RIEMANN SPACES OF CLASS ONE AND THEIR CHARACTERIZATION.

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When one deals with problems of differential geometry having to do with the existence of specified properties of a space one is invariably confronted with the integration of a system of differential equations. It is sometimes possible to reduce this system — frequently the defining equations of the property in question — to an equivalent system of differential equations exhibiting greater simplicity in certain respects. When this has been done the reduced system is usually said to furnish a solution of the problem although in no fundamental sense is this correct since these latter conditions are likewise of differential character. Now it can be shown under very general conditions that the question of the existence of a solution of a system of differential equations can be reduced to the question of the existence of a solution of a system of algebraic equations to which can be applied the highly developed theory of algebraic elimination.[1] This procedure of algebraic elimination will lead to a set of conditions involving polynomials in the fundamental structural functions of the space and their derivatives, necessary and sufficient for the existence of the property under consideration. We embody precise types of such conditions which are of especial interest in the following definition of the algebraic characterization.

In a recent paper entitled *Algebraic characterizations in complex differential geometry*[2] I have considered the question of expressing necessary and sufficient conditions for the existence of a property P of a generalized complex space by conditions of the form

$$F_1 = 0, \quad F_2 \neq 0,$$

where F_1 and F_2 represent sets of definite polynomials in the structural functions of the space and their derivatives to a certain order, and the nonequality sign $\neq$ is interpretated to apply to at least one of the polynomials of the set F_2. If such conditions exist they are said to give an algebraic characterization of the property P. When dealing with a real space it is evident that we must augment the above signs $=$ and $\neq$ by the sign $>$ and even by the combination sign $\geqq$. Thus we shall say that the conditions

$$F_1 = 0, \; F_2 \neq 0, \; F_3 > 0, \; F_4 \geqq 0$$

constitute an algebraic characterization of a property P of a real space, where the F's have the above specified significance, provided that these conditions are necessary and sufficient for the existence of the property P. A simple example of an algebraic characterization is afforded by the equations expressing the vanishing of the curvature tensor of one of the various spaces for which such tensors have been found, these equations giving in fact necessary and sufficient conditions for the space in question to be flat. Other examples only slightly more complicated have been given in a paper by J. Levine and the present author.[3] On the other hand it can be shown that certain spatial properties do

[1] See, T. Y. Thomas, *Un corollaire du théorème de Riquier*, Bull. des Sci. Math. 59, 1935, p. 134.

[2] Trans. Am. Math. Soc., 38, 1935, p. 501.

[3] *On a class of existence theorems in differential geometry*, Bull. Am. Math. Soc. 35, 1934, p. 721.

not admit an algebraic characterization.[1] It is therefore of significance to inquire concerning the existence or non-existence of an algebraic characterization for any specified property P of a space and this question gives rise to a host of interesting and difficult problems in differential geometry.

The following paper deals with the algebraic characterizations of (real) RIEMANN spaces as spaces of class one. In a recent paper by WEISE[2] there are to be found a number of interesting equations expressing necessary conditions for a RIEMANN space to be of class one and some of these have been used in the present paper; but he has not arrived at a true algebraic characterization of such spaces as above defined. In connection with the solution of this problem we have defined an integer invariant of a RIEMANN space which we have called the *type number* of the space (§ 5). If the type number is one the space is flat and hence fails to be of class one. We have excluded those RIEMANN spaces of type number two inasmuch as the discussion of such spaces requires essentially different methods than those of higher type number and it has therefore been thought best to make these spaces the occasion of a separate investigation. For all other cases the algebraic characterizations have been constructed.[3]

The first three sections of the following paper are of an introductory character and as such afford an easy approach to the problem under consideration.[4] They have been added primarily, however, since the precise formulations which they contain are desirable from the standpoint of the later treatment.

1. Fundamental Forms of a Hypersurface.

Let $y^1, \ldots, y^{n+1}$ be the coordinates of a rectangular cartesian coordinate system of an $(n+1)$-dimensional Euclidean space E. Define in E a hypersurface S by the equations

$$y^i = \varphi^i(x^1, \ldots, x^n), \qquad (i = 1, \ldots, n+1), \tag{1.1}$$

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[1] Trans. *loc. cit.* and T. Y. THOMAS, *On the metric representations of affinely connected spaces*, Bull. Am. Math. Soc., 42, 1936, p. 77. These papers contain a proof of the non-existence of an algebraic characterization of the metric spaces in the class of all complex affinely connected spaces.

[2] *Beitrage zum Klassenproblem der quadratischen Differentialformen*, Math. Annalen, 110, 1935, p. 522.

[3] The conditions defining the algebraic characterization are considered to be known if they are definitely obtainable by the recognized procedures of algebraic elimination.

[4] Cf. DUSCHEK-MAYER, *Lehrbuch der Differentialgeometrie*, II, *Riemannsche Geometrie*, Teubner, 1930. EISENHART, *Riemannian Geometry*, Princeton University Press, 1926. LEVI-CIVITA, *The Absolute Differential Calculus*, Blackie and Son, 1929.

where the φ's are continuous and differentiable functions of the variables x^α of any (open and simply connected) neighborhood U of a point x_0^α of the real n-dimensional number space; for the requirements of the following discussion we assume that the functions φ possess continuous partial derivatives in U to the order three inclusive. The condition that S be a hypersurface (regular n-dimensional locus) is expressed analytically by the requirement that the functional matrix

$$\left\| \begin{matrix} \dfrac{\partial \varphi^1}{\partial x^1} \cdots \dfrac{\partial \varphi^{n+1}}{\partial x^1} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \dfrac{\partial \varphi^1}{\partial x^n} \cdots \dfrac{\partial \varphi^{n+1}}{\partial x^n} \end{matrix} \right\|$$

be of rank n in U.

We define the element of distance ds in the Euclidian space E by the quadratic differential form

$$ds^2 = \sum_{i=1}^{n+1} (d\, y^i)^2;$$

when we restrict ourselves to displacements in the hypersurface S this form becomes

$$ds^2 = \sum_{i=1}^{n+1} \sum_{\alpha,\beta=1}^{n} \frac{\partial\, y^i}{\partial x^\alpha} \frac{\partial\, y^i}{\partial x^\beta} d\, x^\alpha\, d\, x^\beta$$

with reference to arbitrary (differential) displacements dx^α in S. Or we may write

$$ds^2 = \sum_{\alpha,\beta=1}^{n} g_{\alpha\beta}(x)\, d\, x^\alpha\, d\, x^\beta, \tag{1.2}$$

where the coefficients $g_{\alpha\beta}$ have the values

$$g_{\alpha\beta} = \sum_{i=1}^{n+1} \frac{\partial \varphi^i}{\partial x^\alpha} \frac{\partial \varphi^i}{\partial x^\beta}. \tag{1.3}$$

The differential form (1.2) is called the *first fundamental form* of the hypersurface S and its coefficients $g_{\alpha\beta}$ are continuous functions of the variables x^α with continuous first and second derivatives in U.

A vector ξ of the space E with components ξ^i is said to be normal to a vector ζ of E having components $\zeta^i = dy^i$ if

$$\sum_{i=1}^{n+1} \xi^i \, dy^i = 0.$$

If the vector ξ is associated with a point of the hypersurface S and if the above condition is satisfied for every set of values dy^i given by (1. 1) the vector ξ is said to be normal to S at the point in question; hence, owing to the arbitrariness of the dx^α, the condition of normality becomes

$$\sum_{i=1}^{n+1} \sum_{\alpha=1}^{n} \xi^i y^i_{,\alpha} \, dx^\alpha = 0, \text{ or } \sum_{i=1}^{n+1} \xi^i y^i_{,\alpha} = 0, \qquad (\alpha = 1, \ldots, n),$$

where the partial derivatives $\partial y^i / \partial x^\alpha$ have been denoted by $y^i_{,\alpha}$. The quantities σ^i are said to be the direction cosines of the vector ξ if they satisfy the last set of equations and are further more so chosen that the sum of their squares is equal to unity, i. e. if

$$\sum_{i=1}^{n+1} \sigma^i y^i_{,\alpha} = 0, \qquad (\alpha = 1, \ldots, n), \tag{1.4}$$

$$\sum_{i=1}^{n+1} \sigma^i \sigma^i = 1. \tag{1.5}$$

Since the functional determinant of (1. 1) is of rank n the equations (1. 4) will have but one solution in their fundamental system and hence the solution of (1. 4) and (1. 5) will be determined uniquely to within algebraic sign corresponding to the ambiguity in the direction of the normal vector ξ to the hypersurface S. On account of the above hypothesis of differentiability the direction cosines σ^i will be continuous functions of the variables x^α with continuous partial derivatives to the second order.

Now consider a second hypersurface S_* defined by the equations

$$y^i_* = \varphi^i(x) + \varepsilon \, \sigma^i(x),$$

where the $\sigma^i(x)$ are the above direction cosines of the normal vectors ξ to the hypersurface S and ε is an infinitesimal; the surfaces S and S_* are said to be *parallel* in the sense that S_* may be thought of as being generated by laying off a distance of constant length ε along the normals ξ to S. The above equations defining S_* specify at the same time a one to one continuous correspondance between the points of S and S_*. By differentiation of these equations we easily deduce the relation

$$ds_*^2 = ds^2 + \sum_{i=1}^{n+1} 2\,\varepsilon\, d y^i\, d\sigma^i + \sum_{i=1}^{n+1} \varepsilon^2 (d\sigma^i)^2,$$

where ds and ds_* are the differential elements of distance between corresponding points. Or

$$ds_*^2 - ds^2 = -\,2\,\varepsilon\,\psi, \text{ where}$$

$$\psi = -\sum_{i=1}^{n+1} dy^i\, d\sigma^i, \tag{1.6}$$

neglecting terms of order higher than the first in the infinitesimal ε. The differential form ψ is called the *second fundamental form* of the hypersurface S. It can be expressed as a quadratic differential form in the arbitrary quantities dx^α; in fact we have

$$dy^i = \sum_{\alpha=1}^{n} y^i_{,\alpha}\, dx^\alpha,\ \ d\sigma^i = \sum_{\alpha=1}^{n} \sigma^i_{,\alpha}\, dx^\alpha$$

so that substituting into (1.6) we obtain

$$\psi = \sum_{\alpha,\beta=1}^{n} b_{\alpha\beta}\, dx^\alpha\, dx^\beta, \text{ where} \tag{1.7}$$

$$b_{\alpha\beta} = b_{\beta\alpha} = -\frac{1}{2}\sum_{i=1}^{n+1} [y^i_{,\alpha}\sigma^i_{,\beta} + y^i_{,\beta}\sigma^i_{,\alpha}]. \tag{1.8}$$

It follows from (1.8) and the preceding observations regarding continuity and differentiability that the coefficients $b_{\alpha\beta}$ of the second fundamental form are continuous functions of the variables x^α possessing continuous first partial derivatives in U.

We shall now derive another expression for ψ which will have application later. Differentiation of (1.4) gives

$$\sum_{i=1}^{n+1} [\sigma^i_{,\beta}\, y^i_{,\alpha} - \sigma^i\, y^i_{,\alpha\beta}] = 0, \qquad (\alpha, \beta = 1, \ldots, n),$$

where the $y^i_{,\alpha\beta}$ denote the second partial derivatives of the functions y^i. Hence

$$-\frac{1}{2}\sum_{i=1}^{n+1}[\sigma^i_{,\alpha}y^i_{,\beta}+\sigma^i_{,\beta}y^i_{,\alpha}]=\sum_{i=1}^{n+1}\sigma^i y^i_{,\alpha\beta}$$

and since the left member of this equation is the same as the right member of (1.8) we obtain

$$b_{\alpha\beta}=\sum_{i=1}^{n+1}\sigma^i y^i_{,\alpha\beta}. \tag{1.9}$$

Now it is evident that the quantities y^i and σ^i can be regarded as *scalars* with respect to transformations of the independent variables x^α or coordinates of the hypersurface S. Adopting this point of view the quantities $y^i_{,\alpha}$ and $\sigma^i_{,\alpha}$ are the components of covariant vectors and the $g_{\alpha\beta}$ as defined by (1.3) are the components of a covariant tensor which is called the *fundamental metric tensor* of the hypersurface S. Likewise the $b_{\alpha\beta}$ defined by (1.8) are the components of a covariant tensor and in fact we easily see the tensor character of all preceding equations. Also it follows readily[1] that the determinant $|g_{\alpha\beta}|$ does not vanish and indeed is positive in U so that the form (1.2) serves as the basis of the process of covariant differentiation in the hypersurface S. Taking the covariant derivative instead of the partial derivative of (1.4) the above process by which the equations (1.9) were deduced will lead to the equations

$$b_{\alpha\beta}=\sum_{i=1}^{n+1}\sigma^i y^i_{,\alpha,\beta}, \tag{1.10}$$

where the $y^i_{,\alpha,\beta}$ are the components of the second covariant derivatives of the scalars y^i and these components are symmetric in their lower indices.

