ON THE CHARACTERISTIC VALUES OF LINEAR INTEGRAL EQUATIONS.

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1. Introduction.

1.1. The present paper is concerned primarily with the following question: *What can be said about the distribution of the characteristic values of the Fredholm integral equation*

$$y(x) = \lambda \int_a^b K(x, \xi)\, y(\xi)\, d\xi \tag{1.1}$$

on the basis of the general analytic properties of the kernel $K(x, \xi)$ such as integrability, continuity, differentiability, analyticity and the like?

The literature where this and analogous questions are treated is very considerable [HELLINGER-TOEPLITZ, 1].[1] A relatively small part of this literature, however, has points of contact with the present paper, the discussion of the majority of papers published on the subject being based on various special properties of the kernels. It is assumed frequently that the kernel belongs to some special class of functions, or that it coincides with the Green's function of a differential or integro-differential boundary value problem. Problems of this sort will be excluded from the scope of our paper although they are interesting from a theoretical point of view and important for the applications.

[1] The quotations in brackets [ ] refer to the list of memoirs at the end of this article.

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The first result concerning the distribution of the characteristic values (abbreviated C.V. in the sequel) of a general kernel is due to FREDHOLM himself. It was shown by Fredholm [I], under the assumption that $K(x, \xi)$ is bounded and integrable, that the set of the C.V. of K coincides with the set of zeros of the »Fredholm determinant of K»

$$d_K(\lambda) \equiv \sum_{\nu=0}^{\infty} (-1)^\nu \delta_\nu(K) \lambda^\nu,$$

where

$$\delta_\nu(K) \equiv \delta_\nu = \frac{1}{\nu!} \int_a^b \cdots \int_a^b K\begin{pmatrix} s_1 s_2 \ldots s_\nu \\ s_1 s_2 \ldots s_\nu \end{pmatrix} ds_1 \ldots ds_\nu$$

and

$$K\begin{pmatrix} s_1 s_2 \ldots s_\nu \\ t_1 t_2 \ldots t_\nu \end{pmatrix} = \det (K(s_i, t_j)) \qquad (i, j = 1, 2, \ldots, \nu).$$

Assuming that

$$|K(x, \xi)| \leqq F$$

on the fundamental square

$$(\mathfrak{S}) \qquad a \leqq x \leqq b, \ a \leqq \xi \leqq b,$$

and using the Hadamard determinant theorem, Fredholm obtains the estimate

$$|\delta_\nu| \leqq \frac{\nu^{\frac{\nu}{2}}}{\nu!} F^\nu,$$

from which, by the general theory of entire functions[1], it follows that the exponent of convergence of the set $\{\lambda_\nu(K)\}$ of the C.V. of K is $\leqq 2$. We assume here and in the sequel that the terms $\lambda_\nu(K) \equiv \lambda_\nu$ of the sequence $\{\lambda_\nu(K)\}$ are repeated according to their multiplicities as the roots of $d_K(\lambda)$, and that they are ordered so that

$$r_1 \leqq r_2 \leqq \cdots \leqq r_\nu \leqq \cdots; \ r_\nu(K) \equiv r_\nu \equiv |\lambda_\nu(K)|.$$

The exponent of convergence of the set $\{\lambda_\nu(K)\}$ will be designated by $\varrho(K) \equiv \varrho$.

[1] We refer to [VALIRON, I and 2] concerning the terminology and the facts of the theory of entire functions, which are used in the present paper.

In the case where K satisfies a Lipschitz condition

$$|K(x, \xi) - K(x, \eta)| \leqq A|\xi - \eta|^{\alpha}, \qquad 0 < \alpha \leqq 1,$$

where A is a constant, Fredholm shows, by an argument of the same nature, that

$$|\delta_\nu| \leqq \frac{(\nu^\nu)^{\frac{1}{2} - \alpha}}{\nu!} A^\nu,$$

whence it follows that[1]

$$\varrho(K) \leqq \frac{2}{2\alpha + 1}.$$

In the case of a symmetric kernel $K(x, \xi) = K(\xi, x)$, WEYL [1; 2, p. 452] obtained a more precise result

$$r_\nu \nu^{-\frac{3}{2}} \to \infty,$$

using certain extremal properties of the C.V., and assuming that $K(x, \xi)$ is continuous on $(\mathfrak{S})$ and that $\frac{\partial K}{\partial x}$ is continuous in the interior of $(\mathfrak{S})$, while the integral

$$\iint_{\mathfrak{S}} \left|\frac{\partial K}{\partial x}\right|^2 dx\, d\xi$$

exists. It was also stated by Weyl that

$$r_\nu \nu^{-s-\frac{1}{2}} \to \infty,$$

provided $K(x, \xi)$ possesses continuous partial derivatives of order s.

By a suitable modification of the original argument of Fredholm, MAZURKIEWICZ [1] was able to show that the estimate

$$r_\nu \nu^{-\frac{3}{2}} \to \infty$$

holds for a general unsymmetric kernel $K(x, \xi)$, provided it is bounded on $(\mathfrak{S})$, and $\frac{\partial K}{\partial x}$ or $\frac{\partial K}{\partial \xi}$ is continuous on $(\mathfrak{S})$.

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[1] See also LALESCO [1, pp. 86—89].

Fredholm's formulas are not applicable, in general, when $K(x, \xi)$ is not bounded on $(\mathfrak{S})$ or not defined on the line $x = \xi$. In the case where

$$(H_\alpha) \qquad K(x, \xi) = O(|x-\xi|^{-\alpha}),\ \alpha < \frac{1}{2},$$

HILBERT [1, p. 31] introduced the modified Fredholm determinant

$$d_K^*(\lambda) \equiv \sum_{\nu=0}^{\infty} (-1)^\nu \delta_\nu^*(K) \lambda^\nu,$$

where $\delta_\nu^*(K)$ is obtained from $\delta_\nu(K)$ if $K(x, x)$ is replaced by 0. This modification will not affect the C.V. of K nor the solutions of the integral equation, while, in case the Fredholm determinant exists, we have the relation

$$d_K^*(\lambda) = d_K(\lambda)\ \exp\left\{\lambda \int_a^b K(x, x)\, dx\right\}$$

[Lalesco, 1, pp. 113—117]. The modified Fredholm determinant $d_K^*(\lambda)$ may exist even when $d_K(\lambda)$ does not. This was shown by Hilbert in the case (H_α), and extended by CARLEMAN [5] to the much more general case where the only assumption concerning $K(x, \xi)$ is the existence of the double integral

$$(L_2) \qquad \iint_{\mathfrak{S}} |K(x, \xi)|^2\, dx\, d\xi.$$

The set of the C.V. of K coincides with the set of roots of $d_K^*(\lambda)$, and Carleman, by ingenious analysis, succeeded in proving, not only that $d_K^*(\lambda)$ is of order $\leqq 2$ and of minimal type if it is of order 2, but also that the series $\Sigma [r_\nu(K)]^{-2}$ converges and

$$\sum_{\nu=1}^{\infty} [r_\nu(K)]^{-2} \leqq \iint_{\mathfrak{S}} |K(x, \xi)|^2\, dx\, d\xi;$$

a result which had already been established by I. SCHUR [1] under more restrictive conditions. In the same paper Carleman gave an estimate for the numerator $d_K^*(x, \xi; \lambda)$ in the expression

$$\mathfrak{R}(x, \xi; \lambda) = \frac{d_K^*(x, \xi; \lambda)}{d_K^*(\lambda)}$$

for the resolvant $\mathfrak{R}(x, \xi; \lambda)$ of K, as well as some important formulas for the coefficients $\delta_\nu(K)$ when $K(x, \xi)$ is a composite kernel

$$K(x, \xi) \equiv \int_a^b \cdots \int_a^b K_1(x, s_1) K_2(s_1, s_2) \ldots K_n(s_{n-1}, \xi) \, ds_1 \ldots ds_{n-1} \equiv (K_1 \ldots K_n)(x, \xi),$$

which proved to be of great use in the subsequent development of the theory.

An important result concerning such composite kernels, namely the convergence of the series

$$\sum_{\nu=1}^{\infty} [r_\nu(K)]^{-1},$$

was stated first by LALESCO [2]. There is no explicit statement of the hypotheses used in Lalesco's paper[1] and his proof can not be considered as complete, at least in the most interesting and natural case when all the »components» of $K \subset L_2$. A rigourous proof of Lalesco's result (under certain restrictive hypotheses) is due to GHEORGHIU [4, p. 35]. In the same paper, which was preceded by three preliminary notes in the Comptes Rendus [1, 2, 3], Gheorghiu derives other interesting properties of composite kernels (on the basis of Carleman's formulas mentioned above) and applies them in estimating the exponent of convergence $\varrho(K)$ under various hypotheses about $K(x, \xi)$ (K is continuous and of bounded variation; K has partial derivatives up to a certain order, or is indefinitely differentiable).

The principal results of the present paper were obtained in the beginning of 1928 and communicated to the Mathematics Club of Princeton University (February 14, 1928) and to the American Mathematical Society (April 6, 1928) [HILLE-TAMARKIN, 1, p. 423], without our knowing about the investigations of Gheorghiu. These results were stated briefly in a note in the Proceedings of the National Academy of Sciences [Hille-Tamarkin, 2]. Our methods, except in proving Lalesco's theorem concerning composite kernels, are entirely different, and our results are more inclusive than those of Gheorghiu.

1.2. Carleman [4] established the existence of continuous kernels for which $\varrho(K)$ equals precisely 2, so that, a fortiori, this limit can not be lowered for the class L_2, although this certainly can be done for more or less wide sub-

[1] Cf. also [Hellinger-Toeplitz, 1, p. 1550].

classes of L_2. On the other hand, it can be proved that for the kerne s of class H_α[1]

$$\varrho(K) \leqq 2\left[\frac{1}{2(1-\alpha)}\right] + 2, \qquad (0 \leqq \alpha < 1),$$

while there exist kernels $\subset H_\alpha$ for which $\varrho(K) \geqq \frac{1}{1-\alpha}$ [Carleman, 3; see also Section 2 below]. There exist other classes of kernels which partially overlap L_2 and to which an extension of the classical Fredholm theory applies [Hobson, 1; Hille-Tamarkin, 3]. The question now is whether such an extension is possible for classes of kernels which are more general than L_2. A natural generalization would be the class of kernels L_p for which the integral

$$(L_p) \qquad \iint_{\Xi} |K(x, \xi)|^p \, dx \, d\xi \qquad (1 \leqq p < 2)$$

exists.[2] It has been shown in a recent paper by the authors [Hille-Tamarkin, 3] that the answer to this question is negative, at least as far as the kernels $\subset L_p$ are concerned. Indeed, we have examples of kernels which are symmetric and admit an arbitrary given denumerable set of real numbers as the set of the C.V., and also unsymmetric kernels for which the set of C.V. covers the whole complex plane (the origin being excluded in both cases), where the number p can be taken as near to 2 as we please. It seems natural therefore to restrict the discussion to the class L_2 and to its various sub-classes. It will be assumed in the sequel, without being stated explicitly, that all the kernels in question $\subset L_2$.

1.3. The principal method on which we base our discussion is a systematic use of infinite determinants. By means of an arbitrary orthonormal complete set

$$\varphi_1(x), \varphi_2(x), \ldots, \varphi_\nu(x), \ldots; \qquad \int_a^b \varphi_i(x)\overline{\varphi_j(x)}\, dx = \delta_{ij},$$

equation (1.1) is readily reduced to an equivalent system in infinitely many unknowns

[1] In a forthcoming paper by the present authors. By $[x]$ we designate, as usual, the greatest integer which is $\leqq x$.

[2] It should be observed that $H_\alpha \subset L_p$ whenever $p < \frac{1}{\alpha}$.

$$y_i = \lambda \sum_{j=1}^{\infty} \varkappa_{ij} y_j \qquad (i = 1, 2, \ldots), \tag{1.2}$$

where

$$y_\nu = \int_a^b y(x) \overline{\varphi_\nu(x)}\, dx, \quad \varkappa_{ij} = \iint_{\mathfrak{S}} K(x, \xi) \overline{\varphi_i(x)} \varphi_j(\xi)\, dx\, d\xi.$$

By multiplying the equations of the system (1.2) by suitable factors we obtain an equivalent system which possesses an absolutely convergent determinant. This determinant replaces in the present theory the determinants $d_K(\lambda)$ and $d_K^*(\lambda)$ of the Fredholm-Hilbert theory. The infinite determinant in question can be readily estimated with the aid of the Hadamard determinant theorem. The whole problem is then reduced to the discussion of the mean quadratic error of the approximation of $K(x, \xi)$ by means of the m-th partial summation of its Fourier series expansion,

$$\iint_{\mathfrak{S}} |K(x, \xi) - T_m(x, \xi)|^2\, dx\, d\xi = \sum_{i=m+1}^{\infty} \sum_{j=1}^{\infty} |\varkappa_{ij}|^2$$

where

$$T_m(x, \xi) = \sum_{i=1}^{m} \varphi_i(x) \int_a^b K(t, \xi) \overline{\varphi_i(t)}\, dt.$$

Corresponding to the various analytic properties of the kernel K, various methods are available in the literature for estimating this mean quadratic error. In each case we obtain an estimate for the order of the determinant of the system (1.2) or of the modified system, as an entire function in λ which at the same time yields an estimate for the exponent of convergence $\varrho(K)$.[1]

From an estimate of the type $\varrho(K) \leqq \gamma$ we conclude immediately that

$$r_n(K)\, n^{-\frac{1}{\gamma+\varepsilon}} \to \infty, \tag{i}$$

[1] It should be observed that the idea of using infinite determinants in the theory of the Fredholm integral equation is far from being new. It was used first by H. v. KOCH [1, 2] and subsequently by PLAS [1], MARTY [1] and MOLLERUP [1, 2]. See also an interesting paper by SZÁSZ [1] where the infinite determinants in question are treated independently of the general theory of H. v. Koch. The application of infinite determinants to the problem of distribution of the C.V. of integral equations, which we give here, is new.

since the series $\Sigma r_\nu^{-\gamma-\varepsilon}$ is convergent and the sequence $\{r_\nu^{-\gamma-\varepsilon}\}$ is non-increasing. In many cases we are able to obtain more precise information, namely to prove that

(ii) $$r_n(K)\, n^{-\frac{1}{\gamma}} \to \infty,$$

or even that

(iii) $$\sum_{\nu=1}^{\infty} r_\nu^{-\gamma} \text{ converges.}$$

It goes without saying that results of this sort are of interest only for kernels which possess infinitely many C.V. No attempt is made here to attack the much more difficult problem of the existence of infinitely many, or even of any C.V. for the given kernel K. In this connection we may mention only the lemma 11.1 below which gives a useful sufficient condition for the existence of infinitely many C.V. in the case of kernels of the type $K(x\,\xi)$.

1.4. The order of the material is as follows. In Section 2 we discuss the kernels of the form $K(x-\xi)$ where $K(t)$ is periodic and of period $(b-a)$.[1] From our present point of view the main interest of these kernels lies in the fact that they are very well fitted for construction of examples and »Gegenbeispiele» in order to illustrate various situations of the general theory. In Section 3 there are collected some facts of the theory of infinite determinants and systems of infinitely many linear equations in infinitely many unknowns, partly known and partly new. The results concerning the bilinear forms represented by bordered determinants are largely new. On the basis of these results we give in Section 4 a fairly simple proof of the Schur-Carleman theorem and of other theorems established by Carleman in the above mentioned paper [5], but in an entirely different way. The result of the next Section 5 is of importance for the proof of the Lalesco theorem in its most general form, the only assumption being that the components of the composite kernel in question should $\subset L_2$. This proof is given in Section 6, the method of the proof being essentially that developed by Gheorghiu [4] under more restrictive assumptions. Section 7 deals with kernels which include as a special case those possessing a derivative of fractional positive order (with respect to x). In Section 8 we con-

[1] The literature on these kernels is very considerable [Hellinger-Toeplitz, pp. 1391, 1534; the important papers by Carleman [1, 4] are not mentioned there, however]. We derive some new results which may be of interest for the theory of trigonometric Fourier series.

sider kernels which satisfy integrated Lipschitz conditions [i. e. which $\subset \mathrm{Lip}(\alpha, p)$ according to the terminology of HARDY-LITTLEWOOD, 2]. The discussion of Sections 7—8 is based upon an application of the YOUNG-HAUSDORFF-RIESZ theorem which is applied directly in Section 7, and on the basis of some results of SZÁSZ [3] in Section 8. In Section 9 we utilize some recent results concerning the approximations by means of Cesàro sums of positive orders [JACOB, 1; references to other papers pertaining to the subject are found there]. Section 10 deals with kernels which are analytic (in x) on (a, b). The discussion of this section is based upon S. BERNSTEIN's theory of polynomial approximations [DE LA VALLÉE POUSSIN, 1]. The kernels which are entire functions (in x) are treated in the next Section 11. Here we utilize the approximation furnished by the Taylor series expansion. The last Section 12 contains a summary of all the results; they are collected in a single table to facilitate comparison.

To simplify the formulas we are using the following symbolic notation:

$$A \cdot f(x) \equiv \int_a^b A(x, \xi) f(\xi)\, d\xi; \qquad (AB)(x, \xi) \equiv \int_a^b A(x, s)\, B(s, \xi)\, ds.$$

2. Periodic Kernels.

We take for simplicity $a = 0$, $b = 2\pi$ and consider the integral equations

$$\text{(I)} \qquad y(x) = f(x) + \lambda K \cdot y(x),$$

and

$$(\mathrm{I_h}) \qquad u(x) = \lambda K \cdot u(x),$$

under the assumption that

$$K(x, \xi) \equiv K(x - \xi),$$

where $K(t) \subset L$ (is integrable) and periodic of period 2π. The same properties will be postulated for the given function $f(x)$ and the »solutions» of (I) and $(\mathrm{I_h})$.

We start with the discussion of the homogeneous equation $(\mathrm{I_h})$. Let

$$K(t) \sim \sum_{\nu=-\infty}^{+\infty} k_\nu e^{i\nu t}, \quad 2\pi k_\nu = \int_0^{2\pi} K(x)\, e^{-i\nu x}\, dx;$$

$$u(t) \sim \sum_{\nu=-\infty}^{+\infty} u_\nu e^{i\nu t}, \quad 2\pi u_\nu = \int_0^{2\pi} u(x) e^{-i\nu x} dx$$

be the formal complex trigonometric Fourier series expansions of $K(t)$ and of $u(x)$. On multiplying ($\mathrm{I_h}$) by e^{-inx}, integrating and using the periodicity of the functions concerned, we get at once

$$u_n = 2\pi\lambda k_n u_n,$$

the necessary interchange of order of integration being readily justified by Fubini's theorem. Hence the C.V. of (I) are

$$\lambda_n = (2\pi k_n)^{-1} \qquad (n=0, \pm 1, \pm 2, \ldots; k_n \neq 0), \tag{2.1}$$

the corresponding fundamental functions being

$$u(x) = e^{inx}.$$

To prove that the limit 2 for $\varrho(K)$ can not be lowered for the class of continuous kernels (Section 1.2) it suffices therefore to exhibit a continuous periodic function $K(t)$ for which the series $\Sigma |k_n|^{2-\varepsilon}$ diverges no matter how small is $\varepsilon > 0$. This was done first by Carleman [4]. Hille [1] indicated a general method for constructing examples of this nature, which is based upon an entirely different principle and is simpler than that of Carleman.

It is easy to construct examples of kernels K for which $\varrho(K)$ assumes any value $0 \leqq \varrho \leqq 2$ if $K \subset L_2$, and any value $2 < \varrho \leqq \infty$ if this restriction is removed. Moreover these examples can be constructed so as to exhibit all three peculiarities i.—iii. mentioned in 1.3. Indeed by a result due to Young [2, pp. 443—444]

$$F_{a,b}(t) \equiv \sum_{n=2}^{\infty} \frac{e^{nit}}{n^a (\log n)^b} = O\left\{t^{a-1}\left(\log\frac{1}{t}\right)^{-b}\right\}, \quad 0 < a < 1.$$

Hence $F_{a,b}(t) \subset L_p$ for any $p < \frac{1}{1-a}$, and even for $p = \frac{1}{1-a}$ provided $b > 1-a$. On the other hand it is obvious that if

$$K(t) = F_{a,b}(t),$$

then $\varrho(K) = \gamma = \frac{1}{a}$, and we have the case i., ii., or iii. according as

$$b \leqq 0, \quad 0 < b \leqq a, \text{ or } b > a.$$

We get the same results if we allow $a > 1$, the kernel $K(t)$ being continuous in this case. The case $\varrho(K) = \infty$ is represented by the kernel

$$K(t) = \sum_{n=2}^{\infty} \frac{\cos nt}{\log n}$$

[Young, 2, pp. 44—45, 48]. The reader will find no difficulty in illustrating the case $\varrho(K) = 0$.

Let us turn now to the non-homogeneous equation (I). If λ is distinct from the C.V. (2. 1), it is known [Hille-Tamarkin, 3, pp. 513, 524] that there exists a uniquely determined solution $y(x) \subset L$ of (I), provided $f(x) \subset L$. On setting

$$y(x) \sim \sum_{\nu=-\infty}^{+\infty} y_\nu e^{i\nu x}, \qquad f(x) \sim \sum_{\nu=-\infty}^{+\infty} f_\nu e^{i\nu x},$$

we get by the same argument as before

$$y_n = f_n + 2\pi\lambda k_n y_n \text{ or } y_n = \frac{f_n}{1 - 2\pi\lambda k_n}.$$

Here $K(t)$ and $f(x)$ are arbitrary functions $\subset L$, while the Fourier series of $y(x)$ is obtained from the Fourier series of $f(x)$ by means of the factor sequence $(1 - 2\pi\lambda k_n)^{-1}$. Using the terminology of M. Riesz[1] we can say that the sequences

(2. 2) $$\{(1 - k_n)^{-1}\}, \quad \{k_n(1 - k_n)^{-1}\}, \quad k_n \neq 1,$$

are of type (1, 1). Since a necessary and sufficient condition that a sequence $\{\mu_n\}$ be a factor sequence of type (1, 1) is that $\left\{\frac{\mu_n}{n}\right\}$, $n \neq 0$, be a sequence of Fourier coefficients of a function of bounded variation, we have

[1] [1, p. 487—488]. Other references concerning various results of the theory of factor sequences are found in this paper.

