MEAN MOTIONS AND ZEROS OF ALMOST PERIODIC FUNCTIONS.

By

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in COPENHAGEN.

Contents.

Introduction.

Lagrange's Problem on Mean Motion.

1. The investigations forming the contents of the present paper have their starting point in the treatment by Lagrange [1], [2] of the perturbations of the large planets. If we denote by $\varrho = \varrho(t)$ and $\varphi = \varphi(t)$ either the excentricity and the longitude of the perihelion or the inclination and the longitude of the ascending node for the orbit of a planet at the time t, we find from the differential equations of the movement for the determination of these functions in first approximation a relation of the form

$$(1) \qquad \varrho(t)\, e^{i\varphi(t)} = F(t) = a_0 e^{i\lambda_0 t} + \cdots + a_N e^{i\lambda_N t},$$

where the function $F(t)$ on the right is an exponential polynomial with complex not vanishing coefficients $a_0, \ldots, a_N$ and real mutually different exponents $\lambda_0, \ldots, \lambda_N$. This exponential polynomial is, therefore, a sum of vectors, each having a constant length and turning with a constant angular velocity. The number of terms $N+1$ equals the number of planets. The study of the variation of the longitude of the perihelion or of the ascending node leads therefore to a study of the *variation of the argument* $\varphi(t)$ *of an exponential polynomial* $F(t)$ of the type described.

As shown by Lagrange, the exponential polynomial in question contains in most cases a *preponderant term*, i. e. a term whose modulus exceeds the sum of the moduli of the remaining terms. This implies that $F(t)$ does not come arbitrarily near to 0, i. e. it satisfies a relation of the form

$$|F(t)| \geqq k > 0. \tag{2}$$

Since the argument of $F(t)$ differs for every t by less than $\frac{1}{2}\pi$ from the argument of the preponderant term we have

$$\varphi(t) = ct + O(1), \tag{3}$$

where c is the exponent of the preponderant term. The argument $\varphi(t)$ is, therefore, in this so-called *Lagrangean case* the sum of a secular term ct and a bounded remainder.

When the polynomial does not contain any preponderant term it will not generally satisfy any relation of the form (2) and it may even take the value 0. In the latter case the continuity of the argument $\varphi(t)$ can only be maintained, if we agree to consider it not mod. 2π but mod. π, and to change the sign of the modulus $\varrho(t)$, when t passes a zero of $F(t)$ of odd order. In case of a planet this means, that we must allow negative values of the excentricity and the inclination, and must consider the line of apsides instead of the perihelion and the nodal line instead of the ascending node. This possibility already occurs in the trivial case $N = 1$, which was also considered by Lagrange. If in the said case $|a_0| > |a_1|$ or $|a_0| < |a_1|$, one term is preponderant, and we have the relation (3) with $c = \lambda_0$ or $c = \lambda_1$ respectively. If, however, $|a_0| = |a_1|$, the function $F(t)$ has equidistant zeros, and using the convention regarding $\varphi(t)$ we easily find that the relation (3) is again true, this time with $c = \frac{1}{2}(\lambda_0 + \lambda_1)$; the remainder is in this case a constant.[1]

Lagrange's treatment did not go beyond the two cases mentioned above, and he added ([2], § 46) the following remark: »Hors de ces deux cas, il est fort difficile et peut être même impossible de prononcer, en général, sur la nature de l'angle φ».

[1] Denoting by a the common value of $|a_0|$ and $|a_1|$ and putting $a_0 = a e^{i\alpha_0}$ and $a_1 = a e^{i\alpha_1}$ we have, in fact,

$$F(t) = a\left(e^{i(\lambda_0 t + \alpha_0)} + e^{i(\lambda_1 t + \alpha_1)}\right)$$

$$= a\left(e^{i\left[\frac{1}{2}(\lambda_0 - \lambda_1)t + \frac{1}{2}(\alpha_0 - \alpha_1)\right]} + e^{i\left[\frac{1}{2}(\lambda_1 - \lambda_0)t + \frac{1}{2}(\alpha_1 - \alpha_0)\right]}\right) e^{i\left[\frac{1}{2}(\lambda_0 + \lambda_1)t + \frac{1}{2}(\alpha_0 + \alpha_1)\right]},$$

so that

$$\varrho(t) = 2a\cos\left[\tfrac{1}{2}(\lambda_0 - \lambda_1)t + \tfrac{1}{2}(\alpha_0 - \alpha_1)\right] \quad \text{and} \quad \varphi(t) = \tfrac{1}{2}(\lambda_0 + \lambda_1)t + \tfrac{1}{2}(\alpha_0 + \alpha_1).$$

2. Having first been made the object of more heuristic considerations by various astronomers, this problem was taken up by Bohl [1], who, besides treating some other cases where the relation (3) is trivial, in detail studied the case $N = 2$.

Suppressing, for the sake of simplicity, the exponential factor $e^{i\lambda_0 t}$ of the first term (which means that $\varphi(t)$ is diminished by the linear term $\lambda_0 t$) we may assume the polynomial to be of the form

$$F(t) = a_0 + a_1 e^{i\lambda_1 t} + \cdots + a_N e^{i\lambda_N t}. \tag{4}$$

One of the trivial cases mentioned is then the one in which the ratio between any two of the exponents $\lambda_1, \ldots, \lambda_N$ is *rational*. In this case $F(t)$ is periodic, whence the relation (3) easily follows, the constant c being the ratio between the variation of the argument in a period and the length of the period.

If $N = 2$, the only interesting case is therefore the one in which the ratio between the exponents λ_1 and λ_2 is *irrational*. In this case the course of the movement can easily be described by applying the theorem that the points of the straight line $(x_1, x_2) = (\lambda_1 t, \lambda_2 t)$ are everywhere densely distributed mod. 2π, a special (and, for the rest, trivial) case of the general Kronecker approximation theorem. It then turns out, that in the non-Lagrangean case the function always comes arbitrarily near to 0, or even takes the value 0. Bohl's main result is now, that in this case we always have

$$\varphi(t) = ct + o(t), \tag{5}$$

i. e. $\varphi(t)$ is again the sum of a secular term ct and a remainder, but the latter is now not necessarily bounded; it is even, as shown by Bernstein [1], generally unbounded[1]. Taking into account a certain uniformity, the method yields a little more, namely the existence of the limit[2]

$$c = \lim_{(\delta-\gamma)\to\infty} \frac{\varphi(\delta) - \varphi(\gamma)}{\delta - \gamma}. \tag{6}$$

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[1] Regarding the importance of these results from an astronomical point of view, see also Bohl [2] and Bernstein [2].

[2] For an arbitrary real function $\varrho(\gamma, \delta)$ defined when $-\infty < \gamma < \delta < +\infty$ we denote by $\liminf\limits_{(\delta-\gamma)\to\infty} \varrho(\gamma, \delta)$ the least upper bound of those numbers r for which there exists a number $T = T(r)$ such that $\varrho(\gamma, \delta) > r$ for $(\delta - \gamma) > T$, and, similarly, by $\limsup\limits_{(\delta-\gamma)\to\infty} \varrho(\gamma, \delta)$ the greatest lower bound of those numbers r for which there exists a number $T = T(r)$ such that $\varrho(\gamma, \delta) < r$ for $(\delta - \gamma) > T$. If these limits are equal, we denote their common value by $\lim\limits_{(\delta-\gamma)\to\infty} \varrho(\gamma, \delta)$. When $\varrho(\gamma, \delta)$ is complex-

In the sequel a complex function $F(t)$ of the real variable t, which may be written in the form $F(t) = \varrho(t) e^{i\varphi(t)}$, where $\varrho(t)$ and $\varphi(t)$ are real and continuous, and $\varrho(t)$ has but isolated zeros (so that $\varphi(t)$ is uniquely determined mod. π), will be said to possess the *mean motion* c, if the relation (6) is true. Bohl's result may then be expressed by saying that an exponential polynomial (4) always possesses a mean motion when $N = 2$. Bohl also gave an expression for the mean motion.

In order to obtain this result, Bohl made use of a refinement of the special case of Kronecker's theorem mentioned above, to the effect that the points of the line $(x_1, x_2) = (\lambda_1 t, \lambda_2 t)$ are not only everywhere densely distributed, but even *equidistributed* mod. 2π. In the form leading to (6) this theorem states that if $l(\gamma, \delta)$ denotes the sum of the lengths of those sub-intervals of $\gamma < t < \delta$ in which the point $(\lambda_1 t, \lambda_2 t)$ belongs mod. 2π to a (sufficiently regular) set not containing equivalent points mod. 2π, then the limit

$$\lim_{(\delta-\gamma)\to\infty} \frac{l(\gamma, \delta)}{\delta - \gamma}$$

exists and is proportional to the area of the set.

3. As shown by Weyl [1], [2], we have equidistribution also in the case of the general Kronecker theorem concerning a straight line $(x_1, \ldots, x_N) = (\lambda_1 t, \ldots, \lambda_N t)$ in the m-dimensional space, for which the numbers $\lambda_1, \ldots, \lambda_N$ are linearly independent[1]. By means of this result he could immediately extend Bohl's investigation to the case $N > 2$ under the assumption that the exponents $\lambda_1, \ldots, \lambda_N$ are linearly independent, with the result that in this case a mean motion always exists. This, however, did not mean a complete solution of Lagrange's problem, since for $N > 2$ the exponents $\lambda_1, \ldots, \lambda_N$ may be linearly dependent even if two of them have an irrational ratio.

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valued, we write $\lim\limits_{(\delta-\gamma)\to\infty} \varrho(\gamma, \delta) = a$ if there exists to every $\varepsilon > 0$ a number $T = T(\varepsilon)$ such that $|\varrho(\gamma, \delta) - a| < \varepsilon$ for $(\delta - \gamma) > T$. For a set of functions $\varrho(\gamma, \delta)$ the limits are said to exist uniformly, if, for an arbitrary ε, the same $T = T(\varepsilon)$ may be used for all functions of the set. — The same notations will be used in cases where the numbers γ and δ are not arbitrary, but are to be chosen from some given set of real numbers.

[1] The numbers $\lambda_1, \ldots, \lambda_N$ are called linearly independent if they satisfy no relation

$$h_1 \lambda_1 + \cdots + h_N \lambda_N = 0$$

with integral coefficients $h_1, \ldots, h_N$ not all vanishing.

Recently the problem has been treated by Hartman, van Kampen, and Wintner [1]; their method, which is also closely related to that of Bohl, shows that, if the exponents and the moduli of the coefficients are given, the formula (5) is valid for almost all sets of values of the arguments of the coefficients. Occasioned by this investigation, Weyl [3] took up his earlier investigation, obtaining in the case of linearly independent exponents an expression of the mean motion, which for $N=2$ is identical with that of Bohl.

All these results are contained in a later investigation by Weyl [4] concerning an arbitrary exponential polynomial $F(t)$, which may now again be of the form (1). The method is a further development of that of Bohl and depends on a representation of the polynomial $F(t)$ well known from earlier papers by Bohl and Esclangon on generalized periodic functions. This expression for $F(t)$ is obtained by introducing an *integral base* of the exponents $\lambda_0, \ldots, \lambda_N$, i. e. a set of linearly independent numbers $\mu_1, \ldots, \mu_m$ such that each λ_n has an expression

$$\lambda_n = h_{n1}\mu_1 + \cdots + h_{nm}\mu_m,$$

where the coefficients h_{nk} are integers. The polynomial then takes the form

$$F(t) = \sum_{n=0}^{N} a_n e^{i(h_{n1}\mu_1 + \cdots + h_{nm}\mu_m)t}$$

and is therefore obtained by considering the function

$$G(x_1, \ldots, x_m) = \sum_{n=0}^{N} a_n e^{i(h_{n1}x_1 + \cdots + h_{nm}x_m)}$$

on the straight line $(x_1, \ldots, x_m) = (\mu_1 t, \ldots, \mu_m t)$, i. e.

$$F(t) = G(\mu_1 t, \ldots, \mu_m t). \tag{7}$$

The study of the argument of $F(t)$ is thus reduced to a study of the function $G(x_1, \ldots, x_m)$, which has the period 2π in all the variables, and an application of the Kronecker-Weyl theorem.

The purpose of Weyl's investigation is to prove the existence of a mean motion in all cases. The proof is, however, valid only under the assumption that the set of zeros of the function $G(x_1, \ldots, x_m)$ contains no manifold of the dimension $m-1$, a restriction not noticed by Weyl. We return later (§ 25) to the meaning of this assumption. Weyl also deduces a simple expression for the mean motion.

Almost Periodic Functions of a Real Variable.

4. It suggests itself to extend Lagrange's problem from exponential polynomials to the more general class of almost periodic functions

$$F(t) \sim \Sigma a_n e^{i \lambda_n t}$$

introduced by Bohr [8], [9]. We briefly recall the fundamental definitions and theorems regarding these functions, referring for details to the original papers or the monographs by Bohr [14], Besicovitch [1], or Favard [1].

A set of real numbers is called *relatively dense*, if, in any interval of a certain length l, it is represented by at least one point. Let $F(t)$ be a continuous complex function defined for $-\infty < t < +\infty$. A number τ is called a *translation number* of $F(t)$ belonging to a given number $\varepsilon > 0$, and is denoted by $\tau(\varepsilon)$ or $\tau_F(\varepsilon)$, if the inequality

$$|F(t+\tau) - F(t)| \leqq \varepsilon$$

holds for all t. Together with τ the number $-\tau$ is, of course, also a translation number of $F(t)$ belonging to ε. The function $F(t)$ is called *almost periodic* if, for any $\varepsilon > 0$, the set of all translation numbers $\tau = \tau(\varepsilon) = \tau_F(\varepsilon)$ is relatively dense.

Every almost periodic function is bounded and uniformly continuous, i. e. the set of translation numbers belonging to a given $\varepsilon > 0$ contains a neighbourhood of the point $\tau = 0$. Further, it possesses a *mean value*

$$\underset{t}{M}\{F(t)\} = \lim_{(\delta - \gamma) \to \infty} \frac{1}{\delta - \gamma} \int_{\gamma}^{\delta} F(t)\, dt.$$

A set of almost periodic functions is called a *uniformity set*, if for any $\varepsilon > 0$ the set of common translation numbers $\tau = \tau(\varepsilon)$ of all functions of the set is relatively dense and contains a neighbourhood of the point $\tau = 0$. For the functions of a uniformity set the mean values exist uniformly.

The sum or the product of two almost periodic functions and the limit of a uniformly convergent sequence of almost periodic functions are again almost periodic.

Corresponding to an arbitrary almost periodic function $F(t)$ there exists only a finite or enumerable set of real numbers λ_n for which the mean value

$$a_n = \underset{t}{M}\{F(t) e^{-i \lambda_n t}\}$$

does not vanish. The series

$$F(t) \sim \Sigma a_n e^{i \lambda_n t}$$

formed with these numbers λ_n as exponents and the corresponding numbers a_n as coefficients is called the *Fourier series* of the function. We have the Parseval formula

$$\underset{t}{M}\{|F(t)|^2\} = \Sigma |a_n|^2.$$

Different functions have different Fourier series.

The Fourier series of the sum or the product of two almost periodic functions or of the limit of a uniformly convergent sequence of almost periodic functions are obtained by performing the corresponding operations on the Fourier series of these functions.

The main theorem of the theory is the *approximation theorem.* According to this theorem the class of almost periodic functions is identical with the class of functions which are the limit of a uniformly convergent sequence of exponential polynomials

$$F_p(t) = \sum_{n=1}^{N_p} a_n^{(p)} e^{i \lambda_n^{(p)} t}.$$

For a given almost periodic function these exponential polynomials may be chosen with exponents among the exponents λ_n of the function.

Besides the ordinary almost periodic functions various classes of generalizations have been considered, for which a similar approximation theorem holds, the uniform convergence being replaced by weaker notions of convergence. As these generalizations are not directly used in the present paper, but are only referred to occasionally, we shall not go into details regarding this subject.

5. By a *modul* M we shall mean a set of real numbers, which, containing a number μ, also contains all integral multiples of μ, and, containing two numbers μ_1 and μ_2, also contains their sum. The smallest modul containing the exponents λ_n of an almost periodic function $F(t)$ is called the modul of the function. It was introduced by Bochner [1]. Evidently it consists of all linear combinations

$$h_1 \lambda_1 + \cdots + h_N \lambda_N$$

of the exponents λ_n with integral coefficients $h_1, \ldots, h_N$.

Between the translation numbers and the exponents of an almost periodic function we have the following important connection: A necessary and sufficient condition that a function $F(t)$ should be almost periodic with exponents from a given modul M, is that to any $\varepsilon > 0$ correspond a finite set of numbers $\lambda_1, \ldots, \lambda_N$ in M and a number $\eta > 0$, such that every number τ satisfying the conditions

$$\left.\begin{array}{l} |\lambda_1 \tau| \leqq \eta \\ \quad \ldots \\ |\lambda_N \tau| \leqq \eta \end{array}\right\} \text{mod. } 2\pi$$

is a $\tau_F(\varepsilon)$.[1]

From this follows: If $F(t)$ is almost periodic, and $G(t)$ denotes a function with the property that to any $\varepsilon > 0$ corresponds a number $\delta > 0$ such that any $\tau_F(\delta)$ is also a $\tau_G(\varepsilon)$, then $G(t)$ is also almost periodic, and its exponents belong to the modul of $F(t)$.

As a special case, we mention that if $F(t)$ is almost periodic and if c is a number with the property that to any $\varepsilon > 0$ corresponds a δ such that any $\tau = \tau_F(\delta)$ satisfies the condition

$$|c\tau| \leqq \varepsilon \quad \text{mod. } 2\pi,$$

then c belongs to the modul of $F(t)$. For the latter condition implies that $|e^{ic\tau} - 1| \leqq \varepsilon$ and hence that τ is a $\tau(\varepsilon)$ of the function e^{ict}, which has the exponent c.

6. The almost periodic function $F(t)$ is *periodic* with the period $p > 0$ if and only if its exponents belong to the discrete modul $M = \left\{ h \frac{2\pi}{p} \right\}$, where h runs through all integers. It is *limit periodic* with limit period p, which means that it is the limit of a uniformly convergent sequence of periodic functions, the periods of which are integral multiples of p, if and only if its exponents belong to the modul $M = \left\{ r \frac{2\pi}{p} \right\}$, where r runs through all rational numbers.

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[1] See Bohr [8], pp. 105—117. The necessity of the condition is his theorem II (p. 110), while the sufficiency is a consequence of his theorems I (p. 105) and B (p. 113). See also Bohr [13], pp. 59—60.

By a theorem of Bohl and Wennberg the set of numbers τ satisfying the conditions $|\lambda_1 \tau| \leqq \eta, \ldots, |\lambda_N \tau| \leqq \eta$ (mod. 2π), where $\lambda_1, \ldots, \lambda_N$ are arbitrary real numbers and $\eta > 0$, is relatively dense. See e.g. Bohr [8], pp. 119—121.

For later reference we notice that even the set of *integers* τ satisfying the conditions is relatively dense; for if δ is chosen sufficiently small, any number τ satisfying the conditions $|\lambda_1 \tau| \leqq \delta, \ldots, |\lambda_N \tau| \leqq \delta$ and $|2\pi\tau| \leqq \delta$ (mod. 2π) differs at most by $\delta/2\pi$ from an integer satisfying the former conditions, and the set of numbers τ satisfying the latter conditions is relatively dense.

An almost periodic function $F(t)$ is said to possess the *finite integral base* $\mu_1, \ldots, \mu_m$, if these numbers form a finite integral base of the exponents of $F(t)$, i. e. if they are linearly independent and the exponents are contained in the modul $M = \{h_1\mu_1 + \cdots + h_m\mu_m\}$, where the set of coefficients $h_1, \ldots, h_m$ runs through all sets of integers. The theory of these functions was developed by Bohl and Esclangon before the general theory of almost periodic functions. Every exponential polynomial is of this kind.

Let $F(t)$ be a function with exponents from M. Allowing terms with the coefficient 0, we may write its Fourier series in the form

$$F(t) \sim \Sigma\, a_{h_1, \ldots, h_m} e^{i(h_1\mu_1 + \cdots + h_m\mu_m)t}.$$

Corresponding to the expression (7) of an exponential polynomial we have in the present general case

$$F(t) = G(\mu_1 t, \ldots, \mu_m t), \tag{8}$$

where $G(x_1, \ldots, x_m)$ is a continuous function with the period 2π in all the variables, and with the Fourier series

$$G(x_1, \ldots, x_m) \sim \Sigma\, a_{h_1, \ldots, h_m} e^{i(h_1 x_1 + \cdots + h_m x_m)}.$$

This function $G(x_1, \ldots, x_m)$ will be called the spatial extension of $F(t)$. Conversely, if $G(x_1, \ldots, x_m)$ is a continuous function with the period 2π in all the variables, the function (8) is an almost periodic function with exponents from M.

An almost periodic function $F(t)$ is said to possess the *infinite* integral base $\mu_1, \mu_2, \ldots$, if these numbers form an infinite integral base of the exponents of $F(t)$, i. e. if they are linearly independent[1] and the exponents are contained in the modul $M = \{h_1\mu_1 + h_2\mu_2 + \cdots\}$, where the sequence of coefficients $h_1, h_2, \ldots$ runs through all sequences of integers of which only a finite number are $\neq 0$. In this case we have a result similar to the preceding one, the spatial extension being now a periodic function of an infinite number of variables.

Finally, an almost periodic function $F(t)$ is said to possess the finite or infinite *rational base* $\mu_1, \ldots, \mu_m$ or $\mu_1, \mu_2, \ldots$, if these numbers form a rational base of the exponents of $F(t)$, i. e. if they are linearly independent and the exponents are contained in the modul $M = \{r_1\mu_1 + \cdots + r_m\mu_m\}$ or $M = \{r_1\mu_1 + r_2\mu_2 + \cdots\}$, where the coefficients are now rational numbers of which, in the case of an

[1] The numbers $\mu_1, \mu_2, \ldots$ are called linearly independent if $\mu_1, \ldots, \mu_m$ are linearly independent for every m.

infinite base, only a finite number are $\neq 0$. In these cases the spatial extension is a limit periodic function of a finite or infinite number of variables.

The above mentioned spatial extensions were used by Bohr [9] to prove the approximation theorem. Later this theorem has been proved more simply. On the basis of the approximation theorem, the spatial extensions may be introduced very simply, as pointed out by Bochner [1]. In the present paper we shall therefore not presuppose any results regarding the spatial extensions.

Mean Motions of Almost Periodic Functions of a Real Variable.

7. It was proved by Bohr [11], [13] that if $F(t)$ is an almost periodic function satisfying a relation of the form (2), then the argument $\varphi(t)$ will be of the form

$$\varphi(t) = ct + \psi(t),$$

where $\psi(t)$ is again almost periodic. Thus the relation (3) is again true. In the Lagrangean case the almost periodicity of the remainder had previously been pointed out by Wintner [2], who had also conjectured the preceding theorem. While in the Lagrangean case the mean motion c is always one of the exponents λ_n, this does not hold good in the case of an arbitrary almost periodic function; yet the mean motion c, and similarly the exponents of $\psi(t)$, always belong to the modul of $F(t)$. Applications of this theorem are given in Wintner [3] (see also Stepanoff [1]) and in Wintner [8], [9].

Bohr's proof of the theorem is founded directly on the definition of almost periodicity. Another proof depending on the approximation theorem has been given by Jessen [4]; it leads to an expression for the mean motion, an expression also found (in an unessentially different form and by other means) by Hartman and Wintner [1]. It was applied in Jessen [4] in the case of an arbitrary almost periodic function $F(t)$, to a study of the mean motion of $F(t) - a$ for different values of a not belonging to the closure of the set of values of $F(t)$. This study was followed by more general investigations on almost periodic movements by Fenchel and Jessen [1] and Fenchel [1]. In the special case where $F(t)$ is an exponential polynomial it was shown by Bohr and Jessen (see Bohr [15]) that the mean motion c always belongs not only to the modul of $F(t)$ but even to a certain finite set, depending only on the exponents of the polynomial.

An extension of Bohr's theorem to almost periodic functions in a group has been given by van Kampen [1].

In Chapter I we give an exposition of the two proofs of Bohr's theorem mentioned above, and of the additional results regarding the values of the mean motion.

Analytic Almost Periodic Functions.

8. If the almost periodic function $F(t)$ does not satisfy any relation of the form (2), the variation of its argument may be very complicated, and if the function has zeros it may even be impossible to fix the argument as a continuous function of t. The problem seems, then, only to be of interest in the case of *analytic* almost periodic functions. We briefly recall the theory of these functions as developed in Bohr [10], referring also to the monographs by Besicovitch [1] and Favard [1].

Let $f(s)$ be a function of the complex variable $s = \sigma + it$, which is regular in a vertical strip $(-\infty \leqq)\alpha < \sigma < \beta(\leqq +\infty)$, denoted briefly by (α, β). A (real) number τ is called a translation number of $f(s)$ belonging to a given number $\varepsilon > 0$ and the given strip, and is denoted by $\tau(\varepsilon; \alpha, \beta)$ or $\tau_f(\varepsilon; \alpha, \beta)$, if the inequality

$$|f(s + i\tau) - f(s)| \leqq \varepsilon$$

holds for all s in the strip. The function $f(s)$ is called almost periodic in (α, β) if, for any $\varepsilon > 0$, the set of all translation numbers $\tau = \tau(\varepsilon; \alpha, \beta) = \tau_f(\varepsilon; \alpha, \beta)$ is relatively dense. The function is called almost periodic in $[\alpha, \beta]$, if it is almost periodic in every reduced strip $(\alpha <)\alpha_1 < \sigma < \beta_1 (< \beta)$, and it is called almost periodic in $[\alpha, \beta)$ or $(\alpha, \beta]$, if it is almost periodic in every strip $(\alpha <)\alpha_1 < \sigma < \beta$ or $\alpha < \sigma < \beta_1 (< \beta)$ respectively. Using throughout square brackets in this manner we have the theorem that a function $f(s)$ almost periodic in $[\alpha, \beta]$ is bounded and uniformly continuous in $[\alpha, \beta]$. This implies that, if $\alpha < \alpha_1 < \beta_1 < \beta$, then the almost periodic functions $F_\sigma(t) = f(\sigma + it)$, where $\alpha_1 \leqq \sigma \leqq \beta_1$, form a uniformity set.

The notation (α, β) will also be used for the interval $\alpha < \sigma < \beta$ and square brackets will be used in this connection in the same manner as for strips. A closed strip or interval $\alpha \leqq \sigma \leqq \beta$ will be denoted briefly by $\{\alpha, \beta\}$

The sum or the product of two functions almost periodic in $[\alpha, \beta]$, and the limit of a sequence of functions almost periodic in $[\alpha, \beta]$ which converges uniformly in $[\alpha, \beta]$, are again almost periodic in $[\alpha, \beta]$. A function obtained from an almost periodic function by replacing s by $ks + l$, where k is real, is almost periodic in the corresponding strip.

To an arbitrary function $f(s)$ almost periodic in $[\alpha, \beta]$ corresponds a *Dirichlet series*

$$f(s) \sim \Sigma a_n e^{\lambda_n s}, \qquad a_n = \underset{t}{M}\{f(\sigma + i t) e^{-\lambda_n (\sigma + i t)}\},$$

with real exponents λ_n and complex coefficients $a_n (\neq 0)$. This series is merely a formal combination of the Fourier series of the functions $F_\sigma(t) = f(\sigma + i t)$, i. e., in the Fourier series $\Sigma a_n^{(\sigma)} e^{i \lambda_n^{(\sigma)} t}$ of the almost periodic function $F_\sigma(t)$, the exponents $\lambda_n^{(\sigma)}$ are independent of σ and the coefficients have the form $a_n^{(\sigma)} = a_n e^{\lambda_n \sigma}$, where the a_n are independent of σ.

The Dirichlet series of the sum or the product, or of the limit of a uniformly convergent sequence of almost periodic functions, or of a function obtained from an almost periodic function replacing s by $ks + l$, where k is real, are obtained by performing the corresponding operations on the Dirichlet series of these functions.

The main theorem of the theory is the approximation theorem, according to which the class of functions almost periodic in a strip $[\alpha, \beta]$ is identical with the class of functions which are the limit of a sequence of exponential polynomials

$$f_p(s) = \sum_{n=1}^{N_p} a_n^{(p)} e^{\lambda_n^{(p)} s}$$

converging uniformly in $[\alpha, \beta]$. For a given almost periodic function these exponential polynomials may be chosen with exponents among the exponents λ_n of the function.

9. The smallest modul containing the exponents of an analytic almost periodic function is called the modul of the function.

The connection between the translation numbers and exponents dealt with in § 5 in the case of almost periodic functions of a real variable may easily be extended to analytic almost periodic functions. This leads to the following result: A necessary and sufficient condition that a function $f(s)$ regular in a strip (α, β) should be almost periodic in $[\alpha, \beta]$ with exponents from a given modul M, is that to any $\varepsilon > 0$ and any reduced strip (α_1, β_1) correspond a finite set of numbers $\lambda_1, \ldots, \lambda_N$ in M and a number $\eta > 0$, such that every number τ satisfying the conditions

$$\left.\begin{array}{l} |\lambda_1 \tau| \leqq \eta \\ \quad \cdots \\ |\lambda_N \tau| \leqq \eta \end{array}\right\} \bmod. 2\pi$$

is a $\tau_f(\varepsilon; \alpha_1, \beta_1)$.

From this follows: If $f(s)$ is almost periodic in $[\alpha, \beta]$, and $g(s)$ denotes a function regular in (α, β) with the property that to any $\varepsilon > 0$ and any reduced strip (α_1, β_1) correspond a $\delta > 0$ and a reduced strip (α_2, β_2) such that any $\tau_f(\delta; \alpha_2, \beta_2)$ is also a $\tau_g(\varepsilon; \alpha_1, \beta_1)$, then $g(s)$ is also almost periodic in $[\alpha, \beta]$, and its exponents belong to the modul of $f(s)$.

We find further: If $f(s)$ is almost periodic in $[\alpha, \beta]$, and c a number with the property that to any $\varepsilon > 0$ correspond a $\delta > 0$ and a reduced strip (α_1, β_1) such that any $\tau_f(\delta; \alpha_1, \beta_1)$ satisfies the condition

$$|c\tau| \leqq \varepsilon \quad \text{mod. } 2\pi,$$

then c belongs to the modul of $f(s)$.

10. It has been proved by Bohr [12] that, if the quotient $h(s) = f(s)/g(s)$ of two functions almost periodic in $[\alpha, \beta]$ is regular in (α, β), then it is also almost periodic in $[\alpha, \beta]$. To every $\varepsilon > 0$ and every reduced strip (α_1, β_1) correspond, in fact, a $\delta > 0$ and a reduced strip (α_2, β_2), such that any common $\tau(\delta; \alpha_2, \beta_2)$ of $f(s)$ and $g(s)$ is a $\tau_h(\varepsilon; \alpha_1, \beta_1)$. This shows, in addition, that if the exponents of $f(s)$ and $g(s)$ all belong to a modul M, then the exponents of $h(s)$ will also belong to M.

By a similar argument it may be proved that if $f(s)$ is almost periodic in $[\alpha, \beta]$, and has only zeros of even order, then an arbitrary branch $g(s) = \sqrt{f(s)}$ of the square root of $f(s)$ is also almost periodic in $[\alpha, \beta]$. To every $\varepsilon > 0$ and every reduced strip (α_1, β_1) correspond, in fact, a $\delta > 0$ and a reduced strip (α_2, β_2), such that for any $\tau = \tau_f(\delta; \alpha_2, \beta_2)$ we have either

$$|g(s + i\tau) - g(s)| \leqq \varepsilon \quad \text{or} \quad |g(s + i\tau) + g(s)| \leqq \varepsilon$$

in the strip (α_1, β_1) and the number 2τ is, therefore, in both cases a $\tau_g(2\varepsilon; \alpha_1, \beta_1)$. This also implies that, if the exponents of $f(s)$ belong to a given modul M, then the exponents of $g(s)$ belong to the modul obtained from M by dividing all numbers by 2. We shall, however, need the more precise result that there exists a number μ in M such that, if we replace $f(s)$ by $f_1(s) = f(s)e^{\mu s}$, then the exponents of an arbitrary branch $g_1(s) = \sqrt{f_1(s)}$ of the square root of $f_1(s)$ belong to M itself.

In order to see this we merely take $\mu = -2\lambda$, where λ is a Dirichlet exponent of $g(s)$. Then the constant term a of the Dirichlet series of $g_1(s)$ is $\neq 0$. Now the constant term of the Dirichlet series of $g_1(s + i\tau) + g_1(s)$ is $2a$; hence the inequality

$$|g_1(s+i\tau)+g_1(s)|\leqq \varepsilon$$

cannot be satisfied in a strip (α_1, β_1) if $\varepsilon < 2|a|$. Thus, corresponding to an $\varepsilon < 2|a|$ and a reduced strip (α_1, β_1), there exist a δ and a reduced strip (α_2, β_2), such that for any $\tau = \tau_{f_1}(\delta;\ \alpha_2, \beta_2)$ we have

$$|g_1(s+i\tau)-g_1(s)|\leqq \varepsilon$$

in the strip (α_1, β_1), i. e. τ is also a $\tau_{g_1}(\varepsilon;\ \alpha_1, \beta_1)$. This shows that the exponents of $g_1(s)$ belong to M.

11. The almost periodic function $f(s)$ is periodic in (α, β) with the period ip, where $p > 0$, if and only if its exponents belong to the modul $M = \left\{h\frac{2\pi}{p}\right\}$, where h runs through all integers. The substitution $e^{\frac{2\pi}{p}s} = z$ shows that the Dirichlet (or Laurent) series is in this case absolutely convergent, and represents the function. The function is limit periodic in $[\alpha, \beta]$ with the limit period ip, which means that it is the limit of a sequence of periodic functions whose periods are integral multiples of ip, converging uniformly in $[\alpha, \beta]$, if and only if its exponents belong to the modul $M = \left\{r\frac{2\pi}{p}\right\}$, where r runs through all rational numbers.

An almost periodic function $f(s)$ will be said to possess the finite or infinite, integral or rational base $\mu_1, \ldots, \mu_m$ or $\mu_1, \mu_2, \ldots$, if these numbers form a base of this kind for the exponents of $f(s)$, i. e. (see § 6) if they are linearly independent and the exponents are contained in the corresponding modul $M = \{h_1\mu_1 + \cdots + h_m\mu_m\}$, $\{h_1\mu_1 + h_2\mu_2 + \cdots\}$, $\{r_1\mu_1 + \cdots + r_m\mu_m\}$, or $\{r_1\mu_1 + r_2\mu_2 + \cdots\}$.

Let $f(s)$ be a function almost periodic in $[\alpha, \beta]$ having a finite integral base $\mu_1, \ldots, \mu_m$. Allowing terms with the coefficient 0, we may write its Dirichlet series in the form

$$f(s) \sim \Sigma a_{h_1, \ldots, h_m} e^{(h_1\mu_1 + \cdots + h_m\mu_m)s}.$$

For an arbitrary s in (α, β) the function $f(s+i\tau)$ has, therefore, considered as a function of the real variable τ, the Fourier series

$$f(s+i\tau) \sim \Sigma a_{h_1, \ldots, h_m} e^{(h_1\mu_1 + \cdots + h_m\mu_m)s} e^{i(h_1\mu_1 + \cdots + h_m\mu_m)\tau}.$$

Denoting its spatial extension by $g(s; x_1, \ldots, x_m)$, we have

$$f(s + i\tau) = g(s; \mu_1 \tau, \ldots, \mu_m \tau).$$

The function $g(s; x_1, \ldots, x_m)$ is easily seen to be an analytic almost periodic function of s in $[\alpha, \beta]$ for arbitrary values of $x_1, \ldots, x_m$, but need not be an analytic function of the variables $x_1, \ldots, x_m$. The formula

$$g(s; x_1, \ldots, x_m) \sim \Sigma a_{h_1, \ldots, h_m} e^{(h_1 \mu_1 + \cdots + h_m \mu_m) s} e^{i(h_1 x_1 + \cdots + h_m x_m)},$$

which for a given s gives the Fourier series of $g(s; x_1, \ldots, x_m)$, considered as a function of $x_1, \ldots, x_m$, will for fixed values of $x_1, \ldots, x_m$ give the Dirichlet series of the function, considered as a function of s.

Similar results hold in the case of an infinite integral base or a finite or infinite rational base.

What will be needed concerning the spatial extensions in the case of analytic almost periodic functions will be developed as easy consequences of the approximation theorem.

Distribution Problems for Almost Periodic Functions.

12. The variation of the argument of an almost periodic function $f(s)$ on vertical lines is closely connected with the distribution of the zeros of the function in vertical strips. This problem, or rather the problem of the distribution of the a-points for an arbitrary a, was first treated in the case of the Riemann zeta function by methods similar to those applied in the case of Lagrange's problem, though without reference to the actual connection with problems of mean motion. Historically, these investigations are at the origin of the theory of almost periodic functions.

As is well known, the zeta function $\zeta(s)$ is a function of the complex variable $s = \sigma + it$ defined in the whole plane and regular, except at the point $s = 1$, where it has a simple pole. In the half-plane $\sigma > 1$, it is given by the two equivalent absolutely convergent representations

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{n=1}^{\infty} \frac{1}{1 - p_n^{-s}},$$

where in the second, the Euler product, p_n runs through the sequence of primes $2, 3, 5, \ldots$. From this representation as a product it immediately follows that $\zeta(s)$ has no zeros in the half-plane $\sigma > 1$.

As shown by Riemann, the zeta function satisfies a simple functional equation connecting the values of the function at the points s and $1-s$. These points lie symmetrically with respect to $s=\frac{1}{2}$, and the function is therefore usually considered in the half-plane $\sigma \geqq \frac{1}{2}$ only. The famous, unproved Riemann hypothesis concerning the zeros of $\zeta(s)$ states that all zeros belonging to this half-plane $\sigma \geqq \frac{1}{2}$ are situated on the boundary line $\sigma=\frac{1}{2}$ itself, i. e. $\zeta(s)$ is different from zero, not only in the half-plane $\sigma>1$, but even in the larger half-plane $\sigma>\frac{1}{2}$.

On account of the Euler product it is convenient, instead of the function $\zeta(s)$ itself, to consider the function $\log \zeta(s)$. In the half-plane $\sigma>1$ a regular branch of this function is given by the expression

$$\log \zeta(s)=\sum_{n=1}^{\infty}-\log(1-p_n^{-s}), \tag{9}$$

where in each term on the right $-\log(1-z)=z+\frac{1}{2}z^2+\cdots$. By $\log\zeta(s)$ in the half-plane $\sigma>\frac{1}{2}$ we shall mean the analytic continuation of this branch in the domain obtained from this half-plane by omitting the segment $\frac{1}{2}<\sigma\leqq 1$, $t=0$, and all segments $\frac{1}{2}<\sigma\leqq\sigma_0$, $t=t_0$, where σ_0+it_0 denote the zeros (if any) of $\zeta(s)$ in $\sigma>\frac{1}{2}$.[1]

13. By means of the expression (9) the closure $M(\sigma)$ of the range of values of $\log\zeta(s)$ on a vertical line with given abscissa $\sigma>1$ was studied in detail by Bohr [2], [3].

From (9) we find

$$\log\zeta(\sigma+it)=\sum_{n=1}^{\infty}-\log(1-p_n^{-\sigma}e^{-i\log p_n t}). \tag{10}$$

Thus, $\log\zeta(\sigma+it)$ is an infinite sum of vectors, each of which describes periodically a certain closed curve. Now, on account of the unique representation of an integer as a product of powers of primes, the numbers $\log p_n$ are linearly independent. It follows therefore if we apply Kronecker's theorem to the partial sums of the series (10), and afterwards pass to the limit, that the closure of the range of values of $\log\zeta(\sigma+it)$ is identical with the range of values of the function

$$G(x_1, x_2, \ldots)=\sum_{n=1}^{\infty}-\log(1-p_n^{-\sigma}e^{ix_n}),$$

[1] In some of the papers quoted below, the function is considered only for $t>0$, which means that the actual results are more precise than those quoted. For the sake of uniformity, we have paid no attention to this difference, which is unimportant from the point of view of method.

where the terms describe, independently of each other, the above mentioned curves. The set $M(\sigma)$ may therefore be described as the vectorial sum of these curves.

From this representation of $M(\sigma)$ and the simple fact that the curves are convex, the following result concerning the shape of the set $M(\sigma)$ was obtained: that $M(\sigma)$ is for each $\sigma > 1$ either a closed domain bounded by a single convex curve $A(\sigma)$, or a closed ring-shaped domain bounded by two convex curves $A(\sigma)$ and $B(\sigma)$, where $B(\sigma)$ lies inside $A(\sigma)$. Some results regarding the variation of these curves with σ were also obtained.

It was further proved that the set $M(\sigma)$ is identical with the set of values actually taken by $\log \zeta(s)$ in points arbitrarily near to the line with the abscissa σ. This means that the range of values of $\log \zeta(s)$ in a vertical strip $\sigma_1 < \sigma < \sigma_2$, where $1 < \sigma_1 < \sigma_2 < +\infty$, is identical with the sum of the corresponding sets $M(\sigma)$.

Quite similar results had been obtained previously by Bohr [1] for the derivative $\zeta'(s)/\zeta(s)$ of the function $\log \zeta(s)$; only in this latter case the situation is simplified by the fact that the convex curves to be added turn out to be circles. The sum is therefore either the closed surface of a circle or a closed concentric circular ring. In this case it was possible by simple computations to prove the existence of a constant $\sigma_0 > 1$, such that for $\sigma \leqq \sigma_0$ we have the case of the circle, and for $\sigma > \sigma_0$ the case of the circular ring. A numerical calculation of σ_0 was given by Burrau [1]. Recently it was shown by Bohr and Jessen [4] that a similar situation (only with a different constant σ_0) holds for the function $\log \zeta(s)$.[1]

14. The corresponding problems in the case $\frac{1}{2} < \sigma \leqq 1$, which are more difficult on account of the divergence of the Euler product, were treated by Bohr [6], the method in question having first been applied to a study of the function $\zeta(s)$ on vertical lines by Bohr and Courant [1].

Regarding the values on a vertical line it was proved that, if $\frac{1}{2} < \sigma \leqq 1$, the values of $\log \zeta(\sigma + it)$ are everywhere densely distributed in the whole plane.

In order to prove this, it was first proved that though the formula (10) does not hold if $\frac{1}{2} < \sigma \leqq 1$, it is true that for any large m the partial sum

$$F_m(t) = \sum_{n=1}^{m} -\log(1 - p_n^{-\sigma} e^{-i \log p_n t})$$

[1] For the problem of the addition of convex curves, and for further results regarding the curves $A(\sigma)$ and $B(\sigma)$ we refer also to Bohr and Jessen [1], Haviland [2], Jessen and Wintner [1], Kerschner and Wintner [2], and Kerschner [1], [2], [3].

approaches $\log \zeta(\sigma + it)$ for most t in the sense that if $L_\varepsilon(T)$ denotes the sum of the lengths of those sub-intervals of $-T<t<T$ in which

(11) $$|\log \zeta(\sigma + it) - F_m(t)| \leqq \varepsilon,$$

then

$$\liminf_{T\to\infty} \frac{L_\varepsilon(T)}{2T}$$

is nearly 1 when m is large (for any given $\varepsilon > 0$).

On the other hand, if $\frac{1}{2} < \sigma \leqq 1$, the closure of the range of values of $F_m(t)$, which is identical with the range of values of the function

$$G_m(x_1, \ldots, x_m) = \sum_{n=1}^{m} - \log(1 - p_n^{-\sigma} e^{i x_n}),$$

is easily seen to converge towards the whole plane when $m \to \infty$.

These two facts are, however, not sufficient to prove that $\log \zeta(\sigma + it)$ comes arbitrarily near to any given value a, since the values of t for which $F_m(t)$ is near to a might all be among those for which $F_m(t)$ does not approach $\log \zeta(\sigma + it)$. This difficulty was overcome by means of the Kronecker-Weyl theorem, which, together with certain results concerning the distribution of the values of the functions $G_m(x_1, \ldots, x_m)$, shows that if $l_\varepsilon(T)$ denotes the sum of the lengths of those sub-intervals of $-T < t < T$ in which

(12) $$|F_m(t) - a| \leqq \varepsilon,$$

then

$$\liminf_{T\to\infty} \frac{l_\varepsilon(T)}{2T}$$

is not small when m is large, so that there exist values of t for which both relations (11) and (12) hold.

Regarding the values actually taken by $\log \zeta(s)$ in a vertical strip (σ_1, σ_2), where $\frac{1}{2} < \sigma_1 < \sigma_2 < 1$, the application of the Kronecker-Weyl theorem made it possible not only to prove that all values are taken in the strip, but even to consider the frequency with which they are taken. It was, in fact, proved that if, for an arbitrary a, we denote by $N_a(\sigma_1, \sigma_2; T)$ the number of a-points[1] of $\log \zeta(\sigma)$ in the rectangle $\sigma_1 < \sigma < \sigma_2$, $-T < t < T$, then

$$\liminf_{T\to\infty} \frac{N_a(\sigma_1, \sigma_2; T)}{2T} > 0.$$

[1] Except when there is an explicit statement to the contrary, multiple a-points are counted according to their order of multiplicity.

This result implies, of course, that if $M_a(\sigma_1, \sigma_2; T)$ denotes the number of a-points of $\zeta(s)$ itself in the same rectangle, then we have also

$$\liminf_{T\to\infty} \frac{M_a(\sigma_1, \sigma_2; T)}{2T} > 0$$

provided $a \neq 0$. In conjunction with the theorem of Bohr and Landau [1], according to which

$$\lim_{T\to\infty} \frac{M_0(\sigma_1, \sigma_2; T)}{2T} = 0,$$

this result showed that, independently of the truth or untruth of the Riemann hypothesis, the number 0 plays an exceptional part for the function $\zeta(s)$ in the half-plane $\sigma > \frac{1}{2}$.

15. Through these results, the possibilities of the method were, however, by no means fully utilized. More precise results were announced by Bohr [7] and proved in detail by Bohr and Jessen [1], [2], [3].

Regarding the distribution of the values of $\log \zeta(s)$ on vertical lines, it was proved that there exists for every $\sigma > \frac{1}{2}$ a continuous function $D(z)$ of the complex variable $z = u + iv$, such that if $l(T)$ denotes the sum of the lengths of those sub-intervals of $-T < t < T$ in which $\log \zeta(\sigma + it)$ belongs to a given rectangle $R(u_1 < u < u_2,\ v_1 < v < v_2)$, then the limit

$$\varphi(R) = \lim_{T\to\infty} \frac{l(T)}{2T}$$

exists, and is equal to the integral of $D(z)$ over the rectangle R:

$$\varphi(R) = \int_{u_1}^{u_2}\int_{v_1}^{v_2} D(z)\, du\, dv. \tag{13}$$

In the case $\frac{1}{2} < \sigma \leqq 1$, it was proved that $D(z) > 0$ for all z.

Regarding the distribution of the values in a vertical strip (σ_1, σ_2), where $\frac{1}{2} < \sigma_1 < \sigma_2$, it was proved that the limit

$$H_a(\sigma_1, \sigma_2) = \lim_{T\to\infty} \frac{N_a(\sigma_1, \sigma_2; T)}{2T}$$

exists and is a continuous function of σ_1 and σ_2 for any fixed a. This limit may be called the relative frequency with which the function $\log \zeta(s)$ takes the value a in the strip (σ_1, σ_2).

