed geodesics which are homotopic to twice the great circle.

Finally let $\bar{R}$ be a plane of non-positive curvature and Φ a non-axial motion of $\bar{R}$ without fixed points. By (6.6) there is a line $\bar{\mathfrak{h}}$ with the property that $\bar{\mathfrak{h}}^+$ is an asymptote $\mathfrak{h}^+\Phi$, therefore Φ preserves the orientation. $\mathfrak{h}$ is not parallel to $\mathfrak{h}\Phi$, because it is readily seen that Φ would then be axial (see the analogous proof for $\bar{L}$ which follows). Since the strip F bounded by $\bar{\mathfrak{h}}^+$ and $\bar{\mathfrak{h}}^+\Phi$ can serve as a fundamental set, $\mathfrak{F} = \{\Phi^i\}$ is the fundamental group of a space R with $\bar{R}$ as universal covering space, and R *is a cylinder.*

R carries straight lines. For if $\bar{x}(t)$ represents $\bar{\mathfrak{h}}^+$ and $x(t) = \bar{x}(t)\Omega$ then a segment T_ν connecting the point $x(-\nu)$ to $x(\nu)$ appears in F either as $T(\bar{x}(-\nu), \bar{x}(\nu))$ or as a set $\Sigma_i T(\bar{x}(t_i), \bar{x}(t_{i+1})\Phi)$ with $\bar{x}(t_1) = \bar{x}(-\nu)$, $\bar{x}(t_{n+1}) = \bar{x}(\nu)\Phi$ (possibly $n = 1$). For $\nu \to \infty$ we have $\bar{x}(t_1)\bar{x}(t_2)\Phi \geq \bar{x}(-\nu)\bar{\mathfrak{h}} \to \infty$ because $\bar{\mathfrak{h}}^-$ is not an asymptote to $\bar{\mathfrak{h}}^-\Phi$. Therefore $x(-\nu)x(\nu) \to \infty$ and a suitable subsequence of T_ν will tend to a straight line L in R. Let $\bar{L}\Omega = L$. Then $\bar{L} \wedge \bar{L}\Phi = 0$ because L has no multiple points (compare (2.4)). It follows that $\bar{L}$ and $\bar{L}\Phi$ bound a fundamental strip. Therefore $\bar{L}$ and $\bar{L}\Phi$ are non-parallel asymptotes to each other.

For if this were not the case and $\bar{y}(t)$ represents $\bar{L}$, then $\bar{y}(t)\bar{y}(t)\Phi$ would reach a positive minimum for same value t'. To any point $\bar{z}'$ not as $\bar{L}$ or $\bar{L}\Phi^i$ there is a point $\bar{z} = \bar{z}'\Phi^i$ between $\bar{L}$ and $\bar{L}\Phi$. Then $\bar{z}\Phi$ is between $\bar{L}\Phi$ and $\bar{L}\Phi^2$, so that $T(\bar{z}, \bar{z}\Phi)$ intersects $\bar{L}\Phi$ in a point $\bar{q}$. Then

$$\bar{y}(t')\bar{y}(t')\Phi \leq \bar{q}\bar{q}\Phi = \bar{q}\Phi^{-1}\bar{q} \leq \bar{q}\Phi^{-1}\bar{z} + \bar{z}\bar{q} = \bar{z}\bar{q} + \bar{q}\bar{z}\Phi = \bar{z}\bar{z}\Phi = \bar{z}'\bar{z}'\Phi$$

and $\bar{z}'\bar{z}'\Phi$ would reach a minimum at $\bar{y}(t')$ or Φ would be axial.

We may therefore assume that $\bar{\mathfrak{h}}$ was chosen such that $\mathfrak{h}$ is a straight line. It is easily seen that the asymptotes to $\bar{\mathfrak{h}}^+$ appear in R as straight lines and that R carries no other straight lines. Also, R has no closed geodesics because no motion Φ^i is axial, see (7.9).