Still another expression for the $b_{\alpha\beta}$ can be obtained by covariant differentiation of (1.4); this gives

$$\sum_{i=1}^{n+1}[\sigma^i_{,\beta}y^i_{,\alpha}+\sigma^i y^i_{,\alpha,\beta}]=0,$$

[1] In fact ds^2 must be *positive* whenever all the dx^α are not equal to zero; for suppose that

$$\sum g_{\alpha\beta}\,dx^\alpha dx^\beta=\sum y^i_{,\alpha}y^i_{,\beta}\,dx^\alpha dx^\beta=\sum\left[\sum y^i_{,\alpha}dx^\alpha\right]\left[\sum y^i_{,\beta}dx^\beta\right]=0.$$

But this implies the vanishing of the bracket expressions and hence all $dx^\alpha=0$ since the rank of the functional matrix $\left\|y^i_{,\alpha}\right\|$ is n. Hence the above form is positive definite and it follows from a theorem in algebra that the determinant $|g_{\alpha\beta}|$ is everywhere greater than zero.

and from these equations and (1. 10) we have

$$b_{\alpha\beta} = -\sum_{i=1}^{n+1} \sigma^i_{,\beta} y^i_{,\alpha}. \tag{1. 11}$$

2. Generalized Gauss and Codazzi Equations.

We shall now derive further *necessary* equations connected with the hypersurface S. Starting with the equations

$$g_{\alpha\beta} = \sum_{i=1}^{n+1} y^i_{,\alpha} y^i_{,\beta}$$

which define the coefficients of the first fundamental form of S we obtain by covariant differentiation the following equations

$$\sum_{i=1}^{n+1} [y^i_{,\alpha} y^i_{,\beta,\gamma} + y^i_{,\alpha,\gamma} y^i_{,\beta}] = 0,$$

$$\sum_{i=1}^{n+1} [y^i_{,\beta} y^i_{,\gamma,\alpha} + y^i_{,\beta,\alpha} y^i_{,\gamma}] = 0,$$

$$\sum_{i=1}^{n+1} [y^i_{,\gamma} y^i_{,\alpha,\beta} + y^i_{,\gamma,\beta} y^i_{,\alpha}] = 0,$$

the second and third of which result by cyclic permutation of the indices in the first equation. By adding the second and third and subtracting the first of these equations we obtain

$$\sum_{i=1}^{n+1} y^i_{,\alpha,\beta} y^i_{,\gamma} = 0. \tag{2. 1}$$

If we keep α and β fixed in (2. 1) we have n equations in the $n + 1$ unknowns $y^i_{,\alpha,\beta}$; and since the matrix $\|y^i_{,\gamma}\|$ of this system has the rank n the equations have but one independent solution. But it follows from (1. 4) that σ^i is a solution of (2. 1). Hence the most general solution of (2. 1) is given by

$$y^i_{,\alpha,\beta} = b_{\alpha\beta}\sigma^i, \tag{2. 2}$$

where the quantities $b_{\alpha\beta}$ are arbitrary functions of the coordinates x^α, symmetric in the indices α and β. However if we multiply (2. 2) by σ^i and sum on the index i we see from (1. 5) and (1. 10) that the $b_{\alpha\beta}$ in the above equations must actually be the coefficients of the second fundamental form of the hypersurface S.

By covariant differentiation of (2. 2) we obtain

$$y^i_{,\alpha,\beta,\gamma} = b_{\alpha\beta,\gamma}\,\sigma^i + b_{\alpha\beta}\,\sigma^i_{,\gamma}. \tag{2. 3}$$

To determine the quantities $\sigma^i_{,\gamma}$ in these equations we consider the system composed of (1. 11) and the equations resulting from covariant differentiation of (1. 5) i. e.

$$\left\{\begin{aligned} &\sum_{i=1}^{n+1} \sigma^i \sigma^i_{,\gamma} = 0, \\ &\sum_{i=1}^{n+1} y^i_{,\beta}\,\sigma^i_{,\gamma} = -\,b_{\beta\gamma}. \end{aligned}\right. \tag{2. 4}$$

The determinant of this system, considered as a system of linear equations for the determination of the unknowns $\sigma^i_{,\gamma}$, is

$$\begin{vmatrix} \sigma^1 & \cdots & \sigma^{n+1} \\ y^1_{,1} & \cdots & y^{n+1}_{,1} \\ \cdot & \cdot & \cdot \\ y^1_{,n} & \cdots & y^{n+1}_{,n} \end{vmatrix};$$

and this determinant is different from zero since its square is the determinant $|g_{\alpha\beta}|$ in consequence of (1. 3), (1. 4) and (1. 5). Hence (2. 4) has a unique solution $\sigma^i_{,\gamma}$ which is in fact easily seen to be given by

$$\sigma^i_{,\gamma} = -\sum_{\mu,\nu=1}^{n} b_{\gamma\mu}\, g^{\mu\nu}\, y^i_{,\nu}. \tag{2. 5}$$

Substituting these values of $\sigma^i_{,\gamma}$ into (2. 3) we have

$$y^i_{,\alpha,\beta,\gamma} = b_{\alpha\beta,\gamma}\,\sigma^i - b_{\alpha\beta} \sum_{\mu,\nu=1}^{n} b_{\gamma\mu}\, g^{\mu\nu}\, y^i_{,\nu}. \tag{2. 6}$$

We now express the fact that the left members of (2. 6) satisfy the identity[1]

$$y^i_{,\alpha,\beta,\gamma} - y^i_{,\alpha,\gamma,\beta} = -\sum_{\nu=1}^{n} y^i_{,\nu} B^\nu_{\alpha\beta\gamma} = -\sum_{\mu,\nu=1}^{n} y^i_{,\nu} g^{\mu\nu} B_{\mu\alpha\beta\gamma},$$

where the B's are the components of the curvature tensor of the hypersurface S. Thus

$$(2.7) \qquad (b_{\alpha\beta,\gamma} - b_{\alpha\gamma,\beta})\sigma^i + \sum_{\mu,\nu=1}^{n} [b_{\alpha\gamma} b_{\beta\mu} - b_{\alpha\beta} b_{\gamma\mu} + B_{\mu\alpha\beta\gamma}] g^{\mu\nu} y^i_{,\nu} = 0.$$

If we multiply these equations by σ^i and make use of (1. 4) and (1. 5) we obtain

$$(2.8) \qquad b_{\alpha\beta,\gamma} = b_{\alpha\gamma,\beta}; \text{ also}$$

$$(2.9) \qquad B_{\alpha\beta\gamma\delta} = b_{\alpha\delta} b_{\beta\gamma} - b_{\alpha\gamma} b_{\beta\delta},$$

i. e. the bracket expression in (2. 7) vanishes in consequence of (2. 8) and the fact that the matrices $\|g^{\mu\nu}\|$ and $\|y^i_{,\nu}\|$ are each of rank n. The equations (2. 9) generalize the equation obtained by Gauss and (2. 8) those obtained by Codazzi for the special case of two dimensional surfaces. In their generalized form these equations were first obtained by Voss; in the following we shall refer to them simply as the Gauss and Codazzi equations of the hypersurface S.

3. Gauss and Codazzi Equations as Conditions for a Riemann Space to be of Class one.

Let us now consider an n-dimensional Riemann space with element of distance defined by a positive definite quadratic differential form

$$(3.1) \qquad ds^2 = \sum_{\alpha,\beta=1}^{n} g_{\alpha\beta}\, dx^\alpha dx^\beta, \qquad (g_{\alpha\beta} = g_{\beta\alpha}),$$

where the g's are continuous functions possessing continuous first and second derivatives in a neighborhood U of a point x^α_0 of the n-dimensional number space. We ask under what conditions this space can be regarded as a hypersurface S in a Euclidean space E of $n+1$ dimensions, i. e. under what conditions will there

[1] See, for example, T. Y. Thomas, *The Differential Invariants of Generalized Spaces*, Cambridge University Press, 1934, p. 44, Eq. (13. 8). Attention is called to the fact that the above components of the curvature tensor are the negatives of those used by some writers.

exist a set of equations (1. 1) defining a hypersurface S in E such that the intrinsic element of distance of S is given precisely by the above form (3. 1). If this hypersurface S exists and is not an n-dimensional plane the RIEMANN space is said to be of *class one.*

The above problem is equivalent to the problem of finding conditions for the form (3. 1) and another form

$$\psi = \sum_{\alpha,\beta=1}^{n} b_{\alpha\beta}\, dx^\alpha dx^\beta, \qquad (b_{\alpha\beta} = b_{\beta\alpha}),$$

where the b's are continuous functions with continuous first derivatives in U, to be the first and second fundamental forms of a hypersurface S defined by equations of the type (1. 1). Approaching the problem from this standpoint we consider the system

$$(3.\ 2)\qquad \begin{cases} \dfrac{\partial y^i}{\partial x^\alpha} = y^i_{,\alpha}, \\[2ex] \dfrac{\partial y^i_{,\alpha}}{\partial x^\beta} = \displaystyle\sum_{\nu=1}^{n} \Gamma^\nu_{\alpha\beta}\, y^i_{,\nu} + b_{\alpha\beta}\,\sigma^i, \\[2ex] \dfrac{\partial \sigma^i}{\partial x^\alpha} = -\displaystyle\sum_{\mu,\nu=1}^{n} b_{\alpha\mu}\, g^{\mu\nu}\, y^i_{,\nu}. \end{cases}$$

The first set of these equations was previously introduced as the equations defining the quantities $y^i_{,\alpha}$, the second set in which the $\Gamma^\nu_{\alpha\beta}$ are CHRISTOFFEL symbols based on the given form (3. 1) is the expanded form of (2. 2); and the last set is identical with (2. 5). We now regard (3. 2) as a system of equations in the unknowns y^i, $y^i_{,\alpha}$ and σ^i and as such this system gives necessary conditions on the quantities $g_{\alpha\beta}$ and $b_{\alpha\beta}$ for them to be the coefficients of the first and second fundamental forms of a hypersurface S. Calculation and simplification of the integrability conditions of (3. 2) leads to the equations

$$(b_{\beta\gamma,\delta} - b_{\beta\delta,\gamma})\,\sigma^i + \sum_{\alpha,\nu=1}^{n} [B_{\alpha\beta\gamma\delta} + b_{\alpha\gamma} b_{\beta\delta} - b_{\alpha\delta} b_{\beta\gamma}]\, g^{\alpha\nu} y^i_{,\nu} = 0,$$

$$\sum_{\alpha,\nu=1}^{n} (b_{\alpha\beta,\gamma} - b_{\alpha\gamma,\beta})\, g^{\alpha\nu} y^i_{,\nu} = 0,$$

which are satisfied identically in consequence of (2. 8) and (2. 9). Hence (3. 2) is completely integrable and by the existence theorem for such systems admits a solution $y^i(x)$, $y^i_{,\alpha}(x)$, $\sigma^i(x)$ defined in the neighborhood U, this solution being uniquely determined by the arbitrary initial values $y^i(x_0)$, $y^i_{,\alpha}(x_0)$, $\sigma^i(x_0)$ of these functions.[1] On account of the above hypothesis of differentiability of the functions $g_{\alpha\beta}$ and $b_{\alpha\beta}$ it is evident that the functions $y^i(x)$ and $\sigma^i(x)$ will be continuous with continuous derivatives to the third and second orders respectively in U.

It remains to show that the equations

$$(3.3) \qquad \begin{cases} [\alpha\beta] \equiv g_{\alpha\beta} - \sum_{i=1}^{n+1} \dfrac{\partial y^i}{\partial x^\alpha} \dfrac{\partial y^i}{\partial x^\alpha} = 0, \\ [\alpha] \equiv \sum_{i=1}^{n+1} \sigma^i \dfrac{\partial y^i}{\partial x^\alpha} = 0, \\ [0] \equiv \sum_{i=1}^{n+1} \sigma^i \sigma^i - 1 = 0, \end{cases}$$

are satisfied in U. For this purpose let us choose the arbitrary initial values of the unknowns which uniquely determine the solution of (3. 2) so that at the initial point $x_0 < U$ the system (3. 3) is satisfied; this is evidently possible.[2] Then the above bracket expressions will be uniquely determined functions of the x^α in U, continuous and with continuous first and second derivatives in this neighborhood, and equal to zero at x_0. To show that these expressions vanish identically in U we differentiate the equations (3. 3) and thereby obtain, after making certain rearrangements by way of simpliciation, the following equations

$$(3.4) \qquad \begin{cases} \dfrac{\partial [\alpha\beta]}{\partial x^\gamma} = \sum_{\nu=1}^{n} \left\{ \Gamma^\nu_{\alpha\gamma} [\nu\beta] + \Gamma^\nu_{\beta\gamma} [\nu\alpha] \right\} - b_{\alpha\gamma} [\beta] - b_{\beta\gamma} [\alpha], \\ \dfrac{\partial [\alpha]}{\partial x^\beta} = \sum_{\mu,\nu=1}^{n} b_{\mu\beta} g^{\mu\nu} [\alpha\nu] + \sum_{\nu=1}^{n} \Gamma^\nu_{\alpha\beta} [\nu] + b_{\alpha\beta} [0], \\ \dfrac{\partial [0]}{\partial x^\alpha} = -2 \sum_{\mu,\nu=1}^{n} b_{\mu\alpha} g^{\mu\nu} [\nu]. \end{cases}$$

[1] See, for example, T. Y. Thomas, *Systems of total differential equations defined over simply connected domains*, Annals of Math., 35, 1934, p. 730.

[2] For example, if we make a linear transformation of the coordinates x^α so that at the initial point we have $g_{\alpha\beta} = \delta^\alpha_\beta$ with respect to the new coordinate system the equations (3. 3) will be

Calculation of the conditions of integrability of (3. 4) shows that they are satisfied identically on account of the GAUSS and CODAZZI equations. Hence (3. 4) possesses a solution $[\alpha\beta]$, $[\alpha]$, $[0]$ uniquely determined in U by the assignment of the arbitrary initial values of these expressions at $x^\alpha = x_0^\alpha$; it follows that the above functions $[\alpha\beta]$, $[\alpha]$, $[0]$ which are defined in U and which vanish at $x^\alpha = x_0^\alpha$ must vanish identically in U.

Since the form (3. 1) is positive definite in U by hypothesis it follows from the first set of equations (3. 3) that the functional matrix $\| y^i_{,\alpha}(x) \|$ will be of rank n in U. Hence the equations

$$y^i = y^i(x), \qquad (x < U), \tag{3. 5}$$

will define a hypersurface S of the Euclidean space E. By the first set of equations (3. 3) the form (3. 1) will be the first fundamental form of S. Also the second set of equations (3. 2) can be solved for the quantities $b_{\alpha\beta}$ by making use of the last equation in (3. 3) and the equations so obtained are identical with (1. 10); hence the above form ψ appears as the second fundamental form of the hypersurface S. We may therefore state the following result:

Two quadratic differential forms

$$\sum_{\alpha,\beta=1}^{n} g_{\alpha\beta}\, dx^\alpha\, dx^\beta, \qquad \sum_{\alpha,\beta=1}^{n} b_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

the coefficients of which are continuous functions of the variables x^α with continuous partial derivatives to the orders two and one respectively in a neighborhood U of a point x_0^α of the n-dimensional number space, the first form being positive definite, will be the first and second fundamental forms of a hypersurface S of the Euclidean space E of $n+1$ dimensions, the surface S being defined by equations of the type (3. 5) *with right members which are continuous functions of the variables x^α and which possess continuous partial derivatives to the third order in U, if, and only if, the* GAUSS *and* CODAZZI *equations are satisfied in the neighborhood U.*

With regard to the original question as to the determination of conditions for a RIEMANN space to be of class one the following modification of the above italicized statement can evidently be made: *A* RIEMANN *space with element of*

satisfied by taking $\sigma^1 = \cdots = \sigma^n = 0$, $\sigma^{n+1} = 1$ and $\partial y^i / \partial x^\alpha = \delta^i_\alpha$. Transforming back to the original coordinates the transformed values of the quantities $\partial y^i/\partial x^\alpha$ and the above values of the scalars σ^i will satisfy the system (3. 3) as required.

distance defined by the positive definite quadratic differential form (3. 1) *the coefficients* $g_{\alpha\beta}$ *of which are continuous functions of the variables* x^α *possessing continuous first and second partial derivatives in the neighborhood* U *will be of class one, if, and only if, the curvature tensor* B *does not vanish identically in* U *and there exists a set of functions* $b_{\alpha\beta}(=b_{\beta\alpha})$ *continuous with continuous first partial derivatives in* U *such that the* GAUSS *and* CODAZZI *equations are satisfied in* U.

