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Global solutions of the gravity-capillary water-wave system in three dimensions

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1. Introduction

The study of the motion of water waves, such as those on the surface of the ocean, is a classical question, and one of the main problems in fluid dynamics. The origins of water-wave theory can be traced back(1) at least to the work of Laplace, Lagrange, Cauchy [11], Poisson, and then Russel, Green, and Airy, among others. Classical studies include those by Stokes [62], Levi-Civita [53], and Struik [63] on progressing waves, the instability analysis of Taylor [65], the works on solitary waves by Friedrichs and Hyers [31], and on steady waves by Gerber [32].

The main questions one can ask about water waves are the typical ones for any physical evolution problem: the local-in-time well-posedness of the Cauchy problem, the regularity of solutions and the formation of singularities, the existence of special solutions (such as solitary waves) and their stability, and the global existence and long-time behavior of solutions. There is a vast body of literature dedicated to all of these aspects. As it would be impossible to give exhaustive references, we will mostly mention works that are connected to our results, and refer to various books and review papers for others (see, e.g., [18], [26], [52], and [64]).

Our main interest here is the existence of global solutions for the initial value problem. In particular, we will consider the full irrotational water-wave problem for a 3-dimensional fluid occupying a region of infinite depth and infinite extent below the graph of a function. This is a model for the motion of waves on the surface of the deep ocean. We will consider such dynamics under the influence of the gravitational force and surface tension acting on particles at the interface. Our main result is the existence of global classical solutions for this problem, for sufficiently small initial data.

1.1. Free boundary Euler equations and water waves

The evolution of an inviscid perfect fluid that occupies a domain $\Omega_t\subset\mathbb{R}^n$, for $n\geqslant 2$, at time $t\in\mathbb{R}$, is described by the free-boundary incompressible Euler equations. If v and p denote respectively the velocity and the pressure of the fluid (with constant density equal to 1) at time t and position $x\in\Omega_t$, these equations are

$$(\partial_t+v\cdot\nabla)v=-\nabla p-ge_n,\quad \nabla\cdot v=0,\quad x\in\Omega_t, \tag{1.1}$$

where g is the gravitational constant. The first equation in (1.1) is the conservation of momentum equation, while the second one is the incompressibility condition. The

(1) We refer to the review paper of Craik [27], and references therein, for more details about these early studies.

free surface $S_t := \partial\Omega_t$ moves with the normal component of the velocity according to the following kinematic boundary condition:

$$\partial_t + v\cdot\nabla \text{ is tangent to } \bigcup_t S_t \subset \mathbb{R}^{n+1}_{x,t}. \tag{1.2}$$

The pressure on the interface is given by

$$p(x,t) = \sigma\varkappa(x,t), \quad x \in S_t, \tag{1.3}$$

where $\varkappa$ is the mean-curvature of S_t and $\sigma \geqslant 0$ is the surface tension coefficient. At liquid-air interfaces, the surface tension force results from the greater attraction of water molecules to each other, rather than to the molecules in the air.

One can also consider the free-boundary Euler equations (1.1)–(1.3) in various types of domains Ω_t (bounded, periodic, unbounded), and study flows with different characteristics (rotational/irrotational, with gravity and/or surface tension), or even more complicated scenarios where the moving interface separates two fluids.

In the case of irrotational flows, $\operatorname{curl} v = 0$, one can reduce (1.1)–(1.3) to a system on the boundary. Indeed, assume also that $\Omega_t \subset \mathbb{R}^n$ is the region below the graph of a function $h: \mathbb{R}^{n-1}_x \times I_t \to \mathbb{R}$, that is

$$\Omega_t = \{(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R} : y \leqslant h(x,t)\} \quad \text{and} \quad S_t = \{(x,y) : y = h(x,t)\}.$$

Let Φ denote the velocity potential, $\nabla_{x,y}\Phi(x,y,t) = v(x,y,t)$ for $(x,y) \in \Omega_t$. If

$$\phi(x,t) := \Phi(x, h(x,t), t)$$

is the restriction of Φ to the boundary S_t, the equations of motion reduce to the following system for the unknowns $h, \phi: \mathbb{R}^{n-1}_x \times I_t \to \mathbb{R}$:

$$\begin{cases} \partial_t h = G(h)\phi, \\ \partial_t \phi = -gh + \sigma \operatorname{div}\left(\dfrac{\nabla h}{(1+|\nabla h|^2)^{1/2}}\right) - \dfrac{1}{2}|\nabla\phi|^2 + \dfrac{(G(h)\phi + \nabla h\cdot\nabla\phi)^2}{2(1+|\nabla h|^2)}. \end{cases} \tag{1.4}$$

Here

$$G(h) := \sqrt{1+|\nabla h|^2}\,\mathcal{N}(h), \tag{1.5}$$

and $\mathcal{N}(h)$ is the Dirichlet–Neumann map associated with the domain Ω_t. Roughly speaking, one can think of $G(h)$ as a first-order, non-local, linear operator that depends nonlinearly on the domain. We refer to [64, Chapter 11] or the book of Lannes [52] for the derivation of (1.4). For sufficiently small smooth solutions, this system admits the conserved energy

$$\begin{aligned}\mathcal{H}(h,\phi) &:= \frac{1}{2}\int_{\mathbb{R}^{n-1}} G(h)\phi\cdot\phi\,dx + \frac{g}{2}\int_{\mathbb{R}^{n-1}} h^2\,dx + \sigma\int_{\mathbb{R}^{n-1}} \frac{|\nabla h|^2}{1+\sqrt{1+|\nabla h|^2}}\,dx \\ &\approx \big\||\nabla|^{1/2}\phi\big\|_{L^2}^2 + \|(g-\sigma\Delta)^{1/2}h\|_{L^2}^2,\end{aligned} \tag{1.6}$$

which is the sum of the kinetic energy corresponding to the L^2 norm of the velocity field and the potential energy due to gravity and surface tension. It was first observed by Zakharov [75] that (1.4) is the Hamiltonian flow associated with (1.6).

One generally refers to the system (1.4) as the gravity water-wave system when $g>0$ and $\sigma=0$, as the capillary water-wave system when $g=0$ and $\sigma>0$, and as the gravity-capillary water-wave system when $g>0$ and $\sigma>0$.

1.2. The main theorem

Our results in this paper concern the gravity-capillary water-wave system (1.4), in the case $n=3$. In this case, h and ϕ are real-valued functions defined on $\mathbb{R}^2\times I$.

To state our main theorem, we introduce some notation. The rotation vector field

$$\Omega := x_1\partial_{x_2} - x_2\partial_{x_1} \tag{1.7}$$

commutes with the linearized system. For $N\geqslant 0$ let H^N denote the standard Sobolev spaces on $\mathbb{R}^2$. More generally, for $N, N'\geqslant 0$ and $b\in\left[-\frac{1}{2},\frac{1}{2}\right]$, $b\leqslant N$, we define the norms

$$\|f\|_{H_\Omega^{N',N}} := \sum_{j\leqslant N'} \|\Omega^j f\|_{H^N} \quad\text{and}\quad \|f\|_{\dot{H}^{N,b}} := \big\|(|\nabla|^N + |\nabla|^b)f\big\|_{L^2}. \tag{1.8}$$

For simplicity of notation, we sometimes let $H_\Omega^{N'} := H_\Omega^{N',0}$. Our main theorem is the following.

Theorem 1.1. (Global regularity) *Let $g,\sigma>0$, let $\delta>0$ be sufficiently small, and N_0, N_1, N_3, and N_4 be sufficiently large*(2) *(for example $\delta=\frac{1}{2000}$, $N_0:=4170$, $N_1:=2070$, $N_3:=30$, and $N_4:=70$; cf. Definition 2.5). Assume that the data (h_0,ϕ_0) satisfies*

$$\begin{aligned}&\|\mathcal{U}_0\|_{H^{N_0}\cap H_\Omega^{N_1,N_3}} + \sup_{2m+|\alpha|\leqslant N_1+N_4} \big\|(1+|x|)^{1-50\delta} D^\alpha\Omega^m\mathcal{U}_0\big\|_{L^2} = \varepsilon_0 \leqslant \bar{\varepsilon}_0, \\ &\mathcal{U}_0 := (g-\sigma\Delta)^{1/2}h_0 + i|\nabla|^{1/2}\phi_0,\end{aligned} \tag{1.9}$$

(2) The values of N_0 and N_1, the total number of derivatives we assume under control, can certainly be decreased by reworking parts of the argument. We prefer, however, to simplify the argument wherever possible, instead of aiming for such improvements. For convenience, we arrange that

$$N_1 - N_4 = \frac{N_0-N_3}{2} - N_4 = \frac{1}{\delta}.$$

where $\bar{\varepsilon}_0$ *is a sufficiently small constant and* $D^\alpha = \partial_1^{\alpha^1}\partial_2^{\alpha^2}$, $\alpha = (\alpha^1, \alpha^2)$. *Then, there is a unique global solution* $(h, \phi) \in C([0,\infty) : H^{N_0+1} \times \dot{H}^{N_0+1/2,1/2})$ *of the system* (1.4), *with* $(h(0), \phi(0)) = (h_0, \phi_0)$. *In addition,*

$$(1+t)^{-\delta^2} \|\mathcal{U}(t)\|_{H^{N_0} \cap H_\Omega^{N_1,N_3}} \lesssim \varepsilon_0 \quad \textit{and} \quad (1+t)^{5/6-3\delta^2} \|\mathcal{U}(t)\|_{L^\infty} \lesssim \varepsilon_0, \tag{1.10}$$

for any $t \in [0,\infty)$, *where* $\mathcal{U} := (g - \sigma\Delta)^{1/2} h + i|\nabla|^{1/2}\phi$.

Remark 1.2. (i) One can derive additional information about the global solution (h, ϕ). Indeed, by rescaling, we may assume that $g=1$ and $\sigma=1$. Let

$$\mathcal{U}(t) := (1-\Delta)^{1/2} h + i|\nabla|^{1/2}\phi, \quad \mathcal{V}(t) := e^{it\Lambda}\mathcal{U}(t), \quad \text{and} \quad \Lambda(\xi) := \sqrt{|\xi| + |\xi|^3}. \tag{1.11}$$

Here, Λ is the linear dispersion relation and $\mathcal{V}$ is the profile of the solution $\mathcal{U}$. The proof of the theorem gives the strong uniform bound

$$\sup_{t\in[0,\infty)} \|\mathcal{V}(t)\|_Z \lesssim \varepsilon_0; \tag{1.12}$$

see Definition 2.5. The pointwise decay bound in (1.10) follows from this and the linear estimates in Lemma 7.5 below.

(ii) The global solution $\mathcal{U}$ scatters in the Z norm as $t \to \infty$, i.e. there is $\mathcal{V}_\infty \in Z$ such that

$$\lim_{t\to\infty} \|e^{it\Lambda}\mathcal{U}(t) - \mathcal{V}_\infty\|_Z = 0.$$

However, the asymptotic behavior is somewhat non-trivial since $|\widehat{\mathcal{U}}(\xi, t)| \gtrsim \log t \to \infty$ for frequencies ξ on a circle in $\mathbb{R}^2$ (the set of space-time resonance outputs) and for some data. This unusual behavior is due to the presence of a large set of space-time resonances.

(iii) The function

$$\mathcal{U} := (g - \sigma\Delta)^{1/2} h + i|\nabla|^{1/2}\phi$$

is called the "Hamiltonian variable", due to its connection to the Hamiltonian (1.6). This variable is important in order to keep track correctly of the relative Sobolev norms of the functions h and ϕ during the proof.

1.3. Background

We now discuss some background on the water-wave system and review some of the history and previous work on this problem.

1.3.1. The equations and the local well-posedness theory

The free-boundary Euler equations (1.1)–(1.3) are a time-reversible system of evolution equations which preserve the total (kinetic plus potential) energy. Under the Rayleigh–Taylor sign condition [65]

$$-\nabla_{n(x,t)}p(x,t)<0, \quad x\in S_t, \tag{1.13}$$

where n is the outward-pointing unit normal to Ω_t, the system has a (degenerate) hyperbolic structure. This structure is somewhat hard to capture because of the moving domain and the quasilinear nature of the problem. Historically, this has made the task of establishing local well-posedness (existence and uniqueness of smooth solutions for the Cauchy problem) non-trivial.

Early results on the local well-posedness of the system include those by Nalimov [55], Yosihara [74], Kano–Nishida [48], and Craig [22]; these results deal with small perturbations of a flat interface for which (1.13) always holds. It was first observed by Wu [71] that in the irrotational case the Rayleigh–Taylor sign condition holds without smallness assumptions, and that local-in-time solutions can be constructed with initial data of arbitrary size in Sobolev spaces [70], [71].

Following the breakthrough of Wu, in the recent years the question of local well-posedness of the water waves and free-boundary Euler equations has been addressed by several authors. Christodoulou–Lindblad [15] and Lindblad [54] considered the gravity problem with vorticity, Beyer–Gunther [9] took into account the effects of surface tension, and Lannes [51] treated the case of non-trivial bottom topography. Subsequent works by Coutand–Shkoller [20] and Shatah–Zeng [59], [60] extended these results to more general scenarios with vorticity and surface tension, including two-fluid systems [12], [60], where surface tension is necessary for well-posedness. Some recent papers that include surface tension and/or low regularity analysis are those by Ambrose–Masmoudi [8], Christianson–Hur–Staffilani [13], Alazard–Burq–Zuily [1], [2], and de Poyferré–Nguyen [56].

Thanks to all the contributions mentioned above, the local well-posedness theory is presently well-understood in a variety of different scenarios. In short, one can say that for sufficiently nice initial configurations, it is possible to find classical smooth solutions on a small time interval, which depends on the smoothness of the initial data.

1.3.2. Asymptotic models

We note that many simplified models have been derived and studied in special regimes, with the goal of understanding the complex dynamics of the water-wave system. These include the Korteweg–de Vries (KdV) equation, the Benjamin–Ono equation, and the Boussinesq and the Kadomtsev-Petviashvili (KP) equations, as well as the non-linear Schrödinger equation. We refer to [7], [19], [22]–[25], [57], [66] and to the book [52] and references therein for more about approximate models.

1.3.3. Previous work on long-time existence

The problem of long time existence of solutions is more challenging, and fewer results have been obtained so far. As in all quasilinear problems, the long-time regularity has been studied in a perturbative (and dispersive) setting, that is in the regime of small and localized perturbations of a flat interface. Large perturbations can lead to breakdown in finite time, see for example the papers on "splash" singularities [10], [21].

The first long-time result for the water-wave system (1.4) is due to Wu [72], who showed almost global existence for the gravity problem ($g>0$ and $\sigma=0$) in two dimensions (1-dimensional interfaces). Subsequently, Germain–Masmoudi–Shatah [34] and Wu [73] proved global existence of gravity waves in three dimensions (2-dimensional interfaces). Global regularity in three dimensions was also proved for the capillary problem ($g=0$ and $\sigma>0$) by Germain–Masmoudi–Shatah [35]. See also the recent work of Wang [67], [69] on the gravity problem in three dimensions over a finite flat bottom.

Global regularity for the gravity water-wave system in two dimensions (the harder case) has been proved by two of the authors in [44] and, independently, by Alazard–Delort [3], [4]. A different proof of Wu's 2-dimensional almost global existence result was later given by Hunter–Ifrim–Tataru [38], and then complemented to a proof of global regularity in [39]. Finally, Wang [68] proved global regularity for a more general class of small data of infinite energy, thus removing the momentum condition on the velocity field that was present in all the previous 2-dimensional results. For the capillary problem in two dimensions, global regularity was proved by two of the authors in [46] and, independently, by Ifrim–Tataru [40] in the case of data satisfying an additional momentum condition.

We remark that all the global regularity results that have been proved so far require three basic assumptions: small data (small perturbations of the rest solution), trivial vorticity inside the fluid, and flat Euclidean geometry. Additional properties are also important, such as the Hamiltonian structure of the equations, the rate of decay of the linearized waves, and the resonance structure of the bilinear wave interactions.

1.4. Main ideas

The classical mechanism to establish global regularity for quasilinear equations has two main components:

(1) propagate control of high frequencies (high-order Sobolev norms);

(2) prove dispersion/decay of the solution over time.

The interplay of these two aspects has been present since the seminal work of Klainerman [49], [50] on non-linear wave equations and vector fields, Shatah [58] on Klein–Gordon and normal forms, Christodoulou–Klainerman [14] on the stability of Minkowski space, and Delort [28] on 1-dimensional Klein–Gordon equations. We remark that, even in the weakly non-linear regime (small perturbations of trivial solutions), smooth and localized initial data can lead to blow-up in finite time, see John [47] on quasilinear wave equations and Sideris [61] on compressible Euler equations.

In the last few years, new methods have emerged in the study of global solutions of quasilinear evolutions, inspired by the advances in semilinear theory. The basic idea is to combine the classical energy and vector-field methods with refined analysis of the Duhamel formula, using the Fourier transform. This is the essence of the "method of space-time resonances" of Germain–Masmoudi–Shatah [33]–[35], see also Gustafson–Nakanishi–Tsai [37], and of the refinements in [29], [30], [36], [41]– [46], using atomic decompositions and more sophisticated norms.

The situation we consider in this paper is substantially more difficult, due to the combination of the following factors:

• Strictly less than $|t|^{-1}$ pointwise decay of solutions. In our case, the dispersion relation is $\Lambda(\xi)=\sqrt{g|\xi|+\sigma|\xi|^3}$, and the best possible pointwise decay, even for solutions of the linearized equation corresponding to Schwartz data, is $|t|^{-5/6}$ (see Figure 1 below).

• Large set of time resonances. In certain cases, one can overcome the slow pointwise decay using the method of normal forms of Shatah [58]. The critical ingredient needed is the absence of time resonances (or at least a suitable "null structure" of the quadratic non-linearity matching the set of time resonances). Our system, however, has a full (codimension-1) set of time resonances (see Figure 2 below) and no meaningful null structures.

We remark that all the previous work on long-term solutions of water-wave models was under the assumption that either $g=0$ or $\sigma=0$. This is not coincidental: in these cases *the combination of slow decay and full set of time resonances was not present*. More precisely, in all the previous global results in three dimensions in [34], [35], [67], [69], [73] it was possible to prove $1/t$ pointwise decay of the non-linear solutions and combine this with high-order energy estimates with slow growth.

On the other hand, in all the 2-dimensional models analyzed in [3], [4], [38]–[40], [44], [46], [68], [72] there were no significant time resonances for the quadratic terms.([3]) As a result, in all of these papers it was possible to prove a *quartic energy inequality* of the form

$$|\mathcal{E}_N(t)-\mathcal{E}_N(0)|\lesssim\int_0^t \mathcal{E}_N(s)\|U(s)\|^2_{W^{N/2+4,\infty}}\,ds,$$

for a suitable functional $\mathcal{E}_N(t)$ satisfying $\mathcal{E}_N(t)\approx\|U(t)\|^2_{H^N}$. The point is to get two factors of $\|U(s)\|_{W^{N/2+4,\infty}}$ in the right-hand side, in order to have suitable decay, and simultaneously avoid loss of derivatives. A quartic energy inequality of this form cannot hold in our case, due to the presence of large resonant sets.

To address these issues, in this paper we use a combination of improved energy estimates and Fourier analysis. The main components of our analysis are the following:

• The energy estimates, which are used to control high Sobolev norms and weighted norms (corresponding to the rotation vector field). They rely on several new ingredients, most importantly on a *strongly semilinear structure* of the space-time integrals that control the increment of energy, and on a *restricted non-degeneracy condition* (see (1.24)) of the time resonant hypersurfaces. The strongly semilinear structure is due to an algebraic correlation (see (1.28)) between the size of the multipliers of the space-time integrals and the size of the modulation, and is related to the Hamiltonian structure of the original system.

• The dispersive estimates, which lead to decay and rely on a partial bootstrap argument in a suitable Z *norm*. We analyze carefully the Duhamel formula, in particular the quadratic interactions related to the slowly decaying frequencies and to the *set of space-time resonances*. The choice of the Z norm in this argument is very important; we use an atomic norm, based on a space-frequency decomposition of the profile of the solution, which depends in a significant way on the location and the shape of the space-time resonant set, thus on the quadratic part of the non-linearity.

We hope that such ideas can be used in other quasilinear problems in two and three dimensions (such as other fluid and plasma models) that involve large resonant sets and slowly decaying solutions. We illustrate some of these main ideas in a simplified model below.

([3]) More precisely, the only time resonances are at the zero frequency, but they are canceled by a suitable null structure. Some additional ideas are needed in the case of capillary waves [46] where certain singularities arise. Moreover, new ideas, which exploit the Hamiltonian structure of the system as in [44], are needed to prove global (as opposed to almost global) regularity.

1.5. A simplified model

To illustrate these ideas, consider the initial-value problem

$$\begin{aligned}(\partial_t+i\Lambda)U &= \nabla V\cdot\nabla U+\tfrac{1}{2}\Delta V\cdot U, \quad U(0)=U_0,\\ \Lambda(\xi) &:= \sqrt{|\xi|+|\xi|^3}, \quad V:=P_{[-10,10]}\,\mathrm{Re}\,U.\end{aligned} \tag{1.14}$$

Compared to the full equation, this model has the same linear part and a quadratic non-linearity leading to similar resonant sets. It is important that V is real-valued, such that solutions of (1.14) satisfy the L^2 conservation law

$$\|U(t)\|_{L^2}=\|U_0\|_{L^2}, \quad t\in[0,\infty). \tag{1.15}$$

The model (1.14) carries many of the difficulties of the real problem and has the advantage that it is much more transparent algebraically. There are, however, significant additional issues when dealing with the full problem; see §1.5.3 below for a short discussion.

The specific dispersion relation $\Lambda(\xi)=\sqrt{|\xi|+|\xi|^3}$ in (1.14) is important. It is radial and has stationary points when $|\xi|=\gamma_0:=(2/\sqrt{3}-1)^{1/2}\approx 0.393$ (see Figure 1 below). As a result, linear solutions can only have $|t|^{-5/6}$ pointwise decay, i.e.

$$\|e^{it\Lambda}\phi\|_{L^\infty}\approx|t|^{-5/6},$$

even for Schwartz functions ϕ whose Fourier transforms do not vanish on the sphere $\{\xi:|\xi|=\gamma_0\}$.

1.5.1. Energy estimates

We would like to control the increment of both high-order Sobolev norms and weighted norms for solutions of (1.14). It is convenient to do all the estimates in the Fourier space, using a quasilinear I-method, as in some of our earlier work. This has similarities with the well-known I-method of Colliander–Keel–Staffilani–Takaoka–Tao [16], [17] used in semilinear problems, and to the energy methods of [3], [4], [33], [38]. Our main estimate is the following partial bootstrap bound:

$$\text{if } \sup_{t\in[0,T]}\big((1+t)^{-\delta^2}\mathcal{E}(t)^{1/2}+\|e^{it\Lambda}U(t)\|_Z\big)\leqslant\varepsilon_1, \text{ then } \sup_{t\in[0,T]}(1+t)^{-\delta^2}\mathcal{E}(t)^{1/2}\lesssim\varepsilon_0+\varepsilon_1^{3/2}, \tag{1.16}$$

where U is a solution on $[0,T]$ of (1.14),

$$\mathcal{E}(t)=\|U(t)\|_{H^N}^2+\|U(t)\|_{H^{N'}_\Omega}^2,$$

Dispersion relation and degenerate frequencies

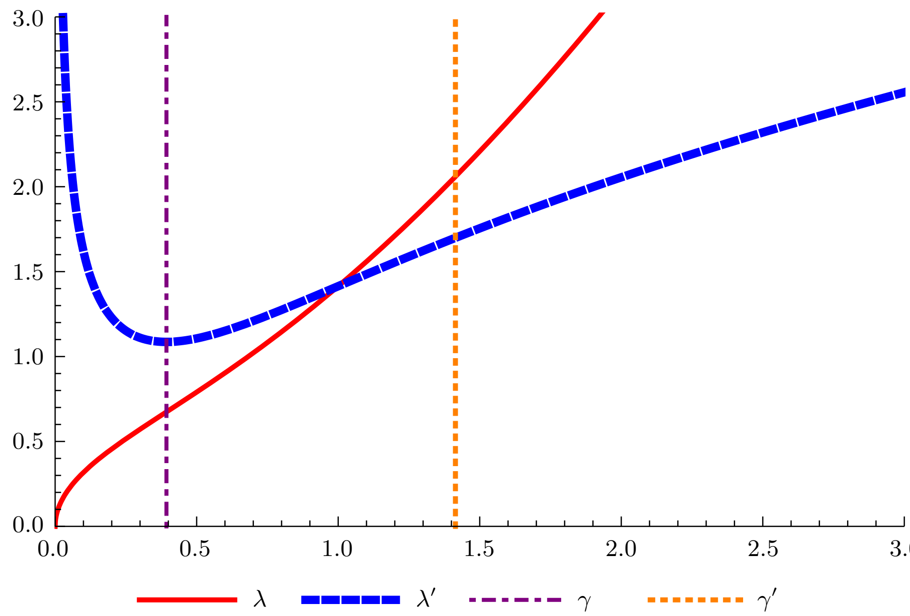

Figure 1. The curves represent the dispersion relation $\lambda(r)=\sqrt{r^3+r}$ and the group velocity λ', for $g=1=\sigma$. Notice that $\lambda''(r)$ vanishes at $r=\gamma_0\approx 0.393$. The frequency $\gamma_1=\sqrt{2}$ corresponds to the sphere of space-time resonant outputs. Notice that while the slower decay at γ_0 is due to some degeneracy in the linear problem, γ_1 is unremarkable from the point of view of the linear dispersion.

and the initial data has small size $\sqrt{\mathcal{E}(0)}+\|U(0)\|_Z\leqslant\varepsilon_0$. The Z norm is important and will be discussed in detail in the next subsection. For simplicity, we focus on the high-order Sobolev norms, and divide the argument into four steps.

Step 1. For N sufficiently large, let

$$W:=W_N:=\langle\nabla\rangle^N U \quad \text{and} \quad E_N(t):=\int_{\mathbb{R}^2}|\widehat{W}(\xi,t)|^2\,d\xi. \tag{1.17}$$

A simple calculation, using the equation and the fact that V is real, shows that

$$\frac{d}{dt}E_N=\int_{\mathbb{R}^2\times\mathbb{R}^2} m(\xi,\eta)\widehat{W}(\eta)\widehat{\overline{W}}(-\xi)\widehat{V}(\xi-\eta)\,d\xi\,d\eta, \tag{1.18}$$

where

$$m(\xi,\eta)=\frac{(\xi-\eta)\cdot(\xi+\eta)}{2}\,\frac{(1+|\eta|^2)^N-(1+|\xi|^2)^N}{(1+|\eta|^2)^{N/2}(1+|\xi|^2)^{N/2}}. \tag{1.19}$$

Notice that $|\xi-\eta|\in[2^{-11},2^{11}]$ in the support of the integral, due to the Littlewood–Paley operator in the definition of V. We notice that m satisfies

$$m(\xi,\eta)=\mathfrak{d}(\xi,\eta)m'(\xi,\eta), \quad \text{where } \mathfrak{d}(\xi,\eta):=\frac{[(\xi-\eta)\cdot(\xi+\eta)]^2}{1+|\xi+\eta|^2} \text{ and } m'\approx 1. \tag{1.20}$$

The *depletion factor* $\mathfrak{d}$ is important in establishing energy estimates, due to its correlation with the modulation function Φ (see (1.28) below). The presence of this factor is related to the exact conservation law (1.15).

Step 2. We would like to estimate now the increment of $E_N(t)$. We use (1.18) and consider only the main case, when $|\xi|, |\eta| \approx 2^k \gg 1$ and $|\xi-\eta|$ is close to the slowly decaying frequency γ_0. So, we need to bound space-time integrals of the form

$$I := \int_0^t \int_{\mathbb{R}^2\times\mathbb{R}^2} m(\xi,\eta)\widehat{P_k W}(\eta,s)\widehat{P_k \overline{W}}(-\xi,s)\widehat{U}(\xi-\eta,s)\chi_{\gamma_0}(\xi-\eta)\,d\xi\,d\eta\,ds,$$

where χ_{γ_0} is a smooth cutoff function supported in the set $\{\xi:\big||\xi|-\gamma_0\big|\ll 1\}$, and we replaced V by U (replacing V by $\overline{U}$ leads to a similar calculation). Notice that it is not possible to estimate $|I|$ by moving the absolute value inside the time integral, due to the slow decay of U in L^∞. So we need to integrate by parts in time; for this, define the profiles

$$u(t) := e^{it\Lambda}U(t) \quad \text{and} \quad w(t) := e^{it\Lambda}W(t). \tag{1.21}$$

Then, decompose the integral in dyadic pieces over the size of the modulation and over the size of the time variable. In terms of the profiles u and w, we need to consider the space-time integrals

$$\begin{aligned} I_{k,m,p} := \int_{\mathbb{R}} q_m(s) \int_{\mathbb{R}^2\times\mathbb{R}^2} & e^{is\Phi(\xi,\eta)} m(\xi,\eta)\widehat{P_k w}(\eta,s)\widehat{P_k \overline{w}}(-\xi,s) \\ & \times \hat{u}(\xi-\eta,s)\chi_{\gamma_0}(\xi-\eta)\varphi_p(\Phi(\xi,\eta))\,d\xi\,d\eta\,ds, \end{aligned} \tag{1.22}$$

where $\Phi(\xi,\eta):=\Lambda(\xi)-\Lambda(\eta)-\Lambda(\xi-\eta)$ is the associated modulation, q_m is smooth and supported in the set $\{s:s\approx 2^m\}$, and φ_p is supported in the set $\{x:|x|\approx 2^p\}$.

Step 3. To estimate the integrals $I_{k,m,p}$, we consider several cases depending on the relative size of k, m, and p. Assume that k and m are large, i.e. $2^k \gg 1$ and $2^m \gg 1$, which is the harder case. To deal with the case of small modulation, when one cannot integrate by parts in time, we need an L^2 bound on the Fourier integral operator

$$T_{k,m,p}(f)(\xi) := \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)}\varphi_k(\xi)\varphi_{\leqslant p}(\Phi(\xi,\eta))\chi_{\gamma_0}(\xi-\eta)f(\eta)\,d\eta,$$

where $s\approx 2^m$ is fixed. The critical bound we prove in Lemma 4.7 ("the main L^2 lemma") is

$$\|T_{k,m,p}(f)\|_{L^2} \lesssim_\varepsilon 2^{\varepsilon m}(2^{(3/2)(p-k/2)}+2^{p-k/2-m/3})\|f\|_{L^2}, \quad \varepsilon>0, \tag{1.23}$$

provided $p-\frac{1}{2}k \in [-0.99m, -0.01m]$. The main gain here is the factor $\frac{3}{2}$ in $2^{(3/2)(p-k/2)}$ in the right-hand side (Schur's test would only give a factor 1).

The proof of (1.23) uses a TT^* argument, which is a standard tool to prove L^2 bounds for Fourier integral operators. This argument depends on a key non-degeneracy property of the function Φ, more precisely on what we call the *restricted non-degeneracy condition*:

$$\Upsilon(\xi,\eta)=\nabla^2_{\xi,\eta}\Phi(\xi,\eta)[\nabla^\perp_\xi\Phi(\xi,\eta),\nabla^\perp_\eta\Phi(\xi,\eta)]\neq 0,\quad \text{if } \Phi(\xi,\eta)=0. \tag{1.24}$$

This condition, which appears to be new, can be verified explicitly in our case, when $\big||\xi-\eta|-\gamma_0\big|\ll 1$. The function Υ does in fact vanish at two points on the resonant set $\{\eta:\Phi(\xi,\eta)=0\}$ (where $\big||\xi-\eta|-\gamma_0\big|\approx 2^{-k}$), but our argument can tolerate vanishing up to order 1.

The non-degeneracy condition (1.24) can be interpreted geometrically: the non-degeneracy of the mixed Hessian of Φ is a standard condition that leads to optimal L^2 bounds on Fourier integral operators. In our case, however, we have the additional cutoff function $\varphi_{\leqslant p}(\Phi(\xi,\eta))$, so we can only integrate by parts in the directions tangent to the level sets of Φ. This explains the additional restriction to these directions in the definition of Υ in (1.24).

Given the bound (1.23), we can control the contribution of small modulations, i.e.

$$p-\tfrac{1}{2}k\leqslant-\tfrac{2}{3}m-\varepsilon m. \tag{1.25}$$

Step 4. In the high-modulation case, we integrate by parts in time in formula (1.22). The main contribution is when the time derivative hits the high-frequency terms, so we focus on estimating the resulting integral

$$\begin{aligned}I'_{k,m,p}:=\int_{\mathbb{R}}q_m(s)\int_{\mathbb{R}^2\times\mathbb{R}^2}&e^{is\Phi(\xi,\eta)}m(\xi,\eta)\frac{d}{ds}(\widehat{P_kw}(\eta,s)\widehat{P_k\overline{w}}(-\xi,s))\\&\times\hat{u}(\xi-\eta,s)\chi_{\gamma_0}(\xi-\eta)\frac{\varphi_p(\Phi(\xi,\eta))}{\Phi(\xi,\eta)}\,d\xi\,d\eta\,ds.\end{aligned} \tag{1.26}$$

Notice that $\partial_t w$ satisfies the equation

$$\partial_t w=\langle\nabla\rangle^N e^{it\Lambda}\big(\nabla V\cdot\nabla U+\tfrac{1}{2}\Delta V\cdot U\big). \tag{1.27}$$

The right-hand side of (1.27) is quadratic. We thus see that replacing w by $\partial_t w$ essentially gains a unit of decay (which is $|t|^{-5/6+}$), but loses a derivative. This causes a problem in some range of parameters, for example when $2^p\approx 2^{k/2-2m/3}$ and $1\ll 2^k\ll 2^m$; cf. (1.25).

We then consider two cases: if the modulation is sufficiently small, then we can use the depletion factor $\mathfrak{d}$ in the multiplier m (see (1.20)), and the following key algebraic correlation:

$$\text{if } |\Phi(\xi,\eta)|\lesssim 1, \text{ then } |m(\xi,\eta)|\lesssim 2^{-k}. \tag{1.28}$$

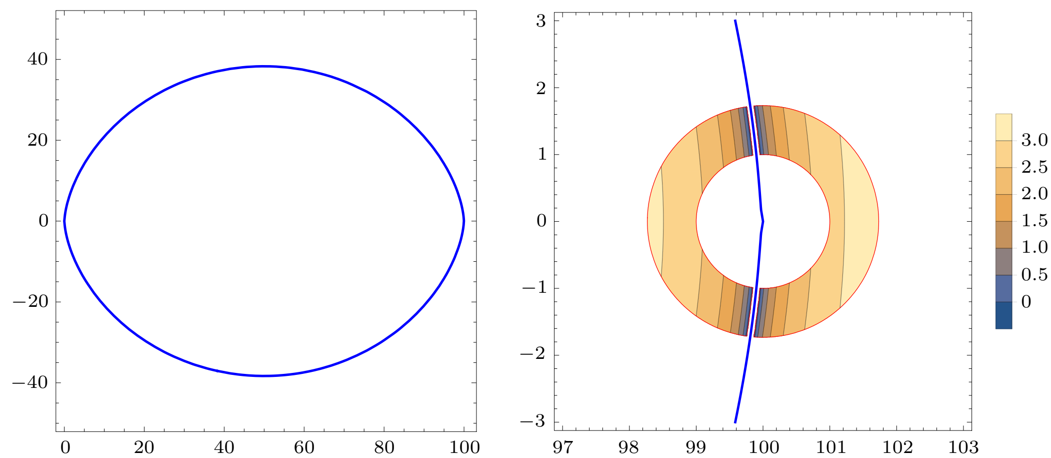

Figure 2. The left picture illustrates the resonant set $\{\eta{:}0{=}\Phi(\xi,\eta){=}\Lambda(\xi)-\Lambda(\eta)-\Lambda(\xi-\eta)\}$ for a fixed large frequency ξ (in the picture $\xi{=}(100,0)$). The picture on the right illustrates the intersection of a neighborhood of this resonant set with the set where $|\xi-\eta|$ is close to γ_0. Note in particular that, near the resonant set, $\xi-\eta$ is almost perpendicular to ξ (see (1.20) and (1.28)). Finally, the colors show the level sets of $\log|\Phi|$.

See Figure 2. As a result, we gain one derivative in the integral $I'_{k,m,p}$, which compensates for the loss of one derivative in (1.27), and the integral can be estimated again using (1.23).

On the other hand, if the modulation is not small, $2^p\geqslant 1$, then the denominator $\Phi(\xi,\eta)$ becomes a favorable factor, and one can use formula (1.27) and reiterate the symmetrization procedure implicit in the energy estimates. This symmetrization avoids the loss of one derivative and gives suitable estimates on $|I'_{k,m,p}|$ in this case. The proof of (1.16) follows.

1.5.2. Dispersive analysis

It remains to prove a partial bootstrap estimate for the Z norm, i.e.

$$\text{if }\sup_{t\in[0,T]}((1+t)^{-\delta^2}\mathcal{E}(t)^{1/2}+\|e^{it\Lambda}U(t)\|_Z)\leqslant\varepsilon_1,\text{ then }\sup_{t\in[0,T]}\|e^{it\Lambda}U(t)\|_Z\lesssim\varepsilon_0+\varepsilon_1^2. \tag{1.29}$$

This complements the energy bootstrap estimate (1.16), and closes the full bootstrap argument.

