THE POISSON INTEGRAL.

A STUDY IN THE UNIQUENESS OF HARMONIC FUNCTIONS.

BY

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in STOCKHOLM.[1]

In this paper I want to deduce some uniqueness theorems for harmonic functions with assigned boundary values in the unit circle.

In this direction there exists a classical result for harmonic functions continuous on the boundary, based on the fact that harmonic functions take their extreme values on the frontier. Here, there was understood by a boundary value what we are going to denote by $u^D(z)$ (cf. 2.1). It was shown that a function, harmonic in a domain and such that $u^D = 0$ at all boundary points, vanishes identically. Even discontinuous boundary values, defined as limits along the radius, have been considered, especially by G. C. Evans, in his book on the logarithmic potential. The harmonic functions had to be restricted by one of the following majorants:

$$|u(r,\theta)| \leq M, \qquad \int_0^{2\pi} |u(r,\theta)|^p \, d\theta < M.$$

The aim of this paper is to consider (i) more general boundary values, such as u_D, defined in 2.2 or limits u_L along the radius, or even more general curves, defined in 6.0; (ii) more general majorants.

Thus we prove in 7.4.6 a result which in a simplified form runs;

If (i) $u(r,\theta)$ is harmonic in the unit circle,

(ii) at every boundary point θ_0, $u(r,\theta)$ converges to zero, if $(r,\theta) \to (1,\theta_0)$ in any sector (cf. def. in 1.0),

(iii) for every $\varepsilon > 0$, there is an $R < 1$ such that $|u(r,\theta)| \leq e^{\varepsilon/(1-r)^m}$ for $r > R$,

then $u \equiv 0$.

[1] Now: Macalester College, St. Paul, Minn., U. S. A.

If $m \leq 1$, then we may, in (ii), require at every boundary point the convergence only in *a* sector, however small, which contains the radius in its interior.

If we define a boundary value not as the limit in a sector, but the limit along a curve, especially the radius, then we are no more able to prove a uniqueness theorem. Even if these boundary values are zero, there can exist boundary points, near which the harmonic function is unbounded. But we can still prove the following theorem.

If (i) $u(r, \theta)$ is harmonic in the unit circle,

(ii) $\overline{\lim\limits_{r \to 1}} \, |u(r, \theta)| < \infty$ for all θ,

(iii) $\underline{\lim\limits_{r \to 1}} \, u(r, \theta) \leq 0 \leq \overline{\lim\limits_{r \to 1}} \, u(r, \theta)$ for almost all θ,

(iv) $u(r, \theta) = o\left(\frac{1}{(1-r)^m}\right)$

then there is a reducible set of points $\mathfrak{R}$, such that $u(r, \theta)$ takes continuously the value zero at every boundary point which does not belong to $\mathfrak{R}$. At an isolated point of this set, the analytic function $f(z)$, for which $\mathcal{R} f(z) = u(r, \theta)$, has a pole of finite order.

We recall that a reducible set of points is such that it contains no part dense in itself. Now, if we study the behaviour of the function in the neighbourhood of an isolated singular point (cf. section 4), we may hope to find the most convenient and best conditions for the non-occurence of an isolated singular boundary point. But a reducible set without isolated points is empty. Hence these last conditions together with the conditions of the above theorem, give a number of very general uniqueness theorems (cf. section 8).

The fundamental theorem, a generalisation of that given above, is enunciated in 7.0. The important notion of a restricted Poisson Integral on an arc of the frontier is defined in 2.9. It describes a, what we may call, normal behaviour of a harmonic function in the neighbourhood of the frontier. It makes it possible to use for such functions the theory of the Poisson integral, of which an account can be found e. g. in Evans' book.

In section 2 there is a number of results from the theory of the Poisson integral. By means of a conformal representation of a general domain on a unit circle, we define the Poisson integral for general domains (cf. 24). The most important results are to be found in 2.14, 2.18, 2.19 and 2.20.

In section 3 we deduce some known lemmas for harmonic and analytic functions. The chief difference from the classical cases is the use of T. Carleman's extension of Lindelöf's theorem and the authors extension of the Pragmen-Lindelöf's theorem.

Section 4 is devoted to the study of a harmonic function which is continuously equal to zero on the boundary everywhere except at one point. We find it convenient to study the problem in a half-plane.

In section 5, we state and deduce lemmas from the theory of conformal representation.

Section 6 is devoted to lemmas and can be considered as the beginning of the proof of the main theorem. We find it stated in 7.0 and proved in 7.1 and 7.2. In 7.01 and 7.0.2 I state two particular cases which are important for the uniqueness problem of trigonometrical expansions. In 7.0.3 and 7.0.4 I state alternative conditions for the validity of 7.0. In 7.4 we prove a result based on a double system of curves, which might prove useful in many cases. We deduce from it, in 7.4.6, a very important result of which I have above stated a particular case.

Section 8 deals with conditions which make isolated singular points impossible and lead to uniqueness theorems. The great variety of different possible uniqueness theorems has not been exhausted and the section presents rather a few examples for constructing such theorems with suitable conditions.

In 9.0 we state our main theorem for general domains.

As regards the conditions in 7.0, the first four are essential. Condition (iii) may be somewhat relaxed by demanding only $\underline{\lim}_{r \to 1} u(r, (Kr, \vartheta)) > -\infty$. Then we have to substitute Poisson Stieltjes integrals in our resoning. If we restrict $\sigma(\vartheta)$ in (iv) to be $< \infty$, the assertion of the theorem remains the same. Otherwise we should get at the end of 7.0 $u = RlPI_C u_C$ (cf. 2.17) instead of $u = RPI_C u_C$. It seems that with our method conditions (v) and (vi) are essential, too. But, by using some other method, they might possibly be improved.

The first result of this kind I have explained in a talk at Professor G. H. Hardy's Conversation Class in Cambridge in December 1937. In their full generality, the results have been made public when I had the honour to be invited by Mittag-Leffler's Institute to deliver two lectures on harmonic functions.

I want to express my deep gratitude to the Swedish government which, by granting me during two years a scholarship, made it possible for me to finish this paper.[1]

1. **Notation.** We denote the complex variable by $z = x + iy = re^{i\theta}$ or by $\zeta = \xi + i\eta = \varrho e^{i\vartheta}$. The functions $f(z)$, $g(z)$, $h(z)$ are supposed to be analytic in certain specified domains. Their real and imaginary parts, or harmonic functions in general, will be denoted by $u(z) = u(x, y) = u(re^{i\theta})$, $v(\xi, \eta)$. For functions transformed by a conformal representation $\zeta = \varphi(z)$, $z = \psi(\zeta)$ we shall use the notation $f(\zeta)$, $u(\zeta)$ etc. The domain $D(z)$ in the z-plane and the domain $D(\zeta)$, in the ζ-plane, are connected by the conformal transformation. The letter C is reserved for the unit-circle and H for the upper half-plane. The domains will be supposed simply-connected and bounded by free rectifiable Jordan curves. The functions $a(z)$, $b(z)$ will be defined only on $F(D)$, the frontier of D. If the unit-circle $C(\zeta)$ is conformally represented by $\zeta = \varphi(z)$ on the domain $D(z)$, and if $u(\zeta)$ or $a(\zeta)$ have a certain property A, then we say that $u(z)$ or $a(z)$ »have the property A in the corresponding unit-circle». Further, $N(z_0)$ will denote a neighbourhood of z_0 and ε a positive number arbitrarly small. If z_0 is a point of the boundary $F(D)$, then we say that z converges to z_0 in a sector, if, denoting by $\varrho(z)$ the distance of z from $F(D)$, we have $\underline{\lim}_{z \to z_0} \frac{\varrho(z)}{|z - z_0|} > 0$. The geometrical significance of this is well known. All boundary functions in C are supposed L-integrable as functions of θ. Generally, a boundary function in D will be supposed to be »L-integrable in the corresponding C».

2.1. **Definition.** Let $u(z)$ be defined in $D(z)$ and $z_0 < F(D)$. Then a is a boundary value of $u(z)$ in z_0, if there exists a sequence $\{z_k\}$ such that $z_k < D$, $z_k \to z_0$ and $u(z_k) \to a$. All such boundary values in a point z_0 are values of the function $u^D(z_0)$.

2.2. **Definition.** The number b is a boundary value in the strict sence, if there exists a sequence $\{z_k\}$ which converges in a sector to z_0 and such that $u(z_k) \to b$. All boundary values in the strict sense at the point z_0 are values of the function $u_D(z_0)$.

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[1] I am also very much indebted to Professor F. Carlson for his kindness and for reading the manuscript and calling my attention to a number of imperfections, and to Professor T. Carleman, who as Director of Mittag-Leffler's Institute has invited me to lecture.

2.3. If $u^D(z)$ is finite in a closed set of $F(D)$, then $u^D(z)$ is there bounded. The analogous result for $u_D(z)$ is false.[1]

2.4. **Definition.** If

2.4.1 $$u(r e^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}\, a(e^{i\varphi})\, d\varphi,$$

then we say that $u(z)$ is the Poisson integral of the boundary function $a(z)$ . . . Shortly we shall write: $u(z) = PI_C\, a(z)$. If $u(z)$ is defined in $D(z)$ and $\zeta = \varphi(z)$ represents conformally $C(\zeta)$ on $D(z)$, then

$$u(z) = PI_D\, a(z)$$

is equivalent with

$$u(\zeta) = PI_C\, a(\zeta).$$

2.5. If $u(z)$ is defined in H, then the two assertions

2.5.1 $$u(z) = PI_H\, a(z),$$

2.5.2 $$u(z) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{a(x')}{(x-x')^2+y^2}\, dx'$$

are equivalent.

The condition

$$\int_{-\infty}^{\infty} \frac{|a(x)|}{1+x^2}\, dx < \infty$$

is equivalent with the L-integrability of $a(x)$ in the corresponding C.

The relation 2.5.2 follows from 2.4.1 by a conformal representation of C on H.

2.6. *If* $u(z) = PI_D\, a(z)$, *then* $u_D(z) = a(z)$ *almost everywhere, and for the upper and lower bounds we get the relations*

$$U.B.\ u(z) \leq U.B.\ a(z)$$
$$L.B.\ u(z) \geq L.B.\ a(z).$$

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[1] Cf. E. Lindelöf, Calcul des résidus, Coll. Borel, Paris 1905, p. 121. From the function $E_\beta(z)$ we can construct a Gegenbeispiel.

For $D = C$ this is a well known property of Poisson integrals.[1] For a general domain, it follows from the definition of the Poisson integral (cf. 2.4). We get the existence of $u_D(z)$ only almost everywhere in the corresponding C. Since $F(D)$ is rectifiable, this is also, by a theorem of Riesz-Privaloff[2], almost everywhere on $F(D)$. The two inequalities for the bounds are easy deductions from 2.4.1.

2.7. *If $u(z)$ is bounded, then $u(z) = PI_D u_D$.*

By 2.4, it is sufficient to prove it for $D = C$. But then, it is a well known result (cf. Evans, p. 52).

2.8. *If*

$$1^\circ \quad u = PI_D\, a(z), \qquad v = PI_D\, b(z)$$

and

$$2^\circ \quad a(z) = b(z)$$

almost everywhere on a rectifiable part K of $F(D)$, then $(u - v)^D = 0$ at all interior points of K.

We may again consider only the case of $D = C$. We have $u - v = = PI_C[a(z) - b(z)]$ and at K, which is a part of the circumference $(\alpha < \theta < \beta)$, $a(z) - b(z) = 0$ almost everywhere. Without changing the functions $u(z)$, $v(z)$ we may suppose $a(z)$ and $b(z)$ to be equal everywhere in (α, β). And it is a classical result that the Poisson integral converges uniformly to its boundary function in any interval which is interior to an interval of continuity of the boundary function (cf. Hobson, Theory of functions, II. Cambridge 1926, p. 633).