Theorem 2.1. *If $K(x)$ is any function $\subset L$, then the sequences (2.2) are factor sequences that transform the Fourier series of an arbitrary function $f(x) \subset L$ into the Fourier series of a function $\subset L$ while the sequences*

$$\left\{\frac{1}{n(1-k_n)}\right\}, \quad \left\{\frac{k_n}{n(1-k_n)}\right\}, \quad k_n \neq 1, \quad n \neq 0,$$

are sequences of Fourier coefficients of functions of bounded variation.

We are not aware of any direct proof of this curious result.

3. Infinite Determinants and Systems of Linear Equations.[1]

In this Section we shall deal with vectors $\mathfrak{a} \equiv (a_1, a_2, \ldots)$ (denoted by small German letters) and matrices $\mathfrak{A} \equiv (a_{ij})$ (denoted by capital German letters) of a complex Hilbert space $\mathfrak{L}_2$, that is such that the series

$$\|\mathfrak{a}\|^2 = \sum_{i=1}^{\infty} |a_i|^2, \quad \|\mathfrak{A}\|^2 = \sum_{i,j=1}^{\infty} |a_{ij}|^2$$

converge. The quantities $\|\mathfrak{a}\|$, $\|\mathfrak{A}\|$ will be designated as the lengths of the vector $\mathfrak{a}$ and of the matrix $\mathfrak{A}$ respectively. The notation $\mathfrak{A}'$ will be used to designate the transposed matrix $\mathfrak{A}' \equiv (a_{ji})$. The vectors that occupy the i-th rows of the matrices $\mathfrak{A}$, $\mathfrak{A}'$ will be denoted by $\mathfrak{a}_i$, $\mathfrak{a}'_i$ respectively. The usual agreements concerning the elementary algebraic operations with matrices and vectors will be assumed without further explanation.

A matrix $\mathfrak{A}$ is said to be of class $\mathfrak{L}'_2$ if the simple and the double series

$$\sigma(\mathfrak{A}) = \sum_{i=1}^{\infty} |a_{ii}|, \quad \|\mathfrak{A}\|^2 = \sum_{i,j=1}^{\infty} |a_{ij}|^2$$

converge. It is obvious that $\mathfrak{L}'_2 \subset \mathfrak{L}_2$.

Lemma 3.1. *If the matrices $\mathfrak{A}$ and $\mathfrak{B} \subset \mathfrak{L}_2$, then their product $\mathfrak{A}\mathfrak{B} \subset \mathfrak{L}'_2$, and*

(3.1) $$\sigma(\mathfrak{A}\mathfrak{B}) \leqq \|\mathfrak{A}\| \cdot \|\mathfrak{B}\|; \quad \|\mathfrak{A}\mathfrak{B}\| \leqq \|\mathfrak{A}\| \cdot \|\mathfrak{B}\|.$$

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[1] We refer to F. Riesz [1], H. v. Koch [1, 2], J. D. Tamarkin [1] as to general properties of absolutely convergent infinite determinants and their applications to systems of linear equations.

Proof. Since the elements of the matrix $\mathfrak{A}\mathfrak{B}$ are given by

$$\sum_{s=1}^{\infty} a_{is}\, b_{sj},$$

formulas (3. 1) are obtained by an immediate application of Cauchy's inequality.

If the matrix $\mathfrak{A} \subset \mathfrak{L}_2$, the infinite determinant

$$A(a) \equiv A = \det(\delta_{ij} - a_{ij}) \qquad (i, j = 1, 2, \ldots)$$

will be designated as the determinant related to the matrix $\mathfrak{A}$. It is well known that the determinant related to a matrix $\mathfrak{A} \subset \mathfrak{L}'_2$ or briefly, a determinant $A \subset \mathfrak{L}'_2$, is absolutely convergent together with all its minors of all orders, and remains so if any number of rows or columns are replaced by vectors $\subset \mathfrak{L}_2$. The same will be true of the determinant

$$A(\mathfrak{x}, \mathfrak{y}) \equiv \begin{vmatrix} 0 & x_1 & x_2 & \ldots \\ y_1 & 1 - a_{11} & -a_{12} & \ldots \\ y_2 & -a_{21} & 1 - a_{22} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

which will be designated as the bordered determinant related to the matrix, provided the vectors $\mathfrak{x}$ and $\mathfrak{y} \subset \mathfrak{L}_2$.

If we know only that $\mathfrak{A} \subset \mathfrak{L}_2$, the related determinant A and the bordered determinant $A(\mathfrak{x}, \mathfrak{y})$ may not exist. Hence we introduce modified (related) determinants and modified bordered (related) determinants of various types. The m-th modified determinant related to the matrix $\mathfrak{A} \subset \mathfrak{L}_2$ is defined as the determinant which is obtained formally from A by multiplying the i-th row $(i = m+1, \ m+2, \ldots)$ by $e^{a_{ii}}$. If we designate by A_m^* the m-th modified determinant of $\mathfrak{A}$, then the matrix $\mathfrak{A}_m^*$ which is related to A_m^* coincides in its first m rows with the matrix $\mathfrak{A}$, the remaining rows $(m+1, \ m+2, \ldots)$ being those of the matrix $\mathfrak{A}^*$ where

$$a_{ij}^* \equiv e^{a_{ii}} a_{ij} \, (i \neq j); \quad a_{ii}^* = 1 - e^{a_{ii}}(1 - a_{ii}).$$

The 0-th modified determinant $A_0^* \equiv A^*$ will be designated simply as the modified determinant related to $\mathfrak{A}$. The m-th modified bordered determinant $A_m^*(\mathfrak{x}, \mathfrak{y})$ is obtained from the related bordered determinant $A(\mathfrak{x}, \mathfrak{y})$ by multiplying the i-th row ($i = m+1,\ m+2, \ldots$) by $e^{a_{ii}}$. It is also obtained from A_m^* by bordering it by means of vectors $\mathfrak{x}$ and $\mathfrak{y}^{*(m)}$ where $\mathfrak{y}^{*(m)}$ coincides with $\mathfrak{y}$ in its first m components, the remaining components being those of the vector

$$\mathfrak{y}^* = (e^{a_{11}} y_1,\ e^{a_{22}} y_2, \ldots).$$

We shall establish several lemmas concerning these determinants; since all the determinants concerned will be absolutely convergent together with all their minors of all orders, the »usual» rules of the theory of finite determinants including the theorem of multiplication and Laplace's expansion theorem, will hold in the present case.

Lemma 3. 2. *If the determinants A and $B \subset \mathfrak{L}'_2$, then their product $C = AB$ also $\subset \mathfrak{L}'_2$. The matrix $\mathfrak{C}$ related to C is expressed in terms of the matrices $\mathfrak{A}$, $\mathfrak{B}$ related to A, B by*

$$\mathfrak{C} = \mathfrak{A} + \mathfrak{B} - \mathfrak{A}\mathfrak{B}.$$

Proof. We have only to apply the theorem of multiplication of determinants and lemma 3. 1.

Lemma 3. 3. *If $\mathfrak{A} \subset \mathfrak{L}'_2$, then*

$$(3.\ 2) \qquad |A|^2 \leqq \left\{ \begin{matrix} \prod_{i=1}^{\infty} \{1 - 2\,\Re(a_{ii}) + \|\mathfrak{a}_i\|^2\} \\ \prod_{j=1}^{\infty} \{1 - 2\,\Re(a_{jj}) + \|\mathfrak{a}'_j\|^2\} \end{matrix} \right\} \leqq \exp\left[-2\,\Re\left(\sum_{i=1}^{\infty} a_{ii}\right) + \|\mathfrak{A}\|^2\right].$$

If A_{ij} denotes the cofactor of the element in the i-th row and the j-th column of A, then

$$|A_{ij}|^2 \leqq \left\{ \begin{matrix} \|\mathfrak{a}_j\|^2 \ \exp\left[-2\,\Re\left(\sum_{i=1}^{\infty} a_{ii}\right) + \|\mathfrak{A}\|^2\right], \\ \|\mathfrak{a}'_i\|^2 \ \exp\left[-2\,\Re\left(\sum_{i=1}^{\infty} a_{ii}\right) + \|\mathfrak{A}\|^2\right], \end{matrix} \right. \qquad (i \neq j)$$

while for A_{ii} we have the same estimate (3. 2) *as for A.*

All these estimates hold for the determinants thdt are obtained from A or A_{ij} by replacing any number (finite or infinite) of the elements by zeros. In particular the same estimates hold for the segments $A^{(n)}$, $A_{ij}^{(n)}$ of the determinants A, A_{ij}, where

$$A^{(n)} \equiv \det(\delta_{ij} - a_{ij}),\ i, j = 1, 2, \ldots, n,$$

and $A_{ij}^{(n)}$ *is the cofactor of the element* $\delta_{ij} - a_{ij}$ *in* $A^{(n)}$.

Lemma 3.4. *If* $\mathfrak{A} \subset \mathfrak{L}_2$, *then the m-th modified determinant* $A_m^* \subset \mathfrak{L}'_2$, *and*

$$(3.3)\qquad |A_m^*(a)| \leqq \Pi_m(a) \equiv \Pi_m,$$

where

$$(3.4)\qquad \begin{aligned} \Pi_m(a) &\equiv \prod_{i=1}^{m} [1 - 2\,\Re(a_{ii}) + \|\mathfrak{a}_i\|^2]^{\frac{1}{2}} \prod_{i=m+1}^{\infty} e^{\Re(a_{ii})} [1 - 2\,\Re(a_{ii}) + \|\mathfrak{a}_i\|^2]^{\frac{1}{2}} \\ &\leqq \prod_{i=1}^{m} [1 - 2\,\Re(a_{ii}) + \|\mathfrak{a}_i\|^2]^{\frac{1}{2}} \exp\left(\frac{1}{2} \sum_{i=m+1}^{\infty} \|\mathfrak{a}_i\|^2\right). \end{aligned}$$

In particular for $m = 0$ *we have*

$$(3.5)\qquad |A^*(a)| \leqq \exp\left(\frac{1}{2}\|\mathfrak{a}\|^2\right).$$

The same estimates hold when any number of the elements a_{ij} *are replaced by zeros.*

Proof. We shall give here only the proof of lemma 3.4; lemma 3.3 will follow from the identity[1]

$$(3.6)\qquad A(a) = A^*(a)\ \exp\left(-\sum_{i=1}^{\infty} a_{ii}\right).$$

Since $\mathfrak{A} \subset \mathfrak{L}_2$ it is seen that $a_{ij} \to 0$. On the other hand

$$a_{ii}^* = 1 - e^{a_{ii}}(1 - a_{ii}) = O(a_{ii}^2),$$

$$a_{ij}^* = e^{a_{ii}} a_{ij} = O(a_{ij}),\ i \neq j,$$

which implies the convergence of the series $\Sigma|a_{ii}^*|$, $\Sigma|a_{ij}^*|^2$. The proof of (3.3) is now readily obtained by means of Hadamard's determinant theorem [Hellinger-Toeplitz, 1, pp. 1356—7] on the basis of the simple inequality

$$1 + x < e^x,\ x \geqq -1.$$

---

[1] Lemma 3.3 is known, in slightly less favorable form, with $|a_{ii}|$, $|a_{jj}|$ instead of $\Re(a_{ii})$, $\Re(a_{jj})$. Cfr. H. v. Koch [1, p. 259], Szász [1, p. 277], Tamarkin [1, p. 131]. v. Koch's formula (13) contains an obvious misprint since the term $2\,\Re(a_{jj})$ is missing there.

We have then

$$|A_m^*(a)|^2 \leqq \prod_{i=1}^{m} \sum_{j=1}^{\infty} |\delta_{ij} - a_{ij}|^2 \cdot \prod_{i=m+1}^{\infty} \sum_{j=1}^{\infty} |\delta_{ij} - a_{ij}^*|^2,$$

while

$$\sum_{j=1}^{\infty} |\delta_{ij} - a_{ij}|^2 = 1 - 2\,\Re(a_{ii}) + \|\mathfrak{a}_i\|^2,$$

$$\sum_{j=1}^{\infty} |\delta_{ij} - a_{ij}^*|^2 = e^{2\Re(a_{ii})}\,[1 - 2\,\Re(a_{ii}) + \|\mathfrak{a}_i\|^2] < \exp(\|\mathfrak{a}_i\|^2).$$

Lemma 3. 5. *If $\mathfrak{A} \subset \mathfrak{L}_2$, then the m-th modified bordered determinant $A_m^*(\mathfrak{x}, \mathfrak{y})$ is a bounded bilinear form as a function of the vectors $\mathfrak{x}$, $\mathfrak{y}$ in the Hilbert space $\mathfrak{L}_2$. The bound of $A_m^*(\mathfrak{x}, \mathfrak{y})$ does not exceed $\sqrt{e}\,\Pi_m(a)$, so that*

$$(3.7)\qquad |A_m^*(\mathfrak{x}, \mathfrak{y})| \leqq \|\mathfrak{x}\|\,\|\mathfrak{y}\|\sqrt{e}\prod_{i=1}^{m}[1 - 2\,\Re(a_{ii}) + \|\mathfrak{a}_i\|^2]^{\frac{1}{2}} \exp\left(\frac{1}{2}\sum_{i=m+1}^{\infty}\|\mathfrak{a}_i\|^2\right).$$

If $A_{m;\,ij}^*$ is the cofactor of the element in the i-th row and the j-th column in A_m^*, we have an absolutely convergent expansion

$$(3.8)\qquad A_m^*(\mathfrak{x}, \mathfrak{y}) = -\sum_{i,j=1}^{\infty} A_{m;\,ij}^*\, y_i^{*\,(m)}\, x_j,$$

where

$$\mathfrak{y}^{*\,(m)} \equiv (y_1^{*\,(m)}, y_2^{*\,(m)}, \ldots).$$

In particular for $m = 0$

$$(3.9)\qquad |A^*(\mathfrak{x}, \mathfrak{y})| \leqq \|\mathfrak{x}\|\,\|\mathfrak{y}\|\sqrt{e}\,\exp\left(\frac{1}{2}\|\mathfrak{A}\|^2\right),$$

$$(3.10)\qquad A^*(\mathfrak{x}, \mathfrak{y}) = -\sum_{i,j=1}^{\infty} A_{ij}^*\, y_i^*\, x_j, \quad y_i^* = e^{a_{ii}}\, y_i.$$

If $\mathfrak{A} \subset \mathfrak{L}_2'$, then the bordered determinant $A(\mathfrak{x}, \mathfrak{y})$ is also a bounded bilinear form in $\mathfrak{L}_2'$, and

$$(3.11)\qquad |A(\mathfrak{x}, \mathfrak{y})| \leqq \|\mathfrak{x}\|\,\|\mathfrak{y}\|\sqrt{e}\,\exp\left(-\Re\left(\sum_{i=1}^{\infty} a_{ii}\right) + \frac{1}{2}\|\mathfrak{A}\|^2\right),$$

(3.12) $$A(\mathfrak{x}, \mathfrak{y}) = -\sum_{i,j=1}^{\infty} A_{ij}\, y_i\, x_j.$$

All these estimates hold if any number of the elements a_{ij} are replaced by zeros.[1]

Proof. By Hadamard's determinant theorem we have

$$|A_m^*(\mathfrak{x}, \mathfrak{y})|^2 \leqq \sum_{i=1}^{\infty} |x_i|^2 \prod_{i=1}^{m} [1 + |y_i|^2 - 2\,\Re(a_{ii}) + \|\mathfrak{a}_i\|^2] \prod_{i=m+1}^{\infty} e^{2\Re(a_{ii})}[\ldots]$$

$$\leqq \sum_{i=1}^{\infty} |x_i|^2 \prod_{i=1}^{\infty} (1 + |y_i|^2)\, \Pi_m^2(a) \leqq \|\mathfrak{x}\|^2 \ \exp\ (\|\mathfrak{y}\|^2)\, \Pi_m^2(a).$$

Hence, on the unit sphere $\|\mathfrak{x}\| = \|\mathfrak{y}\| = 1$,

$$|A_m^*(\mathfrak{x}, \mathfrak{y})| \leqq \sqrt{e}\, \Pi_m(a),$$

which shows that $A_m^*(\mathfrak{x}, \mathfrak{y})$ is a bounded bilinear form whose bound does not exceed $\sqrt{e}\, \Pi_m(a)$. The formulas (3.7) and (3.9) follow at once. The expansions (3.8) and (3.10), and their absolute convergence are known from the general theory of absolutely convergent determinants. Formulas (3.11) and (3.12) are derived in a similar fashion.

Lemma 3.6. *If the matrix $\mathfrak{A}$ and the vector $\mathfrak{c} \subset \mathfrak{L}_2$ then the system*

(S) $$\mathfrak{x} - \mathfrak{A}\,\mathfrak{x} = \mathfrak{c}$$

is equivalent to the system

(S_m^*) $$\mathfrak{x} - \mathfrak{A}_m^*\,\mathfrak{x} = \mathfrak{c}^{*(m)}, \quad c_i^{*(m)} = \begin{cases} c_i, & i = 1, 2, \ldots, m; \\ e^{a_{ii}} c_i, & i = m+1, m+2, \ldots. \end{cases}$$

A necessary and sufficient condition that (S) *should have a unique solution $\mathfrak{x} \subset \mathfrak{L}_2$ is that the determinant $A_m^* \neq 0$. If this condition is satisfied, the solution is given by the usual formulas, and, in addition,*

(3.13) $$\|\mathfrak{x}\| \leqq \|\mathfrak{c}\| \sqrt{e}\, |A^*|^{-1} \ \exp\ \left(\frac{1}{2} \|\mathfrak{A}\|^2\right).$$

---

[1] We refer to Hellinger-Toeplitz [1, §§ 18, 43] concerning the terminology and facts of Hilbert's theory of bounded quadratic and bilinear forms. The fact that $A(\mathfrak{x}, \mathfrak{y})$ is bounded is a special case of a result due to Bóbr [1]. However, Bóbr's method is more complicated and gives an estimate for the bound of $A(\mathfrak{x}, \mathfrak{y})$ which is not suitable for our purposes.

The homogeneous system

$$(\mathrm{S_h}) \qquad \mathfrak{x} - \mathfrak{A}\,\mathfrak{x} = 0$$

has non-trivial solutions $\mathfrak{x} \subset \mathfrak{L}_2$ *when and only when* $A_m^* = 0$. *The classical results concerning the general form of the solutions, the number of linearly independent solutions etc., can be extended to the present case.*

If $\mathfrak{A} \subset \mathfrak{L}'_2$, *all these results hold with* A^* *replaced by* A. *The inequality* (3. 13) *is then replaced by*

$$(3.\,14) \qquad \|\mathfrak{x}\| \leqq \|\mathfrak{c}\| \sqrt{e}\,|A|^{-1} \exp\left(-\Re\left(\sum_{i=1}^{\infty} a_{ii}\right) + \frac{1}{2}\|\mathfrak{A}\|^2\right).$$

Proof. The fact that the system (S) is equivalent to (S_m^*), and all the statements of lemma 3. 6, are known, except for the formulas (3. 13) and (3. 14).[1] To prove (3. 13) we observe that the solution $\mathfrak{x} = (x_1, x_2, \ldots)$ of (S^*) is given by

$$x_i = \frac{1}{A^*} \sum_{j=1}^{\infty} A_{ji}^*\, c_j^*,$$

whence, by (3. 9),

$$\|\mathfrak{x}\|^2 = (A^*)^{-1} \sum_{i,j=1}^{\infty} A_{ji}^*\, c_j^*\, \bar{x}_i = -(A^*)^{-1} A^*(\mathfrak{x}, \mathfrak{c}) \leqq \|\mathfrak{x}\|\,\|\mathfrak{c}\| \sqrt{e}\,|A^*|^{-1} \exp\left(\frac{1}{2}\|\mathfrak{A}\|^2\right).$$

Formula (3. 14) is proved in an analogous fashion.

A case that is frequently met in the applications of the preceding theory is that in which the elements a_{ij} are functions of a parameter λ. Assuming that these functions are analytic in λ we can state

Lemma 3. 7. *In the system*

$$(\mathrm{S}_\lambda) \qquad \mathfrak{x} - \mathfrak{A}(\lambda)\mathfrak{x} = \mathfrak{c}(\lambda)$$

let the coefficients $a_{ij}(\lambda)$ *and the right-hand members* $c_i(\lambda)$ *be analytic in an open domain* Δ *of the complex* λ*-plane. Assume also that* $\|\mathfrak{A}(\lambda)\|$ *and* $\|\mathfrak{c}(\lambda)\|$ *are bounded in every closed sub-region* Δ_0 *of* Δ. *Then* $A_m^*(\lambda)$ *is analytic in* Δ *and, in case* $A_m^*(\lambda)$ *does not vanish identically, the solution of* (S_λ) *is meromorphic in* Δ.[2]

---

[1] F. Riesz [1, p. 39], Tamarkin [1]. The system (S_m^*) is obtained from (S) by multiplying the i-th equation by $e^{a_{ii}}$ $(i = m + 1, \ldots)$.

[2] Tamarkin [1, pp. 135—136]; under less general asumptions v. Koch [2, pp. 268—270].

Proof. It is known [F. Riesz, 1, p. 34] that

$$A_m^*(\lambda) = \lim_{n\to\infty} A_m^{*(n)}(\lambda)$$

where $A_m^{*(n)}(\lambda)$ is the n-th segment of $A_m^*(\lambda)$, that is the determinant which is obtained from $A_m^*(\lambda)$ by replacing by zeros all the elements a_{ij} with i or $j > n$. By lemma 3.4 then $A_m^{*(n)}(\lambda)$ is uniformly bounded in Δ_0 and the analyticity of $A_m^*(\lambda)$ in Δ follows from Montel's theorem. In the same fashion we can prove the analyticity of the numerators in the expressions which give the solutions of (S_λ).