The first main issue is to define an effective Z norm. We use the Duhamel formula, written in terms of the profile u (recall equation (1.14)):

$$\hat{u}(\xi,t)=\hat{u}(\xi,0)-\frac{1}{2}\int_0^t\int_{\mathbb{R}^2}e^{is\Lambda(\xi)}(|\xi|^2-|\eta|^2)\widehat{V}(\xi-\eta,s)e^{-is\Lambda(\eta)}\hat{u}(\eta,s)\,d\eta\,ds. \tag{1.30}$$

For simplicity, consider one of the terms, namely that corresponding to the component U of V (the contribution of $\overline{U}$ is similar). So, we are looking to understand bilinear expressions of the form

$$\begin{aligned}\hat{h}(\xi,t) &:= \int_0^t \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} n(\xi,\eta)\hat{u}(\xi-\eta,s)\hat{u}(\eta,s)\,d\eta\,ds,\\ n(\xi,\eta) &:= (|\xi|^2-|\eta|^2)\varphi_{[-10,10]}(\xi-\eta), \quad \Phi(\xi,\eta)=\Lambda(\xi)-\Lambda(\eta)-\Lambda(\xi-\eta).\end{aligned} \tag{1.31}$$

The idea is to estimate the function $\hat{h}$ by integrating by parts either in s or in η. This is the method of space-time resonances of Germain–Masmoudi–Shatah [34]. The main contribution is expected to come from the *set of space-time resonances* (the stationary points of the integral), that is

$$\mathcal{SR} := \{(\xi,\eta) : \Phi(\xi,\eta)=0 \text{ and } (\nabla_\eta\Phi)(\xi,\eta)=0\}. \tag{1.32}$$

To illustrate how this analysis works in our problem, we consider the contribution of the integral over $s\approx 2^m \gg 1$ in (1.31), and assume that the frequencies are ≈ 1.

Case 1. Start with the contribution of small modulations,

$$\hat{h}_{m,l}(\xi) := \int_{\mathbb{R}} q_m(s) \int_{\mathbb{R}^2} \varphi_{\leqslant l}(\Phi(\xi,\eta)) e^{is\Phi(\xi,\eta)} n(\xi,\eta)\hat{u}(\xi-\eta,s)\hat{u}(\eta,s)\,d\eta\,ds, \tag{1.33}$$

where $l=-m+\delta m$ (δ is a small constant) and $q_m(s)$ restricts the time integral to $s\approx 2^m$. Assume that $u(\cdot,s)$ is a Schwartz function supported at frequency ≈ 1, independent of s (this is the situation at the first iteration). Integration by parts in η (using formula (7.30) to avoid taking η derivatives of the factor $\varphi_{\leqslant l}(\Phi(\xi,\eta))$) shows that the main contribution comes from a small neighborhood of the stationary points where $|\nabla_\eta\Phi(\xi,\eta)| \leqslant 2^{-m/2+\delta m}$, up to negligible errors. Thus, the main contribution comes from space-time resonant points as in (1.32).

In our case, the space-time resonant set is

$$\big\{(\xi,\eta)\in\mathbb{R}^2\times\mathbb{R}^2 : |\xi| = \gamma_1 = \sqrt{2} \text{ and } \eta = \tfrac{1}{2}\xi\big\}. \tag{1.34}$$

Moreover, the space-time resonant points are *non-degenerate* (according to the terminology introduced in [42]), in the sense that the Hessian of the matrix $\nabla^2_{\eta\eta}\Phi(\xi,\eta)$ is non-singular at these points. A simple calculation shows that

$$\hat{h}_{m,l}(\xi) \approx c(\xi)\varphi_{\leqslant -m}(|\xi|-\gamma_1),$$

up to smaller contributions, where we have also ignored factors of $2^{\delta m}$, and c is smooth.

We are now ready to describe more precisely the Z space. This space should include all Schwartz functions. It also has to include functions like $\hat{u}(\xi)=\varphi_{\leqslant -m}(|\xi|-\gamma_1)$, due to the calculation above, for any m large. It should measure localization in both space and frequency, and be strong enough, at least, to recover the $t^{-5/6+}$ pointwise decay.

We use the framework introduced by two of the authors in [41], which was later refined by some of the authors in [30], [36], [42]. The idea is to decompose the profile as a superposition of atoms, using localization in both space and frequency:

$$f=\sum_{j,k} Q_{jk}f, \quad \text{where } Q_{jk}f=\varphi_j(x)\cdot P_k f(x).$$

The Z norm is then defined by measuring suitably every atom. We first define

$$\|f\|_{Z_1}=\sup_{j,k} 2^j\cdot\Big\|\big||\xi|-\gamma_1\big|^{1/2}\widehat{Q_{jk}f}(\xi)\Big\|_{L^2_\xi}, \tag{1.35}$$

up to small corrections (see Definition 2.5 for the precise formula, including the small but important δ-corrections), and then we define the Z norm by applying a suitable number of vector fields D and Ω.

These considerations and (1.30) can also be used to justify the approximate formula

$$(\partial_t\hat{u})(\xi,t)\approx\frac{1}{t}\sum_j g_j(\xi)e^{it\Phi(\xi,\eta_j(\xi))}+\text{lower order terms}, \tag{1.36}$$

as $t\to\infty$, where $\eta_j(\xi)$ denote the stationary points where $\nabla_\eta\Phi(\xi,\eta_j(\xi))=0$. This approximate formula, which holds at least as long as the stationary points are non-degenerate, is consistent with the asymptotic behavior of the solution described in Remark 1.2 (ii). Indeed, at space-time resonances, $\Phi(\xi,\eta_j(\xi))=0$, which leads to logarithmic growth for $\hat{u}(\xi,t)$, while, away from these space-time resonances, the oscillation of $e^{it\Phi(\xi,\eta_j(\xi))}$ leads to convergence.

Case 2. Consider now the case of higher modulations, say $l\geqslant -m+\delta m$. We start from a formula similar to (1.33) and integrate by parts in s. The main case is when d/ds hits one of the profiles u. Using again the equation (see (1.30)), we have to estimate cubic expressions of the form

$$\begin{aligned}\widehat{h'}_{m,l}(\xi):=\int_{\mathbb{R}} q_m(s)\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{\varphi_l(\Phi(\xi,\eta))}{\Phi(\xi,\eta)}e^{is\Phi(\xi,\eta)}n(\xi,\eta)\hat{u}(\xi-\eta,s)\\ \times e^{is\Phi'(\eta,\sigma)}n(\eta,\sigma)\hat{\bar{u}}(\eta-\sigma,s)\hat{u}(\sigma,s)\,d\eta\,d\sigma\,ds,\end{aligned} \tag{1.37}$$

where $\Phi'(\eta,\sigma)=\Lambda(\eta)+\Lambda(\eta-\sigma)-\Lambda(\sigma)$. Assume again that the three functions u are Schwartz functions supported at frequency ≈ 1. We combine Φ and Φ' into a combined phase:

$$\tilde{\Phi}(\xi,\eta,\sigma):=\Phi(\xi,\eta)+\Phi'(\eta,\sigma)=\Lambda(\xi)-\Lambda(\xi-\eta)+\Lambda(\eta-\sigma)-\Lambda(\sigma).$$

We need to estimate $h'_{m,l}$ according to the Z_1 norm. Integration by parts in ξ (approximate finite speed of propagation) shows that the main contribution in $Q_{jk}h'_{m,l}$ is when $2^j \lesssim 2^m$.

We have two main cases: if l is not too small, say $l \geqslant -\frac{1}{14}m$, then we use first multilinear Hölder-type estimates, placing two of the factors $e^{is\Lambda}u$ in L^∞ and one in L^2, together with analysis of the stationary points of $\tilde{\Phi}$ in η and σ. This suffices in most cases, except when all the variables are close to γ_0. In this case we need a key algebraic property: when $|\xi-\eta|$, $|\eta-\sigma|$, and $|\sigma|$ are all close to γ_0, we have that

$$\text{if } \nabla_{\eta,\sigma}\tilde{\Phi}(\xi,\eta,\sigma)=0 \text{ and } \tilde{\Phi}(\xi,\eta,\sigma)=0, \text{ then } \nabla_\xi\tilde{\Phi}(\xi,\eta,\sigma)=0. \tag{1.38}$$

On the other hand, if l is very small, say $l \leqslant -\frac{1}{14}m$, then the denominator $\Phi(\xi,\eta)$ in (1.37) is dangerous. However, we can restrict to small neighborhoods of the stationary points of $\tilde{\Phi}$ in η and σ, and thus to space-time resonances. This is the most difficult case in the dispersive analysis. We need to rely on one more algebraic property, of the form

$$\text{if } \nabla_{\eta,\sigma}\tilde{\Phi}(\xi,\eta,\sigma)=0 \text{ and } |\Phi(\xi,\eta)|+|\Phi'(\eta,\sigma)| \ll 1, \text{ then } \nabla_\xi\tilde{\Phi}(\xi,\eta,\sigma)=0. \tag{1.39}$$

See Lemma 10.6 for the precise quantitative claims for both (1.38) and (1.39).

The point of both (1.38) and (1.39) is that in the resonant region for the cubic integral we have $\nabla_\xi\tilde{\Phi}(\xi,\eta,\sigma)=0$. We call them *slow propagation of iterated resonances* properties; as a consequence, the resulting function is essentially supported when $|x| \ll 2^m$, using the approximate finite speed of propagation. This gain is reflected in the factor 2^j in (1.35).

We remark that the analogous property for quadratic resonances, namely

$$\text{if } \nabla_\eta\Phi(\xi,\eta)=0 \text{ and } \Phi(\xi,\eta)=0, \text{ then} \nabla_\xi\Phi(\xi,\eta)=0,$$

fails. In fact, in our case $|\nabla_\xi\Phi(\xi,\eta)| \approx 1$ on the space-time resonant set.

In proving (1.29), there are, of course, many cases to consider. The full proof covers §8 and §9. The type of arguments presented above are typical in the proof: we decompose our profiles in space and frequency, localize to small sets in the frequency space, keeping track in particular of the special frequencies of size γ_0, γ_1, $\frac{1}{2}\gamma_1$, and $2\gamma_0$, use integration by parts in ξ to control the location of the output, and use multilinear Hölder-type estimates to bound L^2 norms. We remark that the dispersive analysis in the Z norm is much more involved in this paper than in the earlier papers mentioned above.

1.5.3. The special quadratic structure of the full water-wave system

The model (1.14) is useful in understanding the full problem. There are, however, additional difficulties to keep in mind.

In this paper we use *Eulerian coordinates*. The local well-posedness theory, which is non-trivial because of the quasilinear nature of the equations and the hidden hyperbolic structure, then relies on the so-called "good unknown" of Alinhac [1], [4], [5], [6].

In our problem, however, this is not enough. Alinhac's good unknown ω is suitable for the local theory, in the sense that it prevents loss of derivatives in energy estimates. However, for the global theory, we need to adjust the main complex variable U which diagonalizes the system, using a quadratic correction of the form $T_{m'}\omega$ (see (3.4)). In this way, we can identify certain *special quadratic structure*, somewhat similar to the structure in the non-linearity of (1.14). This structure, which appears to be new, is ultimately responsible for the favorable multipliers of the space-time integrals (similar to (1.20)), and leads to global energy bounds.

Identifying this structure is, unfortunately, technically involved. Our main result is in Proposition 3.1, but its proof depends on paradifferential calculus using the Weyl quantization (see Appendix A) and on a suitable paralinearization of the Dirichlet–Neumann operator. We include all the details of this paralinearization in Appendix B, mostly because its exact form has to be properly adapted to our norms and suitable for global analysis. For this, we need some auxiliary spaces: (1) the $\mathcal{O}_{m,p}$ hierarchy, which measures functions, keeping track of both multiplicity (the index m) and smoothness (the index p), and (2) the $\mathcal{M}_r^{l,m}$ hierarchy, which measures the symbols of the paradifferential operators, keeping track also of the order l.

1.5.4. Additional remarks

We list below some other issues one needs to keep in mind in the proof of the main theorem.

(1) Another significant difficulty of the full water-wave system, which is not present in (1.14), is that the "linear" part of the equation is given by a more complicated paradifferential operator T_Σ, not by the simple operator Λ. The operator T_Σ includes non-linear cubic terms that lose $\frac{3}{2}$ derivatives, and an additional smoothing effect is needed.

(2) The very low frequencies $|\xi|\ll 1$ play an important role in all the global results for water-wave systems. These frequencies are not captured in the model (1.14). In our case, there is a suitable *null structure*: the multipliers of the quadratic terms are bounded by $|\xi|\min(|\eta|,|\xi-\eta|)^{1/2}$ (see (7.11)), which is an important ingredient in the dispersive part of the argument.

(3) It is important to propagate energy control of both high Sobolev norms and weighted norms using many copies of the rotation vector field. Because of this control, we can pretend that all the profiles in the dispersive part of the argument are almost

radial and located at frequencies $\lesssim 1$. The linear estimates (in Lemma 7.5) and many of the bilinear estimates are much stronger, because of this almost radiality property of the profiles.

(4) At many stages, it is important that the four spheres, namely the sphere of slow decay $\{\xi:|\xi|=\gamma_0\}$, the sphere of space-time resonant outputs $\{\xi:|\xi|=\gamma_1\}$, the sphere of space-time resonant inputs $\{\xi:|\xi|=\frac{1}{2}\gamma_1\}$, and the sphere $\{\xi:|\xi|=2\gamma_0\}$, are all separated from each other. Such separation property played an important role also in other papers, such as [30], [33], [36].

1.6. Organization

The rest of the paper is organized as follows: in §2 we state the main propositions and summarize the main definitions and notation of the paper.

In §§3–6 we prove Proposition 2.2, which is the main improved energy estimate. The key components of the proof are Proposition 3.1 (derivation of the main quasilinear scalar equation, identifying the special quadratic structure), Proposition 4.1 (the first energy estimate, including the strongly semilinear structure), Proposition 4.2 (reduction to a space-time integral bound), Lemma 4.7 (the main L^2 bound on a localized Fourier integral operator), and Lemma 5.1 (the main interactions in Proposition 4.2). The proof of Proposition 2.2 also uses the material presented in the appendices, in particular the paralinearization of the Dirichlet–Neumann operator in Proposition B.1.

In §§7–9 we prove Proposition 2.3, which is the main improved dispersive estimate. The key components of the proof are the reduction to Proposition 7.1, the precise analysis of the time derivative of the profile in Lemmas 8.1 and 8.2, and the analysis of the Duhamel formula, divided in several cases, in Lemmas 9.4–9.8.

In §10 and §11 we collect estimates on the dispersion relation and the phase functions. The main results are Proposition 10.2 (structure of the resonance sets), Proposition 10.4 (bounds on sublevel sets), Lemma 10.6 (slow propagation of iterated resonances), and Lemmas 11.1–11.3 (restricted non-degeneracy property of the resonant hypersurfaces).

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2. The main propositions

Recall the water-wave system with gravity and surface tension:

$$\begin{cases} \partial_t h = G(h)\phi, \\ \partial_t \phi = -gh + \sigma \,\mathrm{div}\left(\dfrac{\nabla h}{(1+|\nabla h|^2)^{1/2}}\right) - \dfrac{1}{2}|\nabla \phi|^2 + \dfrac{(G(h)\phi + \nabla h \cdot \nabla \phi)^2}{2(1+|\nabla h|^2)}, \end{cases} \tag{2.1}$$

where $G(h)\phi$ denotes the Dirichlet–Neumann operator associated with the water domain. Theorem 1.1 is a consequence of Propositions 2.1–2.3 below.

PROPOSITION 2.1. (Local existence and continuity)

(i) *Let* $N \geqslant 10$. *There is* $\bar{\varepsilon} > 0$ *such that, if*

$$\|h_0\|_{H^{N+1}} + \|\phi_0\|_{\dot{H}^{N+1/2,1/2}} \leqslant \bar{\varepsilon}, \tag{2.2}$$

then there is a unique solution $(h, \phi) \in C([0,1] : H^{N+1} \times \dot{H}^{N+1/2,1/2})$ *of the system* (2.1) *with* $g=1$ *and* $\sigma=1$, *with initial data* (h_0, ϕ_0).

(ii) *Let* $T_0 \geqslant 1$, $N = N_1 + N_3$, *and* $(h, \phi) \in C([0, T_0] : H^{N+1} \times \dot{H}^{N+1/2,1/2})$ *be a solution of the system* (2.1) *with* $g=1$ *and* $\sigma=1$. *With the* Z *norm as in Definition* 2.5 *below and the profile* $\mathcal{V}$ *defined as in* (1.11), *assume that, for some* $t_0 \in [0, T_0]$,

$$\mathcal{V}(t_0) \in H^{N_0} \cap H^{N_1,N_3}_\Omega \cap Z \quad \text{and} \quad \|\mathcal{V}(t_0)\|_{H^N} \leqslant 2\bar{\varepsilon}. \tag{2.3}$$

Then, there is $\tau = \tau(\|\mathcal{V}(t_0)\|_{H^{N_0} \cap H^{N_1,N_3} \cap Z})$ *such that the mapping*

$$t \longmapsto \|\mathcal{V}(t)\|_{H^{N_0} \cap H^{N_1,N_3}_\Omega \cap Z}$$

is continuous on $[0, T_0] \cap [t_0, t_0 + \tau]$, *and*

$$\sup_{t \in [0,T_0] \cap [t_0, t_0+\tau]} \|\mathcal{V}(t)\|_{H^{N_0} \cap H^{N_1,N_3}_\Omega \cap Z} \leqslant 2\|\mathcal{V}(t_0)\|_{H^{N_0} \cap H^{N_1,N_3}_\Omega \cap Z}. \tag{2.4}$$

Proposition 2.1 is a local existence result for the water-wave system. We will not provide the details of its proof in the paper, but only briefly discuss it. Part (i) is a standard well-posedness statement in a sufficiently regular Sobolev space; see for example [1] and [70].

Part (ii) is a continuity statement for the Sobolev norm H^{N_0}, as well as for the H^{N_1,N_3}_Ω and Z norms.[4] Continuity for the H^{N_0} norm is standard. A formal proof of continuity for the H^{N_1,N_3}_Ω and Z norms and of (2.4) requires some adjustments of

[4] Notice that we may assume uniform-in-time smallness of the high Sobolev norm H^N with $N = N_1 + N_3$, due to the uniform control on the Z norm; see Proposition 2.2 and Definition 2.5.

the arguments given in the paper, due to the quasilinear and non-local nature of the equations.

More precisely, we can define ε-truncations of the rotation vector field Ω, that is $\Omega_\varepsilon:=(1+\varepsilon^2|x|^2)^{-1/2}\Omega$, and the associated spaces $H^{N_1,N_3}_{\Omega_\varepsilon}$, with the obvious adaptation of the norm in (1.8). Then, we notice that

$$\Omega_\varepsilon T_a b = T_{\Omega_\varepsilon a} b + T_a \Omega_\varepsilon b + R$$

where R is a suitable remainder bounded uniformly in ε. Because of this, we can adapt the arguments in Proposition 4.1 and in Appendices A and B to prove energy estimates in the ε-truncated spaces $H^{N_1,N_3}_{\Omega_\varepsilon}$. For the Z norm, one can proceed similarly using an ε-truncated version Z_ε (see the proof of [42, Proposition 2.4] for a similar argument) and the formal expansion of the Dirichlet–Neumann operator in §C.2. The conclusion follows from the uniform estimates by letting $\varepsilon\to 0$.

The following two propositions summarize our main bootstrap argument.

PROPOSITION 2.2. (Improved energy control) *Assume that $T\geqslant 1$ and let*

$$(h,\phi)\in C([0,T]:H^{N_0+1}\times \dot H^{N_0+1/2,1/2})$$

be a solution of the system (2.1) *with $g=1$ and $\sigma=1$, with initial data (h_0,ϕ_0). Assume that, with $\mathcal{U}$ and $\mathcal{V}$ defined as in* (1.11),

$$\|\mathcal{U}(0)\|_{H^{N_0}\cap H^{N_1,N_3}_\Omega}+\|\mathcal{V}(0)\|_Z\leqslant \varepsilon_0\ll 1, \tag{2.5}$$

and, for any $t\in[0,T]$,

$$(1+t)^{-\delta^2}\|\mathcal{U}(t)\|_{H^{N_0}\cap H^{N_1,N_3}_\Omega}+\|\mathcal{V}(t)\|_Z\leqslant \varepsilon_1\ll 1, \tag{2.6}$$

where the Z norm is as in Definition 2.5. Then, for any $t\in[0,T]$,

$$(1+t)^{-\delta^2}\|\mathcal{U}(t)\|_{H^{N_0}\cap H^{N_1,N_3}_\Omega}\lesssim \varepsilon_0+\varepsilon_1^{3/2}. \tag{2.7}$$

PROPOSITION 2.3. (Improved dispersive control) *With the same assumptions as in Proposition 2.2 above, in particular* (2.5) *and* (2.6), *we have, for any $t\in[0,T]$,*

$$\|\mathcal{V}(t)\|_Z\lesssim \varepsilon_0+\varepsilon_1^2. \tag{2.8}$$

It is easy to see that Theorem 1.1 follows from Propositions 2.1–2.3 by a standard continuity argument and (7.44) (for the L^∞ bound on $\mathcal{U}$ in (1.10)). The rest of the paper is concerned with the proofs of Propositions 2.2 and 2.3.

2.1. Definitions and notation

We summarize in this subsection some of the main definitions we use in the paper.

2.1.1. General notation

We start by defining several multipliers that allow us to localize in the Fourier space. We fix an even smooth function $\varphi\colon\mathbb{R}\to[0,1]$ supported in $\left[-\frac{8}{5},\frac{8}{5}\right]$ and equal to 1 in $\left[-\frac{5}{4},\frac{5}{4}\right]$. For simplicity of notation, we also let $\varphi\colon\mathbb{R}^2\to[0,1]$ denote the corresponding radial function on $\mathbb{R}^2$. Let

$$\varphi_k(x):=\varphi\left(\frac{|x|}{2^k}\right)-\varphi\left(\frac{|x|}{2^{k-1}}\right)\text{ for any } k\in\mathbb{Z},\quad \varphi_I:=\sum_{m\in I\cap\mathbb{Z}}\varphi_m \text{ for any } I\subseteq\mathbb{R},$$

$$\varphi_{\leqslant B}:=\varphi_{(-\infty,B]},\quad \varphi_{\geqslant B}:=\varphi_{[B,\infty)},\quad \varphi_{<B}:=\varphi_{(-\infty,B)},\quad\text{and}\quad \varphi_{>B}:=\varphi_{(B,\infty)}.$$

For any $a<b\in\mathbb{Z}$ and $j\in[a,b]\cap\mathbb{Z}$, let

$$\varphi_j^{[a,b]}:=\begin{cases}\varphi_j, & \text{if } a<j<b,\\ \varphi_{\leqslant a}, & \text{if } j=a,\\ \varphi_{\geqslant b}, & \text{if } j=b.\end{cases}\tag{2.9}$$

For any $x\in\mathbb{R}$, let $x^+=\max(x,0)$ and $x^-:=\min(x,0)$. Let

$$\mathcal{J}:=\{(k,j)\in\mathbb{Z}\times\mathbb{Z}_+ : k+j\geqslant 0\}.$$

For any $(k,j)\in\mathcal{J}$, let

$$\widetilde{\varphi}_j^{(k)}(x):=\begin{cases}\varphi_{\leqslant -k}(x), & \text{if } k+j=0 \text{ and } k\leqslant 0,\\ \varphi_{\leqslant 0}(x), & \text{if } j=0 \text{ and } k\geqslant 0,\\ \varphi_j(x), & \text{if } k+j\geqslant 1 \text{ and } j\geqslant 1,\end{cases}$$

and notice that, for any $k\in\mathbb{Z}$ fixed, $\sum_{j\geqslant -\min(k,0)}\widetilde{\varphi}_j^{(k)}=1$.

Let P_k, $k\in\mathbb{Z}$, denote the Littlewood–Paley projection operators defined by the Fourier multipliers $\xi\mapsto\varphi_k(\xi)$. Let $P_{\leqslant B}$ (resp. $P_{>B}$) denote the operators defined by the Fourier multipliers $\xi\mapsto\varphi_{\leqslant B}(\xi)$ (resp. $\xi\mapsto\varphi_{>B}(\xi)$). For $(k,j)\in\mathcal{J}$ let Q_{jk} denote the operator

$$(Q_{jk}f)(x):=\widetilde{\varphi}_j^{(k)}(x)\cdot P_kf(x).\tag{2.10}$$

In view of the uncertainty principle, the operators Q_{jk} are relevant only when $2^j2^k\gtrsim 1$, which explains the definitions above.

We will use two sufficiently large constants $\mathcal{D}\gg\mathcal{D}_1\gg 1$ ($\mathcal{D}_1$ is only used in §10 and §11 to prove properties of the phase functions). For $k, k_1, k_2\in\mathbb{Z}$, let

$$\mathcal{D}_{k,k_1,k_2} := \{(\xi,\eta)\in(\mathbb{R}^2)^2 : |\xi|\in[2^{k-4},2^{k+4}], |\eta|\in[2^{k_2-4},2^{k_2+4}], \text{ and } |\xi-\eta|\in[2^{k_1-4},2^{k_1+4}]\}. \tag{2.11}$$

Let $\lambda(r)=\sqrt{|r|+|r|^3}$ and $\Lambda(\xi)=\lambda(|\xi|)=\sqrt{|\xi|+|\xi|^3}$, $\Lambda:\mathbb{R}^2\to[0,\infty)$. Let

$$\mathcal{U}_+ := \mathcal{U}, \quad \mathcal{U}_- := \bar{\mathcal{U}}, \quad \mathcal{V}(t)=\mathcal{V}_+(t) := e^{it\Lambda}\mathcal{U}(t), \quad \text{and} \quad \mathcal{V}_-(t) := e^{-it\Lambda}\mathcal{U}_-(t). \tag{2.12}$$

Let $\Lambda_+=\Lambda$ and $\Lambda_- := -\Lambda$. For $\sigma,\mu,\nu\in\{+,-\}$, we define the associated phase functions

$$\begin{aligned}\Phi_{\sigma\mu\nu}(\xi,\eta) &:= \Lambda_\sigma(\xi)-\Lambda_\mu(\xi-\eta)-\Lambda_\nu(\eta),\\ \tilde{\Phi}_{\sigma\mu\nu\beta}(\xi,\eta,\sigma) &:= \Lambda_\sigma(\xi)-\Lambda_\mu(\xi-\eta)-\Lambda_\nu(\eta-\sigma)-\Lambda_\beta(\sigma).\end{aligned} \tag{2.13}$$

2.1.2. The spaces $O_{m,p}$

We will need several spaces of functions, in order to properly measure linear, quadratic, cubic, quartic, and higher-order terms. In addition, we also need to track the Sobolev smoothness and angular derivatives. Assume that $N_2=40\geqslant N_3+10$ and that N_0 (the maximum number of Sobolev derivatives), N_1 (the maximum number of angular derivatives), and N_3 (additional Sobolev regularity) are as before.

Definition 2.4. Assume $T\geqslant 1$ and let $p\in[-N_3,10]$. For $m\geqslant 1$, we define $O_{m,p}$ as the space of functions $f\in C([0,T]:L^2)$ satisfying

$$\begin{aligned}\|f\|_{O_{m,p}} := \sup_{t\in[0,T]}(1+t)^{(m-1)(5/6-20\delta^2)-\delta^2}\big(\|f(t)\|_{H^{N_0+p}}+\|f(t)\|_{H_\Omega^{N_1,N_3+p}}\\ +(1+t)^{5/6-2\delta^2}\|f(t)\|_{\widetilde{W}_\Omega^{N_1/2,N_2+p}}\big)<\infty,\end{aligned} \tag{2.14}$$

where, with P_k denoting standard Littlewood–Paley projection operators,

$$\|g\|_{\widetilde{W}^N} := \sum_{k\in\mathbb{Z}} 2^{Nk^+}\|P_k g\|_{L^\infty} \quad \text{and} \quad \|g\|_{\widetilde{W}_\Omega^{N',N}} := \sum_{j\leqslant N'}\|\Omega^j g\|_{\widetilde{W}^N}.$$

The spaces $\widetilde{W}^N$ are used in this paper as substitutes of the standard L^∞ based Sobolev spaces, which have the advantage of being closed under the action of singular integrals.

Note that the parameter p in $O_{m,p}$ corresponds to a gain at high frequencies and does not affect the low frequencies. We observe that (see Lemma A.2)

$$O_{m,p}\subseteq O_{n,p} \text{ if } 1\leqslant n\leqslant m \quad \text{and} \quad O_{m,p}O_{n,p}\subseteq O_{m+n,p} \text{ if } 1\leqslant m,n. \tag{2.15}$$

Moreover, by our assumptions (2.6) and Lemma 7.5, the main variables satisfy

$$\|(1-\Delta)^{1/2}h\|_{O_{1,0}}+\big\||\nabla|^{1/2}\phi\big\|_{O_{1,0}}\lesssim\varepsilon_1. \tag{2.16}$$

The spaces $O_{m,p}$ are used mostly in the energy estimates in §3 and in the (elliptic) analysis of the Dirichlet–Neumann operator in Appendix B. However, they are not precise enough for the dispersive analysis of our evolution equation. For this, we need the more precise Z-norm defined below, which is better adapted to our equation.

2.1.3. The Z norm

Let $\gamma_0:=\sqrt{\frac{1}{3}(2\sqrt{3}-3)}$ denote the radius of the sphere of slow decay, and $\gamma_1:=\sqrt{2}$ denote the radius of the space-time resonant sphere. For $n\in\mathbb{Z}$, $I\subseteq\mathbb{R}$, and $\gamma\in(0,\infty)$ we define

$$\begin{gathered}\widehat{A_{n,\gamma}f}(\xi):=\varphi_{-n}\big(2^{100}\big||\xi|-\gamma\big|\big)\hat{f}(\xi),\\ A_{I,\gamma}:=\sum_{n\in I}A_{n,\gamma},\quad A_{\leqslant B,\gamma}:=A_{(-\infty,B],\gamma},\quad\text{and}\quad A_{\geqslant B,\gamma}:=A_{[B,\infty),\gamma}.\end{gathered} \tag{2.17}$$

Given an integer $j\geqslant0$, we define the operators $A^{(j)}_{n,\gamma}$, $n\in\{0,...,j+1\}$ and $\gamma\geqslant2^{-50}$, by

$$A^{(j)}_{j+1,\gamma}:=\sum_{n'\geqslant j+1}A_{n',\gamma},\quad A^{(j)}_{0,\gamma}:=\sum_{n'\leqslant0}A_{n',\gamma},\quad\text{and}\quad A^{(j)}_{n,\gamma}:=A_{n,\gamma}\ \text{ if } 1\leqslant n\leqslant j. \tag{2.18}$$

These operators localize to thin annuli of width 2^{-n} around the circle of radius γ. Most of the times, for us $\gamma=\gamma_0$ or $\gamma=\gamma_1$. We are now ready to define the main Z norm.

Definition 2.5. Let δ, N_0, N_1, and N_4 be as in Theorem 1.1. We define

$$Z_1:=\Big\{f\in L^2(\mathbb{R}^2):\|f\|_{Z_1}:=\sup_{(k,j)\in\mathcal{J}}\|Q_{jk}f\|_{B_j}<\infty\Big\}, \tag{2.19}$$

where

$$\|g\|_{B_j}:=2^{(1-50\delta)j}\sup_{0\leqslant n\leqslant j+1}2^{-(1/2-49\delta)n}\|A^{(j)}_{n,\gamma_1}g\|_{L^2}. \tag{2.20}$$

Then we define, with $D^\alpha:=\partial_1^{\alpha^1}\partial_2^{\alpha^2}$, $\alpha=(\alpha^1,\alpha^2)$,

$$Z:=\Big\{f\in L^2(\mathbb{R}^2):\|f\|_Z:=\sup_{\substack{2m+|\alpha|\leqslant N_1+N_4\\ m\leqslant N_1/2+20}}\|D^\alpha\Omega^m f\|_{Z_1}<\infty\Big\}. \tag{2.21}$$

We remark that the Z norm is used to estimate the *linear profile* of the solution, which is $\mathcal{V}(t):=e^{it\Lambda}\mathcal{U}(t)$, not the solution itself.

2.1.4. Paradifferential calculus

We need some elements of paradifferential calculus in order to be able to describe the Dirichlet–Neumann operator $G(h)\phi$ in (2.1). Our paralinearization relies on the *Weyl quantization*. More precisely, given a symbol $a=a(x,\zeta)$ and a function $f\in L^2$, we define the paradifferential operator $T_a f$ according to

$$\mathcal{F}(T_a f)(\xi)=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\chi\biggl(\frac{|\xi-\eta|}{|\xi+\eta|}\biggr)\tilde{a}\biggl(\xi-\eta,\frac{\xi+\eta}{2}\biggr)\hat{f}(\eta)\,d\eta, \tag{2.22}$$

where $\tilde{a}$ denotes the Fourier transform of a in the first coordinate and $\chi=\varphi_{\leqslant -20}$. In Appendix A we prove several important lemmas related to the paradifferential calculus.

3. The "improved good variable" and strongly semilinear structures

3.1. Reduction to a scalar equation

In this section we assume that $(h,\phi)\colon\mathbb{R}^2\times[0,T]\to\mathbb{R}\times\mathbb{R}$ is a solution of (2.1) satisfying the hypotheses of Proposition 2.2; in particular (see (2.16)),

$$\|\langle\nabla\rangle h\|_{O_{1,0}}+\big\||\nabla|^{1/2}\phi\big\|_{O_{1,0}}\lesssim\varepsilon_1. \tag{3.1}$$

Our goal in this section is to write the system (2.1) as a scalar equation for a suitably constructed complex-valued function (the "improved good variable"). The main result is the following.

PROPOSITION 3.1. *Assume* (3.1) *holds and let* λ_{DN} *be the symbol of the Dirichlet–Neumann operator defined in* (B.5), *let* $\Lambda:=\sqrt{g|\nabla|+\sigma|\nabla|^3}$, *and let*

$$\ell(x,\zeta):=L_{ij}(x)\zeta_i\zeta_j-\Lambda^2 h,\quad L_{ij}:=\frac{\sigma}{\sqrt{1+|\nabla h|^2}}\biggl(\delta_{ij}-\frac{\partial_i h\partial_j h}{1+|\nabla h|^2}\biggr) \tag{3.2}$$

be the mean curvature operator coming from the surface tension. Define the symbol

$$\Sigma:=\sqrt{\lambda_{DN}(g+\ell)} \tag{3.3}$$

and the complex-valued unknown

$$U:=T_{\sqrt{g+\ell}}h+iT_\Sigma T_{1/\sqrt{g+\ell}}\omega+iT_{m'}\omega,\quad m':=\frac{i}{2}\frac{\operatorname{div}V}{\sqrt{g+\ell}}\in\varepsilon_1\mathcal{M}^{-1,1}_{N_3-2}, \tag{3.4}$$

where B, *and* V *and* (*the "good variable"*) $\omega=\phi-T_B h$ *are defined in* (B.3). *Then,*

$$U=\sqrt{g+\sigma|\nabla|^2}h+i|\nabla|^{1/2}\omega+\varepsilon_1^2 O_{2,0} \tag{3.5}$$

and U satisfies the equation

$$(\partial_t + iT_\Sigma + iT_{V\cdot\zeta})U = \mathcal{N}_U + \mathcal{Q}_S + \mathcal{C}_U, \tag{3.6}$$

where

- *the quadratic term $\mathcal{N}_U$ has the special structure*

$$\mathcal{N}_U := T_\gamma(c_1 U + c_2 \overline{U}) \tag{3.7}$$

for some constants $c_1, c_2 \in \mathbb{C}$, where

$$\gamma(x,\zeta) := \frac{\zeta_i\zeta_j}{|\zeta|^2}|\nabla|^{-1/2}\partial_i\partial_j(\mathrm{Im}\, U)(x); \tag{3.8}$$

- *the quadratic terms $\mathcal{Q}_S$ have a gain of one derivative, i.e. they are of the form*

$$\mathcal{Q}_S = A_{++}(U,U) + A_{+-}(U,\overline{U}) + A_{--}(\overline{U},\overline{U}) \in \varepsilon_1^2 O_{2,1}, \tag{3.9}$$

with symbols $a_{\varepsilon_1\varepsilon_2}$ satisfying, for all $k, k_1, k_2 \in \mathbb{Z}$, and $(\varepsilon_1\varepsilon_2) \in \{(++), (+-), (--)\}$,

$$\|a^{k,k_1,k_2}_{\varepsilon_1\varepsilon_2}\|_{S^\infty_\Omega} \lesssim 2^{-\max(k_1,k_2,0)}(1 + 2^{3\min(k_1,k_2)}); \tag{3.10}$$

- *$\mathcal{C}_U$ is a cubic term, $\mathcal{C}_U \in \varepsilon_1^3 O_{3,0}$.*

Let us comment on the structure of the main equation (3.6). In the left-hand side, we have the "quasilinear" part $(\partial_t + iT_\Sigma + iT_{V\cdot\zeta})U$. In the right-hand side we have three types of terms:

(1) a quadratic term $\mathcal{N}_U$ with special structure;
(2) a strongly semilinear quadratic term $\mathcal{Q}_S$, given by symbols of order -1;
(3) a semilinear cubic term $\mathcal{C}_U \in \varepsilon_1^3 O_{3,0}$, whose contribution is easy to estimate.

The key point is the special structure of the quadratic terms, which allows us to obtain favorable energy estimates in Proposition 4.1. This special structure is due to the definition of the variable U, in particular to the choice of the symbol m' in (3.4). We observe that

$$\widetilde{\gamma}(\eta,\zeta) = -\frac{\zeta_i\zeta_j}{|\zeta|^2}\frac{\eta_i\eta_j}{|\eta|^{1/2}}\widehat{\mathrm{Im}\, U}(\eta),$$

and we remark that the angle $\zeta\cdot\eta$ in this expression gives us the strongly semilinear structure that we will use later (see also the factor $\mathfrak{d}$ in (4.6)). For comparison, the use of the standard "good unknown" of Alinhac leads to generic quadratic terms that do not lose derivatives. This would suffice to prove local regularity of the system, but would not be suitable for global analysis.