4. Motions of Hypersurfaces in Euclidean Space.

Let us subject the Euclidean space E to a *motion*, i. e. a point transformation defined by the equations

$$y_*^i = \sum_{k=1}^{n+1} a_k^i y^k + c^i, \tag{4. 1}$$

where the a's are constants forming an orthogonal matrix $\|a_k^i\|$ and the c's are arbitrary constants. The condition that the matrix $\|a_k^i\|$ be orthogonal may conveniently be expressed by equations of the form

$$\sum_{i=1}^{n+1} a_k^i a_m^i = \delta_m^k, \quad \sum_{i=1}^{n+1} a_i^k a_i^m = \delta_m^k, \qquad (k, m = 1, \ldots, n+1), \tag{4. 2}$$

the second set of these equations being in fact an algebraic consequence of the first set. As a result of the motion of E the hypersurface S becomes a hypersurface S_* given by the equations

$$y_*^i = \sum_{k=1}^{n+1} a_k^i y^k(x) + c^i, \qquad (x < U), \tag{4. 3}$$

identical values of the parameters x^α thus defining a one to one correspondance between S and S_*. It is easily seen that the two hypersurfaces S and S_* have the same first and second fundamental forms, namely

$$\sum_{\alpha,\beta=1}^{n} g_{\alpha\beta}(x)\, dx^\alpha\, dx^\beta, \qquad \sum_{\alpha,\beta=1}^{n} b_{\alpha\beta}(x)\, dx^\alpha\, dx^\beta.$$

That the first fundamental forms are the same follows immediately from (1. 3), the corresponding equations for the hypersurface S_* and the condition of orthogonality (4. 2). Also it follows readily from equations of the type (1. 4) and (1. 5) that the direction cosines σ_*^i of the normal vectors to S_* are related to the direction cosines σ^i at corresponding points of S and S_* by the equations

$$\sigma_*^i = \pm \sum_{k=1}^{n+1} a_k^i \sigma^k. \tag{4. 4}$$

We can of course choose the σ_*^i so that the $+$ sign in (4. 4) applies without loss of generality; then it results from (4. 2), (4. 3), (4. 4) and equations of the type (1. 8) that the second fundamental forms are likewise identical for S and S_*.

Now consider two hypersurfaces S and S_* having the same first and second fundamental forms; by this is meant that the coefficients of these forms are either identical functions of the coordinates x^α in a neighborhood U or that they can be made so by a coordinate transformation in one of the hypersurfaces. Assuming this condition to hold we know from the results of § 3 that the hypersurfaces S and S_* are defined by equations of the type (1. 1) the right members of which are given as solutions of the *same* system of equations (3. 2) and are uniquely determined by the values of the quantities $y^i, y^i_{,\alpha}$, and σ^i chosen so as to satisfy (3. 3) at a point $x_0 < U$. For simplicity let us suppose a linear transformation of the coordinates x^α of S and S_* to be made so that at the point x_0 the $g_{\alpha\beta}$ have the values δ^α_β. Then if we put: $y^i_{,\alpha} = \xi^i_\alpha$ and $\sigma^i = \xi^i_{n+1}$ the conditions (3. 3) at $x^\alpha = x^\alpha_0$ can be written in the abbreviated form

$$\sum_{i=1}^{n+1} \xi^i_k \xi^i_m = \delta^k_m, \qquad (k, m = 1, \ldots, n+1). \tag{4. 5}$$

Now it can readily be shown[1] that if $\xi^i_k = \mu^i_k$ and $\xi^i_k = \nu^i_k$ are two solutions of (4. 5) they are related by equations of the form

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[1] We have

$$\Sigma \mu^m_k \mu^m_l = \Sigma \mu^k_m \mu^l_m = \delta^k_l; \quad \Sigma \nu^m_k \nu^m_l = \Sigma \nu^k_m \nu^l_m = \delta^k_l.$$

Hence

$$\Sigma \mu^m_k \mu^m_l = \Sigma \nu^m_k \nu^m_l.$$

Multiplying by μ^i_l and summing on the index l gives

$$\mu^i_k = \Sigma a^i_m \nu^m_k, \quad a^i_m = \Sigma \mu^i_l \nu^m_l.$$

But

$$\Sigma a^i_m a^i_k = \Sigma (\Sigma \mu^i_l \nu^m_l)(\Sigma \mu^i_j \nu^k_j) = \Sigma \nu^k_j \nu^m_j = \delta^k_m,$$

which is the condition that the matrix $\left\|a^i_m\right\|$ be orthogonal.

$$\mu^i_k = \sum_{m=1}^{n+1} a^i_m \nu^m_k, \qquad (i, k = 1, \ldots, n+1),$$

where the a's are the elements of an orthogonal matrix $\| a^i_m \|$. Hence if the functions $y^i(x)$ and $y^i_*(x)$ which define the hypersurfaces S and S_* are such that

$$\left.\begin{aligned} y^i_{,\alpha} &= \nu^i_\alpha, \quad \sigma^i = \nu^i_{n+1} \\ y^i_{*,\alpha} &= \mu^i_\alpha, \quad \sigma^i_* = \mu^i_{n+1} \end{aligned}\right\} \text{ at } x^\alpha = x^\alpha_0$$

we must have the identical relations

$$y^i_*(x) \equiv \sum_{k=1}^{n+1} a^i_k y^k(x) + c^i, \quad y^i_{*,\alpha}(x) \equiv \sum_{k=1}^{n+1} a^i_k y^k_{,\alpha}(x), \quad \sigma^i_*(x) \equiv \sum_{k=1}^{n+1} a^i_k \sigma^k(x),$$

where the constant c^i are chosen so that the first set of these relations holds at $x^\alpha = x^\alpha_0$. This follows from the above existence theorem and the fact that the left and right members of the above equations each constitute a solution of (3. 2) assuming at the point x_0 the same initial values.

We have thus proved the following result: *Two hypersurfaces S and S_* of the Euclidean space E have the same first and second fundamental forms if, and only if, they are transformable by a motion in E.*

5. Types of Hypersurfaces. Intrinsic Rigidity.

A hypersurface S will be said to be of type one if the rank of the matrix $\| b_{\alpha\beta}(x) \|$ is zero or one for $x < U$. It will be said to be of type τ where τ is an integer of the set $2, \ldots, n$ if the rank of the above matrix is τ for $x < U$. We shall now prove the interesting result that *the type number of a hypersurface S is determined by its intrinsic properties*, i. e. *by the first fundamental form.*

First consider the special case of a flat hypersurface S which is characterized by the fact that the curvature tensor vanishes for $x < U$. Then by (2. 9) we have

$$b_{\alpha\delta} b_{\beta\gamma} - b_{\alpha\gamma} b_{\beta\delta} = 0. \tag{5. 1}$$

Now the left members of these equations are the second order minors of the matrix $\| b_{\alpha\beta} \|$. It follows from (5. 1) therefore that $\| b_{\alpha\beta} \|$ has rank zero or one at points $x < U$. Conversely if $\| b_{\alpha\beta} \|$ is of rank zero or one at points $x < U$

then (5. 1) is satisfied and hence by (2. 9) the curvature tensor vanishes in U. Hence, *a hypersurface S is flat if, and only if, it is of type one.*

Now consider the following two systems of equations

$$\sum_{\beta=1}^{n} b_{\alpha\beta} A^{\beta} = 0, \tag{5. 2}$$

$$\sum_{\delta=1}^{n} B_{\alpha\beta\gamma\delta} A^{\delta} = 0. \tag{5. 3}$$

Suppose S is of type n. Then if the matrix $\|B\|$ of the system (5. 3) is of rank $< n$ at a point $x < U$ this system will have a non-trivial solution A^{δ} and it will result from (2. 9) that

$$\sum_{\delta=1}^{n} (b_{\alpha\delta} b_{\beta\gamma} - b_{\alpha\gamma} b_{\beta\delta}) A^{\delta} = 0. \tag{5. 4}$$

But since det. $|b_{\beta\gamma}| \neq 0$ at x by hypothesis it follows from (5. 4) by multiplying by $b^{\beta\gamma}$ and summing on repeated indices that $A^{\delta} = 0$; hence the rank of the matrix $\|B\|$ is n for points $x < U$. Conversely if the rank of $\|B\|$ is n for a point $x_1 < U$ it follows that $\|b_{\alpha\beta}\|$ has rank n at x_1 since otherwise (5. 2) would have a non-trivial solution A^{δ} satisfying (5. 3) in contradiction to the hypothesis on the rank of the matrix $\|B\|$. Hence, *a hypersurface S is of type n if, and only if, the matrix $\|B\|$ has rank n for all points $x < U$.*

Assume finally that S is of type τ where τ is an integer of the set $2, \ldots, n-1$. If the matrix $\|B\|$ has the rank σ at a point $x_1 < U$ then $\sigma \leqq \tau$ since every solution of (5. 2) is likewise a solution of (5. 3). Now transform the coordinates of U so that $\|b_{\alpha\beta}\|$ has the form

$$\left\|\begin{array}{ccc|c} b_{11} & \cdots & b_{1\tau} & \\ \cdot & \cdot & \cdot & 0 \\ b_{\tau 1} & \cdots & b_{\tau\tau} & \\ \hline & 0 & & 0 \end{array}\right\|$$

at the point x_1. Let A^{δ} be a non-trivial solution of (5. 3); then A^{δ} satisfies (5. 4) in consequence of the relations (2. 9). It then follows from (5. 4) in which the indices $\alpha, \beta, \gamma, \delta$ have the values $1, \ldots, \tau$ that $A^1 = \cdots = A^{\tau} = 0$; also we know that one of the quantities A^{δ} for $\delta > \tau$ is different from zero since the above solution is non-trivial. Hence these A's satisfy the system (5. 2), i. e. any solution of (5.3) is a solution of (5.2). Hence $\sigma \geqq \tau$ and it therefore follows that $\sigma = \tau$ or in other words the matrix $\|B\|$ has the rank τ at all points $x < U$. Conversely if the rank of $\|B\|$ is τ at a point $x_1 < U$ the rank of $\|b_{\alpha\beta}\|$ must also be τ at this point since otherweise we would have a contradiction with the results above established. Hence, *a hypersurface S is of type τ where τ is an integer of the set $2, \ldots, n-1$ if, and only if, the matrix $\|B\|$ has rank τ for all points $x < U$.*

It follows from the above italicized statements that the type number of a hypersurface S is completely determined by its intrinsic metric character; *it is an intrinsic integer invariant of S.* A Riemann space will therefore be said to be of type one if the curvature tensor vanishes identically and of type τ where τ is an integer of the set $2, \ldots, n$ if the above matrix $\|B\|$ has rank τ for $x < U$ regardless of whether or not this space can be considered as a hypersurface of the Euclidean space E.

A hypersurface S (or Riemann space of class one) will be said to be *intrinsically rigid* provided that the second fundamental form is uniquely determined (to within algebraic sign) by the first fundamental form and the equations of Gauss and Codazzi. As so defined it is clear that intrinsic rigidity is a *local* property. From the result of § 4 we know that the position of a hypersurface in the Euclidean space E is determined by its first and second fundamental forms to within a motion in E; also it is well known that if two hypersurfaces S and S_* in E are related by such a motion either can be obtained from the other by a rigid displacement in E or else by a rigid displacement combined with a reflection in a plane. It follows that an intrinsically rigid hypersurface S can not be subjected to a continuous deformation in E in such a way that its internal metric properties will be left unaltered throughout the deformation. We proceed to develop certain relations between the type number of a hypersurface and the property of intrinsic rigidity.

Consider the system

$$(5.5) \qquad b_{\alpha\delta} b_{\beta\gamma} - b_{\alpha\gamma} b_{\beta\delta} = \omega_{\alpha\delta} \omega_{\beta\gamma} - \omega_{\alpha\gamma} \omega_{\beta\delta}$$

as equations to be solved for the symmetric quantities $\omega_{\alpha\beta}$. Assume det. $|b_{\alpha\beta}| \neq 0$ at a point $x_1 < U$. It then follows readily from (5.5) that also the det. $|\omega_{\alpha\beta}| \neq 0$ at x_1. In fact if we multiply these equations by $b^{\alpha\delta}$ and sum on the repeated indices α, δ we find

$$(n-1)\,b_{\beta\gamma} = \sum_{\nu=1}^{n}\left[\left(\sum_{\alpha,\,\delta=1}^{n} b^{\alpha\delta}\,\omega_{\alpha\delta}\right)\delta^{\nu}_{\gamma} - \sum_{\alpha=1}^{n} b^{\alpha\nu}\,\omega_{\alpha\gamma}\right]\omega_{\beta\nu};$$

and by taking the determinant of both members of these equations we establish the above result. Now put $\omega_{\alpha\beta} = b_{\alpha\beta} + \lambda_{\alpha\beta}$ and use this substitution to eliminate the ω's from (5. 5) so as to abtain

$$(5.\,6) \qquad b_{\alpha\delta}\,\lambda_{\beta\gamma} + b_{\beta\gamma}\,\lambda_{\alpha\delta} - b_{\alpha\gamma}\,\lambda_{\beta\delta} - b_{\beta\delta}\,\lambda_{\alpha\gamma} + \lambda_{\alpha\delta}\,\lambda_{\beta\gamma} - \lambda_{\alpha\gamma}\,\lambda_{\beta\delta} = 0.$$

1°. Suppose det. $|\lambda_{\alpha\beta}| \neq 0$ at x_1 and multiply (5. 6) by $\lambda^{\beta\gamma}$ summing on repeated indices:

$$(5.\,7) \qquad (n-2)\,b_{\alpha\delta} + (n+b-1)\,\lambda_{\alpha\delta} = 0, \quad b \equiv \sum_{\beta,\,\gamma=1}^{n} \lambda^{\beta\gamma}\,b_{\beta\gamma}.$$

Again multiplying these latter equations by $\lambda^{\alpha\delta}$ we find that $b = -n/2$ and when this value of b is substituted into (5. 7) the resulting equations give $\lambda_{\alpha\delta} = -2\,b_{\alpha\delta}$ for $n \geqq 3$; hence $\omega_{\alpha\beta} = -b_{\alpha\beta}$.