Lemma 3.8. *If the matrix* $\mathfrak{A}(\lambda)$ *of the system* (S_λ) *is a linear function of the parameter* λ,

$$\mathfrak{A}(\lambda) = \mathfrak{A}^{(0)} + \lambda\mathfrak{A}^{(1)},$$

where $\mathfrak{A}^{(0)}$ *and* $\mathfrak{A}^{(1)} \subset \mathfrak{L}_2$, *then all the modified determinants* $A_m^*(\lambda)$ *are entire functions of* λ *of order not exceeding 2, and of minimal type if the order equals 2.*

Proof. In view of the obvious identity

$$A_m^*(a) = \exp\left(-\sum_{i=1}^m a_{ii}\right) A^*(a),$$

it suffices to give the proof for the case $m = 0$ only. Let

$$M(r) = \max_{|\lambda|=r} |A^*(\lambda)|.$$

Then, by lemma 3.4,

$$M(r) \leqq C \exp\left(r^2 ||\mathfrak{A}^{(1)}||^2\right), \quad C = \exp\left(||\mathfrak{A}^{(0)}||^2\right).$$

This shows that the order of $A^*(\lambda)$ is $\leqq 2$. On the other hand we have from (3.3) and (3.4)

$$|A^*(\lambda)| \leqq \Pi_0(a) = \prod_{i=1}^{\infty} e^{\mathfrak{R}(a_{ii})}\left[1 - 2\,\mathfrak{R}(a_{ii}) + ||\mathfrak{A}_i||^2\right]^{\frac{1}{2}} = \prod_{i=1}^{N} \cdot \prod_{i=N+1}^{\infty} = P_1 \cdot P_2.$$

Here we have

$$P_2 \leqq \exp\left(\frac{1}{2}\sum_{i=N+1}^{\infty} ||\mathfrak{a}_i||^2\right) \leqq \exp\left(\sum_{i=N+1}^{\infty} ||\mathfrak{a}_i^{(0)}||^2 + r^2 \sum_{i=N+1}^{\infty} ||\mathfrak{a}_i^{(1)}||^2\right),$$

whence, an arbitrarily small ε being given, we can take N so large that

$$P_2 \leqq C \exp\left(\frac{\varepsilon}{2} r^2\right).$$

The number N being fixed, we can determine now a positive constant C_N so large that

$$P_1 \leqq C_N \exp\left(\frac{\varepsilon}{2} r^2\right).$$

On combining these results we see that

$$|A^*(\lambda)| \leqq P_1 \cdot P_2 = O\,[\exp(\varepsilon r^2)],$$

hence $A^*(\lambda)$ must be of minimal type if it is of order 2.

4. Integral Equations of Class L_2.

The method of infinite determinants can be applied to advantage in solving the integral equation

$$\text{(I)} \qquad y(x) = f(x) + \lambda K \cdot y(x),$$

where $K(x, \xi) \subset L_2$, that is the integral

$$(L_2) \qquad \int_a^b \int_a^b |K(x, \xi)|^2 \, dx \, d\xi \equiv \|K\|^2$$

exists. Let

$$\{\varphi_\nu(x)\}, \ \nu = 1, 2, \ldots; \quad \int_a^b \varphi_i(x) \overline{\varphi_j(x)} \, dx = \delta_{ij},$$

be an arbitrary orthonormal and complete set of functions for the interval (a, b). We shall use the notation

$$f_\nu \equiv \int_a^b f(x) \overline{\varphi_\nu(x)} \, dx, \quad f'_\nu \equiv \int_a^b f(x) \varphi_\nu(x) \, dx$$

to designate the Fourier coefficients of an arbitrary function $f(x)$ with respect to the sets $\{\varphi_\nu(x)\}$ and $\{\overline{\varphi_\nu(x)}\}$. Then for an arbitrary pair of functions $f(x)$, $g(x) \subset L_2$ we have the Parseval identity

$$\int_a^b f(x)\, g(x)\, dx = \sum_{\nu=1}^{\infty} f_\nu g'_\nu.$$

We also set

$$K'(x, \xi) \equiv K(\xi, x); \tag{4. 1}$$

$$k'_i(x) \equiv K \cdot \varphi_i(x), \quad k_i(\xi) \equiv K' \cdot \overline{\varphi_i(\xi)}; \tag{4. 2}$$

$$\varkappa_{ij}(K) \equiv \varkappa_{ij} \equiv \int_a^b \int_a^b K(x, \xi)\, \overline{\varphi_i(x)}\, \varphi_j(\xi)\, dx\, d\xi = (k_i)'_j = (k'_j)_i; \tag{4. 3}$$

$$k_i(\xi) \sim \sum_{j=1}^{\infty} \varkappa_{ij} \overline{\varphi_j(\xi)}; \tag{4. 4}$$

$$k'_i(x) \sim \sum_{j=1}^{\infty} \varkappa_{ji}\, \varphi_j(x); \tag{4. 5}$$

$$K(x, \xi) \sim \sum_{i=1}^{\infty} k_i(\xi) \varphi_i(x) \sim \sum_{i=1}^{\infty} k'_i(x)\, \overline{\varphi_i(\xi)} \sim \sum_{i,\, j=1}^{\infty} \varkappa_{ij} \varphi_i(x) \overline{\varphi_j(\xi)}. \tag{4. 6}$$

Then a repeated application of Parseval's identity reduces (I) to the system of linear equations

$$y_i = f_i + \lambda \sum_{j=1}^{\infty} \varkappa_{ij}\, y_j \qquad (i = 1, 2, \ldots) \tag{4. 7}$$

whose matrix $(\varkappa_{ij}) \subset \mathfrak{L}_2$ since [Tamarkin, 1, p. 138]

$$\sum_{i,j=1}^{\infty} |\varkappa_{ij}|^2 = \sum_i \sum_j = \sum_{i=1}^{\infty} \int_a^b |k_i(\xi)|^2 d\xi = \int_a^b d\xi \left[\sum_{i=1}^{\infty} |k_i(\xi)|^2 \right] = \tag{4. 8}$$

$$= \int_a^b d\xi \int_a^b |K(x, \xi)|^2\, dx = ||K||^2.$$

The results of the previous Sections 2 and 3 are immediately applicable to the system (4. 7) which, under the additional assumption $f(x) \subset L_2$, is equivalent to (I).

If λ is not a C.V. of (I), the resolvant $\mathfrak{K}(x, \xi; \lambda)$ of the kernel $K(x, \xi)$ is defined by

$$K(x, \xi) + \mathfrak{K}(x, \xi; \lambda) = \lambda (K \mathfrak{K})(x, \xi; \lambda) = \lambda (\mathfrak{K} K)(x, \xi; \lambda).$$

It is obtained from (I) by setting $-f(x) = K(x, \xi)$. After a simple computation [Tamarkin, 1, p. 140] we get

$$\mathfrak{K}(x, \xi; \lambda) = \frac{A_m^*(x, \xi; \lambda; K)}{A_m^*(\lambda; K)}, \tag{4.9}$$

where

$$A_m^*(\lambda; K) \equiv A_m^*(\lambda) \equiv \det (\delta_{ij} - \lambda \varkappa_{ij}^{*(m)}), \quad i, j = 1, 2, \ldots; \tag{4.10}$$

$$A_m^*(x, \xi; \lambda; K) \equiv A_m^*(x, \xi; \lambda) \equiv \begin{vmatrix} -K(x, \xi) & k'_1(x) & k'_2(x) \ldots \\ \lambda k_1^{*(m)}(\xi) & 1 - \lambda \varkappa_{11}^{*(m)} & -\lambda \varkappa_{12}^{*(m)} \ldots \\ \lambda k_2^{*(m)}(\xi) & -\lambda \varkappa_{21}^{*(m)} & 1 - \lambda \varkappa_{22}^{*(m)} \ldots \\ \ldots & \ldots & \ldots \end{vmatrix}, \tag{4.11}$$

and

$$\varkappa_{ij}^{*(m)} = \begin{cases} \varkappa_{ij}(K) & \text{if } i = 1, 2, \ldots, m, \\ \varkappa_{ij}^*(K) & \text{if } i = m+1, \ldots; \end{cases} \qquad k_i^{*(m)}(\xi) = \begin{cases} k_i(\xi) & \text{if } i = 1, 2, \ldots, m, \\ k_i^*(\xi) & \text{if } i = m+1, \ldots; \end{cases} \tag{4.12}$$

with

$$\begin{gathered} \varkappa_{ij}^*(K) \equiv \varkappa_{ij}^* = e^{\lambda \varkappa_{ii}} \varkappa_{ij}, \quad i \neq j; \quad 1 - \lambda \varkappa_{ii}^* = e^{\lambda \varkappa_{ii}}(1 - \lambda \varkappa_{ii}); \\ k_i^*(\xi) = e^{\lambda \varkappa_{ii}} k_i(\xi). \end{gathered} \tag{4.13}$$

We shall omit the subscript or superscript m in our formulas in case $m = 0$. Thus

$$\varkappa_{ij}^{*(0)} \equiv \varkappa_{ij}^* \equiv \varkappa_{ij}^*(K); \; k_i^{*(0)}(\xi) \equiv k_i^*(\xi);$$

$$A_0^*(\lambda) \equiv A_0^*(\lambda; K) \equiv A^*(\lambda; K) \equiv A^*(\lambda);$$

$$A_0^*(x, \xi; \lambda) \equiv A_0^*(x, \xi; \lambda; K) \equiv A^*(x, \xi; \lambda; K) \equiv A^*(x, \xi; \lambda).$$

It should be noted that

$$A_m^*(\lambda) = \exp\left(-\lambda \sum_{i=1}^{m} \varkappa_{ii}\right) A^*(\lambda), \tag{4.14}$$

$$A_m^*(x, \xi; \lambda) = \exp\left(-\lambda \sum_{i=1}^{m} \varkappa_{ii}\right) A^*(x, \xi; \lambda). \tag{4.15}$$

We are now prepared to prove the following

Theorem 4.1. *If the kernel $K(x, \xi) \subset L_2$, then with the notation above:*

i. *The resolvant $\mathfrak{R}(x, \xi; \lambda)$ of $K(x, \xi)$ is meromorphic in λ for almost all (x, ξ) on $a \leqq x, \xi \leqq b$, and is given by*

$$\mathfrak{R}(x, \xi; \lambda) = \frac{A_m^*(x, \xi; \lambda; K)}{A_m^*(\lambda; K)} = \frac{A^*(x, \xi; \lambda; K)}{A^*(\lambda; K)}, \tag{4.16}$$

where the denominators as well as the numerators for almost all (x, ξ) are entire functions in λ.

ii. *The totality $\{\lambda_\nu(K)\}$ of the C.V. of (I) coincides with the totality of the zeros of $A^*(\lambda; K)$, and*

$$\sum_{\nu=1}^{\infty} [r_\nu(K)]^{-2} \leqq \|K\|^2, \tag{4.17}$$

each C.V. being repeated according to its multiplicity as a root of $A^*(\lambda; K)$.

iii. *The entire function $A^*(\lambda; K)$ is of genus 1 and is represented by the infinite product*

$$A^*(\lambda; K) = \prod_{\nu=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_\nu}\right) \exp\left(\frac{\lambda}{\lambda_\nu}\right). \tag{4.18}$$

The order of $A^*(\lambda; K)$ does not exceed 2, and if it equals 2, $A^*(\lambda; K)$ is of minimal type.

iv. *We have the estimates*

$$|A_m^*(\lambda; K)| \leqq \Pi_m(r; K) \equiv \Pi_m(r), \tag{4.19}$$

$$|A_m^*(x, \xi; \lambda; K)| \leqq \Pi_m(r; K)\{|K(x, \xi)| + r\sqrt{e}\, \varkappa^{(1)}(x)\, \varkappa^{(2)}(\xi)\}, \tag{4.20}$$

where $r = |\lambda|$ and

$$\Pi_m(r; K) \equiv \prod_{i=1}^{m} \left[1 - 2\,\Re(\lambda \varkappa_{ii}) + r^2 \sum_{j=1}^{\infty} |\varkappa_{ij}|^2\right]^{\frac{1}{2}} \prod_{i=m+1}^{\infty} e^{\Re(\lambda \varkappa_{ii})} \left[1 - 2\,\Re(\lambda \varkappa_{ii}) + r^2 \sum_{j=1}^{\infty} |\varkappa_{ij}|^2\right]^{\frac{1}{2}} \tag{4.21}$$

$$\leqq (1 + r\|K\|)^m \exp \left\{ \frac{r^2}{2} \sum_{i=m+1}^{\infty} \sum_{j=1}^{\infty} |\varkappa_{ij}|^2 \right\}$$

$$= (1 + r\|K\|)^m \exp \left\{ \frac{r^2}{2} \int_a^b d\xi \left[\int_a^b |K(x, \xi)|^2 dx - \sum_{i=1}^{m} |k_i(\xi)|^2 \right] \right\};$$

$$[\varkappa^{(1)}(x)]^2 \equiv \int_a^b |K(x, \xi)|^2 d\xi, \quad [\varkappa^{(2)}(\xi)]^2 \equiv \int_a^b |K(x, \xi)|^2 dx.$$

Proof. Statement i. follows from (4.9) and lemma 3.7. Statement ii.[1] can be proved by a simple limiting process. Let

$$\lambda_1^{(n)}, \ldots, \lambda_n^{(n)}, \ |\lambda_\nu^{(n)}| = r_\nu^{(n)}, \ r_1^{(n)} \leqq r_2^{(n)} \leqq \cdots \leqq r_n^{(n)},$$

be the roots of the n-th segment $A^{*(n)}(\lambda)$ of $A^*(\lambda; K)$. Since

$$A^{*(n)}(\lambda) = \det\,(\delta_{ij} - \lambda \varkappa_{ij}^*), \qquad i, j = 1, 2, \ldots, n,$$

it is readily seen that

$$\sum_{\nu=1}^{n} [\lambda_\nu^{(n)}]^{-1} = \sum_{i=1}^{n} \varkappa_{ii},$$

$$A^{*(n)}(\lambda) = \exp\left(\lambda \sum_{i=1}^{n} \varkappa_{ii}\right) \prod_{\nu=1}^{n} \left(1 - \frac{\lambda}{\lambda_\nu^{(n)}}\right)$$

$$= \prod_{\nu=1}^{n} \left(1 - \frac{\lambda}{\lambda_\nu^{(n)}}\right) \exp\left(\frac{\lambda}{\lambda_\nu^{(n)}}\right).$$

From the uniform convergence of $A^{*(n)}(\lambda)$ to $A^*(\lambda; K)$ on any finite domain of the λ-plane it follows that for a fixed ν

(4.22) $$\lambda_\nu^{(n)} \to \lambda_\nu \text{ as } n \to \infty.$$

It was proved by Schur with the aid of simple algebraic considerations that

$$\sum_{\nu=1}^{n} [r_\nu^{(n)}]^{-2} \leqq \sum_{i,j=1}^{n} |\varkappa_{ij}|^2$$

---

[1] This is the classical theorem of Schur-Carleman referred to in 1.1.

[1, p. 492]. Hence, by (4.22),

$$\sum_{\nu=1}^{N} (r_\nu)^{-2} \leqq \sum_{i,j=1}^{\infty} |\varkappa_{ij}|^2 = ||K||^2$$

for all values of the integer N, which proves (4.17). This result combined with lemma 3.8 shows that the order of $A^*(\lambda; K)$ does not exceed 2 [Valiron, 3, p. 24]. We have then the product representation

$$A^*(\lambda;\ K) = \exp\ (\alpha + \beta\lambda + \gamma\lambda^2) \prod_{\nu=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_\nu}\right) \exp\ \left(\frac{\lambda}{\lambda_\nu}\right).$$

Here $\alpha = 0$ since $A^*(0; K) = 1$. To determine γ we observe that, by a familiar result of Lindelöf [1, p. 11], for infinitely many values of $r \to \infty$

$$\left| \prod_{\nu=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_\nu}\right) \exp\ \left(\frac{\lambda}{\lambda_\nu}\right) \right| > \exp\ (-\varepsilon r^2),\ \varepsilon \to 0.$$

This is compatible with lemma 3.8 only when $\gamma = 0$. Finally since

$$\frac{d}{d\lambda} A^*(\lambda;\ K)\Big|_{\lambda=\infty} = \lim_{n\to\infty} \frac{d}{d\lambda} A^{*(n)}(\lambda)\Big|_{\lambda=0} = 0,$$

we have $\beta = 0$. Statement iii. therefore is proved.[1] To prove statement iv. we observe that

$$\sum_{i=1}^{\infty} |k_i(\xi)|^2 = [\varkappa^{(2)}(\xi)]^2,\ \ \sum_{i=1}^{\infty} |k'_i(x)|^2 = [\varkappa^{(1)}(x)]^2, \tag{4.23}$$

and

$$\begin{aligned} \sum_{i=m+1}^{\infty} \sum_{j=1}^{\infty} |\varkappa_{ij}(K)|^2 &= \sum_{i=m+1}^{\infty} \int_a^b |k_i(\xi)|^2 d\xi = \int_a^b d\xi \sum_{i=m+1}^{\infty} |k_i(\xi)|^2 \\ &= \int_a^b d\xi \left\{ \int_a^b |K(x,\ \xi)|^2\, dx - \sum_{i=1}^{m} |k_i(\xi)|^2 \right\}, \end{aligned} \tag{4.24}$$

while, by Schwarz's inequality,

[1] Cf. an analogous argument of Carleman [5, pp. 216—217; 2].

$$|\varkappa_{ii}(K)| = \left|\int_a^b\int_a^b K(x,\xi)\overline{\varphi_i(x)}\varphi_i(\xi)\,dx\,d\xi\right| \leqq ||K||,$$

so that

$$0 \leqq 1 - 2\Re(\lambda\varkappa_{ii}) + r^2\sum_{j=1}^{\infty}|\varkappa_{ij}|^2 \leqq 1 + 2r||K|| + r^2||K||^2 = (1 + r||K||)^2.$$

It remains only to apply lemmas 3.4 and 3.5.

If we assume Carleman's results [5], the identity of our determinant $A^*(\lambda; K)$ with the Fredholm modified determinant $d_K^*(\lambda)$ follows at once from the infinite product representation (4.18). This identity can also be established directly since it is readily proved that $A^*(\lambda; K)$ and $d_K^*(\lambda)$ are holomorphic functions of the elements $\varkappa_{ij}$ in the Hilbert space $\mathfrak{L}_2$ [in the sense of Gâteaux, 1], and that they coincide whenever the number of the elements $\varkappa_{ij}$ distinct from zero is finite.

If the matrix $(\varkappa_{ij}) \subset \mathfrak{L}'_2$ we shall say that the kernel $K(x,\xi) \subset L'_2$. In this case the series $\Sigma|\varkappa_{ii}|$ converges and we can replace the determinants $A^*(\lambda; K)$ and $A^*(x, \xi; \lambda; K)$ by the determinants

$$A(\lambda; K) \equiv \det(\delta_{ij} - \lambda\varkappa_{ij}) \qquad (i, j = 1, 2, \ldots),$$

and

$$A(x,\xi;\lambda;K) \equiv \begin{vmatrix} -K(x,\xi) & k'_1(x) & k'_2(x) \ldots \\ \lambda k_1(\xi) & 1-\lambda\varkappa_{11} & -\lambda\varkappa_{12} \ldots \\ \lambda k_2(\xi) & -\lambda\varkappa_{21} & 1-\lambda\varkappa_{22} \ldots \\ \ldots & \ldots & \ldots \end{vmatrix}$$

respectively. The identity of the determinants $A(\lambda; K)$ and $d_K(\lambda)$ (classical Fredholm determinant) has been established under certain restrictive assumptions [Marty, 1; Mollerup, 1, 2]. It should be observed that while $A(\lambda; K)$ remains unchanged if the values of $K(x, \xi)$ are modified on an arbitrary set of superficial measure zero, such a modification will affect in general the value of $d_K(\lambda)$ (cf. Section 1.1 above). Hence in general the determinants $A(\lambda; K)$ and $d_K(\lambda)$, even when they both exist, are not equal.[1] However, it can be proved that

[1] For instance, in the case of the Volterra kernel $K(x, \xi) = 1$ or 0 according at $x \geqq \xi$ or $x < \xi$, it is readily found that $d_K(\lambda) = e^{-\lambda}$ while $A(\lambda; K) = A^*(\lambda; K) = 1$; here $a = 0$, $b = 1$.

$$A(\lambda;\, K) = d_K(\lambda)$$

provided (i) the set $\{\varphi_\nu(x)\}$ is uniformly bounded, (ii) the matrix $(\varkappa_{ij}) \subset \mathfrak{L}'_2$ and (iii) the condition $\int_a^b K(x,\, x)\, dx = \sum_{i=1}^{\infty} \varkappa_{ii}$ is satisfied.

The following theorem gives a basis for an estimate of the growth of the C.V. of a kernel $H(x,\, \xi)$ obtained from $K(x,\, \xi)$ by adding a kernel of finite rank.