This proposition is the starting point of our energy analysis. Its proof is technically involved, as it requires the material in Appendices A and B. One can start by understanding the definition A.6 of the decorated spaces of symbols $\mathcal{M}_r^{l,m}$, the simple properties (A.43)–(A.54), and the statement of Proposition B.1 (the proof is not needed). The spaces of symbols $\mathcal{M}_r^{l,m}$ are analogous to the spaces of functions $\mathcal{O}_{m,p}$; for symbols, however, the order l is important (for example a symbol of order 2 counts as two derivatives), but its exact differentiability (measured by the parameter r) is less important.

In Proposition 3.1 we keep the parameters g and σ due to their physical significance.

Remark 3.2. (i) The symbols defined in this proposition can be estimated in terms of the decorated norms introduced in Definition A.6. More precisely, using hypothesis (3.1), the basic bounds (A.43) and (A.45), and definition (B.5), it is easy to verify that

$$\begin{aligned}(g+\ell) &= \frac{(g+\sigma|\zeta|^2)}{\sqrt{1+|\nabla h|^2}}\bigg(1-\frac{\sigma(\zeta\cdot\nabla h)^2}{(g+\sigma|\zeta|^2)}-\frac{\Lambda^2 h}{(g+\sigma|\zeta|^2)}+\varepsilon_1^4\mathcal{M}^{0,4}_{N_3-2}+\varepsilon_1^2\mathcal{M}^{-2,2}_{N_3-2}\bigg),\\ \lambda_{DN} &= |\zeta|\bigg(1+\frac{|\zeta|^2|\nabla h|^2-(\zeta\cdot\nabla h)^2}{2|\zeta|^2}+\frac{|\zeta|^2\Delta h-\zeta_j\zeta_k\partial_j\partial_k h}{2|\zeta|^3}\varphi_{\geqslant 0}(\zeta)\\ &\qquad +\varepsilon_1^4\mathcal{M}^{0,4}_{N_3-2}+\varepsilon_1^3\mathcal{M}^{-1,3}_{N_3-2}\bigg),\end{aligned} \tag{3.11}$$

uniformly for every $t\in[0,T]$. Thus, with $\Lambda=\sqrt{g|\nabla|+\sigma|\nabla|^3}$, we derive the following expansion for Σ:

$$\Sigma=\Lambda+\Sigma_1+\Sigma_{\geqslant 2}, \tag{3.12}$$

with

$$\Sigma_1 := \frac{1}{4}\frac{\Lambda(\zeta)}{|\zeta|}\bigg(\Delta h-\frac{\zeta_i\zeta_j}{|\zeta|^2}\partial_{ij}h\bigg)\varphi_{\geqslant 0}(\zeta)-\frac{1}{2}\frac{|\zeta|}{\Lambda(\zeta)}\Lambda^2 h\in\varepsilon_1\mathcal{M}^{1/2,1}_{N_3-2}$$

and

$$\Sigma_{\geqslant 2}\in\varepsilon_1^2\mathcal{M}^{3/2,2}_{N_3-2}.$$

The formulas are slightly simpler if we disregard quadratic terms, i.e.

$$\begin{aligned}\lambda_{DN}^p &= |\zeta|^p\bigg(1+\frac{p\lambda_1^{(0)}(x,\zeta)}{|\zeta|}+\varepsilon_1^2\mathcal{M}^{0,2}_{N_3-2}\bigg),\\ (g+\ell)^p &= (g+\sigma|\zeta|^2)^p\bigg(1-\frac{p\Lambda^2 h}{g+\sigma|\zeta|^2}+\varepsilon_1^2\mathcal{M}^{0,2}_{N_3-2}\bigg),\\ \Sigma &= \Lambda\bigg(1+\frac{\Sigma_1(x,\zeta)}{\Lambda}+\varepsilon_1^2\mathcal{M}^{0,2}_{N_3-2}\bigg),\end{aligned} \tag{3.13}$$

for $p\in[-2,2]$, where

$$\lambda_1^{(0)}(x,\zeta)=\frac{|\zeta|^2\Delta h-\zeta_j\zeta_k\partial_j\partial_k h}{2|\zeta|^2}\varphi_{\geqslant 0}(\zeta),$$

as in Remark B.2.

In addition, the identity $\partial_t h=G(h)\phi=|\nabla|\omega+\varepsilon_1^2\mathcal{O}_{2,-1/2}$ shows that

$$\begin{aligned}\partial_t\sqrt{g+\ell}&=(g+\sigma|\zeta|^2)^{-1/2}\bigg(\frac{\Delta(g-\sigma\Delta)\omega}{2}\bigg)+\varepsilon_1^2\mathcal{M}^{1,2}_{N_3-4}\in\varepsilon_1\mathcal{M}^{-1,1}_{N_3-4}+\varepsilon_1^2\mathcal{M}^{1,2}_{N_3-4},\\ \partial_t\sqrt{\lambda_{DN}}&=\frac{1}{2\sqrt{|\zeta|}}\partial_t\lambda_1^{(0)}+\varepsilon_1^2\mathcal{M}^{1/2,2}_{N_3-4}\in\varepsilon_1\mathcal{M}^{-1/2,1}_{N_3-4}+\varepsilon_1^2\mathcal{M}^{1/2,2}_{N_3-4},\\ \partial_t\Sigma&=\partial_t\Sigma_1+\varepsilon_1^2\mathcal{M}^{3/2,2}_{N_3-4}\in\varepsilon_1\mathcal{M}^{1/2,1}_{N_3-4}+\varepsilon_1^2\mathcal{M}^{3/2,2}_{N_3-4}.\end{aligned}\tag{3.14}$$

(ii) It follows from Proposition B.1 that $V\in\varepsilon_1\mathcal{O}_{1,-1/2}$. Therefore, $m'\in\varepsilon_1\mathcal{M}^{-1,1}_{N_3-2}$ and the identity (3.5) follows using also Lemma A.7. Moreover, using Proposition B.1 again,

$$\begin{aligned}&V=V_1+V_2,\quad V_1:=|\nabla|^{-1/2}\nabla\operatorname{Im}U,\quad V_2\in\varepsilon_1^2\mathcal{O}_{2,-1/2},\\ &m'=m_1'+\varepsilon_1^2\mathcal{M}^{-1,2}_{N_3-2},\quad m_1'(x,\zeta):=-\frac{i}{2}\frac{|\nabla|^{3/2}\operatorname{Im}U(x)}{\sqrt{g+\sigma|\zeta|^2}}\end{aligned}\tag{3.15}$$

3.2. Symmetrization and special quadratic structure

In this subsection we prove Proposition 3.1. We first write (2.1) as a system for h and ω, and then symmetrize it. We start by combining Proposition B.1 on the Dirichlet–Neumann operator with a paralinearization of the equation for $\partial_t\phi$, to obtain the following lemma.

LEMMA 3.3. (Paralinearization of the system) *With the notation of Propositions B.1 and 3.1, we can rewrite the system (2.1) as*

$$\begin{cases}\partial_t h=T_{\lambda_{DN}}\omega-\operatorname{div}(T_V h)+G_2+\varepsilon_1^3 O_{3,1},\\ \partial_t\omega=-gh-T_\ell h-T_V\nabla\omega+\Omega_2+\varepsilon_1^3O_{3,1},\end{cases}\tag{3.16}$$

where ℓ is given in (3.2) and

$$\Omega_2:=\tfrac12\mathcal{H}(|\nabla|\omega,|\nabla|\omega)-\tfrac12\mathcal{H}(\nabla\omega,\nabla\omega)\in\varepsilon_1^2O_{2,2}.\tag{3.17}$$

Proof. First, we see directly from (2.1) and Proposition B.1 that, for any $t\in[0,T]$,

$$\begin{aligned}&G(h)\phi,B,V,\partial_t h\in\varepsilon_1\mathcal{O}_{1,-1/2},\quad \partial_t\phi\in\varepsilon_1\mathcal{O}_{1,-1},\\ &B=|\nabla|\omega+\varepsilon_1^2\mathcal{O}_{2,-1/2},\quad V=\nabla\omega+\varepsilon_1^2\mathcal{O}_{2,-1/2}.\end{aligned}\tag{3.18}$$

The first equation in (3.16) comes directly from Propostion B.1. To obtain the second equation, we use Lemma A.4 (ii) with $F_l(x)=x_l/\sqrt{1+|x|^2}$ to see that

$$F_l(\nabla h)=T_{\partial_kF_l(\nabla h)}\partial_k h+\varepsilon_1^3\mathcal{O}_{3,3},$$

and hence

$$\sigma\,\mathrm{div}\Big(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\Big)=-T_{L_{jk}\zeta_j\zeta_k}h+\varepsilon_1^3\mathcal{O}_{3,1}.$$

Next, we paralinearize the other non-linear terms in the second equation in (2.1). Recall the definition of V and B in (B.3). We first write

$$-\frac{1}{2}|\nabla\phi|^2+\frac{(G(h)\phi+\nabla h\cdot\nabla\phi)^2}{2(1+|\nabla h|^2)}=-\frac{|V+B\nabla h|^2}{2}+\frac{(1+|\nabla h|^2)B^2}{2}=\frac{B^2-2BV\cdot\nabla h-|V|^2}{2}.$$

Using (2.1), we calculate $\partial_t h=G(h)\phi=B-V\cdot\nabla h$, and

$$\begin{aligned}\partial_t\omega&=\partial_t\phi-T_{\partial_t B}h-T_B\partial_t h\\&=-gh-T_{L_{jk}\zeta_j\zeta_k}h+\tfrac{1}{2}(B^2-2BV\cdot\nabla h-|V|^2)-T_{\partial_t B}h-T_B B+T_B(V\cdot\nabla h)+\varepsilon_1^3\mathcal{O}_{3,1}.\end{aligned}$$

Then, since $V=\nabla\phi-B\nabla h$, we have

$$T_V\nabla\omega=T_V\nabla\phi-T_V(\nabla T_B h)=T_V V+T_V(B\nabla h)-T_V(\nabla T_B h),$$

and we can write

$$\begin{aligned}\partial_t\omega&=-gh-T_{L_{jk}\zeta_j\zeta_k+\partial_t B}h-T_V\nabla\omega+\mathrm{I}+\mathrm{II},\\ \mathrm{I}&:=\tfrac{1}{2}B^2-T_B B-\tfrac{1}{2}|V|^2+T_V V=\tfrac{1}{2}\mathcal{H}(B,B)-\tfrac{1}{2}\mathcal{H}(V,V)=\Omega_2+\varepsilon_1^3\mathcal{O}_{3,1},\\ \mathrm{II}&:=-BV\cdot\nabla h+T_B(V\cdot\nabla h)+T_V(B\nabla h)-T_V(\nabla T_B h)+\varepsilon_1^3\mathcal{O}_{3,1}.\end{aligned}$$

Using (3.18), (B.3), (2.1), and Corollary C.1 (ii), we easily see that

$$L_{jk}\zeta_j\zeta_k+\partial_t B=L_{jk}\zeta_j\zeta_k+|\nabla|\partial_t\phi+\varepsilon_1^2\mathcal{O}_{2,-2}=\ell+\varepsilon_1^2\mathcal{O}_{2,-2}.$$

Moreover, we can verify that II is an acceptable cubic remainder term:

$$\begin{aligned}\mathrm{II}&=-T_{V\cdot\nabla h}B+\mathcal{H}(B,V\cdot\nabla h)+T_V(B\nabla h)-T_V T_B\nabla h-T_V T_{\nabla B}h+\varepsilon_1^3\mathcal{O}_{3,1}\\&=-T_{V\cdot\nabla h}B+T_V T_{\nabla h}B+T_V\mathcal{H}(B,\nabla h)-T_V T_{\nabla B}h+\varepsilon_1^3\mathcal{O}_{3,1}=\varepsilon_1^3\mathcal{O}_{3,1},\end{aligned}$$

and the desired conclusion follows. □

Since our purpose will be to identify quadratic terms as in (3.9)–(3.10), we need a more precise notion of strongly semilinear quadratic errors.

Definition 3.4. Given $t\in[0,T]$, we define $\varepsilon_1^2\mathcal{O}_{2,1}^*$ to be the set of finite linear combinations of terms of the form $S[T_1,T_2]$, where $T_1,T_2\in\{U(t),\overline{U}(t)\}$, and S satisfies

$$\begin{aligned}\mathcal{F}(S[f,g])(\xi)&:=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}s(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta,\\ \|s^{k,k_1,k_2}\|_{S^\infty_\Omega}&\lesssim 2^{-\max(k_1,k_2,0)}(1+2^{3\min(k_1,k_2)}).\end{aligned}\tag{3.19}$$

These correspond precisely to the acceptable quadratic error terms according to (3.10).

We remark that, if S is defined by a symbol as in (3.19) and $p\in[-5,5]$, then

$$S[\mathcal{O}_{m,p},\mathcal{O}_{n,p}]\subseteq\mathcal{O}_{m+n,p+1}. \tag{3.20}$$

This follows by an argument similar to that used in Lemma A.2. As a consequence, given the assumptions (3.1), and with U being defined as in (3.4), we have that $\mathcal{O}^*_{2,1}\subseteq\mathcal{O}_{2,1}$.

Proof of Proposition 3.1. *Step* 1. To diagonalize the principal part of the system (3.16), we define the symmetrizing variables (H,Ψ) by

$$H:=T_{\sqrt{g+\ell}}h \quad\text{and}\quad \Psi:=T_\Sigma T_{1/\sqrt{g+\ell}}\omega+T_{m'}\omega, \tag{3.21}$$

where m' is as in (3.4). Using (3.13) and Lemma A.7, we see that

$$\begin{aligned} &H=\operatorname{Re}(U)+\varepsilon_1^2\mathcal{O}_{2,0}, &&\sqrt{g+\sigma|\nabla|^2}h=\operatorname{Re}(U)+\varepsilon_1^2\mathcal{O}_{2,0},\\ &\Psi=\operatorname{Im}(U)+\varepsilon_1^2\mathcal{O}_{2,0}, &&|\nabla|^{1/2}\omega=\operatorname{Im}(U)+\varepsilon_1^2\mathcal{O}_{2,0}.\end{aligned} \tag{3.22}$$

As a consequence, if $T_1,T_2\in\{U,\overline{U},H,\Psi,(g-\sigma\Delta)^{1/2}h,|\nabla|^{1/2}\omega\}$, and S is as in (3.19), then

$$S[T_1,T_2]\in\varepsilon_1^2\mathcal{O}^*_{2,1}+\varepsilon_1^3\mathcal{O}_{3,0}. \tag{3.23}$$

We will show that

$$\begin{cases}\partial_tH-T_\Sigma\Psi+iT_{V\cdot\zeta}H=-T_\gamma H-\frac12T_{\sqrt{g+\ell}\,\operatorname{div}V}h-T_{m'\Sigma}\omega+\varepsilon_1^2\mathcal{O}^*_{2,1}+\varepsilon_1^3\mathcal{O}_{3,0},\\ \partial_t\Psi+T_\Sigma H+iT_{V\cdot\zeta}\Psi=-\frac12T_\gamma\Psi-T_{m'(g+\ell)}h+\frac12T_{\sqrt{\lambda_{DN}}\,\operatorname{div}V}\omega+\varepsilon_1^2\mathcal{O}^*_{2,1}+\varepsilon_1^3\mathcal{O}_{3,0}.\end{cases} \tag{3.24}$$

Step 2. We examine now the first equation in (3.24). The first equation in (3.16) and the identity $\operatorname{div}T_Vh=\frac12T_{\operatorname{div}V}h+iT_{V\cdot\zeta}h$ show that

$$\begin{aligned}&\partial_tH-T_\Sigma\Psi+iT_{V\cdot\zeta}H+T_\gamma H+\tfrac12T_{\sqrt{g+\ell}\,\operatorname{div}V}h+T_{m'\Sigma}\omega\\ &\quad=(T_{\sqrt{g+\ell}}T_{\lambda_{DN}}-T_\Sigma T_\Sigma T_{1/\sqrt{g+\ell}})\omega-(T_\Sigma T_{m'}-T_{m'\Sigma})\omega\\ &\qquad+i(T_{V\cdot\zeta}H-T_{\sqrt{g+\ell}}T_{V\cdot\zeta}h-iT_\gamma T_{\sqrt{g+\ell}}h)\\ &\qquad+T_{\partial_t\sqrt{g+\ell}}h-\tfrac12(T_{\sqrt{g+\ell}}T_{\operatorname{div}V}-T_{\sqrt{g+\ell}\,\operatorname{div}V})h+T_{\sqrt{g+\ell}}G_2+\varepsilon_1^3T_{\sqrt{g+\ell}}\mathcal{O}_{3,1}.\end{aligned} \tag{3.25}$$

We will treat each line separately. For the first line, we notice that the contribution of low frequencies $P_{\leqslant -9}\omega$ is acceptable. For the high frequencies we use Proposition A.5 to write

$$\begin{aligned}&(T_{\sqrt{g+\ell}}T_{\lambda_{DN}}-T_\Sigma T_\Sigma T_{1/\sqrt{g+\ell}})P_{\geqslant -8}\omega\\ &\quad=\big(T_{\lambda_{DN}\sqrt{g+\ell}}+\tfrac12iT_{\{\sqrt{g+\ell},\lambda_{DN}\}}-(T_{\Sigma^2/\sqrt{g+\ell}}+\tfrac12iT_{\{\Sigma^2,1/\sqrt{g+\ell}\}})\big)P_{\geqslant -8}\omega \qquad (3.26)\\ &\qquad+\bigg(E(\sqrt{g+\ell},\lambda_{DN})-E(\Sigma,\Sigma)T_{1/\sqrt{g+\ell}}-E\bigg(\Sigma^2,\frac{1}{\sqrt{g+\ell}}\bigg)\bigg)P_{\geqslant -8}\omega. \qquad (3.27)\end{aligned}$$

Since

$$\lambda_{DN}\sqrt{g+\ell}=\frac{\Sigma^2}{\sqrt{g+\ell}},\quad \{\sqrt{g+\ell},\lambda_{DN}\}=\Big\{\Sigma^2,\frac{1}{\sqrt{g+\ell}}\Big\},$$

we observe that the expression in (3.26) vanishes. Using (3.13) and Lemma A.8, we see that, up to acceptable cubic terms, we can rewrite the second line of (3.25) as

$$\begin{aligned}&\bigg(E(\sqrt{g+\sigma|\zeta|^2},\lambda_1^{(0)})+E\bigg(-\frac{\Lambda^2 h}{2\sqrt{g+\sigma|\zeta|^2}},|\zeta|\bigg)-(E(\Lambda,\Sigma_1)+E(\Sigma_1,\Lambda))(g-\sigma\Delta)^{-1/2}\\ &\quad -E\bigg(\Lambda^2,\frac{\Lambda^2 h}{2(g+\sigma|\zeta|^2)^{3/2}}\bigg)-2E\bigg(\Lambda\Sigma_1,\frac{1}{\sqrt{g+\sigma|\zeta|^2}}\bigg)-\frac{i}{2}T_{\{\Lambda,m_1'\}}-E(\Lambda,m_1')\bigg)P_{\geqslant -8}\omega\\ &\quad +\varepsilon_1^3\mathcal{O}_{3,0}.\end{aligned}$$

Using (A.39), these terms are easily seen to be acceptable $\varepsilon_1^2\mathcal{O}_{2,1}^*$ quadratic terms.

To control the terms in the second line of the right-hand side of (3.25), we observe that

$$\begin{aligned}&T_{V\cdot\zeta}T_{\sqrt{g+\ell}}h-T_{\sqrt{g+\ell}}T_{V\cdot\zeta}h-iT_\gamma T_{\sqrt{g+\ell}}\\ &\quad =i(T_{\{V\cdot\zeta,\sqrt{g+\ell}\}}-T_{\gamma\sqrt{g+\ell}})h+E(V\cdot\zeta,\sqrt{g+\ell})h\\ &\qquad -E(\sqrt{g+\ell},V\cdot\zeta)h+\tfrac{1}{2}T_{\{\gamma,\sqrt{g+\ell}\}}h-iE(\gamma,\sqrt{g+\ell})h.\end{aligned}\tag{3.28}$$

Using (3.13) and (3.15), we notice that

$$\{V\cdot\zeta,\sqrt{g+\ell}\}=\frac{\zeta_j\partial_k V_j(x)\cdot\sigma\zeta_k}{\sqrt{g+\sigma|\zeta|^2}}+\varepsilon_1^2\mathcal{M}^{1,2}_{N_3-2}=\frac{\sigma\zeta_j\zeta_k\cdot\partial_j\partial_k|\nabla|^{-1/2}\,\mathrm{Im}(U)(x)}{\sqrt{g+\sigma|\zeta|^2}}+\varepsilon_1^2\mathcal{M}^{1,2}_{N_3-2}.$$

Using definition (3.8), it follows that $T_{\{V\cdot\zeta,\sqrt{g+\ell}\}-\gamma\sqrt{g+\ell}}h\in\varepsilon_1^2\mathcal{O}_{2,1}^*+\varepsilon_1^3\mathcal{O}_{3,0}$ as desired. The terms in the second line of (3.28) are also acceptable contributions, as one can see easily by extracting the quadratic parts and using (A.39).

Finally, for the third line, using (3.14), (3.15), and Lemmas A.7 and A.8, we observe that

$$\begin{aligned}T_{\partial_t\sqrt{g+\ell}}h&=T_{(1/2)\Delta(g-\sigma\Delta)\omega/\sqrt{g+\sigma|\zeta|^2}}h+\varepsilon_1^3\mathcal{O}_{3,0},\\ \big(T_{\sqrt{g+\ell}}T_{\operatorname{div}V}-T_{\operatorname{div}V\cdot\sqrt{g+\ell}}\big)h&=(iT_{\{\sqrt{g+\sigma|\zeta|^2},\operatorname{div}V_1\}}+E(\sqrt{g+\sigma|\zeta|^2},\operatorname{div}V_1))h+\varepsilon_1^3\mathcal{O}_{3,0},\\ T_{\sqrt{g+\ell}}G_2+\varepsilon_1^3T_{\sqrt{g+\ell}}\mathcal{O}_{3,1}&=T_{\sqrt{g+\sigma|\zeta|^2}}G_2+\varepsilon_1^3\mathcal{O}_{3,0}.\end{aligned}$$

Using (A.39), the bounds for G_2 in (B.6)–(B.7), and collecting all the estimates above, we obtain the identity in the first line in (3.24).

Step 3. To prove the second identity in (3.24), we first use (3.21) and (3.16) to compute

$$\begin{aligned}\partial_t\Psi+T_\Sigma H+iT_{V\cdot\zeta}\Psi+\tfrac12T_\gamma\Psi+T_{m'(g+\ell)}h-\tfrac12T_{\sqrt{\lambda_{DN}}\,\mathrm{div}\,V}\omega\\ =(T_\Sigma T_{\sqrt{g+\ell}}-T_\Sigma T_{1/\sqrt{g+\ell}}T_{g+\ell})h+(T_{m'(g+\ell)}-T_{m'}T_{g+\ell})h\\ +i\big(T_{V\cdot\zeta}\Psi-\tfrac12 iT_\gamma\Psi-(T_\Sigma T_{1/\sqrt{g+\ell}}+T_{m'})T_{V\cdot\zeta}\omega\big)\\ +\tfrac12(T_\Sigma T_{1/\sqrt{g+\ell}}T_{\mathrm{div}\,V}-T_{\sqrt{\lambda_{DN}}\,\mathrm{div}\,V})\omega+\tfrac12T_{m'}T_{\mathrm{div}\,V}\omega\\ +[\partial_t,T_\Sigma T_{1/\sqrt{g+\ell}}+T_{m'}]\omega+(T_\Sigma T_{1/\sqrt{g+\ell}}+T_{m'})(\Omega_2+\varepsilon_1^3\mathcal{O}_{3,1}).\end{aligned}\tag{3.29}$$

Again, we verify that all lines after the equality sign give acceptable remainders.

For the terms in the first line, using Proposition A.5, (3.13), and Lemma A.8,

$$\begin{aligned}&(T_\Sigma T_{\sqrt{g+\ell}}-T_\Sigma T_{1/\sqrt{g+\ell}}T_{g+\ell})h\\ &\quad=-T_\Sigma E\bigg(\frac{1}{\sqrt{g+\ell}},g+\ell\bigg)h\\ &\quad=\Lambda\bigg(E\bigg(\frac{\Lambda^2h}{2(g+\sigma|\zeta|^2)^{3/2}},g+\sigma|\zeta|^2\bigg)-E\bigg(\frac{1}{\sqrt{g+\sigma|\zeta|^2}},\Lambda^2h\bigg)\bigg)h+\varepsilon_1^3\mathcal{O}_{3,0}.\end{aligned}$$

Using also (A.39), this gives acceptable contributions. In addition,

$$\begin{aligned}(T_{m'}T_{g+\ell}-T_{m'(g+\ell)})h&=\tfrac12 iT_{\{m',g+\ell\}}h+E(m',g+\ell)h\\ &=i\sigma T_{\zeta\cdot\nabla_x m'}h+E(m',\sigma|\zeta|^2)h+\varepsilon_1^3\mathcal{O}_{3,0}.\end{aligned}$$

This gives acceptable contributions, in view of (3.21) and (A.39).

For the terms in the second line of the right-hand side of (3.29), we observe that

$$\begin{aligned}&T_{V\cdot\zeta}\Psi-\tfrac12 iT_\gamma\Psi-(T_\Sigma T_{1/\sqrt{g+\ell}}+T_{m'})T_{V\cdot\zeta}\omega\\ &\quad=(T_{V\cdot\zeta}T_\Sigma T_{1/\sqrt{g+\ell}}-T_\Sigma T_{1/\sqrt{g+\ell}}T_{V\cdot\zeta})\omega-\tfrac12 iT_\gamma T_\Sigma T_{1/\sqrt{g+\ell}}\omega+\varepsilon_1^3\mathcal{O}_{3,0}\\ &\quad=(T_{V_1\cdot\zeta}T_{|\zeta|^{1/2}}-T_{|\zeta|^{1/2}}T_{V_1\cdot\zeta})\omega-\tfrac12 iT_\gamma T_{|\zeta|^{1/2}}\omega+\varepsilon_1^3\mathcal{O}_{3,0}.\end{aligned}$$

Using the definitions (3.15) and (3.8), we notice that, for $p\in[0,2]$,

$$\{V_1\cdot\zeta,|\zeta|^p\}=\gamma\cdot p|\zeta|^p\quad\text{on }\mathbb{R}^2\times\mathbb{R}^2.\tag{3.30}$$

Thus, the terms in the second line of the right-hand side of (3.29) are acceptable

$$\varepsilon_1^2\mathcal{O}_{2,1}^*+\varepsilon_1^3\mathcal{O}_{3,0}$$

contributions.

It is easy to see, using Lemma A.8 and the definitions, that the terms in the third line of the right-hand side of (3.29) are acceptable. Finally, for the last line in (3.29), we observe that

$$\begin{aligned}[\partial_t, T_\Sigma T_{1/\sqrt{g+\ell}}+T_{m'}]\omega &= T_{\partial_t\Sigma}T_{1/\sqrt{g+\ell}}\omega+T_\Sigma T_{\partial_t(1/\sqrt{g+\ell})}\omega+T_{\partial_t m'}\omega\\ &= T_{\partial_t\Sigma_1}T_{(g+\sigma|\zeta|^2)^{-1/2}}\omega-\Lambda T_{\Delta(g-\sigma\Delta)\omega/2(g+\sigma|\zeta|^2)^{3/2}}\omega\\ &\quad+\tfrac{1}{2}iT_{\partial_t(\operatorname{div}V)(g+\sigma|\zeta|^2)^{-1/2}}\omega+\varepsilon_1^3\mathcal{O}_{3,0},\end{aligned}$$

where we used (3.13) and (3.14). Since

$$\partial_t h=|\nabla|\omega+\varepsilon_1^2\mathcal{O}_{2,-1/2}\quad\text{and}\quad \partial_t V=-\nabla(g+\sigma|\nabla|^2)h+\varepsilon_1^2\mathcal{O}_{2,-2}$$

(see Lemma 3.3 and Proposition B.1), it follows that the terms in the formula above are acceptable. Finally, using the relations in Lemma 3.3, we have

$$(T_\Sigma T_{1/\sqrt{g+\ell}}+T_{m'})(\Omega_2)=\varepsilon_1^3\mathcal{O}_{3,0}+\varepsilon_1^2\mathcal{O}^*_{2,1}\quad\text{and}\quad(T_\Sigma T_{1/\sqrt{g+\ell}}+T_{m'})(\varepsilon_1^3\mathcal{O}_{3,1})=\varepsilon_1^3\mathcal{O}_{3,0}.$$

Therefore, all the terms in the right-hand side of (3.29) are acceptable, which completes the proof of (3.24).

Step 4. Starting from the system (3.24), we now want to write a scalar equation for the complex unknown $U=H+i\Psi$ defined in (3.4). Using (3.24), we readily see that

$$\begin{aligned}&\partial_t U+iT_\Sigma U+iT_{V\cdot\zeta}U=Q_U+\mathcal{N}_U+\varepsilon_1^2\mathcal{O}^*_{2,1}+\varepsilon_1^3\mathcal{O}_{3,0},\\ &\qquad Q_U:=\big(-\tfrac{1}{2}T_{\sqrt{g+\ell}\operatorname{div}V}-iT_{m'(g+\ell)}\big)h+\big(-T_{m'\Sigma}+\tfrac{1}{2}iT_{\sqrt{\lambda_{DN}}\operatorname{div}V}\big)\omega=0,\\ &\qquad \mathcal{N}_U:=-T_\gamma H-\tfrac{1}{2}iT_\gamma\Psi=-\tfrac{1}{4}T_\gamma(3U+\overline{U})+\varepsilon_1^3\mathcal{O}_{3,0},\end{aligned}$$

where Q_U vanishes in view of our choice of m', and $\mathcal{N}_U$ has the special structure as claimed. □

3.3. High-order derivatives

To derive higher-order Sobolev and weighted estimates for U, and hence for h and $|\nabla|^{1/2}\omega$, we need to apply (a suitable notion of) derivatives to the equation (3.6). We will then consider quantities of the form

$$\begin{aligned}W_n&:=(T_\Sigma)^n U, && n\in\big[0,\tfrac{2}{2}N_0\big],\\ Y_{m,p}&:=\Omega^p(T_\Sigma)^m U, && p\in[0,N_1]\text{ and }m\in\big[0,\tfrac{2}{3}N_3\big],\end{aligned}\tag{3.31}$$

for U as in (3.4) and Σ as in (3.3). We have the following consequence of Proposition 3.1.

Proposition 3.5. *With the notation above and γ as in (3.8), we have*

$$\partial_t W_n + iT_\Sigma W_n + iT_{V\cdot\zeta} W_n = T_\gamma(c_n W_n + d_n \overline{W}_n) + \mathcal{B}_{W_n} + \mathcal{C}_{W_n} \tag{3.32}$$

and

$$\partial_t Y_{m,p} + iT_\Sigma Y_{m,p} + iT_{V\cdot\zeta} Y_{m,p} = T_\gamma(c_m Y_{m,p} + d_m \overline{Y}_{m,p}) + \mathcal{B}_{Y_{m,p}} + \mathcal{C}_{Y_{m,p}} \tag{3.33}$$

for some complex numbers c_n and d_n. The cubic terms $\mathcal{C}_{W_n}$ and $\mathcal{C}_{Y_{m,p}}$ satisfy the bounds

$$\|\mathcal{C}_{W_n}\|_{L^2} + \|\mathcal{C}_{Y_{m,p}}\|_{L^2} \lesssim \varepsilon_1^3 (1+t)^{-3/2}. \tag{3.34}$$

The quadratic strongly semilinear terms $\mathcal{B}_{W_n}$ have the form

$$\mathcal{B}_{W_n} = \sum_{\iota_1,\iota_2\in\{+,-\}} F^n_{\iota_1\iota_2}[U_{\iota_1}, U_{\iota_2}], \tag{3.35}$$

where $U_+ := U$, $U_- = \overline{U}$, and the symbols $f = f^n_{\iota_1\iota_2}$ of the bilinear operators $F^n_{\iota_1\iota_2}$ satisfy

$$\|f^{k,k_1,k_2}\|_{S^\infty} \lesssim 2^{(3n/2-1)\max(k_1,k_2,0)}(1+2^{3\min(k_1,k_2)}). \tag{3.36}$$

The quadratic strongly semilinear terms $\mathcal{B}_{Y_{m,p}}$ have the form

$$\mathcal{B}_{Y_{m,p}} = \sum_{\iota_1,\iota_2\in\{+,-\}} \bigg(G^{m,p}_{\iota_1\iota_2}[U_{\iota_1}, \Omega^p U_{\iota_2}] + \sum_{\substack{p_1+p_2\leqslant p\\ \max(p_1,p_2)\leqslant p-1}} H^{m,p,p_1,p_2}_{\iota_1\iota_2}[\Omega^{p_1} U_{\iota_1}, \Omega^{p_2} U_{\iota_2}] \bigg), \tag{3.37}$$

where the symbols $g = g^{m,p}_{\iota_1\iota_2}$ and $h = h^{m,p,p_1,p_2}_{\iota_1\iota_2}$ of the operators $G^{m,p}_{\iota_1\iota_2}$ and $H^{m,p,p_1,p_2}_{\iota_1\iota_2}$ satisfy

$$\begin{aligned} \|g^{k,k_1,k_2}\|_{S^\infty} &\lesssim 2^{(3m/2-1)\max(k_1,k_2,0)}(1+2^{3\min(k_1,k_2)}),\\ \|h^{k,k_1,k_2}\|_{S^\infty} &\lesssim 2^{(3m/2+1)\max(k_1,k_2,0)}(1+2^{\min(k_1,k_2)}). \end{aligned} \tag{3.38}$$

We remark that we have slightly worse information on the quadratic terms $\mathcal{B}_{Y_{m,p}}$, than on the quadratic terms $\mathcal{B}_{W_n}$. This is due mainly to the commutator of the operators Ω^p and $T_{V\cdot\zeta}$, which leads to the additional terms in (3.37). These terms can still be regarded as strongly semilinear, because they do not contain the maximum number of Ω derivatives (they do contain, however, two extra Sobolev derivatives, but this is acceptable due to our choice of N_0 and N_1).

Proof. In this proof we need to expand the definition of our main spaces $\mathcal{O}_{m,p}$ to exponents $p<-N_3$. More precisely, we define, for any $t\in[0,T]$,

$$\|f\|_{\mathcal{O}'_{m,p}} := \begin{cases} \|f\|_{\mathcal{O}_{m,p}}, & \text{if } p\geqslant -N_3,\\ \langle t\rangle^{(m-1)(5/6-20\delta^2)-\delta^2}(\|f\|_{H^{N_0+p}}+\langle t\rangle^{5/6-2\delta^2}\|f\|_{\widetilde{W}^{N_2+p}}), & \text{if } p<-N_3; \end{cases} \tag{3.39}$$

compare with (A.7). As in Lemmas A.7 and A.8, we have the basic imbeddings

$$T_a\mathcal{O}'_{m,p}\subseteq \mathcal{O}'_{m+m_1,p-l_1} \quad \text{and} \quad (T_aT_b-T_{ab})\mathcal{O}'_{m,p}\subseteq \mathcal{O}'_{m+m_1+m_2,p-l_1-l_2+1}, \tag{3.40}$$

if $a\in\mathcal{M}^{l_1,m_1}_{20}$ and $b\in\mathcal{M}^{l_2,m_2}_{20}$. In particular, recalling that (see (3.12))

$$\Sigma-\Lambda\in\varepsilon_1\mathcal{M}^{3/2,1}_{N_3-2} \quad\text{and}\quad \Sigma-\Lambda-\Sigma_1\in\varepsilon_1^2\mathcal{M}^{3/2,2}_{N_3-2}, \tag{3.41}$$

it follows from (3.40) that, for any $n\in\left[0,\frac{2}{3}N_0\right]$,

$$T^n_\Sigma U\in\varepsilon_1\mathcal{O}'_{1,-3n/2} \quad\text{and}\quad T^n_\Sigma U-\Lambda^nU=\sum_{l=0}^{n-1}\Lambda^{n-1-l}(T_{\Sigma-\Lambda})T^l_\Sigma U\in\varepsilon_1^2\mathcal{O}'_{2,-3n/2}. \tag{3.42}$$

Step 1. For $n\in\left[0,\frac{2}{3}N_0\right]$, we first prove that the function $W_n=(T_\Sigma)^nU$ satisfies

$$\begin{aligned} (\partial_t+iT_\Sigma+iT_{V\cdot\zeta})W_n &= T_\gamma(c_nW_n+d_n\overline{W}_n)+\mathcal{N}_{S,n}+\varepsilon_1^3\mathcal{O}'_{3,-3n/2},\\ \mathcal{N}_{S,n} &= \sum_{\iota_1,\iota_2\in\{+,-\}} B^n_{\iota_1\iota_2}[U_{\iota_1},U_{\iota_2}]\in\varepsilon_1^2\mathcal{O}'_{2,-3n/2+1},\\ \|(b^n_{\iota_1\iota_2})^{k,k_1,k_2}\|_{S^\infty_\Omega} &\lesssim (1+2^{3\min(k_1,k_2)})(1+2^{\max(k_1,k_2)})^{3n/2-1}. \end{aligned} \tag{3.43}$$

Indeed, the case $n=0$ follows from Proposition 3.1. Assuming that this is true for some $n<\frac{2}{3}N_0$ and applying T_Σ, we find that

$$\begin{aligned} (\partial_t+iT_\Sigma+iT_{V\cdot\zeta})W_{n+1} &= T_\gamma(c_nW_{n+1}+d_n\overline{W}_{n+1})+i[T_{V\cdot\zeta},T_\Sigma]W_n+[\partial_t,T_\Sigma]W_n\\ &\quad+[T_\Sigma,T_\gamma](c_nW_n+d_n\overline{W}_n)+T_\Sigma\mathcal{N}_{S,n}+\varepsilon_1^3T_\Sigma\mathcal{O}'_{3,-3n/2}. \end{aligned}$$

Using (3.40)–(3.42) and (3.14), it follows that

$$[\partial_t,T_\Sigma]W_n=T_{\partial_t\Sigma_1}\Lambda^nU+\varepsilon_1^3\mathcal{O}'_{3,-3(n+1)/2}, \quad T_\Sigma\mathcal{N}_{S,n}=\Lambda\mathcal{N}_{S,n}+\varepsilon_1^3\mathcal{O}'_{3,-3(n+1)/2},$$

and, using also (3.30),

$$\begin{aligned} [T_\Sigma,T_\gamma](c_nW_n+d_n\overline{W}_n) &= [T_\Lambda,T_\gamma](c_n\Lambda^nU+d_n\Lambda^n\overline{U})+\varepsilon_1^3\mathcal{O}'_{3,-3(n+1)/2},\\ [T_{V\cdot\zeta},T_\Sigma]W_n &= [T_{V_1\cdot\zeta},T_\Lambda]W_n+\varepsilon_1^3\mathcal{O}'_{3,-3(n+1)/2}\\ &= \tfrac{3}{2}iT_\gamma W_{n+1}+\mathcal{N}'(\operatorname{Im}U,\Lambda^nU)+\varepsilon_1^3\mathcal{O}'_{3,-3(n+1)/2}, \end{aligned}$$

where $\mathcal{N}'(\operatorname{Im} U, \Lambda^n U)$ is an acceptable strongly semilinear quadratic term as in (3.43). Since $\partial_t h = |\nabla|\omega + \varepsilon_1^2 \mathcal{O}_{2,-1/2}$, and recalling formulas (3.12) and (3.22), it is easy to see that all the remaining quadratic terms are of the strongly semilinear type described in (3.43). This completes the induction step.