2.9. **Definition.** If there is a $z_0 < F(D)$ and a neighbourhood $N(z_0)$, such that $u(z)$ is a PI in $N(z_0) \cdot D$, then we shall say that $u(z)$ is a restricted Poisson integral[3] in z_0, and we shall write $u(z) = RPI_D(z_0)$.

If $D_1 < D$, $K < F(D \cdot D_1)$ and $u = PI_{D_1}$, then we shall say that $u(z)$ is a restricted Poisson integral on K. Shortly $u(z) = RPI_D(K)$.

2.10. *If $u = RPI(z_0)$, then $u = RPI(z')$ for all $z'' < F(D) \cdot N(z_0)$.*

2.11. *If $u = RPI(z_0)$, then u_D is one-valued and finite almost everywhere on $N \cdot F(D)$.*

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[1] G. C. Evans: The Log. Potential, New York, 1927, p. 40. Corallary. Particular case of Fatou's theorem in Acta math. 30. 1906, p. 345.

[2] F. & M. Riesz, Ueber Randwerte analytischer Funktionen, Stockholm Congress 1916; Lusin-Privaloff, Ann. de l'Ecole Normale, T. 42, 1925; the simplest proof F. Riesz, Math. Zeitschrift, Bd. 18. 1923. p. 95.

[3] W. H. Young has introduced the notion of a »restricted Fourier series». Cf. e.g. Hobson II p. 686.

If $u = RPI(K)$ *then* u_D *is one-valued and finite almost everywhere on* K.

This follows from 2.9 and 2.6.

2.12. *If the values of* $u^D(z_0)$ *are finite, then* $u = RPI(z_0)$. *If* $u^D(z)$ *is finite on* $K < F(D)$, *then* $u = RPI(K)$.

This is a consequence of 2.1, 2.7 and 2.9, because the values of $u^D(z)$ form a closed set of points and if it does not contain the infinite point it must be bounded. For the second part of the assertion we use the Heine-Borel theorem.

2.13. *If* $u(z)$ *is harmonic in* D_1, *and* $D < D_1$, *then* $u = RPI(K)$ *for a* K *which lies in the interior of* D_1.

On such a K, $u^D(z)$ is evidently bounded.

2.14. *If* $u(z) = PI_{D_1}$, *and* $D < D_1$, *then we have also* $u(z) = PI_D$.

By a conformal representation we reduce the case to $D = C$. Further $u(z)$ is the difference of two Poisson integrals with positive boundary functions. Hence, without loss of generality, we may suppose that $u(z)$ has a positive boundary function on $F(D_1)$. We put

$$
\begin{aligned}
{}_A u_{D_1}(z) &= u_{D_1}(z) \quad \text{if} \quad u_{D_1}(z) \leq A \\
{}_A u_{D_1}(z) &= A \qquad\quad \text{if} \quad u_{D_1}(z) \geq A.
\end{aligned}
$$

The function ${}_A u(z) = PI_{D_1}({}_A u_{D_1})$ as a function of A is non-decreasing and

$$\lim_{A \to \infty} {}_A u(z) = u(z)$$

in D_1 (cf. 2.4.1). In $D = C$, ${}_A u(z)$ is bounded. We have therefore (cf. 2.7.)

$${}_A u(z) = PI_C({}_A u_C) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - \varrho^2}{1 - 2\varrho\cos(\vartheta - \varphi) + \varrho^2} \cdot {}_A u_C(e^{i\varphi})\, d\varphi.$$

If $A \to \infty$, the limit in the first member exists. Under the integral sign there is a non-decreasing sequence of positive functions. Hence $\lim\limits_{A \to \infty} {}_A u_C(e^{i\varphi})$ exists and is L-integrable. Thus we get

$$u(z) = PI_C(u_C).$$

2.15. *If*

(i) $$K < F(D \cdot D_1)$$

(ii) $$u = PI_D u_D, \quad v = PI_{D_1} v_{D_1}$$

(iii) K *rectifiable and* $u_D = v_{D_1}$ *almost everywhere on* K,

then $(u - v)^D = 0$ *at all inner points of* K.

By 2.14, $u = PI_{D \cdot D_1}$ and $v = PI_{D \cdot D_1}$. Now, the result follows from 2.8.

2.16. *If* $u = RPI(K, a(z))$ *and* $v = RPI(K, a(z))$ *then* $(u - v)^D = 0$ *at all interior points of* K.

This is a corollary of 2.15.

2.17. **Definition.** If

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \varphi) + r^2} \, dU(\varphi)$$

and $U(\varphi) = U_1(\varphi) + U_2(\varphi)$, where U_1 is an integral and U_2 a non-decreasing function, then we shall say that $u(z)$ is a lower Poisson-Stieltjes integral and we shall write

$$u(z) = lPI_C(U).$$

It is known[1] that u_C exists almost everywhere and that it is equal to $\frac{dU}{d\theta}$.

2.17.1. *A sufficient condition for* $u(z)$ *to be a lower Poisson-Stieltjes integral is the existence of an* A *such that*

$$u(z) \geq A, \qquad z < D.$$

2.17.2. *If*

$$u(z) = lPI_D(U),$$

then

$$u \geq PI_D\left(\frac{dU}{d\varphi}\right) = PI_D \, u_D.$$

It is sufficient to prove the inequality for $D = C$. If U is an integral, then $u = PI_D \, u_D$. We may, therefore, suppose that U is a non decreasing function. Then $u(z) \geq 0$.

Using Fatou's theorem we deduce from

$$u(z) = \lim_{\varrho \to 1} \frac{1}{2\pi} \int_0^{2\pi} \frac{\varrho^2 - r^2}{\varrho^2 - 2\varrho r\cos(\theta - \varphi) + r^2} \, u(\varrho e^{i\varphi}) \, d\varphi$$

the desired result

$$u(z) \geq \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \varphi) + r^2} \, u_C(e^{i\varphi}) \, d\varphi.$$

[1] EVANS, p. 40, Corollary.

2.18. *If* $u(z)$ *is harmonic in* D,

2.18.1 $$D_k < D, \quad \Sigma D_k + \Sigma F(D_k) > C,$$

2.18.2 $$u(z) = PI_{D_k} u_{D_k}$$

2.18.3 $$|u| \leq M \ \textit{for} \ z < F(D_k) D, \qquad k = 1, 2, \ldots$$

2.18.4. *there exists a boundary function* $a(z)$, *equal to one of the values of* u_D, *which is integrable in the corresponding* C, *then*

$$u = PI_D u_D.$$

Without lack of generality we may suppose $D = C$, and $D_k \cdot D_l = 0$, $k \neq l$. If the last condition would not be satisfied, it would be easy to construct new D_k^* satisfying this and all the previous conditions.

The set $F(C) \cdot F(D_k) \cdot F(D_l)$ consists of at most two points. Let there exist two points z_1, z_2. Then we can join them by two curves $K_k < D_k$ and $K_l < D_l$. D_l will be contained in a domain bounded by K_k and a circular arc $\widehat{z_1, z_2}$, and $F(D_l)$ can have points on $\widehat{z_1, z_2}$, but not on the complementary arc. Since K_l separates D_k from $\widehat{z_1, z_2}$, $F(D_k)$ cannot have points on $\widehat{z_1, z_2}$. This shows that there are no more common points on $F(C)$.

The points of $F(D_k)$ are, therefore (i) inner points of arcs $F(D_k) \cdot F(C)$, then $u_C = u_{D_k}$; (ii) points which are not on $F(C)$, there $|u_D| \leq M$; and (iii) those of the enumerable set

$$\sum_{k,l} F(C) \cdot F(D_k) \cdot F(D_l).$$

We define

$${}_N a(z) = \max(a(z), -N)$$

and

$${}_N u(z) = PI\, {}_N a(z).$$

At almost all the points of the first kind we shall have ${}_N u_{D_k} \geq u_{D_k}$, since at those points $u = RPI(z)$. At the points of the second kind we have $u_{D_k} \geq -N$, and the enumerable set of the points of the third kind may be disregarded. Hence in virtue of 2.18.2 we shall have ${}_N u - u \geq -N - M$ in all D_k. The same is clearly true at the points of $F(D_k)$. Hence we get ${}_N u - u = lPI_C$. Since $({}_N u - u)_C \geq 0$, 2.17.2. gives ${}_N u - u \geq 0$ or

$$u \leq PI_C\, {}_N u_C.$$

For $N \to \infty$ this becomes

$$u \leq PI_C u_C.$$

By means of a similar reasoning we can deduce the opposite inequality and complete the proof.

2.19. *If*

(i) $$D = \Sigma D_k$$

(ii) *for* ***all*** $z < F(D)$ *there is an* $N(z)$ *and a* k *such that* $D \cdot N(z) < D_k$, *and*

(iii) $$u = PI_{D_k} u_{D_k},$$

then

$$u = PI_D u_D$$

We may again suppose $D = C$. In every $N(z)$, u_C is almost everywhere one-valued and finite and, by (iii) and 5.5, L-integrable on $F(C) \cdot N(z)$. By means of the Heine-Borel theorem it is easy to show that u_C is L-integrable on the whole $F(C)$.

Without loss of generality we may suppose that the $N(z)$ are all circles with centre in z. Since u_C exists one-valued and finite almost everywhere, we can construct, by diminishing, circular neighbourhoods, such that u_C would be finite and one-valued at the points $F(N(z)) \cdot F(C)$. Now, $u(z)$ is bounded on $F(N(z)) \cdot C$. To every $z < F(C)$ we make correspond a closed circular neighbourhood $N'(z)$ whose radius is half of that of $N(z)$. By the Heine-Borel theorem we may cover $F(C)$ by a finite number of $N'(z)$, say $N'(z_k)$, $k = 1, 2, \ldots K$. Then

$$D_{K+1} = C - \sum_{k=1}^{K} N(z_k)$$

is completely interior to C and there is an M such that $|u(z)| \leq M$ in it.

Now it is easy to see that the conditions of theorem 2.18 are satisfied for $D_k = N(z_k)$.

2.20. *If* $u = RPI(z)$ *at all points* z *of* $F(C)$, *then* $u = PI_D u_D$.

Out of these neighbourhoods (s. 2.9) we can choose a finite number of them which completely cover $F(D)$ and such that in the rest of D, u is bounded. Then we can apply 2.19.

2.21. *If* $K < F(D)$ *is a curve with its both endpoints and* $u(z) = RPI(z)$ *for all* $z < K$, *then* $u = RPI(K)$.

The proof follows from 2.9 and 2.20.

2.22. *If* $u = PI_C$, *then*

$$f = u + iv = o\left(\frac{1}{1-r}\right)$$

uniformly in θ.

We start from the expression

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\varphi} + re^{i\theta}}{e^{i\varphi} - re^{i\theta}} u_C(\varphi)\, d\varphi$$

and establish the result by the usual reasoning.

2.23. *If* $f(z) = u + iv$ *and* $u = PI_H(a(x'))$, *where*

$$\int_{-\infty}^{\infty} \frac{a(x')}{1 + x'^2}\, dx' < \infty,$$

then

$$f = o\left(\frac{|z|^2}{y}\right)$$

uniformly for $z \to \infty$ *in* H.

The condition for $a(x')$ is equivalent with the L-integrability of the boundary function in the corresponding C. We may therefore use 2.22. By

$$\zeta = \frac{z - i}{z + i}$$

we represent $C(\zeta)$ on $H(z)$. Hence we get

$$\frac{1}{1-\varrho} = \frac{1}{1 - |\zeta|} \sim \frac{2}{1 - |\zeta|^2} = \frac{1}{y}\,\frac{(x^2 + y^2 + 2y + 1)^2}{2(x^2 + y^2 + y + 1)} \sim \frac{|z|^2}{2y}$$

uniformly for $z \to \infty$. Now, 2.22 gives the required result.