As follows from its representation, the function $\zeta(s)$ is almost periodic in $[1, +\infty)$ and so is, too, the function $\log\zeta(s)$, while a certain generalized almost periodicity is present for $\frac{1}{2} < \sigma \leqq 1$, owing to the approximation property of the partial sums referred to above. This almost periodicity makes the regularity of the distribution less surprising, but was not used in the proofs, which are based directly on the definition.

16. A simplification of the preceding method was used by Jessen [1] to discuss the distribution of the values of an arbitrary almost periodic function $f(s)$ with an infinite Dirichlet series and linearly independent exponents. In this case the series is absolutely convergent in the strip of almost periodicity, and therefore represents the function, so that the discussion may again be based on the explicit representation

$$f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}$$

without direct use of the almost periodic character.

Instead of discussing first the partial sums of this series by means of the Kronecker-Weyl theorem, and then passing to the limit, the series is here discussed directly, the necessary theory of measure and integration in infinitely many dimensions having first been developed. The results are analogous to those obtained for the function $\log\zeta(s)$, with the addition of a certain uniformity, which is also present for $\log\zeta(s)$ in the half-plane $\sigma > 1$. Thus it is proved regarding the a-points of $f(s)$ in a strip (σ_1, σ_2), inside the strip of almost periodicity, that the limit

$$(14) \qquad H_a(\sigma_1, \sigma_2) = \lim_{(\delta-\gamma)\to\infty} \frac{N_a(\sigma_1, \sigma_2;\, \gamma, \delta)}{\delta - \gamma}$$

exists, where $N_a(\sigma_1, \sigma_2; \gamma, \delta)$ denotes the number of a-points of $f(s)$ in the rectangle $\sigma_1 < \sigma < \sigma_2$, $\gamma < t < \delta$. This limit is again a continuous function of σ_1 and σ_2.

17. The distribution of the values of an arbitrary real-valued almost periodic function $F(t)$ of a real variable was studied by Wintner [1], [4], [5] by applying the moment method of the calculus of probability. It was shown by Haviland [1] that this method is also valid if $F(t)$ is complex-valued. The result is, briefly stated, that if $F(t) = U(t) + i\,V(t)$ is an arbitrary almost periodic function and if $l(T)$ denotes the measure of the set in $-T < t < T$ in which $F(t)$ belongs to a given rectangle $R\,(u_1 < u < u_2,\ v_1 < v < v_2)$ in the $z = u + iv$-plane, then the limit

(15) $$\varphi(R) = \lim_{T \to \infty} \frac{l(T)}{2T}$$

exists for all rectangles the sides of which do not lie on a certain enumerable number of lines, and this function $\varphi(R)$, which is called the asymptotic distribution function of $F(t)$, is determined by the moment condition

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u^n v^m \, d_{u,v}\, \varphi(R) = \underset{t}{M}\{U(t)^n V(t)^m\}$$

for all pairs of non-negative integers n and m. It was shown by Bohr [11] that the limit (15) need not exist for all rectangles.

In the special case of linearly independent exponents as considered by Wintner [5], [6] it was essential to work not only with moments but also with Fourier transforms. It was shown by Bochner and Jessen [1] that the whole problem might be treated without recourse to the moment theory, considering only Fourier transforms, a method which also holds in the case of generalized almost periodic functions. The moment condition is then replaced by the condition

(16) $$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{i(xu+yv)} \, d_{u,v}\, \varphi(R) = \underset{t}{M}\{e^{i(xU(t)+yV(t))}\}$$

for all pairs of real numbers x and y.

A systematic exposition of the whole subject, including a new treatment of the functions $\log\zeta(\sigma+it)$ has been given by Jessen and Wintner [1]. The expression (13) of the distribution function of the function $\log\zeta(\sigma+it)$ by means of a continuous density $D(z)$ is here obtained by using formula (16). The method leads to an explicit expression of the density $D(z)$ showing, among other things, that $D(z)$ possesses continuous partial derivatives of arbitrarily high order with respect to the coordinates u and v. A similar result had previously been obtained by Wintner [6] in the case of functions with linearly independent exponents.[1]

18. The method mentioned in § 16 was developed further by Jessen [3] employing the method of Fourier transforms. It is here proved, among other things, that the relative frequency (14) is expressible in the form

$$H_a(\sigma_1, \sigma_2) = \int_{\sigma_1}^{\sigma_2} E_a(\sigma)\, d\sigma,$$

[1] For further results we also refer to Kerschner and Wintner [1], [3], van Kampen and Wintner [1], van Kampen [2], [3], Hartman, van Kampen, and Wintner [2], and Haviland [3].

where the function $E_a(\sigma)$ (for which an explicit expression is obtained) is a continuous function of a and σ. This function may be called the relative frequency with which the function $f(s)$ takes the value a in the neighbourhood of the vertical line with the abscissa σ.

The assumption that the Dirichlet series of $f(s)$ contains an infinite number of terms is here replaced by the weaker assumption that it contains at least five terms.

Mean Motions and Zeros of Analytic Almost Periodic Functions.

19. The distribution of the zeros of an arbitrary function $f(s)$, almost periodic in a strip $[\alpha, \beta]$, in vertical strips has been treated by Jessen [2][1] by establishing a formula analogous to the Jensen formula

$$(17) \qquad \frac{1}{2\pi}\int_0^{2\pi} \log|F(re^{i\theta})|\,d\theta = \log|F(0)| + \sum_{\nu=1}^{n} \log\frac{r}{|z_\nu|}$$

for a function $F(z)$ regular in a circle $|z| \leqq r$, and having the zeros $z_1, \ldots, z_n$ $(\neq 0)$ in this circle. The method was also used for a study of the variation of the argument on such vertical lines on which the function does not come arbitrarily near to 0; an application of this has been given in Jessen [5]. Later it was shown by Hartman [1] that the method can also be used for a study of the variation of the argument on such vertical lines on which the function comes arbitrarily near to, or takes, the value 0.

From a formal point of view, the method is very simple, and may be briefly described as follows.

Setting aside the difficulties arising from the zeros, we consider the function

$$\log f(s) = \log|f(s)| + i \arg f(s).$$

According to the Cauchy-Riemann differential equations we then have

$$(18) \qquad \frac{d}{dt}\arg f(\sigma + it) = \frac{d}{d\sigma}\log|f(\sigma + it)|.$$

---

[1] Some results of a more elementary nature had been obtained previously by Favard [1]. The distribution of the zeros of exponential polynomials (and of more general classes of functions) has been studied by Tamarkin [1], [2], [3], Wilder [1], Pólya [1], Schwengeler [1], and Ritt [1].

Now, if it exists, the mean motion $c(\sigma)$ of the function $f(\sigma+it)$ is evidently equal to the mean value

$$\underset{t}{M}\left\{\frac{d}{dt}\arg f(\sigma+it)\right\},$$

and hence by (18) equal to

$$\underset{t}{M}\left\{\frac{d}{d\sigma}\log|f(\sigma+it)|\right\}.$$

Interchanging in this expression the differentiation and the formation of the mean value, we arrive at the following determination of the mean motion: Corresponding to the function $f(s)$ we form the function

$$\varphi(\sigma)=\underset{t}{M}\{\log|f(\sigma+it)|\}. \tag{19}$$

Then the mean motion $c(\sigma)$ of the function $f(\sigma+it)$ is determined as the derivative

$$c(\sigma)=\varphi'(\sigma). \tag{20}$$

The connection with the distribution of the zeros is, from a formal point of view, equally simple.

Denoting for $\alpha<\sigma_1<\sigma_2<\beta$ by $N(\sigma_1,\sigma_2;\gamma,\delta)$ the number of zeros of $f(s)$ in the rectangle $\sigma_1<\sigma<\sigma_2$, $\gamma<t<\delta$, we define the *relative frequency of zeros* of $f(s)$ in the strip (σ_1,σ_2) as the limit

$$H(\sigma_1,\sigma_2)=\lim_{(\delta-\gamma)\to\infty}\frac{N(\sigma_1,\sigma_2;\gamma,\delta)}{\delta-\gamma},$$

provided this limit exists.

Now the number $N(\sigma_1,\sigma_2;\gamma,\delta)$ is, apart from a factor $1/2\pi$, equal to the variation of the argument of $f(s)$ along the boundary of the rectangle. Setting aside the contributions from the horizontal sides, we therefore find that, apart from the factor $1/2\pi$, it is equal to the difference between the variations of the argument along the vertical sides, both being described in the same direction. The relative frequency $H(\sigma_1,\sigma_2)$ is, therefore, simply the difference $c(\sigma_2)-c(\sigma_1)$ between the mean motions on the two lines $\sigma=\sigma_1$ and $\sigma=\sigma_2$ multiplied by $1/2\pi$, so that by (20)

$$H(\sigma_1,\sigma_2)=\frac{1}{2\pi}(\varphi'(\sigma_2)-\varphi'(\sigma_1)). \tag{21}$$

The left-hand side being as a matter of course always $\geqq 0$, a remarkable consequence of this formula is that $\varphi'(\sigma)$ is an increasing function, or, what amounts to the same, that $\varphi(\sigma)$ is convex.[1]

20. The question now arises as to how far the results obtained by these formal considerations are actually true. The following answer is given in the papers quoted above:

The mean value (19) really exists for all σ in the interval (α, β), though the function $\log |f(\sigma+it)|$ is not necessarily almost periodic, and $\varphi(\sigma)$ is a continuous, convex function. This function is not necessarily differentiable; but if it is differentiable at the point σ, the mean motion $c(\sigma)$ exists and is given by (20), and if it is differentiable at the points σ_1 and σ_2, the relative frequency $H(\sigma_1, \sigma_2)$ exists and is given by (21).

The function $\varphi(\sigma)$ is called the *Jensen function* corresponding to $f(s)$, and the formula (21) is called the *Jensen formula* for almost periodic functions. In the case of a periodic function with the period ip in a half-plane $\sigma < \beta$, converging for $\sigma \to -\infty$ towards a constant $\neq 0$, it reduces itself to the usual Jensen formula (17) by means of the substitution $e^{\frac{2\pi}{p}s} = z$.

A convex function being differentiable everywhere except in a finite or enumerable set of points, these results show in particular, that the mean motion $c(\sigma)$ and the relative frequency $H(\sigma_1, \sigma_2)$ generally exist.

The following result is an easy consequence of the Jensen formula: A necessary and sufficient condition that the function $f(s)$ has no zeros in a strip (α_0, β_0) is that the Jensen function $\varphi(\sigma)$ is linear in the interval (α_0, β_0). Since the function does not come arbitrarily near to 0 on a vertical line in such a strip, this implies, by Bohr's theorem (§ 7), that the constant value of $\varphi'(\sigma)$ in a linearity interval of $\varphi(\sigma)$ always belongs to the modul of the function.

21. In Chapter II we give a detailed exposition of the investigations just described.

Instead of operating for every σ with one continuous branch of the argument of the function $f(\sigma + it)$ determined mod. π, we find it convenient to introduce two arguments, viz. a left argument $\arg^- f(\sigma + it)$, and a right argument $\arg^+ f(\sigma + it)$, both of which are determined mod. 2π. These arguments are

[1] Except when there is an explicit statement to the contrary, these expressions are used in the wide sense (i. e. including functions having intervals of constancy or linearity respectively).

characterized by being continuous except for those values of t for which $f(\sigma+it)$ is zero, and discontinuous with a jump of $-p\pi$ or $p\pi$ respectively, when $\sigma+it$ is a zero of the order p. In describing the line $\sigma+it$ this corresponds to encircling the zeros to the left or the right respectively. In the discontinuity points we define the two arguments by the mean value of the limits from both sides. Obviously the mean value $\frac{1}{2}(\arg^- f(\sigma+it)+\arg^+ f(\sigma+it))$ gives the previous determination of the argument as a continuous function determined mod. π.

Corresponding to these two arguments we consider the four quantities

$$\left.\begin{matrix} \underline{c}^-(\sigma) \\ \bar{c}^-(\sigma) \end{matrix}\right\} = \lim_{(\delta-\gamma)\to\infty} \begin{matrix}\inf \\ \sup\end{matrix} \frac{\arg^- f(\sigma+i\delta)-\arg^- f(\sigma+i\gamma)}{\delta-\gamma}$$

and

$$\left.\begin{matrix} \underline{c}^+(\sigma) \\ \bar{c}^+(\sigma) \end{matrix}\right\} = \lim_{(\delta-\gamma)\to\infty} \begin{matrix}\inf \\ \sup\end{matrix} \frac{\arg^+ f(\sigma+i\delta)-\arg^+ f(\sigma+i\gamma)}{\delta-\gamma},$$

which we call the *lower and upper, left and right mean motions* of $f(\sigma+it)$. If $\underline{c}^-(\sigma)=\bar{c}^-(\sigma)$ or $\underline{c}^+(\sigma)=\bar{c}^+(\sigma)$, that is to say, if the limit in question exists, we call it simply the *left or right mean motion*, and denote it by $c^-(\sigma)$ or $c^+(\sigma)$. If, as previously, we define the mean motion $c(\sigma)$ as the limit

$$c(\sigma)=\lim_{(\delta-\gamma)\to\infty}\frac{\arg f(\sigma+i\delta)-\arg f(\sigma+i\gamma)}{\delta-\gamma},$$

where $\arg f(\sigma+it)$ denotes a continuous branch of the argument determined mod. π, provided this limit exists, it is obvious that the existence of the left and right mean motions $c^-(\sigma)$ and $c^+(\sigma)$ implies the existence of $c(\sigma)$ and that

$$c(\sigma)=\tfrac{1}{2}(c^-(\sigma)+c^+(\sigma)).$$

Being convex the Jensen function $\varphi(\sigma)$ possesses for every value of σ a left derivative $\varphi'(\sigma-0)$ and a right derivative $\varphi'(\sigma+0)$. As a main result we prove for every σ the relations

$$\varphi'(\sigma-0)\leqq\underline{c}^-(\sigma)\leqq\left\{\begin{matrix}\underline{c}^+(\sigma)\\ \bar{c}^-(\sigma)\end{matrix}\right\}\leqq\bar{c}^+(\sigma)\leqq\varphi'(\sigma+0). \tag{22}$$

This implies, of course, that if $\varphi(\sigma)$ is differentiable at the point σ both $c^-(\sigma)$ and $c^+(\sigma)$, and hence also $c(\sigma)$, exist and are equal to $\varphi'(\sigma)$, so that (22) is a generalization of (20).

Regarding the zeros in a strip $(\alpha <) \sigma_1 < \sigma < \sigma_2 (< \beta)$ we consider the two quantities

$$\left.\begin{matrix}\underline{H}(\sigma_1, \sigma_2)\\ \overline{H}(\sigma_1, \sigma_2)\end{matrix}\right\} = \lim_{(\delta - \gamma) \to \infty} \begin{matrix}\inf \\ \sup\end{matrix} \frac{N(\sigma_1, \sigma_2; \gamma, \delta)}{\delta - \gamma},$$

which we call the *lower and upper relative frequencies of zeros* in the strip (σ_1, σ_2). We also find a number of inequalities connecting these two frequencies and the four mean motions. If $\varphi(\sigma)$ is differentiable at the points σ_1 and σ_2, these inequalities are reduced to the Jensen formula.

22. In Chapter III we give a detailed discussion of the distribution of the values of so-called almost periodic sequences, and in Chapter IV we study a special class of analytic almost periodic functions connected with such sequences. The results obtained are a necessary preparation for later constructions of analytic almost periodic functions with prescribed properties, and have, in fact, their origin in earlier constructions of this kind, which will be referred to below.

23. In Chapter V we answer the question as to which sets of six numbers can occur as left and right derivatives of the Jensen function and as lower and upper, left and right mean motions, for a given value of σ, of a function $f(s)$ with exponents from a given modul M.

In the special case where M is discrete, so that the question is about periodic functions, it is easily proved that the left and right mean motions $c^-(\sigma)$ and $c^+(\sigma)$ always exist and are determined by the relations

$$(23) \qquad c^-(\sigma) = \varphi'(\sigma - 0) \quad \text{and} \quad c^+(\sigma) = \varphi'(\sigma + 0),$$

in which both derivatives $\varphi'(\sigma - 0)$ and $\varphi'(\sigma + 0)$ belong to M. If, conversely, d^- and d^+ are given numbers belonging to M, and $d^- \leqq d^+$, there exists a function with exponents from M, such that for the given value of σ we have $\varphi'(\sigma - 0) = d^-$ and $\varphi'(\sigma + 0) = d^+$.

From existing examples (Jessen [2] and Hartman [1]) it follows that the mean motions $c^-(\sigma)$ and $c^+(\sigma)$ do not exist in all cases. We now prove that in the case of an everywhere dense modul M, the above relations (22) are the best possible in the sense that if six given real numbers satisfy the relations

$$d^- \leqq \underline{c}^- \leqq \left\{\begin{matrix}\underline{c}^+\\ \bar{c}^-\end{matrix}\right\} \leqq \bar{c}^+ \leqq d^+,$$

then there exists an almost periodic function with exponents from M, for which these numbers for the given value of σ are equal to the corresponding numbers in the relations (22).

We also prove a theorem concerning the case where the function has no zeros on the vertical line with the abscissa σ.

24. In Chapter VI we consider the problem as to which functions $\varphi(\sigma)$ can occur as the Jensen function of almost periodic functions $f(s)$ with exponents from a given modul M. Necessary conditions for the occurrence of a function $\varphi(\sigma)$ are, by the results already mentioned, that $\varphi(\sigma)$ is convex and that the value of $\varphi'(\sigma)$ in any linearity interval of $\varphi(\sigma)$ belongs to M.

In the special case where M consists of all integral multiples of a number $\frac{2\pi}{p}$, so that the question is about periodic functions with the period ip, the answer can easily be given by means of Weierstrass' product theorem, when, by the transformation $e^{\frac{2\pi}{p}s}=z$, we map the strip in question on a circular ring $a<|z|<b$. The functions $\varphi(\sigma)$ possible are here all stretchwise linear convex functions $\varphi(\sigma)$ for which the values of $\varphi'(\sigma)$ in the linearity intervals belong to M. In the case where M consists of all rational multiples of a number $\frac{2\pi}{p}$, so that the question is about limit periodic functions with the limit period ip, the problem has been solved by Buch [1], [2]. In this case the necessary conditions mentioned above are also sufficient, that is to say that the possible functions $\varphi(\sigma)$ are in this case all convex functions $\varphi(\sigma)$ for which the value of $\varphi'(\sigma)$ in any linearity interval of $\varphi(\sigma)$ belongs to M. We now give a solution of the problem for functions with an arbitrary finite or infinite, integral or rational base, i. e. for all moduls of the form $M=\{h_1\mu_1+\cdots+h_m\mu_m\}$, $\{h_1\mu_1+h_2\mu_2+\cdots\}$, $\{r_1\mu_1+\cdots+r_m\mu_m\}$, or $\{r_1\mu_1+r_2\mu_2+\cdots\}$. The results immediately lead to a characterization of all functions which occur at all as the Jensen function of an almost periodic function. Contrary to what might have been expected, these are not all convex functions. The exact characterization is the following:

A function $\varphi(\sigma)$ in the interval $\alpha<\sigma<\beta$ is the Jensen function of a function $f(s)$ almost periodic in $[\alpha,\beta]$ if and only if it is convex and there correspond to every reduced interval $(\alpha<)\,\alpha_0<\sigma<\beta_0\,(<\beta)$ a finite set of linearly independent numbers $\mu_1,\ldots,\mu_m$ and a positive number k, such that if σ_1 and σ_2, where $\alpha_0<\sigma_1<\sigma_2<\beta_0$, belong to different linearity intervals of $\varphi(\sigma)$, then the difference $\varphi'(\sigma_2)-\varphi'(\sigma_1)$ is of the form

$$\varphi'(\sigma_2) - \varphi'(\sigma_1) = r_1\mu_1 + \cdots + r_m\mu_m,$$

where the coefficients $r_1, \ldots, r_m$ are rational numbers, and

$$\frac{\varphi'(\sigma_2) - \varphi'(\sigma_1)}{\sqrt{r_1^2 + \cdots + r_m^2}} \geqq k.$$

25. The existence of a left and a right mean motion $c^-(\sigma)$ and $c^+(\sigma)$ determined by the relations (23) and the stretchwise linearity of the Jensen function in the case of periodic functions naturally leads to the question whether more general classes of almost periodic functions exist for which results of a similar precision hold.

This problem is treated in Chapter VII, where it is proved that the relations (23) hold for all almost periodic functions $f(s)$ with a finite integral base $\mu_1, \ldots, \mu_m$ for which the spatial extension $g(s; x_1, \ldots, x_m)$ is analytic, not only in s, but in all the variables $s, x_1, \ldots, x_m$. From the relations (23) it follows, of course, that the mean motion $c(\sigma)$ exists, and is given by

$$c(\sigma) = \tfrac{1}{2}(\varphi'(\sigma - 0) + \varphi'(\sigma + 0)). \tag{24}$$

To the functions with a finite integral base and an analytic spatial extension belong in particular all exponential polynomials. As every exponential polynomial $F(t)$ of the real variable t may be written in the form $f(it)$, where $f(s)$ is an exponential polynomial of s, the preceding result contains a complete solution of Lagrange's problem to the effect that the mean motion exists in all cases. This result together with the expression (24) for the mean motion has been stated without proof in Jessen [6].

For functions with a finite integral base and an analytic spatial extension it will also be proved that the Jensen function is stretchwise differentiable. For an exponential polynomial this implies, since the zeros all belong to a finite vertical strip, that the number of non-differentiability points is finite.

The proofs of these general theorems depend on a further elaboration of the Bohl-Weyl method.

Functions with a finite integral base and an analytic spatial extension were also considered by Hartman [1], who showed, among other things, that in the case of an exponential polynomial the mean motion $c(\sigma)$ exists for all σ not belonging to a certain finite set. This exceptional set is exactly the set of those values of σ for which Weyl's proof (§ 3) of the existence of the mean motion fails for the function $f(\sigma + it)$, because the set of zeros of its spatial extension

$g(\sigma; x_1, \ldots, x_m)$ contains a manifold of the dimension $m - 1$. It is easily shown that this exceptional set is identical with the set of non-differentiability points of the Jensen function.

Analogous results are obtained in the case of functions with an infinite integral base, the necessary consideration of analytic functions of an infinite number of variables involving no difficulties. To the functions with an infinite integral base and an analytic spatial extension belong in particular all ordinary Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

in the half-plane $[\alpha, +\infty)$ where the function is almost periodic.

The connection between the variation of the argument and the distribution of the zeros implies that the relations (23) hold for all σ if and only if the relative frequency of zeros exists for every strip (σ_1, σ_2) and is determined by the formula

$$H(\sigma_1, \sigma_2) = \frac{1}{2\pi}(\varphi'(\sigma_2 - 0) - \varphi'(\sigma_1 + 0)).$$

Thus the Jensen formula is valid in this more precise form, not only for periodic functions, but also for all exponential polynomials and ordinary Dirichlet series.

The study of almost periodic functions with a finite integral base and an analytic spatial extension will be continued by Tornehave in another paper, in connection with an extension of the Jensen formula to analytic functions of a finite number of variables (see Tornehave [1]).

26. To the functions with a (finite or infinite) integral base and an analytic spatial extension belong all functions $f(s)$ with linearly independent exponents. The result mentioned in § 18, together with the Jensen formula, shows that if the Dirichlet series contains at least five terms, and if the exponents are linearly independent, the Jensen function $\varphi_a(\sigma)$ of the function $f(s) - a$ is twice differentiable with the second derivative

$$\varphi_a''(\sigma) = E_a(\sigma).$$

In particular, the mean motion of $f(\sigma + it) - a$ exists for all a and σ.

A similar result holds in case of the zeta function in the half-plane $\sigma > 1$. We mention that the whole theory may be extended to generalized almost periodic functions, and that, in particular, results analogous to the preceding ones hold in case of the zeta function in the strip $\frac{1}{2} < \sigma \leqq 1$ also.

CHAPTER I.

Mean Motions of Almost Periodic Functions of a Real Variable.

27. For an arbitrary continuous function $F(t)$, which does not take the value 0, we denote by $\arg F(t)$ an arbitrary continuous branch of the argument of $F(t)$, defined mod. 2π by the condition $F(t) = |F(t)|\, e^{i \arg F(t)}$.

If the continuous function $F(t)$ takes the value 0 it need not be possible to define the argument as a continuous function. We shall consider only the case of functions having but isolated zeros. Such a function $F(t)$ is said to possess a continuous argument if it may be written in the form $F(t) = \varrho(t)\, e^{i\varphi(t)}$, where $\varphi(t)$ is continuous. In every interval in which $F(t)$ has no zeros we have then either $\varrho(t) = |F(t)|$ or $\varrho(t) = -|F(t)|$. The argument $\varphi(t) = \arg F(t)$ is now only determined mod. π. If $F(t)$ is regular for all values of t, it evidently possesses a continuous argument.

If $-\infty < \gamma < \delta < +\infty$ the difference

$$\arg F(\delta) - \arg F(\gamma)$$

is independent of the choice of the branch of the argument. The function $F(t)$ is said to possess the *mean motion*

$$c = \lim_{(\delta-\gamma)\to\infty} \frac{\arg F(\delta) - \arg F(\gamma)}{\delta - \gamma},$$

if this limit exists.

In this chapter we shall confine ourselves to functions which do not come arbitrarily near to 0, i. e. which satisfy a relation of the form

$$(1) \qquad |F(t)| \geqq k > 0$$

for all t. Regarding this case we shall prove the following theorem of Bohr [11], [13].

Theorem 1. *Let $F(t)$ be an arbitrary almost periodic function which does not come arbitrarily near to 0. Then $F(t)$ possesses a mean motion c and, if we put*

$$\arg F(t) = ct + \psi(t),$$

the remainder $\psi(t)$, too, is an almost periodic function.

The mean motion c and the exponents of $\psi(t)$ all belong to the modul of the function $F(t)$.

28. First we shall repeat Bohr's proof in a slightly simplified form.

The function $F(t)$ being uniformly continuous, the condition (1) evidently implies that $\arg F(t)$ is also uniformly continuous. Hence there corresponds to every positive number τ a number K_τ, such that for $|l| \leqq \tau$ and all t

$$|\arg F(t+l) - \arg F(t)| \leqq K_\tau.$$

For an arbitrary positive $\varepsilon < \pi$ we denote by $\tau = \tau(\eta)$ an arbitrary positive translation number of $F(t)$ belonging to $\eta = 2k \sin \frac{1}{2}\varepsilon$. Then the difference $\arg F(t+\tau) - \arg F(t)$ is evidently for every t mod. 2π numerically $\leqq \varepsilon$. Hence there exists an integer n_τ independent of t such that for all t

$$(2) \qquad |\arg F(t+\tau) - \arg F(t) - n_\tau 2\pi| \leqq \varepsilon.$$

Now, for $-\infty < \gamma < \delta < +\infty$, we have $\delta = \gamma + h\tau + l$, where h is a non-negative integer and $0 \leqq l < \tau$. Thus

$$|\arg F(\gamma+\tau) - \arg F(\gamma) - n_\tau 2\pi| \leqq \varepsilon$$

$$|\arg F(\gamma+2\tau) - \arg F(\gamma+\tau) - n_\tau 2\pi| \leqq \varepsilon$$

$$\dots$$

$$|\arg F(\gamma+h\tau) - \arg F(\gamma+(h-1)\tau) - n_\tau 2\pi| \leqq \varepsilon$$

$$|\arg F(\delta) - \arg F(\gamma+h\tau)| \leqq K_\tau,$$

and consequently

$$|\arg F(\delta) - \arg F(\gamma) - h n_\tau 2\pi| \leqq h\varepsilon + K_\tau,$$

so that

$$\left|\arg F(\delta) - \arg F(\gamma) - \frac{\delta-\gamma}{\tau} n_\tau 2\pi\right| \leqq h\varepsilon + K_\tau + |n_\tau| 2\pi \leqq \frac{\delta-\gamma}{\tau}\varepsilon + C_\tau,$$

where $C_\tau = K_\tau + |n_\tau| 2\pi$. Hence

$$\left|\frac{\arg F(\delta) - \arg F(\gamma)}{\delta-\gamma} - \frac{n_\tau 2\pi}{\tau}\right| \leqq \frac{\varepsilon}{\tau} + \frac{C_\tau}{\delta-\gamma}.$$

As τ may be chosen arbitrarily large, this inequality implies the existence of the mean motion

$$c = \lim_{(\delta-\gamma)\to\infty} \frac{\arg F(\delta) - \arg F(\gamma)}{\delta-\gamma},$$

and it also shows that

$$\left|c - \frac{n_\tau 2\pi}{\tau}\right| \leqq \frac{\varepsilon}{\tau}$$

or

(3) $$|c\tau - n_\tau 2\pi| \leqq \varepsilon.$$

By § 5, this relation shows that c belongs to the modul of $F(t)$.

Now, putting $\arg F(t) = ct + \psi(t)$, we find from (2) that

$$|c\tau + \psi(t+\tau) - \psi(t) - n_\tau 2\pi| \leqq \varepsilon,$$

and combining this with (3), we find

(4) $$|\psi(t+\tau) - \psi(t)| \leqq 2\varepsilon.$$

This shows that $\psi(t)$ is almost periodic and, by § 5, that the exponents of $\psi(t)$ belong to the modul of $F(t)$.

This completes the proof of the theorem.

29. For later application we add the following remark.

If $\tau = \tau(\eta)$ is a translation number of $F(t)$ belonging to $\eta = 2k \sin \frac{1}{2}\varepsilon$, it follows from (1) that for all t

(5) $$\Big|\log|F(t+\tau)| - \log|F(t)|\Big| \leqq \frac{\eta}{k} < \varepsilon.$$

Thus the function $\log|F(t)|$ is almost periodic and its exponents belong to the modul of $F(t)$.

It therefore follows from Theorem 1 that the function

$$\log F(t) = \log|F(t)| + i \arg F(t)$$

has the representation

$$\log F(t) = ict + H(t),$$

where $H(t) = \log|F(t)| + i\psi(t)$ is an almost periodic function with exponents from the modul of $F(t)$.

From (4) and (5) it follows that every translation number of $F(t)$ belonging to $\eta = 2k \sin \frac{1}{2}\varepsilon$ is a translation number of $H(t)$ belonging to 3ε. Thus there corresponds to every $\varepsilon > 0$ a number $\delta > 0$ depending on k, but not otherwise on $F(t)$, such that every $\tau_F(\delta)$ is a $\tau_H(\varepsilon)$.

30. In the special case where $F(t)$ is *periodic* with the period $p > 0$ the mean motion c is evidently determined by the expression

(6) $$c = \frac{\arg F(a+p) - \arg F(a)}{p},$$

where a may be arbitrarily chosen. The condition (1) is in this case satisfied if only $F(t)$ does not take the value 0.

For later application we shall prove that if $F(t)$ is an exponential polynomial with the period p, then the mean motion lies between the smallest and the largest exponent (or is equal to one of these), that is to say, if

$$F(t) = \sum_{h=h_1}^{h_2} a_h e^{ih\frac{2\pi}{p}t}, \quad a_{h_1} \neq 0, \quad a_{h_2} \neq 0,$$

then

(7)
$$h_1 \frac{2\pi}{p} \leqq c \leqq h_2 \frac{2\pi}{p}.$$

In order to see this, let us consider the function

$$f(z) = \sum_{h=h_1}^{h_2} a_h z^h$$

of the complex variable z. It follows from (6) that $c = h\frac{2\pi}{p}$, where $h\,2\pi$ denotes the variation of the argument of $f(z)$ along the unit circle $|z| = 1$. Now by Cauchy's theorem we have $h = h_1 + n$, where n denotes the number of zeros of $f(z)$ in $0 < |z| < 1$. As $0 \leqq n \leqq h_2 - h_1$, this implies that $h_1 \leqq h \leqq h_2$, and thus the relation (7).

31. The preceding proof of Theorem 1 is, apart from the last part of the theorem, based directly on the definition of almost periodicity. We shall now repeat a proof by Jessen [4] based on the approximation theorem and leading to a more precise result concerning the value of the mean motion c.

We shall first give an account of almost periodic functions $F(t)$ with a finite integral base $\mu_1, \ldots, \mu_m$, i. e. (see § 6) for which the exponents belong to the modul $M = \{h_1\mu_1 + \cdots + h_m\mu_m\}$, where the numbers $\mu_1, \ldots, \mu_m$ are linearly independent and the set of coefficients $h_1, \ldots, h_m$ runs through all sets of integers. Denoting the inner product $x_1 y_1 + \cdots + x_m y_m$ of two vectors $\boldsymbol{x} = (x_1, \ldots, x_m)$ and $\boldsymbol{y} = (y_1, \ldots, y_m)$ in the m-dimensional space R_m by $\boldsymbol{x}\boldsymbol{y}$, and putting $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)$, we have $M = \{\boldsymbol{h}\boldsymbol{\mu}\}$, where $\boldsymbol{h} = (h_1, \ldots, h_m)$ runs through all vectors of R_m with integral coordinates.

In the discussion an essential part is played by *Kronecker's theorem*, according to which the set of points $\boldsymbol{x} = \boldsymbol{\mu} t = (\mu_1 t, \ldots, \mu_m t), -\infty < t < +\infty$, is mod. 2π everywhere dense in R_m when $\mu_1, \ldots, \mu_m$ are linearly independent. By means of this theorem we shall now deduce the so-called spatial extension of a function $F(t)$ with exponents from M.

Allowing terms with the coefficient 0, we may write the Fourier series of $F(t)$ in the form

$$F(t) \sim \Sigma a_h e^{i h \mu t}.$$

Let us now consider a sequence of exponential polynomials of the form

$$F_p(t) = \Sigma a_h^{(p)} e^{i h \mu t}$$

(where for every p only a finite number of the coefficients $a_h^{(p)}$ are $\neq 0$) converging uniformly towards $F(t)$ as $p \to \infty$. For each function $F_p(t)$ we form the function

$$G_p(x) = \Sigma a_h^{(p)} e^{i h x}, \tag{8}$$

where x runs through R_m. Then $G_p(x)$ is an exponential polynomial of $x_1, \ldots, x_m$ with the period 2π in each variable, and

$$F_p(t) = G_p(\mu t).$$

It follows therefore from Kronecker's theorem that for all p and q

$$\operatorname*{upper\ bound}_{t} |F_p(t) - F_q(t)| = \operatorname*{upper\ bound}_{x} |G_p(x) - G_q(x)|.$$

Thus $G_p(x)$ converges uniformly towards a limit $G(x)$, which is also a continuous function in R_m with the period 2π in each of the variables $x_1, \ldots, x_m$, and we obviously have

$$F(t) = G(\mu t). \tag{9}$$

This function $G(x)$, which is evidently uniquely determined by being continuous, by having the period 2π in each of the variables $x_1, \ldots, x_m$, and by satisfying (9), is called the *spatial extension of* $F(t)$. It follows from (8) that its Fourier series is

$$G(x) \sim \Sigma a_h e^{i h x}.$$

If, conversely, $G(x)$ denotes an arbitrary continuous function in R_m with the period 2π in each of the variables $x_1, \ldots, x_m$, the function $F(t)$ determined by (9) will be an almost periodic function with exponents from the modul M, since the preceding considerations may also be carried through in the opposite direction. We notice that the Fourier series of $F(t)$ is obtained from that of $G(x)$ by replacing x by μt.

Every exponential polynomial

$$F(t) = \sum_{n=1}^{N} a_n e^{i \lambda_n t}$$

possesses a finite integral base $\mu_1, \ldots, \mu_m$, the modul of $F(t)$ being, in fact, of the form $M = \{h_1 \mu_1 + \cdots + h_m \mu_m\} = \{\boldsymbol{h} \boldsymbol{\mu}\}$.

32. We now turn to the second proof of Theorem I and shall first show that it is sufficient to prove the theorem in the special case where the function $F(t)$ is an exponential polynomial.

Let us, then, assume that the theorem has already been proved in this case. For an arbitrary $\varepsilon < \frac{1}{2}\pi$ we choose, corresponding to the given almost periodic function $F(t)$, an exponential polynomial $F^*(t)$ with exponents among the exponents of $F(t)$, such that for all t

$$|F(t) - F^*(t)| \leqq k \sin \varepsilon.$$

Then for all t

$$|F^*(t)| \geqq k - k \sin \varepsilon > 0,$$

and the theorem may therefore be applied to $F^*(t)$. Furthermore, we have for suitably chosen branches of the arguments for all t

$$|\arg F(t) - \arg F^*(t)| \leqq \varepsilon.$$

As, according to our assumption, the function $F^*(t)$ possesses a mean motion, this inequality implies that $F(t)$ has also a mean motion, viz. the same mean motion as $F^*(t)$. Denoting it by c, and putting

$$\arg F(t) = ct + \psi(t) \quad \text{and} \quad \arg F^*(t) = ct + \psi^*(t),$$

we therefore have for all t

$$|\psi(t) - \psi^*(t)| \leqq \varepsilon.$$

The function $\psi^*(t)$ being, by our assumption, almost periodic, this implies, as ε may be chosen arbitrarily small, that $\psi(t)$ is also almost periodic.

From the determination of c as the mean motion of $F^*(t)$ it follows that c belongs to the modul of $F^*(t)$ and hence also to the modul of $F(t)$. As the exponents of $\psi^*(t)$ belong to the modul of $F^*(t)$ and hence also to the modul of $F(t)$, the same will be the case for the exponents of $\psi(t)$.

33. Let $F(t)$ be an exponential polynomial

$$F(t)=\sum_{n=1}^{N} a_n e^{i\lambda_n t},$$

and let its exponents be contained in the modul $M=\{h_1\mu_1+\cdots+h_m\mu_m\}=\{\boldsymbol{h}\boldsymbol{\mu}\}$. Assuming $\lambda_1=\boldsymbol{h}^{(1)}\boldsymbol{\mu}, \ldots, \lambda_N=\boldsymbol{h}^{(N)}\boldsymbol{\mu}$, we have

$$F(t)=\sum_{n=1}^{N} a_n e^{i\boldsymbol{h}^{(n)}\boldsymbol{\mu}t}.$$

From (1) it follows by Kronecker's theorem that the spatial extension

$$G(\boldsymbol{x})=\sum_{n=1}^{N} a_n e^{i\boldsymbol{h}^{(n)}\boldsymbol{x}}$$

satisfies for all $\boldsymbol{x}$ the relation

$$|G(\boldsymbol{x})|\geqq k>0. \tag{10}$$

For an arbitrary continuous branch of the argument $\arg G(\boldsymbol{x})$ the difference

$$\arg G(\ldots, x_l+2\pi, \ldots)-\arg G(\ldots, x_l, \ldots)$$

is for every l an integral multiple of 2π, which is evidently independent of $\boldsymbol{x}$. If we denote it by $h_l 2\pi$ and put $\boldsymbol{h}=(h_1, \ldots, h_m)$ and

$$\arg G(\boldsymbol{x})=\boldsymbol{h}\boldsymbol{x}+\chi(\boldsymbol{x}) \tag{11}$$

the function $\chi(\boldsymbol{x})$ has, therefore, the period 2π in each of the variables $x_1, \ldots, x_m$. Hence it immediately follows that

$$\arg F(t)=\arg G(\boldsymbol{\mu}t)=\boldsymbol{h}\boldsymbol{\mu}t+\chi(\boldsymbol{\mu}t),$$

or

$$\arg F(t)=ct+\psi(t),$$

where $c=\boldsymbol{h}\boldsymbol{\mu}$ belongs to M and $\psi(t)=\chi(\boldsymbol{\mu}t)$ is an almost periodic function with exponents from M.

As M may be chosen as the modul of the function $F(t)$ this implies the theorem in the case of exponential polynomials.

34. As has been shown by Bohr and Jessen (see Bohr [15]), we may from the preceding proof easily deduce a more precise result regarding the mean motion c. We shall prove that the lattice point $\boldsymbol{h}$ occurring in the expression $c=\boldsymbol{h}\boldsymbol{\mu}$ belongs to the convex closure of the set of lattice points $\boldsymbol{h}^{(1)}, \ldots, \boldsymbol{h}^{(N)}$ occurring in the expressions $\lambda_1=\boldsymbol{h}^{(1)}\boldsymbol{\mu}, \ldots, \lambda_N=\boldsymbol{h}^{(N)}\boldsymbol{\mu}$ of the exponents.

If $m = 1$, so that M has the form $M = \{h\mu\}$, where $\mu \neq 0$ and h runs through all integers, and $F(t)$ is periodic with the period $\frac{2\pi}{\mu}$, the statement is that c lies between the smallest and the largest of the exponents (or is equal to one of them), but this has already been proved in § 30.

In the general case where $m > 1$, we have to prove that for every vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \neq (0, \ldots, 0)$ the inequalities

$$\min_n \{\boldsymbol{h}^{(n)} \boldsymbol{\alpha}\} \leqq \boldsymbol{h} \boldsymbol{\alpha} \leqq \max_n \{\boldsymbol{h}^{(n)} \boldsymbol{\alpha}\} \tag{12}$$

hold. For reasons of continuity it is sufficient to consider vectors $\boldsymbol{\alpha}$ with rational coordinates, and since the terms are homogeneous we may even assume all coordinates to be integers.

For such a vector $\boldsymbol{\alpha}$ the function

$$H(t) = G(\boldsymbol{\alpha} t) = \sum_{n=1}^{N} a_n e^{i \boldsymbol{h}^{(n)} \boldsymbol{\alpha} t}$$

is an exponential polynomial whose exponents are the integers $\boldsymbol{h}^{(n)} \boldsymbol{\alpha}$ (or some of these integers, since they need not be mutually different, and some terms therefore may cancel each other). Furthermore it follows from (10) that $|H(t)| \geqq k$ for all t, and from (11) that $H(t)$ has the mean motion $\boldsymbol{h}\boldsymbol{\alpha}$. The relation (12) therefore follows from the validity in the case $m = 1$.

Putting in (12) $\boldsymbol{\alpha} = \boldsymbol{\mu}$, we find that the mean motion c lies between the smallest and the largest exponent (or is equal to one of them) also in the case of an arbitrary exponential polynomial. The convex closure in question containing only a finite number of lattice points $\boldsymbol{h}$, it follows from the result that, for given exponents $\lambda_1, \ldots, \lambda_N$, there is only a finite number of possible values of the mean motion c.

35. A lattice point $\boldsymbol{h}$ belonging to the convex closure of a set of lattice points $\boldsymbol{h}^{(1)}, \ldots, \boldsymbol{h}^{(N)}$ may be written in the form

$$\boldsymbol{h} = r_1 \boldsymbol{h}^{(1)} + \cdots + r_N \boldsymbol{h}^{(N)},$$

where the coefficients $r_1, \ldots, r_N$ are non-negative rational numbers with the sum 1. Thus, it follows from § 34 that, when $F(t)$ is an exponential polynomial with the exponents $\lambda_1, \ldots, \lambda_N$, the mean motion c has the form

$$c = r_1 \lambda_1 + \cdots + r_N \lambda_N,$$

where the coefficients have these properties. On the other hand it will easily be seen that if a number c of this form belongs to the smallest modul containing the numbers $\lambda_1, \ldots, \lambda_N$, then, for any choice of the base $\mu_1, \ldots, \mu_m$, it is of the form $c = \boldsymbol{h}\mu$, where $\boldsymbol{h}$ is a point with integral coordinates belonging to the corresponding convex closure.

36. Let $F(t)$ denote once more an arbitrary almost periodic function which does not come arbitrarily near to 0. If $F(t)$ has the Fourier series

$$F(t) \sim \Sigma a_n e^{i\lambda_n t},$$

it follows from Theorem 1 that the mean motion c may for a sufficiently large N be written in the form

$$c = h_1 \lambda_1 + \cdots + h_N \lambda_N, \tag{13}$$

where the coefficients $h_1, \ldots, h_N$ are integers. Applying this result to the function $F(t) e^{-i\lambda_1 t}$, which has the exponents $\lambda_m - \lambda_1$ and the mean motion $c - \lambda_1$, we find that there even exists a representation of the form (13), for which the coefficients $h_1, \ldots, h_N$ are integers with the sum 1.

On the other hand, it follows from § 32 that in the case of an arbitrary $F(t)$ the mean motion c equals the mean motion of an exponential polynomial $F^*(t)$ with exponents among the exponents of $F(t)$. Together with § 35 this shows that for a sufficiently large N the mean motion c has the form

$$c = r_1 \lambda_1 + \cdots + r_N \lambda_N,$$

where the coefficients $r_1, \ldots, r_N$ are non-negative rational numbers with the sum 1.

We have thus proved the following theorem.

Theorem 2. *The mean motion c of an almost periodic function*

$$F(t) \sim \Sigma a_n e^{i\lambda_n t}$$

with given exponents λ_n, which does not come arbitrarily near to 0, may for a sufficiently large N be written both in the form

$$c = h_1 \lambda_1 + \cdots + h_N \lambda_N,$$

where the coefficients $h_1, \ldots, h_N$ are integers with the sum 1, and in the form

$$c = r_1 \lambda_1 + \cdots + r_N \lambda_N,$$

where the coefficients $r_1, \ldots, r_N$ are non-negative rational numbers with the sum 1

We notice that this result is not the best possible, i. e. a number c expressible in both of these forms is not necessarily the mean motion of an almost periodic function with the exponents λ_n. To see this, we consider the exponents

$$\lambda_1 = \mu_1, \quad \lambda_2 = -2\mu_1, \quad \lambda_3 = \mu_2, \quad \lambda_4 = -3\mu_2,$$

where μ_1 and μ_2 are arbitrary linearly independent numbers. The number $c = 0$ is then expressible in both forms (viz. as $-2\lambda_1 - \lambda_2 + 3\lambda_3 + \lambda_4$ and $\frac{2}{3}\lambda_1 + \frac{1}{3}\lambda_2 + 0\lambda_3 + 0\lambda_4$) but it is not the mean motion of any exponential polynomial

$$F(t) = a_1 e^{i\mu_1 t} + a_2 e^{-2i\mu_1 t} + a_3 e^{i\mu_2 t} + a_4 e^{-3i\mu_2 t}$$

which does not come arbitrarily near to 0. For when t varies, $a_1 e^{i\mu_1 t} + a_2 e^{-2i\mu_1 t}$ and $a_3 e^{i\mu_2 t} + a_4 e^{-3i\mu_2 t}$ describe cyclic curves, the second of which is symmetric with respect to the origin. Since the spatial extension

$$G(x_1, x_2) = a_1 e^{ix_1} + a_2 e^{-2ix_1} + a_3 e^{ix_2} + a_4 e^{-3ix_2}$$

is $\neq 0$, these curves cannot have any point in common, and one must therefore surround the other. Hence the corresponding function $a_1 e^{i\mu_1 t} + a_2 e^{-2i\mu_1 t}$ or $a_3 e^{i\mu_2 t} + a_4 e^{-3i\mu_2 t}$ does not take the value 0, and it determines the mean motion of $F(t)$, which is therefore equal to one of the exponents λ_1, λ_2, λ_3, or λ_4.

CHAPTER II.

The Jensen Function of an Analytic Almost Periodic Function.

Preliminary Description of the Variation of the Argument and the Distribution of the Zeros.

37. Let $f(s)$ denote an arbitrary function of the complex variable $s = \sigma + it$, which is regular in an open domain G and is not identically zero. The function $\arg f(s)$ is then defined mod. 2π, by the condition $f(s) = |f(s)|\, e^{i \arg f(s)}$, for all s in G, with the exception of the zeros of $f(s)$.