There may or may not be geodesics in R other than the asymptotes to $\mathfrak{h}^+$ which tend in the direction of $\mathfrak{h}^+$ to ∞. On ordinary surfaces of revolution E^3 with cylindrical coordinates:$-z = f(r)$, $0 \leq \delta < r < \infty$ where $f(r)$ is a decreasing convex function of r with $f(r) \to \infty$ for $r \to \delta +$, the first case enters for $\delta > 0$. the second for $\delta = 0$. This follows from the well known relation $r \cdot \sin \omega = \text{const}$ for the geodesics where ω is the angle which the geodesic forms with the meridians (see for instance, G. Darboux, Théorie générale dés surfaces, vol. III Paris 1894, p. 3). If $\delta > 0$ then suitable geodesics different from the meridians tend in the direction $z \to \infty$ to infinity and behave essentially like helices on a cylinder.

Every cylinder with non-positive curvature has a plane $\bar{R}$ as universal covering space and its fundamental group is cyclic. Therefore the preceding discussion covers *all* cylinders. We notice in particular:

(10.4) *Any cylinder R of non-positive curvature is generated by a one-parameter family of straight lines.*

(10.5) *A cylinder with negative curvature either carries exactly one great circle $\mathfrak{g}$ and consists of straight lines perpendicular to $\mathfrak{g}$. Or it contains no closed geodesic and its straight lines form a family of non-parallel asymptotes to each other* (of infinitely many types).

11. General Two-Dimensional Spaces.

F. Klein investigated which topological types of two-dimensional manifolds can be provided with Riemann metrics of a given constant curvature.[14] The preceding results (5.7), (9.1), (9.4), (10.1), (10.2) show that Klein's results extend to Finsler spaces whose curvature has constant sign:

(11.1) *All two-dimensional manifolds can be metrized as G-spaces with non-positive curvature except the sphere and the projective plane.*

The plane, cylinder and Moebius strip are the only manifolds that can be provided both with metrics of curvature 0 and of negative curvature.

If a torus or a one-sided torus carry a metric with non-positive curvature, then they have curvature 0.

All other than those five types can be metrized with negative curvature, but not with curvature 0.

The subject of Hadamard's investigations are the *two-dimensional orientable manifolds with finite connectivity and negative curvature, where every free homotopy class contains a closed geodesic.*

Such a manifold R may be represented topologically as an orientable manifold of finite genus p which is punctured at a finite member of points $z_1, \ldots, z_n$. A closed Jordan curve C_i in R which can be contracted to z_i bounds in R a domain D_i' which contains z_i and determines a closed geodesic $\mathfrak{g}_i$. Then the domain D_i bounded by $\mathfrak{g}_i$ and containing z_i is what Hadamard [9] calls »nappe évasée» and Cohn-Vossen [8] calls »eigentlicher Kelch». It behaves exactly like one half of the cylinder described in (10.5) bounded by $\mathfrak{g}$. The part R^* of R which remains after removing the domains D_i is called by Hadamard the »partie finie» of R. A half geodesic $\mathfrak{h}^+$ issuing from an interior point p of R^* falls into one of these three categories: 1) $\mathfrak{h}^+$ may intersect a $\mathfrak{g}_i$. Then the part of $\mathfrak{h}^+$

[14] A concise formulation of F. Klein's results is found in [11].

following this intersection tends on D_i monotonically to infinity just as the first type of geodesic on cylinders with a closed geodesic described in the preceding section. 2) $\mathfrak{h}^+$ may be asymptotic to a $\mathfrak{g}_i$. 3) $\mathfrak{h}^+$ may fall into the angle between the asymptotes to different $\mathfrak{g}_i$. It then stays entirely in R^* and shows a complicated behavior. For the structure of the three sets of geodesics the reader is referred to Hadamard [9].

A compact two-dimensional manifold R always satisfies the hypotheses of finite connectivity and of the existence of a closed geodesic in a given free homotopy class. If R has negative curvature, then all geodesics belong to the last of the three categories enumerated above. Much information can be gained by considering *the universal covering space* $\bar{R}$ and showing that it *has many properties of the hyperbolic plane.* This will be the subject of the remainder of the present section.

Let E denote the euclidean plane with distance $e(x, y)$ and let the interior I of the unitcircle C with center o in E be also metrized by the hyperbolic distance

$$h(x, y) = \log \frac{e(x, u)\, e(y, v)}{e(x, v)\, e(y, u)}$$

where u (or v) is the intersection of the euclidean ray from x through y (from y through x) with C. The open euclidean segments with endpoints on C are the hyperbolic straight lines.