2.24. *If*

2.24.1 $$f(z) = u + iv,$$

2.24.2 $$u = PI_H(a(x'))$$

2.24.3 $$|a(x')| \leq M \quad \text{for} \quad x' > N,$$

then $f(z)$ *is bounded for* $x > N + 2$, $0 < a < y < b$.

Without lack of generality we may suppose $N = 0$. Then

$$u = \frac{y}{\pi}\left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right)\frac{a(x')}{(x-x')^2+y^2}\,dx' = u_1 + u_2$$

where

$$|u_2| \leq \frac{My}{\pi}\int_{0}^{\infty}\frac{dx'}{(x-x')^2+y^2} \leq M$$

and

$$|u_1| \leq \frac{y}{\pi}\int_{-\infty}^{0}\frac{|a(x')|}{x^2+x'^2+y^2}\,dx' \leq \frac{c}{\pi}\int_{-\infty}^{0}\frac{|a(x')|}{1+x'^2}\,dx'$$

for $0 < y < c$. Hence u is bounded in the strip $x > N + 1$ $0 < y < c$, for any arbitrary positive c.

Now we use 3.1 in a way similar to 3.2 and we deduce the boundedness of v in $x > N + 2$, $0 < a < y < b < c$.

3.0. In this section we shall prove some results about harmonic and analytic functions which we shall use later.

3.1. If $|u| \leq M$ in $r < R$, then

$$\left|\frac{\partial u}{\partial r}\right|_{r=0,\theta} \leq \frac{2M}{\varrho}.$$

From

$$u(r e^{i\theta}) = \frac{1}{2\pi}\int_{0}^{2\pi}\frac{\varrho^2 - r^2}{\varrho^2 - 2r\varrho\cos(\theta-\varphi) + r^2}\,u(\varrho e^{i\varphi})\,d\varphi$$

we deduce

$$\left(\frac{\partial u}{\partial r}\right)_{r=0,\theta} = \frac{1}{\pi\varrho}\int_{0}^{2\pi}\cos(\theta-\varphi)\,u(\varrho e^{i\varphi})\,d\varphi.$$

Hence the desired inequality easily follows.

3.2. If 1° $u(z)$ is harmonic in D, defined by

$$\alpha < \arg z < \beta$$

2° $u(z) = o(z^{\varrho})$ for $z \to \infty$, uniformly in D; then

$$f(z) = u + iv = o(z^{\varrho})$$

for $z \to \infty$, uniformly for $\alpha + \varepsilon \leq \arg z \leq \beta - \varepsilon$.

For $\zeta = z_0 \cdot s$, $0 < s < 1$ we get

$$f(z_0) - f(0) = \int_0^{z_0} df = \int_0^{z_0} \left(\frac{\partial u}{\partial \zeta} + i \frac{\partial v}{\partial \zeta}\right) d\zeta = \int_0^{z_0} \left(\frac{\partial u}{\partial \zeta} \pm i \frac{\partial u}{\partial n}\right) d\zeta$$

where $\frac{\partial u}{\partial n}$ is the derivative in the direction of the normal to $\arg z = \arg z_0$. In order to get upper bounds for these derivatives, we use 3.1 for cercles with centres in ζ, and radius $\varepsilon|\zeta|$. The rest of the proof is straightforward.

3.3. If 1° $f(z) = u + iv$ is an entire function, 2° $|u| \leq \exp\left[\varepsilon|z|^{n+\frac{1}{2}}/y^n\right]$ for all $\varepsilon > 0$ and $|z| > R(\varepsilon)$; 3° $u(z) = o(z^{l+1})$ for $z \to \infty$ uniformly in $\alpha \leq \arg z < \beta$, l integral; then $f(z)$ is a polynomial of l-th degree.

Using 3.2 for cercles with centre at z radius $y/2$ we get

$$3.3.1 \qquad |f(z)| \leq \exp\left[\varepsilon\left(|z| + \frac{y}{2}\right)^{n+\frac{1}{2}} \Big/ \left(\frac{y}{2}\right)^n\right] \leq \exp\left[\varepsilon\, 2^{n+2}|z|^{n+\frac{1}{2}}/y^n\right]$$

for all $\varepsilon > 0$ and $|z| > R(\varepsilon)$.

By 3.2, it follows from 3° that

$$f(z) = o(z^{l+1})$$

for $\arg z = \frac{\alpha + \beta}{2}$.

The function

$$g(z) = \left[f(z) - f(0) - \frac{z}{1!} f'(0) - \cdots - \frac{z^l}{l!} f^{(l)}(0)\right] \Big/ z^{l+1}$$

satisfies 3.3.1 and is $o(1)$ on $\arg z = \frac{\alpha + \beta}{2}$. Now, we use the generalized theorem of Phragmén-Lindelöf[1] to the function $g(e^{i(\alpha+\beta)/2}\zeta^2)$ in $\mathcal{J}(\zeta) > 0$. We get

$$f(z) = o(z^{l+1})$$

uniformly for all $z \to \infty$. Hence $f(z)$ must be a polynomial of l-th degree.

3.4. If $f(z)$ is a polynomial of degree l, whose coefficients are real, and there exists a sequence of points $\{a_k\}$ such that 1° $a_k \to \infty$, 2° $\mathcal{J}\{f(a_k)\} = o(a_k)$, 3° $\lim \arg a_k = \pi \varkappa$ where $\varkappa$ is not a fraction, with a denominator less than or equal l, then $f(z)$ is a constant.

[1] F. Wolf, An extension of the P. L. theorem, Journal of the London Math. S. 1939. p. 208.

If, indeed, the highest power in $f(z)$ is $c_k z^k$ $(0 \leq k \leq l)$, then, if 3° is satisfied,

$$\mathcal{J}\{f(a_k)\} \sim c_k |a_k|^k \sin(k\pi x).$$

For $k > 0$, this would be a contradiction to 2°.

3.5. If 1° $f(z)$ is analytic for $\theta_1 \leq \theta \leq \theta_2$, 2° $|f(z)| \leq \exp[\varepsilon r^{\pi/(\theta_2-\theta_1)}]$ for all $\varepsilon > 0$, $\theta_1 \leq \theta \leq \theta_2$ and $|z| > R(\varepsilon)$,

3° $$f(z) = o(z^a) \text{ for } \theta = \theta_1$$
$$f(z) = o(z^b) \text{ for } \theta = \theta_2$$

4° $$\varphi(\theta) = \frac{(\theta - \theta_1) b + (\theta_2 - \theta) a}{\theta_2 - \theta_1}$$

then $f(z) = o(z^{\varphi(\theta)})$ uniformly for $\theta_1 \leq \theta \leq \theta_2$.

We prove it by applying Phragmén-Lindelöf's theorem to $f(z)/z^{\varphi(-i \log z)}$ in $\theta_1 \leq \theta \leq \theta_2$.

4.0. *If*

1° $$u(x, y) \textit{ is harmonic in } H,$$

2° $$u = RPI(z) \textit{ for all } z < F(H)$$

3° $$\int_{-\infty}^{\infty} \frac{|u_H(x, 0)|}{1 + x^2} dx < \infty,$$

then

$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{u_H(\xi, 0)}{(x - \xi)^2 + y^2} d\xi + \mathcal{J}\left[\sum_1^{\infty} a_n z^n\right] = p(x, y) + s(x, y).$$

The series $\sum_1^{\infty} a_n z^n$ *converges for all* z *and* $\mathcal{J}(a_k) = 0$ *for all* k.

By 2° and 2.11, u_H is one-valued and finite almost everywhere. Condition 3° is equivalent with the integrability of u_H in the corresponding C. By 2.5, $p(x, y) = PI_H u_H$ and 2.7 gives

$$(u - p)^H = 0.$$

Hence $s(x, y)$ is continuous and equal to zero on $y = 0$. It can be extended to a function which is harmonic in the whole plane by the equation

4.0.1 $$s(x, -y) = -s(x, y).$$

It may be considered as the imaginary part of an analytic entire function $\sum_1^\infty a_n z^n$ which is real on the real axis.

4.1. Conditions for the non-existence of the singular part.

4.1.1. *If the conditions of* 4.0 *are satisfied and* $u(z) = o(z^2/y)$, *then* $u = PI_H u_H$.

By 2.23, we find that $s = o(z^2/y)$ in H, and by 4.0.1 also in the whole plane. Further 3.1 gives $h(z) = s(z) + i\,t(z) = o(z^2/y)$ which shows that $\mathcal{R}[h(z)] = 0$[1] or $u = PI_H$.

4.1.2. *If the conditions of* 4.0 *are satisfied and* (i) $u(z) = o(z^m)$, (ii) $u(z) = o(z)$ *on a sequence of points* $\{a_k\}$, *such that* $\lim_k |a_k| = \infty$, $\lim_k \arg a_k = \pi\varkappa$ *where* $\varkappa$ *is not a fraction with a denominator less than* m,

then $u = PI_H u_H$.

Condition (i) gives, in the same way as above, $s(z) = \mathcal{J}\left[\sum_1^{k>m} a_k z^k\right]$. Now, we apply 3.4.

4.1.3. *If the conditions of* 4.0 *are satisfied and* (i) $|u| \leq \exp\left[\varepsilon |z|^{n+\frac{1}{2}} / y^n\right]$ *for all* $\varepsilon > 0$ *and* $|z| > R(\varepsilon)$ (ii) $u(z) = o(z^{l+1})$ *for* $z \to \infty$ *uniformly in* $\alpha < \arg z < \beta$, l *integral,* (iii) *there exists a sequence of points* $\{a_k\}$ *such that* 1° $\lim_k |a_k| = \infty$, 2° $u(a_k) = o(a_k)$, 3° $\lim_k \arg a_k = \pi\varkappa$ *where* $\varkappa$ *is not a fraction with a denominator less than or equal to* l, *then*

$$u = PI_H u_H.$$

We deduce the proof in the usual way from 3.3 and 3.5.

4.2. *If*

(i) $u(x, y)$ *is harmonic in* H,

(ii) $u = RPI(x)$ *for* $|x| \geq M$,

(iii) $$\left(\int_{-\infty}^{-M} + \int_M^{\infty}\right) \frac{|u_H(\xi)|}{1 + \xi^2} d\xi < \infty$$

then

$$u(z) = \frac{y}{\pi}\left(\int_{-\infty}^{-M} + \int_M^{\infty}\right) \frac{u_H(\xi)}{(x - \xi)^2 + y^2} d\xi + v(z) + \mathcal{J}\left[\sum_1^\infty a_n z^n\right]$$

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[1] Cf. Carleman's generalization of Lindelöf's th. Acta m. 48.

where $v(z)$ *is harmonic in* $|z| > M$, $v(z) = 0$ *for* $y = 0$, $|x| \geq M$, $v(z) = o\left(\frac{1}{z}\right)$ *uniformly for* $z \to \infty$. *The series* $\sum_{1}^{\infty} a_n z^n$ *converges for all* z *and* $\mathcal{J}(a_k) = 0$ *for all* k.

Similarly to 4.0 we have $(u - p)^H = 0$ for $|x| \geq M$. Hence $q(x, y) = u - p = -q(+x, -y)$ defines a function harmonic in $|z| \geq M$, which may be considered to be the imaginary part of a function $g(z)$ analytic in $|z| \geq M$. This can be written $g(z) = \sum_{-\infty}^{\infty} a_n z^n$, where $\mathcal{J}(a_n) = 0$. We may, evidently, take $a_0 = 0$. Then we get

$$\mathcal{J}(g) = \mathcal{J}\left[\sum_{1}^{\infty} a_n z^n\right] + \mathcal{J}\left[\sum_{-\infty}^{-1} a_n z^n\right]$$

which proves our assertion.