Let L denote a straight line (or segment) belonging to G; we suppose L to be orientated so that we may distinguish between a left and a right side of L. We then define the left argument $\arg^- f(s)$ of $f(s)$ on L as an arbitrary branch of the argument, which is continuous except at the zeros of $f(s)$ on L, while it is discontinuous with a jump of $-p\pi$, when s passes, in the positive direction of L, a zero of $f(s)$ of the order p. Similarly we define the right argument $\arg^+ f(s)$ of $f(s)$ on L as an arbitrary branch of the argument, which is continuous except in the zeros of $f(s)$ on L, while it is discontinuous with a jump of $+p\pi$, when s passes, in the positive direction of L, a zero of $f(s)$ of the order p. In a discontinuity point we use as value the mean value of the limits from the two sides; the two functions $\arg^- f(s)$ and $\arg^+ f(s)$ are then defined for all s on L. Both are, of course, only determined mod. 2π. If $f(s)$ has no zeros on L, each of the functions is identical with a continuous branch of $\arg f(s)$ on the line.

If s_1 and s_2 are points on L, so that the direction from s_1 to s_2 coincides with the positive direction of L, the differences

$$\arg^- f(s_2) - \arg^- f(s_1) \quad \text{and} \quad \arg^+ f(s_2) - \arg^+ f(s_1)$$

are independent of the choice of the branches of the arguments and are called the variation of the argument of $f(s)$ from s_1 to s_2 along the left or right side of L. Obviously they satisfy the inequality

$$\arg^- f(s_2) - \arg^- f(s_1) \leqq \arg^+ f(s_2) - \arg^+ f(s_1).$$

38. Let again G be an open domain in the complex s-plane; let O be a bounded open sub-set of G, whose boundary also belongs to G, and let A be a closed sub-set of O. Let, further, a set of functions $g(s)$ be given, regular and uniformly bounded in G, and not having zero as a limit function, i. e. from which there can be extracted no sequence converging uniformly to zero in every bounded closed sub-set of G. We shall make use of the following (well-known) statements:[1]

(a) There exists a number N, such that the number of zeros in O of every function $g(s)$ of the set is $\leqq N$.

(b) For every number $r > 0$ there exists a constant $m = m(r) > 0$ such that for every function $g(s)$ of the set we have $|g(s)| \geqq m$ at all points s of A having a distance $\geqq r$ from all zeros of $g(s)$ in O.

(c) There exists a constant $k > 0$ with the following property: If for an arbitrary function of the set we denote by $s_1, \ldots, s_{N^*}$ (where $N^* \leqq N$) the zeros of $g(s)$ in O, then the function

$$g^*(s) = \frac{g(s)}{\prod_{n=1}^{N^*} (s - s_n)}$$

satisfies in A the inequality $|g^*(s)| \geqq k$.

[1] For proofs of (a) and (b) see e. g. the proofs of the quite analogous statements in Bohr and Jessen [2], pp. 18—19; (c) and (d) are easy consequences of (a) and (b).

(d) For every number $l > 0$ there exists a constant $v = v(l) > 0$ such that the variation of the argument of every function $g(s)$ of the set along the left or right side of any straight segment of length $\leqq l$ belonging to A is $\leqq v$.

39. Now let $-\infty \leqq \alpha < \alpha_0 < \alpha_1 < \beta_1 < \beta_0 < \beta \leqq +\infty$, and let d denote a positive number smaller than the differences $\alpha_1 - \alpha_0$ and $\beta_0 - \beta_1$. We then choose for the open set G the vertical strip (α_0, β_0), for the open sub-set O of G the rectangle $\alpha_1 - d < \sigma < \beta_1 + d$, $-\frac{1}{2} - d < t < \frac{1}{2} + d$, and for the closed sub-set A of O the rectangle $\alpha_1 \leqq \sigma \leqq \beta_1$, $-\frac{1}{2} \leqq t \leqq \frac{1}{2}$.

Let $f(s)$ be a function almost periodic in the strip $[\alpha, \beta]$ and not identically zero. We may then apply the above theorem on the set of all functions $f(s + it^*)$, $-\infty < t^* < +\infty$. These functions are, in fact, uniformly bounded in G, and as for every σ_0 in the interval (α, β) there exists a constant $h > 0$, such that $|f(\sigma_0 + it)| \geqq h$ for a relatively dense set of values of t, they do not have zero as a limit function.

More generally, if $f_1(s), f_2(s), \ldots$ is a sequence of functions almost periodic in $[\alpha, \beta]$ and converging uniformly in $[\alpha, \beta]$ to a limit function $f_0(s)$, and if none of the functions $f_n(s)$, $n = 0, 1, 2, \ldots$ is identically zero, then the theorem may be applied to the set of all functions $f_n(s + it^*)$, $n = 0, 1, 2, \ldots$, $-\infty < t^* < +\infty$. These functions are in fact uniformly bounded in G, and as there exists a constant $h > 0$ and a bounded closed sub-set R of G such that for every function $f_n(s + it^*)$ of the set the inequality $|f_n(s + it^*)| \geqq h$ is satisfied for some point of R, they do not have zero as a limit function.

We therefore have the following theorem.

Theorem 3. *Let $-\infty \leqq \alpha < \alpha_0 < \alpha_1 < \beta_1 < \beta_0 < \beta \leqq +\infty$, and let d denote a positive number smaller than the two differences $\alpha_1 - \alpha_0$ and $\beta_0 - \beta_1$. Let, further, $f(s)$ be a function almost periodic in $[\alpha, \beta]$ and not identically zero. Then the following statements are valid* (see Fig. 1):

(i) *There exists a number N, such that the number of zeros of $f(s)$ in any rectangle $\alpha_1 - d < \sigma < \beta_1 + d$, $t^* - \frac{1}{2} - d < t < t^* + \frac{1}{2} + d$ is $\leqq N$.*

(ii) *For every number $r > 0$ there exists a constant $m = m(r) > 0$ such that $|f(s)| \geqq m$ at all points s in the closed strip $\{\alpha_1, \beta_1\}$ having a distance $\geqq r$ from all zeros of $f(s)$ in the strip (α_0, β_0).*

(iii) *There exists a constant $k > 0$ with the following property: If, for an arbitrary t^*, we denote by $s_1, \ldots, s_{N^*}$ (where $N^* \leqq N$) the zeros of $f(s)$ in the rectangle $\alpha_1 - d < \sigma < \beta_1 + d$, $t^* - \frac{1}{2} - d < t < t^* + \frac{1}{2} + d$, then the function*

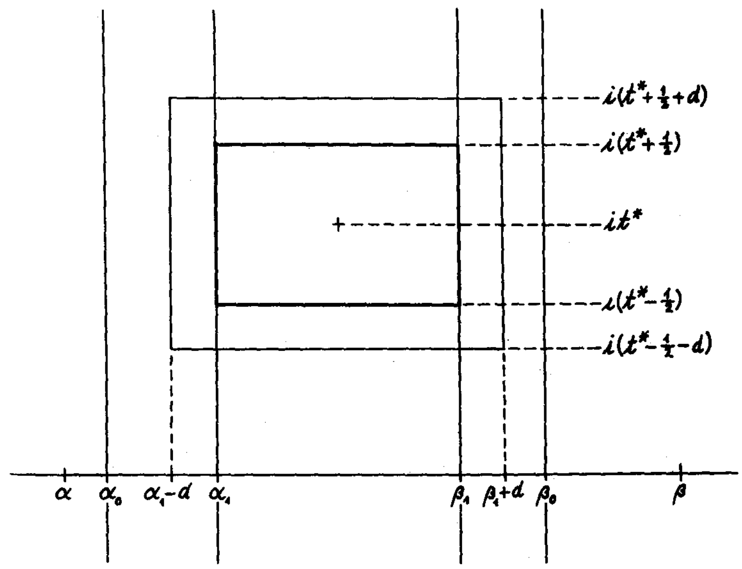

Fig. 1.

$$f^*(s) = \frac{f(s)}{\prod_{n=1}^{N^*}(s-s_n)}$$

satisfies in the rectangle $\alpha_1 \leqq \sigma \leqq \beta_1$, $t^*-\frac{1}{2} \leqq t \leqq t^*+\frac{1}{2}$ *the inequality* $|f^*(s)| \geqq k$.

(iv) *For every number* $l > 0$ *there exists a constant* $v = v(l) > 0$ *such that the variation of the argument of* $f(s)$ *along the left or right side of any straight segment of length* $\leqq l$ *belonging to the strip* $\{\alpha_1, \beta_1\}$ *is* $\leqq v$.

If a sequence of functions $f_1(s)$, $f_2(s)$, ... *almost periodic in* $[\alpha, \beta]$ *converges uniformly in* $[\alpha, \beta]$ *to a limit function* $f_0(s)$, *and if none of the functions* $f_n(s)$, $n=0, 1, 2, \ldots$ *is identically zero, the preceding statements are valid for the functions* $f_n(s)$, $n=0, 1, 2, \ldots$, *with constants* N, $m(r)$, k *and* $v(l)$ *independent of* n.

The Mean Motions and Frequencies of Zeros.

40. We consider again a function $f(s)$, almost periodic in a strip $[\alpha, \beta]$ and not identically zero. The left and right arguments of $f(s)$ on a vertical line $s=\sigma+it$, $-\infty < t < +\infty$, orientated after increasing values of t, will be denoted by $\arg^- f(\sigma+it)$ and $\arg^+ f(\sigma+it)$.

If $-\infty < \gamma < \delta < +\infty$ the variations

$$\arg^- f(\sigma + i\delta) - \arg^- f(\sigma + i\gamma) \quad \text{and} \quad \arg^+ f(\sigma + i\delta) - \arg^+ f(\sigma + i\gamma)$$

of the argument from $\sigma + i\gamma$ to $\sigma + i\delta$ along the left and right side of the line are, considered as functions of σ, continuous from the left and right respectively. We have

$$\arg^- f(\sigma + i\delta) - \arg^- f(\sigma + i\gamma) \leqq \arg^+ f(\sigma + i\delta) - \arg^+ f(\sigma + i\gamma).$$

The four quantities

$$\left.\begin{matrix} \underline{c}^-(\sigma) \\ \bar{c}^-(\sigma) \end{matrix}\right\} = \lim_{(\delta-\gamma)\to\infty} {\inf \atop \sup} \frac{\arg^- f(\sigma + i\delta) - \arg^- f(\sigma + i\gamma)}{\delta - \gamma}$$

and

$$\left.\begin{matrix} \underline{c}^+(\sigma) \\ \bar{c}^+(\sigma) \end{matrix}\right\} = \lim_{(\delta-\gamma)\to\infty} {\inf \atop \sup} \frac{\arg^+ f(\sigma + i\delta) - \arg^+ f(\sigma + i\gamma)}{\delta - \gamma}$$

are called the *lower and upper, left and right mean motions* of $f(s)$ on the line $s = \sigma + it$, $-\infty < t < +\infty$, or, more briefly, of the function $f(\sigma + it)$. From Theorem 3 (iv) it immediately follows that they are finite for every σ and even bounded in $[\alpha, \beta]$. Furthermore

$$\underline{c}^-(\sigma) \leqq \left\{\begin{matrix} \underline{c}^+(\sigma) \\ \bar{c}^-(\sigma) \end{matrix}\right\} \leqq \bar{c}^+(\sigma).$$

If $\underline{c}^-(\sigma) = \bar{c}^-(\sigma)$ or $\underline{c}^+(\sigma) = \bar{c}^+(\sigma)$, i. e. if the limit

$$c^-(\sigma) = \lim_{(\delta-\gamma)\to\infty} \frac{\arg^- f(\sigma + i\delta) - \arg^- f(\sigma + i\gamma)}{\delta - \gamma}$$

or

$$c^+(\sigma) = \lim_{(\delta-\gamma)\to\infty} \frac{\arg^+ f(\sigma + i\delta) - \arg^+ f(\sigma + i\gamma)}{\delta - \gamma}$$

exists, it is called the *left or right mean motion* of $f(\sigma + it)$ respectively.

Denoting by $\arg f(\sigma + it)$ the argument of $f(\sigma + it)$ in the sense of § 27, we have $\arg f(\sigma + it) = \arg^- f(\sigma + it) = \arg^+ f(\sigma + it)$ if $f(\sigma + it)$ has no zeros; otherwise we have $\arg f(\sigma + it) = \frac{1}{2}(\arg^- f(\sigma + it) + \arg^+ f(\sigma + it))$. Thus if the mean motions $c^-(\sigma)$ and $c^+(\sigma)$ both exist, the mean motion

$$c(\sigma) = \lim_{(\delta-\gamma)\to\infty} \frac{\arg f(\sigma + i\delta) - \arg f(\sigma + i\gamma)}{\delta - \gamma}$$

of $f(\sigma + it)$ in the sense of § 27 also exists and is determined by

$$c(\sigma) = \tfrac{1}{2}(c^-(\sigma) + c^+(\sigma)).$$

For $\alpha < \sigma_1 < \sigma_2 < \beta$ and $-\infty < \gamma < \delta < +\infty$ we denote by $N(\sigma_1, \sigma_2; \gamma, \delta)$ the number of zeros of $f(s)$ in the rectangle $\sigma_1 < \sigma < \sigma_2$, $\gamma < t < \delta$. The two quantities

$$\left.\begin{matrix}\underline{H}(\sigma_1, \sigma_2)\\ \overline{H}(\sigma_1, \sigma_2)\end{matrix}\right\} = \lim_{(\delta-\gamma)\to\infty}\begin{matrix}\inf\\ \sup\end{matrix} \frac{N(\sigma_1, \sigma_2; \gamma, \delta)}{\delta - \gamma}$$

are called the *lower and upper relative frequencies of zeros* of $f(s)$ in the strip (σ_1, σ_2). From Theorem 3 (i) it follows that they are always finite. Obviously

$$\underline{H}(\sigma_1, \sigma_2) \leqq \overline{H}(\sigma_1, \sigma_2).$$

If $\underline{H}(\sigma_1, \sigma_2) = \overline{H}(\sigma_1, \sigma_2)$, i. e. if the limit

$$H(\sigma_1, \sigma_2) = \lim_{(\delta-\gamma)\to\infty} \frac{N(\sigma_1, \sigma_2; \gamma, \delta)}{\delta - \gamma}$$

exists, it is called the *relative frequency of zeros* of $f(s)$ in the strip (σ_1, σ_2).

We notice that on account of Theorem 3 it does not make any difference in the preceding definitions if we restrict γ and δ to an arbitrary relatively dense set of numbers.

41. We may now easily prove the following theorem.

Theorem 4. *For any function $f(s)$ almost periodic in $[\alpha, \beta]$ and not identically zero the mean motions and relative frequencies of zeros are for an arbitrary strip (σ_1, σ_2), where $\alpha < \sigma_1 < \sigma_2 < \beta$, connected by the inequalities*

$$\frac{1}{2\pi}(\underline{c}^-(\sigma_2) - \bar{c}^+(\sigma_1)) \leqq \underline{H}(\sigma_1, \sigma_2) \leqq \left\{\begin{matrix}\frac{1}{2\pi}(\bar{c}^-(\sigma_2) - \bar{c}^+(\sigma_1))\\ \frac{1}{2\pi}(\underline{c}^-(\sigma_2) - \underline{c}^+(\sigma_1))\end{matrix}\right\} \leqq \overline{H}(\sigma_1, \sigma_2) \leqq \frac{1}{2\pi}(\bar{c}^-(\sigma_2) - \underline{c}^+(\sigma_1)).$$

To prove this we make use of the remark at the end of § 40 by restricting γ and δ to values for which $f(s)$ has no zeros on the segments $\sigma_1 \leqq \sigma \leqq \sigma_2$, $t = \gamma$ and $\sigma_1 \leqq \sigma \leqq \sigma_2$, $t = \delta$. For such values of γ and δ we have by Cauchy's theorem applied to the rectangle $\sigma_1 < \sigma < \sigma_2$, $\gamma < t < \delta$

(1) $$N(\sigma_1, \sigma_2; \gamma, \delta) =$$

$$\frac{1}{2\pi}[(\arg^- f(\sigma_2 + i\delta) - \arg^- f(\sigma_2 + i\gamma)) - (\arg^+ f(\sigma_1 + i\delta) - \arg^+ f(\sigma_1 + i\gamma)) + R(\sigma_1, \sigma_2; \gamma, \delta)],$$

where the remainder $R(\sigma_1, \sigma_2; \gamma, \delta)$ is the contribution to the variation of the argument from the horizontal sides of the rectangle. By Theorem 3 (iv) this term is bounded for all γ and δ, and the theorem is therefore an immediate consequence of (1).

42. For later use we formulate the following immediate consequences of Theorem 4.

If $c^+(\sigma_1)$ exists, we have for every $\sigma_2 > \sigma_1$ the relations

$$\underline{H}(\sigma_1, \sigma_2) = \frac{1}{2\pi}(\underline{c}^-(\sigma_2) - c^+(\sigma_1)) \quad \text{and} \quad \overline{H}(\sigma_1, \sigma_2) = \frac{1}{2\pi}(\bar{c}^-(\sigma_2) - c^+(\sigma_1)).$$

If $c^-(\sigma_2)$ exists, we have for every $\sigma_1 < \sigma_2$ the relations

$$\underline{H}(\sigma_1, \sigma_2) = \frac{1}{2\pi}(c^-(\sigma_2) - \bar{c}^+(\sigma_1)) \quad \text{and} \quad \overline{H}(\sigma_1, \sigma_2) = \frac{1}{2\pi}(c^-(\sigma_2) - \underline{c}^+(\sigma_1)).$$

If two of the quantities $c^+(\sigma_1)$, $c^-(\sigma_2)$, and $H(\sigma_1, \sigma_2)$ exist, then the third also exists, and we have the relation

$$H(\sigma_1, \sigma_2) = \frac{1}{2\pi}(c^-(\sigma_2) - c^+(\sigma_1)).$$

The Jensen Function.

43. The more detailed study of the mean motions and the frequencies of zeros depends on the Jensen function, the existence of which is proved by the following theorem.

Theorem 5. *For any function $f(s)$ almost periodic in $[\alpha, \beta]$ and not identically zero the mean value*

$$\varphi(\sigma) = \underset{t}{M}\{\log |f(\sigma + it)|\}$$

exists uniformly in $[\alpha, \beta]$, i. e. the function

$$\varphi(\sigma; \gamma, \delta) = \frac{1}{\delta - \gamma}\int_\gamma^\delta \log |f(\sigma + it)|\, dt$$

converges for $(\delta - \gamma) \to \infty$ uniformly in $[\alpha, \beta]$ towards a limit function $\varphi(\sigma)$.

If, for $m > 0$, we put

$$|f(s)|_m = \max\{|f(s)|, m\}$$

the function $\varphi(\sigma)$ is also determined as the limit of the mean value

$$\underset{t}{M}\{\log |f(\sigma + it)|_m\}$$

as $m \to 0$, the convergence being again uniform in $[\alpha, \beta]$.

The function $\varphi(\sigma)$ is called the *Jensen function* of $f(s)$. Since $\varphi(\sigma; \gamma, \delta)$ is continuous, it follows from the theorem that $\varphi(\sigma)$ is continuous.

We repeat the proof given in Jessen [2].

Let $\{\alpha_1, \beta_1\}$ be a closed sub-interval of (α, β). For a given $m > 0$ the function $\log |f(\sigma + it)|_m$ is for every σ in (α, β) an almost periodic function of the real variable t. Further, for $\alpha_1 \leqq \sigma \leqq \beta_1$ these functions form a uniformity set. Thus the mean value

$$\underset{t}{M}\{\log |f(\sigma + it)|_m\} = \lim_{(\delta - \gamma) \to \infty} \frac{1}{\delta - \gamma} \int_{\gamma}^{\delta} \log |f(\sigma + it)|_m \, dt$$

exists uniformly in $\{\alpha_1, \beta_1\}$. As $\log |f(\sigma + it)|_m \geqq \log |f(\sigma + it)|$ it is therefore sufficient to prove that for any $\varepsilon > 0$ there exists an m, such that for $\alpha_1 \leqq \sigma \leqq \beta_1$ and $(\delta - \gamma) > 1$ we have

$$(2) \qquad (0 \leqq) \frac{1}{\delta - \gamma} \int_{\gamma}^{\delta} \log |f(\sigma + it)|_m \, dt - \varphi(\sigma; \gamma, \delta) \leqq \varepsilon,$$

or (what amounts to the same thing)

$$(3) \qquad (0 \leqq) \int_{\gamma}^{\delta} (\log |f(\sigma + it)|_m - \log |f(\sigma + it)|) \, dt \leqq \varepsilon (\delta - \gamma).$$

For (2) implies the first part of the theorem, and if we make $(\delta - \gamma) \to \infty$ it follows from (2) that for $\alpha_1 \leqq \sigma \leqq \beta_1$

$$(4) \qquad (0 \leqq) \underset{t}{M}\{\log |f(\sigma + it)|_m\} - \varphi(\sigma) \leqq \varepsilon,$$

which implies the second part of the theorem. That m may be chosen in accordance with the said condition is proved by means of Theorem 3 in the following manner.

Let $\alpha < \alpha_0 < \alpha_1 < \beta_1 < \beta_0 < \beta$, and let d denote a positive number smaller than the differences $\alpha_1 - \alpha_0$ and $\beta_0 - \beta_1$. To every $r > 0$ there exists by Theorem 3 (ii) a constant $m = m(r) > 0$, which may of course be chosen < 1, such that $|f(s)| \geqq m$, or $|f(s)|_m = |f(s)|$, for all s in the strip $\{\alpha_1, \beta_1\}$ having a distance $\geqq r$ from all zeros of $f(s)$ in (α_0, β_0). Hence on choosing $r < d$ there exists by Theorem 3 (i) a number N independent of r, such that in every integral

$$J = \int_{t^* - \frac{1}{2}}^{t^* + \frac{1}{2}} (\log |f(\sigma + it)|_m - \log |f(\sigma + it)|) \, dt,$$

where $\alpha_1 \leqq \sigma \leqq \beta_1$, the integrand is positive in at most N sub-intervals of $t^* - \frac{1}{2} < t < t^* + \frac{1}{2}$ having a total length $\leqq N 2 r$. Having chosen $m < 1$ we have in these intervals

$$\log |f(\sigma + i t)|_m - \log |f(\sigma + i t)| \leqq - \log^- |f(\sigma + i t)|.^1$$

Hence by Theorem 3 (iii) there exists a constant k independent of r such that denoting by $s_1 = \sigma_1 + i t_1, \ldots, s_{N^*} = \sigma_{N^*} + i t_{N^*}$ (where $N^* \leqq N$) the zeros of $f(s)$ in the rectangle $\alpha_1 - d < \sigma < \beta_1 + d$, $t^* - \frac{1}{2} - d < t < t^* + \frac{1}{2} + d$ we have for $t^* - \frac{1}{2} < t < t^* + \frac{1}{2}$

$$\log |f(\sigma + i t)|_m - \log |f(\sigma + i t)| \leqq - \log^- k - \sum_{n=1}^{N^*} \log^- |s - s_n| \leqq - \log^- k - \sum_{n=1}^{N^*} \log^- |t - t_n|.$$

Thus, for every integral J, we have the estimate

$$(0 \leqq)\, J \leqq - \log^- k \cdot N 2 r - N \int_{-Nr}^{Nr} \log^- |u|\, du,$$

where the quantity on the right converges to 0 as $r \to 0$ (since N and k are independent of r). We now choose r such that this quantity is $\leqq \frac{1}{2} \varepsilon$. Then for the corresponding $m = m(r)$, the inequality (3) is satisfied for $\alpha_1 \leqq \sigma \leqq \beta_1$ and $(\delta - \gamma) > 1$. For, the largest integer $\leqq (\delta - \gamma)$ being denoted by A, the integral on the left in (3) is $\leqq A + 1 \leqq 2A$ integrals J and hence $\leqq \frac{1}{2} \varepsilon\, 2 (\delta - \gamma) = \varepsilon (\delta - \gamma)$.

This completes the proof of the theorem.

44. By means of the inequality (4) we shall now prove the following theorem.

Theorem 6. *The Jensen function $\varphi(\sigma)$ of $f(s)$ depends continuously on $f(s)$ in the following sense: If $f_1(s), f_2(s), \ldots$ are a sequence of functions almost periodic in $[\alpha, \beta]$ and converging uniformly in $[\alpha, \beta]$ to a limit function $f_0(s)$, and if none of the functions $f_n(s)$, $n = 0, 1, 2, \ldots$ is identically zero, then the Jensen function $\varphi_n(\sigma)$ of $f_n(s)$ converges for $n \to \infty$ uniformly in $[\alpha, \beta]$ towards the Jensen function $\varphi_0(\sigma)$ of $f_0(s)$.*

Let $\{\alpha_1, \beta_1\}$ be a closed sub-interval of (α, β). We apply the proof of Theorem 5 simultaneously to all functions $f_n(s)$, $n = 0, 1, 2, \ldots$, choosing (as Theorem 3 permits) the numbers $m = m(r)$, N and k independent of n. We thereby find that for any $\varepsilon > 0$ there exists a positive number $m < 1$ such that in $\{\alpha_1, \beta_1\}$ simultaneously for all $n = 0, 1, 2, \ldots$

$$(0 \leqq) \underset{t}{M} \{\log |f_n(\sigma + i t)|_m\} - \varphi_n(\sigma) \leqq \varepsilon.$$

[1] By $\log^- x$ we denote for $x > 0$ the function $\log^- x = \min \{\log x, 0\}$. The function $-\log^- x$ is non-negative and decreasing; further, if $x = x_1 \ldots x_{N^*}$, we have $-\log^- x \leqq -\log^- x_1 - \cdots - \log^- x_{N^*}$.

For this fixed value of m we obviously have

$$\underset{t}{M}\{\log|f_0(\sigma+it)|_m\} = \lim_{n\to\infty}\underset{t}{M}\{\log|f_n(\sigma+it)|_m\}$$

uniformly in $\{\alpha_1, \beta_1\}$. Thus in the interval $\{\alpha_1, \beta_1\}$ we have

$$|\varphi_0(\sigma) - \varphi_n(\sigma)| \leqq 3\varepsilon$$

for all sufficiently large n; which proves the theorem.

The Connection between the Jensen Function and the Mean Motions and Frequencies of Zeros.

45. We shall now prove the following theorem.

Theorem 7. *For any function $f(s)$ almost periodic in $[\alpha, \beta]$ and not identically zero the corresponding Jensen function $\varphi(\sigma)$ is convex in (α, β), and the four mean motions satisfy for every σ in (α, β) the inequalities*

$$\varphi'(\sigma-0) \leqq \underline{c}^-(\sigma) \leqq \begin{Bmatrix} \underline{c}^+(\sigma) \\ \bar{c}^-(\sigma) \end{Bmatrix} \leqq \bar{c}^+(\sigma) \leqq \varphi'(\sigma+0).$$

Further, the two frequencies of zeros satisfy for every strip (σ_1, σ_2), where $\alpha<\sigma_1<\sigma_2<\beta$, the inequalities

$$\frac{1}{2\pi}(\varphi'(\sigma_2-0)-\varphi'(\sigma_1+0)) \leqq \underline{H}(\sigma_1,\sigma_2) \leqq \overline{H}(\sigma_1,\sigma_2) \leqq \frac{1}{2\pi}(\varphi'(\sigma_2+0)-\varphi'(\sigma_1-0)).$$

It is sufficient to prove the convexity of $\varphi(\sigma)$ and the two inequalities

(5) $$\varphi'(\sigma-0) \leqq \underline{c}^-(\sigma) \quad \text{and} \quad \bar{c}^+(\sigma) \leqq \varphi'(\sigma+0)$$

for then the rest of the theorem follows from § 40 and Theorem 4. Moreover, it is sufficient to prove the convexity of $\varphi(\sigma)$ and the inequalities (5) in any reduced interval (α_1, β_1). Now, on account of the almost periodicity there exist, corresponding to α_1 and β_1, a number $m>0$ and a relatively dense set of real numbers such that $|f(s)| \geqq m$ on every horizontal segment $\alpha_1 \leqq \sigma \leqq \beta_1$, $t=t_0$, where t_0 belongs to this set. Denoting by K the (finite) upper bound of $|f'(s)|$ in the strip $\{\alpha_1, \beta_1\}$, we have on any of these segments the inequality

$$\left|\frac{d\log f(s)}{ds}\right| = \left|\frac{f'(s)}{f(s)}\right| \leqq \frac{K}{m}.$$

When $\alpha_1 < \sigma_1 < \sigma_2 < \beta_1$, and γ and δ belong to the set in question, we therefore have the relation (1), where the remainder term $R(\sigma_1, \sigma_2; \gamma, \delta)$ satisfies the inequality

$$|R(\sigma_1, \sigma_2; \gamma, \delta)| \leqq \frac{2K}{m}(\sigma_2 - \sigma_1).$$

We now consider the function

$$\varphi(\sigma; \gamma, \delta) = \frac{1}{\delta - \gamma} \int_\gamma^\delta \log |f(\sigma + it)| \, dt.$$

For an arbitrary value of σ for which $f(s) \neq 0$ on the vertical segment $s = \sigma + it$, $\gamma \leqq t \leqq \delta$, this function is differentiable with the derivative

$$\varphi'(\sigma; \gamma, \delta) = \frac{\arg f(\sigma + i\delta) - \arg f(\sigma + i\gamma)}{\delta - \gamma}. \tag{6}$$

For in a neighbourhood of the segment we have

$$\log f(s) = \log |f(s)| + i \arg f(s)$$

and therefore by the Cauchy-Riemann differential equations

$$\frac{d}{d\sigma} \log |f(\sigma + it)| = \frac{d}{dt} \arg f(\sigma + it).$$

For given values of γ and δ there are at most a finite number of exceptional values of σ, and, for these values, the right side of (6) has limits from the left and the right, viz.

$$\frac{\arg^- f(\sigma + i\delta) - \arg^- f(\sigma + i\gamma)}{\delta - \gamma} \quad \text{and} \quad \frac{\arg^+ f(\sigma + i\delta) - \arg^+ f(\sigma + i\gamma)}{\delta - \gamma}.$$

Hence the function $\varphi(\sigma; \gamma, \delta)$ is differentiable from the left and the right at these points σ, and for all σ we have for its left and right derivatives $\varphi'(\sigma - 0; \gamma, \delta)$ and $\varphi'(\sigma + 0; \gamma, \delta)$ the expressions

$$\begin{aligned} \varphi'(\sigma - 0; \gamma, \delta) &= \frac{\arg^- f(\sigma + i\delta) - \arg^- f(\sigma + i\gamma)}{\delta - \gamma} \quad \text{and} \\ \varphi'(\sigma + 0; \gamma, \delta) &= \frac{\arg^+ f(\sigma + i\delta) - \arg^+ f(\sigma + i\gamma)}{\delta - \gamma}. \end{aligned} \tag{7}$$

The relation (1) therefore takes the form

$$\frac{N(\sigma_1, \sigma_2; \gamma, \delta)}{\delta - \gamma} = \frac{1}{2\pi}(\varphi'(\sigma_2 - 0; \gamma, \delta) - \varphi'(\sigma_1 + 0; \gamma, \delta) + r(\sigma_1, \sigma_2; \gamma, \delta)),$$

where the remainder $r(\sigma_1, \sigma_2; \gamma, \delta)$ satisfies the inequality

$$|r(\sigma_1, \sigma_2; \gamma, \delta)| \leqq \frac{1}{\delta - \gamma} \frac{2K}{m}(\sigma_2 - \sigma_1).$$

In order to utilize this connection between the function $\varphi(\sigma; \gamma, \delta)$ and the number $N(\sigma_1, \sigma_2; \gamma, \delta)$, we introduce instead of $\varphi(\sigma; \gamma, \delta)$ the function

$$\varphi_1(\sigma; \gamma, \delta) = \varphi(\sigma; \gamma, \delta) + \frac{1}{\delta - \gamma} \frac{K}{m} \sigma^2.$$

Then the last result may also be written in the form

$$\frac{N(\sigma_1, \sigma_2; \gamma, \delta)}{\delta - \gamma} = \frac{1}{2\pi}(\varphi_1'(\sigma_2 - 0; \gamma, \delta) - \varphi_1'(\sigma_1 + 0; \gamma, \delta) + r_1(\sigma_1, \sigma_2; \gamma, \delta)),$$

where the new remainder $r_1(\sigma_1, \sigma_2; \gamma, \delta)$ satisfies the inequalities

$$-\frac{1}{\delta - \gamma} \frac{4K}{m}(\sigma_2 - \sigma_1) \leqq r_1(\sigma_1, \sigma_2; \gamma, \delta) \leqq 0.$$

As $N(\sigma_1, \sigma_2; \gamma, \delta) \geqq 0$, it follows from the last inequality that

$$\varphi_1'(\sigma_1 + 0; \gamma, \delta) \leqq \varphi_1'(\sigma_2 - 0; \gamma, \delta),$$

and hence that $\varphi_1(\sigma; \gamma, \delta)$ is a convex function.

The proof may now be completed in few words. By Theorem 5 we have uniformly in (α_1, β_1) the relation

$$\varphi(\sigma) = \lim_{(\delta - \gamma) \to \infty} \varphi_1(\sigma; \gamma, \delta).$$

The function $\varphi(\sigma)$ is therefore convex. Hence for every σ in (α_1, β_1)

$$\varphi'(\sigma - 0) \leqq \liminf_{(\delta - \gamma) \to \infty} \varphi_1'(\sigma - 0; \gamma, \delta) = \liminf_{(\delta - \gamma) \to \infty} \varphi'(\sigma - 0; \gamma, \delta)$$

and

$$\limsup_{(\delta - \gamma) \to \infty} \varphi'(\sigma + 0; \gamma, \delta) = \limsup_{(\delta - \gamma) \to \infty} \varphi_1'(\sigma + 0; \gamma, \delta) \leqq \varphi'(\sigma + 0).$$

Combining this with (7), we find the inequalities (5).

This completes the proof of the theorem.

46. From Theorem 7 follows that if $\varphi(\sigma)$ is differentiable at the point σ, then the left and right mean motions $c^-(\sigma)$ and $c^+(\sigma)$ of $f(\sigma + it)$ both exist and have the common value

$$c^-(\sigma) = c^+(\sigma) = \varphi'(\sigma).$$

If $\varphi(\sigma)$ is differentiable at the points σ_1 and σ_2, then the relative frequency $H(\sigma_1, \sigma_2)$ of zeros of $f(s)$ in the strip (σ_1, σ_2) exists and has the value

$$H(\sigma_1, \sigma_2) = \frac{1}{2\pi}(\varphi'(\sigma_2) - \varphi'(\sigma_1)).$$

This formula is called the *Jensen formula* for almost periodic functions.

As an immediate consequence of Theorem 7 we have the formula

$$\frac{1}{2\pi}(\varphi'(\sigma + 0) - \varphi'(\sigma - 0)) = \lim_{\varepsilon \to 0} \underline{H}(\sigma - \varepsilon, \sigma + \varepsilon),$$

which shows that the function $\varphi(\sigma)$ is differentiable at the point σ if and only if

$$\lim_{\varepsilon \to 0} \underline{H}(\sigma - \varepsilon, \sigma + \varepsilon) = 0.$$

Strips without Zeros. Periodic Functions.

47. For a function $f(s)$ almost periodic in a strip $[\alpha, \beta]$, the vertical sub-strips $(\alpha \leqq) \alpha_0 < \sigma < \beta_0 (\leqq \beta)$ in which the function has no zeros have a particular interest. Concerning such strips we shall prove the following theorem.

Theorem 8. *A function $f(s)$ almost periodic in $[\alpha, \beta]$ and not identically zero has no zeros in the sub-strip $(\alpha \leqq) \alpha_0 < \sigma < \beta_0 (\leqq \beta)$, if and only if its Jensen function $\varphi(\sigma)$ is linear in the interval (α_0, β_0).*

In this case we have for every reduced strip $(\alpha_0 <) \alpha_1 \leqq \sigma \leqq \beta_1 (< \beta_0)$ that

(8) $$\underset{\alpha_1 \leqq \sigma \leqq \beta_1}{\text{lower bound}} |f(s)| > 0;$$

moreover an arbitrary branch of $\log f(s)$ in (α_0, β_0) has the form

(9) $$\log f(s) = cs + g(s),$$

where c denotes the constant value of $\varphi'(\sigma)$ in the interval (α_0, β_0) and $g(s)$ is almost periodic in $[\alpha_0, \beta_0]$.

The constant c and the exponents of $g(s)$ all belong to the modul of $f(s)$.

If $f(s)$ has no zeros in the strip (α_0, β_0) we have $\overline{H}(\alpha_0, \beta_0) = 0$. Hence, by Theorem 7, we have $\varphi'(\alpha_0 + 0) = \varphi'(\beta_0 - 0)$, which implies that $\varphi(\sigma)$ is linear in the interval (α_0, β_0).

If $f(s)$ has a zero $s_0 = \sigma_0 + i t_0$ in the strip (α_0, β_0), we choose σ_1 and σ_2 such that $\alpha_0 < \sigma_1 < \sigma_0 < \sigma_2 < \beta_0$. Further, we choose a positive number r smaller than the differences $\sigma_0 - \sigma_1$ and $\sigma_2 - \sigma_0$ such that $|f(s)| \neq 0$ on the circle $|s - s_0| = r$. Let m denote the lower bound of $|f(s)|$ on this circle. Then there exists, by Rouché's theorem, for any positive $\varepsilon < m$ and any $\tau = \tau(\varepsilon; \sigma_1, \sigma_2)$ at least one zero of $f(s)$ in the circle $|s - (s_0 + i\tau)| < r$. As the translation numbers are relatively dense, this implies that $\underline{H}(\sigma_1, \sigma_2) > 0$. Hence, by Theorem 7, we have $\varphi'(\sigma_1 - 0) < \varphi'(\sigma_2 + 0)$, which shows that $\varphi(\sigma)$ is not linear in the interval (α_0, β_0).

This completes the proof of the first part of the theorem.

We now assume that $f(s)$ has no zeros in the strip (α_0, β_0). The relation (8) then follows immediately from Theorem 3 (ii) applied for a number r smaller than the differences $\alpha_1 - \alpha_0$ and $\beta_0 - \beta_1$. Thus, the function $F_\sigma(t) = f(\sigma + i t)$ satisfies, for every σ in the interval (α_0, β_0), the conditions of Theorem 1, and we therefore find, employing § 29, that

$$\log f(\sigma + i t) = i c t + H_\sigma(t),$$

where, by Theorem 7, the mean motion c is equal to $\varphi'(\sigma)$, and $H_\sigma(t)$ is almost periodic. Thus we have the representation (9) with

$$g(\sigma + i t) = H_\sigma(t) - c\sigma.$$

Moreover, it follows from the remark at the end of § 29 that, for any $\varepsilon > 0$ and any reduced strip $(\alpha_0 <) \alpha_1 < \sigma < \beta_1 (< \beta_0)$, there exists a $\delta > 0$ such that any $\tau_f(\delta; \alpha_1, \beta_1)$ is a $\tau_{H_\sigma}(\varepsilon)$ for all σ in (α_1, β_1), and therefore a $\tau_g(\varepsilon; \alpha_1, \beta_1)$. This implies that $g(s)$ is almost periodic in $[\alpha_0, \beta_0]$.

The last part of the theorem follows immediately from Theorem 1.

The Jensen formula may be written in the form

$$N(\sigma_1, \sigma_2; \gamma, \delta) = \frac{1}{2\pi}(\varphi'(\sigma_2) - \varphi'(\sigma_1))(\delta - \gamma) + o(\delta - \gamma).$$

It is easily seen that the remainder is bounded if σ_1 and σ_2 belong to linearity intervals of $\varphi(\sigma)$.

48. Let (α_0, β_0) be a strip without zeros, and suppose that $\beta_0 < \beta$. Then, σ_0 having been chosen in the interval (α_0, β_0), the quantities $c^+(\sigma_0) = \varphi'(\sigma_0)$ and $H(\sigma_0, \beta_0) = 0$ both exist. Thus, by § 42, the quantity $c^-(\beta_0)$ also exists, and we have

$$H(\sigma_0, \beta_0) = \frac{1}{2\pi}(c^-(\beta_0) - c^+(\sigma_0)),$$

so that $c^-(\beta_0) = c^+(\sigma_0) = \varphi'(\sigma_0)$ or, $\varphi(\sigma)$ being linear in the interval (α_0, β_0),

$$c^-(\beta_0) = \varphi'(\beta_0 - 0).$$

Similarly, if $\alpha < \alpha_0$, the quantity $c^+(\alpha_0)$ exists, and

$$c^+(\alpha_0) = \varphi'(\alpha_0 + 0).$$

Thus, if two strips without zeros have a common border line $\sigma = \sigma_0$, the mean motions $c^-(\sigma_0)$ and $c^+(\sigma_0)$ both exist and are determined by

(10) $$c^-(\sigma_0) = \varphi'(\sigma_0 - 0) \quad \text{and} \quad c^+(\sigma_0) = \varphi'(\sigma_0 + 0).$$

In the special case where the zeros of $f(s)$ are situated on vertical lines which do not accumulate in the interior of the strip (α, β), the Jensen function is stretchwise linear, with points of non-differentiability in the abscissae of the zeros of $f(s)$. In this case the relations (10) hold for all σ_0. Consequently the relative frequency $H(\sigma_1, \sigma_2)$ always exists and is determined by the formula

(11) $$H(\sigma_1, \sigma_2) = \frac{1}{2\pi}(\varphi'(\sigma_2 - 0) - \varphi'(\sigma_1 + 0)).$$

This particular distribution of the zeros occurs when $f(s)$ is *periodic* with the period ip, where $p > 0$, i. e when the exponents are contained in the discrete modul $M = \left\{h\frac{2\pi}{p}\right\}\cdot$ Thus, in this case, the Jensen function $\varphi(\sigma)$ is stretchwise linear, and the values of $\varphi'(\sigma)$ in the linearity intervals are integral multiples of $\frac{2\pi}{p}\cdot$ In this case the left and right mean motions $c^-(\sigma)$ and $c^+(\sigma)$ and the relative frequency of zeros are, of course, determined by the expressions

$$c^-(\sigma) = \frac{\arg^- f(\sigma + ia + ip) - \arg^- f(\sigma + ia)}{p},$$

$$c^+(\sigma) = \frac{\arg^+ f(\sigma + ia + ip) - \arg^+ f(\sigma + ia)}{p},$$

and

(12) $$H(\sigma_1, \sigma_2) = \frac{N(\sigma_1, \sigma_2; a, a + p)}{p},$$

where a may be arbitrarily chosen except in the last expression, where we must assume $f(s) \neq 0$ on the segment $\sigma_1 < \sigma < \sigma_2$, $t = a$. Similarly, we have for the Jensen function the expression

$$\varphi(\sigma) = \frac{1}{p} \int_{a}^{a+p} \log |f(\sigma + it)| \, dt,$$

where a may be arbitrarily chosen.

From (11) and (12) it immediately follows that the jump $\varphi'(\sigma_0+0)-\varphi'(\sigma_0-0)$ of $\varphi'(\sigma)$ at a vertex of $\varphi(\sigma)$ is equal to $h_0 \frac{2\pi}{p}$, where h_0 denotes the number of zeros of $f(s)$ on a segment $\sigma = \sigma_0$, $a \leqq t < a + p$.

In the particular case where

$$f(s) = \sum_{h=0}^{\infty} a_h e^{h \frac{2\pi}{p} s}, \qquad a_0 \neq 0,$$

so that $f(s)$ is periodic in $(-\infty, \beta)$ and $f(s) \to a_0$ when $\sigma \to -\infty$, the abscissae of the zeros may be arranged in an increasing sequence $\sigma_1, \sigma_2, \ldots$. Then $\varphi(\sigma)$ is linear in $(-\infty, \sigma_1)$ and since $\varphi(\sigma) \to \log |a_0|$ when $\sigma \to -\infty$ we see that $\varphi(\sigma) = \log |a_0|$ for $\sigma < \sigma_1$. Denoting by h_n the number of zeros of $f(s)$ on a segment $\sigma = \sigma_n$, $a \leqq t < a + p$, we therefore have, for every $\sigma < \beta$, the expression

$$\varphi(\sigma) = \log |a_0| + \sum_{\sigma_n \leqq \sigma} h_n (\sigma - \sigma_n).$$

By the substitution $e^{\frac{2\pi}{p} s} = z$ this formula is seen to be equivalent with the usual Jensen formula (§ 19) for a function $F(z)$ with $F(0) \neq 0$ regular in a circle $|z| < \varrho$.

Functions whose Exponents are Bounded Above or Below.

49. Let $f(s)$ once more be almost periodic in $[\alpha, \beta]$, and not identically zero, and let us now assume that its exponents have a finite upper bound $\varLambda$. In this case the function may, according to Bohr [10], be continued in the half-plane $(\alpha, +\infty)$ and will be almost periodic in $[\alpha, +\infty]$. Regarding the behaviour of $f(s)$ for $\sigma \to +\infty$ two different cases should be distinguished.

1) If $\varLambda$ is itself an exponent, we have $f(s) = e^{\varLambda s} g(s)$, where $g(s)$ for $\sigma \to +\infty$ converges uniformly in t towards a constant $A \neq 0$, viz. the coefficient of $e^{\varLambda s}$ in the Dirichlet series of $f(s)$. This implies the existence of a half-plane $\sigma > \sigma_0$, in which $f(s)$ has no zeros.

2) If $\varLambda$ is not an exponent, we have $f(s) = e^{\varLambda s} g(s)$, where $g(s)$ for $\sigma \to +\infty$ converges uniformly in t towards 0. In this case there exists no half-plane $\sigma > \sigma_0$ without zeros of $f(s)$.

In the first case, the Jensen function $\varphi(\sigma)$ is linear for $\sigma > \sigma_0$. Furthermore, in the half-plane $\sigma > \sigma_0$, we have $\log f(s) = \Lambda s + \log g(s)$, and hence

$$\varphi(\sigma) = \underset{t}{M}\{\log |f(\sigma + it)|\} = \Lambda\sigma + \underset{t}{M}\{\log |g(\sigma + it)|\}.$$

Since

$$\underset{t}{M}\{\log |g(\sigma + it)|\} \to \log |A| \quad \text{when} \quad \sigma \to +\infty,$$

this implies that $\varphi(\sigma) = \Lambda\sigma + \log |A|$ for $\sigma > \sigma_0$. Thus the derivative $\varphi'(\sigma)$ has for $\sigma > \sigma_0$ the constant value Λ.

In the second case, the Jensen function $\varphi(\sigma)$ is not linear in any interval $\sigma > \sigma_0$. As furthermore $\varphi(\sigma) = \Lambda\sigma + \underset{t}{M}\{\log |g(\sigma + it)|\}$, where now

$$\underset{t}{M}\{\log |g(\sigma + it)|\} \to -\infty \quad \text{when} \quad \sigma \to +\infty,$$

it is seen that the right derivative $\varphi'(\sigma + 0)$ is $< \Lambda$ for all σ.

Thus we have proved the following theorem.

Theorem 9. *If among the exponents of the function $f(s)$ there is a largest one, say Λ, then, denoting by A the corresponding coefficient, we have for all sufficiently large σ*

$$\varphi(\sigma) = \Lambda\sigma + \log |A|.$$

Thus $\bar{c}^+(\sigma) \leqq \Lambda$ for all σ and $\bar{c}^+(\sigma) = \Lambda$ for all sufficiently large σ.

If the exponents have a finite upper bound Λ, which is not itself an exponent, we have for $\sigma \to +\infty$

$$\varphi(\sigma) - \Lambda\sigma \to -\infty.$$

Thus $\bar{c}^+(\sigma) < \Lambda$ for all σ.

We do not know whether in the second case the relation $\lim\limits_{\sigma \to +\infty} \bar{c}^+(\sigma) < \Lambda$ ever occurs.