2°. Suppose det. $|\lambda_{\alpha\beta}| = 0$ at $x_1 < U$. Let A^{β} be a non-trivial solution of the equations $\Sigma\,\lambda_{\alpha\beta}\,A^{\beta} = 0$. Multiply (5. 6) by $b^{\alpha\delta}$ and sum on repeated indices:

$$(5.\,8) \qquad (n+\lambda-2)\,\lambda_{\beta\gamma} + \lambda\,b_{\beta\gamma} - \sum_{\alpha,\,\delta=1}^{n} b^{\alpha\delta}\,\lambda_{\alpha\gamma}\,\lambda_{\beta\delta} = 0, \quad \lambda \equiv \sum_{\alpha,\,\delta=1}^{n} b^{\alpha\delta}\,\lambda_{\alpha\delta}.$$

Multiplying (5. 8) by A^{γ} and summing on the repeated index γ then gives

$$\left(\sum_{\gamma=1}^{n} b_{\beta\gamma}\,A^{\gamma}\right)\lambda = 0;$$

hence $\lambda = 0$ since the coefficients of λ can not all vanish in these equations as this would be in contradiction to the assumption that det. $|b_{\alpha\beta}| \neq 0$ at x_1. The equations (5. 8) therefore become

$$(n-2)\,\lambda_{\beta\gamma} = \sum_{\alpha,\,\delta=1}^{n} b^{\alpha\delta}\,\lambda_{\alpha\gamma}\,\lambda_{\beta\delta}; \text{ hence}$$

$$(n-2)\,[\omega_{\beta\gamma} - b_{\beta\gamma}] = \sum_{\alpha,\,\delta=1}^{n} b^{\alpha\delta}\,[\omega_{\alpha\gamma} - b_{\alpha\gamma}]\,[\omega_{\beta\delta} - b_{\beta\delta}]$$

and these equations reduce finally to

(5.9) $$n\,\omega_{\beta\gamma} = (n-1)\,b_{\beta\gamma} + \sum_{\alpha,\delta=1}^{n} b^{\alpha\delta}\,\omega_{\alpha\gamma}\,\omega_{\beta\delta};\ \text{hence}$$

(5.10) $$n\,b_{\beta\gamma} = (n-1)\,\omega_{\beta\gamma} + \sum_{\alpha,\delta=1}^{n} \omega^{\alpha\delta}\,b_{\alpha\gamma}\,b_{\beta\delta}$$

since the quantities $\omega_{\beta\gamma}$ and $b_{\beta\gamma}$ in (5.9) are obviously interchangable. It now follows from (5.9) and (5.10) that

$$n\,\omega_{\beta\gamma} = (n-1)\,b_{\beta\gamma} + \sum_{\alpha,\delta=1}^{n} b^{\alpha\delta}\left[\frac{n}{n-1}\,b_{\alpha\gamma} - \sum_{\mu,\nu=1}^{n}\frac{\omega^{\mu\nu}}{n-1}\,b_{\mu\alpha}\,b_{\nu\gamma}\right]\omega_{\beta\delta}$$

and on reduction these equations become $\omega_{\beta\gamma} = b_{\beta\gamma}$ for $n \geqq 3$.

We can now state the following result: If det. $|b_{\alpha\beta}| \neq 0$ at a point $x_1 < U$ then at x_1 the equations (5.5) have $\omega_{\alpha\beta} = \pm\, b_{\alpha\beta}$ as their only solution ($n \geqq 3$). To extend this result assume that $\|b_{\alpha\beta}\|$ has rank $r > 2$ at $x_1 < U$. Transform the coordinates of U so that at the point x_1 the matrix $\|b_{\alpha\beta}\|$ has the form

$$\left\|\begin{array}{ccc|c} b_{11} & \cdots & b_{1r} & \\ \cdot & \cdot & \cdot & 0 \\ b_{r1} & \cdots & b_{rr} & \\ \hline & 0 & & 0 \end{array}\right\|.$$

Then equations (5.5) give $\omega_{\alpha\beta} = \pm\, b_{\alpha\beta}$ $(\alpha, \beta = 1, \ldots, r)$ in consequence of the above result. Also we must have

(5.11) $$\omega_{\alpha\delta}\,\omega_{\beta\gamma} - \omega_{\alpha\gamma}\,\omega_{\beta\delta} = 0$$

if one of the indices in these equations has a value $> r$. Taking $\alpha, \beta, \gamma = 1, \ldots, r$ and $\delta > r$ we can then construct the quantities $\omega^{\beta\gamma}$ by which (5.11) can be multiplied so as to obtain $\omega_{\alpha\delta} = 0$ for $\alpha = 1, \ldots, r$ and $\delta > r$. Now take $\beta, \gamma = 1, \ldots, r$ and $\alpha, \delta = r+1, \ldots, n$ in (5.11); these equations then reduce to $\omega_{\alpha\delta}\,\omega_{\beta\gamma} = 0$ from which $\omega_{\alpha\delta} = 0$ follows immediately. Hence we have that $\omega_{\alpha\beta} = \pm\, b_{\alpha\beta}$ for all values of the indices, i.e. if the matrix $\|b_{\alpha\beta}\|$ has rank > 2 at a point $x_1 < U$ the equations (5.5) at x_1 have $\omega_{\alpha\beta} = \pm\, b_{\alpha\beta}$ as their only symmetric solutions.

The above results establish the following theorem: *A hypersurface S of type $\geqq 3$ is intrinsically rigid.*[1]

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[1] This theorem has been proved by KILLING, *Nicht Euklidische Raumformen*, Leipzig (1885), p. 237 and by later writers; see, for example, L. P. EISENHART, *Riemannian Geometry*, Princeton,

6. The Codazzi Equations as Consequences of the Equations of Gauss.

Consider a Riemann space with element of distance defined by the quadratic differential form (3.1) the coefficients $g_{\alpha\beta}(x)$ of which are continuous functions of the variables x^α in the neighborhood U with continuous first and second derivatives. Let $b_{\alpha\beta}(x)$ be a set of symmetric quantities likewise defined in U with continuous first derivatives in this neighborhood. We assume that the functions $g_{\alpha\beta}(x)$ and $b_{\alpha\beta}(x)$ satisfy the equations of Gauss (2.9) for $x < U$. It will now be shown that in general the above functions will also satisfy the equations of Codazzi (2.8) in U.

By covariant differentiation of (2.9) we obtain

$$(6.1)\qquad \begin{cases} B_{\alpha\beta\gamma\delta,\,\varepsilon} = b_{\alpha\delta,\,\varepsilon}\, b_{\beta\gamma} + b_{\beta\gamma,\,\varepsilon}\, b_{\alpha\delta} - b_{\alpha\gamma,\,\varepsilon}\, b_{\beta\delta} - b_{\beta\delta,\,\varepsilon}\, b_{\alpha\gamma}, \\ B_{\alpha\beta\delta\varepsilon,\,\gamma} = b_{\alpha\varepsilon,\,\gamma}\, b_{\beta\delta} + b_{\beta\delta,\,\gamma}\, b_{\alpha\varepsilon} - b_{\alpha\delta,\,\gamma}\, b_{\beta\varepsilon} - b_{\beta\varepsilon,\,\gamma}\, b_{\alpha\delta}, \\ B_{\alpha\beta\varepsilon\gamma,\,\delta} = b_{\alpha\gamma,\,\delta}\, b_{\beta\varepsilon} + b_{\beta\varepsilon,\,\delta}\, b_{\alpha\gamma} - b_{\alpha\varepsilon,\,\delta}\, b_{\beta\gamma} - b_{\beta\gamma,\,\delta}\, b_{\alpha\varepsilon}, \end{cases}$$

the second and third set of equations being obtained from the first by cyclic permutation of the indices. Adding corresponding members of the above equations we obtain a set of equations the left members of which vanish on account of the Bianchi identities so that we have

$$(6.2)\qquad b_{\alpha\gamma}\,\Phi_{\beta\varepsilon\delta} + b_{\alpha\delta}\,\Phi_{\beta\gamma\varepsilon} + b_{\alpha\varepsilon}\,\Phi_{\beta\delta\gamma} + b_{\beta\gamma}\,\Phi_{\alpha\delta\varepsilon} + b_{\beta\delta}\,\Phi_{\alpha\varepsilon\gamma} + b_{\beta\varepsilon}\,\Phi_{\alpha\gamma\delta} = 0,$$

where

$$\Phi_{\alpha\beta\gamma} = b_{\alpha\beta,\,\gamma} - b_{\alpha\gamma,\,\beta}.$$

Now assume det. $|b_{\alpha\beta}| \neq 0$ at a point $x_1 < U$. We can then construct the quantities $b^{\alpha\gamma}$ at x_1 and multiplying (6.2) by these and summing on the repeated indices we obtain

$$(6.3)\qquad (n-3)\,\Phi_{\beta\varepsilon\delta} + b_{\beta\varepsilon} \sum_{\alpha,\gamma=1}^{n} b^{\alpha\gamma}\,\Phi_{\alpha\gamma\delta} - b_{\beta\delta} \sum_{\alpha,\gamma=1}^{n} b^{\alpha\gamma}\,\Phi_{\alpha\gamma\varepsilon} = 0,$$

account being taken of the skew-symmetry of the Φ's in their last two indices. Similarly multiplying (6.3) by $b^{\beta\varepsilon}$ we have

$$(6.4)\qquad (n-2) \sum_{\alpha,\gamma=1}^{n} b^{\alpha\gamma}\,\Phi_{\alpha\gamma\delta} = 0.$$

(1926), p. 201. The proof which we have given of this theorem although somewhat more lengthy than the proofs of the above authors has the advantage of greater formal simplicity.

Hence if $n \geqq 4$ it follows from (6. 3) and (6. 4) that $\Phi_{\alpha\beta\gamma}$ is equal to zero. That is, if the above functions $g_{\alpha\beta}(x)$ and $b_{\alpha\beta}(x)$ satisfy the equations of GAUSS and if det. $|b_{\alpha\beta}| \neq 0$ for $x < U$ then the equations of CODAZZI are also satisfied in $U\,(n \geqq 4)$.

To extend the above result let us assume that the matrix $\|b_{\alpha\beta}\|$ has rank $r \geqq 4$ at a point $x_1 < U$ and that a coordinate transformation has been made in U so that at x_1 the matrix $\|b_{\alpha\beta}\|$ has the form

$$\left\|\begin{array}{ccc|c} b_{11} & \cdots & b_{1r} & \\ \cdot & \cdot & \cdot & 0 \\ b_{r1} & \cdots & b_{rr} & \\ \hline & 0 & & 0 \end{array}\right\|.$$

At x_1 we then know that $\Phi_{\alpha\beta\gamma} = 0$ if $\alpha, \beta, \gamma = 1, \ldots, r$ by the above result. Now take $\alpha, \beta, \gamma, \delta = 1, \ldots, r$ and $\varepsilon > r$ in (6. 2); multiply these equations by $b_{\alpha\gamma}$ and sum to obtain

$$(6.\ 5) \qquad (r-2)\,\Phi_{\beta\varepsilon\delta} - \beta_{\beta\delta} \sum_{\alpha,\gamma=1}^{n} b^{\alpha\gamma}\,\Phi_{\alpha\gamma\varepsilon} = 0.$$

Again multiply the last set of equations by $b_{\beta\delta}$ giving

$$(r-1) \sum_{\alpha,\gamma=1}^{n} b^{\alpha\gamma}\,\Phi_{\alpha\gamma\varepsilon} = 0.$$

Hence from (6. 5) we have $\Phi_{\beta\varepsilon\delta} = 0$ for $\beta, \delta = 1, \ldots, r$ and $\varepsilon > r$. **Next take** $\beta, \gamma, \delta, \varepsilon = 1, \ldots, r$ and $\alpha > r$ in (6. 2), then multiply these equations by $b^{\beta\gamma}$ and sum on repeated indices; this gives $\Phi_{\alpha\delta\varepsilon} = 0$. We have now shown that the quantities $\Phi_{\alpha\beta\gamma}$ vanish at the point x_1 if two of the indices are in the range $1, \ldots, r$ and the other index is $> r$.

Take $\alpha, \beta, \gamma = 1, \ldots, r$ and $\delta, \varepsilon > r$. Then multiply (6. 2) by $b^{\alpha\gamma}$; we deduce $\Phi_{\beta\varepsilon\delta} = 0$. Take $\alpha, \gamma, \varepsilon = 1, \ldots, r$ and $\beta, \delta > r$ and multiply (6. 2) by $b^{\alpha\gamma}$ to obtain again $\Phi_{\beta\varepsilon\delta} = 0$ but for the last range of indices. Hence the Φ's also vanish if one of the indices has a value $1, \ldots, r$ and the other two indices are $> r$.

Finally take $\alpha, \gamma = 1, \ldots, r$ and $\beta, \delta, \varepsilon > r$ in (6. 2), multiply by $b^{\alpha\gamma}$ and obtain $\Phi_{\beta\varepsilon\delta} = 0$, i. e. the Φ's are equal to zero at x_1 for all indices $> r$. We have now shown that for all values of the indices the quantities $\Phi_{\alpha\beta\gamma}$ vanish in the neighborhood U.