Step 2. We can now prove the proposition. The claims for W_n follow directly from (3.43). It remains to prove the claims for the functions $Y_{m,p}$. Assume that $m \in [0, \frac{2}{3}N_3]$ is fixed. We start from the identity (3.43) with $n=m$, and apply the rotation vector field Ω. Clearly,

$$
\begin{aligned}
(\partial_t + iT_\Sigma + iT_{V\cdot\zeta})Y_{m,p} &= T_\gamma(c_m Y_{m,p} + d_m \overline{Y}_{m,p}) + \Omega^p \mathcal{N}_{S,m} + \varepsilon_1^3 \Omega^p \mathcal{O}_{3,-3m/2} \\
&\quad - i[\Omega^p, T_\Sigma] W_m - i[\Omega^p, T_{V\cdot\zeta}] W_m + [\Omega^p, T_\gamma](c_m W_m + d_m \overline{W}_m).
\end{aligned}
$$

The terms in the first line of the right-hand side are clearly acceptable. It remains to show that the commutators in the second line can also be written as strongly semilinear quadratic terms and cubic terms. Indeed, for $\sigma \in \{\Sigma, V\cdot\zeta, \gamma\}$ and $W \in \{W_m, \overline{W}_m\}$,

$$
[\Omega^p, T_\sigma] W = \sum_{p'=0}^{p-1} c_{p,p'} T_{\Omega_{x,\zeta}^{p-p'}\sigma} \Omega^{p'} W, \tag{3.44}
$$

as a consequence of (A.25). In view of (3.42), we have

$$
\begin{aligned}
\|\Omega^{N_1} W_m\|_{L^2} + \|\langle\nabla\rangle^{N_0-N_3} W_m\|_{L^2} &\lesssim \varepsilon_1 \langle t\rangle^{\delta^2}, \\
\|\Omega^{N_1}(W_m - \Lambda^m U)\|_{L^2} + \|\langle\nabla\rangle^{N_0-N_3}(W_m - \Lambda^m) U\|_{L^2} &\lesssim \varepsilon_1^2 \langle t\rangle^{21\delta^2 - 5/6}
\end{aligned} \tag{3.45}
$$

and, for $q \in \left[0, \frac{1}{2}N_1\right]$,

$$
\begin{aligned}
\|\Omega^q W_m\|_{\widetilde{W}^3} &\lesssim \varepsilon_1 \langle t\rangle^{3\delta^2 - 5/6}, \\
\|\Omega^q (W_m - \Lambda^m U)\|_{\widetilde{W}^3} &\lesssim \varepsilon_1^2 \langle t\rangle^{23\delta^2 - 5/3}.
\end{aligned} \tag{3.46}
$$

By interpolation, and using the fact that $N_0 - N_3 \geqslant \frac{3}{2}N_1$, it follows from (3.45) that

$$
\begin{aligned}
\|\Omega^q \langle\nabla\rangle^{3/2} W_m\|_{L^2} &\lesssim \varepsilon_1 \langle t\rangle^{\delta^2}, \\
\|\Omega^q \langle\nabla\rangle^{3/2} (W_m - \Lambda^m U)\|_{L^2} &\lesssim \varepsilon_1^2 \langle t\rangle^{21\delta^2 - 5/6}
\end{aligned} \tag{3.47}
$$

for $q \in [0, N_1 - 1]$. Moreover, for $\sigma \in \{\Sigma, V\cdot\zeta, \gamma\}$ and $q \in [1, N_1]$, we have

$$
\begin{aligned}
\|\langle\zeta\rangle^{-3/2} \Omega_{x,\zeta}^q \sigma\|_{\mathcal{M}_{20,2}} &\lesssim \varepsilon_1 \langle t\rangle^{2\delta^2}, \\
\|\langle\zeta\rangle^{-3/2} \Omega_{x,\zeta}^q (\sigma - \sigma_1)\|_{\mathcal{M}_{20,2}} &\lesssim \varepsilon_1^2 \langle t\rangle^{22\delta^2 - 5/6},
\end{aligned} \tag{3.48}
$$

while, for $q\in\left[1, \frac{1}{2}N_1\right]$, we also have

$$\begin{aligned} \|\langle\zeta\rangle^{-3/2}\Omega_{x,\zeta}^q\sigma\|_{\mathcal{M}_{20,\infty}} &\lesssim \varepsilon_1\langle t\rangle^{4\delta^2-5/6}, \\ \|\langle\zeta\rangle^{-3/2}\Omega_{x,\zeta}^q(\sigma-\sigma_1)\|_{\mathcal{M}_{20,\infty}} &\lesssim \varepsilon_1^2\langle t\rangle^{24\delta^2-5/3}. \end{aligned} \tag{3.49}$$

See (A.20) for the definition of the norms $\mathcal{M}_{20,q}$. In these estimates σ_1 denotes the linear part of σ, i.e. $\sigma_1\in\{\Sigma_1, V_1\cdot\zeta, \gamma\}$. Therefore, using Lemma A.7 and (3.46)–(3.49),

$$T_{\Omega_{x,\zeta}^{p-p'}\sigma}\Omega^{p'}W = T_{\Omega_{x,\zeta}^{p-p'}\sigma}\Omega^{p'}\Lambda^m U_\pm+\varepsilon_1^3\langle t\rangle^{-8/5}L^2 = T_{\Omega_{x,\zeta}^{p-p'}\sigma_1}\Omega^{p'}\Lambda^m U_\pm+\varepsilon_1^3\langle t\rangle^{-8/5}L^2$$

for $p'\in[0,p-1]$ and $W\in\{W_m,\overline{W}_m\}$. Notice that $T_{\Omega_{x,\zeta}^{p_1}\sigma_1}\Omega^{p_2}\Lambda^m U_\pm$ can be written as

$$H_{\iota_1\iota_2}^{m,p,p_1,p_2}[\Omega^{p_1}U_{\iota_1}, \Omega^{p_2}U_{\iota_2}],$$

with symbols as in (3.38), up to acceptable cubic terms (the loss of one high derivative comes from the case $\sigma_1=V_1\cdot\zeta$). The conclusion of the proposition follows. □

4. Energy estimates I: Setup and the main L^2 lemma

In this section we set up the proof of Proposition 2.2 and collect some of the main ingredients needed in the proof. From now on, we set $g=1$ and $\sigma=1$. With W_n and $Y_{m,p}$ as in (3.31), we define our main energy functional

$$\mathcal{E}_{\mathrm{tot}} := \frac{1}{2}\sum_{0\leqslant n\leqslant 2N_0/3}\|W_n\|_{L^2}^2+\frac{1}{2}\sum_{\substack{0\leqslant m\leqslant 2N_3/3\\ 0\leqslant p\leqslant N_1}}\|Y_{m,p}\|_{L^2}^2. \tag{4.1}$$

We start with the following proposition.

Proposition 4.1. *Assume that* (3.1) *holds. Then,*

$$\|\mathcal{U}(t)\|_{H^{N_0}\cap H_\Omega^{N_1,N_3}}^2 \lesssim \mathcal{E}_{\mathrm{tot}}(t)+\varepsilon_1^3 \quad \textit{and} \quad \mathcal{E}_{\mathrm{tot}}(t)\lesssim \|\mathcal{U}(t)\|_{H^{N_0}\cap H_\Omega^{N_1,N_3}}^2+\varepsilon_1^3, \tag{4.2}$$

where $\mathcal{U}(t)=\langle\nabla\rangle h(t)+i|\nabla|^{1/2}\phi(t)$ *as in Proposition* 2.2*. Moreover,*

$$\frac{d}{dt}\mathcal{E}_{\mathrm{tot}} = \mathcal{B}_0+\mathcal{B}_1+\mathcal{B}_E, \quad \textit{with } |\mathcal{B}_E(t)|\lesssim\varepsilon_1^3(1+t)^{-4/3}. \tag{4.3}$$

The (*bulk*) *terms* $\mathcal{B}_0$ *and* $\mathcal{B}_1$ *are finite sums of the form*

$$\mathcal{B}_l(t) := \sum_{\substack{G\in\mathcal{G}\\ W,W'\in\mathcal{W}_l}}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\mu_l(\xi,\eta)\widehat{G}(\xi-\eta)\widehat{W}(\eta)\widehat{W'}(-\xi)\,d\xi\,d\eta, \tag{4.4}$$

where U and Σ are defined as in Proposition 3.1, $U_+:=U$, $U_-:=\overline{U}$, and

$$\begin{aligned}
&\mathcal{G}:=\{\Omega^a\langle\nabla\rangle^b U_\pm : a\leqslant \tfrac{1}{2}N_1 \text{ and } b\leqslant N_3+2\},\\
&\mathcal{W}_0:=\{\Omega^a T_\Sigma^m U_\pm : \text{either } (a=0 \text{ and } m\leqslant \tfrac{2}{3}N_0) \text{ or } (a\leqslant N_1 \text{ and } m\leqslant \tfrac{2}{3}N_3)\},\\
&\mathcal{W}_1:=\mathcal{W}_0\cup\{(1-\Delta)\Omega^a T_\Sigma^m U_\pm : a\leqslant N_1-1 \text{ and } m\leqslant \tfrac{2}{3}N_3\}.
\end{aligned}\tag{4.5}$$

The symbols $\mu_l=\mu_{l;(G,W,W')}$, $l\in\{0,1\}$, satisfy

$$\begin{aligned}
&\mu_0(\xi,\eta)=c|\xi-\eta|^{3/2}\mathfrak{d}(\xi,\eta),\quad \mathfrak{d}(\xi,\eta):=\chi\bigg(\frac{|\xi-\eta|}{|\xi+\eta|}\bigg)\bigg(\frac{\xi-\eta}{|\xi-\eta|}\cdot\frac{\xi+\eta}{|\xi+\eta|}\bigg)^2 \text{ and } c\in\mathbb{C},\\
&\|\mu_1^{k,k_1,k_2}\|_{S^\infty}\lesssim 2^{-\max(k_1,k_2,0)}2^{3k_1^+},
\end{aligned}\tag{4.6}$$

for any $k,k_1,k_2\in\mathbb{Z}$; see definitions (A.5)–(A.6).

Notice that the a-priori energy estimates we prove here are stronger than standard energy estimates. The terms $\mathcal{B}_0$ and $\mathcal{B}_1$ are strongly semilinear terms, in the sense that they either gain one derivative or contain the depletion factor $\mathfrak{d}$ (which effectively gains one derivative when the modulation is small, compare with (1.28)).

Proof. The bound (4.2) follows from (3.5) and (3.42),

$$\|\langle\nabla\rangle h(t)\|^2_{H^{N_0}\cap H_\Omega^{N_1,N_3}}+\big\||\nabla|^{1/2}\phi(t)\big\|^2_{H^{N_0}\cap H_\Omega^{N_1,N_3}}\lesssim\|U(t)\|^2_{H^{N_0}\cap H_\Omega^{N_1,N_3}}+\varepsilon_1^3\lesssim\mathcal{E}_{\mathrm{tot}}(t)+\varepsilon_1^3.$$

To prove the remaining claims we start from (3.32) and (3.33). For the terms W_n we have

$$\frac{d}{dt}\frac{1}{2}\|W_n\|^2_{L^2}=\mathrm{Re}\langle T_\gamma(c_nW_n+d_n\overline{W}_n),W_n\rangle+\mathrm{Re}\langle\mathcal{B}_{W_n},W_n\rangle+\mathrm{Re}\langle\mathcal{C}_{W_n},W_n\rangle,\tag{4.7}$$

since, as a consequence of Lemma A.3 (ii),

$$\mathrm{Re}\langle iT_\Sigma W_n+iT_{V\cdot\zeta}W_n,W_n\rangle=0.$$

Clearly, $|\langle\mathcal{C}_{W_n},W_n\rangle|\lesssim\varepsilon_1^3\langle t\rangle^{-3/2+2\delta^2}$, so the last term can be placed in $\mathcal{B}_E(t)$. Moreover, using (3.8) and the definitions, $\langle T_\gamma(c_nW_n+d_n\overline{W}_n),W_n\rangle$ can be written in the Fourier space as part of the term $\mathcal{B}_0(t)$ in (4.4).

Finally, $\langle\mathcal{B}_{W_n},W_n\rangle$ can be written in the Fourier space as part of the term $\mathcal{B}_1(t)$ in (4.4) plus acceptable errors. Indeed, given a symbol f as in (3.36), one can write

$$f(\xi,\eta)=\mu_1(\xi,\eta)((1+\Lambda(\xi-\eta)^n)+(1+\Lambda(\eta)^n)),\quad \mu_1(\xi,\eta):=\frac{f(\xi,\eta)}{2+\Lambda(\xi-\eta)^n+\Lambda(\eta)^n}.$$

The symbol μ_1 satisfies the required estimate in (4.6). The summands $1+\Lambda(\xi-\eta)^n$ and $1+\Lambda(\eta)^n$ can be combined with the functions $\widehat{U}_{\iota_1}(\xi-\eta)$ and $\widehat{U}_{\iota_2}(\eta)$, respectively. Recalling that $\Lambda^nU-W_n\in\varepsilon_1^2\mathcal{O}'_{2,-3n/2}$ (see (3.42)), the desired representation (4.4) follows, up to acceptable errors.

The analysis of the terms $Y_{m,p}$ is similar, using (3.37)–(3.38). This completes the proof. □

In view of (4.2), to prove Proposition 2.2 it suffices to prove that

$$|\mathcal{E}_{\rm tot}(t)-\mathcal{E}_{\rm tot}(0)|\lesssim\varepsilon_1^3\langle t\rangle^{2\delta^2}\quad\text{for any } t\in[0,T].$$

In view of (4.3), it suffices to prove that, for $l\in\{0,1\}$,

$$\left|\int_0^t\mathcal{B}_l(s)\,ds\right|\lesssim\varepsilon_1^3(1+t)^{2\delta^2},$$

for any $t\in[0,T]$. Given $t\in[0,T]$, we fix a suitable decomposition of the function $\mathbf{1}_{[0,t]}$, i.e. we fix functions $q_0,...,q_{L+1}:\mathbb{R}\to[0,1]$, $|L-\log_2(2+t)|\leqslant 2$, with the properties

$$\begin{gathered}\operatorname{supp} q_0\subseteq[0,2],\ \ \operatorname{supp} q_{L+1}\subseteq[t-2,t],\ \ \operatorname{supp} q_m\subseteq[2^{m-1},2^{m+1}]\text{ for } m\in\{1,...,L\},\\ \sum_{m=0}^{L+1}q_m(s)=\mathbf{1}_{[0,t]}(s),\ \ q_m\in C^1(\mathbb{R}),\ \text{ and }\ \int_0^t|q_m'(s)|\,ds\lesssim 1\text{ for } m\in\{1,...,L\}.\end{gathered}\tag{4.8}$$

It remains to prove that, for $l\in\{0,1\}$ and $m\in\{0,...,L+1\}$,

$$\left|\int_{\mathbb{R}}\mathcal{B}_l(s)q_m(s)\,ds\right|\lesssim\varepsilon_1^3 2^{2\delta^2 m}.\tag{4.9}$$

In order to be able to use the hypothesis $\|\mathcal{V}(s)\|_Z\leqslant\varepsilon_1$ (see (2.6)), we need to modify slightly the functions G that appear in the terms $\mathcal{B}_l$. More precisely, we define

$$\mathcal{G}':=\big\{\Omega^a\langle\nabla\rangle^b\mathcal{U}_\iota:\iota\in\{+,-\},a\leqslant\tfrac{1}{2}N_1\text{ and } b\leqslant N_3+2\big\},\tag{4.10}$$

where $\mathcal{U}=\langle\nabla\rangle h+i|\nabla|^{1/2}\phi$, $\mathcal{U}_+=\mathcal{U}$, and $\mathcal{U}_-=\bar{\mathcal{U}}$. Then, we define the modified bilinear terms

$$\mathcal{B}_l'(t):=\sum_{\substack{G\in\mathcal{G}'\\ W,W'\in\mathcal{W}_l}}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\mu_l(\xi,\eta)\widehat{G}(\xi-\eta,t)\widehat{W}(\eta,t)\widehat{W'}(-\xi,t)\,d\xi\,d\eta,\tag{4.11}$$

where the sets $\mathcal{W}_0$ and $\mathcal{W}_1$ are as in (4.5), and the symbols μ_0 and μ_1 are as in (4.6). In view of (3.5), $U(t)-\mathcal{U}(t)\in\varepsilon_1^2\mathcal{O}_{2,0}$. Therefore, simple estimates as in the proof of Lemma A.2 show that

$$|\mathcal{B}_l(s)|\lesssim\varepsilon_1^3(1+s)^{-4/5}\quad\text{and}\quad|\mathcal{B}_l(s)-\mathcal{B}_l'(s)|\lesssim\varepsilon_1^3(1+s)^{-8/5}.$$

As a result of these reductions, for Proposition 2.2 it suffices to prove the following.

Proposition 4.2. *Assume that* (h,ϕ) *is a solution of the system* (2.1) *with* $g=1$, $\sigma=1$ *on* $[0,T]$, *and let* $\mathcal{U}=\langle\nabla\rangle h+i|\nabla|^{1/2}\phi$, $\mathcal{V}(t)=e^{it\Lambda}\mathcal{U}(t)$. *Assume that*

$$\langle t\rangle^{-\delta^2}\|\mathcal{U}(t)\|_{H^{N_0}\cap H_\Omega^{N_1,N_3}}+\|\mathcal{V}(t)\|_Z\leqslant\varepsilon_1, \tag{4.12}$$

for any $t\in[0,T]$, *see* (2.6). *Then, for any* $m\in[\mathcal{D}^2,L]$ *and* $l\in\{0,1\}$,

$$\left|\int_{\mathbb{R}}\iint_{\mathbb{R}^2\times\mathbb{R}^2}q_m(s)\mu_l(\xi,\eta)\widehat{G}(\xi-\eta,s)\widehat{W}(\eta,s)\widehat{W'}(-\xi,s)\,d\xi\,d\eta\,ds\right|\lesssim\varepsilon_1^3 2^{2\delta^2 m}, \tag{4.13}$$

where $G\in\mathcal{G}'$ (*see* (4.10)), *and* $W,W'\in\mathcal{W}':=\mathcal{W}_1$ (*see* (4.5)), *and* q_m *are as in* (4.8). *The symbols* μ_0 *and* μ_1 *satisfy the bounds* (*compare with* (4.6))

$$\begin{aligned}&\mu_0(\xi,\eta)=|\xi-\eta|^{3/2}\mathfrak{d}(\xi,\eta),\quad \mathfrak{d}(\xi,\eta):=\chi\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right)\left(\frac{\xi-\eta}{|\xi-\eta|}\cdot\frac{\xi+\eta}{|\xi+\eta|}\right)^2,\\ &\|\mu_1^{k,k_1,k_2}\|_{S^\infty}\lesssim 2^{-\max(k_1,k_2,0)}2^{3k_1^+}.\end{aligned} \tag{4.14}$$

The proof of this proposition will be done in several steps. We remark that both the symbols μ_0 and μ_1 introduce certain *strongly semilinear* structures. The symbols μ_0 contain the depletion factor $\mathfrak{d}$, which counts essentially as a gain of one high derivative in resonant situations. The symbols μ_1 clearly contain a gain of one high derivative.

We will need to further subdivide the expression in (4.13) into the contributions of "good frequencies" with optimal decay and the "bad frequencies" with slower decay. Let

$$\chi_{\mathrm{ba}}(x):=\varphi(2^{\mathcal{D}}(|x|-\gamma_0))+\varphi(2^{\mathcal{D}}(|x|-\gamma_1))\quad\text{and}\quad\chi_{\mathrm{go}}(x):=1-\chi_{\mathrm{ba}}(x), \tag{4.15}$$

where $\gamma_0=\sqrt{\frac{1}{3}(2\sqrt{3}-3)}$ is the radius of the sphere of degenerate frequencies, and $\gamma_1=\sqrt{2}$ is the radius of the sphere of space-time resonances. We then define, for $l\in\{0,1\}$ and $Y\in\{\mathrm{go},\mathrm{ba}\}$,

$$\mathcal{A}_Y^l[F;H_1,H_2]:=\iint_{\mathbb{R}^2\times\mathbb{R}^2}\mu_l(\xi,\eta)\chi_Y(\xi-\eta)\widehat{F}(\xi-\eta)\widehat{H}_1(\eta)\widehat{H}_2(-\xi)\,d\xi\,d\eta. \tag{4.16}$$

In the proof of (4.13) we will need to distinguish between functions G and W that originate from $U=U_+$ and functions that originate from $\overline{U}=U_-$. For this, we define, for $\iota\in\{+,-\}$,

$$\mathcal{G}'_\iota:=\left\{\Omega^a\langle\nabla\rangle^b\mathcal{U}_\iota:a\leqslant\tfrac{1}{2}N_1\text{ and }b\leqslant N_3+2\right\} \tag{4.17}$$

and

$$\begin{aligned}\mathcal{W}'_\iota:=\big\{\langle\nabla\rangle^c\Omega^aT_\Sigma^m U_\iota:\ &\text{either }\big(a=c=0\text{ and }m\leqslant\tfrac{2}{3}2N_0\big)\\ &\text{or }\big(c\in\{0,2\},\,\tfrac{1}{2}c+a\leqslant N_1,\text{ and }m\leqslant\tfrac{2}{3}N_3\big)\big\}.\end{aligned} \tag{4.18}$$

4.1. Some lemmas

In this subsection we collect some lemmas that are used often in the proofs in the next section. We will often use the following Schur's test.

LEMMA 4.3. (Schur's lemma) *Consider the operator T given by*

$$Tf(\xi)=\int_{\mathbb{R}^2} K(\xi,\eta)f(\eta)\,d\eta.$$

Assume that

$$\sup_{\xi}\int_{\mathbb{R}^2}|K(\xi,\eta)|\,d\eta\leqslant K_1 \quad \textit{and} \quad \sup_{\eta}\int_{\mathbb{R}^2}|K(\xi,\eta)|\,d\xi\leqslant K_2.$$

Then,

$$\|Tf\|_{L^2}\lesssim\sqrt{K_1K_2}\|f\|_{L^2}.$$

We will also use a lemma about functions in $\mathcal{G}'_+$ and $\mathcal{W}'_+$.

LEMMA 4.4. (i) *Assume $G\in\mathcal{G}'_+$ (see (4.17)). Then,*

$$\sup_{|\alpha|+2a\leqslant 30}\|D^\alpha\Omega^a[e^{it\Lambda}G(t)]\|_{Z_1}\lesssim\varepsilon_1, \quad \|G(t)\|_{H^{N_1-2}\cap H_\Omega^{N_1/2-1,0}}\lesssim\varepsilon_1\langle t\rangle^{\delta^2} \tag{4.19}$$

for any $t\in[0,T]$. Moreover, G satisfies the equation

$$(\partial_t+i\Lambda)G=\mathcal{N}_G, \quad \|\mathcal{N}_G(t)\|_{H^{N_1-4}\cap H_\Omega^{N_1/2-2,0}}\lesssim\varepsilon_1^2\langle t\rangle^{-5/6+\delta}. \tag{4.20}$$

(ii) *Assume $W\in\mathcal{W}'_+$ ((4.18)). Then,*

$$\|W(t)\|_{L^2}\lesssim\varepsilon_1\langle t\rangle^{\delta^2} \tag{4.21}$$

for any $t\in[0,T]$. Moreover, W satisfies the equation

$$(\partial_t+i\Lambda)W=\mathcal{Q}_W+\mathcal{E}_W, \tag{4.22}$$

where, with $\Sigma_{\geqslant 2}:=\Sigma-\Lambda-\Sigma_1\in\varepsilon_1^2\mathcal{M}^{3/2,2}_{N_3-2}$ as in (3.12),

$$\mathcal{Q}_W=-iT_{\Sigma_{\geqslant 2}}W-iT_{V\cdot\zeta}W, \quad \|\langle\nabla\rangle^{-1/2}\mathcal{E}_W\|_{L^2}\lesssim\varepsilon_1^2\langle t\rangle^{-5/6+\delta}. \tag{4.23}$$

Using Lemma A.3, we see that, for all $k\in\mathbb{Z}$ and $t\in[0,T]$,

$$\begin{aligned}\|(P_kT_{V\cdot\zeta}W)(t)\|_{L^2}&\lesssim\varepsilon_1 2^{k^+}\langle t\rangle^{-5/6+\delta}\|P_{[k-2,k+2]}W(t)\|_{L^2},\\ \|(P_kT_{\Sigma_{\geqslant 2}}W)(t)\|_{L^2}&\lesssim\varepsilon_1^2 2^{3k^+/2}\langle t\rangle^{-5/3+\delta}\|P_{[k-2,k+2]}W(t)\|_{L^2}.\end{aligned} \tag{4.24}$$

Proof. The claims in (4.19) follow from Definition 2.5, the assumptions (4.12), and interpolation (recall that $N_0-N_3=2N_1$). The identities (4.20) follow from (3.4)–(3.6), since $(\partial_t+i\Lambda)\mathcal{U}\in\varepsilon_1^2\mathcal{O}_{2,-2}$. The inequalities (4.21) follow from (3.42). The identities (4.22)–(4.23) follow from Proposition 3.5, since all quadratic terms that lose up to $\frac{1}{2}$ derivatives can be placed into $\mathcal{E}_W$. Finally, the bounds (4.24) follow from (A.22) and (A.48). □

Next, we summarize some properties of the linear profiles of the functions in $\mathcal{G}'_+$.

LEMMA 4.5. *Assume $G\in\mathcal{G}'_+$ as before, and let $f=e^{it\Lambda}G$. Recall the operators Q_{jk} and $A_{n,\gamma}, A_{n,\gamma}^{(j)}$ defined in (2.10)–(2.18). For $(k,j)\in\mathcal{J}$ and $n\in\{0,...,j+1\}$ let*

$$f_{j,k}:=P_{[k-2,k+2]}Q_{jk}f \quad \text{and} \quad f_{j,k,n}:=A_{n,\gamma_1}^{(j)}f_{j,k}.$$

Then, if $m\geqslant 0$, for all $t\in[2^m-1,2^{m+1}]$ we have

$$\begin{aligned}\sup_{|\alpha|+2a\leqslant 30}\|D^\alpha\Omega^a f(t)\|_{Z_1}\lesssim\varepsilon_1, \quad \|f(t)\|_{H^{N_1-2}\cap H_\Omega^{N_1/2-1,0}}\lesssim\varepsilon_1 2^{\delta^2 m},\\ \|P_k\partial_t f(t)\|_{L^2}\lesssim\varepsilon_1^2 2^{-8k^+}2^{-5m/6+\delta m}, \quad \|P_k e^{-it\Lambda}\partial_t f(t)\|_{L^\infty}\lesssim\varepsilon_1^2 2^{-5m/3+\delta m}.\end{aligned}\tag{4.25}$$

Also, the following L^∞ bounds hold for any $k\in\mathbb{Z}$ and $s\in\mathbb{R}$ with $|s-t|\leqslant 2^{m-\delta m}$:

$$\begin{aligned}\|e^{-is\Lambda}A_{\leqslant 2\mathcal{D},\gamma_0}P_k f(t)\|_{L^\infty}\lesssim\varepsilon_1\min(2^{k/2},2^{-4k})2^{-m}2^{52\delta m},\\ \|e^{-is\Lambda}A_{\geqslant 2\mathcal{D}+1,\gamma_0}P_k f(t)\|_{L^\infty}\lesssim\varepsilon_1 2^{-5m/6+3\delta^2 m}.\end{aligned}\tag{4.26}$$

Moreover, we have

$$\begin{aligned}\|e^{-is\Lambda}f_{j,k}(t)\|_{L^\infty}\lesssim\varepsilon_1\min(2^k,2^{-4k})2^{-j+50\delta j},\\ \|e^{-is\Lambda}f_{j,k}(t)\|_{L^\infty}\lesssim\varepsilon_1\min(2^{3k/2},2^{-4k})2^{-m+50\delta j}, \quad \text{if } |k|\geqslant 10.\end{aligned}\tag{4.27}$$

Away from the bad frequencies, we have the stronger bound

$$\|e^{-is\Lambda}A_{\leqslant 2\mathcal{D},\gamma_0}A_{\leqslant 2\mathcal{D},\gamma_1}f_{j,k}(t)\|_{L^\infty}\lesssim\varepsilon_1 2^{-m}\min(2^k,2^{-4k})2^{-j/4},\tag{4.28}$$

provided that $j\leqslant(1-\delta^2)m+\frac{1}{2}|k|$ and $|k|+D\leqslant\frac{1}{2}m$.

Finally, for all $n\in\{0,1,...,j\}$, we have

$$\begin{aligned}\|\hat{f}_{j,k,n}\|_{L^\infty}\lesssim\varepsilon_1 2^{2\delta^2 m}2^{-4k^+}2^{3\delta n}\cdot 2^{-(1/2-55\delta)(j-n)},\\ \Big\|\sup_{\theta\in\mathbb{S}^1}|\hat{f}_{j,k,n}(r\theta)|\Big\|_{L^2(r\,dr)}\lesssim\varepsilon_1 2^{2\delta^2 m}2^{-4k^+}2^{n/2}2^{-j+55\delta j}.\end{aligned}\tag{4.29}$$

Proof. The estimates in the first line of (4.25) follow from (4.19). The estimates (4.26), (4.27), and (4.29) then follow from Lemma 7.5, while the estimate (4.28) follows from (7.53). Finally, the estimate on $\partial_t f$ in (4.25) follows from the bound (8.7). □

We prove now a lemma that is useful when estimating multilinear expression containing a localization in the modulation Φ.

LEMMA 4.6. *Assume that* $k,k_1,k_2\in\mathbb{Z}$, $m\geqslant\mathcal{D}$, $\bar{k}:=\max(k,k_1,k_2)$, $|k|\leqslant\frac{1}{2}m$, $p\geqslant -m$. *Assume that* $\underline{\mu}_0$ *and* $\underline{\mu}_1$ *are symbols supported in the set* $\mathcal{D}_{k_2,k,k_1}$ *and satisfying*

$$\begin{gathered}\underline{\mu}_0(\xi,\eta)=\mu_0(\xi,\eta)n(\xi,\eta),\quad \underline{\mu}_1(\xi,\eta)=\mu_1(\xi,\eta)n(\xi,\eta),\quad \|n\|_{S^\infty}\lesssim 1,\\ \mu_0(\xi,\eta)=|\xi-\eta|^{3/2}\mathfrak{d}(\xi,\eta),\quad \|\mu_1(\xi,\eta)\|_{S^\infty}\lesssim 2^{3k^+-\bar{k}^+}\end{gathered}\tag{4.30}$$

(*compare with* (4.14)). *For* $l\in\{0,1\}$ *and* $\Phi=\Phi_{\sigma\mu\nu}$ *as in* (10.1), *let*

$$\mathcal{I}_p^l[F;H_1,H_2]=\iint_{(\mathbb{R}^2)^2}\underline{\mu}_l(\xi,\eta)\psi_p(\Phi(\xi,\eta))\widehat{P_kF}(\xi-\eta)\widehat{P_{k_1}H_1}(\eta)\widehat{P_{k_2}H_2}(-\xi)\,d\xi\,d\eta,$$

where $\psi\in C_0^\infty(-1,1)$ *and* $\psi_p(x):=\psi(2^{-p}x)$. *Then,*

$$\begin{aligned}|\mathcal{I}_p^0[F;H_1,H_2]|&\lesssim 2^{3k/2}\min(1,2^{-\bar{k}^+}2^{\max(2p,3k^+)}2^{-2k})N(P_kF)\|P_{k_1}H_1\|_{L^2}\,\|P_{k_2}H_2\|_{L^2},\\ |\mathcal{I}_p^1[F;H_1,H_2]|&\lesssim 2^{3k^+-\bar{k}^+}N(P_kF)\|P_{k_1}H_1\|_{L^2}\,\|P_{k_2}H_2\|_{L^2},\end{aligned}\tag{4.31}$$

where

$$N(P_kF):=\sup_{|\varrho|\leqslant 2^{-p}2^{\delta m}}\|e^{i\varrho\Lambda}P_kF\|_{L^\infty}+2^{-10m}\|P_kF\|_{L^2}.\tag{4.32}$$

In particular, if $2^k\approx 1$, *then*

$$\begin{aligned}|\mathcal{I}_p^0[F;H_1,H_2]|&\lesssim\min(1,2^{2p^+-\bar{k}^+})N(P_kF)\|P_{k_1}H_1\|_{L^2}\,\|P_{k_2}H_2\|_{L^2},\\ |\mathcal{I}_p^1[F;H_1,H_2]|&\lesssim 2^{-\bar{k}^+}N(P_kF)\|P_{k_1}H_1\|_{L^2}\,\|P_{k_2}H_2\|_{L^2}.\end{aligned}\tag{4.33}$$

Proof. The proof when $l=1$ is easy. We start from the formula

$$\psi_p(\Phi(\xi,\eta))=C\int_R\hat{\psi}(s)e^{is2^{-p}\Phi(\xi,\eta)}\,ds.\tag{4.34}$$

Therefore,

$$\mathcal{I}_p^1[F;H_1,H_2]=C\int_{\mathbb{R}}\hat{\psi}(s)\iint_{(\mathbb{R}^2)^2}e^{is2^{-p}\Phi(\xi,\eta)}\underline{\mu}_1(\xi,\eta)\widehat{P_kF}(\xi-\eta)\widehat{P_{k_1}H_1}(\eta)\widehat{P_kH_2}(-\xi)\,d\xi\,d\eta.$$

Using Lemma A.1 (i) and (4.30), it follows that

$$|\mathcal{I}_p^1[F;H_1,H_2]|\lesssim\int_{\mathbb{R}}|\hat{\psi}(s)|\,2^{3k^+-\bar{k}^+}\|e^{-is2^{-p}\Lambda_\mu}P_kF\|_{L^\infty}\,\|P_{k_1}H_1\|_{L^2}\,\|P_{k_2}H_2\|_{L^2}\,ds.$$

The bound for $l=1$ in (4.33) follows.

In the case $l=0$, the desired bound follows in the same way unless

$$\bar{k}^{+}+2k \geqslant \max(2p, 3k^{+})+\mathcal{D}. \tag{4.35}$$

On the other hand, if (4.35) holds, then we need to take advantage of the depletion factor $\mathfrak{d}$. The main point is the following:

$$\text{if (4.35) holds and } |\Phi(\xi,\eta)| \lesssim 2^{p}, \text{ then } \mathfrak{d}(\xi,\eta) \lesssim \frac{2^{-\bar{k}}(2^{2p}+2^{3k^{+}})}{2^{2k}}. \tag{4.36}$$

Indeed, if (4.35) holds then $\bar{k}\geqslant\mathcal{D}$ and $p\leqslant\frac{3}{2}\bar{k}-\frac{1}{4}\mathcal{D}$, and we estimate

$$\mathfrak{d}(\xi,\eta) \lesssim \left(\frac{|\xi|-|\eta|}{|\xi-\eta|}\right)^{2} \lesssim \left(\frac{2^{-\bar{k}/2}\big|\lambda(|\xi|)-\lambda(|\eta|)\big|}{2^{k}}\right)^{2} \lesssim \frac{2^{-\bar{k}}(|\Phi(\xi,\eta)|+\lambda(|\xi-\eta|))^{2}}{2^{2k}}$$

in the support of the function $\mathfrak{d}$, which gives (4.36).