Theorem 4.2. *If $K(x,\, \xi)$ and the functions $u_i(x)$, $v_i(\xi) \subset L_2$, $i = 1, 2, \ldots, q$, then the resolvant $\mathfrak{H}(x,\, \xi;\, \lambda)$ of the kernel*

$$H(x,\, \xi) = K(x,\, \xi) + S_q(x,\, \xi),\; S_q(x,\, \xi) = \sum_{i=1}^{q} u_i(x) v_i(\xi), \tag{4.25}$$

can be represented by

$$\mathfrak{H}(x,\, \xi;\, \lambda) = \frac{B_m(x,\, \xi;\, \lambda)}{B_m(\lambda)}, \tag{4.26}$$

where $B_m(\lambda)$ and $B_m(x,\, \xi;\, \lambda)$ for almost all $(x,\, \xi)$ are entire functions in λ, and

$$|B_m(\lambda)| \leqq q^{\frac{q}{2}} [\Pi_m(r;\, K)]^{q+1} [1 + r\, U\, V(1 + r\|K\| + \sqrt{e}\, r^2 \|K\|^2)]^q, \tag{4.27}$$

where

$$U^2 = \max_i \int_a^b |u_i(x)|^2\, dx,\quad V^2 = \max_i \int_a^b |v_i(\xi)|^2\, d\xi. \tag{4.28}$$

Proof. If $\mathfrak{R}(x,\, \xi;\, \lambda)$ is the resolvant of $K(x,\, \xi)$, it is readily found [cf. Bateman, 1] that

$$\mathfrak{H}(x,\, \xi;\, \lambda) = \begin{vmatrix} \mathfrak{R}(x,\, \xi;\, \lambda) & \Theta_1(x) & \ldots & \Theta_q(x) \\ \Omega_1(\xi) & 1 - \lambda b_{11} & \ldots & -\lambda b_{1q} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \Omega_q(\xi) & -\lambda b_{q1} & \ldots & 1 - \lambda b_{qq} \end{vmatrix} : \begin{vmatrix} 1 - \lambda b_{11} & -\lambda b_{12} & \ldots & -\lambda b_{1q} \\ -\lambda b_{21} & 1 - \lambda b_{22} & \ldots & 1 - \lambda b_{2q} \\ \cdot & \cdot & \cdot & \cdot \\ -\lambda b_{q1} & -\lambda b_{q2} & \ldots & 1 - \lambda b_{qq} \end{vmatrix},$$

where

$$b_{ij} = \int_a^b v_i(x)\, u_j(x)\, dx - \lambda \int_a^b v_i(x)\, \mathfrak{R} \cdot u_j(x)\, dx;$$

$$\Theta_i(x) = u_i(x) - \lambda \mathfrak{R} \cdot u_i(x); \quad \Omega_i(\xi) = v_i(\xi) - \lambda \mathfrak{R}' \cdot v_i(\xi).$$

On substituting here the expression (4. 16) for $\mathfrak{R}(x, \xi; \lambda)$ and multiplying the numerator and denominator by $A_m^*(\lambda; K)$ we get a result of type (4. 26) with

$$B_m(\lambda) = A_m^*(\lambda; K) \det (c_{ij}), \quad i, j = 1, 2, \ldots, q;$$

$$c_{ij} = \delta_{ij} A_m^*(\lambda; K) - \lambda A_m^*(\lambda; K) \int_a^b v_i(x) u_j(x)\, dx$$

$$- \lambda^2 \int_a^b \int_a^b v_i(x) A_m^*(x, \xi; \lambda; K) u_j(\xi)\, dx\, d\xi.$$

From (4. 20) it follows by Schwarz's inequality that

$$\left| \int_a^b \int_a^b v_i(x) A_m^*(x, \xi; \lambda; K)\, u_j(\xi)\, dx\, d\xi \right| \leqq$$

$$\leqq \Pi_m(r; K) \left[\int_a^b \int_a^b |v_i(x)\, K(x, \xi)\, u_j(\xi)|\, dx\, d\xi + \sqrt{e}\, r \int_a^b |v_i(x)| \varkappa^{(1)}(x)\, dx \int_a^b |u_i(\xi)| \varkappa^{(2)}(\xi)\, d\xi \right]$$

$$\leqq \Pi_m(r; K) [U V \|K\| + \sqrt{e}\, r\, U V \|K\|^2],$$

and also

$$\left| \int_a^b v_i(x)\, u_j(x)\, dx \right| \leqq U V,$$

whence for the elements of the determinant $\det (c_{ij})$ we have by (4. 19)

$$|c_{ij}| \leqq \Pi_m(r; K) [1 + r U V + r^2 U V \|K\| + \sqrt{e}\, r^3 U V \|K\|^2].$$

Formula (4. 27) is now obtained by the Hadamard determinant theorem.

5. Semi-definite Hermitian Kernels of Class L'_2.

A kernel $K(x, \xi)$ is Hermitian if

$$K'(x, \xi) \equiv K(\xi, x) = \overline{K(x, \xi)},$$

and semi-definite (positive or negative) if the corresponding Hermitian integral form

$$\int_a^b \overline{u(x)}\, K \cdot u(x)\, dx = \int_a^b u(x)\, \overline{K \cdot u(x)}\, dx \geqq 0 \text{ or } \leqq 0.$$

The following theorem is important for our proof of Lalesco's theorem on composite kernels and may be interesting in itself.

Theorem 5.1. *If $K(x, \xi)$ is a Hermitian semi-definite kernel $\subset L'_2$ then:*

i. *The series*

$$\sum_{\nu=1}^{\infty} [r_\nu(K)]^{-1} \tag{5.1}$$

converges.

ii. *The determinant $A(\lambda; K)$ is represented by*

$$A(\lambda; K) = \prod_{\nu=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_\nu}\right), \tag{5.2}$$

and is an entire function in λ of genus zero and of order $\leqq 1$. $A(\lambda; K)$ is of the first class [Valiron, 1, p. 258] *if it is of order 1, hence of minimal type.*

Proof. Without loss of generality we may assume that $K(x, \xi)$ is semi-definite positive. Then all the coefficients

$$\varkappa_{ii}(K) = \int_a^b \overline{\varphi_i(x)}\, K \cdot \varphi_i(x)\, dx \geqq 0,$$

and all the C.V. $\lambda(K) > 0$. Further, the matrix $(\varkappa_{ij}(K)) = \overline{(\varkappa_{ji}(K))}$ is Hermitian. From the formulas

$$\int_a^b \overline{u(x)}\, K \cdot u(x)\, dx = \sum_{i,j=1}^{\infty} \varkappa_{ij} \bar{u}_i u_j, \quad u_\nu = \int_a^b u(x)\, \overline{\varphi_\nu(x)}\, d x,$$

it also follows that the matrix $(\varkappa_{ij}(K))$ is semi-definite positive as well as all its segments

$$(\varkappa_{ij}^{(n)}) \equiv (\varkappa_{ij}(K)); \; i, j = 1, 2, \ldots, n.$$

This implies that the C.V. $\lambda_\nu^{(n)}$ of all these segments are > 0. Since

$$\sum_{\nu=1}^{n} (\lambda_\nu^{(n)})^{-1} = \sum_{i=1}^{n} \varkappa_{ii},$$

and, for a fixed N,

$$\lambda_\nu^{(n)} \to \lambda_\nu(K) \text{ as } n \to \infty, \; \nu \leqq N,$$

it is obvious that the series (5. 1) converges and its sum is $\leqq \sum_{i=1}^{\infty} \varkappa_{ii}$. We shall prove that

$$(5.\,3) \qquad \sum_{\nu=1}^{\infty} [r_\nu(K)]^{-1} = \sum_{i=1}^{\infty} \varkappa_{ii}(K).$$

In order to do this we observe that each member of (5. 3) remains invariant under any unitary transformation of the matrix $(\varkappa_{ij})$. But if this matrix be reduced to diagonal form $(\varkappa'_{ij})$, $\varkappa'_{ij} = 0$ if $i \neq j$, we have

$$[\lambda_\nu(K)]^{-1} = [r_\nu(K)]^{-1} = \varkappa'_{\nu\nu},$$

so that (5. 3) holds for this particular choice of the matrix $(\varkappa_{ij})$. Hence (5. 3) is always true.

We then have by formula (3. 6)

$$A(\lambda; K) = \exp\left[-\lambda \sum_{i=1}^{\infty} \varkappa_{ii}\right] \prod_{i=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_i}\right) \exp\left(\frac{\lambda}{\lambda_i}\right) = \prod_{i=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_i}\right),$$

whence statement ii. follows at once [Valiron, 2, pp. 59—60].

6. Composite Kernels.

By a composite kernel $\subset L_2$ we mean a kernel of the type

$$K(x, \xi) = (K_1 K_2)(x, \xi), \quad K_1, K_2 \subset L_2.$$

The method of infinite determinants is particularly well fitted for the investigation of the growth of the C.V. of such kernels. In this section we give a proof of Lalesco's

Theorem 6.1. *If $K(x, \xi)$ is a composite kernel $\subset L_2$, then:*

i. *The series*

$$\sum_{\nu=1}^{\infty} [\gamma_\nu(K)]^{-1} \tag{6.1.}$$

converges.

ii. *The determinant $\Delta(\lambda; K)$ is identical with the Fredholm determinant $d_K(\lambda)$, and possesses all the properties mentioned in theorem 5.1.*

A rigorous proof of this theorem is due to S. Gheorghiu [4, pp. 35—36] under the restrictive assumption that at least one of the integrals (for each i)

$$\int_a^b |K_i(x, \xi)|^2 \, d\xi, \quad \int_a^b |K_i(x, \xi)|^2 \, dx, \qquad i = 1, 2,$$

is bounded on (a, b). This assumption figures in the proof of Gheorghiu's lemmas (A), (C) and (D). While the proof of lemmas (C) and (D) can be extended to the general case of our theorem 5.1, the proof of lemma (A) is essentially based upon the above assumption. It happens, however, that this lemma (A) is a special case of our theorem 5.1 which, therefore, provides a foundation for the proof of theorem 6.1 in its full generality. The proof that we give here is adapted to the method of infinite determinants. We might refer for some parts of the proof directly to Gheorghiu's paper. We prefer, however, to give a complete development for the reader's convenience since Gheorghiu's paper was published separately in the form of a Thesis, and is not easily accessible.

We shall need several lemmas and a convenient notation. We denote by

$$(\varkappa_{ij}^{(1)}) = (\varkappa_{ij}(K_1)), \ (\varkappa_{ij}^{(2)}) = (\varkappa_{ij}(K_2)), \ \ldots,$$

the matrices $(\varkappa_{ij})$ corresponding to the kernels $K_1, K_2, \ldots$. The coefficient of $(-\lambda)^n$ in the power series expansion of $A(\lambda; K)$ will be denoted by $\alpha_n \equiv \alpha_n(K)$ so that

$$A(\lambda; K) = \sum_{n=0}^{\infty} \alpha_n(-\lambda)^n = \sum_{n=0}^{\infty} \alpha_n(K)(-\lambda)^n. \tag{6.2}$$

We set

$$\Delta_{ij}^{(n)}(K) \equiv \Delta_{ij}^{(n)} = \begin{vmatrix} \varkappa_{i_1 j_1}(K) & \ldots & \varkappa_{i_1 j_n}(K) \\ \cdot & \cdot \; \cdot \; \cdot & \cdot \\ \varkappa_{i_n j_1}(K) & \ldots & \varkappa_{i_n j_n}(K) \end{vmatrix}. \tag{6.3}$$

Finally, the symbols

$$\sum_{(i:n)} \equiv \sum_{(i)}, \quad \sum_{(i,j:n)} \equiv \sum_{(i,j)}, \ldots$$

will be used to designate the summations

$$\sum_{i_1, \ldots, i_n=1}^{\infty}, \quad \sum_{i_1, \ldots, i_n, j_1, \ldots, j_n=1}^{\infty}, \ldots$$

With this notation we have

Lemma 6.1. *If $K(x, \xi) \subset L'_2$, then*

$$\alpha_n(K) = \frac{1}{n!} \sum_{(i:n)} \Delta_{ii}^{(n)}(K). \tag{6.4}$$

[F. Riesz, 1, p. 34].

Lemma 6.2. *If the kernels $K_1(x, \xi)$ and $K_2(x, \xi) \subset L_2$, then the composite kernel*

$$K(x, \xi) = (K_1 K_2)(x, \xi) \subset L'_2, \tag{6.5}$$

and the determinant $A(\lambda; K)$ exists, the coefficients $\alpha_n(K)$ being given by

$$\alpha_n(K) = \frac{1}{(n!)^2} \sum_{(i,j:n)} \Delta_{ij}^{(n)}(K_1) \Delta_{ji}^{(n)}(K_2). \tag{6.6}$$

More generally, if

$$(6.7) \qquad K(x, \xi) = (K_1 K_2 \ldots K_s)(x, \xi); \; K_1, K_2, \ldots, K_s \subset L_2,$$

then $A(\lambda; K)$ *exists, and*

$$(6.8) \qquad \alpha_n(K) = \frac{1}{(n!)^s} \sum_{(i_1, \ldots, i_s : n)} \Delta_{i_1 i_2}^{(n)}(K_1) \Delta_{i_2 i_3}^{(n)}(K_2) \ldots \Delta_{i_s i_1}^{(n)}(K_s).$$

Proof. By Parseval's identity

$$\varkappa_{ij}(K) = \sum_{\nu=1}^{\infty} \varkappa_{i\nu}(K_1) \varkappa_{\nu j}(K_2), \; (\varkappa_{ij}(K)) = (\varkappa_{ij}^{(1)})(\varkappa_{ij}^{(2)}),$$

whence, by lemma 3.1, $(\varkappa_{ij}(K)) \subset \mathfrak{L}'_2$ and $K(x, \xi) \subset L'_2$. Formula (6.6) is proved by an argument familiar in the theory of determinants. Formula (6.8) follows from a repeated application of (6.6).[1]

We introduce now two kernels $N^{(1)}(K)$, $N^{(2)}(K)$ related to K and defined by

$$(6.9) \qquad N^{(1)}(K) \equiv (K \bar{K}')(x, \xi), \; N^{(2)}(K) \equiv (\bar{K}' K)(x, \xi).$$

We shall designate $N^{(1)}(K)$ and $N^{(2)}(K)$ respectively as the first and the second norms of K.[2] When $K(x, \xi)$ is Hermitian its norms coincide and reduce to the iterated kernel $K^{(2)}(x, \xi)$ which, therefore, may be termed the norm of K.

Lemma 6.3. *The norms of a kernel* $K(x, \xi) \subset L_2$ *are Hermitian semi-definite positive kernels* $\subset L'_2$. *The determinants* $A(\lambda; N^{(1)}(K))$ *and* $A(\lambda; N^{(2)}(K))$ *are identical and*

$$(6.10) \qquad \alpha_n(N^{(1)}(K)) = \alpha_n(N^{(2)}(K)) = \frac{1}{(n!)^2} \sum_{(i, j : n)} |\Delta_{ij}^{(n)}(K)|^2 \geqq 0.$$

---

[1] Carleman [5, p. 213] gives formulas which are analogous to (6.6), (6.8) for the Fredholm determinants. For the sake of completeness we may mention also the formula

$$A(\lambda; K_0) = A(\lambda; K_1) A(\lambda; K_2), \; K_0 \equiv K_1 + K_2 - \lambda K_1 K_2,$$

which holds whenever $K_1, K_2 \subset L'_2$ [Fredholm 1, pp. 381—383] and is a direct consequence of the rule of multiplication of determinants.

[2] The kernels $N^{(1)}(K)$, $N^{(2)}(K)$ play an important rôle in E. Schmidt's theory of the »adjoint fundamental functions» of an unsymmetric kernel $K(x, \xi)$ [1, p. 461]. That is why these kernels are designated by Gheorghiu and some other authors as the Schmidt kernels of K. Our terminology is analogous to that used in the theory of bilinear forms in infinitely many variables.

Proof. The kernel $N^{(1)}(K)$ is Hermitian since $(\overline{K\overline{K}'})' = K\overline{K}'$. It is semi-definite positive since

$$\int_a^b \bar{u}\, N^{(1)}(K)\cdot u\, dx = \int_a^b \bar{u}\,(K\overline{K}')\cdot u\, dx = \int_a^b K'\cdot\bar{u}\;\overline{K}'\cdot u\, dx \geqq 0.$$

$N^{(1)}(K) \subset L'_2$ by lemma 6.2. A similar proof holds for $N^{(2)}(K)$. To compute $a_n(N^{(1)}(K))$ we apply lemma 6.2 again and observe that

$$(6.11) \qquad \varkappa_{ij}(\overline{K}') = \overline{\varkappa_{ji}(K)}, \quad \Delta_{ij}^{(n)}(\overline{K}') = \overline{\Delta_{ji}^{(n)}(K)}.$$

Lemma 6.4. *For a composite kernel $\subset L_2$,*

$$K(x, \xi) = (K_1 K_2)(x, \xi); \quad K_1, K_2 \subset L_2,$$

we have

$$(6.12) \qquad A(\lambda;\, N^{(1)}(K)) \equiv A(\lambda;\, K_1 K_2 \overline{K}'_2 \overline{K}'_1) = A(\lambda;\, \overline{K}'_1 K_1 K_2 \overline{K}'_2).$$

Proof. Since the result of the summation in (6.8) is invariant under any cyclic permutation of indices $(i_1, i_2, \ldots, i_s)$ it is seen that $A(\lambda;\, K_1 K_2 \ldots K_s)$ will not change under any cyclic permutation of the components $K_1, K_2, \ldots, K_s$. Now it remains only to put

$$s = 4, \quad K_3 = \overline{K}'_2, \quad K_4 = \overline{K}'_1.$$

We also need some facts from the theory of entire functions, which we collect in

Lemma 6.5. *Let $f(z)$ be an entire function whose zeros, repeated with their multiplicities, are*

$$z_1, z_2, \ldots, z_\nu, \ldots; \; |z_\nu| = r_\nu; \; 0 < r_1 \leqq r_2 \leqq \cdots \leqq r_\nu \leqq \cdots.$$

Let $n(r)$ be the number of the r_ν's with $r_\nu < r$, and

$$M(r) \equiv M(r; f) = \max_{|z|=r} |f(z)|,$$

$$(6.13) \qquad V(r) \equiv V(r; f) = \frac{1}{2\pi}\int_0^{2\pi} \log |f(re^{i\varphi})|\, d\varphi \leqq \log M(r; f).$$

We have then:

i. *The convergence of the integral*

$$\int_0^\infty \frac{n(r)}{r^{1+\tau}}\,dr \qquad (\tau > 0) \tag{6.14}$$

is necessary and sufficient for the convergence of the series

$$\sum_{\nu=1}^{\infty} r_\nu^{-\tau}. \tag{6.15}$$

ii. *If τ is any number $\geqq 0$ and $r_0 > 0$, then*

$$\int_{r_0}^{r} \frac{n(r)}{r^{1+\tau}}\,dr = \frac{V(r)}{r^\tau} - \frac{V(r_0)}{r_0^\tau} + \tau \int_{r_0}^{r} \frac{V(r)}{r^{1+\tau}}\,dr. \tag{6.16}$$

iii. *The integral* (6. 16) *converges whenever the integral*

$$\int_{r_0}^{r} \frac{\log M(r)}{r^{1+\tau}}\,dr \qquad (\tau > 0) \tag{6.17}$$

does so.

iv. *If the order ϱ of $f(z)$ is not an integer, then the convergence of* (6. 15) *for $\tau = \varrho$ is necessary and sufficient for the convergence of* (6. 17) *for $\tau = \varrho$. Or, in Valiron's terminology, if $f(z)$ is of non-integral order, a necessary and sufficient condition that $f(z)$ be of the first class is that $f(z)$ shall be of the inferior class* [Valiron, 1, pp. 258—265].

It is convenient to write

$$M(r;\, K) \equiv \max_{|\lambda| = r} |A(\lambda;\, K)|,$$

and to use the abbreviated notation

$$N_j^{(i)} \equiv N_j^{(i)}(x, \xi) \equiv N^{(i)}(K_j); \quad i, j = 1, 2.$$

We prove now the following lemmas:

Lemma 6. 6. *For a composite kernel*

$$K(x, \xi) = (K_1 K_2)(x, \xi);\; K_1,\, K_2 \subset L_2,$$

we have[1]

$$|\alpha_n(K)| \leqq [\alpha_n(N_1^{(1)})]^{\frac{1}{2}} [\alpha_n(N_2^{(1)})]^{\frac{1}{2}}, \tag{6.18}$$

$$M(r;\, K) \leqq [M(r;\, N_1^{(1)})]^{\frac{1}{2}} [M(r;\, N_2^{(1)})]^{\frac{1}{2}}. \tag{6.19}$$

Proof. Formula (6. 18) follows from lemmas 6. 2, 6. 3 and Cauchy's inequality. To prove (6. 19) we use the inequalities

$$M(r;\, K) \leqq \sum_{n=0}^{\infty} |\alpha_n(K)|\, r^n \leqq \sum_{n=0}^{\infty} [\alpha_n(N_1^{(1)})\, r^n]^{\frac{1}{2}} [\alpha_n(N_2^{(1)})\, r^n]^{\frac{1}{2}}$$

$$\leqq [M(r;\, N_1^{(1)})]^{\frac{1}{2}} [M(r;\, N_2^{(1)})]^{\frac{1}{2}}.$$

Lemma 6. 7. *If* $K(x, \xi) \subset L_2$ *then*

$$M(r;\, K^{(2)}) \leqq M(r;\, N^{(1)}(K));\quad K^{(2)}(x, \xi) = (KK)(x, \xi). \tag{6.20}$$

Proof. This is merely a special case of lemma 6. 6 with $K_1 = K_2 = K$.

Proof of Theorem 6. 1. By lemmas 6. 7 and 6. 4

$$M(r;\, K^{(2)}) \leqq M(r;\, N^{(1)}(K)) = M(r;\, N_1^{(2)} N_2^{(1)}). \tag{6.21}$$

The kernels $N_1^{(2)}$, $N_2^{(1)}$ are Hermitian semi-definite positive by lemma 6. 3. Hence, by lemma 6. 6,

$$M(r;\, N_1^{(2)} N_2^{(1)}) \leqq [M(r;\, N_1^{(2)} N_1^{(2)})]^{\frac{1}{2}} [M(r;\, N_2^{(1)} N_2^{(1)})]^{\frac{1}{2}}. \tag{6.22}$$

Since $N_1^{(2)}, N_2^{(1)} \subset L'_2$ by lemma 6. 2, all the conditions of theorem 5. 1 are satisfied, and the series (with positive terms)

$$\sum_{\nu=1}^{\infty} [\lambda_\nu(N_1^{(2)})]^{-1}, \qquad \sum_{\nu=1}^{\infty} [\lambda_\nu(N_2^{(1)})]^{-1}$$

converge. It is well known, however, that

$$\{\lambda_\nu(N_1^{(2)} N_1^{(2)})\} \equiv \{[\lambda_\nu(N_1^{(2)})]^2\}.$$

[1] From this point on our proof is merely an adaptation of that of Gheorghiu, with non-essential modifications.