First let $\bar{R}$ be any plane (two-dimensional straight space) with non-positive curvature. Fix a point $\bar{q}$ in $\bar{R}$ and map a semicircle $\bar{S}$ of $\bar{q}\bar{x} = 1$ proportionally on a semicircle S of $h(0, x) = 1$ in I. Then map the straight line $\mathfrak{g}(\bar{q}, \bar{x})$, $\bar{x} \varepsilon \bar{S}$ in $\bar{R}$ congruently on the hyperbolic straight line $\mathfrak{g}(0, x)$ in I such that $\bar{q}$ goes into o and $\bar{x}$ into x where x is the image of $\bar{x}$ on S.

With any two points x, y in I we associate as third distance the distance $\bar{x}\bar{y}$ of their originals in $\bar{R}$. With this distance I becomes congruent to $\bar{R}$. Henceforth we identify $\bar{R}$ with I and use letters with bars for points in I. The distances $h(\bar{x}, \bar{y})$ and $\bar{x}\bar{y}$ coincide on a straight line through $0 = \bar{q}$ and $\bar{R}$ is imbedded in E.

Let $\bar{x}(t)$ represent an oriented straight line $\mathfrak{g}^+$ in $\bar{R}$ not through $\bar{q}$. Then the line $\mathfrak{g}(\bar{q}, \bar{x}(t))$ revolves monotonically about $\bar{q}$ as t increases. Therefore $\bar{x}(t)$ converges as point in the euclidean plane E for $t \to \infty$ to a limitpoint e^+ on C, which we call the positive endpoint of $\mathfrak{g}^+$. Similarly $\bar{x}(t)$ approaches for $t \to \infty$ the negative endpoint e^- of $\mathfrak{g}^+$ (or the positive endpoint of $\mathfrak{g}^-$).

(11.2) *The asymptotes to* $\mathfrak{g}^+$ *are exactly those lines which have the same positive endpoint* e^+ *as* $\mathfrak{g}^+$.

For it was just shown that the line $\mathfrak{k}^+$ through $\bar{q}$ with positive endpoint e^+ is an asymptote to $\mathfrak{g}^+$, and that $\mathfrak{k}^+$ cannot be asymptote to any line whose positive endpoint is different from e^+. (11.2) follows therefore from (4.3).

(11.3) *If* $\bar{R}$ *has negative curvature then two points* e^+ *and* e^- *on* C *are the positive and negative endpoints of at most one oriented line in* $\bar{R}$.

For (11.2) shows that different oriented lines with the same endpoints are parallels, which do not exist when $\bar{R}$ has negative curvature, see (4.9).

Negative curvature of R is not sufficient to establish that a line with endpoints e^+ and e^- always exists. For instance, the universal covering space of a surface of revolution $z = f(r)$ $0 \leq \delta < r < \infty$ as discussed in the last section will not have this property when $\delta > 0$. But it will be shown:

11.4) *If* $\bar{R}$ *is the universal covering plane of a compact surface* R *with negative curvature, then any two given points* e^+, e^- *on* C *are endpoints of exactly one straight line.*

$\bar{R}$ admits an axial motion Φ that preserves the orientation. Let G^+ be the oriented axis of Φ with endpoints d^+ and d^-. Call C^+ and C^- the two arcs of C determined by d^+ and d^-. Orient the perpendiculars to G^+ so that their positive endpoints are all on C^+. Let H_0^+ be a perpendicular to G^+ at $\bar{p}$ and let $H_\nu^+ = H_0^+ \Phi^\nu$, and denote the endpoints of H_ν^+ by e_ν^+ and e_ν^-. The points e_ν^+ and $e_{1+\nu}^+$ are different because H_ν^+ and $H_{\nu+1}^+$ are not asymptotes to each other (see (6.8)) and $e_{\nu+1}^+$ follows e_ν^+ on C^+ in the direction of d^+. Therefore $e^+ = \lim_{\nu \to \infty} e_\nu^+$ exists on C^+. Because of (11.2) the mapping Φ induces a mapping of C on itself. Here we need only that e^+ is fixed under all Φ^ν and that therefore a line with positive endpoint e^+ through $\bar{p}$ (the asymptote to $\mathfrak{g}(\bar{q}, e^+)$ through $\bar{p}$) goes again into a line with e^+ as positive endpoint.