There is, of course, a corresponding theorem for functions whose exponents are bounded below, dealing with the behaviour for $\sigma \to -\infty$.

We emphasize as a consequence of these theorems that a lower or upper, left or right mean motion of an almost periodic function $f(s)$ on a vertical line can neither be smaller nor larger than all exponents of the function.

CHAPTER III.

On the Distribution of the Values of Real Almost Periodic Sequences.

Almost Periodic Sequences.

50. In the sequel we shall in a number of cases construct analytic almost periodic functions $f(s)$ with certain prescribed properties. These constructions are all founded on results concerning the distribution of the values of real almost periodic sequences. These results being of a rather complete nature, we have found it convenient to collect them in a separate chapter, together with the analogous results regarding real almost periodic functions, which have been included in order to round off the exposition.

A complex function $U(k)$ defined for all integers $k = \ldots, -2, -1, 0, 1, 2, \ldots$ will briefly be called a *sequence*. An integer $\varkappa$ is called a *translation number* of $U(k)$ belonging to a given number $\varepsilon > 0$, and is denoted by $\varkappa(\varepsilon)$ or $\varkappa_U(\varepsilon)$, if the inequality

$$|U(k+\varkappa) - U(k)| \leqq \varepsilon$$

holds for all k. The sequence $U(k)$ is called *almost periodic* if, for any $\varepsilon > 0$, the set of all translation numbers $\varkappa = \varkappa(\varepsilon) = \varkappa_U(\varepsilon)$ is relatively dense.

Almost periodic sequences have been investigated by Walther [1] and Seynche [1]. They form a special case of von Neumann's [1] general theory of almost periodic functions in a group.

Every almost periodic sequence is bounded and possesses a mean value

$$\underset{k}{M}\{U(k)\} = \lim_{(\delta-\gamma)\to\infty} \frac{1}{\delta-\gamma} \sum_{\gamma}^{\delta} U(k).$$

The sum or the product of two almost periodic sequences and the limit of a uniformly convergent sequence of almost periodic sequences are again almost periodic.

To an arbitrary almost periodic sequence $U(k)$ corresponds a Fourier series

$$U(k) \sim \Sigma\, a_n e^{i\lambda_n k}, \qquad a_n = \underset{k}{M}\{U(k)\, e^{-i\lambda_n k}\},$$

where, of course, the exponents λ_n are only determined mod. 2π. Different sequences have different Fourier series.

The Fourier series of the sum or the product of two almost periodic sequences or of the limit of a uniformly convergent sequence of almost periodic sequences are obtained by performing the corresponding operations on the Fourier series of these sequences.

The main theorem is the approximation theorem. According to this theorem the class of almost periodic sequences is identical with the class of sequences which are the limit of a sequence of exponential polynomials

$$U_p(k) = \sum_{n=1}^{N_p} a_n^{(p)} e^{i \lambda_n^{(p)} k},$$

converging uniformly for all integers k. For a given almost periodic sequence these exponential polynomials may be chosen with exponents among the exponents λ_n of the sequence.

51. A necessary and sufficient condition that a sequence $U(k)$ should be almost periodic with exponents from a given modul M containing the number 2π, is that to any $\varepsilon > 0$ correspond a finite set of numbers $\lambda_1, \ldots, \lambda_N$ in M and a number $\eta > 0$, such that every integer $\varkappa$ satisfying the conditions

$$\left.\begin{array}{l} |\lambda_1 \varkappa| \leqq \eta \\ \ldots \\ |\lambda_N \varkappa| \leqq \eta \end{array}\right\} \text{mod. } 2\pi$$

is a $\varkappa_U(\varepsilon)$.[1]

From this follows: If $U(k)$ is almost periodic with exponents from a given modul M containing the number 2π, and if c is a number with the property that, for some $\varepsilon > 0$, all $\varkappa = \varkappa_U(\varepsilon)$ satisfy the condition

$$c\varkappa \equiv 0 \quad \text{mod. } 2\pi,$$

then c belongs to M.

52. The almost periodic sequence $U(k)$ is periodic with the (integral) period $p > 0$, if and only if its exponents belong to the discrete modul $M = \left\{h \frac{2\pi}{p}\right\}$, where h runs through all integers. The Fourier series is then a finite sum. The sequence is limit periodic, which means that it is the limit of a uniformly convergent sequence of periodic sequences, if and only if its exponents belong to the modul $M = \{r\, 2\pi\}$, where r runs through all rational numbers.

[1] As previously mentioned (see the footnote on p. 145) the set of integers $\varkappa$ satisfying the conditions $|\lambda_1 \varkappa| \leqq \eta, \ldots, |\lambda_N \varkappa| \leqq \eta$ (mod. 2π), where $\eta > 0$ and $\lambda_1, \ldots, \lambda_N$ are arbitrary real numbers, is relatively dense.

In the sequel results will be obtained regarding almost periodic sequences with exponents from a quite arbitrary modul containing the number 2π. These results will be obtained by a reduction to two special types of moduls which will now be considered.

We first consider an arbitrary everywhere dense modul M containing the number 2π, and consisting of rational multiples of 2π. Such a modul may be written in the form $M = \lim_{m\to\infty} \left\{h_m \frac{2\pi}{p_m}\right\}$, where $p_1, p_2, \ldots$ is a sequence of positive integers such that p_{m+1} is, for every m, a proper multiple of p_m, and h_m runs through all integers. (We may for example take $p_1, p_2, \ldots$ as a strictly increasing sub-sequence of the sequence $q_1, q_2, \ldots$, where q_m is the largest divisor of $m!$ for which $\frac{2\pi}{q_m}$ belongs to M.) If, conversely, such a sequence $p_1, p_2, \ldots$ is given, $M = \lim_{m\to\infty} \left\{h_m \frac{2\pi}{p_m}\right\}$ is a modul of the type considered. It is easy to see that a limit periodic sequence $U(k)$ has its exponents in M if and only if it is the limit of a uniformly convergent sequence of periodic sequences, having the periods $p_1, p_2, \ldots$. We express this by saying that $U(k)$ is limit periodic with respect to the periods $p_1, p_2, \ldots$.

Next we consider the case of a modul $M = \{g\,2\pi + h\gamma\}$, where $\gamma/2\pi$ is irrational, and the coefficients g and h run through all integers. Let $U(k)$ be an almost periodic sequence with exponents from M. The exponents being determined only mod. 2π, its Fourier series may be written in the form

$$U(k) \sim \Sigma a_h e^{i h \gamma k}.$$

Using that the points $t = \gamma k$, where k runs through all integers, are everywhere densely distributed mod. 2π, we conclude, by an argument quite similar to that applied in § 31, that $U(k)$ may be written in the form

$$U(k) = F(\gamma k), \tag{1}$$

where $F(t)$ is a uniquely determined continuous function with the period 2π. Its Fourier series is

$$F(t) \sim \Sigma a_h e^{i h t}.$$

If, conversely, $F(t)$ denotes an arbitrary continuous function with the period 2π, the sequence $U(k)$ determined by (1) will be an almost periodic sequence with exponents from M.

Asymptotic Distribution Functions.

53. When speaking of an increasing function $y = \mu(\sigma)$ in an interval $(-\infty \leqq)\,\alpha < \sigma < \beta\,(\leqq +\infty)$, we shall be interested only in the two functions $\mu(\sigma - 0)$ and $\mu(\sigma + 0)$, determined as the limits from the left and the right, and shall in a point of discontinuity consider the function many-valued, ascribing to it all values in the closed interval $\mu(\sigma - 0) \leqq y \leqq \mu(\sigma + 0)$. The notations $\varphi'(\sigma - 0)$ and $\varphi'(\sigma + 0)$ for the left and right derivatives of a convex function $\varphi(\sigma)$ are in accordance with this convention, when, correspondingly, the derivative $\mu(\sigma) = \varphi'(\sigma)$ is considered many-valued in the points where the function is not differentiable.

An increasing function $\mu(\sigma)$ in the interval $-\infty < \sigma < +\infty$ is called a *distribution function*, if it satisfies the conditions

$$\lim_{\sigma \to -\infty} \mu(\sigma) = 0 \quad \text{and} \quad \lim_{\sigma \to +\infty} \mu(\sigma) = 1.$$

We shall here consider only the simple case where there is a finite interval $\{\alpha, \beta\}$ such that $\mu(\sigma) = 0$ for $\sigma < \alpha$ and $\mu(\sigma) = 1$ for $\sigma > \beta$.

For an arbitrary set E of integers we denote by $n(E, \gamma, \delta)$ the number of elements of E belonging to the interval $\gamma \leqq x < \delta$. The two quantities

$$\left.\begin{matrix}\underline{\varrho}(E)\\ \overline{\varrho}(E)\end{matrix}\right\} = \lim_{(\delta-\gamma)\to\infty} \begin{matrix}\inf\\ \sup\end{matrix} \frac{n(E, \gamma, \delta)}{\delta - \gamma}$$

are called the *lower and upper relative frequencies* of the set E. If $\underline{\varrho}(E) = \overline{\varrho}(E)$, i. e. if the limit

$$\varrho(E) = \lim_{(\delta-\gamma)\to\infty} \frac{n(E, \gamma, \delta)}{\delta - \gamma}$$

exists, it is called the *relative frequency* of E.

If E is periodic, $\varrho(E)$ evidently exists, and is equal to the number of elements of E in a period divided by the length of the period.

Similarly, if A denotes a measurable set on the line $-\infty < t < +\infty$, and if $m(A, \gamma, \delta)$ denotes the measure of the part of A belonging to the interval $\gamma \leqq x < \delta$, the quantities

$$\left.\begin{matrix}\underline{r}(A)\\ \overline{r}(A)\end{matrix}\right\} = \lim_{(\delta-\gamma)\to\infty} \begin{matrix}\inf\\ \sup\end{matrix} \frac{m(A, \gamma, \delta)}{\delta - \gamma}$$

are called the *lower and upper relative measures* of A, and if $\underline{r}(A)=\bar{r}(A)$, i. e. if the limit

$$r(A)=\lim_{(\delta-\gamma)\to\infty}\frac{m(A,\gamma,\delta)}{\delta-\gamma}$$

exists, it is called the *relative measure* of A.

If A is periodic, $r(A)$ evidently exists, and is equal to the measure of the part of A belonging to a period divided by the length of the period.

For later application we state the theorem on *equidistribution* mod. 2π of the numbers γk, where $\gamma/2\pi$ is irrational, and k runs through all integers. It says that, if A is periodic with the period 2π, and Jordan measurable, then the set E of all integers k for which $t=\gamma k$ belongs to A has a relative frequency $\varrho(E)$, which is equal to the relative measure $r(A)$ of A.

54. A real sequence $U(k)$ is said to possess an *asymptotic distribution function*, if there exists a distribution function $\mu(\sigma)$ such that, for every σ,

$$\mu(\sigma-0)\leqq\underline{\varrho}(E^-(\sigma))\leqq\left\{\begin{matrix}\underline{\varrho}(E^+(\sigma))\\ \bar{\varrho}(E^-(\sigma))\end{matrix}\right\}\leqq\bar{\varrho}(E^+(\sigma))\leqq\mu(\sigma+0),$$

where $E^-(\sigma)$ and $E^+(\sigma)$ denote the sets of those integers k for which $U(k)<\sigma$ and $U(k)\leqq\sigma$ respectively. The function $\mu(\sigma)$ is then uniquely determined, and for every continuity point of $\mu(\sigma)$ the relative frequencies of $E^-(\sigma)$ and $E^+(\sigma)$ both exist, and are equal to $\mu(\sigma)$.

Similarly, a real measurable function $F(t)$ defined for $-\infty<t<+\infty$ is said to possess an asymptotic distribution function, if there exists a distribution function $\mu(\sigma)$ such that, for every σ,

$$\mu(\sigma-0)\leqq\underline{r}(A^-(\sigma))\leqq\left\{\begin{matrix}\underline{r}(A^+(\sigma))\\ \bar{r}(A^-(\sigma))\end{matrix}\right\}\leqq\bar{r}(A^+(\sigma))\leqq\mu(\sigma+0),$$

where $A^-(\sigma)$ and $A^+(\sigma)$ denote the sets of those points t for which $F(t)<\sigma$ and $F(t)\leqq\sigma$ respectively.

We shall only consider the case of bounded functions $U(k)$ or $F(t)$. The asymptotic distribution function $\mu(\sigma)$ is then, when it exists, of the special type mentioned above. In fact, if the values of $U(k)$ or $F(t)$ all belong to the interval $\{\alpha,\beta\}$, we have $\mu(\sigma)=0$ for $\sigma<\alpha$ and $\mu(\sigma)=1$ for $\sigma>\beta$.

It is easily proved that a bounded function $U(k)$ or $F(t)$ possesses an asymptotic distribution function $\mu(\sigma)$ if and only if

(2) $$\bar{\varrho}(E^+(\sigma_1)) \leqq \underline{\varrho}(E^-(\sigma_2)) \quad \text{for} \quad \sigma_1 < \sigma_2$$

or

(3) $$\bar{r}(A^+(\sigma_1)) \leqq \underline{r}(A^-(\sigma_2)) \quad \text{for} \quad \sigma_1 < \sigma_2$$

respectively.

Asymptotic Distribution Functions of Real Almost Periodic Sequences.

55. In our considerations the following theorem due to Wintner [7] is of fundamental importance.

Theorem 10. *Any real almost periodic sequence $U(k)$ possesses an asymptotic distribution function $\mu(\sigma)$.*

Wintner's proof of this theorem depends on the moment method mentioned in § 17, and leads to a characterization of the distribution function by means of moments. We shall only need the existence of the distribution function, which may be proved quite elementarily as follows.

The sequence $U(k)$ being bounded, it is sufficient to prove that condition (2) is satisfied. Let $\chi(\sigma)$ denote the continuous function which is 1 in the interval $\sigma \leqq \sigma_1$ and 0 in the interval $\sigma \geqq \sigma_2$, and is linear in the interval $\sigma_1 \leqq \sigma \leqq \sigma_2$. Then $\chi(U(k))$ is evidently again almost periodic and possesses therefore a mean value for which we have

$$\bar{\varrho}(E^+(\sigma_1)) \leqq \underset{k}{M}\{\chi(U(k))\} \quad \text{and} \quad \underset{k}{M}\{\chi(U(k))\} \leqq \underline{\varrho}(E^-(\sigma_2)).$$

This implies condition (2).

If $U(k)$ is periodic, the relative frequencies of the sets $E^-(\sigma)$ and $E^+(\sigma)$ exist for all σ, and we have

$$\varrho(E^-(\sigma)) = \mu(\sigma - 0) \quad \text{and} \quad \varrho(E^+(\sigma)) = \mu(\sigma + 0).$$

56. By the same argument, it may be proved that any real almost periodic function $F(t)$ possesses an asymptotic distribution function $\mu(\sigma)$, a theorem which is also due to Wintner [1], [4], [5]. Here condition (3) has to be proved; this condition follows with the same choice of $\chi(\sigma)$ from the relations

$$\bar{r}(A^+(\sigma_1)) \leqq \underset{t}{M}\{\chi(F(t))\} \quad \text{and} \quad \underset{t}{M}\{\chi(F(t))\} \leqq \underline{r}(A^-(\sigma_2)).$$

If $F(t)$ is periodic, the relative measures of the sets $A^-(\sigma)$ and $A^+(\sigma)$ exist for all σ, and we have

$$r(A^-(\sigma)) = \mu(\sigma - 0) \quad \text{and} \quad r(A^+(\sigma)) = \mu(\sigma + 0).$$

Since $A^-(\sigma)$ is open and $A^+(\sigma)$ is closed, this implies that both sets are Jordan measurable for all values of σ for which $\mu(\sigma)$ is continuous.

As pointed out by Wintner, the asymptotic distribution function $\mu(\sigma)$ of an almost periodic function is strictly increasing in the interval $\{\alpha, \beta\}$, where α and β denote the lower and upper bounds of $F(t)$. In fact, if $\alpha \leqq \sigma_1 < \sigma_2 \leqq \beta$, and ε denotes a positive number $< \frac{1}{2}(\sigma_2 - \sigma_1)$, there exists, by the continuity of $F(t)$, an interval $|t - t_0| < \delta$ in which $\sigma_1 + \varepsilon < F(t) < \sigma_2 - \varepsilon$. For an arbitrary $\tau = \tau_F(\varepsilon)$ we therefore have $\sigma_1 < F(t) < \sigma_2$ in the interval $|t + \tau - t_0| < \delta$. The set of these numbers τ being relatively dense, we see that the two sets $A^-(\sigma_2)$ and $A^+(\sigma_1)$ differ by a set of positive lower frequency, whence it follows that $\underline{r}(A^+(\sigma_1)) < \underline{r}(A^-(\sigma_2))$, and thus that $\mu(\sigma_1 - 0) < \mu(\sigma_2 + 0)$.

Conversely, an arbitrary distribution function $\mu(\sigma)$ for which there exists an interval $\{\alpha, \beta\}$ such that $\mu(\sigma) = 0$ for $\sigma < \alpha$ and $\mu(\sigma) = 1$ for $\sigma > \beta$, and $\mu(\sigma)$ is strictly increasing in the interval $\{\alpha, \beta\}$, is the asymptotic distribution function of a real almost periodic function $F(t)$. We may even choose $F(t)$ as a periodic function with a prescribed period p. In order to see this, we consider the inverse function $\sigma = H(y)$ of $y = \mu(\sigma)$, which is continuous for $0 < y < 1$, and hence also in $0 \leqq y \leqq 1$, when we put $H(0) = \alpha$ and $H(1) = \beta$. The function $F(t)$ defined in $0 \leqq t \leqq \frac{1}{2}p$ by $F(t) = H(t/\frac{1}{2}p)$ is now extended to an even function $F(t)$ with the period p, defined for all t; this function will then also be continuous and evidently possesses the asymptotic distribution function $\mu(\sigma)$.

57. For later application we notice that if $F(t)$ is a real continuous function with the period 2π, and if $\gamma/2\pi$ is irrational, then the almost periodic sequence $U(k) = F(\gamma k)$ (see § 52) has the same asymptotic distribution function $\mu(\sigma)$ as $F(t)$.

This follows immediately from the equidistribution of the numbers γk mod. 2π, which shows that, for all values of σ for which the sets $A^-(\sigma)$ and $A^+(\sigma)$ are Jordan measurable, the relative frequencies of the sets $E^-(\sigma)$ and $E^+(\sigma)$ exist and are determined by

$$\varrho(E^-(\sigma)) = r(A^-(\sigma)) = \mu(\sigma - 0) \quad \text{and} \quad \varrho(E^+(\sigma)) = r(A^+(\sigma)) = \mu(\sigma + 0).$$

58. We now return to the consideration of almost periodic sequences and shall first prove the following theorem.

Theorem 11. *The asymptotic distribution function $\mu(\sigma)$ of a real almost periodic sequence $U(k)$ is constant in an interval (α_0, β_0), if and only if $U(k)$ does not take any value from this interval.*

In this case the constant value of $\mu(\sigma)$ in the interval (α_0, β_0) is a rational number r.

If the exponents of $U(k)$ belong to a given modul M containing the number 2π, the number $2\pi r$ also belongs to M.

If $U(k)$ does not take any value from the interval (α_0, β_0), the two sets $E^+(\alpha_0)$ and $E^-(\beta_0)$ are identical. Hence $\underline{\varrho}(E^+(\alpha_0)) = \underline{\varrho}(E^-(\beta_0))$, and therefore $\mu(\alpha_0 + 0) = \mu(\beta_0 - 0)$, which shows that $\mu(\sigma)$ is constant in the interval (α_0, β_0).

If $U(k)$ takes a value σ_0 from the interval (α_0, β_0), i. e. if there exists an integer k_0 such that $U(k_0)=\sigma_0$, we choose σ_1 and σ_2 such that $\alpha_0<\sigma_1<\sigma_0<\sigma_2<\beta_0$. The almost periodicity then implies the existence of a relatively dense set of numbers k for which $\sigma_1<U(k)<\sigma_2$. The two sets $E^-(\sigma_2)$ and $E^+(\sigma_1)$ differ therefore by a relatively dense set, from which it follows that $\underline{\varrho}(E^+(\sigma_1)) < \underline{\varrho}(E^-(\sigma_2))$ and hence that $\mu(\sigma_1 - 0) < \mu(\sigma_2 + 0)$, which shows that $\mu(\sigma)$ is not constant in the interval (α_0, β_0).

We now assume that $U(k)$ does not take any value from the interval (α_0, β_0). If we choose $\varepsilon < \beta_0 - \alpha_0$ it is then obvious that any $\varkappa = \varkappa_U(\varepsilon)$ must be a period for the set $E^+(\alpha_0)$. Now $E^-(\sigma) = E^+(\alpha_0)$ for every σ in the interval (α_0, β_0). Hence $\mu(\sigma) = \varrho(E^-(\sigma))$ must be a rational number having the denominator $\varkappa$. This rational number being denoted by r, we have therefore for every $\varkappa = \varkappa_U(\varepsilon)$

$$2\pi r \varkappa \equiv 0 \mod. 2\pi.$$

If the exponents of $U(k)$ belong to a given modul M containing the number 2π, this shows (see § 51) that $2\pi r$ belongs to M.

59. We shall now give a complete characterization of those distribution functions $\mu(\sigma)$ which may occur as the asymptotic distribution function of a real almost periodic sequence $U(k)$ with exponents from a given modul M containing the number 2π. By Theorem 11, a necessary condition is that the values of $\mu(\sigma)$ in the constancy intervals, multiplied by 2π, belong to M.

If M is discrete and hence of the form $M = \left\{h\frac{2\pi}{p}\right\}$, where p is a positive integer, and h runs through all integers, so that the question is about periodic sequences $U(k)$ with the period p, the answer is obvious. The distribution functions which may occur are then all step-functions whose values in the constancy intervals, multiplied by 2π, belong to M. The points of discontinuity are determined by the finite set of values taken by $U(k)$.

The only case of interest is therefore the one in which M is everywhere dense. Regarding this case we shall prove the following theorem.

Theorem 12. *A distribution function $\mu(\sigma)$ is the asymptotic distribution function of a real almost periodic sequence with exponents from a given everywhere dense modul M containing the number 2π, if and only if there exists an interval $\{\alpha, \beta\}$ such that $\mu(\sigma) = 0$ for $\sigma < \alpha$ and $\mu(\sigma) = 1$ for $\sigma > \beta$ and the values of $\mu(\sigma)$ in the constancy intervals are all rational and, multiplied by 2π, belong to M.*

The necessity of the conditions has already been proved, and we therefore have to prove their sufficiency. We begin by reducing the problem to some special cases.

(i) If the rational multiples of 2π belonging to M form an everywhere dense modul, we may replace M by this sub-modul, which (see § 52) is of the form $\lim_{m\to\infty}\left\{h_m \frac{2\pi}{p_m}\right\}$, where $p_1, p_2, \ldots$ is a sequence of positive integers such that p_{m+1} is for every m a proper multiple of p_m, and h_m runs through all integers.

(ii) If the rational multiples of 2π belonging to M form a discrete modul, this sub-modul has the form $\left\{h \frac{2\pi}{p}\right\}$. A distribution function $\mu(\sigma)$ satisfying the conditions of the theorem may then be written in the form

$$\mu(\sigma) = \frac{1}{p}(\mu_1(\sigma) + \cdots + \mu_p(\sigma)), \tag{4}$$

where each of the functions $\mu_1(\sigma), \ldots, \mu_p(\sigma)$ also satisfies the said conditions, but has no constancy intervals besides those where it is 0 or 1. Hence these functions satisfy the conditions of the theorem corresponding to the modul obtained from M by multiplying all elements by p. Now it is easily proved that a sequence $U(k)$ is almost periodic with exponents from M if and only if each of the sequences

$$U_1(k) = U(1 + kp), \ldots, \ U_p(k) = U(p + kp)$$

is almost periodic with exponents from this new modul. On the other hand, the asymptotic distribution function $\mu(\sigma)$ of $U(k)$ is determined by (4), when $\mu_1(\sigma), \ldots, \mu_p(\sigma)$ denote the asymptotic distribution functions of $U_1(k), \ldots, U_p(k)$.

This implies that we may replace M by the new modul, which amounts to assuming $p=1$. As M is by assumption everywhere dense, it contains a number γ such that $\gamma/2\pi$ is irrational, and we may therefore finally replace M by the submodul $\{g\,2\pi + h\gamma\}$, where g and h run through all integers.

60. We first consider the case $M = \lim_{m\to\infty}\left\{h_m \frac{2\pi}{p_m}\right\}$. That $U(k)$ is almost periodic with exponents from M then means (see § 52) that $U(k)$ is limit periodic with respect to the periods $p_1, p_2, \ldots$. The following construction is an adaptation from Buch [1], [2].

Let $\sigma = H(y)$ denote the inverse function of $y = \mu(\sigma)$. It is defined for $0 \leqq y \leqq 1$, and its discontinuity points, multiplied by 2π, belong to M. For every m we consider the p_m intervals

$$I_{m, q_m}:\ \frac{q_m - 1}{p_m} \leqq y \leqq \frac{q_m}{p_m}, \qquad q_m = 1, \ldots, p_m,$$

into which the interval $0 \leqq y \leqq 1$ may be divided, and the p_m classes of residues mod. p_m, into which the set of all integers k may be divided. Between these intervals and classes of residues we establish for each m a one-to-one correspondence in such a manner that if E_{m, q_m} denotes the class of residues corresponding to the interval I_{m, q_m}, the classes of residues $E_{m+1, q_{m+1}}$ corresponding to the sub-intervals $I_{m+1, q_{m+1}}$ of I_{m, q_m} are just those which are contained in E_{m, q_m}. Together with the intervals I_{m, q_m} we consider the intervals

$$J_{m, q_m}:\ H\left(\frac{q_m - 1}{p_m} + 0\right) \leqq \sigma \leqq H\left(\frac{q_m}{p_m} - 0\right).$$

Denoting by δ_m the maximum of the length of J_{m, q_m}, we evidently have $\delta_m \to 0$ as $m \to \infty$.

Now every integer k belongs to a definite sequence of classes of residues $E_{1, q_1} \supset E_{2, q_2} \supset \cdots$. To this sequence corresponds a definite sequence of intervals $I_{1, q_1} \supset I_{2, q_2} \supset \cdots$ and hence a sequence $J_{1, q_1} \supseteqq J_{2, q_2} \supseteqq \cdots$ converging towards a definite point $U(k)$. We shall now prove that the sequence $U(k)$ thus defined satisfies the conditions of the theorem.

We first notice that, as $U(k)$ belongs to J_{m, q_m} when k belongs to E_{m, q_m}, we have

$$|U(k + \varkappa) - U(k)| \leqq \delta_m$$

for all k, when $\varkappa$ is an arbitrary multiple of p_m. Thus $U(k)$ is almost periodic with exponents from M.

If, further, for an arbitrary σ,

$$\frac{r_m - 1}{p_m} < \mu(\sigma - 0) \leqq \frac{r_m}{p_m} \quad \text{and} \quad \frac{s_m - 1}{p_m} \leqq \mu(\sigma + 0) < \frac{s_m}{p_m},$$

where r_m and s_m are integers, then

$$\sum_{q_m < r_m} E_{m,\,q_m} \subseteqq E^-(\sigma) \subseteqq \sum_{q_m \leqq r_m} E_{m,\,q_m} \quad \text{and} \quad \sum_{q_m < s_m} E_{m,\,q_m} \subseteqq E^+(\sigma) \subseteqq \sum_{q_m \leqq s_m} E_{m,\,q_m}$$

and consequently

$$\frac{r_m - 1}{p_m} \leqq \underline{\varrho}(E^-(\sigma)) \leqq \overline{\varrho}(E^-(\sigma)) \leqq \frac{r_m}{p_m} \quad \text{and} \quad \frac{s_m - 1}{p_m} \leqq \underline{\varrho}(E^+(\sigma)) \leqq \overline{\varrho}(E^+(\sigma)) \leqq \frac{s_m}{p_m}.$$

This shows that the relative frequencies $\varrho(E^-(\sigma))$ and $\varrho(E^+(\sigma))$ both exist and are determined by

$$\varrho(E^-(\sigma)) = \mu(\sigma - 0) \quad \text{and} \quad \varrho(E^+(\sigma)) = \mu(\sigma + 0),$$

so that $U(k)$ possesses the asymptotic distribution function $\mu(\sigma)$.

61. We next consider the case $M = \{g\, 2\pi + h\gamma\}$. That $U(k)$ is almost periodic with exponents from M then means (see § 52) that $U(k) = F(\gamma k)$, where $F(t)$ is a continuous function with the period 2π.

The function $y = \mu(\sigma)$ being strictly increasing in the interval where it is not 0 or 1, its inverse function $\sigma = H(y)$ is continuous in the interval $0 < y < 1$. Let $F(t)$ denote the continuous even function with the period 2π for which $F(t) = H(t/\pi)$ in the interval $0 < t < \pi$. Then $F(t)$ has the asymptotic distribution function $\mu(\sigma)$ and, by § 57, the sequence $U(k) = F(\gamma k)$ therefore also has the asymptotic distribution function $\mu(\sigma)$.

The sets $A^-(\sigma)$ and $A^+(\sigma)$ being here Jordan measurable for all σ, it follows from § 57 that the relative frequencies of the sets $E^-(\sigma)$ and $E^+(\sigma)$ exist for all σ, and are determined by

$$\varrho(E^-(\sigma)) = \mu(\sigma - 0) \quad \text{and} \quad \varrho(E^+(\sigma)) = \mu(\sigma + 0).$$

Detailed Discussion of the Distribution for a Given Abscissa.[1]

62. Let M denote once more an arbitrary modul containing the number 2π, and let σ be an arbitrary number. We shall then give a complete characterization of those sets of six numbers which may occur as the six numbers in the inequalities

$$\mu(\sigma - 0) \leqq \underline{\varrho}(E^-(\sigma)) \leqq \left\{ \begin{matrix} \underline{\varrho}(E^+(\sigma)) \\ \bar{\varrho}(E^-(\sigma)) \end{matrix} \right\} \leqq \bar{\varrho}(E^+(\sigma)) \leqq \mu(\sigma + 0),$$

corresponding to an almost periodic sequence $U(k)$ with exponents from M.

If M is discrete, and hence of the form $M = \left\{ h\frac{2\pi}{p} \right\}$, so that the question is about periodic sequences $U(k)$ with the period p, the answer is obvious. In this case $\mu(\sigma - 0)$ and $\mu(\sigma + 0)$ are integral multiples of $\frac{1}{p}$, and the relative frequencies $\varrho(E^-(\sigma))$ and $\varrho(E^+(\sigma))$ both exist and are, as has already been mentioned, equal to $\mu(\sigma - 0)$ and $\mu(\sigma + 0)$ respectively. If, conversely, f^- and f^+ are given integral multiples of $\frac{1}{p}$, for which $0 \leqq f^- \leqq f^+ \leqq 1$, there exists a periodic sequence $U(k)$ with the period p, for which

$$\varrho(E^-(\sigma)) = \mu(\sigma - 0) = f^- \quad \text{and} \quad \varrho(E^+(\sigma)) = \mu(\sigma + 0) = f^+.$$

As pointed out in §§ 60 and 61, the sequences $U(k)$ there constructed also have the property that the relative frequencies $\varrho(E^-(\sigma))$ and $\varrho(E^+(\sigma))$ exist for every σ and are equal to $\mu(\sigma - 0)$ and $\mu(\sigma + 0)$.

In the case of the analogous problem for real almost periodic functions $F(t)$ of a real variable, where we are concerned with the inequalities

$$\mu(\sigma - 0) \leqq \underline{r}(A^-(\sigma)) \leqq \left\{ \begin{matrix} \underline{r}(A^+(\sigma)) \\ \bar{r}(A^-(\sigma)) \end{matrix} \right\} \leqq \bar{r}(A^+(\sigma)) \leqq \mu(\sigma + 0),$$

it was shown by Bohr [11] by an example that the relative measures $r(A^-(\sigma))$ and $r(A^+(\sigma))$ need not exist. By a further elaboration of the method there applied[2] we shall now prove the following theorem.

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[1] The results of this section are used only in Chapter V.

[2] Bohr's example is a limit periodic function. In extending the construction to functions with exponents from an arbitrary everywhere dense modul we provide an answer to a desideratum mentioned by van Kampen [3].

Theorem 13. *For arbitrary numbers satisfying the conditions*

$$f^- \leqq \underline{e}^- \leqq \begin{Bmatrix} \underline{e}^+ \\ \bar{e}^- \end{Bmatrix} \leqq \bar{e}^+ \leqq f^+,$$

and for an arbitrary σ_0, *there exists a real almost periodic sequence* $U(k)$ *with exponents from a given everywhere dense modul* M *containing the number* 2π, *for which these numbers are equal to the corresponding numbers in the inequalities*

$$\mu(\sigma_0 - 0) \leqq \underline{\varrho}(E^-(\sigma_0)) \leqq \begin{Bmatrix} \underline{\varrho}(E^+(\sigma_0)) \\ \bar{\varrho}(E^-(\sigma_0)) \end{Bmatrix} \leqq \bar{\varrho}(E^+(\sigma_0)) \leqq \mu(\sigma_0 + 0),$$

if and only if either

(a) $0 < f^-$ *and* $f^+ < 1$,

(b) $0 = f^- = \underline{e}^- = \bar{e}^-$ *and* $f^+ < 1$,

(c) $0 < f^-$ *and* $\underline{e}^+ = \bar{e}^+ = f^+ = 1$,

or (d) $0 = f^- = \underline{e}^- = \bar{e}^-$ *and* $\underline{e}^+ = \bar{e}^+ = f^+ = 1$.

The necessity of the conditions is obvious, for if, for an almost periodic sequence $U(k)$, we have $\mu(\sigma_0 - 0) = 0$, the set $E^-(\sigma_0)$ is by Theorem 11 empty, and hence $\underline{\varrho}(E^-(\sigma_0)) = \bar{\varrho}(E^-(\sigma_0)) = 0$; and similarly, if $\mu(\sigma_0 + 0) = 1$, the set $E^+(\sigma_0)$ is the set of all integers, and hence $\underline{\varrho}(E^+(\sigma_0)) = \bar{\varrho}(E^+(\sigma_0)) = 1$.

In our proof of the sufficiency of the conditions we shall restrict ourselves to case (a). Cases (b) and (c) are treated in a quite analogous way, only more simply, and case (d) is trivial. We may suppose without loss of generality that the modul M is of one of the two types $\lim_{m \to \infty} \left\{ h_m \frac{2\pi}{p_m} \right\}$ or $\{g\, 2\pi + h\gamma\}$ considered above; for any everywhere dense modul M contains a modul of one of these forms. Finally we may suppose that $\sigma_0 = 0$.

63. We first consider the case $M = \lim_{m \to \infty} \left\{ h_m \frac{2\pi}{p_m} \right\}$. That $U(k)$ is almost periodic with exponents from M then means (see § 52) that $U(k)$ is limit periodic with respect to the periods $p_1, p_2, \ldots$.

For every positive integer n we choose rational numbers with denominators among the numbers p_m

$$(5) \qquad 0 < s_n^- < \underline{r}_n^- < \begin{Bmatrix} \underline{r}_n^+ \\ \bar{r}_n^- \end{Bmatrix} < \bar{r}_n^+ < s_n^+ < 1,$$

such that the sequences $\{s_n^-\}$, $\{\underline{r}_n^-\}$, and $\{\underline{r}_n^+\}$ are strictly increasing and converge towards f^-, $\underline{e}^-$ and $\underline{e}^+$ respectively, whereas the sequences $\{\bar{r}_n^-\}$, $\{\bar{r}_n^+\}$, and $\{s_n^+\}$ are strictly decreasing and converge towards $\bar{e}^-$, $\bar{e}^+$ and f^+ respectively. If $\bar{e}^- < \underline{e}^+$, we choose $\bar{r}_1^- < \underline{r}_1^+$, which implies $\bar{r}_n^- < \underline{r}_n^+$ for all n. If $\bar{e}^- \geqq \underline{e}^+$, we obviously have $\bar{r}_n^- > \underline{r}_n^+$ for all n.

As we shall now prove, it is then possible for every n to choose among the numbers p_m a common denominator q_n of the numbers (5), as well as periodic sets of integers

$$(6) \qquad S_n^- \subset \underline{R}_n^- \subset \left\{ \begin{matrix} \underline{R}_n^+ \\ \bar{R}_n^- \end{matrix} \right\} \subset \bar{R}_n^+ \subset S_n^+$$

having the period q_n and the corresponding numbers (5) as relative frequencies, such that the sequences $\{S_n^-\}$, $\{\underline{R}_n^-\}$, and $\{\underline{R}_n^+\}$ are strictly increasing, whereas the sequences $\{\bar{R}_n^-\}$, $\{\bar{R}_n^+\}$, and $\{S_n^+\}$ are strictly decreasing, such that, further,

$$(7) \qquad \bar{R}_n^- \begin{smallmatrix} \subseteq \\ \supset \end{smallmatrix} \underline{R}_n^+ \quad \text{according as} \quad \bar{r}_n^- \lesseqgtr \underline{r}_n^+,$$

and, finally, such that for every n the following conditions are satisfied:

(i) The number q_{n+1} is a multiple of q_n and $\frac{q_{n+1}}{q_n} > 3$.

(ii) In the interval $-\frac{1}{2}q_n \leqq k < \frac{1}{2}q_n$ we have

$$S_{n+1}^- = \underline{R}_{n+1}^- = \underline{R}_n^- \quad \text{and} \quad S_{n+1}^+ = \bar{R}_{n+1}^+ = \bar{R}_n^+.$$

(iii) In the interval $\frac{1}{2}q_n \leqq k < \frac{3}{2}q_n$ we have

$$\underline{R}_{n+1}^- = \bar{R}_n^-.$$

(iv) In the interval $-\frac{3}{2}q_n \leqq k < -\frac{1}{2}q_n$ we have

$$\bar{R}_{n+1}^+ = \underline{R}_n^+.$$

64. We shall prove this by induction.

The definition of q_1 and the sets S_1^-, $\underline{R}_1^-$, $\underline{R}_1^+$, $\bar{R}_1^-$, $\bar{R}_1^+$, and S_1^+ in accordance with the conditions presents no difficulties. We may e. g. for q_1 choose an arbitrary common denominator of the numbers s_1^-, $\underline{r}_1^-$, $\underline{r}_1^+$, $\bar{r}_1^-$, $\bar{r}_1^+$, and s_1^+, taken among the numbers p_m, and for the sets S_1^-, $\underline{R}_1^-$, $\underline{R}_1^+$, $\bar{R}_1^-$, $\bar{R}_1^+$, and S_1^+ we may choose the sets determined by the inequalities $0 \leqq k < q_1 a \mod. q_1$, where a denotes s_1^-, $\underline{r}_1^-$, $\underline{r}_1^+$, $\bar{r}_1^-$, $\bar{r}_1^+$, and s_1^+ respectively.

Now, suppose that we have defined q_n and the sets S_n^-, $\underline{R}_n^-$, $\underline{R}_n^+$, $\overline{R}_n^-$, $\overline{R}_n^+$, and S_n^+, and let for the present q_{n+1} denote an arbitrary common denominator of the numbers s_{n+1}^-, $\underline{r}_{n+1}^-$, $\underline{r}_{n+1}^+$, $\bar{r}_{n+1}^-$, $\bar{r}_{n+1}^+$, and s_{n+1}^+, chosen among the numbers p_m and satisfying condition (i).

The sets S_n^-, $\underline{R}_n^-$, $\underline{R}_n^+$, $\overline{R}_n^-$, $\overline{R}_n^+$, and S_n^+ are periodic with the period q_n, and thus consist of classes of residues mod. q_n. Considered as periodic sets with the period q_{n+1} they consist of $q_{n+1} s_n^-$, $q_{n+1} \underline{r}_n^-$, $q_{n+1} \underline{r}_n^+$, $q_{n+1} \bar{r}_n^-$, $q_{n+1} \bar{r}_n^+$, and $q_{n+1} s_n^+$ classes of residues mod. q_{n+1} respectively.

Fig. 2 illustrates the situation in the two cases possible: $\bar{r}_n^- < \underline{r}_n^+$ and $\bar{r}_n^- > \underline{r}_n^+$. The small squares represent the q_{n+1} classes of residues mod. q_{n+1} arranged in q_n columns and $\frac{q_{n+1}}{q_n}$ rows in such a way that each column constitutes a class of residues mod. q_n, and that each of the sets S_n^-, $\underline{R}_n^-$, $\underline{R}_n^+$, $\overline{R}_n^-$, $\overline{R}_n^+$, and S_n^+ consists of all columns to the left of a certain vertical line. These lines are indicated by a thick stroke and marked with the notation of the set in question. Further, we suppose (as we may) that the first row contains all classes of residues mod. q_{n+1} represented in the interval $-\frac{1}{2} q_n \leqq k < \frac{1}{2} q_n$, the second row those represented in the interval $\frac{1}{2} q_n \leqq k < \frac{3}{2} q_n$, and the third row those represented in the interval $-\frac{3}{2} q_n \leqq k < -\frac{1}{2} q_n$.

The sets S_{n+1}^-, $\underline{R}_{n+1}^-$, $\underline{R}_{n+1}^+$, $\overline{R}_{n+1}^-$, $\overline{R}_{n+1}^+$, and S_{n+1}^+ have to be periodic with the period q_{n+1}, i. e. they have to consist of $q_{n+1} s_{n+1}^-$, $q_{n+1} \underline{r}_{n+1}^-$, $q_{n+1} \underline{r}_{n+1}^+$, $q_{n+1} \bar{r}_{n+1}^-$, $q_{n+1} \bar{r}_{n+1}^+$, and $q_{n+1} s_{n+1}^+$ classes of residues mod. q_{n+1} respectively. Thus we have to select the classes of residues mod. q_{n+1} to be added to the sets S_n^-, $\underline{R}_n^-$, and $\underline{R}_n^+$, or subtracted from the sets $\overline{R}_n^-$, $\overline{R}_n^+$, and S_n^+ in order to form the corresponding sets S_{n+1}^-, $\underline{R}_{n+1}^-$, $\underline{R}_{n+1}^+$, $\overline{R}_{n+1}^-$, $\overline{R}_{n+1}^+$, and S_{n+1}^+. The number of these classes is

$$
(8) \qquad \begin{aligned} & q_{n+1}(s_{n+1}^- - s_n^-), \qquad q_{n+1}(\underline{r}_{n+1}^- - \underline{r}_n^-), \qquad q_{n+1}(\underline{r}_{n+1}^+ - \underline{r}_n^+), \\ & q_{n+1}(\bar{r}_n^- - \bar{r}_{n+1}^-), \qquad q_{n+1}(\bar{r}_n^+ - \bar{r}_{n+1}^+), \text{ and } q_{n+1}(s_n^+ - s_{n+1}^+) \end{aligned}
$$

respectively. The classes chosen are indicated in Fig. 2 by thick horizontal strokes connecting the squares in question with the vertical lines indicating the sets to which the classes have to be added, or from which they have to be subtracted.

In the first row we add to S_n^- all the classes belonging to $\underline{R}_n^- - S_n^-$, and subtract from S_n^+ all classes belonging to $S_n^+ - \overline{R}_n^+$. No other changes are made in the first row. By this we evidently obtain that condition (ii) is satisfied. In the second row we add to $\underline{R}_n^-$ the classes belonging to $\overline{R}_n^- - \underline{R}_n^-$, and, if $\bar{r}_n^- > \underline{r}_n^+$,

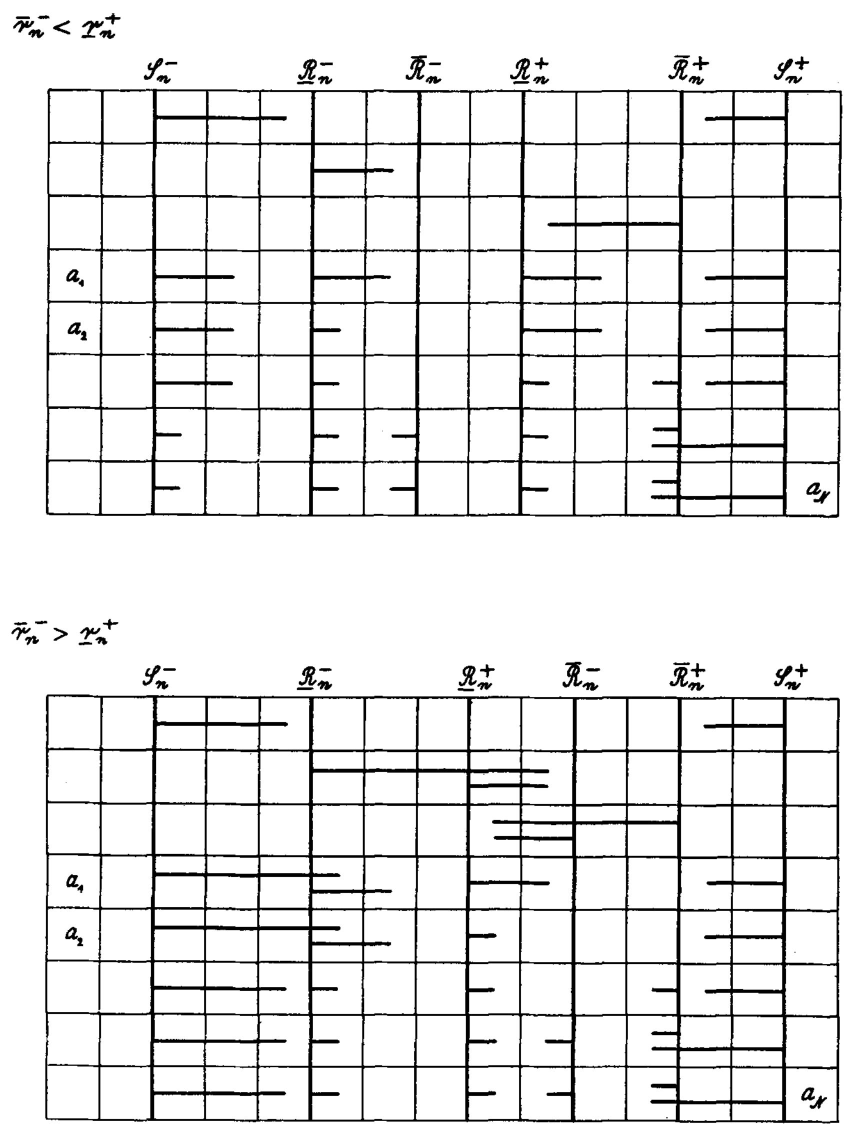

Fig. 2.

we also add to $\underline{R}_n^+$ the classes belonging to $\overline{R}_n^- - \underline{R}_n^+$. By this we obtain that condition (iii) is satisfied. In the third row we subtract from $\overline{R}_n^+$ the classes belonging to $\overline{R}_n^+ - \underline{R}_n^+$, and, if $\bar{r}_n^- > \underline{r}_n^+$, we also subtract from $\overline{R}_n^-$ the classes belonging to $\overline{R}_n^- - \underline{R}_n^+$. By this we obtain that condition (iv) is satisfied.

The numbers of classes in the first three rows to be added to or subtracted from the various sets are independent of q_{n+1} and therefore certainly less than the numbers (8) which are at our disposal, provided q_{n+1} is chosen sufficiently large.