Hence the series

$$\sum_{\nu=1}^{\infty} [\lambda_\nu (N_1^{(2)} N_1^{(2)})]^{-\frac{1}{2}}$$

converges which shows that $A(\lambda; N_1^{(2)} N_1^{(2)})$ is not only of order $\leqq \frac{1}{2}$, but, in addition, of the first class if it is of order $\frac{1}{2}$. By lemma 6. 5, iv., the integral

$$\int_{r_0}^{\infty} \frac{\log M(r; N_1^{(2)} N_1^{(2)})}{r^{1+\frac{1}{2}}} dr, \qquad r_0 > 0,$$

converges. The same conclusion holds for the integral

$$\int_{r_0}^{\infty} \frac{\log M(r; N_2^{(1)} N_2^{(1)})}{r^{1+\frac{1}{2}}} dr,$$

and, by (6. 22), (6. 21), for the integrals

$$\int_{r_0}^{\infty} \frac{\log M(r; N_1^{(2)} N_2^{(1)})}{r^{1+\frac{1}{2}}} dr, \qquad \int_{r_0}^{\infty} \frac{\log M(r; K^{(2)})}{r^{1+\frac{1}{2}}} dr.$$

Lemma 6. 5 shows then that $A(\lambda; K^{(2)})$ is of order $\leqq \frac{1}{2}$ and of the inferior, hence of the first class, whenever it is of order $\frac{1}{2}$. Since

$$\{\lambda_\nu (K^{(2)})\} \equiv \{[\lambda_\nu(K)]^2\},$$

this implies the convergence of the series

$$\sum_{\nu=1}^{\infty} [r_\nu (K^{(2)})]^{-\frac{1}{2}} = \sum_{\nu=1}^{\infty} [r_\nu (K)]^{-1}.$$

Statement i. of theorem 6. 1 is thus established. Statement ii. now becomes obvious.

7. Kernels of Class (β, q).

7. 1. In this section we deal with kernels that satisfy the following condition:

For a given pair of numbers (β, q) *where* $\beta > 0$ *and* $q \geqq 2$ *there exists an integer* $m_0 \geqq 0$ *such that the series*

$$(\beta, q) \qquad \sum_{\nu=m_0+1}^{\infty} \nu^\beta |k_\nu(\xi)|^q \equiv \Omega(\xi)$$

converges for almost all ξ *on* (a, b), *and its sum* $\Omega(\xi)$ *is integrable.*

Theorem 7.1. *If the kernel* $K(x, \xi) \subset L_2$ *and at the same time* $K(x, \xi) \subset (\beta, q)$, *then*

$$(7.1) \qquad r_n(K)\, n^{-\frac{\beta+1}{q}} \to \infty.$$

Proof. To abbreviate we shall write k_ν instead of $k_\nu(\xi)$. We shall use the letters ε for an arbitrarily small fixed positive quantity, not necessarily the same in all the formulas, and m for a fixed positive integer which can be taken arbitrarily large. The letter C will be used as a generic notation for a positive constant which does not depend on ε and m.

Since all the terms of the series (β, q) are $\geqq 0$ an easy application of Lebesgue's theorem will show that

$$(7.2) \qquad \sum_{\nu=m_0+1}^{\infty} \nu^\beta \int_a^b |k_\nu|^q \, d\xi = \int_a^b \Omega(\xi)\, d\xi = \Omega_0.$$

The integer m_0 in (β, q) can be chosen so large that

$$(7.3) \qquad \sum_{\nu=m_0+1}^{\infty} \nu^\beta \int_a^b |k_\nu|^q \, d\xi < \varepsilon.$$

Then for any $n > m_0$ and $n < n' \leqq \infty$

$$(7.4) \qquad \sum_{\nu=n+1}^{n'} \int_a^b |k_\nu|^q \, d\xi \leqq n^{-\beta} \sum_{\nu=n+1}^{n'} \nu^\beta \int_a^b |k_\nu|^q \, d\xi < \varepsilon n^{-\beta}.$$

By theorem 4.1 the C.V. $\{\lambda_\nu(K)\}$ are zeros of $A_m^*(\lambda; K)$ where

$$(7.5) \qquad |A_m^*(\lambda; K)| \leqq \Pi_m(r; K)$$
$$\leqq \prod_{i=1}^{m} \left[1 - 2\Re(\lambda \varkappa_{ii}) + r^2 \int_a^b |k_i|^2 \, d\xi \right]^{\frac{1}{2}} \exp\left[\frac{r^2}{2} \sum_{i=m+1}^{\infty} \int_a^b |k_i|^2 \, d\xi \right].$$

We assume $m > m_0$ and give first an estimate for the second factor

$$P_0 = \exp\left[\frac{r^2}{2}\sum_{i=m+1}^{\infty}\int_a^b |k_i|^2\,d\xi\right].$$

Since for $u \geqq 0$

(7.6) $$e^u \leqq C\exp(u^p),\quad p \geqq 1,$$

we have

$$P_0 < C\exp\left\{2^{-\frac{q}{2}} r^q \sum_{i=m+1}^{\infty}\left[\int_a^b |k_i|^2\,d\xi\right]^{\frac{q}{2}}\right\}.$$

By Hölder's inequality

(7.7) $$\left[\int_a^b |k_i|^2\,d\xi\right]^{\frac{q}{2}} \leqq (b-a)^{\frac{q-2}{2}}\int_a^b |k_i|^q\,d\xi,$$

whence

$$P_0 < C\exp\left\{C r^q \sum_{i=m+1}^{\infty}\int_a^b |k_i|^q\,d\xi\right\},$$

and, by (7.4),

$$P_0 < \exp(\varepsilon r^q m^{-\beta}).$$

We proceed now to the first factor of the right-hand member of (7.5). Here we can write

$$\prod_{i=1}^{m} = \prod_{i=1}^{m_0}\cdot\prod_{i=m_0+1}^{m} = P_1 P_2.$$

As in the proof of theorem 4.1 we have

$$P_1 < (1 + r\,||K||)^{m_0} < \exp(r^\varepsilon).$$

To estimate P_2 we observe that for a fixed σ, $0 < \sigma < 1$, and $u \geqq 0$,

(7.8) $$1 + u < C\exp(u^\sigma).$$

Hence, if τ is a number analogous to σ, we have

$$1 - 2\,\Re(\lambda\,\varkappa_{ii}) + r^2\int_a^b |k_i|^2\,d\xi \leqq (1 + 2\,r\,|\varkappa_{ii}|)\left(1 + r^2\int_a^b |k_i|^2\,d\xi\right)$$

$$\leqq C\exp\,(2^\sigma r^\sigma |\varkappa_{ii}|^\sigma)\,\exp\left[r^{2\tau}\left(\int_a^b |k_i|^2\,d\xi\right)^\tau\right],$$

so that

$$(7.9)\qquad P_2 \leqq C\exp\left[2^\sigma r^\sigma \sum_{i=m_0+1}^{m} |\varkappa_{ii}|^\sigma\right]\,\exp\left[r^{2\tau}\sum_{i=m_0+1}^{m}\left(\int_a^b |k_i|^2\,d\xi\right)^\tau\right] = C P_2' P_2''.$$

Now, again by Hölder's inequality,

$$\sum_{\nu=m_0+1}^{m}\left(\int_a^b |k_\nu|^2\,d\xi\right)^\tau = \sum_{\nu=m_0+1}^{m} \nu^{-\frac{2\beta\tau}{q}}\,\nu^{\frac{2\beta\tau}{q}}\left(\int_a^b |k_\nu|^2\,d\xi\right)^\tau$$

$$\leqq \left\{\sum_{\nu=m_0+1}^{m} \nu^{-\frac{2\beta\tau}{q-2\tau}}\right\}^{1-\frac{2\tau}{q}}\left\{\sum_{\nu=m_0+1}^{m} \nu^\beta\left(\int_a^b |k_\nu|^2\,d\xi\right)^{\frac{q}{2}}\right\}^{\frac{2\tau}{q}},$$

and, if we choose τ subject to the condition $2\tau(\beta+1) < q$, we have

$$\left\{\sum_{\nu=m_0+1}^{m} \nu^{-\frac{2\beta\tau}{q-2\tau}}\right\}^{1-\frac{2\tau}{q}} \leqq C m^{-\frac{2\beta\tau+2\tau-q}{q}}$$

Hence, by (7.3), (7.7) and (7.9),

$$P_2'' < \exp\left\{\varepsilon r^{2\tau} m^{-\frac{2\beta\tau+2\tau-q}{q}}\right\}.$$

To estimate P_2' we observe that

$$\varkappa_{\nu\nu} = \int_a^b k_\nu(\xi)\,\varphi_\nu(\xi)\,d\xi,\quad \int_a^b |\varphi_\nu(\xi)|^2\,d\xi = 1,$$

whence it follows by a repeated application of Hölder's inequality,

$$|\varkappa_{\nu\nu}| \leqq \left[\int_a^b |k_\nu|^q \, d\xi\right]^{\frac{1}{q}} \left[\int_a^b |\varphi_\nu|^{\frac{q}{q-1}} d\xi\right]^{\frac{q-1}{q}}$$

$$\leqq \left[\int_a^b |k_\nu|^q \, d\xi\right]^{\frac{1}{q}} (b-a)^{\frac{q-2}{2q}} \left[\int_a^b |\varphi_\nu|^2 d\xi\right]^{\frac{1}{2}} = C\left[\int_a^b |k_\nu|^q \, d\xi\right]^{\frac{1}{q}};$$

$$\sum_{\nu=m_0+1}^{m} |\varkappa_{\nu\nu}|^\sigma = \sum_{\nu=m_0+1}^{m} \nu^{-\frac{\beta\sigma}{q}} \nu^{\frac{\beta\sigma}{q}} |\varkappa_{\nu\nu}|^\sigma \leqq \left\{\sum_{\nu=m_0+1}^{m} \nu^{-\frac{\beta\sigma}{q-\sigma}}\right\}^{1-\frac{\sigma}{q}} \left\{\sum_{\nu=m_0+1}^{m} \nu^\beta |\varkappa_{\nu\nu}|^q\right\}^{\frac{\sigma}{q}}$$

$$< \varepsilon m^{-\frac{\beta\sigma+\sigma-q}{q}}, \quad \sigma(\beta+1) < q.$$

This shows that

$$P'_2 < \exp\left\{\varepsilon r^\sigma m^{-\frac{\beta\sigma+\sigma-q}{q}}\right\}.$$

If we take for simplicity $\sigma = 2\tau$ and write

$$M(r) = \max_{|\lambda|=r} |A_m^*(\lambda;\, K)|,$$

we get

$$\log M(r) \leqq \log \Pi_m(r;\, K) < r^\varepsilon + \varepsilon\left\{r^q m^{-\beta} + r^\sigma m^{-\frac{\beta\sigma+\sigma-q}{q}}\right\}. \tag{7.10}$$

We can apply now lemma 6.5. Since $A_m^*(0;\, K) = 1$ the lower limit of integration r_0 can be taken so small that $V(r_0) = 0$. From ii. of lemma 6.5 (with $\tau = 0$) it follows that

$$\int_{r_0}^{r} \frac{n(r)}{r}\, dr = V(r) \leqq \log M(r), \tag{7.11}$$

where $n(r)$ is the number of the C.V. $\lambda_\nu(K)$ in the interior of the circle $|\lambda| = r$.

So far the integer m was arbitrary, restricted only by $m > m_0$. For a fixed r we can use this fact in order to obtain as low estimate in (7.10) as possible. The simplest way of doing so is to make the contributions of the two terms in the brackets in (7.10) approximately equal, which leads to the choice

$$m = \left[r^{\frac{q}{\beta+1}}\right],$$

and, after a simple computation, to the estimate

$$\log M(r) \leqq \log \Pi_m(r; K) < \varepsilon r^{\frac{q}{\beta+1}}. \tag{7.12}$$

It follows directly from (7.11) or by other methods familiar in the theory of entire functions [Lindelöf, 1, p. 21; Valiron, 2, pp. 67—71] that

$$n(r) < \varepsilon r^{\frac{q}{\beta+1}},$$

which is equivalent to (7.1).

For our subsequent discussion we shall need the following

Lemma 7.1. *The kernel* $K(x, \xi) \subset L_2$ *if the series*

$$\sum_{\nu=1}^{\infty} \nu^\beta \int_a^b |k_\nu|^q \, d\xi = \Omega_1, \ \beta > 0, \ q \geqq 2, \tag{7.13}$$

converges, and

$$2(\beta + 1) > q. \tag{7.14}$$

Proof. An easy application of Hölder's inequality and formula (7.7) gives

$$\sum_{\nu=n+1}^{n'} \int_a^b |k_\nu|^2 \, d\xi \leqq (n'-n)^{\frac{q-2}{q}} \left\{ \sum_{\nu=n+1}^{n'} \left[\int_a^b |k_\nu|^2 \, d\xi \right]^{\frac{q}{2}} \right\}^{\frac{2}{q}}$$

$$\leqq (n'-n)^{\frac{q-2}{q}} (b-a)^{\frac{q-2}{q}} \left\{ \sum_{\nu=n+1}^{n'} \int_a^b |k_\nu|^q \, d\xi \right\}^{\frac{2}{q}}$$

$$= C(n'-n)^{\frac{q-2}{q}} \left\{ \sum_{\nu=n+1}^{n'} \int_a^b |k_\nu|^q \, d\xi \right\}^{\frac{2}{q}}.$$

[Cf. Szász, 4, p. 533]. On setting here in succession

$$n = 0, \ n' = 1; \ n = 2^\mu, \ n' = 2^{\mu+1}, \ \mu = 0, 1, \ldots,$$

and using the inequality

$$\sum_{\nu=n+1}^{\infty} \int_a^b |k_\nu|^q \, d\xi \leqq n^{-\beta} \sum_{\nu=n+1}^{\infty} \nu^\beta \int_a^b |k_\nu|^q \, d\xi \leqq \Omega_1 n^{-\beta},$$

together with (7. 14), we establish the convergence of the series

$$\sum_{\nu=1}^{\infty} \int_a^b |k_\nu|^2 \, d\xi = \Omega_2.$$

Thus the sequence

$$f_n(\xi) = \sum_{\nu=1}^{n} |k_\nu(\xi)|^2, \qquad n = 1, 2, \ldots$$

is monotone increasing and such that

$$\int_a^b f_n(\xi) \, d\xi \leqq \Omega_2.$$

It follows by Fatou's theorem that its limiting function

$$\sum_{\nu=1}^{\infty} |k_\nu(\xi)|^2 = \int_a^b |K(x, \xi)|^2 \, dx$$

is integrable, whence $K(x, \xi) \subset L_2$.

7. 2. Theorem 7. 1 admits of an important application to the kernels which possess derivatives (of integral or »fractional» orders). Before proceeding to this application we shall recall some facts of the theory of differentiation and integration of fractional order. Following H. Weyl [3, pp. 296—302] we introduce the function $\Psi_\alpha(x)$ whose trigonometric Fourier coefficients are

$$c_0 = 0, \; c_\nu = \Gamma(\alpha) e^{-\frac{\pi i \alpha}{2}} \nu^{-\alpha}, \; c_{-\nu} = \bar{c}_\nu, \qquad \nu = 1, 2, \ldots.$$

For our purposes we need only the case $0 < \alpha \leqq 1$. The function $\Psi_\alpha(x)$ is uniquely defined on $(0, 2\pi)$ and can be extended periodically outside $(0, 2\pi)$ by

$$\Psi_\alpha(x + 2\pi) = \Psi_\alpha(x).$$

It is readily seen that on $(0, 2\pi)$

$$\Psi_\alpha(x) = \begin{cases} 2\pi \lim\limits_{n \to \infty} \left[\sum\limits_{\nu=0}^{n-1} (x + 2\pi\nu)^{\alpha-1} - \frac{1}{\alpha} (2\pi)^{\alpha-1} n^\alpha \right], & 0 < \alpha < 1, \\ \pi - x, \; \alpha = 1. \end{cases}$$

Hence $\Psi_\alpha(x)$ is continuous for $x \neq 2k\pi$, while at $x = 0$

$$\Psi_\alpha(x) = \begin{cases} 2\pi x^{\alpha-1} + \psi_\alpha(x) & \text{for } x > 0, \\ \psi_\alpha(x) & \text{for } x < 0, \end{cases} \qquad (0 < \alpha < 1)$$

where $\psi_\alpha(x)$ is continuous (in fact, analytic) at $x = 0$.

Let $f(x)$ be any periodic function of period 2π integrable over $(0, 2\pi)$. Weyl defines as the α-fold integral of $f(x)$ the following operation[1]

$$(7.15) \qquad f_\alpha(x) = \frac{1}{2\pi\Gamma(\alpha)}\int_0^{2\pi} f(t)\,\Psi_\alpha(x-t)\,dt.$$

Lemma 7. 2. *If $f(x)$ is any integrable periodic function, then:*

i. *The function $f_\alpha(x)$ is integrable and periodic. If $\{f_\nu\}$ is the set of Fourier coefficients of $f(x)$, the set of Fourier coefficients of $f_\alpha(x)$ is given by*

$$(7.16) \qquad (f_\alpha)_\nu = \frac{f_\nu c_\nu}{\Gamma(\alpha)} = \begin{cases} f_\nu e^{-\frac{\pi i \alpha}{2}} \nu^{-\alpha}, & \nu = 1, 2, \ldots, \\ f_\nu e^{\frac{\pi i \alpha}{2}} (-\nu)^{-\alpha}, & \nu = -1, -2, \ldots, \\ 0 & , \nu = 0. \end{cases}$$

ii. *If $f(x)$ has the mean value zero over a period,*

$$(7.17) \qquad \int_0^{2\pi} f(x)\,dx = 0,$$

the operation $f_\alpha(x)$ reduces to

$$(7.18) \qquad I_x^\alpha f(x) \equiv \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x} f(t)\,(x-t)^{\alpha-1}\,dt$$

when $0 < \alpha < 1$, and to a primitive function of $f(x)$ when $\alpha = 1$, the infinite integral in (7. 18) being an improper Lebesgue integral.[2]

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[1] Weyl considers only the case of a continuous $f(x)$. We extend his results to any integrable $f(x)$.

[2] This means that $\int_{-\infty}^{x} = \lim_{a \to -\infty} \int_a^x$.

Proof. The fact that $f^\alpha(x)$ is integrable follows from a known theorem of Young [1]. Furthermore, on interchanging the order of integrations and using the periodicity property of the functions concerned, we have

$$(f_\alpha)_\nu = \frac{1}{2\pi}\int_0^{2\pi} dx\, e^{-i\nu x}\left[\frac{1}{2\pi\Gamma(\alpha)}\int_0^{2\pi} f(s)\Psi_\alpha(x-s)\,ds\right] =$$

$$= \frac{1}{\Gamma(\alpha)}\frac{1}{2\pi}\int_0^{2\pi} e^{-i\nu s} f(s)\,ds\,\frac{1}{2\pi}\int_0^{2\pi} e^{-i\nu(x-s)}\Psi_\alpha(x-s)\,dx = \frac{f_\nu c_\nu}{\Gamma(\alpha)},$$

which proves statement i. When $\alpha = 1$, statement ii. follows immediately from

$$f_1(x) = \frac{1}{2\pi}\left[\int_0^x (\pi - x + s) f(s)\,ds + \int_x^{2\pi} (-\pi - x + s) f(s)\,ds\right]$$

and (7.17). In the case $0 < \alpha < 1$ we observe that $I_x^\alpha f(x)$ exists almost everywhere [Hardy-Littlewood, 2, pp. 566—567]. On the other hand, by Lebesgue's theorem,

$$f_\alpha(x) = \frac{1}{2\pi\Gamma(\alpha)}\int_{-2\pi+x}^{x} f(s)\Psi_\alpha(x-s)\,ds$$

$$= \frac{1}{\Gamma(\alpha)}\int_{-2\pi+x}^{x} f(s)\,ds\,\lim_{n\to\infty}\left[\sum_{\nu=0}^{n-1}(x+2\pi\nu-s)^{\alpha-1} - \frac{1}{\alpha}(2\pi)^{\alpha-1}n^\alpha\right]$$

$$= \lim_{n\to\infty}\frac{1}{\Gamma(\alpha)}\int_{-2\pi+x}^{x} f(s)\,ds\sum_{\nu=0}^{n-1}(x+2\pi\nu-s)^{\alpha-1} = \lim_{n\to\infty}\frac{1}{\Gamma(\alpha)}\int_{-2n\pi+x}^{x} f(s)(x-s)^{\alpha-1}\,ds.$$

7.3. Throughout the remaining part of this section as well as in sections 8 and 9 we shall use for our fundamental orthonormal complete set $\{\varphi_\nu(x)\}$ the set of functions

$$e^{i\nu x},\quad \nu = \ldots, -2, -1, 0, 1, 2, \ldots;$$

and the fundamental interval will always be $(0, 2\pi)$. The fact that for this set the subscript ν ranges over $(-\infty, \infty)$ rather than over $(1, \infty)$ will not cause any trouble and it is hardly necessary to restate theorem 7.1 for such sets $\{\varphi_\nu(x)\}$.

We now introduce a new class of kernels, which is defined as follows:

A kernel $K(x, \xi) \subset (s, \alpha, p_1, p_2)$, *where* s *is any integer* $\geqq 0$; α, p_1, p_2 *are any real numbers such that* $0 < \alpha \leqq 1$ *and* $p_1, p_2 > 1$, *if, as a function of* x, $K(x, \xi)$ *possess for almost all* ξ *the partial derivatives*

$$D_x K(x, \xi), \ldots, D_x^s K(x, \xi),$$

and, in case $s > 0$, *the functions*

$$D_x^\nu K(x, \xi), \nu = 0, 1, \ldots, s-1; \; D_x^0 K \equiv K,$$

are continuous in x *on* $0 \leqq x \leqq 2\pi$ *for almost all* ξ, *while* $D_x^s K(x, \xi)$ *can be represented in the form*

$$(7.19) \qquad D_x^s K(x, \xi) = \begin{cases} \dfrac{1}{2\pi \Gamma(\alpha)} \displaystyle\int_0^{2\pi} G(t, \xi)\, \Psi_\alpha(x-t)\, dt \equiv G_\alpha(x, \xi), & a < 1, \\ \displaystyle\int_0^x G(t, \xi)\, dt + C(\xi), & \alpha = 1. \end{cases}$$

the function $G(x, \xi)$ *being such that the integral*

$$(7.20) \qquad I_{p_1, p_2}(G) \equiv \int_0^{2\pi} \left[\int_0^{2\pi} |G(x, \xi)|^{p_1} dx \right]^{p_2} d\xi$$

exists.