4.3. *If* (i) z_0 *is an isolated singular point on* $F(C)$, *i.e. if there is a circular neighbourhood* $N(z_0)$, *such that at all points* $z < F(C) \cdot N(z_0)$, $z \neq x_0$, $u = RPI(z)$

(ii) $$u(z) = o(z - z_0)^{-m} \text{ in } z < C \cdot N(z_0),$$

then there are constants c_k *such that*

4.3.1 $$u(z) - \mathcal{J}\left[\sum_{k=0}^{m} c_k \Big/ (z - z_0)^k\right] - \frac{1}{2\pi} \int_{F(C) \cdot N(z_0)} \frac{(1 - r^2) u_C(e^{i\varphi})}{1 - 2r\cos(\theta - \varphi) + r^2} d\varphi$$

is a harmonic function which is continuous in $N(z_0) \cdot C$, *zero on* $F(C) \cdot N(z_0)$ *and* $o(z - z_0)$.

If we define the functions

4.3.2 $$S_1 = 1 + 2\sum_{1}^{\infty} r^n \cos n\theta, \quad S_n = \frac{\partial S_{n-1}}{\partial \theta}$$

which are harmonic in C *and* $S_n^D = 0$ *at all points of* $F(C)$ *except at* $\theta = 0$, *then the imaginary part of the sum in* 4.3.1 *may be substituted by a sum of the form:*

4.3.3 $$\sum_{k=1}^{u} d_k S_k(r, \theta - \theta_0), \quad \mathcal{J}(d_k) = 0.$$

Proof. We represent, by

$$\zeta = -i\frac{z + z_0}{z - z_0},$$

$H(\zeta)$ on $C(z)$ and apply 4.2. We get

$$u(\zeta) = \frac{y}{\pi} \int\limits_{F(H)\,N(\infty)} \frac{u_H(\xi^1)}{(\xi - \xi^1) + \eta^2} d\,\xi^1 + v(\zeta) + \mathcal{J}\left[\sum_1^m a_k \zeta^k\right].$$

Now, we have only to prove that the imaginary part of the series can be written in the form which occurs in 4.3.1. But this is evident from

$$\zeta = i + 2\,i\,z_0 \cdot \frac{1}{z - z_0}.$$

Now we show that 4.3.3 is an alternative form.

We have for $z_0 = 1$

$$S_1 = \mathcal{R}\left[1 + 2\sum_1^\infty z^n\right] = \mathcal{R}\left[\frac{z+1}{z-1}\right] = -\mathcal{J}\left[-i\frac{z+1}{z-1}\right] = -\mathcal{J}(\zeta).$$

Similarly

$$S_2 = \frac{\partial S_1}{\partial \theta} = -\mathcal{J}\left[\frac{\partial \zeta}{\partial \theta}\right] = -\mathcal{J}\left[i\frac{d\zeta}{dz}\cdot z\right] = \mathcal{J}\left[\frac{2z}{(z-1)^2}\right] =$$

$$= \mathcal{J}\left[-\frac{1}{2} + \frac{(z+1)^2}{2(z-1)^2}\right] = \mathcal{J}\left[-\frac{1}{2} - \frac{1}{2}\zeta^2\right] = -\frac{1}{2}\mathcal{J}(\zeta^2).$$

Now, we shall suppose that, for all $k \le d - 1$, we have

4.3.4 $$S_k = \sum_{n=1}^{k} c_{k,n}\mathcal{J}(\zeta^n), \quad c_{k,k} = -\frac{(k-1)!}{2^{k-1}}$$

and we shall show that this is true even for $k = l$. We have indeed

$$S_l = \frac{\partial S_{l-1}}{\partial \theta} = \frac{1}{2}\sum_{n=1}^{k-1} n\, c_{l-1,n}\mathcal{J}[\zeta^{n-1}(1 + \zeta^2)]$$

which is of the required form and $c_{l,l} = \frac{l-1}{2} c_{l-1,l-1}$. The equations 4.3.4 are such that it is clearly possible to express $\mathcal{J}(\zeta^n)$ by means of S_n. We have, thus, proved that 4.3.3 can be substituted in 4.3.1.

5.0. In this section we shall formulate some theorems on conformal representation.

5.0.1. Ostrowski has proved the following theorems:[1]

[1] ALEXANDER OSTROWSKI, Acta math. V. 64, 1935, p. 100, 116, 173.

Let $\zeta = \varphi(z)$ represent conformally $D(\zeta)$ on $D(z)$. Further $F(D(z))$ has at z_0 a »corner», i. e. has two half tangents, which form an inner angle $\gamma > 0$. Similarly $F(D(\zeta))$ has a corner at $\zeta_0 = \varphi(z_0)$ with an angle $\gamma_1 > 0$. Then

5.0.2. $$\arg \frac{\zeta - \zeta_0}{z - z_0} = \left(\frac{\gamma_1}{\gamma} - 1\right) \arg(z - z_0) + c + \varepsilon(z - z_0)$$

and

5.0.3. $$\arg \varphi'(z) = \left(\frac{\gamma_1}{\gamma} - 1\right) \arg(z - z_0) + c + \varepsilon_1(z - z_0).$$

Where ε, ε_1 converge to zero, as $z \to z_0$, uniformly in a sector.

5.0.4. We denote by c_1, c_2 two arbitrary positive constants such that $c_1 < c_2$. If, now $z_1 \neq z_2$ are two points in a sector of z_0 such that

$$c_1 \leq \left|\frac{z_1 - z_0}{z_2 - z_0}\right| \leq c_2$$

then

$$\frac{\varphi(z_1) - \varphi(z_0)}{\varphi(z_2) - \varphi(z_0)} \sim \left(\frac{z_1 - z_0}{z_2 - z_0}\right)^{\gamma_1/\gamma}$$

when z_1, z_2 converge towards z_0.

5.0.5. Finally,

$$\frac{\log|\varphi(z) - \varphi(z_0)|}{\log|z - z_0|} \to \frac{\gamma_1}{\gamma}$$

and

$$\frac{\log|\varphi'(z)|}{\log|z - z_0|} \to \frac{\gamma_1}{\gamma} - 1$$

for z converging towards z_0 in a sector.

5.0.6. (Warschawski[1]). We denote by s the length of $F(D)$ from the point z_0 on and by $\theta(s)$ the angle between the real axis and the tangent to $F(D)$ at s. If (i) $\int_0^{\pm\delta} \frac{|\cos\theta(s) - \cos\theta(\pm 0)|}{s} ds$, $\delta > 0$ converges, (ii) $\lim \frac{|z(s) - z_0|}{s}$ exists, (iii) $D(z)$ has an inner angle ϱ at z_0 and (iv) $\zeta = \varphi(z)$ represents $C(\zeta)$ on $D(z)$, then

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[1] Über das Randverhalten etc. Math. Zeit. 35 (1932) p. 427.

$$\lim_{z\to z_0}\frac{\varphi(z)-\varphi(z_0)}{(z-z_0)^{\pi/\varrho}}$$

exists uniformly for $z < D(z)$.

5.0.7. It is easy to show that the conditions are satisfied in the case that $|\theta(s)-\theta(\pm 0)| \leq C|s|^\alpha$, $\alpha > 0$.

For condition (i) this is evident and, as for (ii), we have

$$|z(s)-z_0| = \left|\int_0^s dz(s)\right| = \left|\int_0^s e^{i\theta(s)}\,ds\right| = \left|\int_0^s e^{i(\theta(s)-\theta(\pm 0))}\,ds\right| =$$

$$= \left|\int_0^s ds\right| + O\left(\int_0^s |\theta(s)-\theta(\pm 0)|\,ds\right) = s + O(s^{\alpha+1}).$$

5.0.8. (Warschawski[1]).

Let $F(D)$ have an arc K with a continuous tangent and $z=\psi(\zeta)$, $(\zeta=\varphi(z))$ represent $D(z)$ conformally on $C(\zeta)$. We denote by K' an arc interior to K, and by γ' the corresponding arc of $F(C)$.

A necessary and sufficient condition for the relations

$$|\varphi'(z)-\varphi'(z')| \leq \text{const}\,|z-z'|^\alpha,\quad z, z' < K'$$

$$|\psi'(\zeta)-\psi'(\zeta')| \leq \text{const}\,|\zeta-\zeta'|^\alpha,\quad \zeta, \zeta' < \gamma',$$

is the existence of a k such that

5.0.9. $$|\theta(s)-\theta(s')| \leq k\,|s-s'|^\alpha,\quad s, s' < K.$$

5.1.0. **Definition.** The curve $K(z)$ is said to have the property E at the point z_0, if it is analytic in a certain neighbourhood of z_0, except possibly at the point z_0 itself.

If it is not analytic at z_0, then we suppose moreover $K(z)$ to be such that there exists a domain $D^* = D_1^* + D_2^*$ for which (i) $K = F(D_1^*)\cdot F(D_2^*)$ (ii) $F(D^*)$ has a corner at z_0, and (iii) D_1^*, D_2^* are inverse to each other with respect to the analytic part of $K(z)$.

We see that if $K(z)$ is analytic also at z_0, then the last conditions are superfluous.

5.1.1. If $\zeta=\varphi(z)$ represents conformally $H(\zeta)$ on $D^*(z)$ and $0=\varphi(z_0)$, then $K(\zeta)$ is, by our hypothesis, the upper part of the imaginary axis. Now

[1] l. c. p, 447.

we see easily, by 5.0.2 and 5.0.3, that $K(z)$ is rectifiable near z_0, that it has a tangent at z_0, which is the bissectrice of the corner of $F(D^*)$.

Further, if z_1 is inverse to z_2, then

$$\lim_{z_1 \to 0} (\arg z_1 + \arg z_2) = 2 \lim_{z \subset K,\ z \to z_0} \arg z$$

and, by 5.0.4,

$$\frac{z_1 - z_0}{z_2 - z_0} \sim \left(\frac{\zeta_1}{\zeta_2}\right)^c$$

uniformly, if $z \to z_0$ in a sector.

5.1.2. An immediate consequence of these results is the following.

If (i) $D(z)$ has a corner at z_0, (ii) $K < F(D)$ has the property E at one side of z_0 and (iii) $\zeta = \varphi(z)$ represents conformally $H(\zeta)$ on $D(z)$ so that $0 = \varphi(z_0)$, then $\zeta = \varphi(z)$ represents also conformally $H_1^*(\zeta)\,(> H(\zeta))$, whose inner angle is larger than π, on $D_1(z)$ whose inner angle at z_0 is larger than that of $D(z)$. Here $K(z)$ comes to be in a sector of $D_1(z)$ and the results of 5.0 are valid in an 'one-sided' sector of $D(z)$ whose one side is $K(z)$ itself.

5.1.3. *If* (i) $D(z)$ *has a corner at* z_0, (ii) $K(z)$ *lies in a sector at* z_0 *and has the property* E *at that point and* (iii) $\zeta = \varphi(z)$ *represents* $D(\zeta)$ *on* $D(z)$, *then* $K(\zeta)$ *has the property* E *at* $\zeta_0 = \varphi(z_0)$.

Let $D^*(z) = D_1^*(z) + D_2^*(z)$ be the domain of 5.1.0. Since K lies in a sector at z_0, we may suppose without loss of generality that $D^* < D$. Let $D^*(z)$ be represented conformally on $H(w)$, by $z = \psi(w)$. Then $\zeta = \varphi(z)$ transforms $D^*(z)$ into $D^*(\zeta)\,(< D(\zeta))$ which, by $\zeta = \varphi(\psi(w))$ is conformally represented on $H(w)$. Hence $D^*(\zeta)$ satisfies 5.1. (iii).