As the above requirement that the form (3. 1) be positive definite can evidently be replaced in this discussion by the weaker condition that the det. $|g_{\alpha\beta}| \neq 0$ for $x < U$ we have in fact proved the following

Theorem. *Let*

$$\sum_{\alpha,\beta=1}^{n} g_{\alpha\beta}(x)\,dx^\alpha\,dx^\beta, \qquad \sum_{\alpha,\beta=1}^{n} b_{\alpha\beta}(x)\,dx^\alpha\,dx^\beta$$

be two quadratic differential forms the coefficients of which are continuous functions of the variables x^α of the neighborhood U with continuous partial derivatives to the orders two and one respectively; furthermore let the matrix $\|g_{\alpha\beta}\|$ have rank n and the matrix $\|b_{\alpha\beta}\|$ have rank $\geqq 4$ for $x < U$. Then, if the coefficients of these forms satisfy the equations of Gauss *for $x < U$, the equations of* Codazzi *will automatically be satisfied in the neighborhood U.*

7. Reality Conditions.

We shall now investigate the solutions $b_{\alpha\beta}$ of the Gauss equations (2. 9) considered as a system of algebraic equations, the left members of which are the components of the curvature tensor of a Riemann space ($n \geqq 3$) of type $\tau \geqq 3$ (§ 5) defined in U. In general a solution of these equations will have the complex form

$$b_{\alpha\beta} = p_{\alpha\beta} + \sqrt{-1}\, q_{\alpha\beta}, \tag{7. 1}$$

and as a first step in this investigation we shall determine necessary and sufficient conditions for the solution (7. 1), if such exists, to be real, i. e. for the above quantities $q_{\alpha\beta}$ to vanish.

By a well known theorem in Algebra we have

$$\begin{vmatrix} b_{\alpha\lambda} & b_{\alpha\mu} & b_{\alpha\nu} \\ b_{\beta\lambda} & b_{\beta\mu} & b_{\beta\nu} \\ b_{\gamma\lambda} & b_{\gamma\mu} & b_{\gamma\nu} \end{vmatrix}^2 = \begin{vmatrix} \bar{b}_{\alpha\lambda} & \bar{b}_{\alpha\mu} & \bar{b}_{\alpha\nu} \\ \bar{b}_{\beta\lambda} & \bar{b}_{\beta\mu} & \bar{b}_{\beta\nu} \\ \bar{b}_{\gamma\lambda} & \bar{b}_{\gamma\mu} & \bar{b}_{\gamma\nu} \end{vmatrix},$$

where the $\bar{b}_{\alpha\lambda}$ etc. denote the cofactors of the corresponding elements of the determinant in the left member of this equation. But these cofactors are directly expressible in terms of the components of the curvature tensor on account of the equations (2. 9); making these substitutions we have

$$(7.2)\qquad \left\| \begin{matrix} \bar{b}_{\alpha\lambda} & \bar{b}_{\alpha\mu} & \bar{b}_{\alpha\nu} \\ \bar{b}_{\beta\lambda} & \bar{b}_{\beta\mu} & \bar{b}_{\beta\nu} \\ \bar{b}_{\gamma\lambda} & \bar{b}_{\gamma\mu} & \bar{b}_{\gamma\nu} \end{matrix} \right\| = \left\| \begin{matrix} B_{\beta\gamma\nu\mu} & B_{\beta\gamma\lambda\nu} & B_{\beta\gamma\mu\lambda} \\ B_{\alpha\gamma\mu\nu} & B_{\alpha\gamma\nu\lambda} & B_{\alpha\gamma\lambda\mu} \\ B_{\alpha\beta\nu\mu} & B_{\alpha\beta\lambda\nu} & B_{\alpha\beta\mu\lambda} \end{matrix} \right\|.$$

It therefore follows that

$$(7.3)\qquad \left| \begin{matrix} b_{\alpha\lambda} & b_{\alpha\mu} & b_{\alpha\nu} \\ b_{\beta\lambda} & b_{\beta\mu} & b_{\beta\nu} \\ b_{\gamma\lambda} & b_{\gamma\mu} & b_{\gamma\nu} \end{matrix} \right|^2 = \left| \begin{matrix} B_{\beta\gamma\nu\mu} & B_{\beta\gamma\lambda\nu} & B_{\beta\gamma\mu\lambda} \\ B_{\alpha\gamma\mu\nu} & B_{\alpha\gamma\nu\lambda} & B_{\alpha\gamma\lambda\mu} \\ B_{\alpha\beta\nu\mu} & B_{\alpha\beta\lambda\nu} & B_{\alpha\beta\mu\lambda} \end{matrix} \right| = - \left| \begin{matrix} B_{\alpha\beta\lambda\mu} & B_{\alpha\beta\mu\nu} & B_{\alpha\beta\nu\lambda} \\ B_{\beta\gamma\lambda\mu} & B_{\beta\gamma\mu\nu} & B_{\beta\gamma\nu\lambda} \\ B_{\gamma\alpha\lambda\mu} & B_{\gamma\alpha\mu\nu} & B_{\gamma\alpha\nu\lambda} \end{matrix} \right|.$$

Hence, *the inequalities*

$$(7.4)\qquad - \left| \begin{matrix} B_{\alpha\beta\lambda\mu} & B_{\alpha\beta\mu\nu} & B_{\alpha\beta\nu\lambda} \\ B_{\beta\gamma\lambda\mu} & B_{\beta\gamma\mu\nu} & B_{\beta\gamma\nu\lambda} \\ B_{\gamma\alpha\lambda\mu} & B_{\gamma\alpha\mu\nu} & B_{\gamma\alpha\nu\lambda} \end{matrix} \right| \geqq 0$$

constitute necessary conditions for the solution $b_{\alpha\beta}$ *of the* GAUSS *equations* (2.9) *to be real.*

Now we know from the considerations of § 5 that the (real or complex) solution matrix $\|b_{\alpha\beta}(x)\|$ of the GAUSS equations must have rank τ at points $x < U$. Consider this solution at a particular point $x_1 < U$ where it may have the complex form (7.1). The following cases may then arise:

Case I. Rank of $\|q_{\alpha\beta}\|$ is $\geqq 3$,
Case II. Rank of $\|q_{\alpha\beta}\|$ is 2,
Case III. Rank of $\|q_{\alpha\beta}\|$ is 1,
Case IV. Rank of $\|q_{\alpha\beta}\|$ is 0.

Before proceeding to the discussion of these cases we shall state a number of simple lemmas which will have direct application.

Supposing the above solution $b_{\alpha\beta}$ at x_1 to be pure imaginary it follows from (7.3) that

$$- \left| \begin{matrix} q_{\alpha\lambda} & q_{\alpha\mu} & q_{\alpha\nu} \\ q_{\beta\lambda} & q_{\beta\mu} & q_{\beta\nu} \\ q_{\gamma\lambda} & q_{\gamma\mu} & q_{\gamma\nu} \end{matrix} \right|^2 = - \left| \begin{matrix} B_{\alpha\beta\lambda\mu} & B_{\alpha\beta\mu\nu} & B_{\alpha\beta\nu\lambda} \\ B_{\beta\gamma\lambda\mu} & B_{\beta\gamma\mu\nu} & B_{\beta\gamma\nu\lambda} \\ B_{\gamma\alpha\lambda\mu} & B_{\gamma\alpha\mu\nu} & B_{\gamma\alpha\nu\lambda} \end{matrix} \right|$$

at x_1. But since $\|b_{\alpha\beta}(x_1)\|$ has rank $\geqq 3$ it must be possible to find indices $\alpha, \beta, \gamma, \lambda, \mu, \nu$ so that one of the determinants in the left members of these equations is different from zero; for this selection of indices the left member will then be negative while the right member will be $\geqq 0$ by (7.4). This gives

Lemma I. *The solution $b_{\alpha\beta}$ of the* GAUSS *equations can not be pure imaginary under the conditions* (7.4).

If we substitute (7.1) into the right members of the GAUSS equations and equate to zero the imaginary parts we obtain

$$p_{\alpha\delta} q_{\beta\gamma} + p_{\beta\gamma} q_{\alpha\delta} - p_{\alpha\gamma} q_{\beta\delta} - p_{\beta\delta} q_{\alpha\gamma} = 0. \tag{7.5}$$

Assume the matrix $\|q_{\alpha\beta}\|$ has rank r and give this matrix the form

$$\left\| \begin{array}{c|c} \begin{matrix} q_{11} & \cdots & q_{1r} \\ \cdot & \cdot & \cdot \\ q_{r1} & \cdots & q_{rr} \end{matrix} & 0 \\ \hline 0 & 0 \end{array} \right\|$$

by a linear coordinate transformation in U. If $r = n$ we can construct the quantities $q^{\beta\gamma}$ by which we can multiply (7.5) and sum on the repeated indices; this gives

$$(n-2) p_{\alpha\delta} + p\, q_{\alpha\delta} = 0, \quad p \equiv \sum_{\beta, \gamma=1}^{n} q^{\beta\gamma} p_{\beta\gamma}. \tag{7.6}$$

Similarly multiplying these equations by $q^{\alpha\delta}$ and summing we obtain $p = 0$; hence $p_{\alpha\delta} = 0$ since $n \geqq 3$ by hypothesis. Now suppose $r < n$. Take $\beta, \gamma = 1, \ldots, r$ and $\alpha, \delta > r$ in (7.5). Then (7.5) becomes

$$p_{\alpha\delta} q_{\beta\gamma} = 0$$

from which it follows that $p_{\alpha\delta} = 0$ if $\alpha, \delta > r$. Next take $\alpha, \beta, \gamma = 1, \ldots, r$ and $\delta > r$ in (7.5); then from these equations we have

$$p_{\alpha\delta} q_{\beta\gamma} - p_{\beta\delta} q_{\alpha\gamma} = 0.$$

Multiply the latter equations by $q^{\beta\gamma}$ and sum to obtain

$$(r-1) p_{\alpha\delta} = 0.$$

Hence $p_{\alpha\delta} = 0$ for $\alpha = 1, \ldots, r$ and $\delta > r$ if $r > 1$. Finally if $r > 2$ it follows from the equations (7.6) for $n = r$ that $p_{\alpha\delta} = 0$ if $\alpha, \delta = 1, \ldots, r$. This gives the following lemmas:

Lemma II. *If* $\|q_{\alpha\beta}\|$ *has rank* $\geqq 3$ *then* $p_{\alpha\beta} = 0$ *for all indices.*

Lemma III. *If* $\|q_{\alpha\beta}\|$ *has rank two then* $p_{\alpha\beta} = 0$ *for* $\alpha, \beta > 2$ *and for* $\alpha = 1, 2$; $\beta > 2$.

Lemma IV. *If* $\|q_{\alpha\beta}\|$ *has rank one then* $p_{\alpha\beta} = 0$ *for* $\alpha, \beta > 1$.

Returning now to the above cases we see by Lemma II that if Case I holds the solution $b_{\alpha\beta}$ must be pure imaginary which is in contradiction with Lemma I. If Case II is assumed to hold then by Lemma III the matrix $\|b_{\alpha\beta}\|$ must have the form

$$\left\| \begin{array}{cc|c} (p_{11} + \sqrt{-1}\, q_{11}) & (p_{11} + \sqrt{-1}\, q_{12}) & \\ (p_{21} + \sqrt{-1}\, q_{21}) & (p_{22} + \sqrt{-1}\, q_{22}) & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array} \right\|$$

in contradiction with the fact that this matrix has rank $r \geqq 3$. Similarly under Case III the matrix $\|b_{\alpha\beta}\|$ would have the form

$$\left\| \begin{array}{c|c} (p_{11} + \sqrt{-1}\, q_{11}) & p_{12} \cdots p_{1n} \\ \hline p_{21} & \\ \vdots & 0 \\ p_{n1} & \end{array} \right\|$$

with a maximum rank of two. Hence the remaining Case IV must hold which means that the solution $b_{\alpha\beta}$ of the Gauss equations is real under the above hypotheses.

Theorem. *If the left members of the* Gauss *epuations* (2.9) *are the components of the curvature tensor of a* Riemann *space* ($n \geqq 3$) *of type* $\geqq 3$ *with element of distance* (3.1) *defined in a neighborhood* U *and if these equations have a solution* $b_{\alpha\beta}(x)$ *for* $x < U$, *then this solution will be real at all points* $x < U$ *if, and only if, the conditions* (7.4) *are satisfied.*

8. Algebraic Resultants and the Equations of Gauss.

We have seen in the preceding section that if a Riemann space is of class one the reality conditions (7.4) must be satisfied. If now the Riemann space is of class one and type $\geqq 3$ we know from § 5 that at any point $x_1 < U$ one of the determinants in the left members of (7. 3), the elements of this determinant being the coefficients of the second fundamental form of the corresponding hypersurface, must be different from zero. Hence one of the left members of (7. 4) must actually be > 0 and we can therefore state the following result: *A necessary condition for a* RIEMANN *space* ($n \geqq 3$) *of type* $\tau \geqq 3$ *to be of class one is that the inequality*

$$\sum \begin{vmatrix} B_{\alpha\beta\lambda\mu} & B_{\alpha\beta\mu\nu} & B_{\alpha\beta\nu\lambda} \\ B_{\beta\gamma\lambda\mu} & B_{\beta\gamma\mu\nu} & B_{\beta\gamma\nu\lambda} \\ B_{\gamma\alpha\lambda\mu} & B_{\gamma\alpha\mu\nu} & B_{\gamma\alpha\nu\lambda} \end{vmatrix}^2 > 0, \qquad (x < U), \tag{8. 1}$$

be satisfied, where the summation is to be extended over all possible values of the indices appearing in the above determinant. Further necessary conditions in the form of a system of linear homogeneous equations can be derived as follows. Multiplying both members of the GAUSS equations (2. 9) by $b_{\mu\nu}$ we obtain

$$b_{\mu\nu} B_{\alpha\beta\gamma\delta} = b_{\mu\nu} b_{\alpha\delta} b_{\beta\gamma} - b_{\mu\nu} b_{\alpha\gamma} b_{\beta\delta}; \text{ also}$$
$$b_{\mu\alpha} B_{\nu\beta\gamma\delta} = b_{\mu\alpha} b_{\nu\delta} b_{\beta\gamma} - b_{\mu\alpha} b_{\nu\gamma} b_{\beta\delta}.$$

Subtracting corresponding members of these equations we have

$$b_{\mu\nu} B_{\alpha\beta\gamma\delta} + b_{\mu\alpha} B_{\beta\nu\gamma\delta} + b_{\beta\gamma}(b_{\mu\alpha} b_{\nu\delta} - b_{\mu\nu} b_{\alpha\delta}) + b_{\beta\delta}(b_{\mu\nu} b_{\alpha\gamma} - b_{\mu\alpha} b_{\nu\gamma}) = 0,$$

and when use is made of the substitution (2. 9) the latter equations become

$$b_{\mu\nu} B_{\alpha\beta\gamma\delta} + b_{\mu\alpha} B_{\beta\nu\gamma\delta} + b_{\beta\gamma} B_{\mu\delta\nu\alpha} + b_{\beta\delta} B_{\mu\gamma\alpha\nu} = 0. \tag{8. 2}$$

Let us now write the GAUSS equations (2. 9) in their homogeneous form namely

$$B_{\alpha\beta\gamma\delta}\, t^2 = b_{\alpha\delta} b_{\beta\gamma} - b_{\alpha\gamma} b_{\beta\delta}. \tag{8. 3}$$

Consider the equations (8. 2) and (8. 3) as a system for the determination of the unknowns t and $b_{\alpha\beta}$. Since the B's appearing in this system are the components of the curvature tensor of a RIEMANN space of type $\tau \geqq 3$ the system must admit a solution such that the matrix $\| b_{\alpha\beta} \|$ has rank τ at any point $x < U$ if the RIEMANN space is to be of class one; hence in particular the system composed of (8. 2) and (8. 3) must admit a non-trivial solution (t, $b_{\alpha\beta}$) at any point $x < U$. Now we know from the theory of systems of homogeneous algebraic equations that the above equations (8. 2) and (8. 3) must admit a *resultant system*, i. e. a set of polynomials in the components B such that the vanishing of these polynomials is necessary and sufficient for the existence of a non-trivial solution.[1] Representing the resultant system of (8. 2) and (8. 3) by $R_n(B)$ it follows that

$$R_n(B) = 0, \qquad (x < U), \tag{8. 4}$$

is a necessary condition for the above Riemann space to be a hypersurface of the Euclidean space E.