It follows now that $e^+ = d^+$. For let $\bar{x}(t)$ represent the line through $\bar{p}$ with e^+ as positive endpoint. Then $\bar{x}(t)\Phi$ represents its image and is by (11.2) an asymptote to $\bar{x}(t)$. Hence $\bar{x}(t)\bar{x}(t)\Phi$ is by (4.4 a) bounded for $t \geq 0$. If $x(t)$ did not represent G^+, then $\bar{x}(t)G^+ \to \infty$ for $t \to \infty$ and (6.8) would yield $\bar{x}(t)\bar{x}(t)\Phi \to \infty$. It follows that the endpoints of the perpendiculars to G^+ fill the arcs C^+ and C^- (except for d^+ and d^-).

If H_1^+ and H_2^+ are two perpendiculars to G^+, then it is easy to see that a line L exists such that L^+ is an asymptote to H_1^+ and L^- to H_2^-. Therefore:

(11.5) If $a^+ \varepsilon C^+$ and $a^- \varepsilon C^-$ are given, a line with a^+ and a^- as endpoints exists.

The fundamental group $\mathfrak{F}$ of R contains an axial motion Ψ whose axis H^+ is different from G^+. By (6.4) the images of G^+ and H^+ under the motions of the group generated by Φ and Ψ are axes of motions in $\mathfrak{F}$. The preceding considerations yield readily that any two points a^+ and a^- on C can be separated by the endpoints of an axis of such a motion. (11.5) shows then that a^+ and a^- are endpoints of a straight line in $\bar{R}$. This completes the proof of (11.4).

We show next

(11.6) *If $\bar{R}$ is the universal covering plane of a compact surface R with negative curvature, then for any two open intervals U^+ and U^- on C an axis of a motion in $\mathfrak{F}$ exists which has its positive endpoint in U^+ and its negative endpoint in U^-.* (Compare Nielsen [14, p. 210]).

Proof. Let r_1^+ and r_2^+ be rays from $\bar{q}$ to points e_1^+ and e_2^+ on U^+. If $\bar{x}(t)$ represents the ray from $\bar{q}$ to a point e^+ of U^+ between e_1^+ and e_2^+ then $S(\bar{x}(t), 2\,\delta(R))$ is for large t contained in the angular domain bounded by r_1^+ and r_2^+ because $\bar{x}(t)r_i \to \infty$. By (2.10 c, d) $S(\bar{x}(t), 2\,\delta(R))$ contains a point $\bar{p}$ over any given point p of R. Therefore e^+ is for any point p of R accumulation point of points of the from $\bar{p}\,\Phi_\nu$, $\Phi_\nu\,\varepsilon\,\mathfrak{F}$.

Let $e^+\,\varepsilon\,U^+$ and $e^-\,\varepsilon\,U^-$ be given and choose Φ_ν and $\Phi_{-\nu}$ in $\mathfrak{F}$ such that $\bar{q}\,\Phi_\nu \to e^+$ and $\bar{q}\,\Phi_{-\nu} \to e^-$. The motion $\Phi_{-\nu}^{-1}\Phi_\nu$ determines a class of conjugate elements in $\mathfrak{F}$, therefore a free homotopy class and a closed geodesic G in R. Let p be a simple point of G and $\bar{p}\,\varepsilon\,F(\bar{q})$, $\bar{p}\,\Omega = p$. Then the line $\bar{G}_\nu$ over G through $\bar{p}\,\Phi_{-\nu}^{-1}$ must pass through $\bar{p}\,\Phi_\nu$ and $\bar{G}_\nu$ is the axis of $\Phi_{-\nu}^{-1}\Phi_\nu$ (if $\bar{p}$ does not lie in $H(\bar{q})$ it may be necessary to replace $\bar{p}\,\Phi_{-\nu}^{-1}$ and $\bar{p}\,\Phi_\nu$ by points in $F(\bar{p})\,\Phi$ which are contiguous to $F(\bar{p})\,\Phi_{-\nu}^{-1}$ or $F(\bar{p})\,\Phi_\nu$ but this does not change the conclusion). The endpoints of $\bar{G}_\nu$ tend for $\nu \to \infty$ to e^+ and e^- and lie therefore for large ν on U^+ and U^-.