Theorem 7. 2. *If* $K(x, \xi) \subset L_2$ *and also* $K(x, \xi) \subset \left(s, \alpha, p, \frac{1}{p-1}\right)$ *with* $1 < p \leqq 2$, *then*[1]

$$(7.21) \qquad r_n(K)\, n^{-\left(s+\alpha+1-\frac{1}{p}\right)} \to \infty.$$

Proof. To avoid unnecessary repetitions we agree to consider only such values of ξ for which all the functions concerned are defined. We denote by p' the »conjugate» of p, defined by

[1] The assumption of the theorem concerning $G(x, \xi)$ reduces simply to $G \subset L_2$ when $p=2$. This case (with $s=0$, $\alpha=1$) contains as special cases the theorems of Weyl and Mazurkiewicz mentioned in section 1. 1.

$$\frac{1}{p}+\frac{1}{p'}=1,\ p'=\frac{p}{p-1},\ \frac{p'}{p}=\frac{1}{p-1},$$

and we notice that $p'\geqq 2$ when $p\leqq 2$.

We proceed to the proof of theorem 7.2 in the special case where, in addition to the hypotheses of the theorem, we have[1]

$$J_\nu(\xi;\,K)\equiv J_\nu(\xi)\equiv D_x^\nu K(2\pi,\,\xi)-D_x^\nu K(0,\,\xi)=0,\qquad \nu=0,\,1,\,\ldots,\,s. \tag{7.22}$$

On integrating by parts and using lemma 7.2 we see at once that for $\nu\neq 0$

$$\begin{aligned} k_\nu(\xi) &= \frac{1}{2\pi}\int_0^{2\pi} K(x,\,\xi)\,e^{-i\nu x}\,dx \\ &= \frac{(i\nu)^{-s}}{2\pi}\int_0^{2\pi} D_x^s\,K(x,\,\xi)\,e^{-i\nu x}\,dx=\frac{(i\nu)^{-s}}{2\pi}\int_0^{2\pi} G_\alpha(x,\,\xi)\,e^{-i\nu x}\,dx \\ &= \frac{(i\nu)^{-s}c_\nu}{\Gamma(\alpha)}\,\frac{1}{2\pi}\int_0^{2\pi} G(x,\,\xi)\,e^{-i\nu x}\,dx\equiv\frac{(i\nu)^{-s}c_\nu}{\Gamma(\alpha)}\,g_\nu(\xi), \end{aligned} \tag{7.23}$$

whence

$$|g_\nu(\xi)|=|\nu^{s+\alpha}\,k_\nu(\xi)|. \tag{7.24}$$

The existence of the integral

$$I_p(G)\equiv\int_0^{2\pi}\left[\int_0^{2\pi}|G(x,\,\xi)|^p\,dx\right]^{\frac{p'}{p}}d\xi=I_{p,\,\frac{1}{p-1}}(G) \tag{7.25}$$

implies that of the integral

$$\int_0^{2\pi}|G(x,\,\xi)|^p\,dx$$

for almost all ξ. For such values of ξ we can apply the Young-Hausdorff theorem [cf. Hardy-Littlewood, 1, p. 167] with the result

[1] Conditions $J_{s-1}=0$, $J_s=0$ follow from the hypotheses of the theorem when $\alpha<1$, for $\psi_\alpha(x)$ is periodic and of mean value zero over a period.

$$\sum_{\nu=-\infty}^{+\infty}{}' |\nu^{s+\alpha} k_\nu|^{p'} = \sum_{\nu=-\infty}^{+\infty}{}' |g_\nu(\xi)|^{p'} \leqq \left[\frac{1}{2\pi}\int_0^{2\pi} |G(x,\xi)|^p\, dx\right]^{\frac{p'}{p}},$$

the term with $\nu=0$ being omitted from the summation. The existence of (7.25) implies now that all the conditions of theorem 7.1 (with $\beta = p'(s+\alpha)$, $q = p'$) are satisfied in the present case.

To prove theorem 7.2 in its generality we shall treat the two cases $\alpha = 1$ and $0 < \alpha < 1$ separately.

Let $\alpha = 1$. By subtracting from $K(x,\xi)$ a kernel of finite rank it is always possible to obtain a kernel $H(x,\xi)$ that satisfies the set of conditions

$$(7.26) \qquad J_{-1}(\xi; H) \equiv \int_0^{2\pi} H(x,\xi)\, dx = 0; \; J_\nu(\xi;\, H) = 0, \quad \nu = 0, 1, \ldots, s.$$

Indeed, it suffices to put

$$H(x,\xi) = K(x,\xi) - \sum_{j=0}^{s+1} \frac{x^j}{j!}\,\omega_j(\xi).$$

This yields a set of $(s+2)$ equations

$$(7.27) \qquad 0 = J_\nu(\xi; H) = J_\nu(\xi; K) - \sum_{j=\nu+1}^{s+1} \frac{(2\pi)^{j-\nu}}{(j-\nu)!}\,\omega_j(\xi), \qquad \nu = -1, 0, 1, \ldots, s,$$

which determine $\omega_j(\xi)$ as a linear combination of $J_s(\xi; K), \ldots, J_{j-1}(\xi; K)$ with numerical coefficients. To this kernel $H(x,\xi)$ we can apply the preceding argument provided it is shown that $H(x,\xi) \subset L_2$ and that the integral $I_p(D_x^{s+1} H)$ exists. Using the abbreviated notation

$$\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p\, dx\right)^{\frac{1}{p}},$$

we have, by Minkowski's inequality,

$$\|f+g\|_p \leqq \|f\|_p + \|g\|_p.$$

With this notation we can write

$$(7.28)\quad I_p(D_x^{s+1}H) = \int_0^{2\pi} [\|G(x,\xi) - \omega_{s+1}(\xi)\|_p]^{p'}\, d\xi \leqq \int_0^{2\pi} [\|G\|_p + (2\pi)^{\frac{1}{p}} |\omega_{s+1}|]^{p'}\, d\xi.$$

But from (7.27)

$$\omega_{s+1}(\xi) = \frac{1}{2\pi}[D_x^s K(2\pi, \xi) - D_x^s K(0, \xi)] = \frac{1}{2\pi}\int_0^{2\pi} G(x, \xi)\, dx,$$

whence

$$|\omega_{s+1}(\xi)| \leqq (2\pi)^{-\frac{1}{p}} \|G\|_p.$$

On substituting into (7.28) we get

$$I_p(D_x^{s+1}H) \leqq 2^{p'} \int_0^{2\pi} (\|G\|_p)^{p'}\, d\xi = 2^{p'} I_p(G),$$

which shows the existence of $I_p(D_x^{s+1}H)$.

To prove that $H(x, \xi) \subset L_2$ we observe that its 0-th Fourier coefficent (as a function of x) is zero since $H(x, \xi)$ is of mean value zero over $(0, 2\pi)$. Since $H(x, \xi)$ satisfies conditions (7.26) the same argument as above in case of the kernel K will show the convergence of the series

$$\sum_{\nu=-\infty}^{+\infty} \int_0^{2\pi} |\nu^{s+1} h_\nu(\xi)|^{p'}\, d\xi.$$

The hypotheses of lemma 7.1 (with $\beta = p'(s+1)$, $q = p'$) being satisfied here, we have the desired result $H(x, \xi) \subset L_2$.

The kernel of finite rank

$$S_{s+2}(x, \xi) = \sum_{j=0}^{s+1} \frac{x^j}{j!}\omega_j(\xi) = K(x, \xi) - H(x, \xi) \subset L_2$$

since it is the difference of two kernels $\subset L_2$. This implies that all the functions $\omega_j(\xi) \subset L_2$.[1] Consequently we can apply theorem 4.2 with the rôles of K

[1] This is easily proved when we replace the powers by their expressions in terms of normalized Legendre polynomials for the interval $(0, 2\pi)$. We get then

and H interchanged, and q replaced by $(s+2)$. According to theorem 4.2 the set $\{\lambda_\nu(K)\}$ is a subset of the set of zeros of an entire function $B_m(\lambda)$ for which we have the estimate

$$|B_m(\lambda)| \leqq \{\Pi_m(r; H)\}^{s+3} P(r), \tag{7.29}$$

where $P(r)$ is a polynomial in r. The function $H(x, \xi)$ satisfies the hypotheses of theorem 7.1 with $\beta = p'(s+2)$, $q = p'$. Hence, by the proof of theorem 7.1,

$$\log \Pi_m(r; H) \leqq \varepsilon r^{\frac{p}{p(s+2)-1}}.$$

From (7.29) we obtain an estimate of the same type for

$$\log \max_{|\lambda|=r} |B_m(\lambda)| < \varepsilon r^{\frac{p}{p(s+2)-1}}, \quad m = \left[r^{\frac{p}{p(s+2)-1}}\right].$$

This leads to the desired result

$$r_n(K)\, n^{-\left(s+2-\frac{1}{p}\right)} \to \infty$$

by precisely the same argument as in the corresponding part of the proof of theorem 7.1.

The case $0 < \alpha < 1$ can be treated in an analogous, even simpler way. Here we have to satisfy only the conditions

$$J_\nu(\xi; H) = 0, \quad \nu = -1, 0, 1, \ldots, s-2,$$

which can be accomplished by setting

$$H(x, \xi) = K(x, \xi) - \sum_{j=0}^{s-1} \frac{x^j}{j!}\, \omega_j(\xi).$$

The details of the proof may be left to the reader.

$$S_{s+2}(x, \xi) = \sum_{j=0}^{s+1} p_j(x)\, \omega_j'(\xi),$$

where the $\omega_i'(\xi)$ are linearly independent linear combinations of the $\omega_j(\xi)$. The fact that $\omega_j'(\xi) \subset L_2$, hence that $\omega_j(\xi) \subset L_2$, follows then immediately from the relation

$$\|S_{s+2}\|^2 = \sum_{j=0}^{s+2} \int_0^{2\pi} |\omega_j'(\xi)|^2\, d\xi.$$

7.4. The assumption of the existence of the integral $I_p(G)$ in theorem 7.2 can be replaced by a less restrictive one, with corresponding modifications of the estimate for $r_n(K)$. This is shown by

Theorem 7.3. *Under the hypotheses*

i.
$$K(x, \xi) \subset L_2,$$

ii.
$$K(x, \xi) \subset \left(s, \alpha, p, \frac{q}{p}\right), \quad 1 < p \leqq 2,$$

iii.
$$2 \leqq q < p', \ \beta < (s+\alpha) p', \ \frac{p'(1+\beta)}{1+(s+\alpha)p'} < q < 2(1+\beta),$$

(*the last condition* $q < 2(1+\beta)$ *being unnecessary when* $s = 0$, $0 < \alpha < 1$) *we have*

(7.30)
$$r_n(K)\, n^{-\frac{\beta+1}{q}} \to \infty.$$

Proof. As in the proof of theorem 7.2 assume first that

(7.31)
$$J_\nu(\xi; K) = 0 \qquad (\nu = -1, 0, 1, \ldots, s),$$

these conditions being automatically satisfied when $s = 0$, $0 < \alpha < 1$. We shall prove that under the assumptions of theorem 7.3 and this additional assumption we have $K(x, \xi) \subset (\beta, q)$ which, in view of theorem 7.1, gives the desired result. By hypothesis the integral

$$I_{p, \frac{p}{q}}(G) = \int_0^{2\pi} (\|G\|_p)^q \, d\xi$$

exists, whence $\|G\|_p$ exists for almost all ξ. As in the proof of theorem 7.2 we have

$$\sum_{\nu=-\infty}^{+\infty} |\nu^{s+\alpha} k_\nu|^{p'} \leqq (2\pi)^{-\frac{p'}{p}} (\|G\|_p)^{p'},$$

whence

$$\sum_{\nu=n+1}^{n'} |k_\nu|^{p'} < C\, n^{-(s+\alpha)p'} (\|G\|_p)^{p'}, \qquad 0 \leqq n \leqq n',$$

and

$$\left[\sum_{\nu=n+1}^{n'} |k_\nu|^{p'}\right]^{\frac{q}{p'}} < C n^{-(s+\alpha)q} (\| G \|_p)^q,$$

$$\int_0^{2\pi} \left[\sum_{\nu=n+1}^{n'} |k_\nu|^{p'}\right]^{\frac{q}{p'}} d\xi < C n^{-(s+\alpha)q} I_{p, \frac{p}{q}}(G) = C n^{-(s+\alpha)q}.$$

Now, by Hölder's inequality,

$$\int_0^{2\pi} \left[\sum_{\nu=n+1}^{n'} \nu^\beta |k_\nu|^q\right] d\xi \leqq \left[\sum_{\nu=n+1}^{n'} \nu^{\frac{\beta p'}{p'-q}}\right]^{1-\frac{q}{p'}} \int_0^{2\pi} \left[\sum_{\nu=n+1}^{n'} |k_\nu|^{p'}\right]^{\frac{q}{p'}} d\xi$$

$$< C n^{-(s+\alpha)q} (n')^{\frac{\beta p' + p' - q}{p'}}.$$

On putting here in succession

$$n = 0,\ n' = 1;\ n = 2^\mu,\ n' = 2^{\mu+1},\ \mu = 0, 1, \ldots,$$

and adding the results we conclude that the series

$$\sum_{\nu=1}^{\infty} \nu^\beta \int_0^{2\pi} |k_\nu|^q \, d\xi$$

converges. In the same fashion we prove that the series

$$\sum_{\nu=-\infty}^{-1} |\nu|^\beta \int_0^{2\pi} |k_\nu|^q \, d\xi$$

converges. Since $k_0 = 0$ it follows that $K(x, \xi) \subset (\beta, q)$. It should be observed that the condition $q < 2(\beta + 1)$ is introduced to ensure that $H(x, \xi) \subset L_2$. The general case of theorem 7.3 where the conditions (7.31) are not satisfied can be treated in precisely the same manner as the corresponding case of theorem 7.2.

Remarks. i. The assumptions of theorem 7.3 are obviously less stringent than those of theorem 7.2, for $q < p'$.

ii. The estimates which we have obtained in this section are of the type ii. mentioned in section 1.3. The following examples show that the estimate of theorem 7.2 is the »best possible», not only in the sense that the exponent

$$s+\alpha+1-\frac{1}{p}$$

can not be replaced by any larger one, but also in the sense that our estimate can not be replaced by more precise estimates of the type iii. of 1.3.

(I) The function[1]

$$(7.32)\qquad F(t)\equiv F(t;\,a,\,b,\,c)\equiv\sum_{\nu=2}^{\infty}\nu^{-a}(\log\nu)^{-b}\,\exp\,\{2\pi i\nu\,[(\log\nu)^{c}+t]\}$$

is continuous and its expansion (7.32) converges uniformly provided either

$$a>\tfrac{1}{2},\ c>0,\ b\ \text{arbitrary},$$

or else,

$$a=\tfrac{1}{2},\ c>0,\ b>\tfrac{1}{2}(1+c).$$

Hence, on setting $a=\frac{1}{2}\,K(t)=F_{\frac{1}{2}}(t)$, we get a periodic kernel for which $s=0$, $\alpha=\frac{1}{2}$, $p=2$. We have here $r_n(K)=n\,(\log n)^{b}$ while the series $\Sigma\,[r_n(K)]^{-1}$ diverges if $b\leqq 1$ which is compatible with the conditions above if $c<1$.

(II) Take now the function $F(t)\equiv F_{a,b}(t)$ of section 2. On setting $K(t)=F_1(t)$ we get a periodic kernel with (in the notation of theorem 7.2) $s=0$, $\alpha=1$, $p=\dfrac{1}{1-a}$ provided $b>1-a$. Here we have $\varrho(K)=\dfrac{1}{1+a}$, but the series

$$\sum_{\nu=1}^{\infty}[r_\nu(K)]^{-\varrho}$$

diverges if $b\leqq 1+a$.

8. Kernels of Class Lip (s,α,p,q).

This class of kernels is defined as follows:

Let s be an integer $\geqq 0$, $0<\alpha<1$, $1<p\leqq 2\leqq q$. The kernel $K(x,\xi)\subset$ Lip (s,α,p,q) if, for almost all values of ξ, the partial derivatives

[1] Ingham [1], Hille [2, pp. 181—182].

$$D_x^\nu K(x, \xi), \ \nu = 1, 2, \ldots, s,$$

exist and, in case $s \geqq 1$, *the functions*

$$D_x^\nu K(x, \xi), \ \nu = 0, 1, \ldots, s-1,$$

are continuous in x *for fixed* ξ. *Furthermore, the derivative*

$$D_x^s K(x, \xi) \equiv K_s(x, \xi),$$

considered as a periodic function of x *outside the interval* $(0, 2\pi)$ *satisfies the condition*

$$\int_0^{2\pi} |K_s(x+t, \xi) - K_s(x, \xi)|^p \, dx < g(\xi)\, t^{\alpha p}, \tag{8.1}$$

where $g(\xi) \subset L_q$, *and* t *is* $\geqq 0$ *and sufficiently small.*[1]

Theorem 8.1. *If the kernel satisfies the conditions*

$$K(x, \xi) \subset L_2, \ K(x, \xi) \subset \mathrm{Lip}\left(s, \alpha, p, \frac{p'}{p}\right),$$

then

$$r_n(K) > C n^{s+\alpha+1-\frac{1}{p}} (\log n)^{-s-\alpha}. \tag{8.2}$$

Proof. Assume first that $s > 0$ and

$$J_\nu(\xi; K) \equiv D_x^\nu K(2\pi, \xi) - D_x^\nu K(0, \xi) = 0; \ \nu = 0, 1, \ldots, s-1. \tag{8.3}$$

As in the proof of theorem 7.1 we have[2]

$$\Pi_m(r; K) \leqq \prod_{i=-m}^{m} \left[1 - 2\,\Re(\lambda \varkappa_{ii}) + r^2 \int_0^{2\pi} |k_i|^2 \, d\xi\right]^{\frac{1}{2}} \exp\left\{\frac{r^2}{2} \sum_{i=m+1}^{\infty} \int_0^{2\pi} [|k_i|^2 + |k_{-i}|^2] \, d\xi\right\}. \tag{8.4}$$

---

[1] Conditions of this type were first considered by Szász [3; see also 4], for functions of a single variable. Our notation is analogous to that of Hardy-Littlewood [2]. Condition (L_p) of Szász [4, p. 531] follows from (8.1) by Minkowski's inequality.

[2] The slight difference between this formula and (7.5) is due to the fact that here the subscript ν ranges over $(-\infty, \infty)$ rather than over $(1, \infty)$.

Again as in the proof of theorem 7. 1

$$P_0 = \exp\left\{\frac{r^2}{2}\sum_{i=m+1}^{\infty}\int_0^{2\pi}[|k_i|^2 + |k_{-i}|^2]\,d\xi\right\}$$

$$\leqq C\,\exp\left\{C r^{p'}\sum_{i=m+1}^{\infty}\int_0^{2\pi}[|k_i|^{p'} + |k_{-i}|^{p'}]\,d\xi\right\}.$$

We put for simplicity

$$b_\nu \equiv b_\nu(\xi) \equiv \frac{1}{2\pi}\int_0^{2\pi} K_s(x, \xi)\, e^{-i\nu x}\, d\xi.$$

Then on integrating by parts we get

$$k_\nu = (i\nu)^{-s}\, b_\nu.$$

By an important result due to Szász [4, p. 533]

$$\sum_{\nu=m+1}^{\infty}[|b_\nu|^{p'} + |b_{-\nu}|^{p'}] < C[g(\xi)]^{\frac{p'}{p}} m^{-\alpha p'},$$

whence

(8. 5)
$$\sum_{\nu=m+1}^{\infty}\int_0^{2\pi}[|k_\nu|^{p'} + |k_{-\nu}|^{p'}]\,d\xi < C\, m^{-(\alpha+s)p'},$$

$$P_0 \leqq C\,\exp\{C r^{p'} m^{-(\alpha+s)p'}\}.$$

To estimate the first factor in the right-hand member of (8. 4) we observe that for a suitable choice of the constant C which will depend on K, we have

$$0 < 1 - 2\,\Re(\lambda \varkappa_{ii}) + r^2\int_0^{2\pi}|k_i|^2\,d\xi < C r^2,$$

whence

$$\prod_{i=-m}^{m}(\ldots) < C^{2m+1} r^{4m+2}.$$

This yields the final estimate

$$\log M(r) = \log \max_{|\lambda|=r} |A_m^*(\lambda; K)| < C\{m \log r + r^{p'} m^{-(\alpha+s)p'}\}.$$

To make the contributions of both terms here approximately equal we choose

$$m = \left[r^{\frac{p'}{(\alpha+s)p'+1}} (\log r)^{-\frac{1}{(\alpha+s)p'+1}} \right].$$

Then

$$\log M(r) < C r^{\frac{p}{(\alpha+s+1)p-1}} (\log r)^{\frac{(\alpha+s)p}{(\alpha+s+1)p-1}},$$

and (8. 2) is readily obtained, either directly from the formula (7. 11) or by using a classical formula of Lindelöf [1, p. 21] where we have to put

$$\varrho = \frac{p}{(\alpha+s+1)p-1}, \quad \alpha_1 = \frac{(\alpha+s)p}{(\alpha+s+1)p-1}.$$

The preceding proof holds without any modification in the case $s=0$, even when the condition $J_0(\xi; K) = 0$ is not satisfied. Hence in passing to the general case of theorem 8. 1 we can assume $s>0$. On setting

$$H(x, \xi) = K(x, \xi) - \sum_{j=0}^{s} \frac{x^j}{j!} \omega_j(\xi),$$

we can satisfy the conditions

$$(8.6) \qquad J_\nu(\xi; H) = 0 \qquad (\nu = -1, 0, 1, \ldots, s-1).$$

It is plain that (8. 1) will not change if $K(x, \xi)$ be replaced by $H(x, \xi)$. Consequently our preceding arguments hold for $H(x, \xi)$ provided it is proved that $H(x, \xi) \subset L_2$. To show this we observe that $H(x, \xi)$ as well as its derivatives $D_x H(x, \xi), \ldots, D_x^s H(x, \xi)$ are of mean value zero over $(0, 2\pi)$. On integrating by parts and applying the above mentioned result of Szász we get

$$\sum_{\nu=-\infty}^{\infty} |\nu^s k_\nu|^{p'} < C \int_0^{2\pi} [g(\xi)]^{\frac{p'}{p}} d\xi.$$

The desired result follows then by lemma 7. 1 since $2sp' + 2 > p'$.