To continue the proof, we fix a function $\theta\in C_0^{\infty}(\mathbb{R}^2)$ supported in the ball of radius $2^{k^{+}+1}$ with the property that $\sum_{v\in(2^{k^{+}}\mathbb{Z})^2}\theta(x-v)=1$ for any $x\in\mathbb{R}^2$. For any $v\in(2^{k^{+}}\mathbb{Z})^2$, consider the operator Q_v defined by

$$\widehat{Q_v f}(\xi)=\theta(\xi-v)\hat{f}(\xi).$$

In view of the localization in $(\xi-\eta)$, we have

$$\mathcal{I}_p^0[F;H_1,H_2]=\sum_{|v_1+v_2|\lesssim 2^{k^{+}}}\mathcal{I}^0_{p;v_1,v_2}, \quad \text{with } \mathcal{I}^0_{p;v_1,v_2}:=\mathcal{I}_p^0[F;Q_{v_1}H_1,Q_{v_2}H_2]. \tag{4.37}$$

Moreover, using (4.36), we can insert a factor of $\varphi_{\leqslant\mathcal{D}}(2^{-X}(\xi-\eta)\cdot v_1)$ in the integral defining $\mathcal{I}_p^l[F;Q_{v_1}H_1,Q_{v_2}H_2]$ without changing the integral, where $2^X\approx(2^p+2^{3k^{+}/2})2^{\bar{k}/2}$. Let

$$m_{v_1}(\xi,\eta):=\underline{\mu}_0(\xi,\eta)\varphi_{[k_2-2,k_2+2]}(\xi)\varphi_{[k-2,k+2]}(\xi-\eta)\varphi_{\leqslant k^{+}+2}(\eta-v_1)\varphi_{\leqslant\mathcal{D}}(2^{-X}(\xi-\eta)\cdot v_1).$$

We will show below that, for any $v_1\in\mathbb{R}^2$ with $|v_1|\approx 2^{\bar{k}}$,

$$\|\mathcal{F}^{-1}(m_{v_1})\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)} \lesssim 2^{3k/2}2^{2X}2^{-2k}2^{-2\bar{k}}. \tag{4.38}$$

Assuming this, the desired bound follows as in the case $l=1$ treated earlier. To prove (4.38), we recall that $\|\mathcal{F}^{-1}(ab)\|_{L^1}\lesssim\|\mathcal{F}^{-1}(a)\|_{L^1}\|\mathcal{F}^{-1}(b)\|_{L^1}$. Then, we write

$$(\xi-\eta)\cdot(\xi+\eta)=2(\xi-\eta)\cdot v_1+|\xi-\eta|^2+2(\xi-\eta)\cdot(\eta-v_1).$$

The bound (4.38) follows by analyzing the contributions of the three terms above. □

Our next lemma concerns a linear L^2 estimate on certain localized Fourier integral operators.

LEMMA 4.7. *Assume that* $k\geqslant -100$, $m\geqslant \mathcal{D}^2$,

$$-(1-\delta)m\leqslant p-\tfrac{1}{2}k\leqslant -\delta m, \quad \text{and} \quad 2^{m-2}\leqslant |s|\leqslant 2^{m+2}. \tag{4.39}$$

Given $\chi\in C_0^\infty(\mathbb{R})$ *supported in* $[-1,1]$, *introduce the operator* $L_{p,k}$ *defined by*

$$L_{p,k}f(\xi):=\varphi_{\geqslant -100}(\xi)\int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)}\chi(2^{-p}\Phi(\xi,\eta))\varphi_k(\eta)a(\xi,\eta)f(\eta)\,d\eta, \tag{4.40}$$

where, for some $\mu,\nu\in\{+,-\}$,

$$\begin{gathered}\Phi(\xi,\eta)=\Lambda(\xi)-\Lambda_\mu(\xi-\eta)-\Lambda_\nu(\eta), \quad a(\xi,\eta)=A(\xi,\eta)\chi_{\mathrm{ba}}(\xi-\eta)\hat{g}(\xi-\eta),\\ \|D^\alpha A\|_{L^\infty_{x,y}}\lesssim_{|\alpha|} 2^{|\alpha|m/3}, \quad \text{and} \quad \|g\|_{Z_1\cap H_\Omega^{N_1/3,0}}\lesssim 1.\end{gathered} \tag{4.41}$$

Then,

$$\|L_{p,k}f\|_{L^2}\lesssim 2^{30\delta m}(2^{(3/2)(p-k/2)}+2^{p-k/2-m/3})\|f\|_{L^2}.$$

Remark 4.8. (i) Lemma 4.7, which is proved in §6 below, plays a central role in the proof of Proposition 4.2. A key role in its proof is played by the "curvature" component

$$\Upsilon(\xi,\eta):=(\nabla^2_{\xi,\eta}\Phi)(\xi,\eta)[(\nabla_\xi^\perp\Phi)(\xi,\eta),(\nabla_\eta^\perp\Phi)(\xi,\eta)], \tag{4.42}$$

and in particular by its non-degeneracy close to the bad frequencies γ_0 and γ_1, and to the resonant hypersurface $\Phi(\xi,\eta)=0$. The properties of Υ that we are going to use are described in §11, and in particular in Lemmas 11.1–11.3.

(ii) We can insert S^∞ symbols and bounded factors that depend only on ξ or only on η in the integral in (4.40), without changing the conclusion. We will often use this lemma in the form

$$\begin{aligned}\bigg|\iint_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\Lambda(\xi-\eta)}\chi(2^{-p}\Phi(\xi,\eta))\mu(\xi,\eta)a(\xi,\eta)\widehat{P_{k_1}F_1}(\eta)\widehat{P_kF_2}(-\xi)\,d\xi\,d\eta\bigg|&\\ \lesssim 2^{30\delta m}(2^{(3/2)(p-k/2)}+2^{p-k/2-m/3})\|P_{k_1}F_1\|_{L^2}\,\|P_kF_2\|_{L^2},&\end{aligned} \tag{4.43}$$

provided that $k,k_1\geqslant -80$, (4.39) and (4.41) hold, and $\|\mu\|_{S^\infty}\lesssim 1$. This follows by writing

$$\mu(\xi,\eta)=\iint_{\mathbb{R}^2\times\mathbb{R}^2} P(x,y)e^{-ix\cdot\xi}e^{-iy\cdot\eta}\,d\xi\,d\eta,$$

with $\|P\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}\lesssim 1$, and then combining the oscillatory factors with the functions F_1 and F_2.

5. Energy estimates II: Proof of Proposition 4.2

In this section we prove Proposition 4.2, thus completing the proof of Proposition 2.2. Recall definitions (4.15)–(4.18). For $G\in\mathcal{G}'$ and $W_1, W_2\in\mathcal{W}'$ let

$$\mathcal{A}^l_{Y,m}[G,W_1,W_2]:=\int_{\mathbb{R}} q_m(s)\iint_{\mathbb{R}^2\times\mathbb{R}^2}\mu_l(\xi,\eta)\chi_Y(\xi-\eta)\widehat{G}(\xi-\eta,s)\times\widehat{W_1}(\eta,s)\widehat{W_2}(-\xi,s)\,d\xi\,d\eta\,ds, \tag{5.1}$$

where $l\in\{0,1\}$, $m\in[\mathcal{D}^2,L]$, $Y\in\{\mathrm{go},\mathrm{ba}\}$, and the symbols μ_l are as in (4.14). The conclusion of Proposition 4.2 is equivalent to the uniform bound

$$|\mathcal{A}^l_{Y,m}[G,W_1,W_2]|\lesssim\varepsilon_1^3 2^{2\delta^2 m}. \tag{5.2}$$

In proving this bound, we further decompose the functions W_1 and W_2 dyadically and consider several cases. We remark that the most difficult case (which is treated in Lemma 5.1) is when the "bad" frequencies of G interact with the high frequencies of the functions W_1 and W_2.

5.1. The main interactions

We prove the following lemma.

LEMMA 5.1. *For $l\in\{0,1\}$, $m\in[\mathcal{D}^2,L]$, $G\in\mathcal{G}'$, and $W_1,W_2\in\mathcal{W}'$ we have*

$$\sum_{\min(k_1,k_2)\geqslant -40}|\mathcal{A}^l_{\mathrm{ba},m}[G,P_{k_1}W_1,P_{k_2}W_2]|\lesssim\varepsilon_1^3. \tag{5.3}$$

The rest of the subsection is concerned with the proof of this lemma. We need to further decompose our operators based on the size of the modulation. Assuming that $\overline{W_2}\in\mathcal{W}'_\sigma$, $W_1\in\mathcal{W}'_\nu$, $G\in\mathcal{G}'_\mu$, and $\sigma,\mu,\nu\in\{+,-\}$, see (4.17)–(4.18), we define the associated phase

$$\Phi(\xi,\eta)=\Phi_{\sigma\mu\nu}(\xi,\eta)=\Lambda_\sigma(\xi)-\Lambda_\mu(\xi-\eta)-\Lambda_\nu(\eta). \tag{5.4}$$

Notice that, in proving (5.3), we may assume that $\sigma=+$ (otherwise take complex conjugates) and that the sum is over $|k_1-k_2|\leqslant 50$ (due to localization in $\xi-\eta$).

Some care is needed to properly sum the dyadic pieces in k_1 and k_2. For this, we use frequency envelopes. More precisely, for $k\geqslant -30$, let

$$\begin{aligned}\varrho_k(s)&:=\sum_{i=1}^2\|P_{[k-40,k+40]}W_i(s)\|_{L^2}+2^{5m/6-\delta m}2^{-k/2}\sum_{i=1}^2\|P_{[k-40,k+40]}\mathcal{E}_{W_i}(s)\|_{L^2},\\ \varrho_{k,m}^2&:=\int_{\mathbb{R}}\varrho_k(s)^2(2^{-m}q_m(s)+|q'_m(s)|)\,ds,\end{aligned} \tag{5.5}$$

where $\mathcal{E}_{W_{1,2}}$ are the "semilinear" non-linearities defined in (4.22). In view of (4.21) and (4.23),

$$\sum_{k\geqslant -30}\varrho_{k,m}^2\lesssim \varepsilon_1^2 2^{2\delta^2 m}. \tag{5.6}$$

Given $k\geqslant -30$, let $\underline{p}=\left\lfloor \frac{1}{2}k-\frac{7}{9}m\right\rfloor$ (the largest integer $\leqslant \frac{1}{2}k-\frac{7}{9}m$). We define

$$\begin{aligned}\mathcal{A}_{\mathrm{ba}}^{l,p}[F,H_1,H_2]:=\iint_{\mathbb{R}^2\times\mathbb{R}^2}&\mu_l(\xi,\eta)\varphi_p^{[\underline{p},\infty)}(\Phi(\xi,\eta))\chi_{\mathrm{ba}}(\xi-\eta)\\ &\times\widehat{F}(\xi-\eta)\widehat{H}_1(\eta)\widehat{H}_2(-\xi)\,d\xi\,d\eta,\end{aligned} \tag{5.7}$$

where $p\in[\underline{p},\infty)$ and

$$\varphi_p^{[\underline{p},\infty)}=\begin{cases}\varphi_p, & \text{if } p\geqslant \underline{p}+1,\\ \varphi_{\leqslant\underline{p}}, & \text{if } p=\underline{p}.\end{cases}$$

Assuming that $|k_1-k|\leqslant 30$ and $|k_2-k|\leqslant 30$, let

$$\mathcal{A}_{\mathrm{ba},m}^{l,p}[G,P_{k_1}W_1,P_{k_2}W_2]:=\int_{\mathbb{R}}q_m(s)\mathcal{A}_{\mathrm{ba}}^{l,p}[G(s),P_{k_1}W_1(s),P_{k_2}W_2(s)]\,ds. \tag{5.8}$$

This gives a decomposition $\mathcal{A}_{\mathrm{ba},m}^{l}=\sum_{p\geqslant\underline{p}}\mathcal{A}_{\mathrm{ba},m}^{l,p}$ as a sum of operators localized in modulation. Notice that the sum is either over $p\in\left[\underline{p},\frac{1}{2}k+\mathcal{D}\right]$ (if $\nu=+$ or if $\nu=-$ and $k\leqslant\frac{1}{2}\mathcal{D}$) or over $\left|p-\frac{3}{2}k\right|\leqslant\mathcal{D}$ (if $\nu=-$ and $k>\frac{1}{2}\mathcal{D}$). For (5.3), it remains to prove that

$$|\mathcal{A}_{\mathrm{ba},m}^{l,p}[G,P_{k_1}W_1,P_{k_2}W_2]|\lesssim\varepsilon_1 2^{-\delta m}\varrho_{k,m}^2 \tag{5.9}$$

for any $k\geqslant -30$, $p\geqslant\underline{p}$, and $k_1,k_2\in\mathbb{Z}$ satisfying $|k_1-k|\leqslant 30$ and $|k_2-k|\leqslant 30$.

Using Lemma 4.6 (see (4.33)), we have

$$|\mathcal{A}_{\mathrm{ba}}^{l,p}[G(s),P_{k_1}W_1(s),P_{k_2}W_2(s)]|\lesssim\varepsilon_1 2^{2p^+-k}2^{-5m/6+\delta m}\|P_{k_1}W_1(s)\|_{L^2}\,\|P_{k_2}W_2(s)\|_{L^2}$$

for any $p\geqslant\underline{p}$, due to the L^∞ bound in (4.26). The desired bound (5.9) follows if

$$2p^+-k\leqslant -\tfrac{1}{5}m+\mathcal{D}.$$

Also, using Lemma 4.7, we have

$$|\mathcal{A}_{\mathrm{ba}}^{l,\underline{p}}[G,P_{k_1}W_1,P_{k_2}W_2](s)|\lesssim\varepsilon_1 2^{-m-\delta m}\|P_{k_1}W_1(s)\|_{L^2}\,\|P_{k_2}W_2(s)\|_{L^2},$$

using (4.43), as $2^{\underline{p}-k/2}\lesssim 2^{-7m/9}$ and $\|e^{is\Lambda_\mu}G(s)\|_{Z_1\cap H_\Omega^{N_1/3,0}}\lesssim\varepsilon_1 2^{\delta m}$ (see (4.19)). Thus, (5.9) follows if $p=\underline{p}$. It remains to prove (5.9) when

$$p\geqslant\underline{p}+1\quad\text{and}\quad k\in\left[-30,2p^++\tfrac{1}{5}m\right],\ \ |k_1-k|\leqslant 30,\ \ |k_2-k|\leqslant 30. \tag{5.10}$$

In the remaining range in (5.10) we integrate by parts in s. We define

$$\tilde{\mathcal{A}}^{l,p}_{\mathrm{ba}}[F,H_1,H_2]:=\iint_{\mathbb{R}^2\times\mathbb{R}^2}\mu_l(\xi,\eta)\widetilde{\varphi}_p(\Phi(\xi,\eta))\chi_{\mathrm{ba}}(\xi-\eta)\times\widehat{F}(\xi-\eta)\widehat{H}_1(\eta)\widehat{H}_2(-\xi)\,d\xi\,d\eta, \tag{5.11}$$

where $\widetilde{\varphi}_p(x):=2^px^{-1}\varphi_p(x)$. This is similar to the definition in (5.7), but with φ_p replaced by $\widetilde{\varphi}_p$. Then, we let $W_{k_1}:=P_{k_1}W_1$ and $W_{k_2}:=P_{k_2}W_2$, and write

$$\begin{aligned}0&=\int_{\mathbb{R}}\frac{d}{ds}(q_m(s)\tilde{\mathcal{A}}^{l,p}_{\mathrm{ba}}[G(s),W_{k_1}(s),W_{k_2}(s)])\,ds\\&=\int_{\mathbb{R}}q_m'(s)\tilde{\mathcal{A}}^{l,p}_{\mathrm{ba}}[G(s),W_{k_1}(s),W_{k_2}(s)]\,ds+\mathcal{J}^{l,p}_{\mathrm{ba},0}(k_1,k_2)+\mathcal{J}^{l,p}_{\mathrm{ba},1}(k_1,k_2)+\mathcal{J}^{l,p}_{\mathrm{ba},2}(k_1,k_2)\\&\quad+i2^p\int_{\mathbb{R}}q_m(s)\mathcal{A}^{l,p}_{\mathrm{ba}}[G(s),W_{k_1}(s),W_{k_2}(s)]\,ds,\end{aligned} \tag{5.12}$$

where

$$\begin{aligned}\mathcal{J}^{l,p}_{\mathrm{ba},0}(k_1,k_2)&:=\int_{\mathbb{R}}q_m(s)\tilde{\mathcal{A}}^{l,p}_{\mathrm{ba}}[(\partial_s+i\Lambda_\mu)G(s),W_{k_1}(s),W_{k_2}(s)]\,ds,\\\mathcal{J}^{l,p}_{\mathrm{ba},1}(k_1,k_2)&:=\int_{\mathbb{R}}q_m(s)\tilde{\mathcal{A}}^{l,p}_{\mathrm{ba}}[G(s),(\partial_s+i\Lambda_\nu)W_{k_1}(s),W_{k_2}(s)]\,ds,\\\mathcal{J}^{l,p}_{\mathrm{ba},2}(k_1,k_2)&:=\int_{\mathbb{R}}q_m(s)\tilde{\mathcal{A}}^{l,p}_{\mathrm{ba}}[G(s),W_{k_1}(s),(\partial_s+i\Lambda_{-\sigma})W_{k_2}(s)]\,ds.\end{aligned} \tag{5.13}$$

The integral in the last line of (5.12) is the one we have to estimate. Notice that

$$2^{-p}|\tilde{\mathcal{A}}^{l,p}_{\mathrm{ba}}[G(s),W_{k_1}(s),W_{k_2}(s)]|\lesssim 2^{-p}2^{-5m/6+\delta m}\|W_{k_1}(s)\|_{L^2}\,\|W_{k_2}(s)\|_{L^2},$$

as a consequence of Lemma 4.6 and (4.26). It remains to prove that, if (5.10) holds, then

$$2^{-p}|\mathcal{J}^{l,p}_{\mathrm{ba},0}(k_1,k_2)+\mathcal{J}^{l,p}_{\mathrm{ba},1}(k_1,k_2)+\mathcal{J}^{l,p}_{\mathrm{ba},2}(k_1,k_2)|\lesssim\varepsilon_1 2^{-\delta m}\varrho^2_{k,m}. \tag{5.14}$$

This bound will be proved in several steps in Lemmas 5.2–5.4 below.

5.1.1. Quasilinear terms

We first consider the quasilinear terms appearing in (5.14), which are those where $\partial_t+i\Lambda$ hits the high-frequency inputs W_{k_1} and W_{k_2}. We start with the case when the frequencies k_1 and k_2 are not too large relative to p^+.

LEMMA 5.2. *Assume that (5.10) holds and, in addition,* $k\leqslant\frac{2}{3}p^++\frac{1}{4}m$. *Then,*

$$2^{-p}(|\mathcal{J}^{l,p}_{\mathrm{ba},1}(k_1,k_2)|+|\mathcal{J}^{l,p}_{\mathrm{ba},2}(k_1,k_2)|)\lesssim\varepsilon_1 2^{-\delta m}\varrho^2_{k,m}. \tag{5.15}$$

Proof. It suffices to bound the contributions of $|\mathcal{J}^{l,p}_{\mathrm{ba},1}(k_1,k_2)|$ in (5.15), since the contributions of $|\mathcal{J}^{l,p}_{\mathrm{ba},2}(k_1,k_2)|$ are similar. We estimate, for $s\in[2^{m-1},2^{m+1}]$,

$$\|(\partial_s+i\Lambda_\nu)W_{k_1}(s)\|_{L^2}\lesssim\varepsilon_1 2^{-5m/6+\delta m}(2^{k_1}+2^{3k_1/2}2^{-5m/6})\varrho_k(s), \tag{5.16}$$

using (4.22)–(4.24). As before, we use Lemma 4.6 and the pointwise bound (4.26) to estimate

$$\begin{aligned}&|\tilde{\mathcal{A}}^{l,p}_{\mathrm{ba}}[G(s),(\partial_s+i\Lambda_\nu)W_{k_1}(s),W_{k_2}(s)]|\\&\quad\lesssim\min(1,2^{2p^+-k})\varepsilon_1 2^{-5m/6+\delta m}\|(\partial_s+i\Lambda_\nu)W_{k_1}(s)\|_{L^2}\,\|W_{k_2}(s)\|_{L^2}.\end{aligned} \tag{5.17}$$

The bounds (5.16) and (5.17) suffice to prove (5.15) when $p\geqslant 0$ or when $-\frac{1}{2}m+\frac{1}{2}k\leqslant p\leqslant 0$.

It remains to prove (5.15) when

$$\underline{p}+1\leqslant p\leqslant -\tfrac{1}{2}m+\tfrac{1}{2}k \quad\text{and}\quad k\leqslant\tfrac{1}{5}m. \tag{5.18}$$

For this, we would like to apply Lemma 4.7. We claim that, for $s\in[2^{m-1},2^{m+1}]$,

$$\begin{aligned}&|\tilde{\mathcal{A}}^{l,p}_{\mathrm{ba}}[G(s),(\partial_s+i\Lambda_\nu)W_{k_1}(s),W_{k_2}(s)]|\\&\quad\lesssim 2^{-k}\varepsilon_1 2^{31\delta m}(2^{(3/2)(p-k/2)}+2^{p-k/2-m/3})\|(\partial_s+i\Lambda_\nu)W_{k_1}(s)\|_{L^2}\,\|W_{k_2}(s)\|_{L^2}.\end{aligned} \tag{5.19}$$

Assuming this, and using also (5.16), it follows that

$$\begin{aligned}2^{-p}|\mathcal{J}^{l,p}_{\mathrm{ba},1}(k_1,k_2)|&\lesssim 2^{-p}2^m\varepsilon_1\varrho_{k,m}^2 2^{-5m/6+40\delta m}(2^{(3/2)(p-k/2)}+2^{p-k/2-m/3})\\&\lesssim\varepsilon_1\varrho_{k,m}^2 2^{m/6+40\delta m}(2^{p/2-3k/4}+2^{-k/2-m/3}),\end{aligned}$$

and the desired conclusion follows using also (5.18).

On the other hand, to prove the bound (5.19), we use (4.43). Clearly, with $g=e^{is\Lambda_\mu}G$, we have $\|g\|_{Z_1\cap H^{N_1/3,0}_\Omega}\lesssim\varepsilon_1 2^{\delta^2 m}$ (see (4.25)). The factor 2^{-k} in the right-hand side of (5.19) is due to the symbols μ_0 and μ_1. This is clear for the symbols μ_1, which already contain a factor of 2^{-k} (see (4.14)). For the symbols μ_0, we notice that we can take

$$A(\xi,\eta):=2^k\mathfrak{d}(\xi,\eta)\varphi_{\leqslant 4}(\Phi(\xi,\eta))\varphi_{[k_2-2,k_2+2]}(\xi)\varphi_{[-10,10]}(\xi-\eta).$$

This satisfies the bounds required in (4.41), since $k\leqslant\frac{1}{5}m$. This completes the proof. □

We now look at the remaining cases for the quasilinear terms and prove the following.

Lemma 5.3. *Assume that* (5.10) *holds and, in addition,*

$$p\geqslant 0\quad\textit{and}\quad k\in\left[\tfrac{2}{3}p+\tfrac{1}{4}m,2p+\tfrac{1}{5}m\right]. \tag{5.20}$$

Then,

$$2^{-p}|\mathcal{J}^{l,p}_{\mathrm{ba},1}(k_1,k_2)+\mathcal{J}^{l,p}_{\mathrm{ba},2}(k_1,k_2)|\lesssim\varepsilon_1 2^{-\delta m}\varrho_{k,m}^2. \tag{5.21}$$

Proof. The main issue here is to deal with the case of large frequencies, relative to the time variable, and avoid the loss of derivatives coming from the terms $(\partial_t \pm i\Lambda)W_{1,2}$. For this, we use ideas related to the local existence theory, such as symmetrization. Notice that in Lemma 5.3 we estimate the absolute value of the sum $\mathcal{J}^{l,p}_{\mathrm{ba},1}+\mathcal{J}^{l,p}_{\mathrm{ba},2}$, and not each term separately.

First notice that we may assume $\sigma=\nu=+$, since otherwise $\mathcal{J}^{l,p}_{\mathrm{ba},n}(k_1,k_2)=0$, $n=1,2$, when $k\geqslant \frac{2}{3}p+\frac{1}{4}m$. In particular, $2^p\lesssim 2^{k/2}$. We first deal with the semilinear part of the non-linearity, which is $\mathcal{E}_{W_1}$ in equation (4.22). Using Lemma 4.6 and the definition (5.5),

$$\begin{aligned}|\tilde{\mathcal{A}}^{l,p}_{\mathrm{ba}}[G(s),P_{k_1}\mathcal{E}_{W_1}(s),W_{k_2}(s)]| &\lesssim \varepsilon_1 2^{-5m/6+\delta m}\|P_{k_1}\mathcal{E}_{W_1}(s)\|_{L^2}\,\|W_{k_2}(s)\|_{L^2}\\ &\lesssim \varepsilon_1 2^{-5m/3+2\delta m}2^{k/2}\varrho_k(s)^2.\end{aligned}$$

Therefore,

$$2^{-p}\int_{\mathbb{R}} q_m(s)|\tilde{\mathcal{A}}^{l,p}_{\mathrm{ba}}[G(s),P_{k_1}\mathcal{E}_{W_1}(s),W_{k_2}(s)]|\,ds\lesssim \varepsilon_1 2^{-m/4}\varrho^2_{k,m}.$$

It remains to bound the contributions of $\mathcal{Q}_{W_1}$ and $\mathcal{Q}_{W_2}$. Using again Lemma 4.6, we can easily prove the estimate when $k\leqslant \frac{6}{5}m$ or when $l=1$. It remains to show that

$$\begin{aligned}2^{-p}\int_{\mathbb{R}} q_m(s)|\tilde{\mathcal{A}}^{0,p}_{\mathrm{ba}}[G(s),P_{k_1}\mathcal{Q}_{W_1}(s),W_{k_2}(s)]+\tilde{\mathcal{A}}^{0,p}_{\mathrm{ba}}[G(s),W_{k_1}(s),P_{k_2}\mathcal{Q}_{W_2}(s)]|\,ds&\\ \lesssim \varepsilon_1 2^{-\delta m}\varrho^2_{k,m},&\end{aligned}\tag{5.22}$$

provided that

$$\sigma=\nu=+,\quad k\in\big[2p-\mathcal{D},2p+\tfrac{1}{5}m\big],\quad\text{and}\quad k\geqslant\tfrac{6}{5}m.\tag{5.23}$$

In this case, we consider the full expression and apply a symmetrization procedure to recover the loss of derivatives. Since $W_1\in\mathcal{W}'_+$ and $W_2\in\mathcal{W}'_-$, recall from (4.23) that

$$\mathcal{Q}_{W_1}=-iT_{\Sigma_{\geqslant 2}}W_1-iT_{V\cdot\zeta}W_1\quad\text{and}\quad \mathcal{Q}_{W_2}=i\overline{T_{\Sigma_{\geqslant 2}}\overline{W}_2}+i\overline{T_{V\cdot\zeta}\overline{W}_2}.$$

Therefore, using the definition (5.11),

$$\begin{aligned}\tilde{\mathcal{A}}^{0,p}_{\mathrm{ba}}[G,P_{k_1}\mathcal{Q}_{W_1},W_{k_2}]=\sum_{\sigma\in\{\Sigma_{\geqslant 2},V\cdot\zeta\}}&\iint_{\mathbb{R}^2\times\mathbb{R}^2}\mu_0(\xi,\eta)\widetilde{\varphi}_p(\Phi(\xi,\eta))\chi_{\mathrm{ba}}(\xi-\eta)\\ &\times\widehat{G}(\xi-\eta)\,\varphi_{k_1}(\eta)(-i)\widehat{T_\sigma W_1}(\eta)\,\varphi_{k_2}(\xi)\widehat{W}_2(-\xi)\,d\xi\,d\eta,\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathcal{A}}^{0,p}_{\mathrm{ba}}[G,W_{k_1},P_{k_2}\mathcal{Q}_{W_2}]=\sum_{\sigma\in\{\Sigma_{\geqslant 2},V\cdot\zeta\}}&\iint_{\mathbb{R}^2\times\mathbb{R}^2}\mu_0(\xi,\eta)\widetilde{\varphi}_p(\Phi(\xi,\eta))\chi_{\mathrm{ba}}(\xi-\eta)\\ &\times\widehat{G}(\xi-\eta)\,\varphi_{k_1}(\eta)\widehat{W}_1(\eta)\,\varphi_{k_2}(\xi)\widehat{i\overline{T_\sigma\overline{W}_2}}(-\xi)\,d\xi\,d\eta.\end{aligned}$$

We use definition (2.22) and make suitable changes of variables to write

$$\begin{aligned}&\tilde{\mathcal{A}}^{0,p}_{\mathrm{ba}}[G,P_{k_1}\mathcal{Q}_{W_1},W_{k_2}]+\tilde{\mathcal{A}}^{0,p}_{\mathrm{ba}}[G,W_{k_1},P_{k_2}\mathcal{Q}_{W_2}]\\&\quad=\sum_{\sigma\in\{\Sigma_{\geqslant 2},V\cdot\zeta\}}\frac{-i}{4\pi^2}\iiint_{(\mathbb{R}^2)^3}\widehat{W_1}(\eta)\widehat{W_2}(-\xi)\widehat{G}_{\mathrm{ba}}(\xi-\eta-\alpha)(\delta M)(\xi,\eta,\alpha)\,d\xi\,d\eta\,d\alpha,\end{aligned}$$

where $\widehat{G}_{\mathrm{ba}}:=\chi_{\mathrm{ba}}\widehat{G}$ and

$$\begin{aligned}(\delta M)(\xi,\eta,\alpha)&=\mu_0(\xi,\eta+\alpha)\widetilde{\varphi}_p(\Phi(\xi,\eta+\alpha))\widetilde{\sigma}\biggl(\alpha,\frac{2\eta+\alpha}{2}\biggr)\chi\biggl(\frac{|\alpha|}{|2\eta+\alpha|}\biggr)\varphi_{k_1}(\eta+\alpha)\varphi_{k_2}(\xi)\\&\quad-\mu_0(\xi-\alpha,\eta)\widetilde{\varphi}_p(\Phi(\xi-\alpha,\eta))\widetilde{\sigma}\biggl(\alpha,\frac{2\xi-\alpha}{2}\biggr)\chi\biggl(\frac{|\alpha|}{|2\xi-\alpha|}\biggr)\varphi_{k_1}(\eta)\varphi_{k_2}(\xi-\alpha).\end{aligned}$$

For (5.22), it suffices to prove that, for any $s\in[2^{m-1},2^{m+1}]$ and $\sigma\in\{\Sigma_{\geqslant 2},V\cdot\zeta\}$,

$$\begin{aligned}2^{-p}\biggl|\iiint_{(\mathbb{R}^2)^3}\widehat{W_1}(\eta,s)\widehat{W_2}(-\xi,s)\widehat{G}_{\mathrm{ba}}(\xi-\eta-\alpha,s)(\delta M)(\xi,\eta,\alpha,s)\,d\xi\,d\eta\,d\alpha\biggr|&\\ \lesssim\varepsilon_1\varrho_k(s)^2 2^{-m-\delta m}.&\end{aligned}\tag{5.24}$$

Let

$$\begin{aligned}M(\xi,\eta,\alpha;\theta_1,\theta_2):&=\mu_0(\xi-\theta_1,\eta+\alpha-\theta_1)\widetilde{\varphi}_p(\Phi(\xi-\theta_1,\eta+\alpha-\theta_1))\varphi_{k_2}(\xi-\theta_1)\\&\quad\times\varphi_{k_1}(\eta+\alpha-\theta_1)\widetilde{\sigma}\biggl(\alpha,\eta+\frac{\alpha}{2}+\theta_2\biggr)\chi\biggl(\frac{|\alpha|}{|2\eta+\alpha+2\theta_2|}\biggr).\end{aligned}\tag{5.25}$$

Therefore,

$$\begin{aligned}(\delta M)(\xi,\eta,\alpha)&=M(\xi,\eta,\alpha;0,0)-M(\xi,\eta,\alpha;\alpha,\xi-\eta-\alpha)\\&=\varphi_{\leqslant k-\mathcal{D}}(\alpha)(\alpha\cdot\nabla_{\theta_1}M(\xi,\eta,\alpha;0,0)+(\xi-\eta-\alpha)\cdot\nabla_{\theta_2}M(\xi,\eta,\alpha;0,0))\\&\quad+(eM)(\xi,\eta,\alpha).\end{aligned}$$

Using the formula for μ_0 in (4.14) and recalling that $\sigma\in\varepsilon_1\mathcal{M}^{3/2,1}_{N_3-2}$ (see Definition A.6), it follows that, in the support of the integral,

$$|(eM)(\xi,\eta,\alpha)|\lesssim(1+|\alpha|^2)P(\alpha)2^{-2k}2^{3k/2}\quad\text{and}\quad\|(1+|\alpha|)^8P\|_{L^2}\lesssim 2^{\delta m}.$$

The contribution of (eM) in (5.24) can then be estimated by $2^{-p}2^{\delta m}2^{-k/2}\varepsilon_1\varrho_k(s)^2$, which suffices due to the assumptions (5.23).

We are thus left with estimating the integrals

$$\begin{aligned}\mathrm{I}:=\iiint_{(\mathbb{R}^2)^3}\widehat{G}_{\mathrm{ba}}(\xi-\eta-\alpha)\varphi_{\leqslant k-\mathcal{D}}(\alpha)((\xi-\eta-\alpha)\cdot\nabla_{\theta_2}M(\xi,\eta,\alpha;0,0))&\\ \times\widehat{W_1}(\eta)\widehat{W_2}(-\xi)\,d\alpha\,d\eta\,d\xi,&\end{aligned}$$

$$\mathrm{II}:=\iiint_{(\mathbb{R}^2)^3}\widehat{G}_{\mathrm{ba}}(\xi-\eta-\alpha)\varphi_{\leqslant k-\mathcal{D}}(\alpha)(\alpha\cdot\nabla_{\theta_1}M(\xi,\eta,\alpha;0,0))\widehat{W_1}(\eta)\widehat{W_2}(-\xi)\,d\alpha\,d\eta\,d\xi.$$

If $|\alpha|\ll 2^k$, we have

$$(\xi-\eta-\alpha)\cdot\nabla_{\theta_2}M(\xi,\eta,\alpha;0,0)=\mu_0(\xi,\eta+\alpha)\widetilde{\varphi}_p(\Phi(\xi,\eta+\alpha))\varphi_{k_2}(\xi)\varphi_{k_1}(\eta+\alpha)\\ \times(\xi-\eta-\alpha)\cdot(\nabla_\zeta\widetilde{\sigma})\big(\alpha,\eta+\tfrac{1}{2}\alpha\big).$$

We make the change of variable $\alpha=\beta-\eta$ to rewrite

$$\mathrm{I}=c\iiint_{(\mathbb{R}^2)^3}\widehat{G}_{\mathrm{ba}}(\xi-\beta)\mu_0(\xi,\beta)\widetilde{\varphi}_p(\Phi(\xi,\beta))(\xi-\beta)\cdot\mathcal{F}\{P_{k_1}T_{P_{\leqslant k-\mathcal{D}}\nabla_\zeta\sigma}W_1\}(\beta)\\ \times\widehat{P_{k_2}W_2}(-\xi)\,d\beta\,d\xi.$$

Then, we use Lemma 4.6, (4.26), and (A.22) (recall $\sigma\in\varepsilon_1\mathcal{M}^{3/2,1}_{N_3-2}$) to estimate

$$2^{-p}|I(s)|\lesssim 2^{-p}2^{2p-k}\varepsilon_1 2^{-5m/6+\delta m}\|P_{k_1}T_{P_{\leqslant k-\mathcal{D}}\nabla_\zeta\sigma}W_1(s)\|_{L^2}\,\|P_{k_2}W_2(s)\|_{L^2}\\ \lesssim\varepsilon_1 2^{-3m/2}2^{p-k/2}\varrho_k(s)^2.$$

This is better than the desired bound (5.24). One can estimate $2^{-p}|\mathrm{II}(s)|$ in a similar way, using the flexibility in Lemma 4.6 due to the fact that the symbol $\underline{\mu}_0$ is allowed to contain additional S^∞ symbols. This completes the proof of the bound (5.24) and the lemma. □

5.1.2. Semilinear terms

The only term in (5.12) that remains to be estimated is $\mathcal{J}^{l,p}_0(k_1,k_2)$. This is a semilinear term, since the ∂_t derivative hits the low-frequency component, for which we will show the following lemma.

LEMMA 5.4. *Assume that* (5.10) *holds. Then,*

$$2^{-p}|\mathcal{J}^{l,p}_{\mathrm{ba},0}(k_1,k_2)|\lesssim\varepsilon_1 2^{-\delta m}\varrho^2_{k,m}. \tag{5.26}$$

Proof. Assume first that $p\geqslant-\frac{1}{4}m$. Using integration by parts we can see that, for $\varrho\in\mathbb{R}$,

$$\|\mathcal{F}^{-1}\{e^{i\varrho\Lambda(\xi)}\varphi_{[-20,20]}(\xi)\}\|_{L^1_x}\lesssim 1+|\varrho|. \tag{5.27}$$

Combining this and the bounds in the second line of (4.25), we get

$$\sup_{|\varrho|\leqslant 2^{-p+\delta m}}\|e^{i\varrho\Lambda}[(\partial_s+i\Lambda_\mu)P_{[-10,10]}G(s)]\|_{L^\infty}\lesssim(2^{-p}+1)2^{-5m/3+2\delta m}.$$

Using this in combination with Lemma 4.6 we get

$$|\tilde{\mathcal{A}}^{l,p}_{\mathrm{ba}}[(\partial_s+i\Lambda_\mu)G(s),W_{k_1}(s),W_{k_2}(s)]|\lesssim(2^{-p}+1)2^{-5m/3+2\delta m}\varrho_k(s)^2, \tag{5.28}$$

which leads to an acceptable contribution.