Theorems (11.5) and (11.6) shows that the closed geodesics in R (because they correspond to the axes of motions in $\mathfrak{F}$) are in a very definite sense dense in the set of all geodesics in R. For further exploitation of (11.5) and (11.6) the reader is referred to Nielsen [13]. Here we observe only the following consequence of (6.11) and (11.2), which is essential for a deeper discussion.

(11.7) *No two different axes of motions in $\mathfrak{F}$ have a common endpoint.*

CHAPTER III.

Differentiable Spaces with Non-Positive Curvature.[15]

12. Riemann Spaces.

The connection of the present definition of non-positive curvature with the standard definition can easily be discussed by using the following lemma:

(12.1) If, in a space with non-positive curvature, $x(t)$ and $y(t)$ represent geodesics with $x(0) = y(0)$ then

$$\lim_{t \to 0+} x(\alpha t)\, y(\beta t)/t = \mu(\alpha, \beta), \quad \alpha, \beta \neq 0$$

exists, $\mu(\alpha, \beta) \leq |\alpha| + |\beta|$ and

(12.2) $$x(\alpha t)\, y(\beta t) \geq t\, \mu(\alpha, \beta) \text{ for small positive } t.$$

For $x(\alpha t)\, y(\beta t)$ is a convex function of t and has therefore at $t = 0$ a right hand derivative $\mu(\alpha, \beta)$. The relation $\mu(\alpha, \beta) \leq |\alpha| + |\beta|$ follows from $x(\alpha t)\, y(\beta t) \leq \leq t(|\alpha| + |\beta|)$ for $t > 0$, and (12.2) follows from the fact, that a convex function lies above the right hand tangent at any of its points.

In Riemann spaces non-positive curvature is equivalent to the »*cosine inequality*» (12.4) which can be formulated under very weak differentiability hypotheses.

(12.3) *A Riemann space has non-positive curvature according to the present definition, if and only if every point has a neighborhood $S(p, \varrho)$ such that any geodesic triangle with vertices a, b, c in $S(p, \varrho)$ satisfies the relation*

(12.4) $$\gamma^2 \geq \alpha^2 + \beta^2 - 2\alpha\beta \cos c,$$

where $\alpha = bc$, $\beta = ca$, $\gamma = ab$ and c is the angle at c.

Proof. Let R be a Riemann space with non-positive curvature in the present sense, and let $a, b, c \in S(p, \delta_p)$. If $x(t), y(t)$ are geodesics which represent the segments $\mathfrak{s}(c, b)$ and $\mathfrak{s}(c, a)$ for $0 \leq t \leq \alpha$ and $0 \leq t \leq \beta$ respectively, then $x(0) = y(0) = c$ and, because R is Riemannian,

$$\begin{aligned} \mu^2(\alpha, \beta) &= \lim x(\alpha t)\, y(\beta t)^2 t^{-2} = \lim (\alpha^2 t^2 + \beta^2 t^2 - 2\alpha t \beta t \cos c)\, t^{-2} \\ &= \alpha^2 + \beta^2 - 2\alpha\beta \cos c \end{aligned}$$

and (12.4) follows from (12.2).

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[15] The considerations of Chapter III extend with obvious changes to spaces with non-negative curvature.

For the proof of the converse observe first that in any triangle qrs in $S(p, \varrho)$ with $qr = \varepsilon$, $qs = \delta$, $rs = 2\lambda$, and $qm(r, s) = \mu$ the relation

$$(12.5) \qquad \varepsilon^2 + \delta^2 \geq 2(\lambda^2 + \mu^2)$$

holds. For, if ω denotes the angle $< qm(r, s)r$, then by (12.4)

$$\varepsilon^2 \geq \mu^2 + \lambda^2 - 2\mu\lambda\cos\omega \qquad \text{and}$$
$$\delta^2 \geq \mu^2 + \lambda^2 + 2\mu\lambda\cos\omega.$$

Consider now a triangle abc in $S(p, \varrho)$. Put $a' = m(c, a)$, $b' = m(c, b)$, $ac = 2\alpha$, $bc = 2\beta$, $a'b' = \gamma'$, $ab = \gamma$, and $a'b = \delta$. Applying (12.5) to the triangles $a'bc$ and bca yields

$$(12.6) \qquad 4(\gamma'^2 + \beta^2) \leq 2(\alpha^2 + \delta^2) \leq \gamma^2 + (2\beta)^2.$$

so that $2\gamma' \leq \gamma$ or $2a'b' \leq ab$, q. e. d.