It is easy to see that $D^*(\zeta)$ satisfies also (i) and (ii) and $K(\zeta)$ has therefore the property E.

5.2. We shall say that $K(z)$ has the *property* E^* at z_0 if it has the property E and if,

(iv) denoting by s the length of $K(z)$ from the point z_0 on and by $\theta(s)$ the angle between the tangent and a fixed direction,

5.2.1. $$|\theta(s) - \theta(0)| \leq C s^{\alpha}, \quad \alpha > 0,$$

is fulfilled, for the point s, in a $N(z_0)$.

5.2.2. *If* (i) $D(z)$ *has a corner at* z_0 *and at each of the parts of* $F(D)$, *in a certain neighbourhood of* z_0, 5.2.3. $$|\theta(s)-\theta(s')| \le C|s-s'|^{\alpha}, \quad \alpha > 0$$

is satisfied,

(ii) *$K(z)$ has the property E^* at z_0 and lies in a sector at z_0*

(iii) *$\zeta=\varphi(z)$ represents $D(\zeta)$ conformally on $D(z)$,*

then $K(\zeta)$ has also the property E^* at $\zeta_0=\varphi(z_0)$.

It is, by 5.1.3, evidently sufficient to show that $K(\zeta)$ satisfies 5.2.1. It is easy to verify that, although the constants change, the property E^* is invariant with the transformation $\zeta-\zeta_0=(z-z_0)^{\beta}$.[1]

We may therefore suppose that the two parts of $F(D)$ at z_0 have a common tangent and that, therefore, 5.2.3 is valid not only separately on both sides of z_0, but in a certain two-sided neighbourhood of z_0.

Now, we see that the conditions of 5.0.8 are satisfied. Hence

$$\frac{\varphi'(z)-\varphi'(z_0)}{(z-z_0)^{\alpha}}$$

is bounded, if $z<F(D)\cdot N(z_0)$. By the Phragmén-Lindelöf theorem it must be so even for $z<D\cdot N(z_0)$. And the same deduction can be made for $\psi'(\zeta)= = 1/\varphi'(z)$, in the corresponding $C\cdot N(\zeta_0)$. Hence $\varphi'(z)$ is in absolute value between two positive constants. We have, therefore

$$\arg\varphi'(z)-\arg\varphi'(z_0)=o((z-z_0)^{\alpha}).$$

If, now, $\theta(\sigma)$ has for $K(\zeta)$ the analoguous meaning as $\theta(s)$ has for $K(z)$, then

$$\theta(\sigma)-\theta_{\sigma}(0)=\theta(s)-\theta_s(0)+\arg\varphi'(z)-\arg\varphi'(z_0)=o(s^{\alpha})+o((z-z_0)^{\alpha})$$

or, using 5.0.7,

$$\theta(\sigma)-\theta_{\sigma}(0)=o((z-z_0)^{\alpha})=o((\zeta-\zeta_0)^{\alpha})=o(\sigma^{\alpha}).$$

Thus, we have proved that $K(\zeta)$ satisfies condition 5.2.1.

5.2.9. We have the following result similar to 5.1.2.

If (i) $D(z)$ has a corner of the same kind as in 5.0.6 at z_0 (ii) $K<F(D)$ has the property E^* at one side of z_0 and (iii) $\zeta=\varphi(z)$ represents conformally $H(\zeta)$ on $D(z)$ so that $0=\varphi(z_0)$, then $\zeta=\varphi(z)$ represents conformally also $H_1(\zeta)$, whose inner angle at $\zeta=0$ is larger than π, on $D_1(z)$, whose inner angle at z_0 is larger than that of $D(z)$.

[1] WARSCHAWSKI, l. c. p. 446.

Further, instead for 5.0.5, Warschawski's more precise result 5.0.6. holds in $D(z)$.

5.3. *Let* $\zeta = 0 < D(\zeta) < C(\zeta)$ *and* $(\alpha < \theta < \beta,\ r = 1) < F(D) \cdot F(C)$. *By* $z = \psi(\zeta)$, $C(z)$ *is represented conformally on* $D(\zeta)$, *in such a way that* $0 = \psi(0)$. *Then* $\left|\frac{d\zeta}{d\theta}\right| \geq 1$ *for* $\alpha < \theta < \beta$.

Since $\psi(\zeta)$ is analytic on (α, β), $\psi'(\zeta)$ must exist everywhere. By the theorem of Julia-Caratheodory[1], we have, for $\alpha < \vartheta_0 < \beta$, $e^{i\theta_0} = \psi(e^{i\vartheta_0})$ and $z = \varphi(\zeta)$

$$\frac{1 - |\zeta|^2}{|e^{i\vartheta_0} - \zeta|} \geq |\psi'(e^{i\vartheta_0})| \frac{1 - |z|^2}{|e^{i\vartheta_0} - z|}.$$

If we put $z = \zeta = 0$, we get

$$\left|\frac{d\theta}{d\zeta}\right| = |\psi'(e^{i\vartheta_0})| \leq 1.$$

5.3.1. *If* $a(\zeta)$ *is a boundary function of* $D(\zeta)$, *which is integrable on* (α, β) *and bounded on the rest of* $F(D)$, *then it is integrable in the corresponding* $C(r, \theta)$.

We have indeed

$$\int_{\theta(e^{i\alpha})}^{\theta(e^{i\beta})} |a(\theta)|\, d\theta = \int_{\alpha}^{\beta} |a(s)|\, ds \cdot \left|\frac{d\theta}{d\zeta}\right| \leq \int_{\alpha}^{\beta} |a(s)|\, ds$$

and on the rest of $F(C)$ it is bounded.

5.4. For application of our uniqueness theorem to general domains we shall use a result proved by W. Seidel.[2]

5.4.1. If $z = \psi(\zeta)$ represents conformally a convex $D(z)$ on $C(\zeta)$, then $|z\psi'(z)|$ is increasing on every radius. The boundary function, thus defined, is almost everywhere finite and greater than a certain positive number.

5.4.2. If $z = \psi(\zeta)$ represents conformally a domain with bounded »outer curvature» $D(z)$ on $C(\zeta)$, then $|\psi'(\zeta)|$ is everywhere greater than a certain positive number.

This is a generalization of the preceding result and of another of Seidel's theorems. The outer curvature in a frontier point is the reciprocal value of the radius of the greatest circle which goes through the point and is completely

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[1] Cf. e.g. NEVANLINNA: Eindeutige analytische Funktionen, p. 52.

[2] Ueber Ränderzuordnung bei konformen Abbildungen, Math. Annalen, 104 (1931), pp. 212, 217, 222.

outside D. By a linear transformation, which transforms this circle into the unit circle, we reduce this theorem to the theorem of Julia-Caratheodory (cf. 5.3).

5.4.3. If $a(z)$ is a boundary function, integrable on $F(D)$, and D has a bounded outer curvature, then $a(z)$ is integrable in the corresponding C.

The proof is identical with that given at the end of 5.3.1.

5.5. If $K < F(D)$ is analytic and $K' < K$ completely interior to K, then the L-integrability of $a(z)$ on K' is equivalent with the L-integrability of $a(z)$ on the arc in the corresponding C.

If $\zeta = \varphi(z)$ represents conformally $C(\zeta)$ on $D(z)$, then, on K', $|\varphi'(z)|$ is between two positive constants. Hence the result readily follows.

5.6. Let $D(z)(< C(z))$ be a domain bounded by a curve $K(z)$ joining the points $e^{i\alpha}$, $e^{i\beta}$, and by the circular arc between those points. At $e^{i\alpha}$, $K(z)$ has the property E and $D(z)$ an inner angle ϱ. If $z < D(z)$, $z \to e^{i\alpha}$ and $\zeta = \varphi(z)$ $(z = \psi(\zeta))$ represents $H(\zeta)$ on $D(\zeta)$, $(\infty = \varphi(e^{i\alpha}))$, then

5.6.1. $$\frac{1}{1-|z|} = o\left(\zeta^{\varrho/\pi+\varepsilon}\cdot(\zeta/\eta)\right)$$

for any positive ε.

If, moreover, $K(z)$ has the property E^* at $e^{i\alpha}$, then

5.6.2. $$\frac{1}{1-|z|} = O\left(\zeta^{\varrho/\pi}\cdot(\zeta/\eta)\right).$$

Proof. By applying first the intermediary transformation $\zeta = \zeta_1^{-1}$, we get from 5.0.5

5.6.3. $$\psi(\zeta) - e^{i\alpha} = o\left(\zeta^{-\varrho/\pi+\varepsilon}\right),\quad (\psi'(\zeta))^{-1} = o\left(\zeta^{(\varrho+\pi)/\pi+\varepsilon}\right).$$

Since the circular arc has also the property E^* at $e^{i\alpha}$ these relations are valid uniformly not only in a sector but, by 5.1.2, for all $z < D(z)$.

Now, let $z = re^{i\vartheta}$, $z^* = e^{i\vartheta^*}$ be two points lying on a circle with centre at $e^{i\alpha}$. Then we get

$$\frac{1}{1-|z|} \leqq \frac{\pi}{2}\frac{1}{\operatorname{arc}(z, z^*)} \leq \frac{\pi}{2}\max_{z\subset \operatorname{arc}}|\psi'(\zeta)|\cdot|\zeta-\zeta^*|^{-1} \leq o\left(\zeta^{\varrho/\pi+\varepsilon}\cdot(\zeta/\eta)\right).$$

Thus 5.6.1 is proved.

In order to deduce 5.6.2 we have to use the more precise result of Warschawski (cf. 5.0.6). Hence we shall find instead for 5.6.3, that

$$|(\psi(\zeta) - e^{i\alpha})\,\zeta^{\varrho/\pi}|$$

and

5.6.4 $$|\psi'(\zeta)\,\zeta^{(\varrho+\pi)/\pi}|$$

are between two finite positive constants. The rest of the proof is the same as above.

6.0. *In the unit-circle* $C(r, \theta)$ *we define, by the equation* $\theta = K(r, \lambda)$, *a system of curves* $K(\lambda)$ *which shall satisfy the following conditions:*

1°. *The function* $K(r, \lambda)$ *is continuous as a function of two variables, and periodic, with the period* 2π, *in* λ;

2°. $K(1, \lambda) = \lambda$ *and* $(dK/dr)_{r=1}$ *is finite;*

3°. *The curve* $\theta = K(r, \lambda)$ *has the property* E^* (*cf.* 5.0.1) *at the point* $r = 1, \theta = \lambda$.

6.0.1. A consequence of the definition is the uniform convergence of $K(\lambda_n)$ to $K(\lambda_0)$ as $\lambda_n \to \lambda_0$.

6.0.2. The second condition in 2° signifies geometrically that no curve is tangent to the circumference of the unit-circle.

6.0.3. If we say that $u(z)$ has a property in (α, β), then we mean that it has that property in the domain bounded by the circular arc $\alpha \leq \theta \leq \beta$ and by $K(\alpha)$, $K(\beta)$.

6.1. **Definition.** We shall say that $u(r, \theta)$ satisfies condition A if (i) it is harmonic in C and (ii) $\overline{\lim}_{r \to 1} |u(r, K(r, \lambda))|$ is finite for all λ except for those belonging to an enumerable set $\mathfrak{H}$.