Assume now that

$$\underline{p}+1\leqslant p\leqslant -\tfrac{1}{4}m.$$

Even though there is no loss of derivatives here, the information that we have so far is not sufficient to obtain the bound in this range. The main reason is that some components of $(\partial_s+i\Lambda_\mu)G(s)$ undergo oscillations which are not linear. To deal with this term, we are going to use the following decomposition of $(\partial_s+i\Lambda_\mu)G$, which follows from Lemma 8.3:

$$\chi'_{\mathrm{ba}}(\xi)\mathcal{F}\{(\partial_s+i\Lambda_\mu)G(s)\}(\xi)=g_d(\xi)+g_\infty(\xi)+g_2(\xi) \tag{5.29}$$

for any $s\in[2^{m-1},2^{m+1}]$, where $\chi'_{\mathrm{ba}}(x)=\varphi_{\leqslant 4}(2^{\mathcal{D}}(|x|-\gamma_0))+\varphi_{\leqslant 4}(2^{\mathcal{D}}(|x|-\gamma_1))$ and

$$\begin{aligned}\|g_2\|_{L^2}&\lesssim\varepsilon_1^2 2^{-3m/2+20\delta m},\\ \|g_\infty\|_{L^\infty}&\lesssim\varepsilon_1^2 2^{-m-4\delta m},\\ \sup_{|\varrho|\leqslant 2^{7m/9+4\delta m}}\|\mathcal{F}^{-1}\{e^{i\varrho\Lambda}g_d\}\|_{L^\infty}&\lesssim\varepsilon_1^2 2^{-16m/9-4\delta m}.\end{aligned} \tag{5.30}$$

Clearly, the contribution of g_d can be estimated as in (5.28), using Lemma 4.6. On the other hand, we estimate the contributions of g_2 and g_∞ in the Fourier space, using Schur's lemma. For this, we need to use the volume bound in Proposition 10.4 (i). We have

$$\sup_\xi\|\widetilde{\varphi}_p(\Phi(\xi,\eta))\chi_{\mathrm{ba}}(\xi-\eta)g_\infty(\xi-\eta)\|_{L^1_\eta}\lesssim 2^{(1-\delta)p}\|g_\infty\|_{L^\infty}\lesssim 2^{(1-\delta)p}2^{-(1+4\delta)m}\varepsilon_1^2,$$

and also a similar bound for the ξ integral (keeping η fixed). Therefore, using Schur's lemma, we have

$$|\tilde{\mathcal{A}}^{l,p}_{\mathrm{ba}}[\mathcal{F}^{-1}g_\infty(s),W_{k_1}(s),W_{k_2}(s)]|\lesssim 2^{(1-\delta)p}2^{-(1+4\delta)m}\varepsilon_1^2\varrho_k(s)^2,$$

and the corresponding contribution is bounded as claimed in (5.26). The contribution of g_2 can be bounded in a similar way, using Schur's lemma and the Cauchy–Schwarz inequality. This completes the proof of the lemma. □

5.2. The other interactions

In this subsection we show how to bound all the remaining contributions to the energy increment in (5.1). We remark that we do not use the main L^2 lemma in the estimates in this subsection.

5.2.1. Small frequencies

We consider now the small frequencies and prove the following.

LEMMA 5.5. *For $l\in\{0,1\}$, $m\in[\mathcal{D}^2,L]$, $G\in\mathcal{G}'$, and $W_1,W_2\in\mathcal{W}'$ we have*

$$\sum_{\min(k_1,k_2)\leqslant -40}|\mathcal{A}^l_{\mathrm{ba},m}[G,P_{k_1}W_1,P_{k_2}W_2]|\lesssim\varepsilon_1^3. \tag{5.31}$$

Proof. Let $\underline{k}:=\min\{k_1,k_2\}$. We may assume that $\underline{k}\leqslant -40$, $\max(k_1,k_2)\in[-10,10]$, and $l=1$. We can easily estimate

$$|\mathcal{A}^1_{\mathrm{ba},m}[G,P_{k_1}W_1,P_{k_2}W_2]|\lesssim\sup_{s\in[2^{m-1},2^{m+1}]}2^m2^{\underline{k}}\|G(s)\|_{L^2}\,\|P_{k_1}W_1(s)\|_{L^2}\,\|P_{k_2}W_2(s)\|_{L^2}.$$

By (4.19) and (4.21), this suffices to estimate the sum corresponding to $\underline{k}\leqslant -m-3\delta m$. Therefore, it suffices to show that, if $-(1+3\delta)m\leqslant\bar{k}\leqslant -40$, then

$$|\mathcal{A}^1_{\mathrm{ba},m}[G,P_{k_1}W_1,P_{k_2}W_2]|\lesssim\varepsilon_1^3 2^{-\delta m}. \tag{5.32}$$

As in the proof of Lemma 5.1, let $W_2\in\mathcal{W}'_{-\sigma}$, $W_1\in\mathcal{W}'_{\nu}$, $G\in\mathcal{G}'_{\mu}$, $\sigma,\nu,\mu\in\{+,-\}$, and define the associated phase $\Phi=\Phi_{\sigma\mu\nu}$ as in (5.4). The important observation is that

$$|\Phi(\xi,\eta)|\approx 2^{\underline{k}/2} \tag{5.33}$$

in the support of the integral. We define $\mathcal{A}^{1,p}_{\mathrm{ba}}$ and $\mathcal{A}^{1,p}_{\mathrm{ba},m}$ as in (5.7) and (5.8), by introducing the the cutoff function $\varphi_p(\Phi(\xi,\eta))$. In view of (5.33), we may assume that $\left|p-\frac{1}{2}\underline{k}\right|\lesssim 1$. Then, we integrate by parts as in (5.12) and similarly obtain

$$\begin{aligned}|\mathcal{A}^{1,p}_{\mathrm{ba},m}[G,P_{k_1}W_1,P_{k_2}W_2]|\lesssim 2^{-p}&\left|\int_{\mathbb{R}}q'_m(s)\tilde{\mathcal{A}}^{1,p}_{\mathrm{ba}}[G(s),W_{k_1}(s),W_{k_2}(s)]\,ds\right|\\&+2^{-p}|\mathcal{J}^{1,p}_{\mathrm{ba},0}(k_1,k_2)|+2^{-p}|\mathcal{J}^{1,p}_{\mathrm{ba},1}(k_1,k_2)|\\&+2^{-p}|\mathcal{J}^{1,p}_{\mathrm{ba},2}(k_1,k_2)|;\end{aligned} \tag{5.34}$$

see (5.11) and (5.13) for definitions.

We apply Lemma 4.6 (see (4.33)) to control the terms in the right-hand side of (5.34). Using (4.21) and (4.26) (recall that $2^{-p}\leqslant 2^{-\underline{k}/2+\delta m}\leqslant 2^{m/2+3\delta m}$), the first term is dominated by

$$C\varepsilon_1^3 2^{-p}2^{\delta m}2^{-5m/6+\delta m}\lesssim\varepsilon_1^3 2^{-m/4}.$$

Similarly,

$$2^{-p}|\mathcal{J}^{1,p}_{\mathrm{ba},1}(k_1,k_2)|+2^{-p}|\mathcal{J}^{1,p}_{\mathrm{ba},2}(k_1,k_2)|\lesssim\varepsilon_1^3 2^m2^{-p}2^{-5m/6+\delta m}2^{-5m/6+2\delta m}\lesssim\varepsilon_1^3 2^{-m/10}.$$

For $|\mathcal{J}^{1,p}_{\mathrm{ba},0}(k_1,k_2)|$ we first estimate, using also (5.27) and (4.25),

$$2^{-p}|\mathcal{J}^{1,p}_{\mathrm{ba},0}(k_1,k_2)| \lesssim \varepsilon_1^3 2^m 2^{-p}(2^{-p}2^{-5m/3+\delta m})2^{\delta m} \lesssim \varepsilon_1^3 2^{-2p}2^{-2m/3+2\delta m}.$$

We can also estimate directly in the Fourier space (placing the factor at low frequency in L^1 and the other two factors in L^2),

$$2^{-p}|\mathcal{J}^{1,p}_{\mathrm{ba},0}(k_1,k_2)| \lesssim \varepsilon_1^3 2^m 2^{-p}2^{\underline{k}}2^{-5m/6+3\delta m} \lesssim \varepsilon_1^3 2^p 2^{m/6+3\delta m}.$$

These last two bounds show that $2^{-p}|\mathcal{J}^{1,p}_{\mathrm{ba},0}(k_1,k_2)| \lesssim \varepsilon_1^3 2^{-m/10}$. The desired conclusion (5.32) follows using (5.34). □

5.2.2. The "good" frequencies

We now estimate the contribution of the terms in (5.1), corresponding to the cutoff χ_{go}. One should keep in mind that these terms are similar, but easier than the ones we have already estimated. We often use the sharp decay in (4.28) to bound the contribution of small modulations.

We may assume that $\overline{W}_2 \in \mathcal{W}'_\sigma$, $W_1 \in \mathcal{W}'_\nu$, and $G \in \mathcal{G}'_+$. For (5.2) it suffices to prove that

$$\sum_{k,k_1,k_2\in\mathbb{Z}} |\mathcal{A}^l_{\mathrm{go},m}[P_kG, P_{k_1}W_1, P_{k_2}W_2]| \lesssim \varepsilon_1^3 2^{2\delta^2 m}. \tag{5.35}$$

Recalling the assumptions (4.14) on the symbols μ_l, we have the simple bound

$$|\mathcal{A}^l_{\mathrm{go},m}[P_kG, P_{k_1}W_1, P_{k_2}W_2]| \lesssim 2^m 2^{\min(k,k_1,k_2)} 2^{2k^+} \times \sup_{s\in I_m} \|P_kG(s)\|_{L^2} \|P_{k_1}W_1(s)\|_{L^2} \|P_{k_2}W_2(s)\|_{L^2}.$$

Using now (4.19) and (4.21), it follows that the sum over $k \geqslant 2\delta m$ or $k \leqslant -m-\delta m$ in (5.35) is dominated as claimed. Using also the L^∞ bounds (4.27) and Lemma A.1, we have

$$\begin{aligned}|\mathcal{A}^l_{\mathrm{go},m}[P_kG, P_{k_1}W_1, P_{k_2}W_2]| &\lesssim 2^m 2^{2k^+} \sup_{s\in I_m} \|P_kG(s)\|_{L^\infty} \|P_{k_1}W_1(s)\|_{L^2} \|P_{k_2}W_2(s)\|_{L^2} \\ &\lesssim 2^m 2^{2k^+} \sup_{s\in I_m} \varepsilon_1 2^{k-m+50\delta m} \|P_{k_1}W_1(s)\|_{L^2} \|P_{k_2}W_2(s)\|_{L^2}\end{aligned}$$

if $|k| \geqslant 10$. This suffices to control the part of the sum over $k \leqslant -52\delta m$. Moreover,

$$\sum_{\min(k_1,k_2)\leqslant -\mathcal{D}-|k|} |\mathcal{A}^l_{\mathrm{go},m}[P_kG, P_{k_1}W_1, P_{k_2}W_2]| \lesssim \varepsilon_1^3 2^{-\delta m},$$

if $k\in[-52\delta m, 2\delta m]$. This follows as in the proof of Lemma 5.5, once we notice that $\Phi(\xi,\eta)\approx 2^{\min(k_1,k_2)/2}$ in the support of the integral, so we can integrate by parts in s. After these reductions, for (5.35) it suffices to prove that, for any $k\in[-52\delta m, 2\delta m]$,

$$\sum_{k_1,k_2\in[-\mathcal{D}-|k|,\infty)} |\mathcal{A}^l_{\mathrm{go},m}[P_kG, P_{k_1}W_1, P_{k_2}W_2]| \lesssim \varepsilon_1^3 2^{2\delta^2 m} 2^{-\delta|k|}. \tag{5.36}$$

To prove (5.36) we further decompose in modulation. Let $\bar{k}:=\max(k,k_1,k_2)$ and $\underline{p}:=\lfloor \frac{1}{2}\bar{k}^+ - 110\delta m\rfloor$. We define, as in (5.7) and (5.8),

$$\begin{aligned}\mathcal{A}^{l,p}_{\mathrm{go}}[F,H_1,H_2] := \iint_{\mathbb{R}^2\times\mathbb{R}^2} \mu_l(\xi,\eta)\varphi_p^{[\underline{p},\infty)}(\Phi(\xi,\eta))\chi_{\mathrm{go}}(\xi-\eta)&\\ \times \widehat{F}(\xi-\eta)\widehat{H}_1(\eta)\widehat{H}_2(-\xi)\,d\xi\,d\eta&,\end{aligned} \tag{5.37}$$

and

$$\mathcal{A}^{l,p}_{\mathrm{go},m}[P_kG, P_{k_1}W_1, P_{k_2}W_2] := \int_{\mathbb{R}} q_m(s)\mathcal{A}^{l,p}_{\mathrm{go}}[P_kG(s), P_{k_1}W_1(s), P_{k_2}W_2(s)]\,ds. \tag{5.38}$$

For $p\geqslant \underline{p}+1$ we integrate by parts in s. As in (5.11) and (5.13), let

$$\begin{aligned}\tilde{\mathcal{A}}^{l,p}_{\mathrm{go}}[F,H_1,H_2] := \iint_{\mathbb{R}^2\times\mathbb{R}^2} \mu_l(\xi,\eta)\widetilde{\varphi}_p(\Phi(\xi,\eta))\chi_{\mathrm{go}}(\xi-\eta)&\\ \times \widehat{F}(\xi-\eta)\widehat{H}_1(\eta)\widehat{H}_2(-\xi)\,d\xi\,d\eta&,\end{aligned} \tag{5.39}$$

where $\widetilde{\varphi}_p(x):=2^p x^{-1}\varphi_p(x)$. Let $W_{k_1}=P_{k_1}W_1$, $W_{k_2}=P_{k_2}W_2$, and

$$\begin{aligned}\mathcal{J}^{l,p}_{\mathrm{go},0}(k_1,k_2) &:= \int_{\mathbb{R}} q_m(s)\tilde{\mathcal{A}}^{l,p}_{\mathrm{go}}[P_k(\partial_s+i\Lambda_\mu)G(s), W_{k_1}(s), W_{k_2}(s)]\,ds,\\ \mathcal{J}^{l,p}_{\mathrm{go},1}(k_1,k_2) &:= \int_{\mathbb{R}} q_m(s)\tilde{\mathcal{A}}^{l,p}_{\mathrm{go}}[P_kG(s), (\partial_s+i\Lambda_\nu)W_{k_1}(s), W_{k_2}(s)]\,ds,\\ \mathcal{J}^{l,p}_{\mathrm{go},2}(k_1,k_2) &:= \int_{\mathbb{R}} q_m(s)\tilde{\mathcal{A}}^{l,p}_{\mathrm{go}}[P_kG(s), W_{k_1}(s), (\partial_s+i\Lambda_{-\sigma})W_{k_2}(s)]\,ds.\end{aligned}$$

As in (5.12), we have

$$\begin{aligned}|\mathcal{A}^{l,p}_{\mathrm{go},m}&[P_kG, P_{k_1}W_1, P_{k_2}W_2]|\\ &\lesssim 2^{-p}\left|\int_{\mathbb{R}} q_m'(s)\tilde{\mathcal{A}}^{l,p}_{\mathrm{go}}[P_kG(s), W_{k_1}(s), W_{k_2}(s)]\,ds\right|\\ &\quad +2^{-p}|\mathcal{J}^{l,p}_{\mathrm{go},0}(k_1,k_2)+\mathcal{J}^{l,p}_{\mathrm{go},1}(k_1,k_2)+\mathcal{J}^{l,p}_{\mathrm{go},2}(k_1,k_2)|.\end{aligned} \tag{5.40}$$

Using Lemma 4.6, (4.21), and (4.26), it is easy to see that

$$\sum_{k_1,k_2\in[-\mathcal{D}-|k|,\infty)}\sum_{p\geqslant \underline{p}+1} 2^{-p}\bigg|\int_{\mathbb{R}} q_m'(s)\tilde{\mathcal{A}}^{l,p}_{\mathrm{go}}[P_kG(s),W_{k_1}(s),W_{k_2}(s)]\,ds\bigg|\lesssim\varepsilon_1^3 2^{-\delta m}. \quad (5.41)$$

Using also (5.27) and (4.25), as in the first part of the proof of Lemma 5.4, we have

$$\sum_{k_1,k_2\in[-\mathcal{D}-|k|,\infty)}\sum_{p\geqslant \underline{p}+1} 2^{-p}|\mathcal{J}^{l,p}_{\mathrm{go},0}(k_1,k_2)|\lesssim\varepsilon_1^3 2^{-\delta m}. \quad (5.42)$$

Using Lemma 4.6, (4.26), and (5.16), it follows that

$$\sum_{k_1,k_2\in[-\mathcal{D}-|k|,6m/5]}\sum_{p\geqslant \underline{p}+1} 2^{-p}(|\mathcal{J}^{l,p}_{\mathrm{go},1}(k_1,k_2)|+|\mathcal{J}^{l,p}_{\mathrm{go},2}(k_1,k_2)|)\lesssim\varepsilon_1^3 2^{-\delta m}. \quad (5.43)$$

Finally, a symmetrization argument as in the proof of Lemma 5.3 shows that

$$\sum_{k_1,k_2\in[6m/5-10,\infty)}\sum_{p\geqslant \underline{p}+1} 2^{-p}|\mathcal{J}^{l,p}_{\mathrm{go},1}(k_1,k_2)+\mathcal{J}^{l,p}_{\mathrm{go},2}(k_1,k_2)|\lesssim\varepsilon_1^3 2^{-\delta m}. \quad (5.44)$$

In view of (5.40)–(5.44), to complete the proof of (5.36) it remains to bound the contribution of small modulations. In the case of "bad" frequencies, this is done using the main L^2 lemma. Here we need a different argument.

LEMMA 5.6. *Assume that* $k\in[-52\delta m, 2\delta m]$ *and* $\underline{p}=\lfloor\frac{1}{2}\bar{k}^+-110\delta m\rfloor$. *Then,*

$$\sum_{\min(k_1,k_2)\geqslant-\mathcal{D}-|k|}|\mathcal{A}^{l,\underline{p}}_{\mathrm{go},m}[P_kG,P_{k_1}W_1,P_{k_2}W_2]|\lesssim\varepsilon_1^3 2^{2\delta^2 m}2^{-\delta|k|}. \quad (5.45)$$

Proof. We need to further decompose the function G. Recall that $G\in\mathcal{G}_+'$ and let, for $(k,j)\in\mathcal{J}$,

$$f(s)=e^{is\Lambda}G(s),\quad f_{j,k}=P_{[k-2,k+2]}Q_{jk}f,\quad\text{and}\quad g_{j,k}:=A_{\leqslant 2\mathcal{D},\gamma_0}A_{\leqslant 2\mathcal{D},\gamma_1}f_{j,k}. \quad (5.46)$$

Compare with Lemma 7.5. The functions $g_{j,k}$ are supported away from the bad frequencies γ_0 and γ_1 and $\sum_j g_{j,k}(s)=e^{is\Lambda}G(s)$ away from these frequencies. This induces a decomposition

$$\mathcal{A}^{l,\underline{p}}_{\mathrm{go},m}[P_kG,P_{k_1}W_1,P_{k_2}W_2]=\sum_{j\geqslant\max(-k,0)}\mathcal{A}^{l,\underline{p}}_{\mathrm{go},m}[e^{-is\Lambda}g_{j,k},P_{k_1}W_1,P_{k_2}W_2].$$

Notice that, for $j\leqslant m-\delta m$, we have the stronger estimate (4.28) on $\|e^{-is\Lambda}g_{j,k}\|_{L^\infty}$. Therefore, using Lemma 4.6, if $j\leqslant m-\delta m$ then

$$|\mathcal{A}^{l,\underline{p}}_{\mathrm{go},m}[e^{-is\Lambda}g_{j,k},P_{k_1}W_1,P_{k_2}W_2]|\lesssim\varepsilon_1 2^k 2^{-2k^+}2^{-j/4}\sup_{s\in I_m}\|P_{k_1}W_1(s)\|_{L^2}\,\|P_{k_2}W_2(s)\|_{L^2}.$$

Therefore,(5)

$$\sum_{j\leqslant m-\delta m}\sum_{\min(k_1,k_2)\geqslant -\mathcal{D}-|k|}|\mathcal{A}^{l,\underline{p}}_{\mathrm{go},m}[e^{-is\Lambda}g_{j,k},P_{k_1}W_1,P_{k_2}W_2]|\lesssim \varepsilon_1^3 2^{2\delta^2 m}2^{-\delta|k|}.$$

Similarly, if $j\geqslant m+60\delta m$ then we also have a stronger bound on $\|e^{-is\Lambda}g_{j,k}\|_{L^\infty}$ in the first line of (4.27), and the corresponding contributions are controlled in the same way.

It remains to show that, for any $j\in[m-\delta m, m+60\delta m]$,

$$\sum_{\min(k_1,k_2)\geqslant -\mathcal{D}-|k|}|\mathcal{A}^{l,\underline{p}}_{\mathrm{go},m}[e^{-is\Lambda}g_{j,k},P_{k_1}W_1,P_{k_2}W_2]|\lesssim \varepsilon_1^3 2^{-\delta m}. \tag{5.47}$$

For this, we use Schur's test. As $\min(k,k_1,k_2)\geqslant -53\delta m$, it follows from Proposition 10.4 (i) and the bound $\|\hat{g}_{j,k}\|_{L^2}\lesssim\varepsilon_1 2^{-8k^+}2^{-j+50\delta j}$ that

$$\int_{\mathbb{R}^2}|\mu_l(\xi,\eta)|\varphi_{\leqslant\underline{p}}(\Phi(\xi,\eta))|\hat{g}_{j,k}(\xi-\eta)|\varphi_{[k_1-2,k_1+2]}(\eta)\,d\eta\lesssim\varepsilon_1 2^{(\underline{p}-\bar{k}^+/2)/2+\delta m}2^{-j+50\delta j}$$

for any $\xi\in\mathbb{R}^2$ fixed with $|\xi|\in[2^{k_2-4},2^{k_2+4}]$. The integral in ξ (for η fixed) can be estimated in the same way. Given the choice of $\underline{p}$, the desired bound (5.47) follows using Schur's lemma. □

6. Energy estimates III: Proof of the main L^2 lemma

In this section we prove Lemma 4.7. We divide the proof into several cases. Let

$$\chi_{\gamma_l}(x):=\varphi(2^{\mathcal{D}}(|x|-\gamma_l)),\quad l\in\{0,1\}.$$

We start the most difficult case when $|\xi-\eta|$ is close to γ_0 and $2^k\gg1$. In this case, $\widehat{\Upsilon}$ can vanish up to order 1 (so we can have $2^q\ll1$ in the notation of Lemma 6.1 below).

Lemma 6.1. *The conclusion of Lemma 4.7 holds if $k\geqslant\frac{3}{2}\mathcal{D}_1$ and $\hat{g}$ is supported in the set $\{\xi:\big||\xi|-\gamma_0\big|\leqslant2^{-100}\}$.*

Proof. We will often use the results in Lemma 11.1 below. We may assume that $\sigma=\nu=+$ in the definition of Φ, since otherwise the operator is trivial. We may also assume that $\mu=+$, in view of formula (11.23).

In view of Lemma 11.1 (ii), we may assume that either

$$(\xi-\eta)\cdot\xi^\perp\approx2^k\quad\text{or}\quad-(\xi-\eta)\cdot\xi^\perp\approx2^k$$

(5) This is the only place in the proof of the bound (5.2) where one needs the $2^{2\delta^2 m}$ factor in the right-hand side.

in the support of the integral, due to the factor $\chi(2^{-p}\Phi(\xi,\eta))$. Thus, we may define

$$a^{\pm}(\xi,\eta)=a(\xi,\eta)\mathbf{1}_{\pm}((\xi-\eta)\cdot\xi^{\perp}), \tag{6.1}$$

and decompose the operator $L_{p,k}=L^+_{p,k}+L^-_{p,k}$ accordingly. The two operators can be treated in similar ways, so we will concentrate on the operator $L^+_{p,k}$.

To apply the main TT^* argument, we first need to decompose the operators $L_{p,k}$. For $\varkappa:=2^{-\mathcal{D}^{3/2}}$ (a small parameter) and $\psi\in C^\infty_0(-2,2)$ satisfying $\sum_{v\in\mathbb{Z}}\psi(\cdot+v)\equiv 1$, we write

$$\begin{aligned} L^+_{p,k}&=\sum_{q,r\in\mathbb{Z}}\sum_{j\geqslant 0}L^{r,j}_{p,k,q},\\ L^{r,j}_{p,k,q}f(x)&:=\int_{\mathbb{R}^2}e^{is\Phi(x,y)}\chi(2^{-p}\Phi(x,y))\varphi_q(\widehat{\Upsilon}(x,y))\psi(\varkappa^{-1}2^{-q}\widehat{\Upsilon}(x,y)-r)\\ &\qquad\qquad\times\varphi_k(y)a^+_j(x,y)f(y)\,dy,\\ a^+_j(x,y)&:=A(x,y)\chi_{\gamma_0}(x-y)\mathbf{1}_+((x-y)\cdot x^{\perp})\hat{g}_j(x-y),\\ g_j&:=A_{\geqslant 0,\gamma_0}[\varphi^{[0,\infty)}_j\cdot g]. \end{aligned} \tag{6.2}$$

In other words, we insert the decompositions

$$g=\sum_{j\geqslant 0}g_j\quad\text{and}\quad 1=\sum_{q,r\in\mathbb{Z}}\varphi_q(\widehat{\Upsilon}(x,y))\psi(\varkappa^{-1}2^{-q}\widehat{\Upsilon}(x,y)-r)$$

in formula (4.40) defining the operators $L_{p,k}$. The parameters j and r play a somewhat minor role in the proof (one can focus on the main case $j=0$) but the parameter q is important. Notice that $q\leqslant-\frac{1}{2}\mathcal{D}$, in view of (11.8). The hypothesis $\|g\|_{Z_1\cap H^{N_1/3,0}_\Omega}\lesssim 1$ and Lemma 7.5 (i) show that

$$\|\hat{g}_j\|_{L^\infty}\lesssim 2^{-j(1/2-55\delta)}\quad\text{and}\quad\Big\|\sup_{\theta\in\mathbb{S}^1}|\hat{g}_j(r\theta)|\Big\|_{L^2(r\,dr)}\lesssim 2^{-j(1-55\delta)}. \tag{6.3}$$

Note that, for fixed x (resp. y), the support of integration is included in $\mathcal{S}^{1,-}_{p,q,r}(x)$ (resp. $\mathcal{S}^{2,-}_{p,q,r}(y)$); see (11.11)–(11.12). We can use this to estimate the Schur norm of the kernel. It follows from (11.14) and the first bound in (6.3) that

$$\begin{aligned} \sup_x\int_{\mathbb{R}^2}|\chi(2^{-p}\Phi(x,y))\varphi_q(\widehat{\Upsilon}(x,y))\varphi_k(y)a^+_j(x,y)|\,dy&\lesssim\|a^+_j\|_{L^\infty}|\mathcal{S}^{1,-}_{p,q,r}(x)|\\ &\lesssim 2^{q+p-k/2}2^{-j/3}. \end{aligned} \tag{6.4}$$

A similar estimate holds for the x integral (keeping y fixed). Moreover, using the bounds (11.13)–(11.14) and the second bound in (6.3), we estimate the left-hand side of (6.4) by $C2^{-j+55\delta j}2^{p-k/2}$. In view of Schur's lemma, we have

$$\|L^{r,j}_{p,k,q}\|_{L^2\to L^2}\lesssim\min(2^{q+p-k/2}2^{-j/3},2^{-j+55\delta j}2^{p-k/2}).$$

These bounds suffice to control the contribution of the operators $L^{r,j}_{p,k,q}$, unless

$$q \geqslant \mathcal{D}+\max\big(\tfrac{1}{2}\big(p-\tfrac{1}{2}k\big), -\tfrac{1}{3}m\big) \quad \text{and} \quad 0 \leqslant j \leqslant \min\big(\tfrac{4}{9}m, -\tfrac{2}{3}\big(p-\tfrac{1}{2}k\big)\big). \tag{6.5}$$

Therefore, in the rest of the proof, we may assume that (6.5) holds, so $\varkappa 2^q \gg 2^{p-k/2}$. We use the TT^* argument and Schur's test. It suffices to show that

$$\sup_x \int_{\mathbb{R}^2} |K(x,\xi)|\,d\xi + \sup_\xi \int_{\mathbb{R}^2} |K(x,\xi)|\,dx \lesssim 2^{6\delta^2 m}(2^{3(p-k/2)}+2^{2(p-k/2)}2^{-2m/3}) \tag{6.6}$$

for p, k, q, r, and j fixed (satisfying (4.39) and (6.5)), where

$$\begin{aligned} K(x,\xi) &:= \int_{\mathbb{R}^2} e^{is\Theta(x,\xi,y)}\chi(2^{-p}\Phi(x,y))\chi(2^{-p}\Phi(\xi,y))\psi_{q,r}(x,\xi,y)a_j^+(x,y)\overline{a_j^+(\xi,y)}\,dy, \\ \Theta(x,\xi,y) &:= \Phi(x,y)-\Phi(\xi,y) = \Lambda(x)-\Lambda(\xi)-\Lambda(x-y)+\Lambda(\xi-y), \\ \psi_{q,r}(x,\xi,y) &:= \varphi_q(\widehat{\Upsilon}(x,y))\varphi_q(\widehat{\Upsilon}(\xi,y))\psi(\varkappa^{-1}2^{-q}\widehat{\Upsilon}(x,y)-r)\psi(\varkappa^{-1}2^{-q}\widehat{\Upsilon}(\xi,y)-r)\varphi_k(y)^2. \end{aligned} \tag{6.7}$$

As $K(x,\xi)=\overline{K(\xi,x)}$, it suffices to prove the bound on the first term in the left-hand side of (6.6). The main idea of the proof is to show that K is essentially supported in the set where $\omega:=x-\xi$ is small. Note first that, in view of (11.13), we may assume

$$|\omega| = |x-\xi| \lesssim \varkappa 2^q \ll 1. \tag{6.8}$$

Step 1. We will show in Step 2 below that

$$\text{if } |\omega| \geqslant L := 2^{2\delta^2 m}(2^{p-k/2}2^{-q}+2^{j-q-m}+2^{-2m/3-q}), \text{ then } |K(x,\xi)| \lesssim 2^{-4m}. \tag{6.9}$$

Assuming this, we show now how to prove the bound on the first term in (6.6). Notice that $L \ll 1$, in view of (4.39) and (6.5). We decompose, for fixed x,

$$\int_{\mathbb{R}^2} |K(x,\xi)|\,d\xi \lesssim \int_{|\omega|\leqslant L} |K(x,x-\omega)|\,d\omega + \int_{|\omega|\geqslant L} |K(x,x-\omega)|\,d\omega.$$

Combining (6.8) and (6.9), we obtain a suitable bound for the second integral. We now turn to the first integral, which we bound using Fubini and formula (6.7) by

$$\begin{aligned} C\|a_j^+\|_{L^\infty} \int_{\mathbb{R}^2} |a_j^+(x,y)|\chi(2^{-p}\Phi(x,y))\varphi_q(\widehat{\Upsilon}(x,y))\varphi_k(y)^2 \\ \times\bigg(\int_{|\omega|\leqslant L} |\chi(2^{-p}\Phi(x-\omega,y))|\,d\omega\bigg)\,dy. \end{aligned} \tag{6.10}$$

We observe that, for fixed x and y satisfying $\big||x-y|-\gamma_0\big|\ll 1$ and $|x|\approx 2^k\gg 1$,

$$\int_{|\omega|\leqslant L} |\chi(2^{-p}\Phi(x-\omega,y))|\,d\omega \lesssim 2^{p-k/2}L. \tag{6.11}$$

Indeed, it follows from (11.9) that, if

$$z=(x-y-\omega)=(\varrho\cos\theta,\varrho\sin\theta),\quad |\omega|\leqslant L,\quad \text{and}\quad |\Phi(y+z,y)|\leqslant 2^p,$$

then $\big|\varrho-|x-y|\big|\lesssim L$ and θ belongs to a union of two intervals of length $\lesssim 2^{p-k/2}$. The desired bound (6.11) follows.

Using also (6.4) and $\|a_j\|_{L^\infty}\lesssim 2^{-j/3}$, it follows that the expression in (6.10) is bounded by $C2^{2(p-k/2)}2^{-2j/3}2^qL$. The desired bound (6.6) follows, using also the restrictions (6.5).

Step 2. We now prove (6.9). We define orthonormal frames (e_1,e_2) and (V_1,V_2):

$$\begin{gathered} e_1:=\frac{\nabla_x\Phi(x,y)}{|\nabla_x\Phi(x,y)|},\quad e_2=e_1^\perp,\quad V_1:=\frac{\nabla_y\Phi(x,y)}{|\nabla_y\Phi(x,y)|},\quad V_2=V_1^\perp,\\ \omega=x-\xi=\omega_1e_1+\omega_2e_2. \end{gathered} \tag{6.12}$$

Note that ω_1 and ω_2 are functions of (x,y,ξ). We first make a useful observation: if $|\Theta(x,\xi,y)|\lesssim 2^p$ and $|\omega|\ll 1$, then

$$|\omega_1|\lesssim 2^{-k/2}(2^p+|\omega|^2). \tag{6.13}$$

This follows from a simple Taylor expansion, since

$$|\Phi(x,y)-\Phi(\xi,y)-\omega\cdot\nabla_x\Phi(x,y)|\lesssim|\omega|^2.$$

We now turn to the proof of (6.9). Assuming that x and ξ are fixed with $|x-\xi|\geqslant L$ and using (6.13), we see that, on the support of integration, $|\omega_2|\approx|\omega|$ and

$$\begin{aligned} V_2\cdot\nabla_y\Theta(x,\xi,y) &= V_2\cdot\nabla_y(-\Lambda(x-y)+\Lambda(\xi-y))\\ &= V_2\cdot\nabla^2_{x,y}\Phi(x,y)\cdot(x-\xi)+O(|\omega|^2)\\ &= \omega_2\widehat{\Upsilon}(x,y)+O(|\omega_1|+|\omega|^2). \end{aligned} \tag{6.14}$$

Using (6.5), (6.9), (6.13) and (6.8) (this is where we need $\varkappa\ll 1$), we obtain that

$$|V_2\cdot\nabla_y\Theta(x,\xi,y)|\approx 2^q|\omega_2|\approx 2^q|\omega|$$

in the support of the integral. Using that

$$e^{is\Theta}=\frac{-i}{sV_2\cdot\nabla_y\Theta}V_2\cdot\nabla_y e^{is\Theta}\quad\text{and}\quad |D_y^\alpha\Theta|\lesssim|\omega|,$$

and letting $\Theta_{(1)}:=V_2\cdot\nabla_y\Theta$, after integration by parts we have

$$K(x,\xi)\\ =i\int_{\mathbb{R}^2}e^{is\Theta}\partial_l\biggl(V_2^l\frac{1}{s\Theta_{(1)}}\chi(2^{-p}\Phi(x,y))\chi(2^{-p}\Phi(\xi,y))\psi_{q,r}(x,\xi,y)a_j^+(x,y)\overline{a_j^+(\xi,y)}\biggr)\,dy.$$

We observe that

$$V_2^l\partial_l(\chi(2^{-p}\Phi(x,y))\chi(2^{-p}\Phi(\xi,y)))=-2^{-p}\Theta_{(1)}(\chi(2^{-p}\Phi(x,y))\chi'(2^{-p}\Phi(\xi,y))).$$

This identity is the main reason for choosing V_2 as in (6.12), and this justifies the definition of the function Υ (intuitively, we can only integrate by parts in y along the level sets of the function Φ, due to the very large 2^{-p} factor). Moreover,

$$|D_y^\alpha\psi_{q,r}(x,\xi,y)|\lesssim 2^{-q|\alpha|}\quad\text{and}\quad |D_y^\alpha a_j^+(v,y)|\lesssim_\alpha 2^{|\alpha|j}+2^{|\alpha|m/3},\ \ v\in\{x,\xi\},$$

in the support of the integral defining $K(x,\xi)$. We integrate by parts many times in y as above. At every step we gain a factor of $2^m2^q|\omega|$ and lose a factor of $2^{-p}2^q|\omega|+2^{-q}+2^j+2^{m/3}$. The desired bound in (6.9) follows. This completes the proof. □

We consider now the (easier) case when $|\xi-\eta|$ is close to γ_1 and k is large.

LEMMA 6.2. *The conclusion of Lemma 4.7 holds if* $k\geqslant\frac{3}{2}\mathcal{D}_1$ *and* $\hat{g}$ *is supported in the set* $\bigl\{\xi:\bigl||\xi|-\gamma_1\bigr|\leqslant 2^{-100}\bigr\}$.

Proof. Using (11.8), we see that on the support of integration we have $|\widehat{\Upsilon}(\xi,\eta)|\approx 1$. The proof is similar to the proof of Lemma 6.1 in the case $2^q\approx 1$. The new difficulties come from the less favorable decay in j close to γ_1 and from the fact that the conclusions in Lemma 11.1 (iii) do not apply. We define $a_j^\pm$ as in (6.2) (with γ_1 replacing γ_0 and $g_j:=A_{\geqslant 4,\gamma_1}[\varphi_j^{[0,\infty)}\cdot g]$), and

$$L_{p,k}^{x_0,j}f(x):=\varphi_{\leqslant-\mathcal{D}}(x-x_0)\int_{\mathbb{R}^2}e^{is\Phi(x,y)}\chi(2^{-p}\Phi(x,y))\varphi_k(y)a_j^+(x,y)f(y)\,dy \tag{6.15}$$

for any $x_0\in\mathbb{R}^2$. We have

$$\|\hat{g}_j\|_{L^\infty}\lesssim 2^{6\delta j}\quad\text{and}\quad \Bigl\|\sup_{\theta\in\mathbb{S}^1}\bigl|\widehat{A_{n,\gamma_1}g_j}(r\theta)\bigr|\Bigr\|_{L^2(r\,dr)}\lesssim 2^{(1/2-49\delta)n-j(1-55\delta)} \tag{6.16}$$

for $n\geqslant 1$, as a consequence of Lemma 7.5 (i). Notice that these bounds are slightly weaker than the bounds in (6.3). However, we can still estimate (compare with (6.4))

$$\sup_x \int_{\mathbb{R}^2} |\chi(2^{-p}\Phi(x,y))\varphi_k(y)a_j^+(x,y)|\,dy \lesssim 2^{p-k/2}2^{-(1-55\delta)j}. \tag{6.17}$$

Indeed, we use only the second bound in (6.16), decompose the integral as a sum of integrals over the dyadic sets $\big||x-y|-\gamma_1\big|\approx 2^{-n}$, $n\geqslant 1$, and use (11.9) and the Cauchy–Schwarz in each dyadic set. As a consequence of (6.17), it remains to consider the sum over $j\leqslant\frac{4}{9}m$.