(12.7) *A Riemann space has non-positive curvature in the present sense if and only if it has non-positive curvature in the usual sence.*

Proof. If a Riemann space has non-positive curvature in the usual sense, then (12.4) holds locally (compare [7, p. 261], where it is proved that (12.4) holds in the large for simply connected spaces. This implies, of course, that it holds in the small for general spaces). By (12.3) the space has non-positive curvature in the present sense.

The converse can be proved by using (12.3) to establish (12.4) and then tracing Cartan's steps back. But it is simpler and geometrically more convincing the proceed as follows: let $x(t)$ and $y(t)$ represent two geodesics which form at $x(0) = y(0)$ the angle γ. Then as in the preceding proof

$$\lim x(t)y(t)/t = \mu(1, 1) = [2(1 - \cos\gamma)]^{1/2} = 2\sin(\gamma/2)$$

hence by (12.2)

$$(12.8) \qquad x(t)y(t) \geq 2t\sin(\gamma/2).$$

That R has at p non-positive curvature means this (see [7, pp. 191—199]): If P is any two dimensional surface element at p then the two dimensional surface M formed by all geodesics through p and tangent to P has at p a non-positive Gauss curvature. Take ν geodesics $x_\nu(t)$ in M through p such that the angle formed by $x_\nu(t)$ and $x_{\nu+1}(t)$ at p is $2\pi/\nu$. If λ_t is the length of the circle with radius t about p in M then by (12.8)

$$\lambda_t > \Sigma x_\nu(t)\, x_{\nu+1}(t) \geq 2\nu t \sin(\pi/\nu),$$

whence $\lambda_t \geq 2\pi t$ for $\nu \to \infty$. The well known expression of Bertrand and Puiseux (see [7, p. 240]) for the Gauss curvature K

$$K = 3\pi^{-1} \lim_{t \to 0+} (2\pi t - \lambda_t)\, t^{-3}$$

shows that $K \leq 0$.

(9.2) shows that also Riemann spaces of curvature 0 in the usual sense are identical with Riemann spaces of curvature 0 in the present sense. However, the corresponding statement *for negative curvature* is not true, *the present definition being a little wider*. For theorem (3.14) shows that $2a'b' < ab$ in smale non-degenerate triangles abc of a Riemann space with negative curvature in the usual sense, which therefore also has negative curvature in the present sense. But, clearly, the relation $2a'b' < ab$ may still be true for non-collinear points when the curvature vanishes at certain sets of two dimensional elements, for instance at all two-dimensional elements in isolated points.

In any case the preceding considerations on G-spaces of non-positive or negative curvature apply to Riemann spaces of non-positive or negative curvature in the usual sense.

13. Inequalities for Volume and Area of Spheres.

The explicite definition of a Finsler space in the usual sense is not needed here. It suffices for the following to know that introduction of *normal coordinates* at a point p may be formulated as follows: (see [3, II § 2]).

In $S(p, \eta_2(p))$ a Minkowskian metric $m(a, b)$ topologically equivalent to the given metric ab can be introduced such that

(13.1) $ab = m(a, b)$ *for points on the same diameter of* $S(p, \eta_2(p))$.[16]

(13.2) *If* $a_\nu \to p$, $b_\nu \to p$, *and* $a_\nu \neq b_\nu$, *then* $a_\nu b_\nu / m(a_\nu, b_\nu) \to 1$.