The remaining numbers of classes to be added or subtracted are chosen in the last $\frac{q_{n+1}}{q_n} - 3$ rows. The classes in these rows, arranged lexicographically according to column- and row-numbers, are denoted by $a_1, a_2, \ldots, a_N$ (where $N = q_{n+1} - 3q_n$). To S_n^- we then add the number of classes still missing, beginning with the first element of the sequence $a_1, a_2, \ldots, a_N$ not belonging to S_n^-, and proceeding successively until the required number has been reached. In just the same way we select the classes which are still to be added to $\underline{R}_n^-$ and $\underline{R}_n^+$. By the choice of the classes to be subtracted from S_n^+ we begin with the last element of the sequence $a_1, a_2, \ldots, a_N$ belonging to S_n^+, and move backwards successively until the required number has been reached. In just the same way we select the classes which are still to be subtracted from $\overline{R}_n^+$ and $\overline{R}_n^-$.

It now only remains to secure that the sets constructed satisfy the conditions corresponding to (6) and (7), viz.

$$S_{n+1}^- \subset \underline{R}_{n+1}^- \subset \begin{Bmatrix} \underline{R}_{n+1}^+ \\ \overline{R}_{n+1}^- \end{Bmatrix} \subset \overline{R}_{n+1}^+ \subset S_{n+1}^+ \tag{9}$$

and

$$\overline{R}_{n+1}^- \overset{\subset}{\supset} \underline{R}_{n+1}^+ \quad \text{according as} \quad \bar{r}_{n+1}^- \lesseqgtr \underline{r}_{n+1}^+. \tag{10}$$

Since $\bar{r}_{n+1}^- \lesseqgtr \underline{r}_{n+1}^+$ according as $\bar{r}_n^- \lesseqgtr \underline{r}_n^+$, it is obvious that these conditions are satisfied in the first three rows. Now let $A \subset B$ denote any one of the relations included in (9) or (10). For the corresponding relative frequencies a and b we then have $a < b$. The condition of having $A \subset B$ is obviously that the number of classes in the first three rows belonging to B but not to A does not exceed the difference between the total number of classes in B and A, which is $q_{n+1}(b-a)$. This is evidently true for all possible pairs A, B provided q_{n+1} is chosen sufficiently large.

65. We now choose a strictly decreasing sequence $\delta_1, \delta_2, \ldots$ of positive numbers converging towards 0, and, denoting by S_0^- the empty set and by S_0^+ the set of all integers, we put

$$U(k) = \begin{cases} -\delta_n \text{ in the set } S_n^- - S_{n-1}^- \\ 0 \text{ in the set } \lim\,(S_n^+ - S_n^-) \\ \delta_n \text{ in the set } S_{n-1}^+ - S_n^+. \end{cases}$$

We shall then prove that the sequence $U(k)$ satisfies the conditions of the theorem.

That $U(k)$ is almost periodic with exponents from M is obvious, since for every n all integral multiples of q_n are translation numbers of $U(k)$ belonging to $2\,\delta_{n+1}$.

The asymptotic distribution function $\mu(\sigma)$ of $U(k)$ is determined by

$$\mu(\sigma)=\begin{cases} 0 & \text{for } \sigma<-\delta_1 \\ s_n^- & \text{for } -\delta_n<\sigma<-\delta_{n+1} \\ s_n^+ & \text{for } \delta_{n+1}<\sigma<\delta_n \\ 1 & \text{for } \sigma>\delta_1. \end{cases}$$

Hence

$$\mu(-0)=\lim s_n^-=f^- \quad \text{and} \quad \mu(+0)=\lim s_n^+=f^+.$$

The sets $E^-(0)$ and $E^+(0)$ are determined by

$$E^-(0)=\lim S_n^- \quad \text{and} \quad E^+(0)=\lim S_n^+.$$

On account of condition (i) the sequence $\{q_n\}$ is strictly increasing. Condition (ii) therefore implies that in the interval $-\frac{1}{2}q_n\leqq k<\frac{1}{2}q_n$ we have

$$S_m^-=\underline{R}_m^-=\underline{R}_n^- \quad \text{and} \quad S_m^+=\overline{R}_m^+=\overline{R}_n^+$$

for all $m>n$, and hence

$$\lim S_m^-=\lim \underline{R}_m^-=\underline{R}_n^- \quad \text{and} \quad \lim S_m^+=\lim \overline{R}_m^+=\overline{R}_n^+.$$

Since $q_n\to\infty$, this shows that

$$(11) \qquad E^-(0)=\lim \underline{R}_n^- \quad \text{and} \quad E^+(0)=\lim \overline{R}_n^+$$

and hence

$$(12) \qquad \underline{R}_n^-\subset E^-(0)\subset \overline{R}_n^- \quad \text{and} \quad \underline{R}_n^+\subset E^+(0)\subset \overline{R}_n^+$$

for all n. Moreover, it is seen that in the interval $-\frac{1}{2}q_n\leqq k<\frac{1}{2}q_n$ we have

$$(13) \qquad \underline{R}_n^-=E^-(0) \quad \text{and} \quad E^+(0)=\overline{R}_n^+.$$

From (11) and conditions (iii) and (iv) it further follows that in the intervals $\frac{1}{2}q_n\leqq k<\frac{3}{2}q_n$ and $-\frac{3}{2}q_n\leqq k<-\frac{1}{2}q_n$ we have

$$(14) \qquad E^-(0)=\overline{R}_n^- \quad \text{and} \quad \underline{R}_n^+=E^+(0)$$

respectively.

From the relations (12) it follows that for every n we have

$$\underline{r}_n^- \leqq \underline{\varrho}(E^-(0)) \leqq \bar{\varrho}(E^-(0)) \leqq \bar{r}_n^- \quad \text{and} \quad \underline{r}_n^+ \leqq \underline{\varrho}(E^+(0)) \leqq \bar{\varrho}(E^+(0)) \leqq \bar{r}_n^+.$$

Hence

$$(15) \qquad \begin{aligned} \underline{\varrho}(E^-(0)) &\geqq \lim \underline{r}_n^- = \underline{e}^-, & \bar{\varrho}(E^-(0)) &\leqq \lim \bar{r}_n^- = \bar{e}^-, \\ \underline{\varrho}(E^+(0)) &\geqq \lim \underline{r}_n^+ = \underline{e}^+, & \bar{\varrho}(E^+(0)) &\leqq \lim \bar{r}_n^+ = \bar{e}^+. \end{aligned}$$

Moreover, it follows from the validity of the relations (13) in the interval $-\frac{1}{2}q_n \leqq k < \frac{1}{2}q_n$ that

$$\frac{n(E^-(0), -\frac{1}{2}q_n, \frac{1}{2}q_n)}{q_n} = \underline{r}_n^- \quad \text{and} \quad \frac{n(E^+(0), -\frac{1}{2}q_n, \frac{1}{2}q_n)}{q_n} = \bar{r}_n^+,$$

and from the validity of the relations (14) in the intervals $\frac{1}{2}q_n \leqq k < \frac{3}{2}q_n$ and $-\frac{3}{2}q_n \leqq k < -\frac{1}{2}q_n$ respectively that

$$\frac{n(E^-(0), \frac{1}{2}q_n, \frac{3}{2}q_n)}{q_n} = \bar{r}_n^- \quad \text{and} \quad \frac{n(E^+(0), -\frac{3}{2}q_n, -\frac{1}{2}q_n)}{q_n} = \underline{r}_n^+.$$

Since $q_n \to \infty$, we conclude from this that

$$(16) \qquad \begin{aligned} \underline{\varrho}(E^-(0)) &\leqq \underline{e}^-, & \bar{\varrho}(E^+(0)) &\geqq \bar{e}^+, \\ \bar{\varrho}(E^-(0)) &\geqq \bar{e}^-, & \underline{\varrho}(E^+(0)) &\leqq \underline{e}^+. \end{aligned}$$

From (15) and (16) it finally follows that

$$\underline{\varrho}(E^-(0)) = \underline{e}^-, \quad \bar{\varrho}(E^-(0)) = \bar{e}^-, \quad \underline{\varrho}(E^+(0)) = \underline{e}^+, \quad \bar{\varrho}(E^+(0)) = \bar{e}^+.$$

66. We next consider the case $M = \{g\,2\pi + h\gamma\}$. That $U(k)$ is almost periodic with exponents from M then means (see § 52) that $U(k) = F(\gamma k)$, where $F(t)$ is a continuous function with the period 2π.

The construction of such a sequence $U(k)$, satisfying the conditions of the theorem, will be carried out in close analogy to the preceding construction. In order to make this analogy as clear as possible we shall, for an arbitrary positive integer p, call the set of integers k satisfying an inequality

$$\left|\gamma k - \nu\frac{2\pi}{p}\right| < \frac{\pi}{p} \qquad \text{mod. } 2\pi,$$

where ν is an integer, a *class* mod. p. There exist p such classes, corresponding to the p intervals

$$\left|t - \nu\frac{2\pi}{p}\right| < \frac{\pi}{p} \qquad \text{mod. } 2\pi.$$

The limits of these intervals being different from the points γk, it is obvious that the classes mod. p, taken together, form the set of all integers k. If q is a multiple of p, each class mod. p consists of q/p classes mod. q.

From the equidistribution of the numbers γk mod. 2π it follows that any class mod. p has the relative frequency $1/p$.

For every positive integer n we choose rational numbers

$$(17) \qquad 0 < s_n^- < \underline{r}_n^- < \begin{Bmatrix} \underline{r}_n^+ \\ \bar{r}_n^- \end{Bmatrix} < \bar{r}_n^+ < s_n^+ < 1,$$

such that the sequences $\{s_n^-\}$, $\{\underline{r}_n^-\}$, and $\{\underline{r}_n^+\}$ are strictly increasing and converge towards f^-, $\underline{e}^-$, and $\underline{e}^+$ respectively, whereas the sequences $\{\bar{r}_n^-\}$, $\{\bar{r}_n^+\}$, and $\{s_n^+\}$ are strictly decreasing and converge towards $\bar{e}^-$, $\bar{e}^+$, and f^+ respectively. If $\bar{e}^- < \underline{e}^+$, we choose $\bar{r}_1^- < \underline{r}_1^+$, which implies $\bar{r}_n^- < \underline{r}_n^+$ for all n. If $\bar{e}^- \geqq \underline{e}^+$, we obviously have $\bar{r}_n^- > \underline{r}_n^+$ for all n.[1]

As we shall now prove, it is then possible for every n to choose a common denominator p_n of the numbers (17) and two positive integers q_n and p_n^*, as well as sets of integers

$$S_n^- \subset \underline{R}_n^- \subset \begin{Bmatrix} \underline{R}_n^+ \\ \bar{R}_n^- \end{Bmatrix} \subset \bar{R}_n^+ \subset S_n^+$$

consisting of classes mod. p_n and having the corresponding numbers (17) as relative frequencies, such that the sequences $\{S_n^-\}$, $\{\underline{R}_n^-\}$, and $\{\underline{R}_n^+\}$ are strictly increasing, whereas the sequences $\{\bar{R}_n^-\}$, $\{\bar{R}_n^+\}$, and $\{S_n^+\}$ are strictly decreasing, such that, further,

$$\bar{R}_n^- \subseteqq \underline{R}_n^+ \quad \text{according as} \quad \bar{r}_n^- \lesseqgtr \underline{r}_n^+,$$

and, finally, such that for every n the following conditions are satisfied:

(i) The number p_n^* is a multiple of p_n and the number p_{n+1} a multiple of p_n^* such that $\frac{p_{n+1}}{p_n^*} > 3$.

(ii) $$q_n > q_{n-1}.$$[2]

---

[1] Thus the numbers s_n^-, $\underline{r}_n^-$, $\underline{r}_n^+$, $\bar{r}_n^-$, $\bar{r}_n^+$, and s_n^+ are chosen just as in the preceding case, except that now there is no restriction on their denominators.

[2] This condition must be left out, when $n = 1$, since q_{n-1} does not then exist.

(iii)
$$\frac{n(\underline{R}_n^-, -\frac{1}{2}q_n, \frac{1}{2}q_n)}{q_n} < \underline{r}_{n+1}^-, \qquad \frac{n(\overline{R}_n^+, -\frac{1}{2}q_n, \frac{1}{2}q_n)}{q_n} > \overline{r}_{n+1}^+,$$
$$\frac{n(\overline{R}_n^-, \frac{1}{2}q_n, \frac{3}{2}q_n)}{q_n} > \overline{r}_{n+1}^-, \qquad \frac{n(\underline{R}_n^+, -\frac{3}{2}q_n, -\frac{1}{2}q_n)}{q_n} < \underline{r}_{n+1}^+.$$

(iv) The numbers of the interval $-\frac{3}{2}q_n \leqq k < \frac{3}{2}q_n$ belong to different classes mod. p_n^*.

(v) In the interval $-\frac{1}{2}q_n \leqq k < \frac{1}{2}q_n$ we have

$$S_{n+1}^- = \underline{R}_{n+1}^- = \underline{R}_n^- \quad \text{and} \quad S_{n+1}^+ = \overline{R}_{n+1}^+ = \overline{R}_n^+.$$

(vi) In the interval $\frac{1}{2}q_n \leqq k < \frac{3}{2}q_n$ we have

$$\underline{R}_{n+1}^- = \overline{R}_n^-.$$

(vii) In the interval $-\frac{3}{2}q_n \leqq k < -\frac{1}{2}q_n$ we have

$$\overline{R}_{n+1}^+ = \underline{R}_n^+.$$

67. We shall prove this by induction.

The definition of p_1 and the sets S_1^-, $\underline{R}_1^-$, $\underline{R}_1^+$, $\overline{R}_1^-$, $\overline{R}_1^+$, and S_1^+ in accordance with the conditions presents no difficulties. We may e. g. for p_1 choose an arbitrary common denominator of the numbers s_1^-, $\underline{r}_1^-$, $\underline{r}_1^+$, $\overline{r}_1^-$, $\overline{r}_1^+$, and s_1^+, and for the sets S_1^-, $\underline{R}_1^-$, $\underline{R}_1^+$, $\overline{R}_1^-$, $\overline{R}_1^+$, and S_1^+ we may choose the sets determined by the inequalities $|\gamma k| < \pi a$ mod. 2π, where a denotes s_1^-, $\underline{r}_1^-$, $\underline{r}_1^+$, $\overline{r}_1^-$, $\overline{r}_1^+$, and s_1^+ respectively.

Now suppose that we have defined q_{n-1}, p_{n-1}^*, and p_n as well as the sets S_n^-, $\underline{R}_n^-$, $\underline{R}_n^+$, $\overline{R}_n^-$, $\overline{R}_n^+$, and S_n^+.[1]

We then begin by choosing q_n such that conditions (ii) and (iii) are satisfied. Since

$$\varrho(\underline{R}_n^-) = \underline{r}_n^- < \underline{r}_{n+1}^-, \qquad \varrho(\overline{R}_n^+) = \overline{r}_n^+ > \overline{r}_{n+1}^+,$$
$$\varrho(\overline{R}_n^-) = \overline{r}_n^- > \overline{r}_{n+1}^-, \qquad \varrho(\underline{R}_n^+) = \underline{r}_n^+ < \underline{r}_{n+1}^+,$$

this will be the case, if only q_n is chosen sufficiently large.

Next we choose p_n^* in accordance with condition (i) such that condition (iv) is satisfied. This is evidently possible.

The sets S_n^-, $\underline{R}_n^-$, $\underline{R}_n^+$, $\overline{R}_n^-$, $\overline{R}_n^+$, and S_n^+ consist of classes mod. p_n and hence also of classes mod. p_n^*. Denoting for the present by p_{n+1} an arbitrary common

[1] For $n=1$ the numbers q_{n-1} and p_{n-1}^* do not exist.

denominator of the numbers s_{n+1}^-, $\underline{r}_{n+1}^-$, $\underline{r}_{n+1}^+$, $\bar{r}_{n+1}^-$, $\bar{r}_{n+1}^+$, and s_{n+1}^+ satisfying condition (i), the situation may again be illustrated by Fig. 2, where the small squares now represent the p_{n+1} classes mod. p_{n+1} arranged in p_n^* columns and $\frac{p_{n+1}}{p_n^*}$ rows in such a way that each column constitutes a class mod. p_n^* and that each of the sets S_n^-, $\underline{R}_n^-$, $\underline{R}_n^+$, $\bar{R}_n^-$, $\bar{R}_n^+$, and S_n^+ consists of all columns to the left of a certain vertical line. In consequence of condition (iv) we may further suppose that all classes mod. p_{n+1} containing a number in the interval $-\frac{1}{2} q_n \leqq k < \frac{1}{2} q_n$ are placed in the first row, that those containing a number in the interval $\frac{1}{2} q_n \leqq k < \frac{3}{2} q_n$ are placed in the second row, and that those containing a number in the interval $-\frac{3}{2} q_n \leqq k < -\frac{1}{2} q_n$ are placed in the third row.

The number p_{n+1} and the sets S_{n+1}^-, $\underline{R}_{n+1}^-$, $\underline{R}_{n+1}^+$, $\bar{R}_{n+1}^-$, $\bar{R}_{n+1}^+$, and S_{n+1}^+ may now be defined by the same procedure as was applied in § 64, and thus we obtain that conditions (v), (vi) and (vii) are satisfied.

68. Denoting by S_0^- the empty set and by S_0^+ the set of all integers, and placing $s_0^- = 0$ and $s_0^+ = 1$, we now consider the two sets

$$\lim S_n^- = \sum_{n=1}^{\infty} (S_n^- - S_{n-1}^-) \quad \text{and} \quad S_0^+ - \lim S_n^+ = \sum_{n=1}^{\infty} (S_{n-1}^+ - S_n^+). \tag{18}$$

Each of the sets $S_n^- - S_{n-1}^-$ and $S_{n-1}^+ - S_n^+$ consists of certain classes mod. p_n

$$\left| \gamma k - \nu \frac{2\pi}{p_n} \right| < \frac{\pi}{p_n} \qquad \text{mod. } 2\pi,$$

each of which, in its turn, corresponds to an interval

$$\left| t - \nu \frac{2\pi}{p_n} \right| < \frac{\pi}{p_n} \qquad \text{mod. } 2\pi. \tag{19}$$

The sums of the intervals (19) thus corresponding to the sets (18) will be denoted by A and B. These sets A and B are evidently disjunct open sets with the period 2π, and their relative measures are

$$r(A) = \sum_{n=1}^{\infty} (s_n^- - s_{n-1}^-) = f^- \quad \text{and} \quad r(B) = \sum_{n=1}^{\infty} (s_{n-1}^+ - s_n^+) = 1 - f^+. \tag{20}$$

An integer k belongs to $\lim S_n^-$ if and only if γk belongs to A, and to $S_0^+ - \lim S_n^+$ if and only if γk belongs to B.

We now define a real function $F(t)$ with the period 2π as follows: If t belongs to one of the intervals (19) constituting $A+B$, and if δ denotes the shortest distance from t to the limits of the interval, we put $F(t)=-\delta$ or $F(t)=\delta$ according as the interval belongs to A or B. If t does not belong to $A+B$, we put $F(t)=0$. Then $F(t)$ is evidently a continuous function, and we shall now prove that the corresponding sequence $U(k)=F(\gamma k)$ satisfies the conditions of the theorem.

By § 57, the sequence $U(k)$ has the same asymptotic distribution function $\mu(\sigma)$ as $F(t)$. The sets $A^-(0)$ and $A^+(0)$ in which $F(t)<0$ and $F(t)\leqq 0$ respectively are, however, the sets A and the complementary set of B. Using (20), we therefore find

$$\mu(-0)=r(A^-(0))=f^- \quad\text{and}\quad \mu(+0)=r(A^+(0))=f^+.$$

The sets $E^-(0)$ and $E^+(0)$ are determined as the sets of all integers k for which $F(\gamma k)<0$ and $F(\gamma k)\leqq 0$ respectively, i. e. for which γk belongs to the set A and the complementary set of B respectively. Hence

$$E^-(0)=\lim S_n^- \quad\text{and}\quad E^+(0)=\lim S_n^+.$$

On account of condition (ii) the sequence $\{q_n\}$ is strictly increasing. Condition (v) therefore implies that in the interval $-\frac{1}{2}q_n\leqq k<\frac{1}{2}q_n$ we have

$$S_m^-=\underline{R}_m^-=\underline{R}_n^- \quad\text{and}\quad S_m^+=\overline{R}_m^+=\overline{R}_n^+$$

for all $m>n$, and hence

$$\lim S_m^-=\lim \underline{R}_m^-=\underline{R}_n^- \quad\text{and}\quad \lim S_m^+=\lim \overline{R}_m^+=\overline{R}_n^+.$$

Since $q_n\to\infty$, this shows that

(21) $$E^-(0)=\lim \underline{R}_n^- \quad\text{and}\quad E^+(0)=\lim \overline{R}_n^+$$

and hence

(22) $$\underline{R}_n^-\subset E^-(0)\subset \overline{R}_n^- \quad\text{and}\quad \underline{R}_n^+\subset E^+(0)\subset \overline{R}_n^+$$

for all n. Moreover, it is seen that in the interval $-\frac{1}{2}q_n\leqq k<\frac{1}{2}q_n$ we have

(23) $$\underline{R}_n^-=E^-(0) \quad\text{and}\quad E^+(0)=\overline{R}_n^+.$$

From (21) and conditions (vi) and (vii) it further follows that in the intervals $\frac{1}{2}q_n\leqq k<\frac{3}{2}q_n$ and $-\frac{3}{2}q_n\leqq k<-\frac{1}{2}q_n$ we have

(24) $$E^-(0)=\overline{R}_n^- \quad\text{and}\quad \underline{R}_n^+=E^+(0)$$

respectively.

As in § 65, it follows, from the relations (22), that

$$\underline{\varrho}(E^-(0)) \geqq \underline{e}^-, \quad \bar{\varrho}(E^-(0)) \leqq \bar{e}^-, \quad \underline{\varrho}(E^+(0)) \geqq \underline{e}^+, \quad \bar{\varrho}(E^+(0)) \leqq \bar{e}^+.$$

Moreover, it follows from (23), (24), and condition (iii), that

$$\frac{n(E^-(0), -\frac{1}{2}q_n, \frac{1}{2}q_n)}{q_n} < \underline{r}^-_{n+1}, \qquad \frac{n(E^+(0), -\frac{1}{2}q_n, \frac{1}{2}q_n)}{q_n} > \bar{r}^+_{n+1},$$

$$\frac{n(E^-(0), \frac{1}{2}q_n, \frac{3}{2}q_n)}{q_n} > \bar{r}^-_{n+1}, \qquad \frac{n(E^+(0), -\frac{3}{2}q_n, -\frac{1}{2}q_n)}{q_n} < \underline{r}^+_{n+1}.$$

Since $q_n \to \infty$, we conclude from this that

$$\underline{\varrho}(E^-(0)) \leqq \underline{e}^-, \quad \bar{\varrho}(E^-(0)) \geqq \bar{e}^-, \quad \underline{\varrho}(E^+(0)) \leqq \underline{e}^+, \quad \bar{\varrho}(E^+(0)) \geqq \bar{e}^+.$$

Thus it is proved that

$$\underline{\varrho}(E^-(0)) = \underline{e}^-, \quad \bar{\varrho}(E^-(0)) = \bar{e}^-, \quad \underline{\varrho}(E^+(0)) = \underline{e}^+, \quad \bar{\varrho}(E^+(0)) = \bar{e}^+,$$

which completes the proof of the theorem.

69. If the real almost periodic sequence $U(k)$ does not take the value σ, i. e. if $U(k) \neq \sigma$ for all k, the two sets $E^-(\sigma)$ and $E^+(\sigma)$ are identical and may be briefly denoted by $E(\sigma)$. Our considerations are then restricted to the four quantities

$$\mu(\sigma - 0) \leqq \underline{\varrho}(E(\sigma)) \leqq \bar{\varrho}(E(\sigma)) \leqq \mu(\sigma + 0).$$

Regarding this case we shall prove the following theorem.

Theorem 14. *For arbitrary numbers satisfying the inequalities*

$$f^- \leqq \underline{e} \leqq \bar{e} \leqq f^+,$$

and for an arbitrary σ_0, there exists a real almost periodic sequence $U(k)$ with exponents from a given everywhere dense modul M containing the number 2π, which does not take the value σ_0, and for which these numbers are equal to the corresponding numbers in the inequalities

$$\mu(\sigma_0 - 0) \leqq \underline{\varrho}(E(\sigma_0)) \leqq \bar{\varrho}(E(\sigma_0)) \leqq \mu(\sigma_0 + 0),$$

if and only if either

(a) $0 < f^-$ *and* $f^+ < 1$,

(b) $0 = f^- = \underline{e} = \bar{e}$ *and* $f^+ < 1$,

or (c) $0 < f^-$ *and* $\underline{e} = \bar{e} = f^+ = 1$.

The necessity of the conditions is obvious. In the proof of the sufficiency we shall restrict ourselves to case (a), the two other cases being simpler. We may suppose without loss of generality that the modul M is of one of the types $\lim_{m\to\infty}\left\{h_m\frac{2\pi}{p_m}\right\}$ or $\{g\,2\pi+h\gamma\}$, and that $\sigma_0=0$.

70. We first consider the case $M=\lim_{m\to\infty}\left\{h_m\frac{2\pi}{p_m}\right\}$.

Following the procedure of § 63 we choose, for every positive integer n, rational numbers with denominators among the numbers p_m,

$$0<s_n^-<\underline{r}_n<\bar{r}_n<s_n^+<1, \tag{25}$$

such that the sequences $\{s_n^-\}$ and $\{\underline{r}_n\}$ are strictly increasing and converge towards f^- and $\underline{\varrho}$ respectively, whereas the sequences $\{\bar{r}_n\}$ and $\{s_n^+\}$ are strictly decreasing and converge towards $\bar{\varrho}$ and f^+ respectively.

For every n it is then possible to choose among the numbers p_m a common denominator q_n of the numbers (25), as well as periodic sets of integers

$$S_n^-\subset\underline{R}_n\subset\bar{R}_n\subset S_n^+$$

having the period q_n and the corresponding numbers (25) as relative frequencies, such that the sequences $\{S_n^-\}$ and $\{\underline{R}_n\}$ are strictly increasing, whereas the sequences $\{\bar{R}_n\}$ and $\{S_n^+\}$ are strictly decreasing, and, further, such that for every n the following conditions are satisfied:

(i) The number q_{n+1} is a multiple of q_n and $\frac{q_{n+1}}{q_n}>2$.

(ii) In the interval $0\leq k<q_n$ we have

$$S_{n+1}^-=\underline{R}_{n+1}=\bar{R}_{n+1}=S_{n+1}^+=\underline{R}_n$$

(iii) In the interval $-q_n\leq k<0$ we have

$$S_{n+1}^-=\underline{R}_{n+1}=\bar{R}_{n+1}=S_{n+1}^+=\bar{R}_n.$$

The proof that these conditions may all be satisfied is quite analogous to the proof in § 64.

We now define $U(k)$ as in § 65. Then $U(k)$ is again almost periodic with exponents from M, and satisfies the conditions

$$\mu(-0)=f^-\quad\text{and}\quad\mu(+0)=f^+.$$

Further, the sets $E^{-}(0)$ and $E^{+}(0)$ are again determined by

$$E^{-}(0) = \lim S_n^{-} \quad \text{and} \quad E^{+}(0) = \lim S_n^{+}.$$

On account of (i) the sequence $\{q_n\}$ is strictly increasing. From (ii) and (iii) it therefore follows that in the intervals $0 \leqq k < q_n$ and $-q_n \leqq k < 0$ we have

$$S_m^{-} = S_m^{+} = \underline{R}_n \quad \text{and} \quad S_m^{-} = S_m^{+} = \overline{R}_n$$

respectively for all $m > 0$, and hence that in these intervals

$$E^{-}(0) = E^{+}(0) = \underline{R}_n \quad \text{and} \quad E^{-}(0) = E^{+}(0) = \overline{R}_n$$

respectively. Since $q_n \to \infty$, this shows that $E^{-}(0) = E^{+}(0)$, i. e. $U(k)$ does not take the value 0. Introducing the notation $E(0)$, we have therefore in the intervals $0 \leqq k < q_n$ and $-q_n \leqq k < 0$

$$E(0) = \underline{R}_n \quad \text{and} \quad E(0) = \overline{R}_n$$

respectively, which implies that

$$\underline{R}_n \subset E(0) \subset \overline{R}_n$$

for all n. From these properties of $E(0)$ it follows that

$$\underline{\varrho}(E(0)) = \underline{e} \quad \text{and} \quad \bar{\varrho}(E(0)) = \bar{e}.$$

71. We next consider the case $M = \{g\, 2\pi + h\gamma\}$.

Following the procedure of § 66 we choose, for every positive integer n, rational numbers

(26) $$0 < s_n^{-} < \underline{r}_n < \bar{r}_n < s_n^{+} < 1,$$

such that the sequences $\{s_n^{-}\}$ and $\{\underline{r}_n\}$ are strictly increasing and converge towards f^{-} and $\underline{e}$ respectively, whereas the sequences $\{\bar{r}_n\}$ and $\{s_n^{+}\}$ are strictly decreasing and converge towards $\bar{e}$ and f^{+} respectively.

It is then possible for every n to choose a common denominator p_n of the numbers (26) and two positive integers q_n and p_n^{*}, as well as sets of integers

$$S_n^{-} \subset \underline{R}_n \subset \overline{R}_n \subset S_n^{+}$$

consisting of classes mod. p_n and having the corresponding numbers (26) as relative frequencies, such that the sequences $\{S_n^{-}\}$ and $\{\underline{R}_n\}$ are strictly increasing, whereas the sequences $\{\overline{R}_n\}$ and $\{S_n^{+}\}$ are strictly decreasing, and, further, such that for every n the following conditions are satisfied:

(i) The number p_n^* is a multiple of p_n and the number p_{n+1} a multiple of p_n^* such that $\frac{p_{n+1}}{p_n^*} > 2$.

(ii)
$$q_n > q_{n-1}.$$

(iii)
$$\frac{n(\underline{R}_n, 0, q_n)}{q_n} < \underline{r}_{n+1} \quad \text{and} \quad \frac{n(\overline{R}_n, -q_n, 0)}{q_n} > \overline{r}_{n+1}.$$

(iv) The numbers of the interval $-q_n \leqq k < q_n$ belong to different classes mod. p_n^*.

(v) In the interval $0 \leqq k < q_n$ we have

$$S_{n+1}^- = \underline{R}_{n+1} = \overline{R}_{n+1} = S_{n+1}^+ = \underline{R}_n.$$

(vi) In the interval $-q_n \leqq k < 0$ we have

$$S_{n+1}^- = \underline{R}_{n+1} = \overline{R}_{n+1} = S_{n+1}^+ = \overline{R}_n.$$

The proof that these conditions may all be satisfied is quite analogous to the proof in § 67.

We now define $U(k)$ as in § 68. Then $U(k)$ is almost periodic with exponents from M and satisfies the conditions

$$\mu(-0) = f^- \quad \text{and} \quad \mu(+0) = f^+.$$

The sets $E^-(0)$ and $E^+(0)$ are again determined by

$$E^-(0) = \lim S_n^- \quad \text{and} \quad E^+(0) = \lim S_n^+.$$

On account of (ii) the sequence $\{q_n\}$ is strictly increasing. From (v) and (vi) it therefore follows that in the intervals $0 \leqq k < q_n$ and $-q_n \leqq k < 0$ we have

$$S_m^- = S_m^+ = \underline{R}_n \quad \text{and} \quad S_m^- = S_m^+ = \overline{R}_n$$

respectively for all $m > n$, and hence that in these intervals

$$E^-(0) = E^+(0) = \underline{R}_n \quad \text{and} \quad E^-(0) = E^+(0) = \overline{R}_n$$

respectively. Since $q_n \to \infty$, this shows that $E^-(0) = E^+(0)$, i. e. $U(k)$ does not take the value 0. Introducing the notation $E(0)$, we have therefore in the intervals $0 \leqq k < q_n$ and $-q_n \leqq k < 0$

$$E(0) = \underline{R}_n \quad \text{and} \quad E(0) = \overline{R}_n$$

respectively, which implies that

$$\underline{R}_n \subset E(0) \subset \overline{R}_n$$

for all n. From these properties of $E(0)$ together with (iii) it follows that

$$\underline{\varrho}(E(0)) = \underline{e} \quad \text{and} \quad \overline{\varrho}(E(0)) = \bar{e}.$$

This completes the proof of the theorem.

72. By means of Theorem 13 we can easily prove an analogous theorem on real almost periodic functions $F(t)$, viz. that for arbitrary numbers satisfying the conditions

$$(27) \qquad f^- \leqq \underline{e}^- \leqq \begin{Bmatrix} \underline{e}^+ \\ \bar{e}^- \end{Bmatrix} \leqq \bar{e}^+ \leqq f^+,$$

and for an arbitrary σ_0, there exists a real almost periodic function $F(t)$ with exponents from a given everywhere dense modul M, for which these numbers are equal to the corresponding numbers in the inequalities

$$(28) \qquad \mu(\sigma_0 - 0) \leqq \underline{r}(A^-(\sigma_0)) \leqq \begin{Bmatrix} \underline{r}(A^+(\sigma_0)) \\ \bar{r}(A^-(\sigma_0)) \end{Bmatrix} \leqq \bar{r}(A^+(\sigma_0)) \leqq \mu(\sigma_0 + 0),$$

if and only if either

(a) $0 < f^-$ and $f^+ < 1$,

(b) $0 = f^- = \underline{e}^- = \bar{e}^-$ and $f^+ < 1$,

(c) $0 < f^-$ and $\underline{e}^+ = \bar{e}^+ = f^+ = 1$,

or (d) $0 = f^- = \underline{e}^- = \bar{e}^-$ and $\underline{e}^+ = \bar{e}^+ = f^+ = 1$.

The necessity of the conditions is again obvious. In the proof of the sufficiency we may suppose that M contains the number 2π, since otherwise we may replace the desired function $F(t)$ by $F\left(\frac{2\pi}{\gamma} t\right)$, where $\gamma \neq 0$ is a number of M, at the same time multiplying the elements of M by $\frac{2\pi}{\gamma}$.

The above conditions being the same as in Theorem 13 there exists a real almost periodic sequence $U(k)$ with exponents from M, for which the numbers (27) are equal to the corresponding numbers in the inequalities

$$(29) \qquad \mu(\sigma_0 - 0) \leqq \underline{\varrho}(E^-(\sigma_0)) \leqq \begin{Bmatrix} \underline{\varrho}(E^+(\sigma_0)) \\ \overline{\varrho}(E^-(\sigma_0)) \end{Bmatrix} \leqq \overline{\varrho}(E^+(\sigma_0)) \leqq \mu(\sigma_0 + 0).$$

The function $F(t)$ which is $= U(k)$ when $t = k$ and $= \sigma_0$ when $t = k + \frac{1}{2}$ and is linear in all intervals $k \leqq t \leqq k + \frac{1}{2}$ and $k + \frac{1}{2} \leqq t \leqq k + 1$, is then (by §§ 5 and 51) also almost periodic with exponents from M, and for this function $F(t)$ the numbers in the inequalities (28) are easily seen to be equal to the corresponding numbers in the inequalities (29) for the sequence $U(k)$.

If the real almost periodic function $F(t)$ takes the value σ_0 only in an enumerable set (or, more generally, in a set of relative measure 0) we have

$$\underline{r}(A^-(\sigma_0)) = \underline{r}(A^+(\sigma_0)) \quad \text{and} \quad \bar{r}(A^-(\sigma_0)) = \bar{r}(A^+(\sigma_0)).$$

Using Theorem 14 we find that for arbitrary numbers satisfying the conditions

$$f^- \leqq \underline{e} \leqq \bar{e} \leqq f^+,$$

and for an arbitrary σ_0, there exists a real almost periodic function $F(t)$ with exponents from a given everywhere dense modul M containing the number 2π, which takes the value σ_0 only in an enumerable set, and for which these numbers are equal to the numbers

$$\mu(\sigma_0 - 0), \quad \underline{r}(A^-(\sigma_0)) = \underline{r}(A^+(\sigma_0)), \quad \bar{r}(A^-(\sigma_0)) = \bar{r}(A^+(\sigma_0)), \quad \mu(\sigma_0 + 0),$$

if and only if either

(a) $0 < f^-$ and $f^+ < 1$,

(b) $0 = f^- = \underline{e} = \bar{e}$ and $f^+ < 1$,

or (c) $0 < f^-$ and $\underline{e} = \bar{e} = f^+ = 1$.

CHAPTER IV.

Analytic Almost Periodic Functions Connected with Almost Periodic Sequences.

73. The application of the results of the preceding chapter to analytic almost periodic functions will depend on the consideration of functions $f(s)$ almost periodic in $[-\infty, +\infty]$ possessing on each line $t = k$, where k is an integer, one simple zero $s_k = U(k) + ik$ belonging to a finite vertical strip, but otherwise different from zero. The zeros of such a function $f(s)$ are therefore determined by a bounded sequence $U(k)$.

From these conditions it follows that the Jensen function $\varphi(\sigma)$ of the function is linear in the intervals $(-\infty, \alpha)$ and $(\beta, +\infty)$, where α and β denote the lower and upper bounds of $U(k)$, and further that $\varphi'(\sigma_2) - \varphi'(\sigma_1) = 2\pi$ for $\sigma_1 < \alpha$ and $\sigma_2 > \beta$, since the relative frequency $H(\sigma_1, \sigma_2)$ of zeros is then 1.

We begin by proving the following theorem.

Theorem 15. *The points $s_k = U(k) + ik$, where $U(k)$ is a real bounded sequence, are the zeros of a function $f(s)$ almost periodic in $[-\infty, +\infty]$ if and only if $U(k)$ is almost periodic.*

They are the zeros of a function $f(s)$ almost periodic in $[-\infty, +\infty]$ with exponents from a given modul M if and only if M contains the number 2π and the exponents of $U(k)$ belong to M.

For a given sequence $U(k)$ the corresponding function $f(s)$ may be chosen such that its Jensen function $\varphi(\sigma)$ is constant for $\sigma <$ lower bound $U(k)$.

The proof is, in the main, an adaptation from Buch [1], [2].

74. We first prove that the conditions are necessary. Let, therefore, $f(s)$ be a function of our class with the zeros $s_k = U(k) + ik$. We shall then prove that $U(k)$ is almost periodic.

According to our assumption the zeros belong to a finite strip $\{\alpha, \beta\}$. For an arbitrary positive $\varepsilon < \frac{1}{2}$ there exists by Theorem 3 (ii) a number $m > 0$, such that $|f(s)| \geqq m$ in the part of the strip $\{\alpha - 1, \beta + 1\}$ which does not belong to the circles $|s - s_k| < \varepsilon$. From Rouché's theorem it therefore follows that, if τ is a translation number of $f(s)$ belonging to m and the strip $(\alpha - 1, \beta + 1)$, the function $f(s + i\tau)$ possesses a zero in each of these circles. Thus τ differs by less than ε from a translation number $\varkappa$ of $U(k)$ belonging to ε. This proves that $U(k)$ is almost periodic.

If the exponents of $f(s)$ belong to a given modul M, this modul must contain the number 2π, since $\varphi'(\sigma_2) - \varphi'(\sigma_1) = 2\pi$ when $\sigma_1 < \alpha$ and $\sigma_2 > \beta$. Further, since an arbitrary integral translation number $\varkappa$ of $f(s)$ belonging to m and the strip $(\alpha - 1, \beta + 1)$ is a translation number of $U(k)$ belonging to ε, it follows from §§ 9 and 51 that the exponents of $U(k)$ belong to M.

75. Next we prove that the conditions are sufficient. Let, therefore, $U(k)$ be a real almost periodic sequence. We shall then prove the existence of a function $f(s)$ almost periodic in $[-\infty, +\infty]$ with the zeros $s_k = U(k) + ik$. Since $U(k)$ is almost periodic, it is bounded, say $\alpha \leqq U(k) \leqq \beta$.

The infinite product

$$f_q(s) = \prod_k (1 - e^{(s-s_k)^2/q}) \tag{1}$$

is for an arbitrary $q > 0$ uniformly absolutely convergent in every bounded domain in the s-plane. Since $1 - e^{s^2/q}$ has a double zero at $s = 0$ and no further zeros in the strip $(-\sqrt{q\pi}, \sqrt{q\pi})$, it is seen that, when $\sqrt{q\pi} > \beta - \alpha$, the function $f_q(s)$ has double zeros at the points s_k, and has no further zeros in the strip $(\beta - \sqrt{q\pi}, \alpha + \sqrt{q\pi})$.

We shall now prove that $f_q(s)$ is almost periodic in $[-\infty, +\infty]$. Let $\varkappa$ denote a translation number of $U(k)$ belonging to a given $\varepsilon > 0$. Introducing $s + i\varkappa$ in (1) instead of s, and at the same time replacing k by $k + \varkappa$, we obtain

$$f_q(s+i\varkappa) = \prod_k (1 - e^{(s+i\varkappa - s_{k+\varkappa})^2/q}) = \prod_k (1 - e^{(s - s_k - \varepsilon_k)^2/q}),$$

where for the sake of brevity we have put $s_{k+\varkappa} - (s_k + i\varkappa) = U(k+\varkappa) - U(k) = \varepsilon_k$, so that $|\varepsilon_k| \leqq \varepsilon$ for all k. Hence[1]

$$f_q(s + i\varkappa) - f_q(s) = \sum_l (e^{(s-s_l-\varepsilon_l)^2/q} - e^{(s-s_l)^2/q})\, K_l(s),$$

where

$$K_l(s) = \prod_{k<l} (1 - e^{(s-s_k-\varepsilon_k)^2/q}) \prod_{k>l} (1 - e^{(s-s_k)^2/q}).$$

Since $s_k + \varepsilon_k = s_{k+\varkappa} - i\varkappa$, each of these factors $K_l(s)$ is a product of the form

$$\prod_k{}' (1 - e^{(s-s_k^*)^2/q}),$$

where each s_k^* lies on the segment $\alpha \leqq \sigma \leqq \beta, t = k$, and where the dot indicates that one factor is missing. This implies, however, that for every finite strip (α_1, β_1) there exists a constant K, independent of ε and $\varkappa$, such that

$$|K_l(s)| \leqq K$$

in the strip (α_1, β_1) for all l. In the strip (α_1, β_1) we therefore have

$$|f_q(s+i\varkappa) - f_q(s)| \leqq K \sum_l |e^{(s-s_l-\varepsilon_l)^2/q} - e^{(s-s_l)^2/q}| \leqq L\varepsilon,$$

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[1] By the formula

$$\prod_k a_k - \prod_k b_k = \sum_l (a_l - b_l) \prod_{k<l} a_k \prod_{k>l} b_k,$$

valid for arbitrary convergent products.

where L is independent of ε and x. Thus x is a translation number of $f_q(s)$ belonging to $L\varepsilon$ and the strip (α_1, β_1), and this proves that $f_q(s)$ is almost periodic in $[-\infty, +\infty]$.

If the exponents of $U(k)$ belong to a given modul M containing the number 2π, it follows from the preceding proof, if we use §§ 9 and 51, that the exponents of $f_q(s)$ belong to M.[1]

76. We now consider the sequence of functions $f_q(s)$, where $q = Q + 1$, $Q + 2, \ldots$, the number $Q > 0$ being chosen such that $\sqrt{Q\pi} > \beta - \alpha$. Each of the functions

$$\frac{f_{q+1}(s)}{f_q(s)} \tag{2}$$

is then (§ 10) almost periodic and $\neq 0$ in the strip $[\beta - \sqrt{q\pi}, \alpha + \sqrt{q\pi}]$ and has therefore, by Theorem 8, in this strip the form

$$\frac{f_{q+1}(s)}{f_q(s)} = e^{c_q s + g_q(s)}, \tag{3}$$

where c_q denotes a constant and $g_q(s)$ is almost periodic in $[\beta - \sqrt{q\pi}, \alpha + \sqrt{q\pi}]$. It is therefore possible to choose an exponential polynomial $h_q(s)$ such that

$$\left| \frac{f_{q+1}(s)}{f_q(s)} e^{-c_q s - h_q(s)} - 1 \right| < \frac{1}{q^2}$$

in the strip $(\beta - \sqrt{(q-1)\pi}, \alpha + \sqrt{(q-1)\pi})$.

We now consider the sequence of partial products of the infinite product

$$f_{Q+1}(s) \prod_{q=Q+1}^{\infty} \frac{f_{q+1}(s)}{f_q(s)} e^{-c_q s - h_q(s)}.$$

For every finite strip (α_1, β_1) containing the strip (α, β) these partial products are, from a certain stage, regular and almost periodic in (α_1, β_1) and have double zeros at the points s_k but otherwise no zeros in the strip (α_1, β_1). Moreover, the sequence

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[1] In fact, by § 51 there correspond to the given ε numbers $\lambda_1, \ldots, \lambda_N$ in M and an η, such that any integer x satisfying the conditions $|\lambda_1 x| \leq \eta, \ldots, |\lambda_N x| \leq \eta \pmod{2\pi}$ is a $x_U(\varepsilon)$ and hence a $\tau_{f_q}(L\varepsilon; \alpha_1, \beta_1)$. Now, if δ is sufficiently small, any real number τ satisfying the conditions $|\lambda_1 \tau| \leq \delta, \ldots, |\lambda_N \tau| \leq \delta$, and $|2\pi\tau| \leq \delta \pmod{2\pi}$ will differ at most by $\delta/2\pi$ from an integer x satisfying the previous conditions. On account on the uniform continuity of $f_q(s)$ in (α_1, β_1) (which is a consequence of the almost periodicity) we may therefore choose δ such that any τ satisfying the latter conditions is a $\tau_{f_q}(2L\varepsilon; \alpha_1, \beta_1)$. Since M contains 2π this shows, by § 9, that the exponents of $f_q(s)$ belong to M.

is uniformly convergent in (α_1, β_1). The limit function $p(s)$ is therefore regular in the whole plane and almost periodic in $[-\infty, +\infty]$, and has double zeros at the points s_k, but no further zeros. An arbitrary branch $f(s) = \sqrt{p(s)}$ of the square root of $p(s)$ is therefore (§ 10) almost periodic in $[-\infty, +\infty]$ and has the zeros s_k.

More generally we may put $f(s) = \sqrt{p(s)\, e^{\mu s}}$, where μ is an arbitrary real number.

If the exponents of $U(k)$ belong to a given modul M containing the number 2π, the exponents of the functions $f_q(s)$ will, as has already been mentioned, also belong to M. The modul M then also contains the exponents of the functions (2) and therefore also the constants c_q and the exponents of the functions $g_q(s)$ occurring in (3). It is therefore possible to choose the exponential polynomials $h_q(s)$ with exponents from M. The modul M then also contains the exponents of $p(s)$. We may therefore (§ 10) choose the number μ in M such that the exponents of $f(s) = \sqrt{p(s)\, e^{\mu s}}$ also belong to M.

Finally we may choose $f(s)$ such that its Jensen function $\varphi(\sigma)$ is constant for $\sigma <$ lower bound $U(k)$. For, if this property is not obtained by the first choice of μ, we replace $f(s)$ by $f(s)\, e^{-cs}$, where c denotes the constant value of $\varphi'(\sigma)$ for $\sigma <$ lower bound $U(k)$.

This completes the proof of the theorem.

77. Now let $f(s)$ denote an arbitrary function almost periodic in $[-\infty, +\infty]$, possessing on each line $t = k$, where k is an integer, one simple zero $s_k = U(k) + ik$ belonging to a finite vertical strip, but otherwise different from zero, and suppose that $\varphi(\sigma)$ is constant, i. e. $\varphi'(\sigma) = 0$, for $\sigma < \alpha =$ lower bound $U(k)$. Then $\varphi'(\sigma) = 2\pi$ for $\sigma > \beta =$ upper bound $U(k)$. Hence the increasing function

$$\mu(\sigma) = \frac{1}{2\pi}\varphi'(\sigma)$$

satisfies the conditions $\mu(\sigma) = 0$ for $\sigma < \alpha$ and $\mu(\sigma) = 1$ for $\sigma > \beta$, and is therefore a distribution function.