6.2. If $u(r, \theta)$ satisfies condition A, then for every perfect set $\mathfrak{E}$, there is a number M_0 and a section $\mathfrak{E} \cdot (a, b)$, such that

$$|u(r, K(r, \lambda))| \leq M_0 \text{ for } \lambda < \mathfrak{E} \cdot (a, b).$$

Since $u(r, K(r, \lambda))$ is a continuous function of r and λ, the set

$$\mathfrak{A}(M) = \lim_{\varrho \to 1} \underset{\lambda}{E}[|u(r, K(r, \lambda))| \leq M, r \leq \varrho]$$

is closed. From (ii) it follows that

$$\lim_{M \to \infty} \mathfrak{A}(M) > C(\mathfrak{H})$$

and in particular[1]

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[1] We put $\mathfrak{H} = \sum_1^\infty \theta_n$.

$$\lim_{M\to\infty}\left[\mathfrak{A}(M)+\sum_1^\infty \theta_n\right]\cdot\mathfrak{E}=\mathfrak{E}.$$

Hence follows the existence of a M_0 such that $\mathfrak{A}(M_0)+\sum_1^{M_0}\theta_n$ is dense in a section of $\mathfrak{E}$. But, then, even $\mathfrak{A}(M_0)$ must be dense in a section $\mathfrak{E}\cdot(a, b)$. The rest follows from the closure of $\mathfrak{A}(M_0)$.

6. 3. If $u(r, \theta)$ satisfies the condition A, then the inner points of the set of points α, for which $u = RPI(e^{i\alpha})$, form an open everywhere dense set $\mathfrak{O}$. By $\mathfrak{F}$, we shall denote the closed set complementary to $\mathfrak{O}$.

If we take $\mathfrak{E}=(a, b)$ in 6. 2, we see that u^D is finite at a partial interval of (a, b), and since this is arbitrary it follows that u^D is finite at an everywhere dense set of points. By 2. 12, $\mathfrak{O}$ is, therefore, also everywhere dense.

6. 4. If $u(r, \theta)$ satisfies the condition A and $\mathfrak{F}=\mathfrak{P}+\mathfrak{R}$, where $\mathfrak{P}$ is perfect and $\mathfrak{R}$ reducible, then, if $\mathfrak{P}$ is not empty, we can find a closed interval (a, b), containing a section $\mathfrak{P}_1$ of $\mathfrak{P}$, and having no points of $\mathfrak{R}$ in its interior. Further, there exists an M, such that

6. 4. 1. $$|u(r, K(r, \lambda))| \leq M$$

for $\lambda < \mathfrak{P}_1$.

In one of the intervals complementary to $\mathfrak{R}$, there exists a section of $\mathfrak{P}$. To this we apply 6. 2.

7. 0. **Theorem.** *If*

(i) $u(r,\theta)$ *is harmonic in* C,

(ii) *the curve* $K(\vartheta)$, *defined in* 6. 0, *forms with the circumference the angle* $\beta(\theta)$

$$\left(\frac{\pi}{2}\leq\beta<\pi\right),$$

(iii) $\overline{\lim}_{r\to 1}|u(r, K(r, \vartheta))|$ *is finite for all* ϑ, *except for the points of an enumerable set* $\mathfrak{H}$.

(iv) $\underline{\lim}_{r\to 1} u(r, K(r, \vartheta)) \leq \sigma(\vartheta) \leq \overline{\lim}_{r\to 1} u(r, K(r, \vartheta))$ *where* $\sigma(\vartheta)$ *is L-integrable,*

(v) *for every* ϑ *and* ε, *there is a* $N(e^{i\vartheta})$, *such that*

$$|u(r,\theta)| \leq \exp\left[\varepsilon/(1-r)^{\pi/2\beta(\vartheta)}\right] \text{ for } (r,\theta) < N(e^{i\vartheta}),$$

(vi) *for every* ϑ *there exists an angle* $A(\vartheta)$: $c_1(1-r) \leq \theta - \vartheta \leq c_2(1-r)$ *having* $K(\vartheta)$ *in its interior, such that*

$$u(r,\theta) = o\left(\frac{1}{(1-r)^{n(\vartheta)}}\right)$$

$$\textit{for } (r,\theta) < A(\vartheta),\ n(\vartheta) \textit{ integral},$$

then $u = RPI(z)$ (*cf.* 2.1) *for all* $z < F(C)$, *except for the points of a reducible set* $\mathfrak{R}$.

Remark. Without changing the conclusions, it is possible to introduce a reducible set $\mathfrak{R}_1$ for the points of which no conditions need to be satisfied. We may, indeed, by the following reasoning, show that in every complementary interval of $\mathfrak{R}_1$, there is only a reducible set of singular points. The final set of singular points is again reducible.

7.0.1. In order to illustrate by simpler examples the meaning of the theorem, we enunciate two particular cases.

If (i) $u(r,\theta)$ *is harmonic in the unit circle* $C(r,\theta)$,

(ii) $\overline{\lim}_{r\to 1} |u(r,\theta)|$ *is finite, except at the points of an enumerable set* $\mathfrak{H}$,

(iii) $\underline{\lim}_{r\to 1} u(r,\theta) \leq \sigma(\theta) \leq \overline{\lim}_{r\to 1} u(r,\theta)$ *where* $\sigma(\theta)$ *is L-integrable*,

(iv) $u(r,\theta) = o\left(\frac{1}{(1-r)^m}\right)$ *uniformly for all* θ,

then $u = RPI(z)$ *for all* $z < F(C)$ *except for the points of a reducible set* $\mathfrak{R}$.

This theorem is fundamental for application to the uniqueness theory of summable trigonometrical series (cf. F. Wolf, On (C, k) summable trigonometrical series).[1]

7.0.2. *If* (i) $u(x,y)$ *is harmonic in the upper half-plane* H,

(ii) $\overline{\lim}_{y\to 0} |u(x,y)|$ *is finite except at the points of an enumerable set of points* $\mathfrak{H}$,

(iii) $\underline{\lim}_{y\to 0} u(x,y) \leq \sigma(x) \leq \overline{\lim}_{y\to 0} u(x,y)$, *where* $\int_{-\infty}^{\infty} \frac{|\sigma(x)|}{1+x^2}\,dx < \infty$,

(iv) $u(x,y) = o(y^{-m(X)})$ *uniformly in any finite interval* $(-X, X)$;

then $u = RPI(z)$ *for all* $z < F(H)$ *except for the points of a reducible set of points* $\mathfrak{R}$.

This theorem is equally fundamental for the uniqueness theory of (C, k) summable trigonometrical integrals.

It is deduced from 7.0 by the conformal representation of $H(\zeta)$ on $C(z)$. It is easy to define suitable $K(\vartheta)$.

[1] Proc. London Math. Soc. S. 2, Vol. 45, 1939, p. 328.

7.0.3. If the first four conditions of 7.0 are satisfied and

$$u(r,\theta) = \frac{\partial^n v(r,\theta)}{\partial r^m \partial^{n-m}\theta}$$

where $v(r,\theta)$ is a harmonic function continuous in $C + F(C)$, then the result of 7.0 holds good, with $n(\vartheta) = n$.

From 3.1 it follows indeed that,

$$\left|\frac{\partial v(z_0)}{\partial r}\right|, \left|\frac{\partial v(z_0)}{\partial \theta}\right| \leq \frac{1}{\pi(1-|z_0|)^2} \int\limits_{|z-z_0|=1-|z_0|} |v(z) - v(z_0)|\, dz = o\left(\frac{1}{1-|z_0|}\right);$$

similarly for higher derivatives

$$\frac{\partial^n v}{\partial r^m\, \partial \theta^{n-m}} = o(1 - |z_0|)^{-n}.$$

Hence 7.0 (v) and (vi) are satisfied.

7.0.4. If the first four conditions of 7.0 are satisfied and

(i) $$\lim_{R\to 1} \int_0^R (1-r)^{mp}\, r\, dr \int_0^{2\pi} |u(r,\theta)|^p\, d\theta < \infty, \qquad p \geq 1.$$

then the result of 7.0 holds good, with

$$n(\vartheta) = m + \frac{2}{p}.$$

We have indeed

$$|u(0,\theta)| = \left|\frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\, d\theta\right| = \left|\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi} u(r,\theta)\, d\theta\, r\, dr\right|$$

$$\leq \left[\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi} |u(r,\theta)|^p\, r\, d\theta\, dr\right]^{\frac{1}{p}}.$$

Further for an arbitrary $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\iint\limits_D (1-r)^{mp} |u(r,\theta)|^p\, r\, dr\, d\theta < \varepsilon$$

if the area of D is less than δ.

Now, we take a R such that the area of $R < r < 1$ is less than δ. Then, for all $r > \frac{2R+1}{3}$, the circle C_r round (r, θ) with the radius $\frac{1-r}{2}$ is in the domain and we get

$$u(r,\theta) \leq \left[\frac{4}{\pi(1-r)^2}\iint\limits_{C_r} |u(r,\theta)|^p \, r\,dr\,d\theta\right]^{\frac{1}{p}}$$

$$\leq \left[\frac{2^{2+mp}}{\pi(1-r)^{2+mp}}\iint\limits_{C_r} (1-r)^{mp}\,|u(r,\theta)|^p\, r\,dr\,d\theta\right]^{\frac{1}{p}}$$

$$\leq \frac{2^{m+\frac{2}{p}}\,\varepsilon^{\frac{1}{p}}}{\pi^{1/p}(1-r)^{m+\frac{2}{p}}}$$

or

$$u(r,\theta) = o(1-r)^{-m-2/p}.$$

7.1 If

(i) (α, β) is an interval of $\mathfrak{D}$ (cf. 6.3),

(ii) $u(r, K(r, \alpha))$ is bounded,

(iii) γ such that $\alpha < \gamma < \beta$, and $u_C(\gamma)$ is finite and therefore $u(r, K(r, \gamma))$ bounded,

(iv) $D = \underset{r,\theta}{E}[K(r,\alpha) < \theta < K(r,\gamma),\ 0 < r < 1] + \underset{r,\theta}{E}[r < R]$[1], and $z = \varphi(\zeta)$ is the conformal representation of $D(z)$ on $H(\zeta)$, such that

$$e^{i\alpha} = \varphi(\infty), \qquad 0 = \varphi(0),$$

then

7.1.1 $$u(\zeta) = p(\zeta) + s(\zeta)$$

where

7.1.2 $$p(\zeta) = PI_H u_H$$

and

7.1.3 $$s(\zeta) = \mathcal{J}\Big[\sum_{k=1}^{k \leq n(\alpha)\cdot\beta(\alpha)/\pi - 1} a_k \zeta^k\Big].$$

7.1.4. Further there is an angle $B(\alpha)$: $-\delta_1 < \vartheta < \delta_1$ in which $p(\zeta) = o(\zeta)$ uniformly for $\zeta \to \infty$.

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[1] $R < 1$, but so large that D should be connected.

The function $u(z)$ is an RPI at the points of $K(\alpha)$ and $K(\beta)$ (cf. 2.12) except possibly at their endpoints $e^{i\alpha}$, $e^{i\beta}$. From the definition of $\mathfrak{D}$ it follows that $u = RPI_C(e^{i\delta})$ $\alpha < \delta < \beta$. By 2.9 and 2.14 we have also $u = RPI_D(e^{i\gamma})$. The only possible singular point of $u(z)$ on $F(D)$ is, therefore, $e^{i\alpha}$. Condition (iv) shows that 4.0 (ii) is satisfied.

The domain D satisfies the conditions in 5.3 and by 7.0 (iv) and 5.3.1 u_D, which exists almost everywhere on $F(D)$, is integrable in the corresponding C. Hence 4.0 (iii) is satisfied, and 4.0 gives 7.1.1 and 7.1.2.

Let the inner angle of D at $e^{i\alpha}$ be denoted by ϱ ($\leq \beta(\alpha)$, cf. 7.0 (ii)). Now, we use the second part of 5.6 and we get from 7.0 (v)

7.1.5
$$|u(\zeta)| = |u(z)| \leq \exp\left[\varepsilon/(1-r)^{\pi/2\beta(\theta)}\right] \leq$$
$$\leq \exp\left[\varepsilon\, \zeta^{\varrho/2\beta} \sin^{-\pi/2\beta}(\arg\zeta)\right]$$
$$\leq \exp\left[\varepsilon\, \zeta^{-\frac{1}{2}} \sin^{-1}\arg\zeta\right].$$

We deduce in a similar way from 7.0 (vi) the existence of an angle $0 < \arg\zeta < \delta$ in which

7.1.6
$$u(\zeta) = o(\zeta^{\varrho\, n(\alpha)/\pi}).$$

From 7.1.1, 7.1.2 and 2.23 it follows that 7.1.5 and 7.1.6 are valid also for $s(\zeta)$. Now, 3.3 gives 7.1.3.