Assuming (8. 4) to be satisfied let $(t, b_{\alpha\beta})$ be a non-trivial solution of (8. 2) and (8. 3) at a particular point $x_1 < U$. *Suppose* $t = 0$ *in this solution.* Then

$$b_{\alpha\delta} b_{\beta\gamma} - b_{\alpha\gamma} b_{\beta\delta} = 0. \tag{8. 5}$$

Now multiply (8. 2) by $b_{\zeta\eta}$, interchange ζ and μ and subtract; when use is made of the conditions (8. 5) and the identities satisfied by the components of the curvature tensor we then obtain

$$b_{\zeta\eta} b_{\beta\gamma} B_{\alpha\nu\delta\mu} + b_{\mu\eta} b_{\beta\gamma} B_{\alpha\nu\zeta\delta} + b_{\zeta\eta} b_{\beta\delta} B_{\alpha\nu\mu\gamma} + b_{\mu\eta} b_{\beta\delta} B_{\alpha\nu\gamma\zeta} = 0. \tag{8. 6}$$

Putting $\gamma = \mu$ in (8. 6) we have

$$\left\{\begin{array}{l} b_{\zeta\eta} b_{\beta\mu} B_{\alpha\nu\delta\mu} + b_{\mu\eta} b_{\beta\delta} B_{\alpha\nu\mu\zeta} + b_{\mu\eta} b_{\beta\mu} B_{\alpha\nu\zeta\delta} = 0, \\ b_{\zeta\eta} b_{\beta\mu} B_{\nu\omega\delta\mu} + b_{\mu\eta} b_{\beta\delta} B_{\nu\omega\mu\zeta} + b_{\mu\eta} b_{\beta\mu} B_{\nu\omega\zeta\delta} = 0, \\ b_{\zeta\eta} b_{\beta\mu} B_{\omega\alpha\delta\mu} + b_{\mu\eta} b_{\beta\delta} B_{\omega\alpha\mu\zeta} + b_{\mu\eta} b_{\beta\mu} B_{\omega\alpha\zeta\delta} = 0, \end{array}\right. \tag{8. 7}$$

the second and third of these sets of equations being obtained from the first by replacing the indices $(\alpha\nu)$ by $(\nu\omega)$ and $(\omega\alpha)$ respectively. The determinant of the above system is

$$\begin{vmatrix} B_{\alpha\nu\delta\mu} & B_{\alpha\nu\mu\zeta} & B_{\alpha\nu\zeta\delta} \\ B_{\nu\omega\delta\mu} & B_{\nu\omega\mu\zeta} & B_{\nu\omega\zeta\delta} \\ B_{\omega\alpha\delta\mu} & B_{\omega\alpha\mu\zeta} & B_{\omega\alpha\zeta\delta} \end{vmatrix}. \tag{8. 8}$$

By (8. 1) one of determinants (8. 8) must be different from zero. Let the indices $\alpha, \nu, \omega, \delta, \mu, \zeta$ be chosen so that the above determinant (8. 8) is not equal to zero; then from (8. 7) we have

$$b_{\zeta\eta} b_{\beta\mu} = b_{\mu\eta} b_{\beta\delta} = b_{\mu\eta} b_{\beta\mu} = 0,$$

where δ, μ, ζ are *determined* and β, η are *arbitrary.* Putting $\eta = \beta$ we see from

[1] See B. L. van der Waerden, *Moderne Algebra*, II, Berlin, Springer, 1931, p. 14.

the above equations that $b_{\mu\beta} b_{\beta\mu} = 0$ from which it follows that $b_{\mu\beta} = 0$ for β arbitrary. Making use of this fact and replacing the free index γ in (8. 2) by the determined index ζ these equations yield the system

$$
(8.9) \qquad \begin{cases} b_{\beta\zeta} B_{\alpha\nu\delta\mu} + b_{\beta\delta} B_{\alpha\nu\mu\zeta} = 0, \\ b_{\beta\zeta} B_{\nu\omega\delta\mu} + b_{\beta\delta} B_{\nu\omega\mu\zeta} = 0, \\ b_{\beta\zeta} B_{\omega\alpha\delta\mu} + b_{\beta\delta} B_{\omega\alpha\mu\zeta} = 0, \end{cases}
$$

with matrix

$$
\begin{Vmatrix} B_{\alpha\nu\delta\mu} & B_{\alpha\nu\mu\zeta} \\ B_{\nu\omega\delta\mu} & B_{\nu\omega\mu\zeta} \\ B_{\omega\alpha\delta\mu} & B_{\omega\alpha\mu\zeta} \end{Vmatrix}.
$$

The rank of this matrix must be two since otherwise the determinant (8. 8) would be equal to zero contrary to hypothesis. Hence (8. 9) gives $b_{\beta\zeta} = 0$ and $b_{\beta\delta} = 0$ for ζ, δ determined and β arbitrary. It now follows from equations (8. 2) that

$$
\begin{cases} b_{\beta\gamma} B_{\alpha\nu\delta\mu} = 0, \\ b_{\beta\gamma} B_{\nu\omega\delta\mu} = 0, \\ b_{\beta\gamma} B_{\omega\alpha\delta\mu} = 0, \end{cases}
$$

where the indices β and γ are arbitrary. Now one of the B's in these equations must be different from zero since otherwise the determinant (8. 8) would be equal to zero. Hence $b_{\beta\gamma} = 0$ for arbitrary values of the indices β, γ and since we have assumed that $t = 0$ it follows that the solution $(t, b_{\alpha\beta})$ of the system (8. 2) and (8. 3) is trivial contrary to hypothesis. Hence we must have $t \neq 0$ so that the quantities $b_{\alpha\beta}/t$ can be defined and these constitute a solution of the Gauss equations (2. 9). We thus arrive at the following

Theorem. *If the left members of the* Gauss *equations* (2. 9) *are the components of the curvature tensor of a* Riemann *space* ($n \geqq 3$) *of type* $\tau \geqq 3$ *with element of distance* (3. 1) *defined in a neighborhood* U *then these equations have a solution* $b_{\alpha\beta}(x)$ *for* $x < U$ *if, and only if, the inequality* (8. 1) *and the equations* (8. 4) *are satisfied.*

When the conditions (7. 4) are likewise imposed it follows from the theorem at the end of § 7 that the above solution $b_{\alpha\beta}(x)$ will be *real.* In this case the polynomial inequality (8. 1) can be replaced by the polynomial inequality

$$- \sum \begin{vmatrix} B_{\alpha\beta\lambda\mu} & B_{\alpha\beta\mu\nu} & B_{\alpha\beta\nu\lambda} \\ B_{\beta\gamma\lambda\mu} & B_{\beta\gamma\mu\nu} & B_{\beta\gamma\nu\lambda} \\ B_{\gamma\alpha\lambda\mu} & B_{\gamma\alpha\mu\nu} & B_{\gamma\alpha\nu\lambda} \end{vmatrix} > 0, \qquad (x < U) \tag{8.10}$$

of lower degree, the summation in this inequality and in (8. 1) having the same significance. Hence we have the

Theorem. *If the left members of the* GAUSS *equations* (2. 9) *are the components of the curvature tensor of a* RIEMANN *space* ($n \geqq 3$) *of type* $\tau \geqq 3$ *with element of distance* (3. 1) *defined in a neighborhood* U *then these equations will have a real solution* $b_{\alpha\beta}$ *for* $x < U$ *if, and only if, the inequalities* (7. 4) *and* (8. 10) *and the equations* (8. 4) *are satisfied.*

It follows from the results of § 5 that the solution $b_{\alpha\beta}(x)$ of the GAUSS equations which exists in accordance with the above theorem will be determined uniquely to within algebraic sign at each point $x < U$.

9. Explicit Determination of the Solutions of the Gauss Equations. Continuity and Differentiability Properties.

Since the real solution $b_{\alpha\beta}(x)$ of the GAUSS equations is determined to within algebraic sign at each point $x < U$ under the conditions of the last theorem in the preceding section we are now faced with the problem of showing how these algebraic signs can be selected at the various points of U so that the functions $b_{\alpha\beta}(x)$ will be continuous and differentiable in U. We shall first derive a set of equations which will have application in the discussion of this problem after which it will be treated in detail under a number of special cases.

Consider the determinant

$$\Delta(\alpha\beta\gamma\lambda\mu\nu) = \begin{vmatrix} b_{\alpha\lambda} & b_{\alpha\mu} & b_{\alpha\nu} \\ b_{\beta\lambda} & b_{\beta\mu} & b_{\beta\nu} \\ b_{\gamma\lambda} & b_{\gamma\mu} & b_{\gamma\nu} \end{vmatrix}, \tag{9.1}$$

and also its adjoint, i. e. the determinant

$$\bar{\Delta}(\alpha\beta\gamma\lambda\mu\nu) = \begin{vmatrix} \bar{b}_{\alpha\lambda} & \bar{b}_{\alpha\mu} & \bar{b}_{\alpha\nu} \\ \bar{b}_{\beta\lambda} & \bar{b}_{\beta\mu} & \bar{b}_{\beta\nu} \\ \bar{b}_{\gamma\lambda} & \bar{b}_{\gamma\mu} & \bar{b}_{\gamma\nu} \end{vmatrix}, \tag{9.2}$$

wher $\bar{b}_{\alpha\beta}$ is the cofactor of the corresponding element in $\varDelta$. By a theorem in Algebra[1] any element of the determinant $\varDelta$, multiplied by $\varDelta$, is equal to the cofactor of the corresponding element of the adjoint determinant $\bar{\varDelta}$. Hence by (7. 2) we have

$$(9.3)\quad \varDelta(\alpha\beta\gamma\lambda\mu\nu)\,b_{\alpha\lambda} = \begin{vmatrix} B_{\alpha\gamma\nu\lambda} & B_{\alpha\gamma\lambda\mu} \\ B_{\alpha\beta\lambda\nu} & B_{\alpha\beta\mu\lambda} \end{vmatrix}, \quad \varDelta(\alpha\beta\gamma\lambda\mu\nu)\,b_{\alpha\mu} = \begin{vmatrix} B_{\alpha\gamma\lambda\mu} & B_{\alpha\gamma\mu\nu} \\ B_{\alpha\beta\mu\lambda} & B_{\alpha\beta\nu\mu} \end{vmatrix}, \ldots$$

Case I. $n = \tau = 3$. In this special case the determinant $\varDelta$ (123123) does not vanish in U; let us therefore choose the above algebraic signs so that at any point $x < U$ the determinant $\varDelta$ (123123) is positive. Now the determinant $\varDelta$ can be expressed in terms of the B's on account of (7. 3); when this substitution is made for $\varDelta$ (123123) the equations (9. 3) give

$$(9.4)\quad \begin{cases} b_{11} = \begin{vmatrix} B_{1331} & B_{1312} \\ B_{1213} & B_{1221} \end{vmatrix} \Bigg/ \sqrt{-\begin{vmatrix} B_{1212} & B_{1223} & B_{1231} \\ B_{2312} & B_{2323} & B_{2331} \\ B_{3112} & B_{3123} & B_{3131} \end{vmatrix}}, \\ b_{12} = \begin{vmatrix} B_{1312} & B_{1323} \\ B_{1221} & B_{1232} \end{vmatrix} \Bigg/ \sqrt{-\begin{vmatrix} B_{1212} & B_{1223} & B_{1231} \\ B_{2312} & B_{2323} & B_{2331} \\ B_{3112} & B_{3123} & B_{3131} \end{vmatrix}}, \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \end{cases}$$

Since the coefficients of the form (3. 1) are assumed to be continuous with continuous first and second derivatives in U and also since the above equations (9. 4) are valid throughout U it follows immediately that the functions $b_{\alpha\beta}(x)$ are continuous in U. If furthermore the coefficients of the form (3. 1) are such that the B's possess continuous partial derivatives the functions $b_{\alpha\beta}(x)$ will have continuous partial derivatives to the corresponding order.