The transition from the kernel $H(x, \xi)$ back to the kernel $K(x, \xi)$ can now be achieved in precisely the same fashion as in the proof of theorem 7. 2.

This will yield an estimate for $r_n(K)$ of the same type as in the special case above.

Certain limiting cases of theorem 8.1 are of interest. They present themselves when $\alpha = 0$, or $\alpha = 1$, or $p = 1$. We shall classify these cases as follows:

(1) $p > 1$, $\alpha = 0$, $s > 0$; (2) $p > 1$, $\alpha = 1$; (3) $p = 1$, $\alpha = 0$, $s > 0$; (4) $p = 1$, $\alpha = 1$.

In case (1) by modifying suitably a proof of Hardy-Littlewood [2, p. 566] we can show that the corresponding class of kernels $\subset \left(s-1,\ 1,\ p,\ \frac{p'}{p}\right)$ so that theorem 7.2 can be immediately applied here with the result

$$r_n(K)\, n^{-\left(s+1-\frac{1}{p}\right)} \to \infty. \tag{8.7}$$

In case (2) it is readily shown by imitating another proof of Hardy-Littlewood [2, pp. 599—600] that our kernel $\subset \left(s,\ 1,\ p,\ \frac{p'}{p}\right)$ so that

$$r_n(K)\, n^{-\left(s+2-\frac{1}{p}\right)} \to \infty. \tag{8.8}$$

In case (3) we see that $D_x^{s-1} K(x, \xi)$ is absolutely continuous in x for every fixed ξ which does not belong to an exceptional set of measure zero. Assume for simplicity that

$$\int_0^{2\pi} |K_s(x, \xi)|\, dx \subset L_2. \tag{8.9}$$

In this case it is readily proved that

$$\int_0^{2\pi} |b_\nu|^2\, d\xi = \frac{1}{4\pi^2} \int_0^{2\pi} d\xi \left| \int_0^{2\pi} K_s(x, \xi)\, e^{-i\nu x}\, dx \right|^2 \to 0 \text{ as } |\nu| \to \infty.$$

This follows immediately from Lebesgue's theorem since the integrand

$$\left| \int_0^{2\pi} K_s(x, \xi)\, e^{-i\nu x}\, dx \right|^2$$

tends to zero for almost all ξ and is dominated by a fixed integrable function

$$\left[\int_0^{2\pi} |K_s(x, \xi)|\, dx\right]^2.$$

Hence for a given ε there exists a positive integer m_0 so large that

$$\int_0^{2\pi} |b_\nu|^2\, d\xi < \varepsilon, \quad |\nu| > m_0.$$

Assuming again that conditions (8.6) are satisfied we see at once that

$$\sum_{\nu=m+1}^{\infty} \int_0^{2\pi} [|k_\nu|^2 + |k_{-\nu}|^2]\, d\xi < \varepsilon \sum_{\nu=m+1}^{\infty} \nu^{-2s} = \varepsilon\, m^{-2s+1}.$$

Repeating the same argument as in the corresponding case of theorem 8.1 and choosing

$$m = \left[r^{\frac{1}{s}} (\log r)^{-\frac{1}{2s}}\right]$$

we arrive at the conclusion

$$r_n(K)\, n^{-s} (\log n)^{s-\frac{1}{2}} \to \infty. \tag{8.10}$$

The case where conditions (8.6) are not satisfied can be treated in the same fashion as before, with the same estimate (8.10) for $r_n(K)$.

Finally, in case (4) the function $K_s(x, \xi)$ is of bounded variation in x [Hardy-Littlewood, 2, pp. 599—600] for every fixed ξ which does not belong to an exceptional set of measure zero. Let $V(\xi)$ be the total variation of $K_s(x, \xi)$ over $(0, 2\pi)$ and let $V(\xi) \subset L_2$. An easy application of the second law of the mean shows that

$$|b_\nu(\xi)| \leqq \frac{4}{|\nu|} V(\xi) \qquad (\nu \neq 0),$$

whence, again under the hypothesis (8.6),

$$\sum_{\nu=m+1}^{\infty} \int_0^{2\pi} [|k_\nu|^2 + |k_{-\nu}|^2]\, d\xi < C m^{-2s-1}.$$

The same argument as in the previous case (3) will show that

$$(8.11)\qquad r_n(K) > C n^{s+1} (\log n)^{-s-\frac{1}{2}}.$$

Remarks. i. By analogy with theorem 7.3 the condition of theorem 8.1 that $g(\xi) \subset L_{\frac{p'}{p}}$ can be replaced by a weaker one, viz.

$$g(\xi) \subset L_q,\ q \geqq 2,$$

with corresponding modifications in the estimate of $r_n(K)$.

ii. We have excluded from consideration the cases (1′) $p > 1$, $\alpha = s = 0$ and (3′) $p = 1$, $\alpha = s = 0$. These cases lead to interesting classes of kernels for which the integral

$$\int_0^{2\pi} \left[\int_0^{2\pi} |K(x, \xi)|^p \, dx \right]^{\frac{p'}{p}} d\xi,\quad 1 \leqq p \leqq 2,$$

exists, where the case $p = 1$ ($p' = \infty$) should be interpreted in the sense that

$$\int_0^{2\pi} |K(x, \xi)|^p \, dx$$

should be bounded. The case (3′) has been investigated (without discussion of the growth of the C.V.) in a recent paper [Hille-Tamarkin, 3], the case (1′) will be treated in a forthcoming paper by the present authors.

iii. The estimate of theorem 8.3 is more precise than an estimate of the type i. of 1.3 (with $\varrho = \gamma$) but less precise than an estimate of the type ii. Still our estimate is the »best possible» in the sense that it can not be replaced by an estimate of the type ii. This is shown by the example of the periodic kernel

$$K(t) = \sum_{\nu=2}^{\infty} \nu^{-1} \exp\ [i\nu(\log \nu + t)]$$

[Hardy-Littlewood 3, p. 632; Hille, 2]. This kernel satisfies uniformly a Lipschitz condition of order 1/2, whence $K(t) \subset \mathrm{Lip}\ (0, \frac{1}{2}, 2, 1)$. From theorem 8.1 we get $r_n > Cn(\log n)^{-\frac{1}{2}}$ while actually $r_n = n$.

The periodic kernels

$$\sum_{\nu=2}^{\infty} \frac{\sin \nu t}{\nu \log \nu}, \quad \sum_{\nu=1}^{\infty} \frac{\sin \nu t}{\nu}$$

can be used to illustrate the limiting cases (3) and (4) respectively, and to show that the values

$$\varrho(K) \leqq \frac{1}{s}, \quad \varrho(K) \leqq \frac{1}{s+1}$$

of the exponents of convergence can not be improved. It is very probable, however, that the presence of the logarithmic factors in the estimates (8. 2), (8. 10), (8. 11) is due to the imperfection of the method used, and that actually these factors should be removed or even replaced by logarithmic factors with exponents of opposite signs.

9. Kernels of Class $C(s, l, \alpha)$.

The result of Szász which was used in the preceding section is based on the theory of approximation in mean of a function $f(x)$ by the Fejér (or C_1) means of its Fourier series. Due to several recent investigations analogous results are available for the approximation in mean by the C_δ means. These investigations have been summarized and completed in a recent paper by M. Jacob [1] to which we refer for further bibliography of the subject. A scrutiny of the results and proofs of Jacob will show the truth of the following

Lemma 9. 1. *Let $G(x, \xi)$ be a function of two variables defined on the square $0 \leqq x, \xi \leqq 2\pi$ and extended periodically in each variable outside of it. Assume that:*

i. $G(x, \xi) \subset L_2$.

ii. *On setting*

$$(9.1) \qquad \begin{cases} g_0(x, \xi, t) = G(x+2t, \xi) + G(x-2t, \xi) - 2G(x, \xi); \\ g_i(x, \xi, t) = \int_0^t g_{i-1}(x, \xi, u)\, du, \quad i = 1, 2, \ldots, l, \ l > 0; \\ G_i(x, \xi, t) = i!\, t^{-i} g_i(x, \xi, t); \end{cases}$$

we have for almost all (x, ξ)

$$\int_0^\tau |G_l(x, \xi, t)|\, dt < \gamma_l(x, \xi; \tau)\, \tau^{1+\alpha} \qquad 0 \leqq \alpha \leqq 1, \tag{9.2}$$

where $\gamma_l(x, \xi; \tau) \subset L_2$ *for* $\tau > 0$, *and*

$$\|\gamma_l\|^2 \equiv \int_0^{2\pi}\int_0^{2\pi} |\gamma_l(x, \xi; \tau)|^2\, dx\, d\xi$$

is bounded as $\tau \to 0$.

Then the mean quadratic error of approximation of $G(x, \xi)$ *by the n-th* C_δ *mean of its Fourier series with respect to x can be estimated by*

$$\int_0^{2\pi}\int_0^{2\pi} |G(x, \xi) - C_\delta(s_n)|^2\, dx\, d\xi < \begin{cases} C n^{-2\alpha} \text{ if } \alpha < 1,\ \delta > l + \alpha, \\ C n^{-2} (\log n)^2 \text{ if } \alpha = 1,\ \delta = l + 1, \end{cases} \tag{9.3}$$

where

$$C_\delta(s_n) \equiv C_\delta[s_n(x, \xi)] \equiv \frac{1}{A_n^{(\delta)}} \sum_{\nu=-n}^{+n} A_{n-|\nu|}^{(\delta)} g_\nu(\xi)\, e^{i\nu x},\ g_\nu(\xi) = \frac{1}{2\pi}\int_0^{2\pi} G(x, \xi)\, e^{-i\nu x}\, dx,$$

and

$$A_n^{(\delta)} = \binom{n+\delta}{n}.$$

We now introduce the class $C(s, l, \alpha)$ of kernels which is defined as follows:

Let s and l be integers $\geqq 0$ *and* $0 \leqq \alpha \leqq 1$. *A kernel* $K(x, \xi) \subset C(s, l, \alpha)$ *if for almost all* ξ *the partial derivatives*

$$D_x^\nu K(x, \xi), \quad \nu = 1, 2, \ldots, s,$$

exist and, in case $s > 0$, *the derivatives*

$$D_x^\nu K(x, \xi), \quad \nu = 0, 1, \ldots, s-1,$$

are continuous in x for ξ *fixed, whereas the derivative*

$$D_x^s K(x, \xi) \equiv K_s(x, \xi) = G(x, \xi)$$

satisfies the conditions of lemma 9.1.

With this notation we have

Theorem 9.1. *If* $K(x, \xi) \subset L_2$ *and also* $K(x, \xi) \subset C(s, l, \alpha)$, *then*

$$r_n(K) > C n^{s+\alpha+\frac{1}{2}} (\log n)^{-s-\alpha} \qquad \text{when } 0 \leqq \alpha < 1, \tag{9.4}$$

and

$$r_n(K) > C n^{s+\frac{3}{2}} (\log n)^{-s-2} \qquad \text{when } \alpha = 1. \tag{9.5}$$

Proof. We start with the familiar assumption

$$J_\nu(\xi; K) = 0, \quad \nu = 0, 1, \ldots, s-1. \tag{9.6}$$

Then, since the polynomial

$$s_n(x, \xi) = \sum_{\nu=-n}^{n} g_\nu(\xi) e^{i\nu x}, \quad g_\nu(\xi) \equiv b_\nu(\xi),$$

gives the minimum mean quadratic error,

$$\sum_{\nu=m+1}^{\infty} \int_0^{2\pi} [|g_\nu|^2 + |g_{-\nu}|^2] \, d\xi = 2\pi \int_0^{2\pi}\int_0^{2\pi} |G(x, \xi) - s_m(x, \xi)|^2 \, dx \, d\xi$$

$$\leqq 2\pi \int_0^{2\pi}\int_0^{2\pi} |G(x, \xi) - C_\delta(s_m)|^2 \, dx \, d\xi.$$

Hence, when $0 \leqq \alpha < 1$, we get

$$\sum_{\nu=m+1}^{\infty} \int_0^{2\pi} [|k_\nu|^2 + |k_{-\nu}|^2] \, d\xi = \sum_{\nu=m+1}^{\infty} \nu^{-2s} \int_0^{2\pi} [|g_\nu|^2 + |g_{-\nu}|^2] \, d\xi < C m^{-2(s+\alpha)},$$

which is of the same type as (8.5) in the proof of theorem 8.1, with $p = 2$. This leads immediately to the result (9.4).

When $\alpha = 1$ we have instead

$$\sum_{\nu=m+1}^{\infty} \int_0^{2\pi} [|k_\nu|^2 + |k_{-\nu}|^2] \, d\xi < C m^{-2(s+1)} (\log m)^2,$$

whence

$$\log M(r) < C\{m \log r + r^2 m^{-2(s+1)} (\log m)^2\}.$$

A simple computation shows that the contributions of the two terms will be asymptotically the same if

$$m = \left[r^{\frac{2}{2s+3}} (\log r)^{\frac{1}{2s+3}} \right].$$

With this choice of m we get

$$\log M(r) < C r^{\frac{2}{2s+3}} (\log r)^{\frac{2s+4}{2s+3}}.$$

The result (9.5) then follows from (7.11) or from Lindelöf's formula [1, p. 21], mentioned above.

The general case of theorem 9.1 can be reduced to the special case above in precisely the same fashion as in the proof of theorem 8.1 and the remaining details may be left to the reader.

Remark. The case where $K_s(x, \xi)$ satisfies uniformly a Lipschitz condition of order α can be considered as a special case of theorems 9.1 and 8.1, with $p = 2$. It is well known [de la Vallée Poussin, 1, p. 52] that in this case $k_n = O(n^{-s-\alpha})$ so that, whereas Gheorghiu found $\varrho(K) \leqq \frac{1}{s+\alpha}$ [4, pp. 51—52], thorems 8.1 and 9.1 yield the estimate $\varrho(K) \leqq \frac{2}{2(s+\alpha)+1}$ which is the best possible of its kind.

10. Analytic Kernels.

In this section we shall assume that the fundamental interval (a, b) reduces to $(-1, 1)$ and that the complete orthonormal set $\{\varphi_\nu(x)\}$ coincides with the set of normalized Legendre polynomials so that

$$\varphi_\nu(x) = \sqrt{\frac{2\nu+1}{2}} P_\nu(x); \quad \nu = 0, 1, 2, \ldots.$$

Theorem 10.1. *Assume that $K(x, \xi)$ for almost all ξ is analytic in x in the interior of an ellipse in the complex x-plane, whose foci are at the points ± 1 and whose sum of semi-axes is R, and that for all such values of x,*

$$|K(x, \xi)| < M(\xi), \quad M(\xi) \subset L_2. \tag{10.1}$$

Then

$$r_n(K) > R^{\frac{(1-\varepsilon)n}{4}}. \tag{10.2}$$

Proof. We shall consider only »non-exceptional» values of ξ. It is well known [de la Vallée Poussin, 1, pp. 123—124] that, under the hypotheses of the theorem, $K(x, \xi)$ can be approximated by a polynomial in x, $Q_m(x, \xi)$, of degree $\leqq m$, such that

$$|K(x, \xi) - Q_m(x, \xi)| \leqq \frac{2\, M(\xi)}{R^m (R-1)}. \tag{10.3}$$

Since the m-th partial sum of the Legendre series of $K(x, \xi)$,

$$T_m(x, \xi) \equiv \sum_{\nu=1}^{m} k_\nu(\xi)\, \varphi_\nu(x),$$

gives the minimum mean quadratic error for a given m, it follows at once that

$$\sum_{\nu=m+1}^{\infty} \int_{-1}^{+1} |k_\nu|^2\, d\xi = \int_{-1}^{+1}\int_{-1}^{+1} |K(x, \xi) - T_m(x, \xi)|^2\, dx\, d\xi \leqq C_0 R^{-2m},$$

$$C_0 = \frac{4}{(R-1)^2} \int_{-1}^{+1} [M(\xi)]^2\, d\xi.$$

In view of (10.3) and (4.21) this gives an estimate of the form

$$\log \max_{|\lambda|=r} |A_m^*(\lambda; K)| \equiv \log M(r) < (1+\varepsilon)\left(m \log r + \frac{C_0}{2} r^2 R^{-2m}\right).$$

If we put here

$$m = \left[\frac{\log r}{\log R}\right],$$

we find after a simple computation

$$\log M(r) < (1+\varepsilon) \frac{(\log r)^2}{\log R}.$$

The desired result is now obtained either directly from (7.11) or by applying a formula of R. Mattson [1, p. 57].

Remark. The estimate furnished by theorem 10.1 is crude and is in a rather loose connection with the properties of the kernel. It can not be considerably improved, however, in the sense that the exponential function of n as a lower bound for the growth of the C.V. can not be replaced by a more rapidly increasing function. This is shown by the example of the periodic kernel

$$K(x-\xi)=\sum_{\nu=1}^{\infty} e^{-\nu} e^{i\nu(x-\xi)}$$

for which $R=1+\sqrt{2}$ so that, by theorem 10.1,

$$r_n(K)>(1+\sqrt{2})^{\frac{(1-\varepsilon)n}{4}},$$

while actually $r_n(K)=e^n$.

11. Entire Kernels.

We start with a discussion of kernels $K(x, \xi)$ which are entire functions in x of a finite order $\varrho>0$,[1] for almost all ξ.

For greater simplicity we assume that the fundamental interval reduces to $(0, 1)$ and that the set $\{\varphi_\nu(x)\}$ coincides with the set of normalized Legendre polynomials for the interval $(0, 1)$. We shall use the power series expansion

$$K(x, \xi)=\sum_{\nu=0}^{\infty} \varkappa_\nu(\xi) x^\nu.$$

If ξ is fixed, a necessary and sufficient condition that $K(x, \xi)$ be an entire function of order ϱ is that [Valiron, 2, p. 40]

$$\varliminf_{\nu\to\infty}\left(\frac{-\log|\varkappa_\nu(\xi)|}{\nu\log\nu}\right)=\frac{1}{\varrho}.$$

It is natural, therefore, to assume that

$$|\varkappa_n(\xi)|<\gamma(\xi)\exp(-\sigma n\log n),\ \sigma=\frac{1}{\varrho+\varepsilon}.$$

[1] Throughout this section the letter ϱ will be used to designate the order of $K(x, \xi)$ as an entire function in x. No confusion can arise with the symbol $\varrho(K)$ of our previous notation for the exponent of convergence of the set of C.V., since this exponent reduces to zero in sections 10 and 11.

We shall assume in addition that $\gamma(\xi) \subset L_2$. As an approximating polynomial we can now take

$$Q_m(x, \xi) \equiv \sum_{\nu=0}^{m} \varkappa_\nu(\xi) x^\nu.$$

It is readily found that the mean quadratic error of this approximation can be estimated by

$$\int_0^1 \int_0^1 |K(x, \xi) - Q_m(x, \xi)|^2 \, dx \, d\xi < C \exp(-2\sigma m \log m),$$

so that, for the same reason as in the proof of theorem 10. 1,

$$\sum_{\nu=m+1}^{\infty} \int_0^1 |k_\nu|^2 \, d\xi < C \exp(-2\sigma m \log m),$$

and

$$\log M(r) < (1+\varepsilon)\{m \log r + Cr^2 \exp(-2\sigma m \log m)\}.$$

Let μ be the solution of the equation (for fixed r)

$$Cr^2 \exp(-2\sigma\mu \log \mu) = 1.$$

It is readily seen that, for large values of r,

$$m = [\mu] + 1 \sim \frac{\log r}{\sigma \log_2 r}; \quad \log_2 r = \log \log r,$$

whence

$$\log M(r) < (\varrho + \varepsilon) \frac{(\log r)^2}{\log_2 r}.$$

By the formula of Mattson, mentioned above, we get

$$r_n(K) > \exp\left(\frac{1-\varepsilon}{4\varrho} n \log n\right).$$

Let us now turn to the case where $K(x, \xi)$ is an entire function in x of zero order. We may then write[1]

[1] This does not exhaust all the possible cases, but will suffice as an illustration of our method. The same remark should be made below, in connection with our treatment of the case where $K(x, \xi)$ is an entire function of infinite order.

$$|\varkappa_n(\xi)| < \gamma(\xi) \exp\{-n\, e_{k-1}(n^\sigma)\}, \qquad \sigma = \frac{1}{\tau+\varepsilon}, \quad \gamma(\xi) \subset L_2,$$

where k is a positive integer, and, as usual,

$$e_0(t) \equiv t, \ e_k(t) \equiv \exp(e_{k-1}(t)), \qquad k = \cdots, -2, -1, 0, 1, 2, \ldots$$
$$\log_k t \equiv e_{-k}(t).$$

We now have

$$\log M(r) < (1+\varepsilon)\{m \log r + Cr^2 \exp(-2\, m\, e_{k-1}(m^\sigma))\}.$$

We shall treat separately the two cases $k=1$ and $k>1$, assuming in each case that $m = [\mu]+1$, where μ is the solution of the equation

$$Cr^2 \exp\{-2\,\mu e_{k-1}(\mu^\sigma)\} = 1.$$

When $k=1$ we have

$$m \sim (\log r)^{\frac{1}{1+\sigma}},$$

$$\log M(r) \sim (1+\varepsilon)(\log r)^{\frac{\sigma+2}{\sigma+1}},$$

and, by Mattson's formula, since $\sigma = \frac{1}{\tau+\varepsilon}$,

$$r_n(K) > \exp\left\{\tau(\tau+1)^{\frac{\tau+1}{\tau}}(2\tau+1)^{-\frac{2\tau+1}{\tau}} n^{\frac{\tau+1}{\tau}-\varepsilon}\right\}.$$

When $k>1$ we have

$$m \sim (\log_k r)^{\frac{1}{\sigma}},$$

$$\log M(r) < (1+\varepsilon) \log r\, (\log_k r)^{\frac{1}{\sigma}},$$

whence

$$r_n(K) > e_k\left(n^{\frac{1}{\tau}-\varepsilon}\right)$$

There remains the case where $K(x, \xi)$ is an entire function in x of infinite order. Here we assume

$$|\varkappa_\nu(\xi)| < \gamma(\xi) \exp(-\sigma n \log_k n), \ k \geqq 2, \ \sigma = \frac{1}{\tau+\varepsilon},$$
$$\gamma(\xi) \subset L_2.$$

This gives

$$\log M(r) < (1+\varepsilon)\{m \log r + Cr^2 \exp(-2\sigma m \log_k m)\}$$

$$< (\tau+\varepsilon)\frac{(\log r)^2}{\log_{k+1} r}, \qquad m = [\mu] + 1,$$

where μ is the solution of the equation

$$Cr^2 \exp(-2\sigma\mu \log_k \mu) = 1.$$

In view of Mattson's formula this yields the result

$$r_n(K) > \exp\left(\frac{1-\varepsilon}{4\tau} n \log_k n\right).$$

The results of the preceding discussion can be summarized in the following

Theorem 11.1. *If $K(x, \xi)$ for almost all ξ is an entire function of x, given by the expansion*

(11.1) $$K(x, \xi) = \sum_{\nu=0}^{\infty} \varkappa_\nu(\xi) x^\nu,$$

then

(11.2) $$r_n(K) > \exp\left(\frac{1-\varepsilon}{4\tau} n \log_k n\right), \qquad k = 1, 2, \ldots;$$

(11.3) $$r_n(K) > \exp\left\{\tau(\tau+1)^{\frac{\tau+1}{\tau}}(2\tau+1)^{-\frac{2\tau+1}{\tau}} n^{\frac{\tau+1}{\tau}-\varepsilon}\right\};$$

(11.4) $$r_n(K) > e_k\left(n^{\frac{1}{\tau}-\varepsilon}\right), \qquad k = 2, 3, \ldots;$$

according as

(11.5) $$|\varkappa_n(\xi)| < \gamma(\xi) \exp\left(-\frac{1}{\tau+\varepsilon} n \log_k n\right);$$

(11.6) $$|\varkappa_n(\xi)| < \gamma(\xi) \exp\left(-n^{1+\frac{1}{\tau+\varepsilon}}\right);$$

(11.7) $$|\varkappa_n(\xi)| < \gamma(\xi) \exp\left\{-n e_{k-1}\left(n^{\frac{1}{\tau+\varepsilon}}\right)\right\};$$

where $\gamma(\xi) \subset L_2$.