By 5.2.9, $\zeta = \varphi(z)$ represents a $D_1(z) > D(z)$ on $D_1(\zeta) > H(\xi)$. Further $D_1(\zeta) - H(\zeta)$ has a positive angle at ∞ below the real positive axis and may be supposed to be so small that $D_1(z) - D(z) < A(\alpha)$ (cf. 7.0 (vi)). Then, by 7.0 (vi) and 5.0.6, there is an angle $-\delta < \arg\zeta < \delta$ in $D_1(\zeta)$ in which $u = o(\zeta^{\varrho\, n(\alpha)/\pi})$. In $-\delta/2 < \arg\zeta < \delta/2$ we have, by 3.2, $f(\zeta) = u(\zeta) + iv(\zeta) = o(\zeta^{\varrho\, n(\alpha)/\pi})$. Further, by 7.1.3, $h(\zeta) = s(\zeta) + i\,t(\zeta) = o(\zeta^{\varrho\, n(\alpha)/\pi})$ and therefore also $g(\zeta) = p(\zeta) + i\,q(\zeta) = o(\zeta^{\varrho\, n(\alpha)/\pi})$ in the same angle.

Since we know from 2.24 and condition (ii) that $g(\zeta) = o(1)$ for any constant positive $\mathcal{J}(z)$, we can apply 3.5. We get $g(\zeta) = o(\zeta)$ in a certain partial angle $B(\alpha)$: $-\delta_1 < \arg\zeta < \delta_1$. Thus, 7.1.4 is proved.

7.2. Proof of 7.0. If the conditions of 7.0 are satisfied, then so is condition A of 6.1. If the set of exceptional points is not reducible, then, by 6.4, there is an interval (a, b), such that the exceptional points in (a, b) form a perfect set $\mathfrak{P}_1$, and there is a constant N such that

7.2.1
$$|u(r, K(r, \lambda))| \leq N$$

for $\lambda < \mathfrak{P}_1$. We shall show that this is impossible, by deducing that $u = RPI(a, b)$, which is an obvious contradiction to the existence of $\mathfrak{P}_1$ (cf. 6. 3).

Let $\alpha < \mathfrak{P}_1$ be a left-hand endpoint of an interval of $(a, b) - \mathfrak{P}_1$. Then all the conditions of 7. 1 are satisfied. We can find a sequence of points $\{\alpha_k\}$ such that $\alpha_k < \mathfrak{P}_1$, $\alpha_k < \alpha_{k+1}$ and $\lim \alpha_k = \alpha$. From 6. 0. 1 we know that $K(\alpha_k)$ converges uniformly to $K(\alpha)$. Hence for all sufficiently large k, $K(\alpha_k)$ intersect the nearer side S of the angle $B(\alpha, z)$ (cf. end of 7. 1). We call $C(\alpha, z)$ the angle formed by S and the bissectrice of that part of B which lies outside D. $\left(\text{Then } B(\alpha, \zeta) \text{ is asymptotically } -\delta_1 \leq \arg \zeta \leq -\frac{\delta_1}{2}\right)$. The parts of $K(\alpha_k)$ which lie in $C(\alpha, z)$ converge uniformly to $e^{i\alpha}$. In $H(\zeta)$ there will correspond to the sides of $C(\alpha, z)$ two curves K_1, K_2, which asymptotically form an angle $\delta_1/2$, and to the parts of $K(\alpha_k)$ correspond curves K_k^* which join K_1 and K_2 and converge uniformly to infinity. On K_k^*, we have, by 7. 2. 1, $|u| \leq N$ and by 7. 1. 4, $s(\zeta) = o(\zeta)$. But, by 7. 1. 3, $s(\zeta)$ is a polynomial to which we may apply 3. 4. This shows that $s(\zeta)$ is a constant which is, by 7. 1. 3 equal to zero.

Hence

$$u(z) = p(z) = PI_D u_D.$$

A similar reasoning we can apply to a right-hand endpoint of any interval of $(a, b) - \mathfrak{P}_1$. Hence we have proved that $u = RPI(z_0)$ for all $z_0 < F(D(\alpha, \beta))$, where $D(\alpha, \beta)$ denotes the domain bounded by $K(\alpha)$, $K(\beta)$ and the circular arc α, β. By 2. 20 this is equivalent to $u = PI_{D(\alpha, \beta)}$. The conditions of 2. 18 are satisfied for $D(a, b)$ (cf. 5. 3). Hence we get the desired contradiction $u = PI_{D(a, b)}$.

7. 3. The theorem 7. 0 remains valid, if we substitute (ii) by

(ii a) *the curve* $K(\vartheta)$, *defined in* 6. 0, *has the property* E^1 *and forms with the circumference an angle which is* **less** *than* $\beta(\theta)$.

We have used the property E^* in using 5. 2. 1 to establish 5. 6. 2 and 7. 1. 5. Now we use 5. 6. 1 in the same way and we get

$$|u(\zeta)| \leq \exp\left[\varepsilon\, \zeta^{(\varrho+\varepsilon_1)/2\beta(\theta)} \sin^{-1}(\arg \zeta)\right].$$

Since $\beta(\theta)$ is this time *less* than ϱ, we take ε_1 so small as to make $(\varrho + \varepsilon_1)/2\beta(\theta) \leq 1/2$ and we get again

$$|u(\zeta)| \leq \exp\left[\varepsilon\, \zeta^{1/2} \cdot \sin^{-1} \arg \zeta\right].$$

[1] Instead of E^*.

7.4. Let $L_1(\lambda)$, $L_2(\lambda)$ be two systems of curves in $C(r, \theta)$ given by the equations $\theta = L_k(r, \lambda)$ which satisfy the following conditions:

(i) The functions $L_k(r, \lambda)$, $k = 0, 1$ are continuous as functions of both variables and periodic with the period 2π in λ,

(ii) $L_k(1, \lambda) = \lambda$, and $\left(\frac{d L_1}{d r}\right)_{r=1}$, $\left(\frac{d L_2}{d r}\right)_{r=1}$ are finite and different from each other. That means that the two curves have at $(1, \lambda)$ tangents, different from each other and from that of the circumference.

7.4.1. **Theorem.**[1] *If*

(i) *$u(r, \theta)$ is harmonic in $C(r, \theta)$*

(ii) *$\overline{\lim\limits_{r \to 1}} \, |u(r, L_k(r, \lambda))|$ is finite for $k = 0, 1$ except for λ in an enumerable set of points $\mathfrak{H}$*

(iii) *there is a function $\sigma(\theta)$ integrable in $(0, 2\pi)$, lying between the larger upper and the smaller lower limit of $u(r, L_k(r, \theta))$, $k = 0, 1$ as $r \to 1$*

(iv) *for every ϑ and $\varepsilon > 0$ there is a $N(e^{i\vartheta})$ such that $|u| \leq \exp[\varepsilon/(1 - r)^{\pi/2\varrho}]$, where $\varrho(\vartheta)$ is larger than the largest angle between a L_K and the circumference;*

then $u = RPI(z)$ at all $z < F(D)$ except at the points of a reducible set $\mathfrak{R}$.

7.4.2. We shall say that $u(r, \theta)$ satisfies condition B, if $\overline{\lim\limits_{r \to 1}} \, |u(r, L_k(r, \vartheta))|$ exists finite for $k = 0, 1$ at all ϑ except those of an enumerable set of points $\mathfrak{H}$.

7.4.3. If $u(r, \theta)$ satisfies condition B, then for every perfect set $\mathfrak{E}$, there exists a M_0 and a section $\mathfrak{E} \cdot (a, b)$ such that

$$|u(r, L_k(r, \vartheta))| \leq M_0$$

for $\vartheta < \mathfrak{E} \cdot (a, b)$ and $k = 0, 1$.

Since $u(r, L_k(r, \vartheta))$ is continuous in (r, ϑ)

$$\mathfrak{A}(M) = \lim_{\varrho \to 1} \underset{\vartheta}{E}[|u(r, L_k(r, \vartheta))| \leq M, k = 0, 1, r \leq \varrho]$$

is closed. The rest of the proof is similar to that of 6.2.

7.4.4. If u satisfies condition B, then the results of 6.3. and 6.4. hold good. In 6.4.1, $K(r, \vartheta)$ should be substituted by $L_k(r, \vartheta)$, $k = 0, 1$.

7.4.5. Let a be a left-hand endpoint of an interval of $(a, b) - \mathfrak{P}_1$. By 7.4 (ii), one of the two $L_k(r, a)$ is less than the other in a certain neighbour-

[1] Note that this theorem has a much simpler proof than 7.0. It does not require the results of § 5.

hood of $r = 1$. We shall suppose that $L_1(r, \alpha) < L_2(r, \alpha)$, for $R \leq r < 1$. We define

$$D = \underset{(r, \theta)}{E}[L_1(r, \alpha) < \theta < L_2(r, \gamma),\ R \leq r < 1] + \underset{(r, \theta)}{E}[r < R]$$

where γ is chosen in the same way as in 7. 1. 3. Now, 7. 1. 1 and 7. 1. 2 are deduced in the same manner as in 7. 1.

If $\alpha_k < \mathfrak{P}_1$ and $\alpha_k \to \alpha$, then for sufficiently large k there exists domains

$$D_k = \underset{(r, \theta)}{E}[L_1(r, \alpha) < \theta < L_2(r, \alpha_k),\ R < r < 1].$$

From the hypothesis made about $L_k(r, \theta)$ it follows that D_k converges uniformly to $D_0 = \underset{(r, \theta)}{E}[L_1(r, \alpha) < \theta < L_2(r, \alpha),\ R < r < 1]$. Further, D_k has evidently no point in common with $F(C)$ and $u(z)$ is therefore continuous in D_k. Since it is on $L_2(\alpha_k)$, $\alpha_k < \mathfrak{P}_1$ and on $L_1(r, \alpha)$ in absolute value less than M, it is so in D_k and hence in D_0 also. If we represent $D(z)$ on $H(\zeta)$ in the same way as in 7. 1 then $D_0(\zeta)$ will be a domain with a positive angle at infinity. From the boundedness of u in $D_0(\zeta)$ it is easy to deduce, by means of 3.3 and 2.23, that $s(\zeta) = 0$. The rest of the proof is the same as in 7.2.

7.4.6. *If* (i) $u(r, \theta)$ *is harmonic in* $C(r, \theta)$, (ii) $u_C(\theta)$ *is finite and there exists an L-integrable function* $\sigma(\theta)$ *lying between the largest and the smallest value of* $u_C(\theta)$ *and* (iii) *for every* ϑ *and* $\varepsilon > 0$ *there is a* $N(e^{i\vartheta})$ *and* $M(\vartheta)$, *such that* $|u| \leq \exp[\varepsilon/(1-r)^M]$ *for* $(r, \theta) < N(e^{i\vartheta})$; *then* $u(r, \theta) = P I_C u_C(\theta)$.

It is easy to see that the conditions of 7.4. 5 are satisfied, if we choose e. g. $L_1(\lambda)$, $L_2(\lambda)$ to be the two straight lines going through $e^{i\lambda}$ and forming with the radius the angle $\pi(M-1)/2M$. In the angle $A\,(=\pi/2M(\alpha))$ between the circumference and the nearest $L_k(\alpha)$, 7. 1 gives $u = R\,P\,I_A(e^{i\alpha})$. Between $L_1(\alpha)$ and $L_2(\alpha)$, $u(z)$ is bounded. Hence we get $u = R\,P\,I_C(e^{i\alpha})$ for all α.