For an arbitrary σ we denote by $E^-(\sigma)$ and $E^+(\sigma)$ the sets of those values of k for which $U(k) < \sigma$ and $U(k) \leqq \sigma$ respectively. Then, if $\sigma_1 < \alpha$ and $\sigma_2 > \beta$ are chosen such that $\sigma_1 < \sigma < \sigma_2$, the lower and upper relative frequencies of zeros of $f(s)$ in the strips (σ_1, σ) and (σ, σ_2) are determined by

$$\underline{H}(\sigma_1, \sigma) = \underline{\varrho}(E^-(\sigma)), \qquad \overline{H}(\sigma_1, \sigma) = \bar{\varrho}(E^-(\sigma)),$$

$$\underline{H}(\sigma, \sigma_2) = 1 - \bar{\varrho}(E^+(\sigma)), \qquad \overline{H}(\sigma, \sigma_2) = 1 - \underline{\varrho}(E^+(\sigma)).$$

On the other hand, since, by § 46, $c^+(\sigma_1)$ and $c^-(\sigma_2)$ exist and are equal to 0 and 2π respectively, it follows from § 42 that

$$\underline{H}(\sigma_1, \sigma) = \frac{1}{2\pi}\underline{c}^-(\sigma), \qquad \overline{H}(\sigma_1, \sigma) = \frac{1}{2\pi}\bar{c}^-(\sigma),$$

$$\underline{H}(\sigma, \sigma_2) = \frac{1}{2\pi}(2\pi - \bar{c}^+(\sigma)), \qquad \overline{H}(\sigma, \sigma_2) = \frac{1}{2\pi}(2\pi - \underline{c}^+(\sigma)).$$

Hence

$$\frac{1}{2\pi}\underline{c}^-(\sigma) = \underline{\varrho}(E^-(\sigma)), \qquad \frac{1}{2\pi}\bar{c}^-(\sigma) = \bar{\varrho}(E^-(\sigma)),$$

$$\frac{1}{2\pi}\underline{c}^+(\sigma) = \underline{\varrho}(E^+(\sigma)), \qquad \frac{1}{2\pi}\bar{c}^+(\sigma) = \bar{\varrho}(E^+(\sigma)).$$

This implies by Theorem 7 that

$$\frac{1}{2\pi}\varphi'(\sigma - 0) \leqq \underline{\varrho}(E^-(\sigma)) \leqq \begin{Bmatrix} \underline{\varrho}(E^+(\sigma)) \\ \bar{\varrho}(E^-(\sigma)) \end{Bmatrix} \leqq \bar{\varrho}(E^+(\sigma)) \leqq \frac{1}{2\pi}\varphi'(\sigma + 0),$$

which shows that $\mu(\sigma) = \frac{1}{2\pi}\varphi'(\sigma)$ is the asymptotic distribution function of $U(k)$

Thus we have proved the following theorem.

Theorem 16. *If a function $f(s)$ almost periodic in $]-\infty, +\infty[$ has the zeros $s_k = U(k) + ik$, belonging to a finite vertical strip, and if its Jensen function $\varphi(\sigma)$ is constant for $\sigma <$* lower bound *$U(k)$, then the sequence $U(k)$ has the asymptotic distribution function $\mu(\sigma) = \frac{1}{2\pi}\varphi'(\sigma)$.*

For an arbitrary σ the lower and upper, left and right mean motions of $f(s)$, multiplied by $\frac{1}{2\pi}$, are equal to the lower and upper relative frequencies of the sets $E^-(\sigma)$ and $E^+(\sigma)$ of those values of k for which $U(k) < \sigma$ and $U(k) \leqq \sigma$ respectively, i. e.

$$\frac{1}{2\pi}\underline{c}^-(\sigma) = \underline{\varrho}(E^-(\sigma)), \qquad \frac{1}{2\pi}\bar{c}^-(\sigma) = \bar{\varrho}(E^-(\sigma)),$$

$$\frac{1}{2\pi}\underline{c}^+(\sigma) = \underline{\varrho}(E^+(\sigma)), \qquad \frac{1}{2\pi}\bar{c}^+(\sigma) = \bar{\varrho}(E^+(\sigma)).$$

CHAPTER V.

Detailed Discussion of the Mean Motions on a Given Vertical Line.

78. By Theorem 7 we have for an arbitrary function $f(s)$ almost periodic in a strip $[\alpha, \beta]$ and not identically zero, and for an arbitrary σ in the interval (α, β), the inequalities

$$\varphi'(\sigma - 0) \leqq \underline{c}^-(\sigma) \leqq \left\{ \begin{matrix} \underline{c}^+(\sigma) \\ \bar{c}^-(\sigma) \end{matrix} \right\} \leqq \bar{c}^+(\sigma) \leqq \varphi'(\sigma + 0)$$

connecting the left and right derivatives of the Jensen function and the four mean motions at the point σ.

If $f(s)$ is periodic with the period ip, that is to say, if its exponents belong to the discrete modul $M = \left\{ h \frac{2\pi}{p} \right\}$, the mean motions $c^-(\sigma_0)$ and $c^+(\sigma_0)$ exist, according to § 48, for an arbitrary σ_0 and are equal to $\varphi'(\sigma_0 - 0)$ and $\varphi'(\sigma_0 + 0)$, which in this case belong to M. Conversely, if d^- and d^+ are given numbers in M such that $d^- \leqq d^+$, and σ_0 is a given number, there exists a periodic function $f(s)$ with the period ip, for which $c^-(\sigma_0) = \varphi'(\sigma_0 - 0) = d^-$ and $c^+(\sigma_0) = \varphi'(\sigma_0 + 0) = d^+$. If $d^- = d^+ = d$ we may take $f(s) = e^{ds}$; if $d^- < d^+$ we may take $f(s) = e^{d^-(s-\sigma_0)} + e^{d^+(s-\sigma_0)}$.[1]

For functions $f(s)$ with exponents from an everywhere dense modul M we shall now prove the following theorem.

Theorem 17. *For arbitrary numbers satisfying the relations*

$$(1) \qquad d^- \leqq \underline{c}^- \leqq \left\{ \begin{matrix} \underline{c}^+ \\ \bar{c}^- \end{matrix} \right\} \leqq \bar{c}^+ \leqq d^+,$$

and for an arbitrary σ_0, there exists a function $f(s)$ almost periodic in $[-\infty, +\infty]$ with exponents from a given everywhere dense modul M, for which these numbers are equal to the corresponding numbers in the inequalities

$$(2) \qquad \varphi'(\sigma_0 - 0) \leqq \underline{c}^-(\sigma_0) \leqq \left\{ \begin{matrix} \underline{c}^+(\sigma_0) \\ \bar{c}^-(\sigma_0) \end{matrix} \right\} \leqq \bar{c}^+(\sigma_0) \leqq \varphi'(\sigma_0 + 0).$$

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[1] For since all zeros of $f(s)$ have the abscissa σ_0, the function $\varphi(\sigma)$ is linear for $\sigma \leqq \sigma_0$ and $\sigma \geqq \sigma_0$. For $\sigma < \sigma_0$ the first term is preponderant; hence $\varphi'(\sigma) = d^-$ for $\sigma < \sigma_0$, and therefore $\varphi'(\sigma_0 - 0) = d^-$. For $\sigma > \sigma_0$ the second term is preponderant; hence $\varphi'(\sigma) = d^+$ for $\sigma > \sigma_0$, and therefore $\varphi'(\sigma_0 + 0) = d^+$.

If the exponents of an almost periodic function $f(s)$ belong to a given modul M, the exponents of the function $f_1(s) = f(ks)$, where $k > 0$, will belong to the modul obtained from M by multiplying all numbers by k. Further, when multiplied by k, the numbers in the inequalities (2) will be equal to the corresponding numbers formed for the function $f_1(s)$ and the value $k\sigma_0$. We may therefore in the proof of the theorem suppose that M contains the number 2π.

When this condition is satisfied, and the exponents of $f(s)$ belong to M, the exponents of the function $f_1(s) = e^{2\pi k_1 s} f(k_2 s)$, where k_1 and $k_2 > 0$ are integers, will also belong to M, and the numbers in the inequalities (2) will, after multiplication by k_2 and the addition of $2\pi k_1$, be equal to the corresponding numbers formed for the function $f_1(s)$ and the value $k_2\sigma_0$. We may therefore in the proof of the theorem suppose that

$$0 < d^- \quad \text{and} \quad d^+ < 2\pi.$$

In this case there exists by Theorem 13 a real almost periodic sequence $U(k)$ with exponents from M, for which the numbers (1), divided by 2π, are equal to the corresponding numbers in the inequalities

$$\mu(\sigma_0 - 0) \leqq \underline{\varrho}(E^-(\sigma_0)) \leqq \begin{Bmatrix} \underline{\varrho}(E^+(\sigma_0)) \\ \bar{\varrho}(E^-(\sigma_0)) \end{Bmatrix} \leqq \bar{\varrho}(E^+(\sigma_0)) \leqq \mu(\sigma_0 + 0). \tag{3}$$

By Theorem 15 there exists a function $f(s)$ almost periodic in $[-\infty, +\infty]$ with exponents from M, which has the zeros $U(k) + ik$, and for which the Jensen function $\varphi(\sigma)$ is constant for $\sigma <$ lower bound $U(k)$. This function $f(s)$ satisfies the conditions of the theorem, since by Theorem 16 the numbers (3) are equal to the corresponding numbers in the inequalities (2), divided by 2π.

79. If the almost periodic function $f(s)$ has no zeros on the vertical line with the abscissa σ the left and right arguments $\arg^- f(\sigma + it)$ and $\arg^+ f(\sigma + it)$ are identical. Hence the two lower mean motions $\underline{c}^-(\sigma)$ and $\underline{c}^+(\sigma)$ are equal and may be denoted briefly by $\underline{c}(\sigma)$, and the two upper mean motions $\bar{c}^-(\sigma)$ and $\bar{c}^+(\sigma)$ are equal and may be denoted briefly by $\bar{c}(\sigma)$. Our considerations are then restricted to the four quantities

$$\varphi'(\sigma - 0) \leqq \underline{c}(\sigma) \leqq \bar{c}(\sigma) \leqq \varphi'(\sigma + 0).$$

By means of Theorem 14 we find the following theorem regarding this case

Theorem 18. *For arbitrary numbers satisfying the relations*

$$d^- \leqq \underline{c} \leqq \bar{c} \leqq d^+,$$

and for an arbitrary σ_0, there exists a function $f(s)$ almost periodic in $[-\infty, +\infty]$ with exponents from a given everywhere dense modul M, which has no zeros on the vertical line with the abscissa σ_0, and for which these numbers are equal to the corresponding numbers in the inequalities

$$\varphi'(\sigma_0 - 0) \leqq \underline{c}(\sigma_0) \leqq \bar{c}(\sigma_0) \leqq \varphi'(\sigma_0 + 0).$$

This theorem shows that even if the function $f(\sigma_0 + it)$ has no zeros it need not have a mean motion.

CHAPTER VI.

Detailed Study of the Jensen Function.

Periodic and Limit Periodic Functions.

80. We shall now consider the problem as to what conditions a function $\varphi(\sigma)$ in a given interval (α, β) must satisfy to be the Jensen function of a function $f(s)$ almost periodic in $[\alpha, \beta]$ with exponents from a given modul M. We shall give a solution of this problem for functions with an arbitrary finite or infinite, integral or rational base, i. e. for all moduls of the form $M = \{h_1\mu_1 + \cdots + h_m\mu_m\}$, $\{h_1\mu_1 + h_2\mu_2 + \cdots\}$, $\{r_1\mu_1 + \cdots + r_m\mu_m\}$, or $\{r_1\mu_1 + r_2\mu_2 + \cdots\}$.

We shall first prove an auxiliary theorem.

Theorem 19. *Let M denote an arbitrary modul, and $\varphi(\sigma)$ a function in the interval (α, β), which may be written in the form*

$$\varphi(\sigma) = \sum_{n=1}^{\infty} \varphi_n(\sigma),$$

where each $\varphi_n(\sigma)$ is the Jensen function of a function $f_n(s)$ almost periodic in $[\alpha, \beta]$ with exponents from M, and where in any reduced interval $(\alpha <) \alpha_1 \leqq \sigma \leqq \beta_1 (< \beta)$ the functions $\varphi_n(\sigma)$ all vanish from a certain stage. Then, also $\varphi(\sigma)$ is the Jensen function of a function $f(s)$ almost periodic in $[\alpha, \beta]$ with exponents from the modul M.

According to our assumption there exist a number N and a sequence of intervals (α_n, β_n), $n = N+1, N+2, \ldots$, converging increasingly towards (α, β), such that $\varphi_n(\sigma) = 0$ in the interval (α_n, β_n). In $(\alpha_{N+1}, \beta_{N+1})$ we choose a sub-interval (α_N, β_N).

By Theorem 8 we have in the strip (α_n, β_n), where $n > N$, the representation

$$f_n(s) = e^{g_n(s)},$$

where $g_n(s)$ is almost periodic in $[\alpha_n, \beta_n]$ with exponents from M. Since $\log |f_n(s)|$ is the real part of $g_n(s)$, it follows from the definition of the Jensen function that the constant term of the Dirichlet series of $g_n(s)$ is purely imaginary. We may therefore choose an exponential polynomial $h_n(s)$ with exponents from M, and likewise with a purely imaginary constant term, such that

$$|f_n(s)\, e^{-h_n(s)} - 1| < \frac{1}{n^2}$$

in the strip $(\alpha_{n-1}, \beta_{n-1})$. The function $f_n(s)\, e^{-h_n(s)}$ then also possesses the Jensen function $\varphi_n(\sigma)$. Moreover, the infinite product

$$f(s) = \prod_{n=1}^{N} f_n(s) \prod_{n=N+1}^{\infty} f_n(s)\, e^{-h_n(s)}$$

is uniformly convergent in $[\alpha, \beta]$. The function $f(s)$ is therefore almost periodic in $[\alpha, \beta]$ with exponents from M and has, by Theorem 6, the Jensen function $\varphi(\sigma)$.

81. In the particular case where M is the discrete modul $M = \left\{ h \frac{2\pi}{p} \right\}$, so that we are dealing with *periodic* functions with the period ip, the solution of our problem is easy.

Necessary conditions of a function $\varphi(\sigma)$ in the interval (α, β) being the Jensen function of a periodic function $f(s)$ with the period ip are, by § 48, that $\varphi(\sigma)$ is convex and stretchwise linear and that the values of $\varphi'(\sigma)$ in the linearity intervals belong to M.

These conditions are, however, also sufficient. In proving this we may, by Theorem 19, restrict ourselves to the case where $\varphi(\sigma)$ is either linear or is composed of two linear pieces. If $\varphi(\sigma)$ is linear, say $\varphi(\sigma) = c\sigma + d$, where c belongs to M, it is the Jensen function of the function $f(s) = e^{cs+d}$, which has the period ip. If $\varphi(\sigma)$ is not linear, say $\varphi(\sigma) = c_1(\sigma - \sigma_0) + d$ for $\sigma \leqq \sigma_0$ and $\varphi(\sigma) = c_2(\sigma - \sigma_0) + d$ for $\sigma \geqq \sigma_0$, where c_1 and c_2 belong to M and $c_1 < c_2$, it is the Jensen function of the function $e^{c_1(s-\sigma_0)+d} + e^{c_2(s-\sigma_0)+d}$, which also has the period ip.

82. In the case where M consists of all rational multiples of a given number, so that we are dealing with limit periodic functions, the solution of our problem is given by the following theorem of Buch [1], [2].

Theorem 20. *A function $\varphi(\sigma)$ in the interval (α, β) is the Jensen function of a function $f(s)$ almost periodic in $[\alpha, \beta]$ with exponents from the modul $M = \left\{ r \frac{2\pi}{p} \right\}$, i. e. of a limit periodic function with the limit period ip, if and only if it satisfies the following conditions:*

(i) *It is convex in the interval (α, β).*

(ii) *The value of $\varphi'(\sigma)$ in any linearity interval of $\varphi(\sigma)$ belongs to M.*

The necessity of the conditions follows from Theorems 7 and 8. To prove their sufficiency we may, by Theorem 19, restrict ourselves to the case where $\varphi(\sigma)$ is linear in two intervals (α, α_1) and (β_1, β), and it is then no restriction to suppose that $\alpha = -\infty$ and $\beta = +\infty$.

If $\varphi(\sigma)$ is linear, say $\varphi(\sigma) = c\sigma + d$, the function $f(s) = e^{cs+d}$ is a solution. If $\varphi(\sigma)$ is not linear we denote by c_1 and c_2 the values of $\varphi'(\sigma)$ in the intervals $(-\infty, \alpha_1)$ and $(\beta_1, +\infty)$. Then $c_1 < c_2$. Without loss of generality we may suppose that $c_1 = 0$ since otherwise we replace the desired function $f(s)$ by $f(s)e^{-c_1 s}$. Further we may suppose that $c_2 = 2\pi$ since otherwise we replace $f(s)$ by $f\left(\frac{2\pi}{c_2} s\right)$, at the same time multiplying the elements of M by $\frac{2\pi}{c_2}$. We then simply have $M = \{r\, 2\pi\}$; furthermore, the function $\mu(\sigma) = \frac{1}{2\pi}\varphi'(\sigma)$ is a distribution function, for which $\mu(\sigma) = 0$ for $\sigma < \alpha_1$ and $\mu(\sigma) = 1$ for $\sigma > \beta_1$, and the values of $\mu(\sigma)$ in the constancy intervals are all rational.

By Theorem 12 there exists a real almost periodic sequence $U(k)$ with exponents from M, that is to say a limit periodic sequence $U(k)$, with the asymptotic distribution function $\mu(\sigma)$. By Theorem 15 there exists a function $f(s)$ almost periodic in $[-\infty, +\infty]$ with exponents from M and with the zeros $U(k) + ik$, whose Jensen function $\psi(\sigma)$ is constant for $\sigma < \alpha_1$. We may assume $\psi(\sigma)$ to be equal to $\varphi(\sigma)$ for $\sigma < \alpha_1$ since otherwise we multiply $f(s)$ by a properly chosen constant. The function $f(s)$ then has the Jensen function $\varphi(\sigma)$; for by Theorem 16 we have $\mu(\sigma) = \frac{1}{2\pi}\psi'(\sigma)$, and two convex functions are identical when they are equal for one value of σ and their derivatives are identical.

Functions with a Finite Integral Base.

83. We shall first give an account of analytic almost periodic functions $f(s)$ with a finite integral base $\mu_1, \ldots, \mu_m$, i. e. with exponents from the modul $M = \{h_1\mu_1 + \cdots + h_m\mu_m\}$, where the numbers $\mu_1, \ldots, \mu_m$ are linearly independent, and the set of coefficients $h_1, \ldots, h_m$ runs through all sets of integers. Denoting, as in § 31, the inner product $x_1 y_1 + \cdots + x_m y_m$ of two vectors $\boldsymbol{x} = (x_1, \ldots, x_m)$ and $\boldsymbol{y} = (y_1, \ldots, y_m)$ in the m-dimensional space R_m by $\boldsymbol{x}\boldsymbol{y}$, and putting $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)$, we have $M = \{\boldsymbol{h}\boldsymbol{\mu}\}$, where $\boldsymbol{h} = (h_1, \ldots, h_m)$ runs through all vectors of R_m with integral coordinates. The null-vector $(0, \ldots, 0)$ will be denoted by $\mathbf{0}$.

Let $f(s)$ be a function almost periodic in $[\alpha, \beta]$ with exponents from M. We shall then deduce a representation of $f(s)$ analogous to the spatial extension considered in § 31 in the case of functions of a real variable. We arrive at this representation by considerations similar to those in § 31. Allowing terms with the coefficient 0, we may write the Dirichlet series of $f(s)$ in the form

$$f(s) \sim \Sigma a_{\boldsymbol{h}} e^{\boldsymbol{h}\boldsymbol{\mu} s}.$$

Let us now consider a sequence of exponential polynomials of the form

$$f_p(s) = \Sigma a_{\boldsymbol{h}}^{(p)} e^{\boldsymbol{h}\boldsymbol{\mu} s}$$

(where for every p only a finite number of the coefficients $a_{\boldsymbol{h}}^{(p)}$ are $\neq 0$) converging uniformly towards $f(s)$ in $[\alpha, \beta]$ as $p \to \infty$. For each function $f_p(s)$ we form the function

$$g_p(s; \boldsymbol{x}) = \Sigma a_{\boldsymbol{h}}^{(p)} e^{i\boldsymbol{h}\boldsymbol{x}} e^{\boldsymbol{h}\boldsymbol{\mu} s},$$

where $\boldsymbol{x}$ runs through R_m. In each of the variables $x_1, \ldots, x_m$ it has the period 2π. Further $f_p(s) = g_p(s; \mathbf{0})$ and

$$g_p(s + i\tau; \boldsymbol{x}) = g_p(s; \boldsymbol{x} + \boldsymbol{\mu}\tau),$$

so that in particular

$$f_p(s + i\tau) = g_p(s; \boldsymbol{\mu}\tau). \tag{1}$$

As $f_p(s)$ converges uniformly towards $f(s)$ in $[\alpha, \beta]$, the function $f_p(s + i\tau)$ converges uniformly towards $f(s + i\tau)$ for s in $[\alpha, \beta]$ and all τ[1]; from (1) it follows

[1] i. e., of course, in any domain $(\alpha <)\alpha_1 < \sigma < \beta_1 (< \beta)$, $-\infty < \tau < +\infty$ in the (s, τ)-space.

therefore, by Kronecker's theorem, that $g_p(s; \boldsymbol{x})$ converges uniformly towards a certain limit function $g(s; \boldsymbol{x})$ for s in $[\alpha, \beta]$ and all $\boldsymbol{x}$. This limit function $g(s; \boldsymbol{x})$ is evidently uniformly continuous for s in $[\alpha, \beta]$ and all $\boldsymbol{x}$. In each of the variables $x_1, \ldots, x_m$ it has the period 2π. For every $\boldsymbol{x}$ it is, considered as a function of s, almost periodic in $[\alpha, \beta]$, and its Dirichlet series is

$$g(s; \boldsymbol{x}) \sim \Sigma a_h e^{i h \boldsymbol{x}} e^{h \mu s}.$$

We moreover find $f(s) = g(s; \mathbf{0})$ and

$$(2) \qquad g(s + i\tau; \boldsymbol{x}) = g(s; \boldsymbol{x} + \mu\tau),$$

so that in particular

$$f(s + i\tau) = g(s; \mu\tau).$$

This relation shows that the function $g(s; \boldsymbol{x})$ is for every fixed s, considered as a function of $\boldsymbol{x}$, the *spatial extension* of the function $f(s + i\tau)$ considered as a function of τ.

It further shows that for every τ the function $g(s; \mu\tau)$ has the same Jensen function $\varphi(\sigma)$ as $f(s)$. By Kronecker's theorem and Theorem 6 this implies that, for every $\boldsymbol{x}$, the function $g(s; \boldsymbol{x})$ has the Jensen function $\varphi(\sigma)$, so that in particular any strip (α', β') containing a zero of one of the functions $g(s; \boldsymbol{x})$ also contains a zero of $f(s)$.

In introducing the function $g(s; \boldsymbol{x})$ we have actually introduced one real variable more than necessary, since on account of (2) we have

$$g(\sigma + i t; \boldsymbol{x}) = g(\sigma; \boldsymbol{x} + \mu t),$$

which shows how $g(s; \boldsymbol{x})$ is expressed by means of $g(\sigma; \boldsymbol{x})$. It would, however, be inconvenient to work only with the function $g(\sigma; \boldsymbol{x})$.

84. We emphasize that the spatial extension $g(s; \boldsymbol{x})$ need not always be regular in the variables $x_1, \ldots, x_m$. This is shown by the following example, where $m = 2$.

Let the integer $a > 1$ and $0 < b < 1$ be values corresponding to a non-differentiable Weierstrass function

$$H(\xi) = \sum_{n=1}^{\infty} b^n e^{i a^n \xi}.$$

Denoting by μ an irrational number and by $[y]$, where y is an arbitrary real number, the largest integer $\leqq y$, we will consider the function

$$f(s) = \sum_{n=1}^{\infty} b^n e^{(a^n \mu - [a^n \mu]) s}.$$

Since $0 < a^n \mu - [a^n \mu] < 1$ for all n, the series is absolutely convergent for all s and represents therefore an almost periodic function in $[-\infty, +\infty]$ with exponents from the modul $M = \{h_1 \mu + h_2\}$. For the spatial extension we find

$$g(s; \boldsymbol{x}) = g(s; x_1, x_2) = \sum_{n=1}^{\infty} b^n e^{i(a^n x_1 - [a^n \mu] x_2)} e^{(a^n \mu - [a^n \mu]) s}.$$

Hence

$$g(0; x_1, 0) = \sum_{n=1}^{\infty} b^n e^{i a^n x_1} = H(x_1),$$

which shows that $g(s; \boldsymbol{x})$ is not regular in x_1.

85. The solution of our problem is given in the case of functions with a finite integral base by the following theorem.

Theorem 21. *A function $\varphi(\sigma)$ in the interval (α, β) is the Jensen function of a function $f(s)$ almost periodic in $[\alpha, \beta]$ with the finite integral base $\mu_1, \ldots, \mu_m$, where $m \geqq 2$, i. e. with exponents from the modul $M = \{h_1 \mu_1 + \cdots + h_m \mu_m\}$, if and only if it satisfies the following conditions:*

(i) *It is convex in the interval (α, β).*

(ii) *The value of $\varphi'(\sigma)$ in any linearity interval of $\varphi(\sigma)$ belongs to M*

(iii) *Any reduced interval $(\alpha <) \alpha_1 < \sigma < \beta_1 (< \beta)$ contains at most a finite number of linearity intervals of $\varphi(\sigma)$.*

86. The necessity of conditions (i) and (ii) follows from Theorems 7 and 8. The necessity of (iii) will be established by proving that, if $f(s)$ is a function almost periodic in $[\alpha, \beta]$ with exponents from M, any reduced strip (α_1, β_1) contains at most a finite number of strips without zeros of $f(s)$. In the proof we shall use the spatial extension $g(s; \boldsymbol{x})$ of the function $f(s + i\tau)$, introduced in § 83.

For an arbitrary point $\boldsymbol{x}_0$ in R_m we choose a rectangle $S(\boldsymbol{x}_0)$ in the s-plane with sides parallel to the real and imaginary axes and containing the segment $\alpha_1 < \sigma < \beta_1$, $t = 0$, such that $g(s; \boldsymbol{x}_0) \neq 0$ and hence that

$$|g(s; \boldsymbol{x}_0)| \geqq \text{(some) } k = k(\boldsymbol{x}_0) > 0$$

on the boundary of $S(\boldsymbol{x}_0)$. Next we choose an open interval $I(\boldsymbol{x}_0)$ in R_m con taining $\boldsymbol{x}_0$, such that $|g(s; \boldsymbol{x}) - g(s; \boldsymbol{x}_0)| < k$ on the boundary of $S(\boldsymbol{x}_0)$ when $\boldsymbol{x}$ belongs to $I(\boldsymbol{x}_0)$. By Rouché's theorem the functions $g(s; \boldsymbol{x})$ have then, for $\boldsymbol{x}$

belonging to $I(x_0)$, all the same number $p = p(x_0)$ of zeros in $S(x_0)$, and the set of p points formed by these zeros depends continuously on x. If the abscissae of the zeros are $\sigma_1(x) \leqq \cdots \leqq \sigma_p(x)$, each of the functions $\sigma_1(x), \ldots, \sigma_p(x)$ is therefore a continuous function of x. This shows that the set $K(x_0)$ consisting of all values of these abscissae is composed of at most p intervals.

By Borel's covering theorem the space R_m is mod. 2π covered by a finite number of the intervals $I(x_0)$. The sum K of the corresponding sets $K(x_0)$ is therefore composed of a finite number of intervals.

Now the abscissa σ of any zero $\sigma + it$ of $f(s)$ in (α_1, β_1) belongs to K; for $f(\sigma + it) = g(\sigma; \mu t)$, and μt belongs mod. 2π to one of the covering intervals $I(x_0)$, so that σ belongs to the corresponding set $K(x_0)$. On the other hand any point of K belonging to (α_1, β_1) is either itself the abscissa of a zero of $f(s)$ or an accumulation point for abscissae of zeros; for by § 83 any strip (α', β') containing a zero of one of the functions $g(s; x)$ contains a zero of $f(s)$. Since K is composed of a finite number of intervals this implies that the strip (α_1, β_1) contains only a finite number of strips without zeros of $f(s)$.

87. To prove the sufficiency of conditions (i)—(iii) we may, by Theorem 19, restrict ourselves to the case where $\varphi(\sigma)$ is either linear in (α, β) or linear in two intervals (α, α_1) and (β_1, β), but not in any sub-interval of (α_1, β_1). When this latter condition is satisfied we may suppose that $\alpha = -\infty$ and $\beta = +\infty$.

If $\varphi(\sigma)$ is linear, say $\varphi(\sigma) = c\sigma + d$, the function $f(s) = e^{cs+d}$ is a solution. If $\varphi(\sigma)$ is not linear we denote by c_1 and c_2 the values of $\varphi'(\sigma)$ in the intervals $(-\infty, \alpha_1)$ and $(\beta_1, +\infty)$. Without loss of generality we may suppose that $c_1 = 0$, since otherwise we replace the desired function $f(s)$ by $f(s)e^{-c_1 s}$. Further we may suppose that $c_2 = 2\pi$, since otherwise we replace $f(s)$ by $f\left(\frac{2\pi}{c_2}s\right)$, at the same time multiplying the elements of M by $\frac{2\pi}{c_2}\cdot$ The modul M then contains the number 2π; furthermore, the function $\mu(\sigma) = \frac{1}{2\pi}\varphi'(\sigma)$ is a distribution function, for which $\mu(\sigma) = 0$ for $\sigma < \alpha_1$ and $\mu(\sigma) = 1$ for $\sigma > \beta_1$, and which is not constant in any sub-interval of (α_1, β_1).

The existence of a function $f(s)$ almost periodic in $[-\infty, +\infty]$ with exponents from M and with the Jensen function $\varphi(\sigma)$ is now established by applying Theorems 12, 15, and 16 as in § 82.

Functions with an Infinite Integral Base.

88. The preceding results may without difficulty be extended to almost periodic functions $f(s)$ with an infinite integral base $\mu_1, \mu_2, \ldots$, i. e. with exponents from the modul $M = \{h_1\mu_1 + h_2\mu_2 + \cdots\}$, where the numbers $\mu_1, \mu_2, \ldots$ are linearly independent, and the sequence of coefficients $h_1, h_2, \ldots$ runs through all sequences of integers of which only a finite number are $\neq 0$. Denoting the inner product $x_1y_1 + x_2y_2 + \cdots$ of two vectors $\boldsymbol{x} = (x_1, x_2, \ldots)$ and $\boldsymbol{y} = (y_1, y_2, \ldots)$ in the infinite-dimensional space R_ω, the first of which has only a finite number of coordinates $\neq 0$, by $\boldsymbol{xy}$, and putting $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots)$, we have $M = \{\boldsymbol{h\mu}\}$, where $\boldsymbol{h} = (h_1, h_2, \ldots)$ runs through all vectors with integral coordinates of which only a finite number are $\neq 0$.

89. In the space R_ω we use the following well-known definitions of an interval, a limit point, continuity, etc., developed in detail in Bohr [9] (see also Bochner [1] and Jessen [3]).

An (open) interval is the set of points $\boldsymbol{x} = (x_1, x_2, \ldots)$ defined by a finite number of inequalities $a_i < x_i < b_i$, while the remaining coordinates are unrestricted. A sequence of points $\boldsymbol{x}^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots)$ is said to converge towards the limit point $\boldsymbol{x} = (x_1, x_2, \ldots)$ if any interval containing $\boldsymbol{x}$ contains all except a finite number of the points $\boldsymbol{x}^{(n)}$. Evidently $\boldsymbol{x}^{(n)} \to \boldsymbol{x}$ if and only if $x_i^{(n)} \to x_i$ for every i. The terms closed set, open set, everywhere dense set, etc., are to be understood in accordance with this definition. A set is called bounded if it is contained in a set defined by a set of inequalities $a_i < x_i < b_i$, where $i = 1, 2, \ldots$, and all a and b_i are finite. We have the Borel covering theorem, stating that if a closed and bounded set in R_ω is covered by a system of intervals it is covered by a finite number of these intervals.

A function $F(\boldsymbol{x})$ is called continuous if $F(\boldsymbol{x}^{(n)}) \to F(\boldsymbol{x})$ whenever $\boldsymbol{x}^{(n)} \to \boldsymbol{x}$. This is equivalent to saying that to every point $\boldsymbol{x}$ and every $\varepsilon > 0$ there corresponds an interval I containing $\boldsymbol{x}$, such that $|F(\boldsymbol{y}) - F(\boldsymbol{x})| \leqq \varepsilon$ for all points $\boldsymbol{y}$ in I. The function $F(\boldsymbol{x})$ is called uniformly continuous if to every $\varepsilon > 0$ there corresponds an interval I containing the point $\boldsymbol{0} = (0, 0, \ldots)$, such that $|F(\boldsymbol{y}) - F(\boldsymbol{x})| \leqq \varepsilon$ whenever $\boldsymbol{y} - \boldsymbol{x} = (y_1 - x_1, y_2 - x_2, \ldots)$ belongs to I. A continuous function defined in a closed and bounded set is always uniformly continuous.

If a function $F(\boldsymbol{x})$ defined in the whole of R_ω is continuous and periodic in each variable x_i with a given period p_i, it satisfies the condition

$$F(x_1 + h_1 p_1, x_2 + h_2 p_2, \ldots) = F(x_1, x_2, \ldots)$$

for any sequence of integers $h_1, h_2, \ldots$; being uniformly continuous in the set $0 \leqq x_i \leqq p_i$, $i = 1, 2, \ldots$, the function is uniformly continuous in R_ω.

Of essential importance in the treatment of almost periodic functions with exponents from the modul $M = \{h_1 \mu_1 + h_2 \mu_2 + \cdots\} = \{\boldsymbol{h\mu}\}$ is an extension of Kronecker's theorem to the space R_ω, which states that the set of points $\boldsymbol{x} = \boldsymbol{\mu} t = (\mu_1 t, \mu_2 t, \ldots)$, $-\infty < t < +\infty$, is mod. 2π everywhere dense in R_ω when $\mu_1, \mu_2, \ldots$ are linearly independent. This follows immediately from the theorem in the case of a finite number of linearly independent numbers.

The definitions of continuity, etc. may also be applied to functions $g(s; \boldsymbol{x})$ of a complex variable s describing a strip (α, β) and the real variables $x_1, x_2, \ldots$, since $g(s; \boldsymbol{x})$ may be considered as a function of the real variables $\sigma, t, x_1, x_2, \ldots$.

90. Let $f(s)$ be a function almost periodic in $[\alpha, \beta]$ with exponents from M. Allowing terms with the coefficient 0, we may write its Dirichlet series in the form

$$f(s) \sim \Sigma a_{\boldsymbol{h}} e^{\boldsymbol{h\mu} s}.$$

Let us now consider a sequence of exponential polynomials of the form

$$f_p(s) = \Sigma a_{\boldsymbol{h}}^{(p)} e^{\boldsymbol{h\mu} s}$$

(where for every p only a finite number of the coefficients $a_{\boldsymbol{h}}^{(p)}$ are $\neq 0$) converging uniformly towards $f(s)$ in $[\alpha, \beta]$ as $p \to \infty$ For each function $f_p(s)$ we form the function

$$g_p(s; \boldsymbol{x}) = \Sigma a_{\boldsymbol{h}}^{(p)} e^{i \boldsymbol{h x}} e^{\boldsymbol{h\mu} s},$$

where $\boldsymbol{x}$ runs through R_ω. This function actually depends on only a finite number of the variables $x_1, x_2, \ldots$, and has the period 2π in each of the variables. Further $f_p(s) = g_p(s; \boldsymbol{0})$ and

$$g_p(s + i\tau; \boldsymbol{x}) = g_p(s; \boldsymbol{x} + \boldsymbol{\mu}\tau),$$

so that in particular

$$f_p(s + i\tau) = g_p(s; \boldsymbol{\mu}\tau). \tag{3}$$

As $f_p(s)$ converges uniformly towards $f(s)$ in $[\alpha, \beta]$, the function $f_p(s + i\tau)$ converges uniformly towards $f(s + i\tau)$ for s in $[\alpha, \beta]$ and all τ; from (3) it follows therefore, by Kronecker's theorem, that $g_p(s; \boldsymbol{x})$ converges uniformly towards a certain limit function $g(s; \boldsymbol{x})$ for s in $[\alpha, \beta]$ and all $\boldsymbol{x}$. This limit function $g(s; \boldsymbol{x})$

is evidently uniformly continuous for s in $[\alpha, \beta]$ and all $\boldsymbol{x}$. It is periodic with the period 2π in each of the variables $x_1, x_2, \ldots$. For every $\boldsymbol{x}$ it is, considered as a function of s, almost periodic in $[\alpha, \beta]$, and its Dirichlet series is

$$g(s; \boldsymbol{x}) \sim \Sigma a_h e^{i h x} e^{h \mu s}.$$

We moreover find $f(s) = g(s; \mathbf{0})$ and

$$g(s + i\tau; \boldsymbol{x}) = g(s; \boldsymbol{x} + \mu\tau),$$

so that in particular

$$f(s + i\tau) = g(s; \mu\tau).$$

The function $g(s; \boldsymbol{x})$ is therefore for every fixed s called the *spatial extension* of the function $f(s+i\tau)$, considered as a function of τ. As in § 83, it is shown that, for every $\boldsymbol{x}$, the function $g(s; \boldsymbol{x})$ has the same Jensen function $\varphi(\sigma)$ as $f(s)$, and hence that any strip (α', β') containing a zero of one of the functions $g(s; \boldsymbol{x})$ also contains a zero of $f(s)$.

91. The solution of the present case of our problem is given by the following theorem.

Theorem 22. *A function $\varphi(\sigma)$ in the interval (α, β) is the Jensen function of a function $f(s)$ almost periodic in $[\alpha, \beta]$ with the infinite integral base $\mu_1, \mu_2, \ldots$, i. e. with exponents from the modul $M = \{h_1\mu_1 + h_2\mu_2 + \cdots\}$, if and only if it satisfies the following conditions:*

(i) *It is convex in the interval (α, β).*

(ii) *The value of $\varphi'(\sigma)$ in any linearity interval of $\varphi(\sigma)$ belongs to M.*

(iii) *Any reduced interval $(\alpha <) \alpha_1 < \sigma < \beta_1 (< \beta)$ contains at most a finite number of linearity intervals of $\varphi(\sigma)$.*

The proof is quite analogous to that of Theorem 21, the only difference being that R_m is here replaced by R_ω.

Further Results Concerning Functions with a Finite Integral Base.

92. As a preliminary to the study of functions with a finite or infinite rational base we shall study in greater detail the case of functions with a finite integral base $\mu_1, \ldots, \mu_m$, i. e. of functions with exponents from the modul $M = \{h_1\mu_1 + \cdots + h_m\mu_m\}$.

Let $f(s)$ be a function almost periodic in $[\alpha, \beta]$ with exponents from M and not identically zero. Let σ_1 and σ_2, where $\sigma_1 < \sigma_2$, belong to different linearity intervals of its Jensen function $\varphi(\sigma)$. We shall then consider the difference

$$\varphi'(\sigma_2) - \varphi'(\sigma_1),$$

which is evidently positive and belongs to M. Denoting by

$$(4) \qquad \cdots \leqq t_{-2} \leqq t_{-1} \leqq t_0 \leqq t_1 \leqq t_2 \leqq \cdots$$

the ordinates of the zeros of $f(s)$ in the strip (σ_1, σ_2), each one being written so often as the number of zeros with this ordinate indicates, and by $N(\gamma, \delta)$ the number of these ordinates contained in the interval $\gamma < t < \delta$, we have the expression

$$\varphi'(\sigma_2) - \varphi'(\sigma_1) = 2\pi \lim_{(\delta-\gamma)\to\infty} \frac{N(\gamma, \delta)}{\delta - \gamma}.$$

93. Our considerations will be based on the spatial extension $g(s; \boldsymbol{x})$ of the function $f(s + i\tau)$. By § 83 the function $g(s; \boldsymbol{x})$ has for every $\boldsymbol{x}$ in R_m, considered as a function of s, the same Jensen function $\varphi(\sigma)$ as $f(s)$, and has therefore in particular no zeros on the lines $\sigma = \sigma_1$ and $\sigma = \sigma_2$.

We shall now consider the set C of those points $\boldsymbol{x}$ for which the function $g(s; \boldsymbol{x})$ possesses at least one zero on the segment $\sigma_1 < \sigma < \sigma_2$, $t = 0$. If $g(s; \boldsymbol{x})$ has p zeros on this segment the point $\boldsymbol{x}$ will be considered a p-fold point of C. The set C has, of course, the period 2π in each of the variables $x_1, \ldots, x_m$. Since $f(\sigma + it) = g(\sigma; \boldsymbol{\mu} t)$ the ordinates (4) are just those values of t for which the point $\boldsymbol{\mu} t$ belongs to C, each being written as often as the multiplicity of the point indicates.

In order to examine this set C let us write an arbitrary point $\boldsymbol{x}$ of R_m in the form

$$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{\mu} t,$$

where $\boldsymbol{y} = (0, y_2, \ldots, y_m)$ is a point of the $(m-1)$-dimensional sub-space $x_1 = 0$ of R_m, and then use $\boldsymbol{y}$ and t as coordinates in R_m along with $x_1, \ldots, x_m$. Since $g(\sigma + it; \boldsymbol{y}) = g(\sigma; \boldsymbol{y} + \boldsymbol{\mu} t)$ the ordinates of the zeros of $g(s; \boldsymbol{y})$ in the strip (σ_1, σ_2) are just those values of t for which the point $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{\mu} t$ belongs to C, the number of zeros with a given ordinate being equal to the multiplicity of the point.

For every $\boldsymbol{x}^* = \boldsymbol{y}^* + \boldsymbol{\mu} t^*$, where $\boldsymbol{y}^* = (0, y_2^*, \ldots, y_m^*)$, we choose a rectangle $\sigma_1 \leqq \sigma \leqq \sigma_2$, $|t - t^*| \leqq \eta = \eta(\boldsymbol{x}^*)$ in which $g(s; \boldsymbol{y}^*)$ possesses no zero outside the segment $\sigma_1 < \sigma < \sigma_2$, $t = t^*$. On its boundary we then have $|g(s; \boldsymbol{y}^*)| \geqq k = k(\boldsymbol{x}^*) > 0$. Next, in the sub-space $x_1 = 0$, we choose an interval $J(\boldsymbol{y}^*)$ containing $\boldsymbol{y}^*$, such that $|g(s; \boldsymbol{y}) - g(s; \boldsymbol{y}^*)| < k$ on the boundary of the rectangle when $\boldsymbol{y}$ belongs to $J(\boldsymbol{y}^*)$. Let $p = p(\boldsymbol{x}^*) (\geqq 0)$ denote the number of zeros of $g(s; \boldsymbol{y}^*)$ on the segment $\sigma_1 < \sigma < \sigma_2$, $t = t^*$; then, by Rouché's theorem, the function $g(s; \boldsymbol{y})$ has p zeros in the rectangle when $\boldsymbol{y}$ belongs to $J(\boldsymbol{y}^*)$. If their ordinates are $v_1(\boldsymbol{y}) \leqq \cdots \leqq v_p(\boldsymbol{y})$, each of the functions $v_1(\boldsymbol{y}), \ldots, v_p(\boldsymbol{y})$ is a continuous function of $\boldsymbol{y}$ in $J(\boldsymbol{y}^*)$.

Hence, if $U(\boldsymbol{x}^*)$ denotes the neighbourhood of the point $\boldsymbol{x}^*$ consisting of all points $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{\mu} t$ for which $\boldsymbol{y}$ belongs to $J(\boldsymbol{y}^*)$ and $|t - t^*| < \eta$, the part of the set C contained in $U(\boldsymbol{x}^*)$ consists of the $p = p(\boldsymbol{x}^*)$ continuous $(m-1)$-dimensional surface elements $t = v_1(\boldsymbol{y}), \ldots, t = v_p(\boldsymbol{y})$.

94. By Borel's covering theorem, the space R_m is mod. 2π covered by a finite number of the neighbourhoods $U(\boldsymbol{x}^*)$. For two overlapping neighbourhoods the surface elements having points in their common part will uniquely combine so as to form larger surface elements. Thus the set C divides into components obtained by continuation of the surface elements, each component being determined by an equation $t = t(\boldsymbol{y})$, where $t(\boldsymbol{y})$ is a continuous function defined for all $\boldsymbol{y}$. To $\boldsymbol{y} = \boldsymbol{0}$ corresponds the line $\boldsymbol{x} = \boldsymbol{\mu} t$, on which the set C contains the points determined by the values (4). Thus C divides into an infinite number of components

$$\ldots, C_{-2}, C_{-1}, C_0, C_1, C_2, \ldots,$$

where C_j is determined by an equation $t = t_j(\boldsymbol{y})$ such that $t_j(\boldsymbol{0}) = t_j$, and where, for all $\boldsymbol{y}$,

$$\cdots \leqq t_{-2}(\boldsymbol{y}) \leqq t_{-1}(\boldsymbol{y}) \leqq t_0(\boldsymbol{y}) \leqq t_1(\boldsymbol{y}) \leqq t_2(\boldsymbol{y}) \leqq \cdots.$$

Fig. 3 illustrates the situation in the case $m = 2$.

95. By the translation $2\pi \boldsymbol{h}$, where $\boldsymbol{h} = (h_1, \ldots, h_m)$ is a vector with integral coordinates, the component C_0 is taken into a component C_i, where $i = i(\boldsymbol{h})$. By the same translation the component C_j must then for every j be taken into the component C_{j+i}. Evidently $i(\boldsymbol{h}' + \boldsymbol{h}'') = i(\boldsymbol{h}') + i(\boldsymbol{h}'')$. The function $i(\boldsymbol{h})$ is therefore a linear transformation of the m-dimensional lattice G of vectors $\boldsymbol{h}$ into a certain arithmetical progression $\ldots, -d, 0, d, \ldots$. In the case illustrated in Fig. 3 we have $d = 2$.