Remarks. i. It is easily proved that if we can take $\varepsilon = 0$ in (11.6), (11.7) then formulas (11.3), (11.4) can be replaced by more precise ones, viz.

$$(11.8) \qquad r_n(K) > \exp\left\{(1-\varepsilon)\tau(\tau+1)^{\frac{\tau+1}{\tau}}(2\tau+1)^{-\frac{2\tau+1}{\tau}} n^{\frac{\tau+1}{\tau}}\right\},$$

$$(11.9) \qquad r_n(K) > e_k\left\{(1-\varepsilon)n^{\frac{1}{\tau}}\right\}.$$

ii. A curious example of an application of the formula (11.2) is presented by the kernel

$$K(x, \xi) = \sin 2\pi x\xi, \qquad 0 \leqq x, \xi \leqq 1.$$

This kernel is symmetric and closed since the equation

$$\int_0^1 \sin 2\pi x\xi\, \varphi(\xi)\, d\xi = 0, \qquad \varphi(x) \subset L_2,$$

whose left-hand member is analytic in x, implies for $x = n/2$

$$\int_0^1 \sin n\pi\xi\, \varphi(\xi)\, d\xi = 0,$$

whence $\varphi(x) \equiv 0$. Since our kernel is not of finite rank it must possess infinitely many C.V. [Hellinger-Toeplitz, 1, p. 1513]. On the other hand $K(x, \xi)$ is not definite since $K(x, x) = \sin 2\pi x^2$ changes the sign on (0, 1) [ibidem, p. 1510]. Hence not all the C.V. are of the same sign. On the basis of the expansion

$$K(x, \xi) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu (x\xi)^{2\nu+1}}{(2\nu+1)!}$$

and of theorem 11.1 we conclude at once that

$$r_n(K) > \exp\left(\frac{1-\varepsilon}{4} n \log n\right).$$

We are not aware of any previous proof of this result.

iii. An example where the kernel is an entire function of zero order is presented by

$$K(x, \xi) = \sum_{\nu=0}^{\infty} e^{-\nu^2} (x\xi)^\nu, \qquad 0 \leqq x,\ \xi \leqq 1.$$

Since

$$\int_0^1 \overline{u(x)}\, K \cdot u(x)\, dx = \sum_{\nu=0}^{\infty} e^{-\nu^2} \left| \int_0^1 x^\nu u(x)\, dx \right|^2 > 0$$

unless $u(x) \equiv 0$, our kernel is definite positive, hence closed. It has infinitely many C.V., all positive. By theorem 11.1 and remark i. we have

$$r_n(K) > \exp\left[(1-\varepsilon)\frac{4n^2}{27}\right].$$

The following lemma may be of use in discussing more general kernels of the type $K(x, \xi) = K(x\xi)$:

Lemma 11.1. *Let* $K(x, \xi) = K(x\xi)$, *where* $K(z)$ *is analytic in the circle* $|z| \leqq R$. *In order that* $K(x\xi)$ *be closed for the interval* (a, b), $0 \leqq a < b \leqq \sqrt{R}$, *with respect to functions* $\subset L_2$, *it is sufficient that in the power series expansion*

$$K(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$$

the coefficients a_{p_k} *are* $\neq 0$, $k = 1, 2, \ldots$, *where* $\sum_{k=1}^{\infty} (p_k)^{-1}$ *diverges.*

Remark. We can allow a to be < 0 if the sequence $\{p_k\}$ satisfies additional restrictions.

Proof. It is no restriction to assume $a = 0$. Let $u(x) \subset L_2$. It is well known that

$$\int_0^b x^\nu u(x)\, dx \to 0 \quad \text{as} \quad \nu \to \infty.$$

Hence $K \cdot u(x)$ is analytic at least in the circle $|x| \leqq R$. Now, if $K \cdot u(x) \equiv 0$, the coefficients of all powers of x in the expansion of $K \cdot u(x)$ must vanish. Since, by hypothesis, $a_{p_k} \neq 0$, we must have

$$\int_0^b \xi^{p_k} u(\xi)\, d\xi = 0, \qquad k = 1, 2, \ldots$$

$\sum_{k=1}^{\infty} (p_k)^{-1}$ being divergent, this implies $u(x) = 0$ for almost all x [cf. Szász, 2, p. 488; this paper gives extensive references to the literature on this question].

iv. The following two examples show that the estimate (11.2) for $k = 1$, that is in the case where $K(x, \xi)$ is an entire function of finite order, can not be improved, in the sense that the exponent $\frac{1-\varepsilon}{4\varrho} n \log n$ can not be replaced by a more rapidly increasing function. They also illustrate the fact that this exponent is inversely proportional to the order ϱ of $K(x, \xi)$. We take

$$K_1(x, \xi) = J_0(\sqrt{(1-x^2)(1-\xi^2)})\, e^{ix\xi} = \sqrt{2\pi} \sum_{n=0}^{\infty} i^n J_{n+\frac{1}{2}}(1)\, \varphi_n(x)\, \varphi_n(\xi)$$

[Watson, 1, p. 370] and

$$K_2(x, \xi) = J_0(\sqrt{(1-x)(1+\xi)})\, J_0(\sqrt{(1+x)(1-\xi)}) = 2 \sum_{n=0}^{\infty} J_{2n+1}(2)\, \varphi_n(x) \varphi_n(\xi)$$

[Bateman, 2, p. 135] where

$$\varphi_n(x) = \sqrt{\frac{2n+1}{2}}\, P_n(x), \quad n = 0, 1, 2, \ldots$$

are the normalized Legendre polynomials for the interval $(-1, 1)$.

It is readily seen that the first of these kernels is an entire function of order 1 while the second is of order 1/2. On the basis of theorem 11.1 and remark i. we should get the estimates

$$r_n(K_1) > \exp\left(\frac{1-\varepsilon}{4} n \log n\right), \quad r_n(K_2) > \exp\left(\frac{1-\varepsilon}{2} n \log n\right),$$

while actually we have

$$r_n(K_1) \sim \sqrt{2} \exp\,[(n+1) \log n - n(1 - \log 2)],$$

$$r_n(K_2) \sim 2\sqrt{\pi} \exp\left[\left(2n + \frac{3}{2}\right) \log n - 2n(1 - \log 2)\right].$$

12. Summary of the results.

For the convenience of the reader we collect in this section our main results concerning the growth of the C.V. for kernels of various classes. All the kernels in the table below are assumed to $\subset L_2$. We recall the definitions of the following classes of kernels:

Class L'_2, pp. 21, 26.

Class (β, q), pp. 21, 38.

Class (s, α, p_1, p_2), pp. 44, 46. This class can be characterized briefly as the class of kernels for which $D_x^s K(x, \xi)$ is an α-th integral.

Class Lip (s, α, p, q), pp. 53—54.

Class $C(s, l, \alpha)$, p. 61.

Limiting cases of Lip $\left(s, \alpha, p, \frac{1}{p-1}\right)$:

Lip (1) $[p > 1,\ \alpha = 0,\ s > 0] \subset \left(s-1, 1, p, \frac{1}{p-1}\right)$, p. 57.

Lip (2) $[p > 1,\ \alpha = 1,\ s \geqq 0] \subset \left(s, 1, p, \frac{1}{p-1}\right)$, p. 57.

Lip (3) $[p = 1,\ \alpha = 0,\ s > 0]$, $D_x^{s-1} K(x, \xi)$ is absolutely continuous in x, p. 57.

Lip (4) $[p = 1,\ \alpha = 1,\ s \geqq 0]$, $D_x^s K(x, \xi)$ is of bounded variation in x, p. 58.

We set

$$\sigma = s + \alpha + 1 - \frac{1}{p}; \quad \sigma_0 = s + 1 - \frac{1}{p}; \quad \sigma_1 = s + 2 - \frac{1}{p}.$$

	Properties of the kernel	Properties of the C.V.
1.	$K(x, \xi) \subset L_2$	Σr_n^{-2} converges
2.	$K(x, \xi) = (K_1 K_2)(x, \xi)$, $K_1, K_2 \subset L_2$	Σr_n^{-1} converges
3.	$K(x, \xi) \subset L'_2$, is Hermitian semi-definite	Σr_n^{-1} converges
4.	$K(x, \xi) \subset (\beta, q)$	$r_n n^{-\frac{\beta+1}{q}} \to \infty$

	Properties of the kernel	Properties of the C.V.
5.	$K(x, \xi) \subset \left(s, \alpha, p, \frac{1}{p-1}\right)$	$r_n n^{-\sigma} \to \infty$
6.	$K(x, \xi) \subset \mathrm{Lip}\left(s, \alpha, p, \frac{1}{p-1}\right)$	$r_n > C n^{\sigma} (\log n)^{-s-\alpha}$
7.	$K(x, \xi) \subset \mathrm{Lip}\ (1)$	$r_n n^{-\sigma_0} \to \infty$
8.	$K(x, \xi) \subset \mathrm{Lip}\ (2)$	$r_n n^{-\sigma_1} \to \infty$
9.	$K(x, \xi) \subset \mathrm{Lip}\ (3)$	$r_n n^{-s} (\log n)^{s+\frac{1}{2}} \to \infty$
10.	$K(x, \xi) \subset \mathrm{Lip}\ (4)$	$r_n > C n^{s+1} (\log n)^{-s-\frac{1}{2}}$
11.	$K(x, \xi) \subset C(s, l, \alpha)$ $0 \leqq \alpha < 1$	$r_n > C n^{s+\alpha+\frac{1}{2}} (\log n)^{-s-\alpha}$
	$\alpha = 1$	$r_n > C n^{s+\frac{3}{2}} (\log n)^{-s-2}$
12.	$K(x, \xi)$ is analytic in x in an ellipse with foci at $(-1, 1)$ and sum of semi-axes R, pp. 63—64.	$r_n > R^{\frac{1-\varepsilon}{4} n}$
13.	$K(x, \xi) = \sum_{\nu=0}^{\infty} \varkappa_\nu(\xi) x^\nu$, $\gamma(\xi) \subset L_2$, $\lvert\varkappa_\nu(\xi)\rvert < \gamma(\xi) \exp\left[-\frac{\nu}{\tau+\varepsilon} \log_k \nu\right]$	$r_n > \exp\left[\frac{1-\varepsilon}{4\tau} n \log_k n\right]$, $k = 1, 2, \ldots$
14.	$K(x, \xi) = \sum_{\nu=0}^{\infty} \varkappa_\nu(\xi) x^\nu$, $\gamma(\xi) \subset L_2$, $\lvert\varkappa_\nu(\xi)\rvert < \gamma(\xi) \exp\left[-\nu^{1+\frac{1}{\tau+\varepsilon}}\right]$	$r_n > \exp\left[\tau_0 n^{\frac{\tau+1}{\tau}-\varepsilon}\right]$, $\tau_0 = \tau(\tau+1)^{\frac{\tau+1}{\tau}} (2\tau+1)^{-\frac{2\tau+1}{\tau}}$
15.	$K(x, \xi) = \sum_{\nu=0}^{\infty} \varkappa_\nu(\xi) x^\nu$, $\gamma(\xi) \subset L_2$, $\lvert\varkappa_\nu(\xi)\rvert < \gamma(\xi) \exp\left[-\nu\, e_{k-1}\left(\nu^{\frac{1}{\tau+\varepsilon}}\right)\right]$	$r_n > e_k\left(n^{\frac{1}{\tau}-\varepsilon}\right)$, $k = 2, 3, \ldots$

List of References.

BATEMAN H. 1. A formula for the solving function of a certain integral equation of the second kind. Mess. of Math., 37 (1908) pp. 179—187.

——. 2. Notes on integral equations. IV. Ibidem, 39 (1910) pp. 129—135.

BÓBR, ST. 1. Eine Verallgemeinerung des v. Kochschen Satzes über die absolute Konvergenz der unendlichen Determinanten. Math. Z., 10 (1921) pp. 1—11.

CARLEMAN, T. 1. Über das Neumann-Poincarésche Problem für ein Gebiet mit Ekken. Inaugural-Dissertation. Uppsala, 1916. IV + 195 pp.

——. 2. Sur le genre du dénominateur $D(\lambda)$ de Fredholm. Arkiv f. Mat., Astr. och Fysik, 12 (1917) N° 15, 5 pp.

——. 3. Sur la distribution des valeurs singulières d'une classe des noyaux infinis. Ibidem, 13 (1918) N° 6, 7 pp.

——. 4. Über die Fourierkoeffizienten einer stetigen Funktion. Acta math., 41 (1918) pp. 377—384.

——. 5. Zur Theorie der linearen Integralgleichungen. Math. Z., 9 (1921) pp. 196—217.

FREDHOLM, I. 1. Sur une classe d'équations fonctionnelles. Acta math., 27 (1903) pp. 365—390.

GÂTEAUX, R. 1. Fonctions d'une infinité de variables indépendantes. Bull. Soc. Math. de France, 47 (1919) pp. 70—96.

GHEORGHIU, S. A. 1. Sur la croissance du dénominateur $D(\lambda)$ de Fredholm. C.R. 184 (1927) pp. 864—865.

——. 2. ——. Ibidem, pp. 1309—1311.

——. 3. Sur la croissance des dénominateurs $D(\lambda)$ de Fredholm. Ibidem, 186 (1928) pp. 838—840.

——. 4. Sur l'équation de Fredholm. Thèse. Paris, 1928. 70 + II pp.

HARDY, G. H. and LITTLEWOOD, J. E. 1. Some new properties of Fourier constants. Math. Ann. 97 (1927) pp. 159—209.

——. 2. Some properties of fractional integrals. I. Math. Z., 27 (1928) pp. 565—606.

——. 3. A convergence criterion for Fourier series. Ibidem, 28 (1928) pp. 612—634.

HELLINGER, E. und TOEPLITZ, O. 1. Integralgleichungen und Gleichungen mit unendlich vielen Unbekannten. Enc. der Math. Wiss., II. 3_2 (1923—1927) pp. 1335—1601.

HILBERT, D. 1. Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen. Leipzig, 1912, XXVI + 282 pp.; or Göttinger Nachr. (1904) pp. 49—91.

HILLE, E. 1. Note on the behavior of certain power series on the circle of convergence with application to a problem of Carleman. Proc. Nat. Ac. of Sc., 14 (1928) pp. 217—220.

HILLE, E. 2. Note on a power series considered by Hardy and Littlewood. Journ. Lond. Math. Soc., 4 (1929) pp. 176—182.

HILLE, E. and TAMARKIN, J. D. 1. Contributions to the theory of integral equations. Bull. Amer. Math. Soc., 34 (1928) p. 423 (abstract).

——. 2. On the characteristic values of linear integral equations. Proc. Nat. Ac. of Sc., 14 (1928) pp. 911—914.

——. 3. On the theory of linear integral equations. I. Ann. of Math., (2) 31 (1930) pp. 479—528.

HOBSON, E. W. 1. On the linear integral equation. Proc. Lond. Math. Soc., (2) 13 (1913—1914) pp. 307—340.

INGHAM, A. E. 1. Note on a certain power series. Ann. of Math., (2) 31 (1930) pp. 241—250.

JACOB, M. 1. Über die Summierbarkeit von Fourierschen Reihen und Integralen. Math. Z., 29 (1929) pp. 20—33.

KOCH, H. V. 1. Sur la convergence des déterminants infinis. Rend. Circ. Mat. Pal., 28 (1909) pp. 255—266.

——. 2. Sur un théorème de M. Hilbert. Math. Ann., 69 (1910) pp. 266—283.

LALESCO, T. 1. Introduction à la théorie des équations intégrales. Paris, 1912, VII + 152 pp.

——. 2. Un théorème sur les noyaux composés. Bull. de la Sect. sc. de l'Ac. Roumaine, 3 (1914—1915) pp. 271—272.

LINDELÖF, E. 1. Mémoire sur la théorie des fonctions entières de genre fini. Acta Soc. Sc. Fenn., 31 (1902) N° 1, 79 pp.

MARTY, J. 1. Transformation d'un déterminant infini en un déterminant de Fredholm. Bull. Sc. Math., (2) 33 (1909) pp. 296—300.

MATTSON, R. 1. Contributions à la théorie des fonctions entières. Thèse. Upsal, 1905, 97 pp.

MAZURKIEWICZ, ST. 1. Sur le déterminant de Fredholm. II. Les noyaux dérivables. C.R. des séances de la Soc. des Sc. de Varsovie, 8 (1915) pp. 805—810.

MOLLERUP, J. 1. La détermination du système orthogonal complet d'un noyau donné par les déterminants infinis de M. H. von Koch. Bull. Sci. Math., (2) 35 (1911) pp. 266—273.

——. 2. Sur l'identité du déterminant de Fredholm et d'un déterminant infini de von Koch. Ibidem, (2) 36 (1912) pp. 130—136.

——. 3. Om Identiteten af den Fredholmske Determinant og en uendelig v. Koch'sk Determinant. Beretning om den anden Skand. Mat.-kongres i Kjöbenhavn 1911, Copenhagen, 1912, pp. 81—87.

PLAS, H. M. 1. De functionaalverglijking van Fredholm, opgelost met behulp van cosinusreeksen. Diss. Groningen, 1911. XII + 113 pp.

RIESZ, F. 1. Les systèmes d'équations linéaires à une infinité d'inconnues. Paris, 1913. VI + 182 pp.

RIESZ, M. 1. Sur les maxima des formes bilinéaires et sur les fonctionnelles linéaires. Acta math., 49 (1926) pp. 465—497.

SCHMIDT, E. 1. Zur Theorie der linearen und nichtlinearen Integralgleichungen. I. Teil. Math. Ann. 63 (1907) pp. 433—476.

SCHUR, I. 1. Über die charakteristischen Wurzeln einer linearen Substitution mit einer Anwendung auf die Theorie der Integralgleichungen. Math. Ann., 66 (1909) pp. 488—510.

SZÁSZ, O. 1. A végtelen determinánsok elméletéhez. Math. és Phys. Lapok, 21 (1912) pp. 224—295.

——. 2. Über die Approximation stetiger Funktionen durch lineare Aggregate von Potenzen. Math. Ann., 77 (1916) pp. 482—496.

——. 3. Über den Konvergenzexponenten der Fourierschen Reihen gewisser Funktionenklassen. Sitz. Ber. Münch. Ak. Math.-Phys. Kl. 1922, pp. 135—150.

—— 4. Über die Fourierschen Reihen gewisser Funktionenklassen. Math. Ann., 100 (1928) pp. 530—536.

TAMARKIN, J. D. 1. On Fredholm's integral equations whose kernels are analytic in a parameter. Ann. of Math., (2) 28 (1927) pp. 127—152.

VALIRON, G. 1. Sur les fonctions entières d'ordre fini. Bull. Sc. Math., (2) 45 (1921) pp. 258—270.

——. 2. Lectures on the general theory of integral functions. Toulouse, 1923. XI + 208 pp.

——. 3. Fonctions entières et fonctions méromorphes d'une variable. Mém. des Sc. Math., II (1925) 58 pp.

VALLÉE POUSSIN, C. DE LA. 1. Leçons sur l'approximation des fonctions d'une variable réelle. Paris, 1919. VI + 151 pp.

WATSON, G. N. 1. The theory of Bessel functions. Cambridge, 1922. VIII + 804 pp.

WEYL, H. 1. Über die asymptotische Verteilung der Eigenwerte. Gött. Nachr., 1911, pp. 110—117.

——. 2. Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen. Math. Ann., 71 (1912) pp. 441—479.

——. 3. Bemerkungen zum Begriff des Differentialquotienten gebrochener Ordnung. Viertj. der Naturf. Ges. Zürich. 62 (1917) pp. 296—302.

YOUNG, H. W. 1. Sur la généralisation du théorème de Parseval. C.R., 155 (1912) pp. 30—33.

——. 2. On the Fourier series of bounded functions. Proc. Lond. Math. Soc., (2) 12 (1913) pp. 41—70.

——. 3. On the mode of oscillation of a Fourier series and of its allied series. Proc. Lond. Math. Soc., (2) 12 (1913) pp. 433—452.