A property which is a little weaker than (13.2) but sufficient for the present purposes can be deduced from a simple geometric postulate of differentiability

[16] A diameter of $S(p, \eta_2(p))$ is a segment with center p and length $2\eta_2(p)$ without the endpoints.

and regularity. To formulate it let $\beta = \min(\eta_2(p)/2, 1)$. For $a, b \varepsilon S(p, \beta)$ denote by $\mathfrak{r}(a, b)$ the oriented segment with length 2β and origin a that contains b. Then we require:

(13.3) *If* a_ν, b_ν, c_ν *tend to* p *in such a way that* $\mathfrak{r}(a_\nu, b_\nu)$ *and* $\mathfrak{r}(a_\nu, c_\nu)$ *converge, but to different limits, and if* $\lim a_\nu b_\nu / a_\nu c_\nu$ *exists* (∞ *admitted*), *then* $\mathfrak{r}(b_\nu, c_\nu)$ *converges.*

Moreover, $(b_\nu a_\nu + a_\nu c_\nu)/b_\nu c_\nu \to 1$ *if and only if*

$$\lim \mathfrak{r}(b_\nu, a_\nu) = \lim \mathfrak{r}(b_\nu, c_\nu) \text{ or } \lim \mathfrak{r}(a_\nu, c_\nu) = \lim \mathfrak{r}(b_\nu, c_\nu).$$

This is, except for slightly different notations, the postulate $\Delta(p)$ of [3, II § 3]. Its two parts are of a different nature, the first is merely a differentiability hypothesis, the second corresponds to the regularity of the integrand (or the strict convexity of the indicatrix) in ordinary Finsler spaces. It insures that the resulting local Minkowski metric satisfies Axiom V of Section 1. According to [3, II §§ 3, 4, 5] the condition (13.3) implies that in $S(p, \eta_2(p))$ a Minkowski metric $m(a, b)$ can be introduced[17] which satisfies (13.1) and

(13.2 a) *If* $a_\nu \to p$, $b_\nu \to p$ *and* $(a_\nu p + p b_\nu)/a_\nu b_\nu < \zeta < \infty$ *then* $a_\nu b_\nu / m(a_\nu, b_\nu) \to 1$.

Assume now that the space has non-positive curvature and that (13.3) holds at the point p. Let $x(t)$ and $y(t)$ represent different geodesics with $x(0) = y(0) = p$. Then

(13.4) $m[x(\alpha t), y(\beta t)] = t m[x(\alpha), y(\beta)]$ for $0 \leq t < \eta_2(p)$, $|\alpha| < 1$, $|\beta| < 1$

because $x(t)$ and $y(t)$ represent diameters of $S(p, \eta_2(p))$ both for the Minkowski and the given metric. Because of [3, II.2 Theorem 2, p. 52]

$$[x(\alpha t)p + p y(\beta t)]/x(\alpha t) y(\beta t) = (|\alpha| t + |\beta| t)/x(\alpha t) y(\beta t) < \zeta < \infty.$$

Therefore by (13.2 a)

$$\lim x(\alpha t) y(\beta t)/t = \mu(\alpha, \beta) = m(x(\alpha), x(\beta))$$

and by (12.2) $x(\alpha) y(\beta) \geq m(x(\alpha), y(\beta))$ hence

(13.5) $$ab \geq m(a, b) \text{ for } a, b \varepsilon S(p, \eta_2(p)).$$

Remember that $\eta_2(p) = \infty$ for simply connected spaces.

The inequalities for volume and area which are the subject of this section follow from (13.5). Fortunately the question, which area we are going to use

[17] A space which satisfies (13.3) at one point p is because of [4, Theorem (4.12)] a topological manifold and has the property of domain invariance.

proves unimportant in the present case: The inequalities hold for k-dimensional Hausdorff measure defined by coverings with spheres only or with arbitrary sets as well as for intrinsic area (compare [5, Sections 1, 2, 3]).[18] We denote by $|M|_k$ and $|M, m|_k$ the k-dimensional Hausdorff measures defined by arbitrary sets with respect to the given metric xy and to $m(x,y)$ respectively. For the sets M considered here the measure $|M, m|_k$ equals the corresponding other Hausdorff measure or intrinsic area, so that using M_k yields the strongest inequalities. By Kolmogoroff's principle (see [5, (1.14)])

$$|M|_k \geq |M, m|_k. \tag{13.6}$$

The spheres $S(p, \varrho)$ with respect to ab and $m(a, b)$ are identical pointsets. Since a sphere in a Minkowski metric has the same volume as in a euclidean metric (see [5, (2.5)]) we have if the space is n-dimensional

$$|S(p, \varrho)|_n \geq k^{(n)} \varrho^n, \ k^{(n)} = \pi^{n/2} \Gamma^{-1}\left(\frac{n}{2} + 1\right), \ \varrho \leq \eta_2(p). \tag{13.7}$$

This relation generalizes, under surprisingly weak differentiability assumptions a well known inequality for Riemann spaces to Finsler spaces.