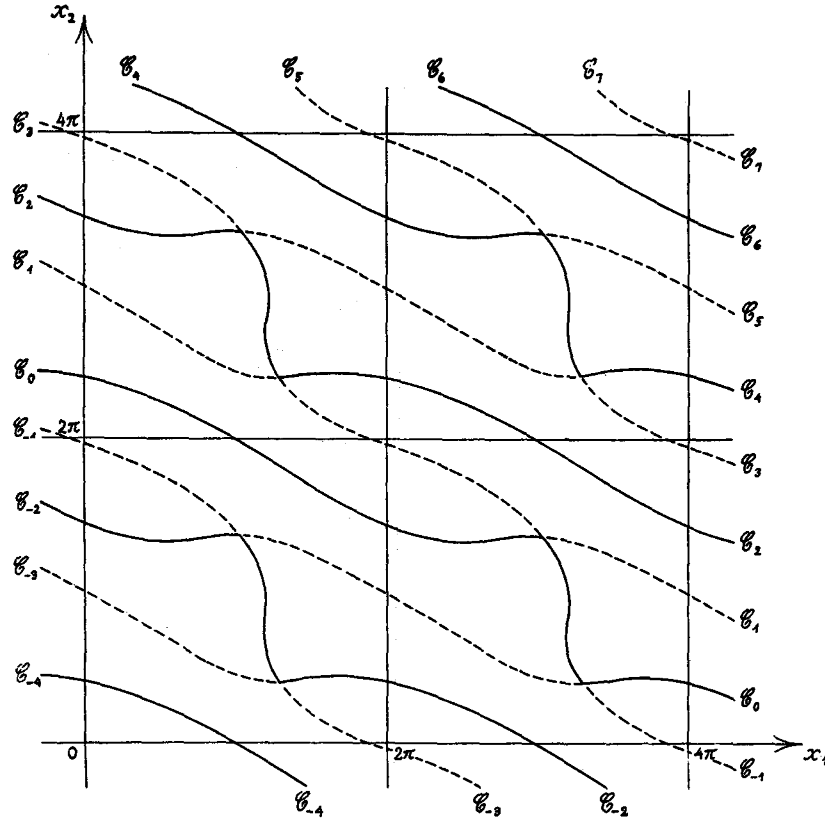

Fig. 3.

If we denote by G_n the part of G in which $i(\boldsymbol{h}) = nd$, it follows that G_0 is a lattice, and that G_n is derived from G_0 by the translation $n\,2\pi\boldsymbol{h}^{(1)}$, where $\boldsymbol{h}^{(1)} = (h_1^{(1)}, \ldots, h_m^{(1)})$ denotes an arbitrary vector in G_1. This again implies that the lattice G_0 is $(m-1)$-dimensional. Let $\boldsymbol{h}^{(2)} = (h_1^{(2)}, \ldots, h_m^{(2)}), \ldots, \boldsymbol{h}^{(m)} = (h_1^{(m)}, \ldots, h_m^{(m)})$ denote a base of G_0. Then $\boldsymbol{h}^{(1)}, \ldots, \boldsymbol{h}^{(m)}$ form a base of G.

Since the component C_0 is taken into itself by the translations $2\pi\boldsymbol{h}^{(2)}, \ldots, 2\pi\boldsymbol{h}^{(m)}$ it possesses a parametric representation

$$\boldsymbol{x} = z_2\boldsymbol{h}^{(2)} + \cdots + z_m\boldsymbol{h}^{(m)} + H(z_2, \ldots, z_m)\,\boldsymbol{\mu},$$

where $z_2, \ldots, z_m$ are real variables, and $H(z_2, \ldots, z_m)$ is a continuous function with the period 2π in each of the variables $z_2, \ldots, z_m$. For the component C_{nd} we find the representation

$$\boldsymbol{x} = n\,2\pi\boldsymbol{h}^{(1)} + z_2\boldsymbol{h}^{(2)} + \cdots + z_m\boldsymbol{h}^{(m)} + H(z_2, \ldots, z_m)\,\boldsymbol{\mu}.$$

96. Let K denote the maximum of $|H(z_2, \ldots, z_m)|$. It then follows from the preceding result that $|t_{nd} - t'_{nd}| \leqq K$, where t'_{nd} denotes the value of t for which the point μt belongs to the plane

$$\boldsymbol{x} = 2\pi n \boldsymbol{h}^{(1)} + z_2 \boldsymbol{h}^{(2)} + \cdots + z_m \boldsymbol{h}^{(m)}.$$

Eliminating the parameters $z_2, \ldots, z_m$, we find for this plane the equation

$$\begin{vmatrix} x_1 & h_1^{(2)} & \ldots & h_1^{(m)} \\ \vdots & \vdots & & \vdots \\ x_m & h_m^{(2)} & \ldots & h_m^{(m)} \end{vmatrix} = 2\pi n \begin{vmatrix} h_1^{(1)} & h_1^{(2)} & \ldots & h_1^{(m)} \\ \vdots & \vdots & & \vdots \\ h_m^{(1)} & h_m^{(2)} & \ldots & h_m^{(m)} \end{vmatrix}$$

or

$$\boldsymbol{\lambda x} = 2\pi n,$$

where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ denotes the vector whose coordinates are the complements of the elements of the first column of the determinant on the left, divided by the value of the determinant on the right, which is either $+1$ or -1.

Hence the point of intersection with the line $\boldsymbol{x} = \boldsymbol{\mu} t$ is determined by

$$t'_{nd} = \frac{2\pi n}{\boldsymbol{\lambda\mu}},$$

and we have, therefore,

$$\left| t_{nd} - \frac{2\pi n}{\boldsymbol{\lambda\mu}} \right| \leqq K$$

for all n.

By (4) this implies that

$$N(\gamma, \delta) = d(\delta - \gamma)\frac{\boldsymbol{\lambda\mu}}{2\pi} + O(1),$$

so that we find

$$\varphi'(\sigma_2) - \varphi'(\sigma_1) = 2\pi \lim_{(\delta-\gamma)\to\infty} \frac{N(\gamma, \delta)}{\delta - \gamma} = d\boldsymbol{\lambda\mu}.$$

Putting for the sake of brevity $d\boldsymbol{\lambda} = \boldsymbol{\nu} = (\nu_1, \ldots, \nu_m)$ we finally find

$$\varphi'(\sigma_2) - \varphi'(\sigma_1) = \boldsymbol{\nu\mu} = \nu_1\mu_1 + \cdots + \nu_m\mu_m. \tag{5}$$

Since $\boldsymbol{\lambda}$ has integral coordinates the coordinates of $\boldsymbol{\nu}$ are also integers, and the representation (5) shows therefore, once more, that $\varphi'(\sigma_2) - \varphi'(\sigma_1)$ belongs to M.

For later application we notice that each of the components C_j lies between two planes orthogonal to the vector $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_m)$.

97. Denoting the length $\sqrt{x_1^2 + \cdots + x_m^2}$ of a vector $x = (x_1, \ldots, x_m)$ in R_m by $\|x\|$, we shall now assume that $g(\sigma: x) \neq 0$ in the interval $\sigma_1 < \sigma < \sigma_2$ for all x belonging to a system of spheres

$$\left\| x - \left(x^* + \frac{2\pi h}{N} \right) \right\| < \frac{a}{N}, \tag{6}$$

where N is a positive integer, $a < \pi$ a positive number, $x^* = (x_1^*, \ldots, x_m^*)$ a point of R_m, and $h = (h_1, \ldots, h_m)$ runs through all vectors of R_m with integral coordinates. Thus these spheres all have the common radius $\frac{a}{N}$ and their centres form a lattice with the edge-length $\frac{2\pi}{N}$.

It will be shown that under this assumption the vector v occurring in (5) satisfies a relation

$$\frac{v \mu}{\|v\| \, \|\mu\|} \geqq b,$$

where $b > 0$ depends only on a and the direction of μ $\left(\text{i. e. on } a \text{ and } \frac{\mu}{\|\mu\|}\right)$, and is therefore independent of N, the function $f(s)$ in question, and the values σ_1 and σ_2. Denoting the angle between two vectors $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ different from the null-vector, by $[x, y]$, we have

$$\cos [x, y] = \frac{x y}{\|x\| \, \|y\|}, \qquad 0 \leqq [x, y] \leqq \pi.$$

A statement equivalent to the preceding one is, therefore, that the vector v satisfies a relation

$$[v, \mu] \leqq \tfrac{1}{2}\pi - \theta, \tag{7}$$

where $\theta > 0$ depends only on a and the direction of μ.

98. For an arbitrary vector $h = (h_1, \ldots, h_m)$ with integral coordinates we denote the sphere (6) by S_h. When S_h is translated in the direction of μ it describes a tube T_h consisting of all points x having a distance $< \frac{a}{N}$ from the half-line

$$x = x^* + \frac{2\pi h}{N} + \mu t, \qquad t \geqq 0.$$

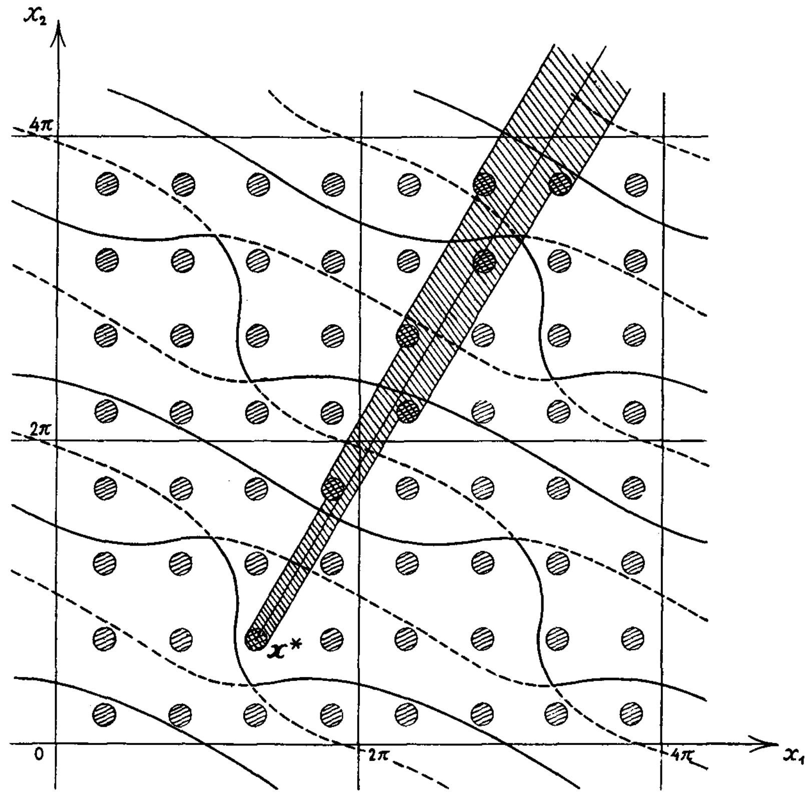

Fig. 4.

We shall now consider the set U obtained by taking first the tube T_0 corresponding to the sphere S_0, next joining to this tube all tubes T_h for which the corresponding sphere S_h has a point in T_0, then joining to the set thus obtained all tubes T_h for which the corresponding sphere S_h has a point in this set, and so on. In other words, the set U is the set in R_m which is passed through when we let first the sphere S_0 move in the direction of μ, next let every sphere S_h which it hits move along with it, then in the same manner every sphere S_h hit by these, and so on.

Fig. 4 illustrates the situation in the case $m = 2$, $N = 4$.

We shall now prove that the set U contains a cone

$$[x - x^*, \mu] < \theta, \tag{8}$$

where $\theta > 0$ depends only on a and the direction of μ.

For this purpose we denote by $\varDelta(t)$, for an arbitrary $t \geqq 0$, the radius of the largest sphere which has the centre $x^* + \mu t$ and is contained in U. Then $\varDelta(t)$ is an increasing function of t, and $\varDelta(t) \geqq \frac{a}{N}$ for all t. Let S'_h denote the sphere which is concentric with S_h and has the radius $\frac{a}{2N}$. By Kronecker's theorem there exists a number $l > 0$ depending only on a and the direction of μ, and such that any segment

$$x = x_0 + \mu t, \quad 0 < t < \frac{l}{\|\mu\| N},$$

has at least one point in common with some sphere S'_h. Thus, if the point x_0 belongs to U, the sphere

$$\left\| x - \left(x_0 + \frac{\mu l}{\|\mu\| N} \right) \right\| < \frac{a}{2N}$$

also belongs to U. The function $\varDelta(t)$ therefore satisfies the condition

$$\varDelta\left(t + \frac{l}{\|\mu\| N} \right) \geqq \varDelta(t) + \frac{a}{2N};$$

which, together with the relation $\varDelta(t) \geqq \frac{a}{N}$, shows that $\varDelta(t) \geqq \frac{a}{2l} \|\mu\| t$ for all $t \geqq 0$. This means that U contains the cone (8) when $\theta = \arcsin \frac{a}{2l}$, $0 < \theta < \frac{1}{2}\pi$.

99. We shall now prove that the relation (7) holds for this value of θ. For this purpose we notice that in the coordinates $y = (0, y_2, \ldots, y_m)$ and t introduced in § 93 the cone (8) is determined by an inequality

$$t > t(y),$$

where $t(y)$ is a continuous function of y. Together with this function we consider the function $t = t_j(y)$ which determines the component C_j of the set C. Evidently we may choose j such that the centre x^* of the sphere S_0 belongs to the part A of R_m determined by the inequality $t > t_j(y)$.

The assumption made in § 97 means that the set C does not contain any point of the spheres S_h. That x^* belongs to A implies therefore first that S_0, and hence the whole tube T_0, belongs to A, next that every sphere S_h having a point in T_0, and hence the corresponding tubes T_h, belong to A, and so on. Thus the whole set U and *a fortiori* the cone (8) belong to A. We therefore have

$$t_j(y) \leqq t(y)$$

for all y.

On the other hand, by § 96, the component C_j lies between two planes orthogonal to the vector $\boldsymbol{v}$. Thus there exists a plane orthogonal to $\boldsymbol{v}$ containing no point of the cone (8). Since $\boldsymbol{v}\boldsymbol{\mu} > 0$ and hence $[\boldsymbol{v}, \boldsymbol{\mu}] < \frac{1}{2}\pi$ this implies the relation (7).

100. The preceding results will suffice as preliminaries to the treatment of functions with a finite rational base. In studying functions with an infinite rational base we shall need some more properties of the vector $\boldsymbol{v} = (\nu_1, \ldots, \nu_m)$.

Denoting by $m_0 < m$ a positive integer, we consider together with R_m the space R_{m_0} of vectors $\boldsymbol{x}' = (x_1, \ldots, x_{m_0})$. For an arbitrary vector $\boldsymbol{x} = (x_1, \ldots, x_m)$ of R_m the corresponding vector $\boldsymbol{x}' = (x_1, \ldots, x_{m_0})$ is called the projection of $\boldsymbol{x}$ upon R_{m_0}. Using without change for vectors of R_{m_0} the previous notations for inner products, lengths, and angles of vectors, the inequalities

$$(9) \qquad \left\| \boldsymbol{x}' - \left(\boldsymbol{x}^{*\prime} + \frac{2\pi \boldsymbol{h}'}{N} \right) \right\| < \frac{a}{N},$$

where N is a positive integer, $a < \pi$ a positive number, $\boldsymbol{x}^{*\prime}$ a point of R_{m_0}, and $\boldsymbol{h}'$ runs through all vectors of R_{m_0} with integral coordinates, define a system of spheres in R_{m_0}. All points $\boldsymbol{x}$ of R_m whose projections $\boldsymbol{x}'$ upon R_{m_0} belong to one of these spheres form a cylinder in R_m, and the inequalities (9) may therefore also be said to define a system of cylinders in R_m.

We shall now assume that $g(\sigma; \boldsymbol{x}) \neq 0$ in the interval $\sigma_1 < \sigma < \sigma_2$ for all $\boldsymbol{x}$ belonging to this system of cylinders. It will then be shown that the $m - m_0$ last coordinates $\nu_{m_0+1}, \ldots, \nu_m$ of the vector $\boldsymbol{v}$ are equal to 0 and that its projection $\boldsymbol{v}' = (\nu_1, \ldots, \nu_{m_0})$ satisfies the relation

$$\frac{\boldsymbol{v}' \boldsymbol{\mu}'}{\|\boldsymbol{v}'\| \, \|\boldsymbol{\mu}'\|} \geq b,$$

where $b > 0$ depends only on a and the direction of the projection $\boldsymbol{\mu}' = (\mu_1, \ldots, \mu_{m_0})$ of $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)$, and is therefore independent of $\mu_{m_0+1}, \ldots, \mu_m$, as well as of N, the function $f(s)$ in question, and the values σ_1 and σ_2.

The number b is here simply the number determined, according to § 97, by a and the direction of $\boldsymbol{\mu}'$ if m is replaced by m_0.

101. The proof is closely analogous to the proof of §§ 98—99, the only differences being the following:

Instead of the spheres S_h we consider the cylinders determined by the inequalities (9). The set U is therefore replaced by a set, which must contain all points x for which the projection x' belongs to the cone

$$[x' - x^{*\prime}, \mu'] < \theta$$

in R_{m_0}, where θ is the number determined, according to § 98, by a and the direction of μ', if m is replaced by m_0. The set of these points x may be called a wedge in R_m.

There exists a plane orthogonal to $\boldsymbol{v}$ containing no point of this wedge. This implies that the $m - m_0$ last coordinates $v_{m_0+1}, \ldots, v_m$ are all 0, and further, that, since $\boldsymbol{v}\boldsymbol{\mu} > 0$, the projection $\boldsymbol{v}'$ satisfies the relation

$$[\boldsymbol{v}', \boldsymbol{\mu}'] \leqq \tfrac{1}{2}\pi - \theta.$$

This establishes the desired result.

Functions with a Finite Rational Base.

102. We now turn to the study of almost periodic functions $f(s)$ with a finite rational base $\mu_1, \ldots, \mu_m$, i. e. with exponents from the modul $M = \{r_1 \mu_1 + \cdots + r_m \mu_m\}$, where the numbers $\mu_1, \ldots, \mu_m$ are linearly independent, and the set of coefficients $r_1, \ldots, r_m$ runs through all sets of rational numbers. Using the vectorial notation, we have $M = \{\boldsymbol{r}\boldsymbol{\mu}\}$, where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)$, and $\boldsymbol{r} = (r_1, \ldots, r_m)$ runs through all vectors with rational coordinates.

Let $f(s)$ be a function almost periodic in $[\alpha, \beta]$ with exponents from M. Allowing terms with the coefficient 0, we may write its Dirichlet series in the form

$$f(s) \sim \Sigma a_{\boldsymbol{r}} e^{\boldsymbol{r}\boldsymbol{\mu} s}.$$

Let us now consider a sequence of exponential polynomials of the form

$$f_p(s) = \Sigma a_{\boldsymbol{r}}^{(p)} e^{\boldsymbol{r}\boldsymbol{\mu} s}$$

(where for every p only a finite number of the coefficients $a_{r}^{(p)}$ are $\neq 0$) converging uniformly towards $f(s)$ in $[\alpha, \beta]$ as $p \to \infty$. For each function $f_p(s)$ we form the function

$$g_p(s; x) = \Sigma a_{r}^{(p)} e^{i r x} e^{r \mu s},$$

where x runs through R_m. In each of the variables $x_1, \ldots, x_m$ this function has the period $2\pi N_p$, where N_p denotes a common denominator of the coordinates of those vectors r for which $a_{r}^{(p)} \neq 0$. Moreover

$$f_p(s + i\tau) = g_p(s; \mu\tau). \tag{10}$$

As $f_p(s)$ converges uniformly towards $f(s)$ in $[\alpha, \beta]$, the function $f_p(s + i\tau)$ converges uniformly towards $f(s+i\tau)$ for s in $[\alpha, \beta]$ and all τ. Since any two of the functions $g_p(s; x)$ have a common period $2\pi N$ in all the variables $x_1, \ldots, x_m$, it follows therefore from (10), if we use Kronecker's theorem, that $g_p(s; x)$ converges uniformly towards a certain limit function $g(s; x)$ for s in $[\alpha, \beta]$ and all x. We have then obviously

$$f(s + i\tau) = g(s; \mu\tau).$$

103. The solution of the present case of our problem is given by the following theorem.

Theorem 23. *A function $\varphi(\sigma)$ in the interval (α, β) is the Jensen function of a function $f(s)$ almost periodic in $[\alpha, \beta]$ with the finite rational base $\mu_1, \ldots, \mu_m$, i. e. with exponents from the modul $M = \{r_1\mu_1 + \cdots + r_m\mu_m\}$, if and only if it satisfies the following conditions:*

(i) *It is convex in the interval (α, β).*

(ii) *The value of $\varphi'(\sigma)$ in any linearity interval of $\varphi(\sigma)$ belongs to M.*

(iii) *To any reduced interval $(\alpha <)\, \alpha_0 < \sigma < \beta_0 \,(< \beta)$ corresponds a number $k > 0$ such that if σ_1 and σ_2, where $\alpha_0 < \sigma_1 < \sigma_2 < \beta_0$, belong to different linearity intervals of $\varphi(\sigma)$, and the element $\varphi'(\sigma_2) - \varphi'(\sigma_1)$ of M has the representation*

$$\varphi'(\sigma_2) - \varphi'(\sigma_1) = r_1\mu_1 + \cdots + r_m\mu_m,$$

we have the inequality

$$\frac{\varphi'(\sigma_2) - \varphi'(\sigma_1)}{\sqrt{r_1^2 + \cdots + r_m^2}} \geqq k.$$

For $m = 1$ this theorem reduces to Theorem 20, since in this case condition (iii) is implied by (i) and (ii).

104. The necessity of conditions (i) and (ii) follows from Theorems 7 and 8. The necessity of condition (iii) will be established by means of the results of §§ 92—99.

Let $f(s)$ be a function almost periodic in $[\alpha, \beta]$ with exponents from M, which is not identically zero. Using the notations of § 102, we shall consider the functions

$$\Phi_p(\boldsymbol{x}) = \min_{\alpha_0 \leqq \sigma \leqq \beta_0} |g_p(\sigma; \boldsymbol{x})|$$

and

$$\Phi(\boldsymbol{x}) = \min_{\alpha_0 \leqq \sigma \leqq \beta_0} |g(\sigma; \boldsymbol{x})|.$$

Obviously $\Phi_p(\boldsymbol{x})$ converges uniformly towards $\Phi(\boldsymbol{x})$ in R_m as $p \to \infty$. Further $\Phi_p(\boldsymbol{x})$ is periodic with the period $2\pi N_p$ in each of the variables $x_1, \ldots, x_m$, and $\Phi(\boldsymbol{x})$ is not identically zero since

$$\Phi(\boldsymbol{\mu}\tau) = \min_{\alpha_0 \leqq \sigma \leqq \beta_0} |g(\sigma; \boldsymbol{\mu}\tau)| = \min_{\alpha_0 \leqq \sigma \leqq \beta_0} |f(\sigma + i\tau)|.$$

Hence there exist a positive number η and a system of spheres

$$(11) \qquad \|\boldsymbol{x} - (\boldsymbol{x}^* + 2\pi N \boldsymbol{h})\| \leqq a N,$$

where N is a positive integer, $a < \pi$ a positive number, $\boldsymbol{x}^*$ a point of R_m, and $\boldsymbol{h}$ runs through all vectors of R_m with integral coordinates, such that

$$\Phi(\boldsymbol{x}) \geqq \eta$$

for all $\boldsymbol{x}$ belonging to these spheres.

For arbitrary values of σ_1 and σ_2 for which $\alpha_0 < \sigma_1 < \sigma_2 < \beta_0$, and which belong to different linearity intervals of $\varphi(\sigma)$, we may now choose a number p satisfying the following conditions:

(a) The lines $\sigma = \sigma_1$ and $\sigma = \sigma_2$ belong to strips without zeros of the function $f_p(s)$, and $f_p(s)$ has the same mean motion as $f(s)$ on these lines (cf. § 32). This means that σ_1 and σ_2 belong to linearity intervals of the Jensen function $\varphi_p(\sigma)$ of $f_p(s)$, and that

$$\varphi_p'(\sigma_1) = \varphi'(\sigma_1) \quad \text{and} \quad \varphi_p'(\sigma_2) = \varphi'(\sigma_2).$$

(b) For all $\boldsymbol{x}$ belonging to the spheres (11) we have $\Phi_p(\boldsymbol{x}) > 0$, which means that $g_p(\sigma; \boldsymbol{x}) \neq 0$ for all σ in the interval $\alpha_0 < \sigma < \beta_0$.

By the definition of N_p the numbers $\frac{\mu_1}{N_p}, \ldots, \frac{\mu_m}{N_p}$ and hence also the numbers

$\frac{\mu_1}{N_p N}, \ldots, \frac{\mu_m}{N_p N}$ form an integral base for the function $f_p(s)$. The spatial extension of $f_p(s + i\tau)$ corresponding to this base is by (10) the function $g_p(s; \boldsymbol{x} N_p N)$, with the period 2π in each of the variables $x_1, \ldots, x_m$. By condition (a) we have

$$\varphi'(\sigma_2) - \varphi'(\sigma_1) = \varphi_p'(\sigma_2) - \varphi_p'(\sigma_1)$$

and by condition (b) we have $g_p(\sigma; \boldsymbol{x} N_p N) \neq 0$ for all σ in the interval $\alpha_0 < \sigma < \beta_0$ and all $\boldsymbol{x}$ belonging to the spheres

$$\left\| \boldsymbol{x} - \left(\frac{\boldsymbol{x}^*}{N_p N} + \frac{2\pi \boldsymbol{h}}{N_p} \right) \right\| \leqq \frac{a}{N_p}.$$

It follows therefore from §§ 96—97 that

$$\varphi'(\sigma_2) - \varphi'(\sigma_1) = \boldsymbol{v} \frac{\boldsymbol{\mu}}{N_p N} = \nu_1 \frac{\mu_1}{N_p N} + \cdots + \nu_m \frac{\mu_m}{N_p N},$$

where the vector $\boldsymbol{v} = (\nu_1, \ldots, \nu_m)$ has integral coordinates and satisfies the condition

$$\frac{\boldsymbol{v} \dfrac{\boldsymbol{\mu}}{N_p N}}{\| \boldsymbol{v} \| \left\| \dfrac{\boldsymbol{\mu}}{N_p N} \right\|} \geqq b,$$

where $b > 0$ denotes a constant depending only on a and the direction of $\frac{\boldsymbol{\mu}}{N_p N}$, i. e. the direction of $\boldsymbol{\mu}$. Putting

$$\varphi'(\sigma_2) - \varphi'(\sigma_1) = r_1 \mu_1 + \cdots + r_m \mu_m = \boldsymbol{r} \boldsymbol{\mu}$$

we therefore have $\boldsymbol{r} = \frac{\boldsymbol{v}}{N_p N}$ and hence

$$\frac{\boldsymbol{r} \boldsymbol{\mu}}{\| \boldsymbol{r} \| \, \| \boldsymbol{\mu} \|} \geqq b,$$

which proves condition (iii) with $k = b \, \| \boldsymbol{\mu} \|$.

105. To prove the sufficiency of conditions (i)—(iii) we may, by Theorem 19, restrict ourselves to the case where $\varphi(\sigma)$ is either linear in (α, β) or linear in two intervals (α, α_1) and (β_1, β) but not in (α, β).

If $\varphi(\sigma)$ is linear, say $\varphi(\sigma) = c\sigma + d$, the function $f(s) = e^{cs+d}$ is a solution. If $\varphi(\sigma)$ is linear in (α, α_1) and (β_1, β) but not in (α, β) condition (iii) takes the following form: There exists a number $\theta > 0$ such that if σ_1 and σ_2, where $\alpha < \sigma_1 < \sigma_2 < \beta$, belong to different linearity intervals of $\varphi(\sigma)$, and the element $\varphi'(\sigma_2) - \varphi'(\sigma_1)$ has the representation

$$\varphi'(\sigma_2) - \varphi'(\sigma_1) = r_1 \mu_1 + \cdots + r_m \mu_m = \boldsymbol{r}\boldsymbol{\mu},$$

then

$$[\boldsymbol{r}, \boldsymbol{\mu}] \leqq \tfrac{1}{2}\pi - \theta.$$

Corresponding to this θ we now introduce another rational base $\lambda_1, \ldots, \lambda_m$ of the modul M, consisting of the positive elements

$$\lambda_1 = \boldsymbol{r}^{(1)}\boldsymbol{\mu}, \ldots, \lambda_m = \boldsymbol{r}^{(m)}\boldsymbol{\mu}$$

of M, where the vectors $\boldsymbol{r}^{(1)}, \ldots, \boldsymbol{r}^{(m)}$ are chosen such that the cone

$$[\boldsymbol{x}, \boldsymbol{\mu}] \leqq \tfrac{1}{2}\pi - \theta \tag{12}$$

belongs to the part of R_m determined by the parametric representation

$$\boldsymbol{x} = \boldsymbol{r}^{(1)} z_1 + \cdots + \boldsymbol{r}^{(m)} z_m, \quad z_1 > 0, \ldots, z_m > 0. \tag{13}$$

If then σ_1 and σ_2, where $\alpha<\sigma_1<\sigma_2<\beta$, belong to different linearity intervals of $\varphi(\sigma)$, and the element $\varphi'(\sigma_2)-\varphi'(\sigma_1)$ of M by means of the new base has the representation

$$\varphi'(\sigma_2) - \varphi'(\sigma_1) = q_1\lambda_1 + \cdots + q_m\lambda_m,$$

we have the inequalities

$$q_1 > 0, \ldots, q_m > 0.$$

This result may also be expressed as follows: If for an arbitrary σ belonging to a linearity interval of $\varphi(\sigma)$ the element $\varphi'(\sigma)$ of M has the representation

$$\varphi'(\sigma) = q_1(\sigma)\lambda_1 + \cdots + q_m(\sigma)\lambda_m, \tag{14}$$

then the functions $q_1(\sigma), \ldots, q_m(\sigma)$, so far defined only in the linearity intervals of $\varphi(\sigma)$, are increasing functions of σ, each of which assumes different values in different linearity intervals.

106. The representation (14) will now be extended to a representation of $\varphi'(\sigma)$ valid for all σ in (α, β) by means of increasing functions $q_1(\sigma), \ldots, q_m(\sigma)$.

For a value σ_0 not belonging to the linearity intervals of $\varphi(\sigma)$, let us consider the largest closed interval $\{\alpha', \beta'\}$ containing σ_0 but not containing points of any linearity interval of $\varphi(\sigma)$; we then define $q_1(\sigma_0), \ldots, q_m(\sigma_0)$ such that the two $(m+1)$-dimensional vectors

$$\{\varphi'(\sigma_0) - \varphi'(\alpha' - 0), \quad q_1(\sigma_0) - q_1(\alpha' - 0), \ldots, q_m(\sigma_0) - q_m(\alpha' - 0)\}$$

and

$$\{\varphi'(\beta' + 0) - \varphi'(\alpha' - 0), \; q_1(\beta' + 0) - q_1(\alpha' - 0), \ldots, q_m(\beta' + 0) - q_m(\alpha' - 0)\}$$

are proportional. The relation (14) is then valid for all σ.

If $\alpha' = \beta'$, the functions $q_1(\sigma), \ldots, q_m(\sigma)$ will all be continuous or all discontinuous at the point σ_0 according as $\varphi'(\sigma)$ is continuous or discontinuous at σ_0. If $\alpha' < \beta'$, we must have $\varphi'(\beta' + 0) - \varphi'(\alpha' - 0) > 0$, and the vector

$$(q_1(\beta' + 0) - q_1(\alpha' - 0))\, \boldsymbol{r}^{(1)} + \cdots + (q_m(\beta' + 0) - q_m(\alpha' - 0))\, \boldsymbol{r}^{(m)}$$

is therefore not the null-vector; being the limit of vectors belonging to the cone (12), it belongs itself to this cone, and hence to the part of R_m determined by (13), i. e. we have

$$q_1(\beta' + 0) - q_1(\alpha' - 0) > 0, \ldots, q_m(\beta' + 0) - q_m(\alpha' - 0) > 0.$$

This implies that none of the functions $q_1(\sigma), \ldots, q_m(\sigma)$ has any other constancy intervals than the constancy intervals of $\varphi'(\sigma)$.

107. By integration of (14) we arrive at a representation

$$\varphi(\sigma) = \varphi_1(\sigma) + \cdots + \varphi_m(\sigma),$$

where the functions $\varphi_1(\sigma), \ldots, \varphi_m(\sigma)$ are convex functions having the same linearity intervals as $\varphi(\sigma)$. Further, the values of $\varphi_l'(\sigma)$ in the linearity intervals belong to the modul $M_l = \{q_l \lambda_l\}$, where q_l runs through all rational numbers. Hence, by Theorem 20, there exists for each l a function $f_l(s)$ almost periodic in $[\alpha, \beta]$ with exponents from M_l, i. e. a limit periodic function with the limit period $\frac{2\pi}{\lambda_l}$, having the Jensen function $\varphi_l(\sigma)$. The product

$$f(s) = f_1(s) \ldots f_m(s)$$

is then almost periodic in $[\alpha, \beta]$ with exponents from M, and has the Jensen function $\varphi(\sigma)$.

This completes the proof of the theorem.

Functions with an Infinite Rational Base.

108. Finally we turn to the study of almost periodic functions $f(s)$ with an infinite rational base $\mu_1, \mu_2, \ldots$, i. e. with exponents from the modul $M = \{r_1\mu_1 + r_2\mu_2 + \cdots\}$, where the numbers $\mu_1, \mu_2, \ldots$ are linearly independent, and the sequence of coefficients $r_1, r_2, \ldots$ runs through all sequences of rational numbers of which only a finite number are $\neq 0$.

Using in the infinite-dimensional space R_ω the vectorial notation introduced in § 88 we have $M = \{\boldsymbol{r}\boldsymbol{\mu}\}$, where $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots)$, and $\boldsymbol{r} = (r_1, r_2, \ldots)$ runs through all vectors of R_ω with rational coordinates, of which only a finite number are $\neq 0$.

Let $f(s)$ be a function almost periodic in $[\alpha, \beta]$ with exponents from M. Allowing terms with the coefficient 0 we may write its Dirichlet series in the form

$$f(s) \sim \Sigma\, a_{\boldsymbol{r}}\, e^{\boldsymbol{r}\boldsymbol{\mu} s}.$$

Let us now consider a sequence of exponential polynomials of the form

$$f_p(s) = \Sigma\, a_{\boldsymbol{r}}^{(p)}\, e^{\boldsymbol{r}\boldsymbol{\mu} s}$$

where for every p only a finite number of the coefficients $a_{\boldsymbol{r}}^{(p)}$ are $\neq 0$) converging uniformly towards $f(s)$ in $[\alpha, \beta]$ as $p \to \infty$. For each function $f_p(s)$ we consider the function

$$g_p(s;\, \boldsymbol{x}) = \Sigma\, a_{\boldsymbol{r}}^{(p)}\, e^{i\boldsymbol{r}\boldsymbol{x}}\, e^{\boldsymbol{r}\boldsymbol{\mu} s},$$

where $\boldsymbol{x}$ runs through R_ω. This function actually depends only on a finite number of the variables $x_1, x_2, \ldots$ and has in each of these variables the period $2\pi N_p$, where N_p denotes a common denominator of the coordinates of those vectors $\boldsymbol{r}$ for which $a_{\boldsymbol{r}}^{(p)} \neq 0$. Further

$$f_p(s + i\tau) = g_p(s;\, \boldsymbol{\mu}\tau).$$

As $f_p(s)$ converges uniformly towards $f(s)$ in $[\alpha, \beta]$, the function $f_p(s + i\tau)$ converges uniformly towards $f(s + i\tau)$ for s in $[\alpha, \beta]$ and all τ. Since any two of the functions $g_p(s;\, \boldsymbol{x})$ have a common period $2\pi N$ in all the variables $x_1, x_2, \ldots$, it follows therefore, by Kronecker's theorem, that $g_p(s;\, \boldsymbol{x})$ converges uniformly towards a certain limit function $g(s;\, \boldsymbol{x})$ for s in $[\alpha, \beta]$ and all $\boldsymbol{x}$. We have then obviously

$$f(s + i\tau) = g(s;\, \boldsymbol{\mu}\tau).$$

109. The solution of the present case of our problem is given by the following theorem.

Theorem 24. *A function $\varphi(\sigma)$ in the interval (α, β) is the Jensen function of a function $f(s)$ almost periodic in $[\alpha, \beta]$ with the infinite rational base $\mu_1, \mu_2, \ldots$, i. e. with exponents from the modul $M = \{r_1\mu_1 + r_2\mu_2 + \cdots\}$, if and only if it satisfies the following conditions:*

(i) *It is convex in the interval (α, β).*

(ii) *The value of $\varphi'(\sigma)$ in any linearity interval of $\varphi(\sigma)$ belongs to M.*

(iii) *To any reduced interval $(\alpha <)\, \alpha_0 < \sigma < \beta_0 (< \beta)$ there correspond a positive integer m_0 and a number $k > 0$ such that if σ_1 and σ_2, where $\alpha_0 < \sigma_1 < \sigma_2 < \beta_0$, belong to different linearity intervals of $\varphi(\sigma)$, then the difference $\varphi'(\sigma_2) - \varphi'(\sigma_1)$ belongs to the modul $M_{m_0} = \{r_1\mu_1 + \cdots + r_{m_0}\mu_{m_0}\}$ with the finite rational base $\mu_1, \ldots, \mu_{m_0}$, and if its representation is*

$$\varphi'(\sigma_2) - \varphi'(\sigma_1) = r_1\mu_1 + \cdots + r_{m_0}\mu_{m_0}$$

we have the inequality

$$\frac{\varphi'(\sigma_2) - \varphi'(\sigma_1)}{\sqrt{r_1^2 + \cdots + r_{m_0}^2}} \geqq k.$$

110. The necessity of conditions (i) and (ii) follows from Theorems 7 and 8. The necessity of condition (iii) will be established by means of the results of §§ 100—101.

Let $f(s)$ be a function almost periodic in $[\alpha, \beta]$ with exponents from M, and not identically zero. Using the notations of § 108, we consider the functions

$$\Phi_p(x) = \min_{\alpha_0 \leqq \sigma \leqq \beta_0} |g_p(\sigma; x)|$$

and

$$\Phi(x) = \min_{\alpha_0 \leqq \sigma \leqq \beta_0} |g(\sigma; x)|.$$

Obviously $\Phi_p(x)$ converges uniformly towards $\Phi(x)$ in R_ω as $p \to \infty$. Moreover $\Phi_p(x)$ actually depends on only a finite number of the variables $x_1, x_2, \ldots$ and has in each of these variables the period $2\pi N_p$, and $\Phi(x)$ is not identically zero, since

$$\Phi(\mu\tau) = \min_{\alpha_0 \leqq \sigma \leqq \beta_0} |g(\sigma; \mu\tau)| = \min_{\alpha_0 \leqq \sigma \leqq \beta_0} |f(\sigma + i\tau)|.$$

Hence there exist a positive integer m_0 and a positive number η such that

$$\Phi(x) \geqq \eta$$

for all points $x = (x_1, x_2, \ldots)$ in R_ω for which the corresponding point $x' = (x_1, \ldots, x_{m_0})$ in R_{m_0}, which is called the projection of x on R_{m_0}, belongs to a system of spheres

$$\|x' - (x^{*\prime} + 2\pi N h')\| \leqq a N, \tag{15}$$

where N is a positive integer, $a < \pi$ a positive number, $x^{*\prime}$ a point of R_{m_0}, and h' runs through all vectors of R_{m_0} with integral coordinates.

For arbitrary values of σ_1 and σ_2 for which $\alpha_0 < \sigma_1 < \sigma_2 < \beta_0$, and which belong to different linearity intervals of $\varphi(\sigma)$, we may now choose a number p satisfying the following conditions:

(a) The lines $\sigma = \sigma_1$ and $\sigma = \sigma_2$ belong to strips without zeros of the function $f_p(s)$, and $f_p(s)$ has the same mean motion as $f(s)$ on these lines (cf. § 32). This means that σ_1 and σ_2 belong to linearity intervals of the Jensen function $\varphi_p(\sigma)$ of $f_p(s)$, and that

$$\varphi_p'(\sigma_1) = \varphi'(\sigma_1) \quad \text{and} \quad \varphi_p'(\sigma_2) = \varphi'(\sigma_2).$$

(b) For all x for which the projection x' belongs to the spheres (15) we have $\Phi_p(x) > 0$, which means that $g_p(\sigma; x) \neq 0$ for all σ in the interval $\alpha_0 < \sigma < \beta_0$.

Let $m > m_0$ be chosen such that $g_p(s; x)$ is actually a function of the variables s and $x_1, \ldots, x_m$ alone. By an argument analogous to that of § 104, only using the results of § 100 instead of those of § 97, we find that

$$\varphi'(\sigma_2) - \varphi'(\sigma_1) = \nu_1 \frac{\mu_1}{N_p N} + \cdots + \nu_{m_0} \frac{\mu_{m_0}}{N_p N} = \nu' \frac{\mu'}{N_p N},$$

where the vector $\nu' = (\nu_1, \ldots, \nu_{m_0})$ has integral coordinates and satisfies the condition

$$\frac{\nu' \dfrac{\mu'}{N_p N}}{\|\nu'\| \left\| \dfrac{\mu'}{N_p N} \right\|} \geqq b,$$

where $b > 0$ denotes a constant depending only on a and the direction of $\dfrac{\mu'}{N_p N}$, i. e. the direction of μ', and thus in particular not on m. Hence $\varphi'(\sigma_2) - \varphi'(\sigma_1)$ belongs to the modul $M_{m_0} = \{r_1 \mu_1 + \cdots + r_{m_0} \mu_{m_0}\}$, and putting

$$\varphi'(\sigma_2) - \varphi'(\sigma_1) = r_1 \mu_1 + \cdots + r_{m_0} \mu_{m_0} = r' \mu'$$

we have $r' = \frac{v'}{N_p N}$ and hence

$$\frac{r' \mu'}{\|r'\| \, \|\mu'\|} \geqq b,$$

which proves condition (iii) with $k = b \|\mu'\|$.

111. To prove the sufficiency of conditions (i)—(iii) we may by Theorem 19 restrict ourselves to the case where $\varphi(\sigma)$ is either linear in (α, β) or linear in two intervals (α, α_1) and (β_1, β), but not in (α, β).

If $\varphi(\sigma)$ is linear, say $\varphi(\sigma) = c\sigma + d$, the function $f(s) = e^{cs+d}$ is a solution. If $\varphi(\sigma)$ is linear in (α, α_1) and (β_1, β) but not in (α, β) we may by condition (iii) choose a positive integer m such that the difference $\varphi'(\sigma_2) - \varphi'(\sigma_1)$, where σ_1 and σ_2 belong to arbitrary linearity intervals of $\varphi(\sigma)$, belongs to the modul $M_m = \{r_1 \mu_1 + \cdots + r_m \mu_m\}$, and by condition (ii) we may assume that m has been chosen so large that the value of $\varphi'(\sigma)$ for some linearity interval of $\varphi(\sigma)$ also belongs to M_m. The function $\varphi(\sigma)$ then satisfies conditions (i)—(iii) of Theorem 23 with respect to the modul M_m and is therefore the Jensen function of a function $f(s)$ almost periodic in $[\alpha, \beta]$ with exponents from M_m.

Functions with Arbitrary Exponents.

112. For all moduls of the forms $M = \{h_1 \mu_1 + \cdots + h_m \mu_m\}$, $\{h_1 \mu_1 + h_2 \mu_2 + \cdots\}$, $\{r_1 \mu_1 + \cdots + r_m \mu_m\}$, or $\{r_1 \mu_1 + r_2 \mu_2 + \cdots\}$ we have now characterized those functions $\varphi(\sigma)$ which may occur as the Jensen function of an analytic almost periodic function with exponents from the modul in question. Our last Theorem makes it possible also to characterize those functions $\varphi(\sigma)$ which may occur as the Jensen function of an analytic almost periodic function with arbitrary exponents. The result is given by the following theorem.

Theorem 25. *A function $\varphi(\sigma)$ in the interval (α, β) is the Jensen function of a function $f(s)$ almost periodic in $[\alpha, \beta]$ if and only if it satisfies the following conditions:*

(i) *It is convex in the interval (α, β).*

(ii) *For any reduced interval $(\alpha <) \alpha_0 < \sigma < \beta_0 (< \beta)$ there exist a finite set of linearly independent numbers $\mu_1, \ldots, \mu_m$ and a number $k > 0$, such that if σ_1 and σ_2, where $\alpha_0 < \sigma_1 < \sigma_2 < \beta_0$, belong to different linearity intervals of $\varphi(\sigma)$, then the difference $\varphi'(\sigma_2) - \varphi'(\sigma_1)$ is of the form*

$$\varphi'(\sigma_2) - \varphi'(\sigma_1) = r_1 \mu_1 + \cdots + r_m \mu_m,$$

where the coefficients $r_1, \ldots, r_m$ *are rational numbers and*

$$\frac{\varphi'(\sigma_2) - \varphi'(\sigma_1)}{\sqrt{r_1^2 + \cdots + r_m^2}} \geqq k.$$

The necessity of the conditions follows immediately from Theorem 24, since any almost periodic function possesses an infinite rational base.

In order to prove the sufficiency of the conditions we first notice that if a set of positive numbers has a finite rational base $\mu_1, \ldots, \mu_m$ such that for all numbers $a = r_1 \mu_1 + \cdots + r_m \mu_m = \boldsymbol{r} \boldsymbol{\mu}$ of the set the ratio $\frac{a}{\|\boldsymbol{r}\|}$ exceeds a positive constant, then it has this property with respect to any finite rational base $\lambda_1, \ldots, \lambda_l$ of the set. To see this we consider the set of all numbers

$$r_1 \mu_1 + \cdots + r_m \mu_m + q_1 \lambda_1 + \cdots + q_l \lambda_l$$

with arbitrary rational coefficients. This set contains both a finite rational base obtained by enlarging the set $\mu_1, \ldots, \mu_m$ and a finite rational base obtained by enlarging the set $\lambda_1, \ldots, \lambda_l$, and these two bases must contain the same number of elements. It is therefore sufficient to consider the case where $l = m$. Now if in this case $r_1 \mu_1 + \cdots + r_m \mu_m = \boldsymbol{r} \boldsymbol{\mu}$ and $q_1 \lambda_1 + \cdots + q_m \lambda_m = \boldsymbol{q} \boldsymbol{\lambda}$ are the expressions of the same number a of the set, the vectors $\boldsymbol{r}$ and $\boldsymbol{q}$ are connected by a linear substitution, and the ratio $\frac{\|\boldsymbol{r}\|}{\|\boldsymbol{q}\|}$ therefore exceeds a positive constant. This proves the above statement.

If now $\varkappa_1, \varkappa_2, \ldots$ denotes an infinite rational base for the values of $\varphi'(\sigma)$ in the linearity intervals of $\varphi(\sigma)$ it follows from condition (ii) that the values $\varphi'(\sigma)$ belonging to the linearity intervals contained in a reduced interval $(\alpha <)\, \alpha_0 < \sigma < \beta_0\, (< \beta)$ have the finite rational base $\varkappa_1, \ldots, \varkappa_{m_0}$ for some m_0. Hence with respect to the base $\varkappa_1, \varkappa_2, \ldots$ the function $\varphi(\sigma)$ satisfies the conditions of Theorem 24 and is therefore the Jensen function of a function $f(s)$ almost periodic in $[\alpha, \beta]$.

113. A consequence of Theorem 25 is the existence of convex functions $\varphi(\sigma)$ which are not the Jensen function of an analytic almost periodic function. It is sufficient to consider a convex function $\varphi(\sigma)$ having in a reduced interval an infinity of linearity intervals for which the corresponding values of $\varphi'(\sigma)$ are linearly independent.

114. As is easily seen, the preceding theorems admit of the following uniform formulation.

Denoting by M an arbitrary modul of one of the forms $\{h_1\mu_1 + \cdots + h_m\mu_m\}$, $m \geqq 2$, $\{h_1\mu_1 + h_2\mu_2 + \cdots\}$, $\{r_1\mu_1 + \cdots + r_m\mu_m\}$, or $\{r_1\mu_1 + r_2\mu_2 + \cdots\}$, or the modul of all real numbers, a function $\varphi(\sigma)$ in the interval (α, β) is the Jensen function of a function $f(s)$ almost periodic in $[\alpha, \beta]$ with exponents from M if and only if it satisfies the following conditions.

(i) It is convex in the interval (α, β).