8. 0. In order to deduce from 7. 0 uniqueness theorems, it is sufficient to add conditions which make an isolated singular point impossible (cf. 4. 1). Indeed, if there is no isolated point, then the reducible set of singular points must be empty.

Let $z_0 < F(C)$ be an isolated point of the exceptional set $\mathfrak{R}$. We construct a neighbourhood $N(z_0)$ which contains no other exceptional point and is such that u_C is finite at $F(C) \cdot F(N)$. Then u is bounded on $F(N) \cdot C$. Hence, the only possible exceptional point on $F(N \cdot C)$ is z_0. Condition 4. 0 (ii) is therefore satisfied in the corresponding H. Further, in virtue of 7. 0 (iv), u_C is integrable on $F(C) \cdot N$, bounded on $F(N) \cdot C$ and, by 5.3, it is, therefore, integrable in the corresponding C. Hence 4.0 (iii) is also satisfied.

Now, we can apply 4.1 in the corresponding H. The conformal representation $\zeta = \varphi(z)$ of $H(\zeta)$ on $N \cdot C$, is analytic in $z = z_0$. We have, therefore,

8.0.1 $$\lim_{z \to z_0} (z - z_0)\zeta = \text{const}$$

and hence, by 5.6.2 we get

8.0.2. $$\lim_{|\zeta| \to \infty} (1 - r)|\zeta|^2/\eta = \text{const.}$$

8.1. At an isolated point of $\mathfrak{R}$ which does not belong to $\mathfrak{H}$, we cannot have $n(\alpha) = 1$ (cf. 7.0 (vi)).

Let α be the point. Then 7.1 shows that $s(\zeta)$ is a polynomial of degree $n(\alpha)\beta(\alpha)/\pi - 1$. If, now, we apply the same conformal representation as in 8.0 we find by 8.0.1 that in $A(\alpha, \zeta)$, which corresponds to $A(\alpha)$, $s(\zeta) = o(\zeta^{n(\alpha)})$. Hence $s(\zeta)$ is a polynomial of degree at most $n(\alpha) - 1$. If $n(\alpha) = 1$, then $u = RPI(e^{i\alpha})$ and α cannot be a point of $\mathfrak{R}$.

8.1.1. At an isolated point α of $\mathfrak{R}$ at which to every ε, there is a $N(e^{i\alpha})$, in which $|u(z)| \leq \exp[\varepsilon/(1 - r)^{1/2}]$ $n(\alpha)$ cannot be equal to 1.

If we represent, as in 8.0, $N \cdot C$ conformally on H, then it is easy to see that $s(\zeta)$ satisfies the conditions of 3.3. Hence $s(\zeta)$ is a polynomial. If $n(\alpha) = 1$ then, as in 8.1, α cannot be a point of $\mathfrak{R}$.

8.2. The point α cannot be both an isolated point of $\mathfrak{R}$ and satisfy one of the following conditions:

(i) if α belongs to $\mathfrak{H}$, then to every $\varepsilon > 0$ there is a $N(e^{i\alpha})$ in which $|u(z)| \leq \exp[\varepsilon/(1 - r)^{1/2}]$ and $n(\alpha) = 1$.

(ii) if α does not belong to $\mathfrak{H}$, then there exists a sequence of points $\{a_K(\alpha)\}$ such that (a) $\lim_{K \to \infty} a_K(\alpha) = e^{i\alpha}$, (b) $\lim_{K \to \infty} \arg(a_K - e^{i\alpha}) - \alpha = \pi\varkappa$, $\varkappa$ no fraction with a denominator less than $n(\alpha)$ (c) $u(a_K) = o(1/(a_K - e^{i\alpha}))$.

At the point α, $s(\zeta)$ will be again a polynomial of degree $n(\alpha) - 1$. Now, 4.2 gives $u = RPI(e^{i\alpha})$.

8.3.1. *If the conditions of* 7.0 *are satisfied, and those of* 8.2 *are satisfied for all* α, *then* $u = PI_C u_C$.

8.3.2. *If the conditions of* 7.0 *are satisfied and for all* $\alpha < \mathfrak{H}$, *we can find to every* ε *a* $N(e^{i\alpha})$ *in which* $|u(z)| \leq \exp[\varepsilon/(1 - r)^{1/2}]$ *and* $n(\alpha) = 1$; *for all the* *other* α: *if* $\beta(\alpha) = \pi\varkappa$ (cf 7.0 (11)), $\varkappa$ *not a fraction with a denominator less than* $n(\alpha)$, *then* $u = P I_C u_C$.

For α not belonging to $\mathfrak{H}$, the points of $K(\alpha)$ have all the properties of the sequence required in 8.2.

8.3.3. *If the conditions of* 7.0 *are satisfied with* $n(\vartheta) = 2$ *and, moreover, for all* $\alpha < \mathfrak{H}$, *we can find to every* ε *a* $N(e^{i\alpha})$ *in which* $|u(z)| \leq \exp[\varepsilon/(1-r)^{1/2}]$ *and* $n(\alpha) = 1$, *then* $u = P I_C u_C$.

This is a simple corollary of 8.3.2, since $\beta(\alpha) \neq \pi$.

8.3.4. *If* $K(\alpha)$ *are straight lines through a point* $z_0 < C$, $z_0 \neq 0$, $\mathfrak{H}$ *is empty and* $n(\vartheta) < N$, *then* $\mathfrak{R}$ *has at most* $2N(N-1) \cdot \text{arc sin} |z_0|/\pi + 2$ *points*.

If $z_0 = r_0 e^{i\theta_0}$, then

$$\beta(\theta) = \frac{\pi}{2} + \text{arc tg} \frac{r_0 \sin(\theta - \theta_0)}{1 - r_0 \cos(\theta - \theta_0)}$$

for $\theta_0 \leq \theta \leq \theta_0 + \pi$. For $\theta_0 - \pi \leq \theta \leq \theta_0$, we have $\beta(\theta) = \beta(2\theta_0 - \theta)$. The function increases from $\beta(\theta_0) = \frac{\pi}{2}$ to $\beta(\theta_0 + \text{arc cos}\, r_0) = \frac{\pi}{2} + \text{arc sin}\, r_0$ and decreases back to $\beta(\theta_0 + \pi) = \frac{\pi}{2}$. Hence it takes every value four times in the interval $(0, 2\pi)$. A point can be a singular point if $\beta(\theta) = \frac{m}{n}\pi$, $n < N$. Further $\left|\frac{m}{n}\right| \pi \leq \text{arc sin}\, r_0$. There are, therefore, not more than $2N(N-1)\,\text{arc sin}\, r_0/\pi + 2$ points of R.

8.4.1. *If* $u(z)$ *satisfies the first four conditions of* 7.0 *and it is the first derivative of a function continuous in the closed* C, *then* $u = P I_C u_C$.

This is a consequence of 7.0.3 and 8.1.

8.4.2. *If* $u(z)$ *satisfies the first four conditions of* 7.0, *if at the points of* $\mathfrak{H}$, $n(\vartheta) = 1$ *and if* $u(z)$ *is the second derivative of a function continuous in the closed* C, *then* $u = P I_C u_C$.

This follows from 7.0.3 and 8.3.3.

8.4.3. *If the first four conditions of* 7.0 *are satisfied, and* 7.0.4 (i) *is fulfilled with* $m + 2/p \leq 1$, *then* $u = P I_C u_C$.

This follows from 7.0.4 and 8.1.

8.4.4. *If* $u(z)$ *satisfies the first four conditions of* 7.0 *and* 7.0.4 (i) *is fulfilled with* $m + 2/p \leq 2$ *and* $H = 0$, *then* $u = P I_C u_C$.

This follows from 7.0.4 and 8.2.3.

Remark. The two last results can be slightly generalized by means of 4.3. It is possible to introduce into the first integral of 7.0.4 (i) $\log^p \frac{1}{1-r}$ without impairing the results.

9.0. **Theorem.** *If*

(i) *$u(z)$ is harmonic in D;*

(ii a) *except for the points of a reducible set $\mathfrak{R}_1$, $F(D)$ has at all points two half-tangents, forming a positive angle; there is a system of curves $K(\zeta)$, $\zeta < F(D)$ given by the function $z = K(\zeta, t)$, $0 \leq t \leq 1$, which is continuous in both variables. Further $\zeta = K(\zeta, 1)$, $z_0 = K(\zeta, 0) < D$, $K(\zeta)$ is analytic in a neighbourhood of ζ and forms with $F(D)$ angles which are positive and less than $\beta(\zeta)$.*

(iii) *$\overline{\lim\limits_{t\to 1}} |u(K(\zeta, t))|$ is finite for all ζ, except for the points of an enumerable set $\mathfrak{H}$;*

(iv) *$\underline{\lim\limits_{t\to 1}}\, u(K(\zeta, t)) \leq \sigma(\zeta) \leq \overline{\lim\limits_{t\to 1}}\, u(K(\zeta, t))$ where $\sigma(\zeta)$ is L-integrable in the corresponding C,*

(v) *For every $\zeta < F(D) - \mathfrak{R}_1$ and $\varepsilon > 0$, there is a $N(\zeta)$ such that $|u(z)| \leq$ $\leq \exp[\varepsilon/\varrho^{\pi/2\beta(\zeta)}]$ for $z < N(\zeta) \cdot D$, $\varrho = \min\limits_{\zeta_1 \subset F(D)} |z - \zeta_1|$;*

(vi) *for every $\zeta < F(D) - \mathfrak{R}$, there exists an angle $A(\zeta)$ having $K(\zeta)$ in its interior, such that*

$$u(z) = o(\varrho^{-n(\zeta)})$$

for $z < A(\zeta)$.

Then $u = RPI(\zeta)$ for all $\zeta < F(D)$ except for the points of a reducible set $\mathfrak{R}$.

9.0.1. (ii) *If we add the condition that $F(D)$ should satisfy 5.2.3, then we may take $\beta(\zeta)$ equal to the greater angle which $K(\zeta)$ forms with $F(D)$ at ζ. This relaxes somewhat* (v).

9.0.2. If we represent conformally $D(z)$ on $C(z')$, by $z = \varphi(z')$, then all the respective conditions of 7.0 are satisfied.

We can disregard the reducible exceptional set of points, since we can make the reasoning for each of the complementary interval, and at the end get the same result. Since $F(D)$ has a corner at all points, $K(\zeta')$ will still have the property E, by 5.1.3 and (ii), respectively (ii a) (cf. 7.3) will be satisfied. The transformation of conditions (iii) and (iv) is immediate.

Conditions (v) and (vi) must be transformed by means of 5.0.5 in a way very similar to 5.6 and go over into the corresponding conditions of 7.0.

9.1. Now, we want to give a few criterii for the L-integrability of $\sigma(\zeta)$ in the corresponding C.

9.1.1. *If D is convex and $\sigma(\zeta)$ is integrable on $F(D)$, then $\sigma(\zeta)$ is integrable in the corresponding C.*

This follows from 5.4.1 in a way similar to 5.3.

9.1.2. *If $F(D)$ has bounded outer curvature and $\sigma(\zeta)$ is integrable on $F(D)$, then $\sigma(\zeta)$ is integrable in the corresponding C.*

We use 5.4.2 instead of 5.4.1.

9.1.3. It is also possible to suppose that on certain parts of $F(D)$, u^D is bounded, the rest satisfying 9.1.1 or 9.1.2. From the results in § 7 and § 8, it is easy to deduce a number of theorems adapted to different individual cases. The only difficulty lies evidently in the field of conformal representation.