Before proceeding to the next case we shall make some general remarks which will have application whenever the rank τ of the matrix $\|b_{\alpha\beta}(x)\|$ is $< n$. By hypothesis then the matrix $\|b_{\alpha\beta}(x)\|$ has rank $\tau < n$ in U. Consider an arbitrary point $x_1 < U$. By a theorem in Algebra there must be a diagonal determinant of $\|b_{\alpha\beta}(x_1)\|$ of order τ which is different from zero since τ is the rank of the matrix $\|b_{\alpha\beta}(x_1)\|$; for definiteness we suppose this non-vanishing determinant to be situated in the upper left hand corner of $\|b_{\alpha\beta}(x_1)\|$, i. e. we suppose the determinant

[1] See M. BOCHER, *Introduction to Higher Algebra*, MacMillan, 1929, p. 33.

$$\left|\begin{matrix} b_{11} & \cdots & b_{1\tau} \\ \cdot & \cdot & \cdot \\ b_{\tau 1} & \cdots & b_{\tau\tau} \end{matrix}\right| \tag{9.5}$$

to be different from zero at x_1. Then at x_1 there must exist identical relations of the form

$$b_{\mu d} = \sum_{c=1}^{\tau} b_{cd} A^c_\mu, \qquad \begin{pmatrix} d = 1, \ldots, \tau \\ \mu = \tau + 1, \ldots, n \end{pmatrix}, \tag{9.6}$$

$$b_{\mu\nu} = \sum_{c=1}^{\tau} b_{c\nu} A^c_\mu = \sum_{c,\, d=1}^{\tau} b_{cd} A^c_\mu A^d_\nu, \qquad (\mu, \nu = \tau + 1, \ldots, n). \tag{9.7}$$

Now multiply (9.6) by b_{ef} interchange the indices e and d in these equations and subtract. When use is made of the Gauss equations (2.9) we then obtain the *consistent* system

$$B_{def\mu} = \sum_{c=1}^{\tau} B_{defc} A^c_\mu, \qquad \begin{pmatrix} d, e, f = 1, \ldots, \tau \\ \mu = \tau + 1, \ldots, n \end{pmatrix}, \tag{9.8}$$

with matrix

$$\left\|\begin{matrix} B_{def1} & \cdots & B_{def\tau} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix}\right\|. \tag{9.9}$$

Now consider that set of the Gauss equations which contains in its left members the elements of the above matrix, i. e. the equations

$$B_{defc} = b_{dc} b_{ef} - b_{df} b_{ec}, \qquad (d, e, f, c = 1, \ldots, \tau). \tag{9.10}$$

Since $\tau \geqq 3$ and the determinant (9.5) is different from zero at x_1 we know from the consideration of the system equivalent to (9.10) in § 5 that at x_1 the matrix (9.9) has rank τ, the maximum possible rank of this matrix in U. Hence the equations (9.8) give a determination of the quantities A^c_μ as rational expressions in the components of the curvature tensor B, these expressions being valid in a neighborhood V such that $x_1 < V < U$.

Case II. $n > 3$, $\tau = 3$. At an arbitrary point $x_1 < U$ the matrix $\| b_{\alpha\beta}(x) \|$ will have rank three and hence will contain a non-vanishing diagonal determinant of order three; without loss of generality we may suppose this to be the above determinant Δ (123123) and we may furthermore suppose the algebraic sign of the functions $b_{\alpha\beta}(x)$ at $x = x_1$ to be selected so that Δ (123123) is positive. As we have already mentioned it then follows from § 5 that the corresponding matrix (9. 9) will have rank three at x_1. Let $R(B)$ denote a third order determinant in (9. 9) which does not vanish at x_1; then the equations (9. 8) corresponding to this determinant will give a unique determination of the quantities A^c_μ by rational expressions in the B's valid in a neighborhood V such that $x_1 < V < U$. Now the square of Δ (123123) is equal to the expression underneath the radical in (9. 4) which is $\geqq 0$ by (7. 4). But since Δ (123123) is positive at x_1 the above expression underneath the radical will also be positive at x_1 and hence positive in a neighborhood $W < V$ of x_1. Hence Δ (123123) will be different from zero in W and we can therefore choose the algebraic signs of the functions $b_{\alpha\beta}(x)$ for $x < W$ so that the determinant Δ (123123) is positive in W. Then $b_{\alpha\beta}(x)$ for $x < W$ and $\alpha, \beta = 1, 2, 3$ will be given by (9. 4). Denote the polynomial in the B's which appears underneath the radical in the denominators of (9. 4) by $Q(B)$. Then substituting (9. 4) for the b_{cd} and also the above rational expression for the A^c_μ into the right members of (9. 6) and (9. 7) we have the $b_{\alpha\beta}$ given, for all values of the indices, by expressions of the form

$$\frac{P(B)}{[Q(B)]^{1/2}} \quad \text{or} \quad \frac{P(B)}{[Q(B)]^{1/2} R(B)}, \tag{9.11}$$

where $P(B)$ denotes any polynomial in the B's, and these expressions are valid in the neighborhood W.

Case III. $n > 3$, $\tau > 3$. The matrix $\|b_{\alpha\beta}\|$ has a fixed rank τ at points $x < U$ and hence at any point x there will be a diagonal determinant of order τ which does not vanish. At a particular point $x_1 < U$ let us suppose for definiteness that the determinant

$$\begin{vmatrix} b_{11} & \cdots & b_{1\tau} \\ \cdot & \cdot\;\cdot & \cdot \\ b_{\tau 1} & \cdots & b_{\tau\tau} \end{vmatrix} \tag{9.12}$$

is different from zero. As we have seen above we may now determine the A^c_μ by (9. 8) as rational expressions in the B's, these expressions being valid in a neighborhood $V < U$ of the point x_1. It remains to be shown that expressions can be found for the elements b_{cd} of the above determinant (9. 12) corresponding to the first expression in (9. 11) after which the equations (9. 6) and (9. 7) can be applied as in the preceding case. In this connection we shall make use of the following lemma which we shall state without proof since it obviously admits a simple formal demonstration.

Lemma. *Let Q be a quadratic form with matrix M of rank $\sigma \geqq 3$. It is then possible by means of a non-singular transformation of the variables to transform Q into a form Q' such that in the matrix M' of the coefficients of Q' every minor determinant of order three will be different from zero.*

By this lemma it is possible to make a linear transformation $x \rightarrow y$ of the coordinates of U in consequence of which the matrix $\|b_{\alpha\beta}\|$ at x_1 will have the form

$$\left\| \begin{array}{ccc|c} b_{11} & \cdots & b_{1\tau} & \\ . & . & . & 0 \\ b_{\tau 1} & \cdots & b_{\tau\tau} & \\ \hline & 0 & & 0 \end{array} \right\|$$

at the corresponding point y_1 and all of the third order minors $\Delta(\alpha\beta\gamma\lambda\mu\nu)$ in the determinant (9. 12) appearing in the upper left hand corner of the above matrix will be different from zero. Now by (7. 3) these third order minors are given by

$$\Delta(\alpha\beta\gamma\lambda\mu\nu) = \pm \sqrt{-\begin{vmatrix} B_{\alpha\beta\lambda\mu} & B_{\alpha\beta\mu\nu} & B_{\alpha\beta\nu\lambda} \\ B_{\beta\gamma\lambda\mu} & B_{\beta\gamma\mu\nu} & B_{\beta\gamma\nu\lambda} \\ B_{\gamma\alpha\lambda\mu} & B_{\gamma\alpha\mu\nu} & B_{\gamma\alpha\nu\lambda} \end{vmatrix}}. \tag{9. 13}$$

Since all the expressions underneath the radical will be positive at y_1 they will therefore be positive in a certain neighborhood D of y_1. Hence in D all third order minors of (9. 12) will be different from zero. Now choose the algebraic signs of the functions $b_{\alpha\beta}$ at points of D so that the determinant Δ (123123) will be positive in D. Then Δ (123123) is given in D by the corresponding equation (9. 13) in the right member of which the $+$ sign is to be taken; hence the functions $b_{\alpha\beta}$ for $\alpha, \beta = 1, 2, 3$ will be given by (9. 4) in D and will therefore be continuous in this neighborhood. Now consider a third order minor of (9. 12) containing one of the rows or columns of Δ (123123), for example, the determinant

$$\Delta\ (234123) = \begin{vmatrix} b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{vmatrix}.$$

Suppose that the equation (9. 13) which contains this determinant in its left member involves in its right member the $-$ sign at y_1 and the $+$ sign at some other point $y_2 < D$. Join y_1 and y_2 by a continuous curve C in D. Then since the function Δ (234123) does not vanish on C and has opposite algebraic signs at the end points of C it must be discontinuous at some point P of C. At P one of the elements in the first row of the determinant Δ (234123) must be different from zero since this determinant does not vanish at P; for definiteness suppose $b_{21} \neq 0$ at P. Then by (9. 3) we have

$$\Delta\,(234123) = \begin{vmatrix} B_{2431} & B_{2412} \\ B_{2313} & B_{2321} \end{vmatrix} \Big/ b_{21}.$$

Hence the left member of this equation is continuous at P since $b_{21} \neq 0$ at P and is continuous in D. It follows that the equation (9. 13) which gives Δ (234123) in D must involve only the $-$ sign in its right member if this sign is valid at the point y_1. From (9. 3) we can now obtain expressions analogous to (9. 4) for the functions in the last row of Δ (234123) and these expressions will be valid throughout D. Continuing we can thus show that the functions $b_{\alpha\beta}$ for $\alpha, \beta = 1, \ldots, \tau$ are given in D by definite expressions in the B's of the general form of the first expression in (9. 11); from these and the rational expressions for the A^c_μ it follows by (9. 6) and (9. 7) that the functions $b_{\alpha\beta}$ are given, for all values of the indices, by definite expressions in the B's of the general form (9. 11) valid throughout D. Hence all $b_{\alpha\beta}$ are continuous in D and are likewise differentiable in this neighborhood in accordance with the differentiability properties of the components of the curvature tensor B. If, now, we make the linear transformation $y \rightarrow x$ to the original coordinates of U the above properties of continuity and differentiability of the functions $b_{\alpha\beta}$ will persist in a neighborhood $W < U$ of the point x_1.

Under Case II and Case III we have now shown that corresponding to any point $x_1 < U$ there exists a neighborhood $W < U$ containing x_1 such that in W the algebraic signs of the functions $b_{\alpha\beta}(x)$ can be chosen so that these functions are continuous. To indicate the association of the above neighborhood W with the point x_1 we shall henceforth employ the notation $W(x_1)$. Now *the algebraic signs of the functions $b_{\alpha\beta}(x)$ in $W(x_1)$ are uniquely determined by the selection of sign at an arbitrary point $x' < W(x_1)$ and the requirement that the functions $b_{\alpha\beta}(x)$ be continuous in $W(x_1)$.* In fact we saw from the preceding considerations that the continuous functions $b_{\alpha\beta}(x)$ were determined uniquely in $W(x_1)$ by a selection of algebraic sign at x_1; and it was also clear that if the other selection of sign were made at x_1 the functions $b_{\alpha\beta}(x)$ so determined would be the negatives of those previously found. Hence there are only the two possibilities $\pm b_{\alpha\beta}(x)$ and from this fact the above statement follows immediately.

We must now show that we can choose the algebraic signs of the functions $b_{\alpha\beta}(x)$ in the various neighborhoods $W(x_1) < U$ so that these functions will be continuous throughout U. For this purpose we assume $b_{\alpha\beta}(x)$ continuous in $W(x_1)$ where x_1 is any point of U. Let x' be any other point of U not in $W(x_1)$. Join x_1 to x' by a continuous curve C. We may now suppose for simplicity that the neighborhoods $W(x)$ are *spherical* neighborhoods in U in the definition of which the Euclidean measure of distance in U may be adopted. We can now cover the points of C by a finite number of spherical neighborhoods $W(x_1)$, $W(x_2), \ldots, W(x_n)$, $W(x')$ which can be taken so that two consecutive neighborhoods alone have points in common. By taking the algebraic signs of the functions $b_{\alpha\beta}(x)$ for $x < W(x_2)$ so that at a point x'' of the intersection $W(x_1) \wedge W(x_2)$ the values of the above functions are identical with the values of the functions $b_{\alpha\beta}(x)$ for $x < W(x_1)$ we secure the identity of $b_{\alpha\beta}(x)$ for $x < W(x_1)$ and $x < W(x_2)$ at all points of the intersection; this follows from the above italicized statement. Then $b_{\alpha\beta}(x)$ is continuous for $x < W(x_1) + W(x_2)$. Proceeding we define the $b_{\alpha\beta}(x)$ as continuous functions for $x < N$ where

$$N = W(x_1) + \cdots + W(x_n) + W(x').$$

If, now, we join x_1 to x' by another continuous curve C' and proceed as for the curve C the values of the functions $b_{\alpha\beta}(x)$ at $x = x'$ will be the same for this second determination. In fact since U is simply connected by hypothesis we can pass from C' to C by a continuous deformation in U thus sweeping out a surface Ψ. Now it is possible to pass from C to C' by a finite number of continuous curves $C, C_1, \ldots, C_m, C'$ on the surface Ψ possessing the property that any curve of the set lies wholly within the spherical neighborhoods W by which the preceding curve is covered; as this construction is evidently possible the details will be omitted. Proceeding along the neighborhoods by which C_1 is covered we define the continuous functions $b_{\alpha\beta}(x)$ for $x < N_1, \ldots$, and finally the continuous functions $b_{\alpha\beta}(x)$ for $x < N'$. Thus we arrive at the continuous functions $b_{\alpha\beta}(x)$ for $x < N^*$ where

$$N^* = N + N_1 + \cdots + N_m + N'$$

and hence the determination of the values of $b_{\alpha\beta}(x)$ at $x = x'$ is independent of the curve C as above stated. It is therefore possible to choose the algebraic signs so that the functions $b_{\alpha\beta}(x)$ will be continuous for $x < U$ and in fact uniquely determined by the selection of algebraic sign at an arbitrary point $x_1 < U$.

For brevity in the statement of our results let us denote by R_2 a RIEMANN space with element of distance defined by the (positive definite) quadratic differential form

$$d s^2 = \sum_{\alpha, \beta=1}^{n} g_{\alpha\beta}(x)\, d x^\alpha\, d x^\beta,$$

having coefficients $g_{\alpha\beta}(x)$ which are continuous and possessing continuous first and second derivatives in an open simply connected neighborhood U of a point of the $n (\geqq 2)$ dimensional (real) number space. If furthermore the components of the curvature tensor B possess continuous first derivatives in U the space will be denoted by R_2^*. We may now state the following

Theorem. *Let the left members of the* GAUSS *equations* (2. 9) *be the components of the curvature tensor B of a space R_2 and denote by $b_{\alpha\beta}(x)$ the real solution of these equations which is determined to within algebraic sign at the various points $x < U$ under the conditions stated in the Theorem at the end of* § 8. *Then it is possible to select these algebraic signs so that the functions $b_{\alpha\beta}(x)$ are continuous in U being uniquely determined by the selection of algebraic sign at an arbitrary point $x_1 < U$ and having the general form $\pm\, b_{\alpha\beta}(x)$ in U. If the space R_2 is also an R_2^* then the continuous functions $b_{\alpha\beta}(x)$ will have continuous first partial derivatives in U.*

10. Algebraic Characterizations.

By the result of § 5 we know that the space R_2 will be of type $\tau \geqq 2$, if, and only if, the matrix

$$\begin{Vmatrix} B_{\alpha\beta\gamma 1} & \cdots & B_{\alpha\beta\gamma n} \\ B_{\lambda\mu\nu 1} & \cdots & B_{\lambda\mu\nu n} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot\;\cdot\;\cdot & \cdot \end{Vmatrix}$$

has rank τ at all points $x < U$. To state this in the form of an algebraic characterization as defined in the introduction to this paper let us denote by $H_\tau^{(1)}, H_\tau^{(2)}, \ldots$ the minor determinants of order $\tau + 1$ and by $G_\tau^{(1)}, G_\tau^{(2)}, \ldots$ the minor determinants of order τ of the above matrix. Then the algebraic characterization of Riemann spaces R_2 of type τ is given by the following[1]

Theorem. *A space R_2 is of type $\tau \geqq 2$ if, and only if, the following conditions*

$$H_{\tau}^{(1)} = H_{\tau}^{(2)} = \cdots = 0, \quad \sum_i [G_{\tau}^{(i)}]^2 \neq 0$$

are satisfied at all points $x < U$.