The Finsler area of the surface $K(p, \varrho): m(p, x) = \varrho$ is not a function of ϱ alone but depends on the metric as a whole. The analogue to (13.7) will thereford be more complicated.

If $e(a, b)$ is a euclidean metric associated to $m(a, b)$ (compare [5, section 2]), denote by $\sigma(\nu)$ the euclidean $(n-1)$-dimensional area of the intersection of the Minkowski unitsphere $S(p, 1)$ with a hyperplane whose normal has direction ν. Then

$$|K(p, 1), m|_{n-1} = k^{(n-1)} \int \sigma^{-1}(\nu)\, d\, S(\nu) \tag{13.8}$$

where $d\, S(\nu)$ is the euclidean surface element of $K(p, 1)$ at the point where the normal to $K(p, 1)$ has direction ν (compare [5, (7.6)]). Since the spheres $K(p, \varrho)$ are at the same time Minkowski spheres and are homothetic with respect to the Minkowski-metric, we find (see [5, (1.14)])

$$|K(p, \varrho)|_{n-1} \geq k^{(n-1)} \varrho^{n-1} \int \sigma^{-1}(\nu)\, d\, S(\nu), \ \varrho \leq \eta_2(p). \tag{13.9}$$

There are similar relations for lower dimensions. As an example we discuss the two-dimensional case. Let P be a two dimensional surface element through

[18] Whether lower semicontinuous areas can be used is a question which is beyond our present stage of knowledge on area in Finsler spaces.

p and consider the manifold M formed by geodesics through p to tangent P. Under the present weak differentiability hypothesis we mean with P a two-dimensional plane in the Minkowski space $m(a, b)$ (or the euclidean space $e(a, b)$) and the geodesics in P through p, which are the Minkowski (or euclidean) straight lines. On every geodesic through P we lay off a segment of length ϱ and obtain a circle C_ϱ in M which bounds a set S_ϱ in M. Then as before

$$|S_\varrho|_2 \geq \pi \varrho^2.$$

The relations corresponding to (13.8, 9) can easily be found, but they have only in the two-dimensional case a nice geometric interpretation. (For the following compare [6].) Let K_1 be the curve in M which originates from C_1 be a polar reciprocity in $e(p, x) = 1$ with respect to the associated euclidean metric $e(a, b)$ and a subsequent revolution about p through $\pi/2$. *This curve* is interesting because it *has an intrinsic Minkowskian significance*, for it solves the isoperimetric problem for the Minkowski metric in M. Then $|C_1, m|_1 = 2A(C_1, K_1)$ where $A(C_1, K_1)$ is the mixed area of C_1 and K_1 (with respect to the same associated euclidean metric). Hence

$$|C_\varrho|_1 \geq 2\varrho A(C_1, K_1) \geq 2\pi^{1/2} A(K_1)^{1/2} \cdot \varrho$$

where $A(K_1)$ is the euclidean area bounded by K_1 (see [6, (3)]). The right side of (13.9) cannot in general be expressed as a mixed volume, because $\sigma^{-1}(\nu)$ is not always a convex function of ν.

It would be desirable to find an »infinitesimal» condition instead of the finite condition (3.2) for non-positive curvature in Finsler spaces. It is not hard to use (12.1) to find a condition for the derivatives of F, but the condition obtained in that way does not seem to be related to the known invariants of a Finsler metric.

Berwald [1] contains geometric interpretations of the invariants for two-dimensional spaces. In this theory the Finsler metric is approximated by a Riemann metric in the neighborhood of one line element only and cannot be applied directly because in (3.2) even if restricted to »narrow» triangles, (that is triangles for which the directions of $\mathfrak{s}(a, b)$ and $\mathfrak{s}(a, c)$ are close together) the direction of $\mathfrak{s}(b, c)$ is still arbitrary and can in no case be assumed to be close to $\mathfrak{s}(a, b)$.

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