(ii) The value of $\varphi'(\sigma)$ in any linearity interval of $\varphi(\sigma)$ belongs to M.

(iii) For any reduced interval $(\alpha <)\, \alpha_0 < \sigma < \beta_0 (< \beta)$ there exist a finite set of linearly independent numbers $\lambda_1, \ldots, \lambda_l$ and a number $k > 0$, such that if σ_1 and σ_2, where $\alpha_0 < \sigma_1 < \sigma_2 < \beta_0$, belong to different linearity intervals of $\varphi(\sigma)$, then the difference $\varphi'(\sigma_2) - \varphi'(\sigma_1)$ is of the form

$$\varphi'(\sigma_2) - \varphi'(\sigma_1) = q_1\lambda_1 + \cdots + q_l\lambda_l,$$

where the coefficients $q_1, \ldots, q_l$ are rational numbers and

$$\frac{\varphi'(\sigma_2) - \varphi'(\sigma_1)}{\sqrt{q_1^2 + \cdots + q_l^2}} \geqq k.$$

It is natural to ask whether this theorem remains true for an arbitrary everywhere dense modul M. The necessity of the conditions is obvious in all cases, and in some cases other than those considered above the sufficiency may be proved by similar arguments, for instance in the case of all »mixed» moduls, i. e. moduls of the type $\{g_1\mu_1 + \cdots + g_m\mu_m\}$ or $\{g_1\mu_1 + g_2\mu_2 + \cdots\}$, where the numbers μ_i are linearly independent while some of the coefficients g_i run through all integers and the rest through all rational numbers. We do not know whether the conditions are sufficient in all cases.

CHAPTER VII.

Analytic Almost Periodic Functions with Integral Base and Analytic Spatial Extension.

Functions with a Finite Integral Base.

115. Let $f(s)$ be a function almost periodic in $[\alpha, \beta]$ with a finite integral base $\mu_1, \ldots, \mu_m$ and let $g(s; x) = g(s; x_1, \ldots, x_m)$ denote the corresponding spatial extension introduced in § 83. We shall now consider the case where this function $g(s; x)$ is a regular function not only of the complex variable s for given values of the real variables $x_1, \ldots, x_m$ but of all the variables $s, x_1, \ldots, x_m$.[1] This property is easily seen to be independent of the particular choice of the base, but this is not important for the following considerations which are founded on a definite choice of the base. We express the said property briefly by saying that $f(s)$ has an *analytic spatial extension.*

We shall now prove the following theorem.

Theorem 26. *Let $f(s)$ be a function almost periodic in $[\alpha, \beta]$ with a finite integral base and an analytic spatial extension, and not identically zero. Then the mean motions $c^-(\sigma)$ and $c^+(\sigma)$ exist for every σ in (α, β) and are determined by*

$$c^-(\sigma) = \varphi'(\sigma - 0) \quad \textit{and} \quad c^+(\sigma) = \varphi'(\sigma + 0). \tag{1}$$

Further the frequency $H(\sigma_1, \sigma_2)$ of zeros exists for every strip (σ_1, σ_2), where $\alpha < \sigma_1 < \sigma_2 < \beta$, and is determined by

$$H(\sigma_1, \sigma_2) = \frac{1}{2\pi}(\varphi'(\sigma_2 - 0) - \varphi'(\sigma_1 + 0)). \tag{2}$$

In every reduced interval $(\alpha <) \alpha_1 < \sigma < \beta_1 (< \beta)$ there exist at most a finite number of values of σ for which $\varphi(\sigma)$ is not differentiable.

By a remark in § 40 this theorem implies that the mean motion $c(\sigma)$ of the function $f(\sigma + it)$, according to the definition in § 27, exists for every σ in (α, β) and is determined by the mean derivative

$$c(\sigma) = \tfrac{1}{2}(\varphi'(\sigma - 0) + \varphi'(\sigma + 0)).$$

[1] Since $x_1, \ldots, x_m$ are real variables the regularity in all the variables $s, x_1, \ldots, x_m$ means more than regularity in each of the variables $s, x_1, \ldots, x_m$ for fixed values of the remaining variables.

116. The proof is based on the *Kronecker-Weyl theorem*, according to which the points of the line $x = \mu t = (\mu_1 t, \ldots, \mu_m t)$, $-\infty < t < +\infty$, in R_m are not only everywhere densely distributed mod. 2π when $\mu_1, \ldots, \mu_m$ are linearly independent, but even equidistributed mod. 2π. We shall use this theorem in the following form: Let $P(x) = P(x_1, \ldots, x_m)$ denote a function in R_m with the period 2π in each of the variables, which is integrable in the Riemann sense and for which the function $P(\mu t)$ of the real variable t is also integrable in the Riemann sense. Then the mean value

$$\underset{t}{M}\{P(\mu t)\} = \lim_{(\delta-\gamma)\to\infty} \frac{1}{\delta-\gamma}\int_{\gamma}^{\delta} P(\mu t)\,dt$$

exists and is equal to the mean value

$$\underset{x}{M}\{P(x)\} = \frac{1}{(2\pi)^m}\int_0^{2\pi}\cdots\int_0^{2\pi} P(x_1, \ldots, x_m)\,dx_1 \ldots dx_m.$$

We begin the proof with the remark that the existence of the left mean motion $c^-(\sigma)$ for a given value of σ is equivalent to the existence of a mean value of the stretchwise continuous function $\arg^- f(\sigma + i(t+\frac{1}{2})) - \arg^- f(\sigma + i(t-\frac{1}{2}))$, and that when it exists $c^-(\sigma)$ is equal to this mean value:

$$c^-(\sigma) = \underset{t}{M}\{\arg^- f(\sigma + i(t+\tfrac{1}{2})) - \arg^- f(\sigma + i(t-\tfrac{1}{2}))\}.$$

This follows from Theorem 3 (iv), which shows that, except for a bounded remainder, the quantity

$$\int_{\gamma}^{\delta}[\arg^- f(\sigma + i(t+\tfrac{1}{2})) - \arg^- f(\sigma + i(t-\tfrac{1}{2}))]\,dt = -\int_{\gamma-\frac{1}{2}}^{\gamma+\frac{1}{2}} + \int_{\delta-\frac{1}{2}}^{\delta+\frac{1}{2}} \arg^- f(\sigma + it)\,dt$$

is equal to $\arg^- f(\sigma + i\delta) - \arg^- f(\sigma + i\gamma)$. Similarly

$$c^+(\sigma) = \underset{t}{M}\{\arg^+ f(\sigma + i(t+\tfrac{1}{2})) - \arg^+ f(\sigma + i(t-\tfrac{1}{2}))\},$$

where the two sides exist simultaneously.

Let us now consider the spatial extension $g(s; x)$ and for an arbitrary x in R_m denote by $a^-(\sigma; x)$ and $a^+(\sigma; x)$ the variation of the argument of $g(s; x)$ along the left or right side of the straight segment from $\sigma - i\frac{1}{2}$ to $\sigma + i\frac{1}{2}$. Since $f(s + i\tau) = g(s; \mu\tau)$ the preceding relations then take the form

$$c^-(\sigma) = \underset{t}{M}\{a^-(\sigma; \mu t)\} \quad \text{and} \quad c^+(\sigma) = \underset{t}{M}\{a^+(\sigma; \mu t)\},$$

where in both cases the two sides exist simultaneously.

From § 38 it follows that $a^-(\sigma; x)$ and $a^+(\sigma; x)$ are bounded for σ in $[\alpha, \beta]$ and all x. It will be shown below that for an arbitrary σ they are Riemann integrable functions of x. If we assume this for the moment it at once follows from the Kronecker-Weyl theorem that the mean motions $c^-(\sigma)$ and $c^+(\sigma)$ both exist and are given by

$$c^-(\sigma) = \underset{x}{M}\{a^-(\sigma; x)\} \quad \text{and} \quad c^+(\sigma) = \underset{x}{M}\{a^+(\sigma; x)\}. \tag{3}$$

From these expressions we easily arrive at the expressions (1). For the functions $a^-(\sigma; x)$ and $a^+(\sigma; x)$ are for any fixed x, considered as functions of σ, continuous from the left and right respectively; it therefore follows from (3) (by the theorem on bounded convergence) that $c^-(\sigma)$ and $c^+(\sigma)$ are continuous from the left and right respectively, and since both are equal to $\varphi'(\sigma)$ at the points where $\varphi(\sigma)$ is differentiable, this implies the relations (1). The existence of $H(\sigma_1, \sigma_2)$ and the relation (2) is an immediate consequence of (1), on account of § 42.

117. Still arguing on the basis of the Riemann integrability of the functions $a^-(\sigma; x)$ and $a^+(\sigma; x)$ we deduce from (1) and (3) the relation

$$\varphi'(\sigma + 0) - \varphi'(\sigma - 0) = \underset{x}{M}\{a^+(\sigma; x)\} - \underset{x}{M}\{a^-(\sigma; x)\} = \underset{x}{M}\{a^+(\sigma; x) - a^-(\sigma; x)\}.$$

But

$$a^+(\sigma; x) - a^-(\sigma; x) = 2\pi n(\sigma; x),$$

where $n(\sigma; x)$ denotes the number of zeros of $g(s; x)$ on the segment $s = \sigma + it$, $-\frac{1}{2} \leqq t \leqq \frac{1}{2}$, a zero at an end-point of the interval being counted with only half of its order[1]. The function $n(\sigma; x)$ is therefore also Riemann integrable, and we have

$$\varphi'(\sigma + 0) - \varphi'(\sigma - 0) = 2\pi \underset{x}{M}\{n(\sigma; x)\}, \tag{4}$$

which shows that $\varphi(\sigma)$ is differentiable at the point σ if and only if

$$\underset{x}{M}\{n(\sigma; x)\} = 0.$$

The theorem will therefore be proved if we prove first that for every σ in the interval (α, β) the functions $a^-(\sigma; x)$ and $a^+(\sigma; x)$ are Riemann integrable, and secondly that in every reduced interval (α_1, β_1) the mean value $\underset{x}{M}\{n(\sigma; x)\}$ is positive only for a finite number of values of σ.

[1] Thus $n(\sigma; x)$ is not necessarily an integer.

118. In order to prove the Riemann integrability of $a^-(\sigma; x)$ and $a^+(\sigma; x)$ for a given σ it is sufficient to prove the Riemann integrability of $n(\sigma; x)$.

To make this clear we use the theorem that a function is Riemann integrable if and only if it is bounded and is discontinuous only in a (Lebesgue) null-set. Since in our case the functions are known to be bounded it is sufficient to prove that if $n(\sigma; x)$ is discontinuous only in a null-set the functions $a^-(\sigma; x)$ and $a^+(\sigma; x)$ are also discontinuous only in null-sets. Here we may neglect the set of points x for which $g(s; x)$ has a zero at one of the end-points of the segment $s = \sigma + it$, $-\frac{1}{2} \leqq t \leqq \frac{1}{2}$, i. e. for which either $g(\sigma - i\frac{1}{2}; x) = 0$ or $g(\sigma + i\frac{1}{2}; x) = 0$. For the function $g(s_0; x)$ is for every fixed s_0 an analytic function of $x_1, \ldots, x_m$ and not identically zero, since this would imply $f(s_0 + i\tau) = g(s_0; \mu\tau) = 0$ for all τ, which is impossible; hence $g(s_0; x) = 0$ only in a null-set. For a point x for which $g(\sigma - i\frac{1}{2}; x) \neq 0$ and $g(\sigma + i\frac{1}{2}; x) \neq 0$ it is, however, obvious that the continuity of $n(\sigma; x)$ implies the continuity of $a^-(\sigma; x)$ and $a^+(\sigma; x)$.

Thus it now only remains to be proved first that for every σ in the interval (α, β) the function $n(\sigma; x)$ is discontinuous only in a null-set, and secondly that in every reduced interval (α_1, β_1) the mean value $\underset{x}{M}\{n(\sigma; x)\}$ is positive only for a finite number of values of σ.

119. For a given reduced interval (α_1, β_1) and an arbitrary point x_0 in R_m we choose a rectangle

$$S(x_0):\ \alpha_1 - \varepsilon \leqq \sigma \leqq \beta_1 + \varepsilon,\ \ -\tfrac{1}{2} - \varepsilon \leqq t \leqq \tfrac{1}{2} + \varepsilon,$$

where $0 < \varepsilon = \varepsilon(x_0) < \min(\alpha_1 - \alpha, \beta - \beta_1)$, on the boundary of which $g(s; x_0)$ is $\neq 0$, and therefore has a positive lower bound $k = k(x_0)$. Next we choose an open interval $I(x_0)$ in R_m containing x_0 such that $|g(s; x) - g(s; x_0)| < k$ on the boundary of $S(x_0)$ when x belongs to $I(x_0)$. By Rouché's theorem the functions $g(s; x)$ have then for all x belonging to $I(x_0)$ the same number $p = p(x_0)$ $(\geqq 0)$ of zeros in $S(x_0)$. Let

$$(5) \qquad s^p + A_1(x)s^{p-1} + \cdots + A_p(x)$$

denote the polynomial having these zeros, so that the coefficients $A_1(x), \ldots, A_p(x)$ are the elementary symmetric functions of the zeros. Since $g(s; x)$ is regular in all the variables $s, x_1, \ldots, x_m$ the functions $A_1(x), \ldots, A_p(x)$ are then, according to Weierstrass' 'Vorbereitungssatz', regular functions of $x_1, \ldots, x_m$ in $I(x_0)$.[1]

[1] The 'Vorbereitungssatz' deals only with the neighbourhood of a zero. The above statement is, however, an easy consequence. To prove the functions $A_1(x), \ldots, A_p(x)$ to be regular in the

We now put $s=\sigma+it$ in the polynomial (5) and develop with respect to t; after division by i^n it then takes the form

$$t^p + B_1(\sigma; x)\, t^{p-1} + \cdots + B_p(\sigma; x), \tag{6}$$

where the coefficients $B_1(\sigma; x), \ldots, B_p(\sigma; x)$ are polynomials of σ whose coefficients are regular functions of $x_1, \ldots, x_m$ in $I(x_0)$. Thus the number $n(\sigma; x)$ is for every σ in (α_1, β_1) and every x in $I(x_0)$ determined as the number of zeros of the polynomial (6) in the interval $-\frac{1}{2} \leqq t \leqq \frac{1}{2}$, a zero at an end-point of the interval being counted with only half of its order.

By multiplying (6) with the conjugate polynomial

$$t^p + \overline{B_1(\sigma; x)}\, t^{p-1} + \cdots + \overline{B_p(\sigma; x)}$$

we arrive at a polynomial

$$t^{2p} + C_1(\sigma; x)\, t^{2p-1} + \cdots + C_{2p}(\sigma; x),$$

where the coefficients $C_1(\sigma; x), \ldots, C_{2p}(\sigma; x)$ are polynomials of σ whose coefficients are *real* regular functions of $x_1, \ldots, x_m$ in $I(x_0)$. Thus the number $n(\sigma; x)$ is half the number of zeros of this polynomial in the interval $-\frac{1}{2} \leqq t \leqq \frac{1}{2}$, a zero at an end-point of the interval being counted with only half of its order.

120. We now apply the following lemma, which is an easy consequence of Sturm's theorem:

For every positive integer n and every interval $t_1 \leqq t \leqq t_2$ there exist a finite number of polynomials $P_1(a_1, \ldots, a_n), \ldots, P_q(a_1, \ldots, a_n)$ with real coefficients such that the number of zeros of a real polynomial

$$Q(t) = t^n + a_1 t^{n-1} + \cdots + a_n$$

in the interval $t_1 \leqq t \leqq t_2$, where a zero at an end-point is to be counted with only half of its order, depends only on the signs $\operatorname{sign} P_1(a_1, \ldots, a_n), \ldots, \operatorname{sign} P_q(a_1, \ldots, a_n)$ of the values of these polynomials. By the sign of a real number ξ we mean $+1$, 0, or -1 according as $\xi > 0$, $\xi = 0$, or $\xi < 0$.[1]

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neighbourhood of a given point x^* of $I(x_0)$ we need only apply the theorem to each of the zeros of $g(s; x^*)$ in $S(x_0)$. Our statement may also be proved directly by an immediate extension of the usual proof (see e. g. Osgood [1]) of the 'Vorbereitungssatz'.

[1] Sturm's theorem gives, in fact, a definite procedure for the determination of this number; this procedure consists of the following steps: (i) Decision as to whether t_1 or t_2 are roots of $Q(t)$ and of what order, and division of $Q(t)$ by the corresponding powers of $t-t_1$ and $t-t_2$. (ii) Determination of the number of zeros of the quotient $R(t)$ in $t_1 < t < t_2$, multiple zeros being counted only once, by means of Sturm's chain formed from the functions $R(t)$ and $R'(t)$. (iii) Determina-

On applying this lemma we see that there exist a finite number of functions

(7) $$D_1(\sigma; x), \ldots, D_l(\sigma; x),$$

each of which is a polynomial in σ whose coefficients are real regular functions of $x_1, \ldots, x_m$ in $I(x_0)$, such that for an arbitrary σ in (α_1, β_1) and an arbitrary x in $I(x_0)$ the number $n(\sigma; x)$ depends only on the signs of the values of these functions. This means that there exists a function $\Psi(\eta_1, \ldots, \eta_l)$ defined for the 3^l combinations of l signs $\eta_1, \ldots, \eta_l$, such that

(8) $$n(\sigma; x) = \Psi(\text{sign } D_1(\sigma; x), \ldots, \text{sign } D_l(\sigma; x))$$

for all σ in (α_1, β_1) and all x in $I(x_0)$.

Evidently we may assume that none of the functions (7) is identically zero in σ and x.

121. In order to complete the proof of the theorem it is now, by Borel's covering theorem, sufficient to prove first that, for every σ in (α_1, β_1), the sub-set of $I(x_0)$ in which $n(\sigma; x)$ is discontinuous is a null-set, and secondly that the integral

(9) $$\int_{I(x_0)} n(\sigma; x)\, dx_1 \ldots dx_m$$

is positive only for a finite number of values of σ in (α_1, β_1).

The first of these statements is an immediate consequence of (8) since for a given σ in (α_1, β_1) each of the functions sign $D_j(\sigma; x)$ is discontinuous only in a null-set, the function $D_j(\sigma; x)$ being a regular function of $x_1, \ldots, x_m$ in $I(x_0)$ and therefore either identically zero or zero only in a null-set.

In order to prove the second statement we have to prove that for each set of signs $\eta_1, \ldots, \eta_l$ for which $\Psi(\eta_1, \ldots, \eta_l) > 0$ the measure of the set of points in $I(x_0)$ defined by the relations

(10) $$\text{sign } D_1(\sigma; x) = \eta_1, \ldots, \text{sign } D_l(\sigma; x) = \eta_l$$

is positive only for a finite number of values of σ in (α_1, β_1).

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tion of the number of zeros of $R(t)$ in $t_1 < t < t_2$ of multiplicity $\geqq 2$ by means of Sturm's chain formed from the functions $R_1(t)$ and $R_1'(t)$, where $R_1(t)$ is the greatest common divisor of $R(t)$ and $R'(t)$. (iv) Determination of the number of zeros of $R(t)$ in $t_1<t<t_2$ of multiplicity $\geqq 3$, $\geqq 4$, etc. by repetition of this process. — This procedure involves only rational calculations and the determination of the signs of the values of polynomials at the points t_1 and t_2, and there are altogether only a finite number of possibilities for its course; which shows the truth of our lemma.

If all $\eta_j \neq 0$ the set is empty for all σ in (α_1, β_1), for otherwise there would, on account of the continuity of the functions $D_j(\sigma; x)$, exist a sub-interval (α^*, β^*) of (α_1, β_1) and a sub-interval I^* of $I(x_0)$ such that the equations (10) would be satisfied for all σ in (α^*, β^*) and all x in I^*; hence the integral (9) would be positive for all σ in (α^*, β^*), which is impossible since on account of (4) it is positive at most for an enumerable number of values σ.

It is therefore sufficient to prove that for each j the measure of the set of points in $I(x_0)$ defined by the relation $\operatorname{sign} D_j(\sigma; x) = 0$, i. e. by the relation $D_j(\sigma; x) = 0$, is positive only for a finite number of values of σ. This, however, is clear. For the relation $D_j(\sigma; x) = 0$ is for a given value of σ satisfied in a set in $I(x_0)$ of positive measure only if $D_j(\sigma; x)$ is identically zero in x for this value of σ, and as $D_j(\sigma; x)$ is a polynomial in σ that may happen for at most as many values of σ as the degree indicates.

This completes the proof of the theorem.[1]

122. An arbitrary *exponential polynomial*

$$f(s) = \sum_{n=0}^{N} a_n e^{\lambda_n s}. \tag{11}$$

possesses a finite integral base $\mu_1, \ldots, \mu_m$. Let

$$\lambda_n = h_{n1}\mu_1 + \cdots + h_{nm}\mu_m$$

be the expressions of the exponents by means of the base. The spatial extension is then

$$g(s; x) = \sum_{n=0}^{N} a_n e^{i(h_{n1}x_1 + \cdots + h_{nm}x_m)} e^{\lambda_n s},$$

which is evidently a regular function of all the variables $s, x_1, \ldots, x_m$. The preceding theorem is therefore applicable, and gives in particular the following solution of Lagrange's problem.

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[1] An immediate consequence of Theorem 26 is that if $N(\sigma; \gamma, \delta)$ denotes the number of zeros of $f(s)$ on the segment $s = \sigma + it$, $\gamma < t < \delta$, then the limit

$$\lim_{(\delta-\gamma)\to\infty} \frac{N(\sigma; \gamma, \delta)}{\delta - \gamma}$$

exists for every σ in (α, β) and is equal to

$$\frac{1}{2\pi}\big(\varphi'(\sigma+0) - \varphi'(\sigma-0)\big).$$

See in this connection Kac, van Kampen, and Wintner [1].

Theorem 27. *An arbitrary exponential polynomial*

$$F(t) = \sum_{n=0}^{N} a_n e^{i\lambda_n t}$$

of the real variable t possesses a mean motion c determined by the expression

$$c = \tfrac{1}{2}(\varphi'(-0) + \varphi'(+0)),$$

where $\varphi(\sigma)$ is the Jensen function of

$$f(s) = \sum_{n=0}^{N} a_n e^{\lambda_n s}.$$

In the special case mentioned in § 1, where $N = 1$ and $|a_0| = |a_1|$, so that none of the terms in $F(t)$ is preponderant, the theorem again gives Lagrange's expression $c = \frac{1}{2}(\lambda_0 + \lambda_1)$ for the mean motion.[1] [2]

Regarding the general properties of the Jensen function $\varphi(\sigma)$ of an exponential polynomial (11) we notice that if the notations are chosen such that $\lambda_0 < \cdots < \lambda_N$, then, by Theorem 9 (and the corresponding theorem for functions whose exponents are bounded below), we have $\varphi(\sigma) = \lambda_0\sigma + \log|a_0|$ for all $\sigma <$ (some) α_0 and $\varphi(\sigma) = \lambda_N\sigma + \log|a_N|$ for all $\sigma >$ (some) β_0. It follows therefore from Theorems 21 and 26 that in the whole interval $(-\infty, +\infty)$ the function $\varphi(\sigma)$ possesses only a finite number of linearity intervals and a finite number of points of non-differentiability.

A more precise result regarding the linearity intervals follows from Theorems 2 and 8, which show that the values of $\varphi'(\sigma)$ in these intervals belong to the finite set of numbers which may be written both in the form $h_0\lambda_0 + \cdots + h_N\lambda_N$ with integral coefficients h_n with the sum 1 and in the form $r_0\lambda_0 + \cdots + r_N\lambda_N$ with non-negative rational coefficients r_n with the sum 1.

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[1] For in this case the function $f(s)$ has no zeros outside the line $\sigma = 0$ so that $\varphi(\sigma)$ is linear for $\sigma < 0$ and $\sigma > 0$. If, for instance, we assume $\lambda_0 < \lambda_1$ the first term is preponderant for $\sigma < 0$, whereas the second is preponderant for $\sigma > 0$. Hence $\varphi'(\sigma) = \lambda_0$ for $\sigma < 0$ and $\varphi'(\sigma) = \lambda_1$ for $\sigma > 0$, so that $\varphi'(-0) = \lambda_0$ and $\varphi'(+0) = \lambda_1$. Thus $c = \frac{1}{2}(\lambda_0 + \lambda_1)$.

[2] The remainder $\psi(t)$ in the formula $\arg F(t) = ct + \psi(t)$ has been studied by Wintner [10] who has proved that, if the exponents and the moduli of the coefficients are given, then $\psi'(t)$ is almost periodic in Besicovitch's generalized sense for almost all sets of values of the arguments of the coefficients. Our method easily shows that the difference $\psi(t+\frac{1}{2}) - \psi(t-\frac{1}{2})$ is almost periodic in Weyl's generalized sense in all cases. By a simple transformation we obtain the more general result that $\psi(t+k) - \psi(t-k)$ is almost periodic in Weyl's sense for an arbitrary k. On the other hand, it may be shown without difficulty that

$$\limsup_{(\delta-\gamma)\to\infty} \frac{1}{\delta-\gamma} \int_{\gamma}^{\delta} \left| \frac{\psi(t+k) - \psi(t-k)}{2k} - \psi'(t) \right| dt \to 0 \quad \text{as} \quad k \to 0.$$

Hence, also $\psi'(t)$ is almost periodic in Weyl's sense.

123. An extensive class of almost periodic functions with a finite integral base and an analytic spatial extension is given in the following theorem.

Theorem 28. *If the almost periodic function*

$$f(s) \sim \Sigma a_n e^{\lambda_n s}$$

possesses a finite integral base $\mu_1, \ldots, \mu_m$ of negative numbers such that in the expressions

$$\lambda_n = h_{n1}\mu_1 + \cdots + h_{nm}\mu_m$$

of the exponents the integral coefficients h_{nl} are all non-negative, then $f(s)$ has an analytic spatial extension.

Since the exponents λ_n are all $\leqq 0$ the function $f(s)$ may according to Bohr [10], if it is almost periodic in $[\alpha, \beta]$, be continued in the half-plane $(\alpha, +\infty)$, and is almost periodic in $[\alpha, +\infty)$. For $\sigma \to +\infty$ it converges uniformly in t towards the constant term of the Dirichlet series. A sequence of exponential polynomials

$$f_p(s) = \Sigma a_n^{(p)} e^{\lambda_n s}$$

(where for every p only a finite number of the coefficients $a_n^{(p)}$ are $\neq 0$) converging uniformly towards $f(s)$ in $[\alpha, \beta]$ will also converge uniformly in $[\alpha, +\infty)$, and the sequence

$$g_p(s; x_1, \ldots, x_m) = \Sigma a_n^{(p)} e^{i(h_{n1}x_1 + \cdots + h_{nm}x_m)} e^{\lambda_n s}$$

will converge uniformly towards the spatial extension $g(s; x_1, \ldots, x_m)$ for s in $[\alpha, +\infty)$ and all $x_1, \ldots, x_m$.

Now the function

$$h_p(s; z_1, \ldots, z_m) = \Sigma a_n^{(p)} z_1^{h_{n1}} \ldots z_m^{h_{nm}} e^{\lambda_n s}$$

is for every fixed s a polynomial of the complex variables $z_1, \ldots, z_m$ and we therefore find by the maximum modulus principle that for arbitrary p and q

$$\underset{x_1, \ldots, x_m}{\text{upper bound}} |g_p - g_q| = \underset{|z_1| \leqq 1, \ldots, |z_m| \leqq 1}{\text{upper bound}} |h_p - h_q|.$$

Hence $h_p(s; z_1, \ldots, z_m)$ converges uniformly for s in $[\alpha, +\infty)$ and $|z_1| \leqq 1, \ldots, |z_m| \leqq 1$ towards a limit function $h(s; z_1, \ldots, z_m)$ regular for $\sigma > \alpha$ and $|z_1| < 1, \ldots, |z_m| < 1$.

For every $\delta > 0$ we have

$$g_p(s; x_1, \ldots, x_m) = h_p(s - \delta; e^{\mu_1 \delta} e^{i x_1}, \ldots, e^{\mu_m \delta} e^{i x_m}).$$

Since the numbers $\mu_1, \ldots, \mu_m$ are negative this implies that for $\sigma > \alpha + \delta$ and all $x_1, \ldots, x_m$

$$g(s; x_1, \ldots, x_m) = h(s - \delta; e^{\mu_1 \delta} e^{i x_1}, \ldots, e^{\mu_m \delta} e^{i x_m}).$$

Hence $g(s; x_1, \ldots, x_m)$ is regular for $\sigma > \alpha + \delta$ and all $x_1, \ldots, x_m$. As $\delta > 0$ is arbitrary the theorem is hereby proved.[1]

Functions with an Infinite Integral Base.

124. In order to extend the preceding results to functions with an infinite integral base, we must first to the definitions of § 89 (of continuity etc. of functions of an infinite number of variables) add a definition of regularity.

A function $F(x) = F(x_1, x_2, \ldots)$ defined in an interval in R_ω (or the whole of R_ω) is called a regular function of all the variables $x_1, x_2, \ldots$ if it is continuous and if for every m it is a regular function of $x_1, \ldots, x_m$ for arbitrary fixed values of $x_{m+1}, x_{m+2}, \ldots$. Similarly a function $g(s; x) = g(s; x_1, x_2, \ldots)$ of a complex variable s describing a strip (α, β) and the real variables $x_1, x_2, \ldots$ is called regular in all the variables $s, x_1, x_2, \ldots$ if it is continuous and if for every m it is regular in the variables $s, x_1, \ldots, x_m$ for arbitrary fixed values of $x_{m+1}, x_{m+2}, \ldots$.

Concerning this notion of regularity, which, though very weak, is sufficient for our purpose, we notice that it is not invariant even under very simple linear transformations. This is seen from the following example.

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[1] The power series of $h(s; z_1, \ldots, z_m)$, considered as a function of $z_1, \ldots, z_m$ for a fixed s in $(\alpha, +\infty)$, is obtained from the expression of $h_p(s; x_1, \ldots, x_m)$ by a formal passage to the limit, and therefore, for $|z_1| < 1, \ldots, |z_m| < 1$, we have

$$h(s; z_1, \ldots, z_m) = \sum a_n z_1^{h_{n1}} \ldots z_m^{h_{nm}} e^{\lambda_n s},$$

where the series is absolutely convergent. If we put

$$z_1 = e^{\mu_1 \delta}, \ldots, z_m = e^{\mu_m \delta},$$

where $\delta > 0$, this shows that the Dirichlet series of $f(s)$ is absolutely convergent for $\sigma > \alpha + \delta$, and hence, since $\delta > 0$ is arbitrary, for $\sigma > \alpha$.

Let the integer $a > 1$ and $0 < b < 1$ be values corresponding to a non-differentiable Weierstrass function

$$H(\xi) = \sum_{n=1}^{\infty} b^n e^{i a^n \xi}.$$

The function

$$F(x) = F(x_1, x_2, \ldots) = \sum_{n=1}^{\infty} b^n e^{i a^n x_n}$$

is then evidently a regular function of $x_1, x_2, \ldots$, but by the substitution

$$x_1 = y_1, \quad x_n = y_1 + y_n \quad \text{for} \quad n > 1$$

we obtain the function $F(y_1, y_1 + y_2, y_1 + y_3, \ldots)$, which is no regular function of $y_1, y_2, \ldots$ since for $y_2 = y_3 = \cdots = 0$ it reduces to $F(y_1, y_1, y_1, \ldots) = H(y_1)$.

125. Let $f(s)$ be a function almost periodic in $[\alpha, \beta]$ with an infinite integral base $\mu_1, \mu_2, \ldots$, and let $g(s; x) = g(s; x_1, x_2, \ldots)$ be the spatial extension introduced in § 90. If this function $g(s; x)$ is a regular function of all the variables $s, x_1, x_2, \ldots$ we will say that $f(s)$ has an analytic spatial extension with respect to the base $\mu_1, \mu_2, \ldots$.

This does not imply that with respect to any other infinite integral base the spatial extension is also analytic, as is seen from the example

$$f(s) = \sum_{n=1}^{\infty} b^n e^{a^n \mu_n s},$$

where a and b are chosen as in § 124, and where $\mu_1, \mu_2, \ldots$ are linearly independent numbers such that the sequence $a^n \mu_n$ is bounded. The series is then uniformly convergent in $[-\infty, +\infty]$, so that $f(s)$ is almost periodic in $[-\infty, +\infty]$; further $f(s)$ has the infinite integral base $\mu_1, \mu_2, \ldots$ and the corresponding spatial extension is

$$g(s; x) = \sum_{n=1}^{\infty} b^n e^{i a^n x_n} e^{a^n \mu_n s},$$

which is a regular function of all the variables $s, x_1, x_2, \ldots$. But $f(s)$ has also the infinite integral base $\lambda_1, \lambda_2, \ldots$, where

$$\mu_1 = \lambda_1, \quad \mu_n = \lambda_1 + \lambda_n \quad \text{for} \quad n > 1,$$

and with respect to this base the spatial extension is $g(s; y_1, y_1 + y_2, y_1 + y_3, \ldots)$, which for $s = 0$ and $y_2 = y_3 = \cdots = 0$ reduces to $g(0; y_1, y_1, y_1, \ldots) = H(y_1)$, and is therefore not a regular function of all the variables $s, y_1, y_2, \ldots$.

This dependence on the base is not important for our considerations, which are founded on a definite choice of the base.

We shall now prove the following theorem.

Theorem 29. *Let $f(s)$ be a function almost periodic in $[\alpha, \beta]$ with an infinite integral base and analytic spatial extension with respect to some such base, and not identically zero. Then the mean motions $c^-(\sigma)$ and $c^+(\sigma)$ exist for every σ in (α, β) and are determined by*

$$c^-(\sigma) = \varphi'(\sigma - 0) \quad \text{and} \quad c^+(\sigma) = \varphi'(\sigma + 0).$$

Further the frequency $H(\sigma_1, \sigma_2)$ of zeros exists for every strip (σ_1, σ_2), where $\alpha < \sigma_1 < \sigma_2 < \beta$, and is determined by

$$H(\sigma_1, \sigma_2) = \frac{1}{2\pi}(\varphi'(\sigma_2 - 0) - \varphi'(\sigma_1 + 0)).$$

In every reduced interval $(\alpha <) \alpha_1 < \sigma < \beta_1 (< \beta)$ there exist at most a finite number of values of σ for which $\varphi(\sigma)$ is not differentiable.

126. The proof is directly analogous to the proof of Theorem 26, and is based on the theory of measure and integration in infinitely many dimensions as developed in Jessen [3]. We shall now proceed to give a summary of the parts of this theory which we require.

The theory deals with measure and integration in the space R_ω considered mod. 1, which we denote by Q_ω. Thus a point of Q_ω is actually a class of points in R_ω which are equivalent mod. 1. The notation $x = (x_1, x_2, \ldots)$ for a point in R_ω will also be used for the corresponding point of Q_ω. By an interval I in Q_ω we mean, of course, the set of points $x = (x_1, x_2, \ldots)$ for which a finite number of the coordinates belong mod. 1 to given intervals of lengths $\leqq 1$, while the remaining coordinates are unrestricted. The product of the lengths of these intervals is called the measure $m(I)$ of the interval.

For an arbitrary set A in Q_ω we now consider all coverings of A with a (finite or) enumerable number of intervals I and we determine for each such covering the sum of the measures of the covering intervals. The lower bound of the set of these sums, which is evidently $\leqq 1$, is called the exterior Lebesgue measure $m_e(A)$ of A, while the interior Lebesgue measure $m_i(A)$ is defined as $m_i(A) = 1 - m_e(Q_\omega - A)$. If both are equal the set is called measurable in the Lebesgue sense with the measure $m(A) = m_e(A) = m_i(A)$. It is easily proved that intervals are measurable sets and that their measure is equal to that already defined.

This measure has all the general properties of the usual Lebesgue measure. The integral based on it, for which we shall use the notation

$$\int_A F(x)\, m(d\, Q_\omega)$$

has therefore all the general properties of the usual Lebesgue integral.

Denoting by Q_n and $Q_{n,\omega}$ the spaces with $(x_1, \ldots, x_n)$ and $(x_{n+1}, x_{n+2}, \ldots)$ as variable points, and by m_n and $m_{n,\omega}$ the corresponding measures, we have in analogy with Fubini's theorem

$$\int_{Q_\omega} F(x)\, m(d\, Q_\omega) = \int_{Q_{n,\omega}} m_{n,\omega}(d\, Q_{n,\omega}) \int_{Q_n} F(x)\, m_n(d\, Q_n),$$

where the inner integral on the right exists except in a null-set of $Q_{n,\omega}$.

The usual definition of the Riemann integral based on divisions into a finite number of intervals and formation of the corresponding lower and upper sums is immediately extended to Q_ω, and we have the theorem that a function $F(x)$ in Q_ω is Riemann integrable if and only if it is bounded and continuous except in a null-set. The Kronecker-Weyl theorem is immediately extended to Q_ω. It says that if $F(x)$ is Riemann integrable in Q_ω, and if $\mu = (\mu_1, \mu_2, \ldots)$ is a point of R_ω with linearly independent coordinates, such that the function $F(\mu t)$ of the real variable t is also Riemann integrable, then the mean value $\underset{t}{M}\{F(\mu t)\}$ exists and is equal to the integral of $F(x)$ over Q_ω:

$$\underset{t}{M}\{F(\mu t)\} = \int_{Q_\omega} F(x)\, m(d\, Q_\omega).$$

127. On the basis of this theory the extension of the proof of Theorem 26 to the present case requires only a few remarks.

Let $\mu_1, \mu_2, \ldots$ be an infinite integral base of the function $f(s)$ with respect to which it has an analytic spatial extension $g(s; x)$. This function is periodic with the period 2π in all the variables $x_1, x_2, \ldots$, while the preceding theory deals with functions with the period 1 in all the variables. We therefore apply throughout the substitution $x = 2\pi y$. The mean values with respect to x of the various functions considered are then defined as the integrals over Q_ω of the corresponding functions of y. This is the only change required in §§ 116—117.

In extending § 118 we shall need the theorem that if $F(x)$ is regular in all the variables $x_1, x_2, \ldots$ and has the period 1 in all the variables, then it is either identically zero or zero only in a null-set. To see this we notice that since $F(x)$ is continuous the set of points where $F(x) = 0$ is a closed set in Q_ω and therefore necessarily measurable. Suppose that its measure is positive. It then follows from the extension of Fubini's theorem mentioned in § 126 that, for every n, there exist values of $x_{n+1}, x_{n+2}, \ldots$ for which $F(x)$, considered as a function of $x_1, \ldots, x_n$, is zero in a set of positive measure. As $F(x)$ is regular in $x_1, \ldots, x_n$ this implies that $F(x)$ is identically zero in $x_1, \ldots, x_n$ for the values of $x_{n+1}, x_{n+2}, \ldots$ in question. Thus it is possible for an arbitrary point $x = (x_1, x_2, \ldots)$ and an arbitrary n by a change of the coordinates $x_{n+1}, x_{n+2}, \ldots$ to arrive at a point $x^{(n)}$ for which $F(x^{(n)}) = 0$. Since $x^{(n)} \to x$ as $n \to \infty$ this implies that $F(x)$ is identically zero.

This result is immediately extended to the case where $F(x)$ is defined only in an interval; this case is needed later in the proof.

In § 119 the only point requiring a change is that where Weierstrass' 'Vorbereitungssatz' is applied. Here we have to prove that when for a given point x_0 in R_ω the rectangle $S(x_0)$ in the s-plane and the interval $I(x_0)$ in R_ω have been chosen, then the elementary symmetric functions $A_1(x), \ldots, A_p(x)$ of the zeros of $g(s; x)$ in $S(x_0)$ are regular functions of all the variables $x_1, x_2, \ldots$ in $I(x_0)$. That they are continuous is evident since the set of zeros depends continuously on x; and that they are regular functions of $x_1, \ldots, x_n$ for given values of $x_{n+1}, x_{n+2}, \ldots$ is also clear, since $g(s; x)$ is for these values of $x_{n+1}, x_{n+2}, \ldots$ a regular function of the variables $s, x_1, \ldots, x_n$.

In §§ 120—121 no new arguments are required.

128. We shall now prove the following theorem, which is an immediate extension of Theorem 28.

Theorem 30. *If the almost periodic function*

$$f(s) \sim \Sigma a_n e^{\lambda_n s}$$

possesses an infinite integral base $\mu_1, \mu_2, \ldots$ of negative numbers such that in the expressions

$$\lambda_n = h_{n1}\mu_1 + h_{n2}\mu_2 + \cdots$$

of the exponents the integral coefficients h_{ni} are all non-negative, then $f(s)$ has with respect to this base an analytic spatial extension.

As in the case of a finite base, the function $f(s)$ may, if it is almost periodic in $[\alpha, \beta]$, be continued in the half-plane $(\alpha, +\infty)$, and is almost periodic in $[\alpha, +\infty)$. A sequence of exponential polynomials

$$f_p(s) = \Sigma a_n^{(p)} e^{\lambda_n s}$$

(where for every p only a finite number of the coefficients $a_n^{(p)}$ are $\neq 0$) converging uniformly towards $f(s)$ in $[\alpha, \beta]$ will also converge uniformly in $[\alpha, +\infty)$, and the sequence

$$g_p(s; x_1, x_2, \ldots) = \Sigma a_n^{(p)} e^{i(h_{n1} x_1 + h_{n2} x_2 + \cdots)} e^{\lambda_n s}$$

will converge uniformly towards the spatial extension $g(s; x_1, x_2, \ldots)$ for s in $[\alpha, +\infty)$ and all $x_1, x_2, \ldots$.

Now the function

$$h_p(s; z_1, z_2, \ldots) = \Sigma a_n^{(p)} z_1^{h_{n1}} z_2^{h_{n2}} \ldots e^{\lambda_n s}$$

is for every fixed s a polynomial of the complex variables $z_1, z_2, \ldots$ (depending actually on a finite number of the variables only); by the maximum modulus principle we therefore find that for arbitrary p and q

$$\underset{x_1, x_2, \ldots}{\text{upper bound}} |g_p - g_q| = \underset{|z_1| \leqq 1,\ |z_2| \leqq 1, \ldots}{\text{upper bound}} |h_p - h_q|.$$

Hence $h_p(s; z_1, z_2, \ldots)$ converges uniformly for s in $[\alpha, +\infty)$ and $|z_1| \leqq 1$, $|z_2| \leqq 1, \ldots$ towards a limit function $h(s; z_1, z_2, \ldots)$, which is continuous[1] and regular in $s, z_1, \ldots, z_m$ for $\sigma > \alpha$ and $|z_1| < 1, \ldots, |z_m| < 1$ for given values of $z_{m+1}, z_{m+2}, \ldots$.

For every $\delta > 0$ we have

$$g_p(s; x_1, x_2, \ldots) = h_p(s - \delta; e^{\mu_1 \delta} e^{i x_1}, e^{\mu_2 \delta} e^{i x_2}, \ldots).$$

Since the numbers $\mu_1, \mu_2, \ldots$ are negative this implies that for $\sigma > \alpha + \delta$ and all $x_1, x_2, \ldots$

$$g(s; x_1, x_2, \ldots) = h(s - \delta; e^{\mu_1 \delta} e^{i x_1}, e^{\mu_2 \delta} e^{i x_2}, \ldots).$$

This shows that $g(s; x_1, x_2, \ldots)$ is regular for $\sigma > \alpha + \delta$ and all $x_1, x_2, \ldots$. As $\delta > 0$ is arbitrary the proof is hereby complete.

[1] In the usual sense, i.e. $h(s^{(n)}; z_1^{(n)}, z_2^{(n)}, \ldots) \to h(s; z_1, z_2, \ldots)$ if $s^{(n)} \to s$ and $z_i^{(n)} \to z_i$ for all i.

129. For an *ordinary Dirichlet series*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{n=1}^{\infty} a_n e^{-(\log n)s}$$

possessing a half-plane of convergence (and hence also a half-plane of absolute convergence) the abscissa α of uniform convergence is defined as the lower bound of all numbers α_1 for which the series is uniformly convergent in the half-plane $(\alpha_1, +\infty)$. According to a theorem of Bohr [4], this abscissa α is also the lower bound of all numbers α_2 for which $f(s)$ is regular and bounded in the half-plane $(\alpha_2, +\infty)$. Thus the function $f(s)$ is almost periodic in $[\alpha, +\infty)$, but not in any larger half-plane, and its Dirichlet series in the sense of the theory of almost periodic functions is the above series, or rather this series after the omission of terms with the coefficient 0.

In this case the function has the infinite integral base $-\log p_1$, $-\log p_2, \ldots$, where $p_1, p_2, \ldots$ denote the prime numbers, and the conditions of Theorem 30 are evidently satisfied for this base. Thus $f(s)$ has an analytic spatial extension with respect to this base[1]. Theorem 29 is therefore applicable.

Regarding the general properties of the Jensen function $\varphi(\sigma)$ in this case we notice that if n_0 is the smallest value of n for which $a_n \neq 0$, we have, by Theorem 9, that $\varphi(\sigma) = -(\log n_0)\sigma + \log|a_{n_0}|$ for all $\sigma >$ (some) σ_0. Hence $\varphi(\sigma)$ is a decreasing function and possesses, by Theorems 22 and 29, on every half-line $\sigma > \alpha_1 (> \alpha)$ only a finite number of linearity intervals and a finite number of points of non-differentiability.

A more precise result regarding the linearity intervals follows from Theorems 2 and 8, which show that the value of $\varphi'(\sigma)$ in a linearity interval is expressible in the form $k_1(-\log p_1) + k_2(-\log p_2) + \cdots$ both with integral coefficients $k_1, k_2, \ldots$ and with non-negative rational coefficients. Since on account of the linear independence of the numbers $-\log p_1$, $-\log p_2, \ldots$ the two expressions must be identical, we find that the value must be one of the numbers $-\log n$, $n = 1, 2, \ldots$. We notice that the corresponding coefficient a_n need not be $\neq 0$, so that the value of $\varphi'(\sigma)$ is not necessarily one of the proper exponents of $f(s)$.

We may collect our results on ordinary Dirichlet series in the following theorem.

[1] A closer study of an unessentially different form of this spatial extension has been given by Bohr [5].

Theorem 31. *For an ordinary Dirichlet series*

$$f(s) = \sum_{n=n_0}^{\infty} \frac{a_n}{n^s}, \quad a_{n_0} \neq 0,$$

with the uniform convergence abscissa α, *the Jensen function* $\varphi(\sigma)$ *possesses on every half-line* $\sigma > \alpha_1 (> \alpha)$ *only a finite number of linearity intervals and a finite number of points of non-differentiability. The values of* $\varphi'(\sigma)$ *in the linearity intervals belong to the set of numbers* $-\log n$, $n \geqq n_0$. *For* $\sigma >$ *(some)* σ_0 *we have*

$$\varphi(\sigma) = -(\log n_0)\sigma + \log |a_{n_0}|.$$

For an arbitrary $\sigma > \alpha$ *the mean motions* $c^-(\sigma)$ *and* $c^+(\sigma)$ *both exist and are determined by*

$$c^-(\sigma) = \varphi'(\sigma - 0) \quad \text{and} \quad c^+(\sigma) = \varphi'(\sigma + 0).$$

For an arbitrary strip (σ_1, σ_2), *where* $\alpha < \sigma_1 < \sigma_2 < +\infty$, *the relative frequency* $H(\sigma_1, \sigma_2)$ *of zeros exists and is determined by*

$$H(\sigma_1, \sigma_2) = \frac{1}{2\pi}(\varphi'(\sigma_2 - 0) - \varphi'(\sigma_1 + 0)).$$

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