Let us now consider the algebraic characterization of spaces R_2^* for which $n \geqq 3$, $\tau = 3$ as spaces of class one. We assume first of all that the conditions of the Theorem at the end of the preceding section are satisfied so that the Gauss equations admit a real solution $b_{\alpha\beta}(x)$ continuous and differentiable in U. If $n = 3$ the conditions that the Codazzi equations (2. 8) be satisfied are obtained immediately by substituting into these equations the values of the $b_{\alpha\beta}$ given by (9. 4). Denoting the polynomial in the B's which appears underneath the radical in (9. 4) by $Q(B)$ the equations resulting from the above substitution will involve terms of the form

$$\frac{P(B)\,\Gamma}{[Q(B)]^{1/2}}, \quad \frac{P(B, B')}{[Q(B)]^{1/2}}, \quad \frac{P(B)\,B'}{[Q(B)]^{3/2}},$$

where $P(B)$ is used to denote any polynomial in the B's and $P(B, B')$ to denote any polynomial in the B's and their first partial derivatives B'; also it is observed that those terms of the first of the above types contain the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$. Multiplying the above equations by $[Q(B)]^{3/2}$ and transposing all terms to the left members these become polynomials in the quantities Γ, B and the first partial derivatives of the B's which are denoted by B'. Let us represent this set of polynomials by $F_3(B, B', \Gamma)$.

If $n > 3$ the proceedure while analogous to the above is somewhat more complicated. Select *any* third order determinant $R(B)$ from the matrix (9. 9) with $\tau = 3$ and use the corresponding equations (9. 8) to determine the A^c_{μ}. Substitute these A^c_{μ} and the b_{cd} given by (9. 4) into (9. 6) and (9. 7) to determine the remaining $b_{\alpha\beta}$. We shall then have the $b_{\alpha\beta}$ given by expressions of the form

[1] Although we have previously defined the algebraic characterization in terms of conditions involving a set of polynomials in the structural functions $g_{\alpha\beta}$ of the Riemann space and their derivatives there is no objection to stating these conditions in terms of polynomials in the components of the curvature tensor B since we can immediately pass from these latter conditions to the former. Analogous remarks apply to the algebraic characterizations given by the following theorems.

$$\frac{P(B)}{[Q(B)]^{1/2}}, \quad \frac{P(B)}{[Q(B)]^{1/2} R(B)}. \tag{10.1}$$

Substituting these expressions for the $b_{\alpha\beta}$ into the CODAZZI equations (2.8) the resulting equations will involve a sum of terms of the form

$$\begin{cases} \dfrac{P(B)\Gamma}{[Q(B)]^{1/2}}, & \dfrac{P(B)\Gamma}{[Q(B)]^{1/2} R(B)}, \quad \dfrac{P(B, B')}{[Q(B)]^{1/2}}, \quad \dfrac{P(B) B'}{[Q(B)]^{3/2}}, \\ \dfrac{P(B, B')}{[Q(B)]^{1/2} R(B)}, & \dfrac{P(B) B'}{[Q(B)]^{3/2} R(B)}, \quad \dfrac{P(B) B'}{[Q(B)]^{1/2} [R(B)]^2}. \end{cases}$$

Multiply these equations by $[Q(B)]^{3/2} [R(B)]^2$ and transpose all terms to the left members; then these left members will be polynomials in the Γ's the B's and the first partial derivatives B' of the B's. Denote the set of these polynomials by $F_n^{(1)}(B, B', \Gamma)$.

Now for the case under discussion it is possible for $Q(B)$ or $R(B)$ to vanish at some point $x_1 < U$. But if $Q(B) = 0$ at x_1 the expressions (10.1) at x_1 will be indeterminate, i. e. the numerators $P(B)$ will likewise vanish at this point; similarly if $R(B) = 0$ at x_1 the second set of terms (10.1) must be indeterminate. This follows from the above assumption that the GAUSS equations have a solution $b_{\alpha\beta}$ in U and the fact that without this indeterminacy we would have a contradiction with this assumption. Hence if either $Q(B)$ or $R(B)$ is equal to zero at x_1 we see that at this point the polynomials of the set $F_n^{(1)}$ must likewise vanish. In the contrary case, i. e. if neither $Q(B)$ nor $R(B)$ is equal to zero at x_1 then $F_n^{(1)} = 0$ express necessary conditions on the space R_2^*.

Owing to the possibility that $Q(B)$ or $R(B)$ may vanish at some point $x_1 < U$ with the consequence that the equations $F_n^{(1)} = 0$ will fail to express the condition that the CODAZZI equations are satisfied at x_1 we must repeat the above process by which the expressions $F_n^{(1)}$ were determined for all selections of third order determinants of the matrix (9.9) and for all sets of equations analogous to (9.4). In this connection it may be observed that the selection of algebraic sign before the denominators in (9.4), or the analogous equations, is immaterial since a reversal of sign at this place will merely reverse the sign of all quantities $b_{\alpha\beta}$ which are thereby determined. We thus arrive at a finite set of polynomial expressions $F_n^{(1)}, F_n^{(2)}, \ldots$ which for brevity we shall denote by $F_n(B, B', \Gamma)$. From what we have said above we now see that the equations

$$F_n(B, B', \Gamma) = 0, \qquad (x < U), \tag{10. 2}$$

give necessary conditions for R_2^* to be of class one. Moreover $F_n = 0$ is sufficient for the CODAZZI equations to be satisfied; this follows from the fact that $F_n = 0$ must contain one set of equations $F_n^{(i)} = 0$ in the construction of which the determinant selected from the matrix (9. 9) as well as the determinant which appears in the denominators of the equations of the type (9. 4) will be different from zero at the point $x < U$. This result in conjunction with the result of § 3 and the Theorems at the end of § 8 and § 9 now gives us the following theorem.

Theorem. *A space R_2^* of dimensionality $n \geqq 3$ and type $\tau = 3$ is immersible in a Euclidean space E of $n + 1$ dimensions i. e. it is of class one if, and only if, the conditions* (7. 4), (8. 4), (8. 10) *and* (10. 2) *are satisfied.*

If the space R_2^* is of type $\tau > 3$ the CODAZZI equations are satisfied automatically under the conditions stated in the Theorem at the end of § 6. The following theorem results immediately.

Theorem. *A space R_2^* of dimensionality $n \geqq 4$ and type $\tau \geqq 4$ is immersible in a Euclidean space E of $n + 1$ dimensions i. e. it is of class one if, and only if, the conditions* (7. 4), (8. 4) *and* (8. 10) *are satisfied.*

By combining the conditions of the first Theorem of this section with those of the following Theorems we obtain directly the algebraic characterization of the space R_2^* as a space of type $\tau \geqq 3$ and class one.

11. Extension to Topological Spaces.

Consider a space in which neighborhoods are defined satisfying the four axioms of HAUSDORFF (topological space) the neighborhoods N of the space being homeomorphic to the above neighborhood U. In consequence of this homeomorphism the space is covered by one or more systems of coordinates x^α and we shall assume for our present requirements that the coordinate transformation which is thereby defined in the intersection $N_1 \wedge N_2$ of any two intersecting neighborhoods N_1 and N_2 is continuous and possesses continuous partial derivatives to the order three inclusive. A topological space of this character will be denoted by H_3. The extension of the preceding discussion to *closed* spaces H_3 is of particular interest from a geometrical standpoint. In the following we state certain results concerning topological spaces H_3 which can be obtained from the foregoing developments without the necessity of additional calculations.

Let $y^1(x), \ldots, y^{n+1}(x)$ be a set of $n + 1$ scalar functions defined over H_3 these functions being continuous and having continuous partial derivatives to the third order; it is to be observed that these properties of continuity and differentiability of the functions $y(x)$ will be retained in the transition from one neighborhood N_1 of H_3 to an other neighborhood N_2 owing to the corresponding properties of continuity and differentiability of the coordinate relationships between the coordinates of intersecting neighborhoods of H_3. We assume that the functional matrix $\|\partial y^i(x)/\partial x^\alpha\|$ has rank n at any point of H_3 so that the equations

$$y^i = y^i(x), \qquad (x < U), \tag{11.1}$$

taken over H_3 define a hypersurface S of the $n + 1$ dimensional Euclidean space E. Now let (3.1) be a positive definite quadratic differential form defined over H_3; that is more specifically (3.1) is defined in the coordinate neighborhoods U by which H_3 is covered in such a way that the coefficients $g_{\alpha\beta}(x)$ of this form enjoy the tensor law of transformation under transformations of coordinates in H_3. We shall say that a Riemann space R_2 is defined over H_3 if the above coefficients $g_{\alpha\beta}(x)$ are continuous and possess continuous first and second partial derivatives in the coordinate neighborhoods U; the space R_2 will be called a Riemann space R_2^* over H_3 if the components of the curvature tensor B possess continuous first partial derivatives throughout H_3. In a corresponding manner we may extend the definition of the type number τ defined in § 5 to a Riemann space R_2 over H_3. Now it is evident that the discussion in § 1 to § 9 inclusive extends immediately to spaces R_2 or R_2^* of type τ defined over H_3 owing to the invariant character of the underlying equations of these sections provided that the topological space H_3 is simply connected, the property of simple connectivity being necessary since use was made of this property of the neighborhoods U on several occasions. We can therefore state immediately the following

Theorem. *A Riemann space R_2^* of dimensionality $n \geqq 3$ defined over a simply connected topological space H_3 is of class one and type three if, and only if, the conditions* (7.4), (8.4), (8.10), (10.2) *and*

$$H_3^{(1)} = H_3^{(2)} = \cdots = 0, \quad \sum_i [G_3^{(i)}]^2 \neq 0, \qquad (x < U),$$

are satisfied over H_3. Similarly the space R_2^* is of class one and type $\tau \geqq 4$ if, and only if, the conditions (7. 4), (8. 4), (8. 10) *and*

$$H_\tau^{(1)} = H_\tau^{(2)} = \cdots = 0, \quad \sum_i [G_\tau^{(i)}]^2 \neq 0, \qquad (x < U),$$

are satisfied over H_3.

We shall say that the Riemann space R_2 is of *variable type* if the rank of the matrix $\|B\|$ defined in § 5 is not constant over H_3; in particular R_2 will be said to be of variable type $\tau \geqq \sigma$ if the rank of the matrix $\|B\|$ is not less than σ over H_3. If a space R_2 is of variable type $\geqq 3$ over H_3 it follows from the Theorem at the end of § 8 that under the (necessary) conditions (7. 4), (8. 4) and (8. 10) the Gauss equations admit a real solution $b_{\alpha\beta}$ over H_3 this solution being determined to within algebraic sign at the points of H_3 (subject of course to the indeterminacy due to coordinate transformations). To prove that these algebraic signs can be chosen so that functions $b_{\alpha\beta}(x)$ are continuous over a simply connected space H_3 we select any point $p < H_3$ at which the matrix $\|b_{\alpha\beta}\|$ has rank $\geqq 3$ and by the Lemma of § 9 transform the coordinates of the neighborhood $U > p$ so that at this point, with respect to the coordinates introduced, all third order minors of $\|b_{\alpha\beta}\|$ are different from zero. We first choose the algebraic signs so that the functions $b_{\alpha\beta}(x)$ are continuous in some neighborhood of the point p using the method of § 9, which was there however applied only to those elements of the matrix $\|b_{\alpha\beta}\|$ appearing in the determinant (9. 12), for this purpose.[1] We then extend this selection of algebraic signs throughout the simply connected space H_3 by the process employed in § 9 with reference to the coordinate neighborhood U. We thus arrive at the conclusion that *the algebraic signs can be selected so that the functions $b_{\alpha\beta}(x)$ for a space R_2 of variable type $\geqq 3$ are continuous in H_3 and in fact that these functions possess continuous first partial derivatives in case we are dealing with a space R_2^* of variable type $\geqq 3$ over H_3.* Since the continuous functions $b_{\alpha\beta}(x)$ for a space R_2^* of variable type > 3 over H_3 necessarily satisfy the Codazzi equations by the Theorem at the end of § 6 the following theorem has now been proved.

[1] It is here to be noted that we have avoided the use of the equations (9. 6) and (9. 7) which are applicable only when the space R_2 is of definite type τ over H_3. The method of § 9 involving the use of the above equations (9. 6) and (9. 7) enables us however to avoid the transformation of coordinates in the Lemma of § 9 in the rigorous derivation of the continuity and differentiability properties of the solutions $b_{\alpha\beta}$ of the Gauss equations for spaces R_2^* of type three; also the above equations are clearly necessary in the derivation of the conditions (10. 2).

Theorem. *A* Riemann *space* R_2^* *of dimensionality* $n > 3$ *defined over a simply connected topological space* H_3 *is of class one and variable type* $\tau > 3$ *if, and only if, the conditions* (7. 4), (8. 4), (8. 10) *and*

$$\sum_i [H_3^{(i)}]^2 \neq 0, \qquad (x < U),$$

are satisfied over H_3.

If the conditions of any one of the above theorems are satisfied so that the Riemann space R_2^* over H_3 can be regarded as a hypersurface S in the Euclidean space E then, since the second fundamental form ψ of the hypersurface S is (to within algebraic sign) determined uniquely, we know from the result of § 4 that S is determined in E to within a motion of this latter space.

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