We can then proceed as in the proof of Lemma 6.1. Using the TT^* argument for the operators $L_{p,k}^{x_0,j}$ and Schur's lemma, it suffices to prove bounds similar to those in (6.6). Let $\omega=x-\xi$, and notice that $|\omega|\leqslant 2^{-\mathcal{D}+10}$. This replaces the diameter bound (6.8) and is the main reason for adding the localization factors $\varphi_{\leqslant-\mathcal{D}}(x-x_0)$ in (6.15). The main claim is that

$$\text{if } |\omega|\geqslant L:=2^{2\delta^2 m}(2^{p-k/2}+2^{j-m}+2^{-2m/3}), \text{ then } |K(x,\xi)|\lesssim 2^{-4m}. \tag{6.18}$$

The same argument as in Step 1 in the proof of Lemma 6.1 shows that this claim suffices. Moreover, this claim can be proved using integration by parts, as in Step 2 in the proof of Lemma 6.1. The conclusion of the lemma follows. □

Finally, we now consider the case of low frequencies.

LEMMA 6.3. *The conclusion of Lemma 4.7 holds if* $k\in\big[-100,\frac{7}{4}\mathcal{D}_1\big]$.

Proof. For small frequencies, the harder case is when $|\xi-\eta|$ is close to γ_1, since the conclusions of Lemma 11.3 are weaker than the conclusions of Lemma 11.2, and the decay in j is less favorable. So, we will concentrate on this case.

We first need to decompose our operator. For $j\geqslant 0$ and $l\in\mathbb{Z}$ we define

$$\begin{aligned} a_{j,l}^{\pm}(x,y) &:= A(x,y)\chi_{\gamma_1}(x-y)\varphi_l^{\pm}((x-y)\cdot x^{\perp})\hat{g}_j(x-y), \\ g_j &:= A_{\geqslant 4,\gamma_1}[\varphi_j^{[0,\infty)}\cdot P_{[-8,8]}g], \end{aligned} \tag{6.19}$$

where $\varphi_l^{\pm}(v):=\mathbf{1}_{\pm}(v)\varphi_l(v)$. This is similar to (6.2), but with the additional dyadic decomposition in terms of the angle $|(x-y)\cdot x^{\perp}|\approx 2^l$. Then we decompose, as in (6.2),

$$L_{p,k}=\sum_{q,r\in\mathbb{Z}}\sum_{j\geqslant 0}\sum_{l\in\mathbb{Z}}\sum_{\iota\in\{+,-\}} L_{p,k,q}^{r,j,l,\iota}, \tag{6.20}$$

where, with $\varkappa=2^{-\mathcal{D}^{3/2}}$ and $\psi\in C_0^\infty(-2,2)$ satisfying $\sum_{v\in\mathbb{Z}}\psi(\cdot-v)\equiv 1$ as before,

$$\begin{aligned} L_{p,k,q}^{r,j,l,\iota}f(x) := \varphi_{\geqslant -100}(x)\int_{\mathbb{R}^2} & e^{is\Phi(x,y)}\chi(2^{-p}\Phi(x,y))\varphi_q(\Upsilon(x,y)) \\ & \times\psi(\varkappa^{-1}2^{-q}\Upsilon(x,y)-r)\varphi_k(y)a_{j,l}^{\iota}(x,y)f(y)\,dy. \end{aligned} \tag{6.21}$$

We consider two main cases, depending on the size of q.

Case 1: $q\leqslant -\mathcal{D}_1$. As a consequence of (11.25), the operators $L^{r,j,l,\iota}_{p,k,q}$ are non-trivial only if $2^k\approx 1$ and $2^l\approx 1$. Using also (11.24), it follows that

$$\begin{aligned}|\nabla_x\Phi|\approx 1,\quad |\nabla_x\Upsilon\cdot\nabla_x^\perp\Phi|\approx 1,\\ |\nabla_y\Phi|\approx 1,\quad |\nabla_y\Upsilon\cdot\nabla_y^\perp\Phi|\approx 1,\end{aligned}\tag{6.22}$$

in the support of the integrals defining the operators $L^{r,j,l,\iota}_{p,k,q}$.

Step 1. The proof proceeds as in Lemma 6.1. For simplicity, we assume that $\iota=+$. Let

$$\begin{aligned}\mathcal{S}^1_{p,q,r,l}(x):=\big\{z:\big||z|-\gamma_1\big|\leqslant 2^{-\mathcal{D}+1},\ |\Phi(x,x-z)|\leqslant 2^{p+1},\ |\Upsilon(x,x-z)|\leqslant 2^{q+2},\\ |\Upsilon(x,x-z)-r\varkappa 2^q|\leqslant 10\varkappa 2^q,\ \text{and } z\cdot x^\perp\in[2^{l-2},2^{l+2}]\big\}.\end{aligned}\tag{6.23}$$

Recall that, if $z=(\varrho\cos\theta,\varrho\sin\theta)$ and $x=(|x|\cos\alpha,|x|\sin\alpha)$, then

$$\Phi(x,x-z)=\lambda(|x|)-\mu\lambda(\varrho)-\nu\lambda\big(\sqrt{|x|^2+\varrho^2-2\varrho|x|\cos(\theta-\alpha)}\,\big).\tag{6.24}$$

It follows from (6.22) and the change-of-variable argument in the proof of Lemma 11.1 (iii) that

$$|\mathcal{S}^1_{p,q,r,l}(x)|\lesssim 2^{p+q}\quad\text{and}\quad \operatorname{diam}(\mathcal{S}^1_{p,q,r,l}(x))\lesssim 2^p+\varkappa 2^q,\tag{6.25}$$

if $|x|\approx 1$ and $2^l\approx 1$. Moreover, using (6.24), for any x and ϱ,

$$|\{\theta:z=(\varrho\cos\theta,\varrho\sin\theta)\in\mathcal{S}^1_{p,q,r,l}(x)\}|\lesssim 2^p.\tag{6.26}$$

Therefore, using (6.16) and these last two bounds, if $|x|\approx 1$ then

$$\int_{\mathbb{R}^2}|\chi(2^{-p}\Phi(x,y))\varphi_q(\Upsilon(x,y))\varphi_k(y)a^+_{j,l}(x,y)|\,dy\lesssim\min(2^{p+q}2^{6\delta j},2^p2^{-j+55\delta j}).\tag{6.27}$$

One can prove a similar bound for the x integral, keeping y fixed. In view of Schur's lemma, it remains to bound the contribution of the terms for which

$$q\geqslant\mathcal{D}+\max\big(\tfrac12 p,-\tfrac13 m\big)\quad\text{and}\quad 0\leqslant j\leqslant\min\big(\tfrac49 m,-\tfrac23 p\big).\tag{6.28}$$

Step 2. Assuming (6.28), we use the TT^* argument and Schur's test. It suffices to show that

$$\sup_x\int_{\mathbb{R}^2}|K(x,\xi)|\,d\xi+\sup_\xi\int_{\mathbb{R}^2}|K(x,\xi)|\,dx\lesssim 2^{6\delta m}(2^{3p}+2^{2p-2m/3})\tag{6.29}$$

for p, k, q, r, j, and l fixed satisfying (6.28), where

$$K(x,\xi):=\varphi_{\geqslant -100}(x)\varphi_{\geqslant -100}(\xi)\int_{\mathbb{R}^2}e^{is\Theta(x,\xi,y)}\chi(2^{-p}\Phi(x,y))\chi(2^{-p}\Phi(\xi,y))\\ \times\psi_{q,r}(x,\xi,y)a^+_{j,l}(x,y)\overline{a^+_{j,l}(\xi,y)}\,dy, \tag{6.30}$$

and, as in (6.7),

$$\Theta(x,\xi,y):=\Phi(x,y)-\Phi(\xi,y)=\Lambda(x)-\Lambda(\xi)-\Lambda_\mu(x-y)+\Lambda_\mu(\xi-y),$$
$$\psi_{q,r}(x,\xi,y):=\varphi_q(\Upsilon(x,y))\varphi_q(\Upsilon(\xi,y))\psi(\varkappa^{-1}2^{-q}\Upsilon(x,y)-r)\psi(\varkappa^{-1}2^{-q}\Upsilon(\xi,y)-r)\varphi_k(y)^2.$$

Let $\omega:=x-\xi$. As in the proof of Lemma 6.1, the main claim is that

$$\text{if } |\omega|\geqslant L:=2^{2\delta^2 m}(2^{p-q}+2^{j-q-m}+2^{-q-2m/3}),\ \text{then } |K(x,\xi)|\lesssim 2^{-4m}. \tag{6.31}$$

The same argument as in Step 1 in the proof of Lemma 6.1, using (6.27), shows that this claim suffices. Moreover, this claim can be proved using integration by parts, as in Step 2 in the proof of Lemma 6.1. The desired bound (6.29) follows.

Case 2: $q\geqslant-\mathcal{D}_1$. There is one new issue in this case, namely when the angular parameter 2^l is very small and bounds like (6.26) fail. As in the proof of Lemma 6.2, we also need to modify the main decomposition (6.20). Let

$$L^{x_0,j,l}_{p,k,q}f(x):=\varphi_{\leqslant-\mathcal{D}}(x-x_0)\int_{\mathbb{R}^2}e^{is\Phi(x,y)}\chi(2^{-p}\Phi(x,y))\\ \times\varphi_q(\Upsilon(x,y))\varphi_k(y)a^+_{j,l}(x,y)f(y)\,dy. \tag{6.32}$$

Here $x_0\in\mathbb{R}^2$, $|x_0|\geqslant 2^{-110}$, and the localization factor on $x-x_0$ leads to a good upper bound on $|x-\xi|$ in the TT^* argument below. It remains to prove that, if $q\geqslant-\mathcal{D}_1$, then

$$\|L^{x_0,j,l}_{p,k,q}\|_{L^2\to L^2}\lesssim 2^{\delta^2 l}2^{-\delta^2 j}2^{30\delta m}(2^{(3/2)p}+2^{p-m/3}). \tag{6.33}$$

Step 1. We start with a Schur bound. For $x\in\mathbb{R}^2$ with $|x|\in[2^{-120},2^{\mathcal{D}_1+10}]$ let

$$\mathcal{S}^1_{p,q,l}(x):=\big\{z:\big||z|-\gamma_1\big|\leqslant 2^{-\mathcal{D}+1},\ |\Phi(x,x-z)|\leqslant 2^{p+1},\\ |\Upsilon(x,x-z)|\in[2^{q-2},2^{q+2}],\ \text{and } z\cdot x^\perp\in[2^{l-2},2^{l+2}]\big\}. \tag{6.34}$$

The condition $|\Upsilon(x,x-z)|\geqslant 2^{-\mathcal{D}_1-4}$ shows that

$$|\nabla_z(\Phi(x,x-z))|\in[2^{-4\mathcal{D}_1},2^{\mathcal{D}_1}]\quad\text{for } z\in\mathcal{S}^1_{p,q,l}(x).$$

Formula (6.24) shows that

$$|\{\theta:z=(\varrho\cos\theta,\varrho\sin\theta)\in\mathcal{S}^1_{p,q,l}(x)\}|\lesssim 2^{p-l}. \tag{6.35}$$

Moreover, we claim that, for any x,

$$|\mathcal{S}^1_{p,q,l}(x)| \lesssim 2^{p+l}. \tag{6.36}$$

Indeed, this follows from (6.35) if $l \geqslant -\mathcal{D}$. On the other hand, if $l \leqslant -\mathcal{D}$ then

$$\partial_\theta(\Phi(x, x-z)) \leqslant 2^{-\mathcal{D}/2}$$

(due to (6.24)), so

$$\partial_\varrho(\Phi(x, x-z)) \geqslant 2^{-5\mathcal{D}_1}$$

(due to the inequality $|\nabla_z(\Phi(x, x-z))| \in [2^{-4\mathcal{D}_1}, 2^{\mathcal{D}_1}]$). Recalling also (6.16), it follows from these last two bounds that

$$\int_{\mathbb{R}^2} |\chi(2^{-p}\Phi(x,y))\varphi_q(\Upsilon(x,y))\varphi_k(y)a^+_{j,l}(x,y)|\,dy \lesssim \min(2^{6\delta j}2^{p+l}, 2^{-j+55\delta j}2^{p-l}) \tag{6.37}$$

if $|x| \in [2^{-120}, 2^{\mathcal{D}_1+10}]$. In particular, the integral is also bounded by $C2^p2^{-j/2+31\delta j}$. The integral in x, keeping y fixed, can be estimated in a similar way. The desired bound (6.33) follows unless

$$j \leqslant \min\big(\tfrac{2}{3}m, -p\big) - \mathcal{D} \quad \text{and} \quad l \geqslant \max\big(\tfrac{1}{2}p, -\tfrac{1}{3}m\big) + \mathcal{D}. \tag{6.38}$$

Step 2. Assuming (6.38), we use the TT^* argument and Schur's test. It suffices to show that

$$\sup_x \int_{\mathbb{R}^2} |K(x,\xi)|\,d\xi \lesssim 2^{55\delta m}(2^{3p} + 2^{2p-2m/3}) \tag{6.39}$$

for p, k, q, x_0, j, and l fixed, where $\Theta(x,\xi,y) = \Phi(x,y) - \Phi(\xi,y)$ and

$$\begin{aligned} K(x,\xi) := \varphi_{\leqslant -\mathcal{D}}(x-x_0)\varphi_{\leqslant -\mathcal{D}}(\xi-x_0) \int_{\mathbb{R}^2} e^{is\Theta(x,\xi,y)}\chi(2^{-p}\Phi(x,y))\chi(2^{-p}\Phi(\xi,y)) \\ \times \varphi_q(\Upsilon(x,y))\varphi_q(\Upsilon(\xi,y))\varphi_k(y)^2 a^+_{j,l}(x,y)\overline{a^+_{j,l}(\xi,y)}\,dy. \end{aligned} \tag{6.40}$$

Let $\omega = x - \xi$. As before, the main claim is that

$$\text{if } |\omega| \geqslant L := 2^{2\delta^2 m}(2^p + 2^{j-m} + 2^{-2m/3}), \text{ then } |K(x,\xi)| \lesssim 2^{-4m}. \tag{6.41}$$

To see that this claim suffices, we use an argument similar to the one in Step 1 in the proof of Lemma 6.1. Indeed, up to acceptable errors, the left-hand side of (6.39) is bounded by

$$\begin{aligned} C\|a^+_{j,l}\|_{L^\infty} \sup_{|x-x_0| \leqslant 2^{-\mathcal{D}+2}} \int_{\mathbb{R}^2} |a^+_{j,l}(x,y)|\chi(2^{-p}\Phi(x,y))\varphi_q(\Upsilon(x,y)) \\ \times \bigg(\int_{|\omega| \leqslant L} |\chi(2^{-p}\Phi(x-\omega, y))|\,d\omega \bigg)\,dy. \end{aligned} \tag{6.42}$$

Notice that, if $|\Upsilon(x,y)|\geqslant 2^{-\mathcal{D}_1-2}$, then $|(\nabla_x\Phi)(x,y)|\geqslant 2^{-4\mathcal{D}_1}$, and thus

$$|(\nabla_w\Phi)(x-w,y)|\geqslant 2^{-4\mathcal{D}_1-1}$$

if $|\omega|\leqslant L\leqslant 2^{-\mathcal{D}}$. Therefore, the integral in ω in the expression above is bounded by $C2^pL$. Using also (6.37), the expression in (6.42) is bounded by

$$C2^{6\delta j}2^pL2^p2^{-j/2+32\delta j}\lesssim 2^{\delta m}2^{3p}+2^{40\delta m}2^{2p+j/2-m}+2^{\delta m}2^{2p-2m/3}.$$

The desired bound (6.39) follows using also that $j\leqslant\frac{2}{3}m$; see (6.38).

The claim (6.41) follows by the same integration-by-part argument as in Step 2 in the proof of Lemma 6.1, once we recall that $|(\nabla_x\Phi)(x,y)|\geqslant 2^{-4\mathcal{D}_1}$ and $|(\nabla_y\Phi)(x,y)|\geqslant 2^{-4\mathcal{D}_1}$ in the support of the integral, while $|\omega|\leqslant 2^{-\mathcal{D}+4}$. This completes the proof of the lemma. □

7. Dispersive analysis I: Setup and the main proposition

7.1. The Duhamel formula and the main proposition

In this section we start the proof of Proposition 2.3. With $\mathcal{U}=\langle\nabla\rangle h+i|\nabla|^{1/2}\phi$, assume that $\mathcal{U}$ is a solution of the equation

$$(\partial_t+i\Lambda)\mathcal{U}=\mathcal{N}_2+\mathcal{N}_3+\mathcal{N}_{\geqslant 4} \tag{7.1}$$

on some time interval $[0,T]$, $T\geqslant 1$, where $\mathcal{N}_2$ is a quadratic non-linearity in $\mathcal{U}$ and $\bar{\mathcal{U}}$, $\mathcal{N}_3$ is a cubic non-linearity, and $\mathcal{N}_{\geqslant 4}$ is a higher-order non-linearity. Such an equation will be verified below (see §C.2) starting from the main system (2.1) and using the expansion of the Dirichlet–Neumann operator in §B.1. The non-linearity $\mathcal{N}_2$ is of the form

$$\begin{aligned}\mathcal{N}_2&=\sum_{\mu,\nu\in\{+,-\}}\mathcal{N}_{\mu\nu}(\mathcal{U}_\mu,\mathcal{U}_\nu),\\ (\mathcal{F}\mathcal{N}_{\mu\nu}(f,g))(\xi)&=\int_{\mathbb{R}^2}\mathfrak{m}_{\mu\nu}(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta,\end{aligned} \tag{7.2}$$

where $\mathcal{U}_+=\mathcal{U}$ and $\mathcal{U}_-=\bar{\mathcal{U}}$. The cubic non-linearity is of the form

$$\begin{aligned}\mathcal{N}_3&=\sum_{\mu,\nu,\beta\in\{+,-\}}\mathcal{N}_{\mu\nu\beta}(\mathcal{U}_\mu,\mathcal{U}_\nu,\mathcal{U}_\beta),\\ (\mathcal{F}\mathcal{N}_{\mu\nu\beta}(f,g,h))(\xi)&=\int_{\mathbb{R}^2\times\mathbb{R}^2}\mathfrak{n}_{\mu\nu\beta}(\xi,\eta,\sigma)\hat{f}(\xi-\eta)\hat{g}(\eta-\sigma)\hat{h}(\sigma)\,d\eta\,d\sigma.\end{aligned} \tag{7.3}$$

The multipliers $\mathfrak{m}_{\mu\nu}$ and $\mathfrak{n}_{\mu\nu\beta}$ satisfy suitable symbol-type estimates. We define the profiles $\mathcal{V}_\sigma(t)=e^{it\Lambda_\sigma}\mathcal{U}_\sigma(t)$, $\sigma\in\{+,-\}$, as in (1.11). The Duhamel formula is

$$(\partial_t\widehat{\mathcal{V}})(\xi,s)=e^{is\Lambda(\xi)}\widehat{\mathcal{N}}_2(\xi,s)+e^{is\Lambda(\xi)}\widehat{\mathcal{N}}_3(\xi,s)+e^{is\Lambda(\xi)}\widehat{\mathcal{N}}_{\geqslant 4}(\xi,s), \tag{7.4}$$

or, in integral form,

$$\widehat{\mathcal{V}}(\xi,t)=\widehat{\mathcal{V}}(\xi,0)+\widehat{W_2}(\xi,t)+\widehat{W_3}(\xi,t)+\int_0^t e^{is\Lambda(\xi)}\widehat{\mathcal{N}}_{\geqslant 4}(\xi,s)\,ds, \tag{7.5}$$

where, with the definitions in (2.13),

$$\widehat{W_2}(\xi,t):=\sum_{\mu,\nu\in\{+,-\}}\int_0^t\int_{\mathbb{R}^2} e^{is\Phi_{+\mu\nu}(\xi,\eta)}\mathfrak{m}_{\mu\nu}(\xi,\eta)\widehat{\mathcal{V}}_\mu(\xi-\eta,s)\widehat{\mathcal{V}}_\nu(\eta,s)\,d\eta\,ds, \tag{7.6}$$

$$\begin{aligned}\widehat{W_3}(\xi,t):=\sum_{\mu,\nu,\beta\in\{+,-\}}\int_0^t\int_{\mathbb{R}^2\times\mathbb{R}^2} & e^{is\tilde{\Phi}_{+\mu\nu\beta}(\xi,\eta,\sigma)}\mathfrak{n}_{\mu\nu\beta}(\xi,\eta,\sigma)\\ &\times\widehat{\mathcal{V}}_\mu(\xi-\eta,s)\widehat{\mathcal{V}}_\nu(\eta-\sigma,s)\widehat{\mathcal{V}}_\beta(\sigma,s)\,d\eta\,d\sigma\,ds.\end{aligned} \tag{7.7}$$

The vector field Ω acts on the quadratic part of the non-linearity according to the identity

$$\Omega_\xi\widehat{\mathcal{N}}_2(\xi,s)=\sum_{\mu,\nu\in\{+,-\}}\int_{\mathbb{R}^2}(\Omega_\xi+\Omega_\eta)(\mathfrak{m}_{\mu\nu}(\xi,\eta)\widehat{\mathcal{U}}_\mu(\xi-\eta,s)\widehat{\mathcal{U}}_\nu(\eta,s))\,d\eta.$$

A similar formula holds for $\Omega_\xi\widehat{\mathcal{N}}_3(\xi,s)$. Therefore, for $1\leqslant a\leqslant N_1$, letting

$$\mathfrak{m}^b_{\mu\nu}:=(\Omega_\xi+\Omega_\eta)^b\mathfrak{m}_{\mu\nu}\quad\text{and}\quad\mathfrak{n}^b_{\mu\nu\beta}:=(\Omega_\xi+\Omega_\eta+\Omega_\sigma)^b\mathfrak{n}_{\mu\nu\beta},$$

we have

$$\Omega^a_\xi(\partial_t\widehat{\mathcal{V}})(\xi,s)=e^{is\Lambda(\xi)}\Omega^a_\xi\widehat{\mathcal{N}}_2(\xi,s)+e^{is\Lambda(\xi)}\Omega^a_\xi\widehat{\mathcal{N}}_3(\xi,s)+e^{is\Lambda(\xi)}\Omega^a_\xi\widehat{\mathcal{N}}_{\geqslant 4}(\xi,s), \tag{7.8}$$

where

$$\begin{aligned}e^{is\Lambda(\xi)}\Omega^a_\xi\widehat{\mathcal{N}}_2(\xi,s)=\sum_{\mu,\nu\in\{+,-\}}\sum_{a_1+a_2+b=a}\int_{\mathbb{R}^2} & e^{is\Phi_{+\mu\nu}(\xi,\eta)}\mathfrak{m}^b_{\mu\nu}(\xi,\eta)\\ &\times(\Omega^{a_1}\widehat{\mathcal{V}}_\mu)(\xi-\eta,s)(\Omega^{a_2}\widehat{\mathcal{V}}_\nu)(\eta,s)\,d\eta\end{aligned} \tag{7.9}$$

and

$$\begin{aligned}e^{is\Lambda(\xi)}\Omega^a_\xi\widehat{\mathcal{N}}_3(\xi,s)=\sum_{\mu,\nu,\beta\in\{+,-\}}\sum_{a_1+a_2+a_3+b=a}\int_{\mathbb{R}^2\times\mathbb{R}^2} & e^{is\tilde{\Phi}_{+\mu\nu\beta}(\xi,\eta,\sigma)}\mathfrak{n}^b_{\mu\nu\beta}(\xi,\eta,\sigma)\\ \times(\Omega^{a_1}\widehat{\mathcal{V}}_\mu)(\xi-\eta,s)(\Omega^{a_2}\widehat{\mathcal{V}}_\nu)(\eta-\sigma,s)&(\Omega^{a_3}\widehat{\mathcal{V}}_\beta)(\sigma,s)\,d\eta\,d\sigma.\end{aligned} \tag{7.10}$$

To state our main proposition, we need to make suitable assumptions on the non-linearities $\mathcal{N}_2$, $\mathcal{N}_3$, and $\mathcal{N}_{\geqslant 4}$. Recall the class of symbols S^∞ defined in (A.5).

- Concerning the multipliers defining $\mathcal{N}_2$, we assume that $(\Omega_\xi+\Omega_\eta)m(\xi,\eta)\equiv 0$ and

$$\begin{aligned} \|m^{k,k_1,k_2}\|_{S^\infty} &\lesssim 2^k 2^{\min(k_1,k_2)/2},\\ \|D_\eta^\alpha m^{k,k_1,k_2}\|_{L^\infty} &\lesssim_{|\alpha|} 2^{(|\alpha|+3/2)\max(|k_1|,|k_2|)},\\ \|D_\xi^\alpha m^{k,k_1,k_2}\|_{L^\infty} &\lesssim_{|\alpha|} 2^{(|\alpha|+3/2)\max(|k|,|k_1|,|k_2|)}, \end{aligned} \tag{7.11}$$

for any $k,k_1,k_2\in\mathbb{Z}$ and $m\in\{\mathfrak{m}_{\mu\nu}:\mu,\nu\in\{+,-\}\}$, where

$$m^{k,k_1,k_2}(\xi,\eta):=m(\xi,\eta)\varphi_k(\xi)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta).$$

- Concerning the multipliers defining $\mathcal{N}_3$, we assume that $(\Omega_\xi+\Omega_\eta+\Omega_\sigma)n(\xi,\eta,\sigma)\equiv 0$ and

$$\begin{aligned} \|n^{k,k_1,k_2,k_3}\|_{S^\infty} &\lesssim 2^{\min(k,k_1,k_2,k_3)/2} 2^{3\max(k,k_1,k_2,k_3,0)},\\ \|D_{\eta,\sigma}^\alpha n^{k,k_1,k_2,k_3;l}\|_{L^\infty} &\lesssim_{|\alpha|} 2^{|\alpha|\max(|k_1|,|k_2|,|k_3|,|l|)} 2^{(7/2)\max(|k_1|,|k_2|,|k_3|)},\\ \|D_\xi^\alpha n^{k,k_1,k_2,k_3}\|_{L^\infty} &\lesssim_{|\alpha|} 2^{(|\alpha|+7/2)\max(|k|,|k_1|,|k_2|,|k_3|)}, \end{aligned} \tag{7.12}$$

for any $k,k_1,k_2,k_3,l\in\mathbb{Z}$ and $n\in\{\mathfrak{n}_{\mu\nu\beta}:\mu,\nu\in\{+,-\}\}$, where

$$\begin{aligned} n^{k,k_1,k_2,k_3}(\xi,\eta,\sigma) &:= n(\xi,\eta,\sigma)\varphi_k(\xi)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta-\sigma)\varphi_{k_3}(\sigma),\\ n^{k,k_1,k_2,k_3;l}(\xi,\eta,\sigma) &:= n(\xi,\eta,\sigma)\varphi_k(\xi)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta-\sigma)\varphi_{k_3}(\sigma)\varphi_l(\eta). \end{aligned}$$

Our main result is the following.

PROPOSITION 7.1. *Assume that $\mathcal{U}$ is a solution of the equation*

$$(\partial_t+i\Lambda)\mathcal{U}=\mathcal{N}_2+\mathcal{N}_3+\mathcal{N}_{\geqslant 4}, \tag{7.13}$$

on some time interval $[0,T]$, $T\geqslant 1$, with initial data $\mathcal{U}_0$. Define, as before, $\mathcal{V}(t)=e^{it\Lambda}\mathcal{U}(t)$ and $\mathcal{V}_0=\mathcal{U}_0$. With δ as in Definition 2.5, assume that

$$\|\mathcal{U}_0\|_{H^{N_0}\cap H_\Omega^{N_1,N_3}}+\|\mathcal{V}_0\|_Z\leqslant\varepsilon_0\ll 1 \tag{7.14}$$

and

$$\begin{aligned} (1+t)^{-\delta^2}\|\mathcal{U}(t)\|_{H^{N_0}\cap H_\Omega^{N_1,N_3}}+\|\mathcal{V}(t)\|_Z &\leqslant\varepsilon_1\ll 1,\\ (1+t)^2\|\mathcal{N}_{\geqslant 4}(t)\|_{H^{N_0-N_3}\cap H_\Omega^{N_1,0}}+(1+t)^{1+\delta^2}\|e^{it\Lambda}\mathcal{N}_{\geqslant 4}(t)\|_Z &\leqslant\varepsilon_1^2, \end{aligned} \tag{7.15}$$

for all $t\in[0,T]$. Moreover, assume that the non-linearities $\mathcal{N}_2$ and $\mathcal{N}_3$ satisfy (7.2)–(7.3) and (7.11)–(7.12). Then, for any $t\in[0,T]$,

$$\|\mathcal{V}(t)\|_Z\lesssim\varepsilon_0+\varepsilon_1^2. \tag{7.16}$$

We will show in §C.2 below how to use this proposition and a suitable expansion of the Dirichlet–Neumann operator to complete the proof of Proposition 2.3.

7.2. Some lemmas

In this subsection we collect several important lemmas which are used often in the proofs in the next two sections. Let $\Phi=\Phi_{\sigma\mu\nu}$ be as in (2.13).

7.2.1. Integration by parts

In this subsection we state two lemmas that are used in the paper in integration-by-part arguments. We start with an oscillatory integral estimate. See [42, Lemma 5.4] for the proof of (i), and the proof of (ii) is similar.

LEMMA 7.2. (i) *Assume that* $0<\varepsilon\leqslant 1/\varepsilon\leqslant K$, $N\geqslant 1$ *is an integer, and* $f,g\in C^N(\mathbb{R}^2)$. *Then,*

$$\left|\int_{\mathbb{R}^2} e^{iKf} g\,dx\right| \lesssim_N (K\varepsilon)^{-N}\bigg(\sum_{|\alpha|\leqslant N} \varepsilon^{|\alpha|}\|D_x^\alpha g\|_{L^1}\bigg), \tag{7.17}$$

provided that f *is real-valued,*

$$|\nabla_x f|\geqslant \mathbf{1}_{\operatorname{supp} g}, \quad \textit{and} \quad \|D_x^\alpha f\cdot\mathbf{1}_{\operatorname{supp} g}\|_{L^\infty}\lesssim_N \varepsilon^{1-|\alpha|}, \;\; 2\leqslant|\alpha|\leqslant N+1. \tag{7.18}$$

(ii) *Similarly, if* $0<\varrho\leqslant 1/\varrho\leqslant K$, *then*

$$\left|\int_{\mathbb{R}^2} e^{iKf} g\,dx\right| \lesssim_N (K\varrho)^{-N}\bigg(\sum_{m\leqslant N} \varrho^{m}\|\Omega^m g\|_{L^1}\bigg), \tag{7.19}$$

provided that f *is real-valued,*

$$|\Omega f|\geqslant \mathbf{1}_{\operatorname{supp} g}, \quad \textit{and} \quad \|\Omega^m f\cdot\mathbf{1}_{\operatorname{supp} g}\|_{L^\infty}\lesssim_N \varrho^{1-m}, \;\; 2\leqslant m\leqslant N+1. \tag{7.20}$$

We will need another result about integration by parts using the vector field Ω. This lemma is more subtle. It is needed many times in the next two sections to localize and then estimate bilinear expressions. The point is to be able to take advantage of the fact that our profiles are "almost radial" (due to the bootstrap assumption involving many copies of Ω), and prove that for such functions one has better localization properties than for general functions.

LEMMA 7.3. *Assume that* $N\geqslant 100$, $m\geqslant 0$, $p,k,k_1,k_2\in\mathbb{Z}$,

$$2^{-k_1}\leqslant 2^{2m/5}, \quad 2^{\max(k,k_1,k_2)}\leqslant U\leqslant U^2\leqslant 2^{m/10}, \quad \textit{and} \quad U^2+2^{3|k_1|/2}\leqslant 2^{p+m/2}. \tag{7.21}$$

For some $A\geqslant\max(1,2^{-k_1})$, *assume that*

$$\begin{aligned}\sup_{0\leqslant a\leqslant 100}(\|\Omega^a g\|_{L^2}+\|\Omega^a f\|_{L^2})+\sup_{|\alpha|\leqslant N} A^{-|\alpha|}\|D^\alpha f\|_{L^2}\leqslant 1,\\ \sup_{\xi,\eta}\sup_{|\alpha|\leqslant N}(2^{-m/2}|\eta|)^{|\alpha|}|D_\eta^\alpha m(\xi,\eta)|\leqslant 1.\end{aligned} \tag{7.22}$$

Fix $\xi\in\mathbb{R}^2$ *and let, for* $t\in[2^m-1,2^{m+1}]$,

$$I_p(f,g):=\int_{\mathbb{R}^2} e^{it\Phi(\xi,\eta)}m(\xi,\eta)\varphi_p(\Omega_\eta\Phi(\xi,\eta))\varphi_k(\xi)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta)f(\xi-\eta)g(\eta)\,d\eta.$$

If $2^p\leqslant U2^{|k_1|/2+100}$ *and* $A\leqslant 2^m U^{-2}$, *then*

$$|I_p(f,g)|\lesssim_N (2^{p+m})^{-N}U^{2N}[2^{m/2}+A2^p]^N+2^{-10m}. \tag{7.23}$$

In addition, assuming that $\big(1+\frac{1}{4}\delta\big)\nu\geqslant -m$, *the same bound holds when* I_p *is replaced by*

$$\tilde{I}_p(f,g):=\int_{\mathbb{R}^2} e^{it\Phi(\xi,\eta)}\varphi_\nu(\Phi(\xi,\eta))m(\xi,\eta)\varphi_p(\Omega_\eta\Phi(\xi,\eta))\\ \times\varphi_k(\xi)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta)f(\xi-\eta)g(\eta)\,d\eta.$$

A slightly simpler version of this integration by parts lemma was used recently in [30]. The main interest of this lemma is that we have essentially no assumption on g and very mild assumptions on f.

Proof of Lemma 7.3. We decompose first

$$f=\mathcal{R}_{\leqslant m/10}f+[I-\mathcal{R}_{\leqslant m/10}]f \quad \text{and} \quad g=\mathcal{R}_{\leqslant m/10}g+[I-\mathcal{R}_{\leqslant m/10}]g,$$

where the operators $\mathcal{R}_{\leqslant L}$ are defined in polar coordinates by

$$(\mathcal{R}_{\leqslant L}h)(r\cos\theta,r\sin\theta):=\sum_{n\in\mathbb{Z}}\varphi_{\leqslant L}(n)h_n(r)e^{in\theta} \quad \text{if} \quad h(r\cos\theta,r\sin\theta):=\sum_{n\in\mathbb{Z}}h_n(r)e^{in\theta}. \tag{7.24}$$

Since Ω corresponds to $d/d\theta$ in polar coordinates, using (7.22) we have,

$$\|[I-\mathcal{R}_{\leqslant m/10}]f\|_{L^2}+\|[I-\mathcal{R}_{\leqslant m/10}]g\|_{L^2}\lesssim 2^{-10m}.$$

Therefore, using the Hölder inequality,

$$|I_p([I-\mathcal{R}_{\leqslant m/10}]f,g)|+|I_p(\mathcal{R}_{\leqslant m/10}f,[I-\mathcal{R}_{\leqslant m/10}]g)|\lesssim 2^{-10m}.$$

It remains to prove a similar inequality for $I_p:=I_p(f_1,g_1)$, where

$$f_1:=\varphi_{[k_1-2,k_1+2]}\,\mathcal{R}_{\leqslant m/10}f \quad \text{and} \quad g_1:=\varphi_{[k_2-2,k_2+2]}\,\mathcal{R}_{\leqslant m/10}g.$$

It follows from (7.22) and the definitions that

$$\|\Omega^a g_1\|_{L^2}\lesssim_a 2^{am/10} \quad \text{and} \quad \|\Omega^a D^\alpha f_1\|_{L^2}\lesssim_a 2^{am/10}A^{|\alpha|}, \tag{7.25}$$

for any $a\geqslant 0$ and $|\alpha|\leqslant N$. Integration by parts gives

$$I_p=c\varphi_k(\xi)\int_{\mathbb{R}^2}e^{it\Phi(\xi,\eta)}\Omega_\eta\bigg(\frac{m(\xi,\eta)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta)}{t\Omega_\eta\Phi(\xi,\eta)}\varphi_p(\Omega_\eta\Phi(\xi,\eta))f_1(\xi-\eta)g_1(\eta)\bigg)\,d\eta.$$

Iterating N times, we obtain an integrand made of a linear combination of terms like

$$\begin{aligned}e^{it\Phi(\xi,\eta)}\varphi_k(\xi)\bigg(\frac{1}{t\Omega_\eta\Phi(\xi,\eta)}\bigg)^N\Omega_\eta^{a_1}(m(\xi,\eta)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta))\\ \times\Omega_\eta^{a_2}f_1(\xi-\eta)\cdot\Omega_\eta^{a_3}g_1(\eta)\cdot\Omega_\eta^{a_4}\varphi_p(\Omega_\eta\Phi(\xi,\eta))\cdot\frac{\Omega_\eta^{a_5+1}\Phi}{\Omega_\eta\Phi}\dots\frac{\Omega_\eta^{a_q+1}\Phi}{\Omega_\eta\Phi},\end{aligned}$$

where $\sum_i a_i=N$. The desired bound follows from the pointwise bounds

$$\begin{aligned}|\Omega_\eta^a\{m(\xi,\eta)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta)\}|&\lesssim 2^{am/2},\\ |\Omega_\eta^a\varphi_p(\Omega_\eta\Phi(\xi,\eta))|+\bigg|\frac{\Omega_\eta^{a+1}\Phi}{\Omega_\eta\Phi}\bigg|&\lesssim U^{2a}2^{am/2},\end{aligned}\tag{7.26}$$

which hold in the support of the integral, and the L^2 bounds

$$\begin{aligned}\|\Omega_\eta^a g_1(\eta)\|_{L^2}&\lesssim 2^{am/4},\\ \|\Omega_\eta^a f_1(\xi-\eta)\varphi_k(\xi)\varphi_{[k_2-2,k_2+2]}(\eta)\varphi_{\leqslant p+2}(\Omega_\eta\Phi(\xi,\eta))\|_{L^2_\eta}&\lesssim U^{2a}(2^{m/2}+A2^p)^a.\end{aligned}\tag{7.27}$$

The first bound in (7.26) is direct (see (7.21)). For the second bound we notice that

$$\begin{aligned}\Omega_\eta(\xi\cdot\eta^\perp)=-\xi\cdot\eta,\quad \Omega_\eta(\xi\cdot\eta)=\xi\cdot\eta^\perp,\quad \Omega_\eta\Phi(\xi,\eta)=\frac{\lambda_\mu'(|\xi-\eta|)}{|\xi-\eta|}(\xi\cdot\eta^\perp),\\ |\Omega_\eta^a\Phi(\xi,\eta)|\lesssim\lambda(|\xi-\eta|)(|\xi-\eta|^{-2a}|\xi\cdot\eta^\perp|^a+|\xi-\eta|^{-a}U^a).\end{aligned}\tag{7.28}$$

Since $\lambda'(|\xi-\eta|)\approx 2^{|k_1|/2}$ in the support of the integral, we have

$$|\xi-\eta|^{-2}|\xi\cdot\eta^\perp|\approx 2^p2^{-k_1-|k_1|/2}.$$

The second bound in (7.26) follows, once we recall the assumptions in (7.21).

We now turn to the proof of (7.27). The first bound follows from the construction of g_1. For the second bound, if $2^p\gtrsim 2^{|k_1|/2+\min(k,k_2)}$, then we have the simple bound

$$\|\Omega_\eta^a f_1(\xi-\eta)\varphi_k(\xi)\varphi_{[k_2-2,k_2+2]}(\eta)\|_{L^2_\eta}\lesssim(A2^{\min(k,k_2)}+2^{m/10})^a,$$

which suffices. On the other hand, if $2^p\ll 2^{|k_1|/2+\min(k,k_2)}$, then we may assume that $\xi=(s,0)$, with $s\approx 2^k$. The identities (7.28) show that $\varphi_{\leqslant p+2}(\Omega_\eta\Phi(\xi,\eta))\neq 0$ only if

$$|\xi\cdot\eta^\perp|\leqslant 2^{p+20}2^{k_1-|k_1|/2},$$

which gives

$$|\eta_2| \leqslant 2^{p+30} 2^{k_1-|k_1|/2} 2^{-k}.$$

Therefore $|\eta_2| \ll 2^{k_1}$, so we may assume that $|\eta_1 - s| \approx 2^{k_1}$.

We now write

$$-\Omega_\eta f_1(\xi-\eta) = (\eta_1\partial_2 f_1 - \eta_2\partial_1 f_1)(\xi-\eta) = \frac{\eta_1}{s-\eta_1}(\Omega f_1)(\xi-\eta) - \frac{s\eta_2}{s-\eta_1}(\partial_1 f_1)(\xi-\eta).$$

By iterating this identity, we see that $\Omega_\eta^a f_1(\xi-\eta)$ can be written as a sum of terms of the form

$$P(s,\eta)\left(\frac{1}{s-\eta_1}\right)^{c+d+e}\left(\frac{s\eta_2}{s-\eta_1}\right)^{|b|-d}(D^b\Omega^c f_1)(\xi-\eta),$$

where $|b|+c+d+e \leqslant a$, $|b|, c, d, e \in \mathbb{Z}_+$, $|b| \geqslant d$, and $P(s,\eta)$ is a polynomial of degree at most a in s and at most a in (η_1, η_2). The second bound in (7.27) follows using the bounds on f_1 in (7.25) and the bounds proved earlier: $|s\eta_2| \lesssim 2^p 2^{k_1-|k_1|/2}$ and $|\eta_1 - s| \approx 2^{k_1}$.

The last claim follows using formula (7.30), as in Lemma 7.4 below. □

7.2.2. Localization in modulation

Our lemma in this subsection shows that localization with respect to the phase is often a bounded operation.

LEMMA 7.4. *Let $s \in [2^m-1, 2^{m+1}]$, $m \geqslant 0$, and $-p \leqslant m - 2\delta^2 m$. Let $\Phi = \Phi_{\sigma\mu\nu}$ be as in (2.13) and assume that $\frac{1}{2} = 1/q + 1/r$ and χ is a Schwartz function. Then, if $\|m\|_{S^\infty} \leqslant 1$,*

$$\begin{aligned}\bigg\|\varphi_{\leqslant 10m}(\xi)\int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} m(\xi,\eta)\chi(2^{-p}\Phi(\xi,\eta))\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta\bigg\|_{L^2_\xi} \\ \lesssim \sup_{|\varrho| \leqslant 2^{-p+\delta^2 m}} \|e^{-i(s+\varrho)\Lambda_\mu} f\|_{L^q} \|e^{-i(s+\varrho)\Lambda_\nu} g\|_{L^r} + 2^{-10m}\|f\|_{L^2}\|g\|_{L^2},\end{aligned} \tag{7.29}$$

where the constant in the inequality only depends on the function χ.

Proof. We may assume that $m \geqslant 10$ and use the Fourier transform to write

$$\chi(2^{-p}\Phi(\xi,\eta)) = c\int_{\mathbb{R}} e^{i\varrho 2^{-p}\Phi(\xi,\eta)}\widehat{\chi}(\varrho)\,d\varrho. \tag{7.30}$$

The left-hand side of (7.29) is dominated by

$$C\int_{\mathbb{R}} |\widehat{\chi}(\varrho)| \bigg\|\varphi_{\leqslant 10m}(\xi)\int_{\mathbb{R}^2} e^{i(s+2^{-p}\varrho)\Phi(\xi,\eta)} m(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta\bigg\|_{L^2_\xi} d\varrho.$$

Using (A.2), the contribution of the integral over $|\varrho| \leqslant 2^{\delta^2 m}$ is dominated by the first term in the right-hand side of (7.29). The contribution of the integral over $|\varrho| \geqslant 2^{\delta^2 m}$ is arbitrarily small and is dominated by the second term in the right-hand side of (7.29). □

7.2.3. Linear estimates

We first note the straightforward estimates

$$\|P_k f\|_{L^2} \lesssim \min(2^{(1-50\delta)k}, 2^{-Nk})\|f\|_{Z_1\cap H^N} \quad \text{for } N>0. \tag{7.31}$$

We now prove several linear estimates for functions in $Z_1\cap H^N_\Omega$. As in Lemma 7.3, it is important to take advantage of the fact that our functions are "almost radial". The bounds we prove here are much stronger than the bounds one would normally expect for general functions with the same localization properties, and this is important in the next two sections.

LEMMA 7.5. *Assume that* $N\geqslant 10$ *and*

$$\|f\|_{Z_1} + \sup_{\substack{k\in\mathbb{Z}\\ a\leqslant N}} \|\Omega^a P_k f\|_{L^2} \leqslant 1. \tag{7.32}$$

Let $\delta' := 50\delta + 1/2N$. *For any* $(k,j)\in\mathcal{J}$ *and* $n\in\{0,...,j+1\}$ *let* (*recall the notation* (2.9))

$$f_{j,k} := P_{[k-2,k+2]}Q_{jk}f \quad \textit{and} \quad \hat f_{j,k,n}(\xi) := \varphi^{[-j-1,0]}_{-n}(2^{100}(|\xi|-\gamma_1))\hat f_{j,k}(\xi). \tag{7.33}$$

For any $\xi_0\in\mathbb{R}^2\setminus\{0\}$ *and* $\varkappa,\varrho\in[0,\infty)$ *let* $\mathcal{R}(\xi_0;\varkappa,\varrho)$ *denote the rectangle*

$$\mathcal{R}(\xi_0;\varkappa,\varrho) := \left\{\xi\in\mathbb{R}^2 : \left|\frac{(\xi-\xi_0)\cdot\xi_0}{|\xi_0|}\right| \leqslant \varrho \ \textit{and} \ \left|\frac{(\xi-\xi_0)\cdot\xi_0^\perp}{|\xi_0|}\right| \leqslant \varkappa\right\}. \tag{7.34}$$

(i) *Then, for any* $(k,j)\in\mathcal{J}$, $n\in[0,j+1]$, *and* $\varkappa,\varrho\in(0,\infty)$ *satisfying* $\varkappa+\varrho\leqslant 2^{k-10}$,

$$\Big\|\sup_{\theta\in\mathbb{S}^1} |\hat f_{j,k,n}(r\theta)|\Big\|_{L^2(r\,dr)} \lesssim 2^{(1/2-49\delta)n-(1-\delta')j}, \tag{7.35}$$

$$\int_{\mathbb{R}^2} |\hat f_{j,k,n}(\xi)| \mathbf{1}_{\mathcal{R}(\xi_0;\varkappa,\varrho)}(\xi)\,d\xi \lesssim \varkappa 2^{-j+\delta' j}2^{-49\delta n}\min(1, 2^n\varrho 2^{-k})^{1/2}, \tag{7.36}$$

$$\|\hat f_{j,k,n}\|_{L^\infty} \lesssim \begin{cases} 2^{(\delta+(1/2N))n}2^{-(1/2-\delta')(j-n)}, & \textit{if } |k|\leqslant 10,\\ 2^{-\delta' k}2^{-(1/2-\delta')(j+k)}, & \textit{if } |k|\geqslant 10,\end{cases} \tag{7.37}$$

and

$$\|D^\beta \hat f_{j,k,n}\|_{L^\infty} \lesssim_{|\beta|} \begin{cases} 2^{|\beta|j}2^{(\delta+1/2N)n}2^{-(1/2-\delta')(j-n)}, & \textit{if } |k|\leqslant 10,\\ 2^{|\beta|j}2^{-\delta' k}2^{-(1/2-\delta')(j+k)}, & \textit{if } |k|\geqslant 10.\end{cases} \tag{7.38}$$

(ii) (Dispersive bounds) *If* $m\geqslant 0$ *and* $|t|\in[2^m-1, 2^{m+1}]$, *then*

$$\|e^{-it\Lambda}f_{j,k,n}\|_{L^\infty} \lesssim \|\hat f_{j,k,n}\|_{L^1} \lesssim 2^k 2^{-j+50\delta j}2^{-49\delta n}, \tag{7.39}$$

$$\|e^{-it\Lambda}f_{j,k,0}\|_{L^\infty} \lesssim 2^{3k/2}2^{-m+50\delta j}, \quad \textit{if } |k|\geqslant 10. \tag{7.40}$$

Recall the operators A_{n,γ_0} *defined in* (2.17). *If* $j\leqslant(1-\delta^2)m+\frac{1}{2}|k|$ *and* $|k|+\mathcal{D}\leqslant\frac{1}{2}m$, *then we have the more precise bounds*

$$\|e^{-it\Lambda}A_{\leqslant 0,\gamma_0}f_{j,k,n}\|_{L^\infty}\lesssim\begin{cases}2^{-m+2\delta^2 m}2^{-(j-n)(1/2-\delta')}2^{n(\delta+1/2N)}, & \text{if } n\geqslant 1,\\ 2^{-m+2\delta^2 m}2^{k}2^{-(1/2-\delta')j}, & \text{if } n=0.\end{cases}\tag{7.41}$$

Moreover, for $l\geqslant 1$,

$$\|e^{-it\Lambda}A_{l,\gamma_0}f_{j,k,0}\|_{L^\infty}\lesssim\begin{cases}2^{-m+2\delta^2 m}2^{\delta' j}2^{m/2-j/2-l/2-\max(j,l)/2}, & \text{if } 2l+\max(j,l)\geqslant m,\\ 2^{-m+2\delta^2 m}2^{\delta' j}2^{(l-j)/2}, & \text{if } 2l+\max(j,l)\leqslant m.\end{cases}\tag{7.42}$$

In particular, if $j\leqslant(1-\delta^2)m+\frac{1}{2}|k|$ *and* $|k|+\mathcal{D}\leqslant\frac{1}{2}m$, *then*

$$\begin{aligned}\|e^{-it\Lambda}A_{\leqslant 0,\gamma_0}f_{j,k}\|_{L^\infty}&\lesssim 2^{-m+2\delta^2 m}2^{k}2^{j(\delta+1/2N)},\\ \sum_{l\geqslant 1}\|e^{-it\Lambda}A_{l,\gamma_0}f_{j,k}\|_{L^\infty}&\lesssim 2^{-m+2\delta^2 m}2^{\delta' j}2^{(m-3j)/6}.\end{aligned}\tag{7.43}$$

For all $k\in\mathbb{Z}$ *we have the bounds*

$$\begin{aligned}\|e^{-it\Lambda}A_{\leqslant 0,\gamma_0}P_kf\|_{L^\infty}&\lesssim(2^{k/2}+2^{2k})2^{-m}(2^{51\delta m}+2^{m(2\delta+1/2N)}),\\ \|e^{-it\Lambda}A_{\geqslant 1,\gamma_0}P_kf\|_{L^\infty}&\lesssim 2^{-5m/6+2\delta^2 m}.\end{aligned}\tag{7.44}$$

Proof. (i) The hypothesis gives

$$\|f_{j,k,n}\|_{L^2}\lesssim 2^{(1/2-49\delta)n-(1-50\delta)j}\quad\text{and}\quad\|\Omega^N f_{j,k,n}\|_{L^2}\lesssim\|\Omega^N P_kf\|_{L^2}\lesssim 1.\tag{7.45}$$

The bounds (7.35) follow using the general interpolation inequality

$$\Big\|\sup_{\theta\in\mathbb{S}^1}|h(r\theta)|\Big\|_{L^2(r\,dr)}\lesssim L^{1/2}\|h\|_{L^2}+L^{1/2-N}\|\Omega^N h\|_{L^2},\tag{7.46}$$

for any $h\in L^2(\mathbb{R}^2)$ and $L\geqslant 1$, which easily follows using the operators $\mathcal{R}_{\leqslant L}$ defined in (7.24).

Inequality (7.36) follows from (7.35). Indeed, the left-hand side is dominated by

$$\begin{aligned}C(\varkappa 2^{-k})\sup_{\theta\in\mathbb{S}^1}&\int_{\mathbb{R}}|\hat f_{j,k,n}(r\theta)|\mathbf{1}_{\mathcal{R}(\xi_0;\varkappa,\varrho)}(r\theta)r\,dr\\ &\lesssim\sup_{\theta\in\mathbb{S}^1}\|\hat f_{j,k,n}(r\theta)\|_{L^2(r\,dr)}(\varkappa 2^{-k})(2^k\min(\varrho,2^{k-n}))^{1/2},\end{aligned}$$

which gives the desired result.

We now consider (7.37). For any fixed $\theta\in\mathbb{S}^1$ we have

$$\begin{aligned}\|\hat f_{j,k,n}(r\theta)\|_{L^\infty}&\lesssim 2^{j/2}\|\hat f_{j,k,n}(r\theta)\|_{L^2(dr)}+2^{-j/2}\|(\partial_r\hat f_{j,k,n})(r\theta)\|_{L^2(dr)}\\ &\lesssim 2^{j/2}2^{-k/2}\|\hat f_{j,k,n}(r\theta)\|_{L^2(r\,dr)},\end{aligned}$$

using the support property of $Q_{jk}f$ in the physical space. The desired bound follows using (7.35) and the observation that $\hat{f}_{j,k,n}=0$ unless $n=0$ or $k\in[-10,10]$. The bound (7.38) also follows since differentiation in the Fourier space essentially corresponds to multiplication by factors of 2^j, due to space localization.

(ii) The bound (7.39) follows directly from Hausdorff–Young and (7.45). To prove (7.40), if $|k|\geqslant 10$ then the standard dispersion estimate

$$\left|\int_{\mathbb{R}^2} e^{-it\lambda(|\xi|)}\varphi_k(\xi)e^{ix\cdot\xi}\,d\xi\right|\lesssim 2^{2k}(1+|t|2^{k+|k|/2})^{-1} \tag{7.47}$$

gives

$$\|e^{-it\Lambda}f_{j,k,n}\|_{L^\infty}\lesssim \frac{2^{2k}}{1+|t|2^{k/2}}\|f_{j,k,n}\|_{L^1}\lesssim \frac{2^{2k}}{1+|t|2^{k/2}}2^{50\delta j}. \tag{7.48}$$

The bound (7.40) follows (in case $m\leqslant 10$ and $k\geqslant 0$, one can use (7.39)).

We now prove (7.41). The operator $A_{\leqslant 0,\gamma_0}$ is important here, because the function λ has an inflection point at γ_0; see (10.3). Using Lemma 7.2 (i) and the observation that $|(\nabla\Lambda)(\xi)|\approx 2^{|k|/2}$ if $|\xi|\approx 2^k$, it is easy to see that

$$|(e^{-it\Lambda}A_{\leqslant 0,\gamma_0}f_{j,k,n})(x)|\lesssim 2^{-10m}\quad \text{unless } |x|\approx 2^{m+|k|/2}.$$

Also, letting $f'_{j,k,n}:=\mathcal{R}_{\leqslant m/5}f_{j,k,n}$ (see (7.24)), we have $\|f_{j,k,n}-f'_{j,k,n}\|_{L^2}\lesssim 2^{-mN/5}$, and thus

$$\|e^{-it\Lambda}A_{\leqslant 0,\gamma_0}(f_{j,k,n}-f'_{j,k,n})\|_{L^\infty}\lesssim \|\hat{f}_{j,k,n}-\widehat{f'_{j,k,n}}\|_{L^1}\lesssim 2^{-2m}2^k. \tag{7.49}$$

On the other hand, if $|x|\approx 2^{m+|k|/2}$ then, using again Lemma 7.2 and (7.38),

$$\begin{aligned}&(e^{-it\Lambda}A_{\leqslant 0,\gamma_0}f'_{j,k,n})(x)\\ &\quad =C\int_{\mathbb{R}^2} e^{i\Psi(\xi)}\varphi(\varkappa_r^{-1}\nabla_\xi\Psi)\varphi(\varkappa_\theta^{-1}\Omega_\xi\Psi)\widehat{f'_{j,k,n}}(\xi)\varphi_{\geqslant -100}(|\xi|-\gamma_0)\,d\xi+O(2^{-10m}),\end{aligned} \tag{7.50}$$

where

$$\begin{aligned}\Psi&:=-t\Lambda(\xi)+x\cdot\xi,\\ \varkappa_r&:=2^{\delta^2 m}(2^{(m+|k|/2-k)/2}+2^j),\\ \varkappa_\theta&:=2^{\delta^2 m}2^{(m+k+|k|/2)/2}.\end{aligned} \tag{7.51}$$

We notice that the support of the integral in (7.50) is contained in a $\varkappa\times\varrho$ rectangle in the direction of the vector x, where

$$\varrho\lesssim\frac{\varkappa_r}{2^{m+|k|/2-k}},\quad \varkappa\lesssim\frac{\varkappa_\theta}{2^{m+|k|/2}},\quad\text{and}\quad \varkappa\lesssim\varrho.$$

This is because the function λ'' does not vanish in the support of the integral, and so $\lambda''(|\xi|)\approx 2^{|k|/2-k}$. Therefore, we can estimate the contribution of the integral in (7.50) using either (7.36) or (7.37). More precisely, if $j\leqslant\frac{1}{2}\big(m+\frac{1}{2}|k|-k\big)$, then we use (7.37), while if $j\geqslant\frac{1}{2}\big(m+\frac{1}{2}|k|-k\big)$, then we use (7.36) (and estimate $\min(1,2^n\varrho 2^{-k})\leqslant 2^n\varrho 2^{-k}$); in both cases the desired estimate follows.

We now prove (7.42). We may assume that $|k|\leqslant 10$ and $m\geqslant\mathcal{D}$. As before, we may assume that $|x|\approx 2^m$ and replace $f_{j,k,0}$ by $f'_{j,k,0}$. As in (7.50), we have

$$\begin{aligned}&(e^{-it\Lambda}A_{l,\gamma_0}f_{j,k,0})(x)\\&\quad=C\int_{\mathbb{R}^2}e^{i\Psi(\xi)}\varphi(2^{-m/2-\delta^2m}\Omega_\xi\Psi)\widehat{f'_{j,k,0}}(\xi)\varphi_{-l-100}(|\xi|-\gamma_0)\,d\xi+O(2^{-2m}),\end{aligned}\tag{7.52}$$

where Ψ is as in (7.51). The support of the integral above is contained in a $\varkappa\times\varrho$ rectangle in the direction of the vector x, where $\varrho\lesssim 2^{-l}$ and $\varkappa\lesssim 2^{-m/2+\delta^2m}$. Since

$$|\widehat{f'_{j,k,0}}(\xi)|\lesssim 2^{-j/2+\delta' j}$$

in this rectangle (see (7.37)), the bound in the first line of (7.42) follows if $l\geqslant j$. On the other hand, if $l\leqslant j$ then we use (7.36) to show that the absolute value of the integral in (7.52) is dominated by $C2^{-j+\delta' j}\varkappa\varrho^{1/2}$, which gives again the bound in the first line of (7.42).

It remains to prove the stronger bound in the second line of (7.42) in the case $2l+\max(j,l)\leqslant m$. We notice that $\lambda''(|\xi|)\approx 2^{-l}$ in the support of the integral. Assume that $x=(x_1,0)$, with $x_1\approx 2^m$, and notice that we can insert an additional cutoff function of the form

$$\varphi[\varkappa_r^{-1}(x_1-t\lambda'(|\xi_1|)\,\mathrm{sgn}\,(\xi_1))],\quad\text{where } \varkappa_r:=2^{\delta^2m}(2^{(m-l)/2}+2^j+2^l),$$

in the integral in (7.52), at the expense of an acceptable error. This can be verified using Lemma 7.2 (i). The support of the integral is then contained in a $\varkappa\times\varrho$ rectangle in the direction of the vector x, where $\varrho\lesssim\varkappa_r 2^{-m}2^l$ and $\varkappa\lesssim 2^{-m/2+\delta^2m}$. The desired estimate then follows as before, using the L^∞ bound (7.37) if $2j\leqslant m-l$ and the integral bound (7.36) if $2j\geqslant m-l$.

The bounds in (7.43) follow from (7.41) and (7.42) by summation over n and l, respectively. Finally, the bounds in (7.44) follow by summation (use (7.39) if $j\geqslant(1-\delta^2)m$ or $m\leqslant 4\mathcal{D}$, use (7.40) if $j\leqslant(1-\delta^2)m$ and $|k|\geqslant 10$, and use (7.43) if $j\leqslant(1-\delta^2)m$ and $|k|\leqslant 10$). □

Remark 7.6. We notice that we also have the bound (with no loss of $2^{2\delta^2m}$)

$$\|e^{-it\Lambda}A_{\leqslant 0,\gamma_0}f_{j,k,0}\|_{L^\infty}\lesssim 2^{-m}2^k2^{-(1/2-\delta'-\delta)j},\tag{7.53}$$

provided that $j\leqslant(1-\delta^2)m+\frac{1}{2}|k|$ and $|k|+\mathcal{D}\leqslant\frac{1}{2}m$. Indeed, this follows from (7.41) if $j\geqslant\frac{1}{10}m$. On the other hand, if $j\leqslant\frac{1}{10}m$, then we can decompose (compare with (7.50)),

$$(e^{-it\Lambda}A_{\leqslant 0,\gamma_0}f_{j,k,0})(x)=\sum_{p\geqslant 0}C\int_{\mathbb{R}^2}e^{i\Psi(\xi)}\varphi_p^{[0,\infty)}(\varkappa^{-1}\nabla_\xi\Psi)\hat{f}_{j,k,0}(\xi)\varphi_{\geqslant -100}(|\xi|-\gamma_0)\,d\xi,$$

where $\varkappa:=2^{(m+|k|/2-k)/2}$. The contribution of $p=0$ is estimated as before, using (7.37), while for $p\geqslant 1$ we can first integrate by parts at most three times, and then estimate the integral in the same way.

8. Dispersive analysis II: The function $\partial_t\mathcal{V}$

In this section we prove several lemmas describing the function $\partial_t\mathcal{V}$. These lemmas rely on the Duhamel formula (7.8),

$$\Omega_\xi^a(\partial_t\widehat{\mathcal{V}})(\xi,s)=e^{is\Lambda(\xi)}\Omega_\xi^a\widehat{\mathcal{N}}_2(\xi,s)+e^{is\Lambda(\xi)}\Omega_\xi^a\widehat{\mathcal{N}}_3(\xi,s)+e^{is\Lambda(\xi)}\Omega_\xi^a\widehat{\mathcal{N}}_{\geqslant 4}(\xi,s), \tag{8.1}$$

where

$$\begin{aligned}&e^{is\Lambda(\xi)}\Omega_\xi^a\widehat{\mathcal{N}}_2(\xi,s)\\&\quad=\sum_{\mu,\nu\in\{+,-\}}\sum_{a_1+a_2=a}\int_{\mathbb{R}^2}e^{is\Phi_{+\mu\nu}(\xi,\eta)}\mathfrak{m}_{\mu\nu}(\xi,\eta)(\Omega^{a_1}\widehat{\mathcal{V}}_\mu)(\xi-\eta,s)(\Omega^{a_2}\widehat{\mathcal{V}}_\nu)(\eta,s)\,d\eta\end{aligned} \tag{8.2}$$

and

$$\begin{aligned}&e^{is\Lambda(\xi)}\Omega_\xi^a\widehat{\mathcal{N}}_3(\xi,s)\\&\quad=\sum_{\mu,\nu,\beta\in\{+,-\}}\sum_{a_1+a_2+a_3=a}\int_{\mathbb{R}^2\times\mathbb{R}^2}e^{is\tilde{\Phi}_{+\mu\nu\beta}(\xi,\eta,\sigma)}\mathfrak{n}_{\mu\nu\beta}(\xi,\eta,\sigma)\\&\qquad\qquad\times(\Omega^{a_1}\widehat{\mathcal{V}}_\mu)(\xi-\eta,s)(\Omega^{a_2}\widehat{\mathcal{V}}_\nu)(\eta-\sigma,s)(\Omega^{a_3}\widehat{\mathcal{V}}_\beta)(\sigma,s)\,d\eta\,d\sigma.\end{aligned} \tag{8.3}$$

Recall also the assumptions on the non-linearity $\mathcal{N}_{\geqslant 4}$ and the profile $\mathcal{V}$ (see (7.15)),

$$\begin{aligned}\|\mathcal{V}(t)\|_{H^{N_0}\cap H_\Omega^{N_1,N_3}}\leqslant\varepsilon_1(1+t)^{\delta^2},\quad\|\mathcal{V}(t)\|_Z\leqslant\varepsilon_1,\\ \|\mathcal{N}_{\geqslant 4}(t)\|_{H^{N_0-N_3}\cap H_\Omega^{N_1}}\lesssim\varepsilon_1^2(1+t)^{-2},\end{aligned} \tag{8.4}$$

and the symbol-type bounds (7.11) on the multipliers $\mathfrak{m}_{\mu\nu}$. Given $\Phi=\Phi_{\sigma\mu\nu}$ as in (2.13), let

$$\begin{aligned}\Xi=\Xi_{\mu\nu}(\xi,\eta)&:=(\nabla_\eta\Phi_{\sigma\mu\nu})(\xi,\eta)=(\nabla\Lambda_\mu)(\xi-\eta)-(\nabla\Lambda_\nu)(\eta), &&\Xi:\mathbb{R}^2\times\mathbb{R}^2\longrightarrow\mathbb{R}^2,\\ \Theta=\Theta_\mu(\xi,\eta)&:=(\Omega_\eta\Phi_{\sigma\mu\nu})(\xi,\eta)=\frac{\lambda_\mu'(|\xi-\eta|)}{|\xi-\eta|}(\xi\cdot\eta^\perp), &&\Theta:\mathbb{R}^2\times\mathbb{R}^2\longrightarrow\mathbb{R}.\end{aligned} \tag{8.5}$$

In this section we prove three lemmas describing the function $\partial_t\mathcal{V}$.

LEMMA 8.1. (i) *Assume* (8.1)–(8.4), $m\geqslant 0$, $s\in[2^m-1,2^{m+1}]$, $k\in\mathbb{Z}$, *and* $\sigma\in\{+,-\}$. *Then,*

$$\|(\partial_t\mathcal{V}_\sigma)(s)\|_{H^{N_0-N_3}\cap H^{N_1}_\Omega}\lesssim\varepsilon_1^2 2^{-5m/6+6\delta^2 m}, \tag{8.6}$$

$$\sup_{\substack{a\leqslant N_1/2+20\\ 2a+|\alpha|\leqslant N_1+N_4}}\|e^{-is\Lambda_\sigma}P_kD^\alpha\Omega^a(\partial_t\mathcal{V}_\sigma)(s)\|_{L^\infty}\lesssim\varepsilon_1^2 2^{-5m/3+6\delta^2 m}. \tag{8.7}$$

(ii) *In addition, if* $a\leqslant\frac{1}{2}N_1+20$ *and* $2a+|\alpha|\leqslant N_1+N_4$, *then we may decompose*

$$P_kD^\alpha\Omega^a(\partial_t\mathcal{V}_\sigma)=\varepsilon_1^2\sum_{\substack{a_1+a_2=a\\ \alpha_1+\alpha_2=\alpha\\ \mu,\nu\in\{+,-\}}}\sum_{[(k_1,j_1),(k_2,j_2)]\in X_{m,k}}A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}+\varepsilon_1^2P_kE^{a,\alpha}_\sigma, \tag{8.8}$$

where

$$\|P_kE^{a,\alpha}_\sigma(s)\|_{L^2}\lesssim 2^{-3m/2+5\delta m}. \tag{8.9}$$

Moreover, with $\mathfrak{m}_{+\mu\nu}(\xi,\eta):=\mathfrak{m}_{\mu\nu}(\xi,\eta)$ *and* $\mathfrak{m}_{-\mu\nu}(\xi,\eta):=\overline{\mathfrak{m}_{(-\mu)(-\nu)}(-\xi,-\eta)}$, *we have*

$$\mathcal{F}\{A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}\}(\xi,s):=\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}\mathfrak{m}_{\sigma\mu\nu}(\xi,\eta)\varphi_k(\xi)\widehat{f^\mu_{j_1,k_1}}(\xi-\eta,s)\widehat{f^\nu_{j_2,k_2}}(\eta,s)\,d\eta, \tag{8.10}$$

where

$$f^\mu_{j_1,k_1}=\varepsilon_1^{-1}P_{[k_1-2,k_1+2]}Q_{j_1k_1}D^{\alpha_1}\Omega^{a_1}\mathcal{V}_\mu\quad\text{and}\quad f^\nu_{j_2,k_2}=\varepsilon_1^{-1}P_{[k_2-2,k_2+2]}Q_{j_2k_2}D^{\alpha_2}\Omega^{a_2}\mathcal{V}_\nu.$$

Let $N_0'=N_1-N_4=1/\delta$. *The sets* $X_{m,k}$ *and the functions* $A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}$ *have the following properties*:

(1) $X_{m,k}=\varnothing$, *unless* $m\geqslant\mathcal{D}^2$, $k\in\big[-\frac{3}{4}m,m/N_0'\big]$, *and*

$$X_{m,k}\subseteq\big\{[(k_1,j_1),(k_2,j_2)]\in\mathcal{J}\times\mathcal{J}:k_1,k_2\in\big[-\tfrac{3}{4}m,m/N_0'\big],\ \max(j_1,j_2)\leqslant 2m\big\}. \tag{8.11}$$

(2) *If* $[(k_1,j_1),(k_2,j_2)]\in X_{m,k}$ *and* $\min(k_1,k_2)\leqslant-2m/N_0'$, *then*

$$\max(j_1,j_2)\leqslant(1-\delta^2)m-|k|,\quad \max(|k_1-k|,|k_2-k|)\leqslant 100,\quad \mu=\nu, \tag{8.12}$$

and

$$\|A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}(s)\|_{L^2}\lesssim 2^{2k}2^{-m+6\delta^2 m}. \tag{8.13}$$

(3) *If* $[(k_1,j_1),(k_2,j_2)]\in X_{m,k}$, $\min(k_1,k_2)\geqslant-5m/N_0'$, $k\leqslant\min(k_1,k_2)-200$, *then*

$$\max(j_1,j_2)\leqslant(1-\delta^2)m-|k|,\quad \max(|k_1|,|k_2|)\leqslant 10,\quad \mu=-\nu, \tag{8.14}$$

and

$$\|A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}(s)\|_{L^2}\lesssim 2^k 2^{-m+4\delta m}. \tag{8.15}$$

(4) *If* $[(k_1,j_1),(k_2,j_2)]\in X_{m,k}$ *and* $\min(k,k_1,k_2)\geqslant -6m/N_0'$, *then*

$$\textit{either}\quad j_1\leqslant \tfrac{5}{6}m \quad \textit{or}\quad |k_1|\leqslant 10, \tag{8.16}$$

$$\textit{either}\quad j_2\leqslant \tfrac{5}{6}5m \quad \textit{or}\quad |k_2|\leqslant 10, \tag{8.17}$$

and

$$\min(j_1,j_2)\leqslant (1-\delta^2)m. \tag{8.18}$$

Moreover,

$$\|A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}(s)\|_{L^2}\lesssim 2^k 2^{-m+4\delta m}, \tag{8.19}$$

and

$$\textit{if}\ \max(j_1,j_2)\geqslant (1-\delta^2)m-|k|,\ \textit{then}\ \|A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}(s)\|_{L^2}\lesssim 2^{-4m/3+4\delta m}. \tag{8.20}$$

(iii) *As a consequence of* (8.9), (8.13), (8.15), *and* (8.19), *if*

$$a\leqslant \tfrac{1}{2}N_1+20 \quad \textit{and}\quad 2a+|\alpha|\leqslant N_1+N_4,$$

then we have the L^2 *bound*

$$\|P_kD^\alpha\Omega^a(\partial_t\mathcal{V}_\sigma)\|_{L^2}\lesssim \varepsilon_1^2[2^k2^{-m+5\delta m}+2^{-3m/2+5\delta m}]. \tag{8.21}$$

Proof. (i) We first consider the quadratic part of the non-linearity. Let $I^{\sigma\mu\nu}$ denote the bilinear operator defined by

$$\begin{gathered}\mathcal{F}\{I^{\sigma\mu\nu}[f,g]\}(\xi):=\int_{\mathbb{R}^2} e^{is\Phi_{\sigma\mu\nu}(\xi,\eta)}m(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta,\\ \|m^{k,k_1,k_2}\|_{S^\infty}\leqslant 2^k2^{\min(k_1,k_2)/2},\quad \|D^\alpha_\eta m^{k,k_1,k_2}\|_{L^\infty}\lesssim_{|\alpha|} 2^{(|\alpha|+3/2)\max(|k_1|,|k_2|)},\end{gathered} \tag{8.22}$$

where, for simplicity of notation, $m=\mathfrak{m}_{\sigma\mu\nu}$. For simplicity, we often write Φ, Ξ, and Θ instead of $\Phi_{\sigma\mu\nu}$, $\Xi_{\mu\nu}$, and Θ_μ in the rest of this proof.

We define the operators P_k^+ for $k\in\mathbb{Z}_+$ by $P_k^+:=P_k$ for $k\geqslant 1$ and $P_0^+:=P_{\leqslant 0}$. In view of Lemma A.1 (ii), (8.4), and (7.44), for any $k\geqslant 0$ we have

$$\begin{aligned}\|P_k^+ I^{\sigma\mu\nu}[\mathcal{V}_\mu,\mathcal{V}_\nu](s)\|_{H^{N_0-N_3}}&\\ \lesssim 2^{(N_0-N_3)k}\sum_{\substack{0\leqslant k_1\leqslant k_2\\ k_2\geqslant k-10}} 2^k2^{k_1/2}\|P^+_{k_2}\mathcal{V}(s)\|_{L^2}\|e^{-is\Lambda}P^+_{k_1}\mathcal{V}(s)\|_{L^\infty}&\\ \lesssim \varepsilon_1^2 2^{-k}2^{-5m/6+6\delta^2 m},&\end{aligned} \tag{8.23}$$

which is consistent with (8.6). Similarly,

$$\|P_k^+ I^{\sigma\mu\nu}[\Omega^{a_2}\mathcal{V}_\mu, \Omega^{a_3}\mathcal{V}_\nu](s)\|_{L^2} \lesssim 2^{-k}\varepsilon_1^2 2^{-5m/6+6\delta^2 m}, \quad a_2+a_3 \leqslant N_1, \tag{8.24}$$

by placing the factor with less than $\frac{1}{2}N_1$ Ω-derivatives in L^∞, and the other factor in L^2. Finally, using L^∞ estimates on both factors,

$$\|e^{-is\Lambda_\sigma} P_k^+ I^{\sigma\mu\nu}[D^{\alpha_2}\Omega^{a_2}\mathcal{V}_\mu, D^{\alpha_3}\Omega^{a_3}\mathcal{V}_\nu](s)\|_{L^\infty} \lesssim \begin{cases} \varepsilon_1^2 2^{-5m/3+6\delta^2 m}, & \text{if } k \leqslant 20, \\ \varepsilon_1^2 2^{4k} 2^{-11m/6+52\delta m}, & \text{if } k \geqslant 20, \end{cases} \tag{8.25}$$

provided that $a_2+a_3=a$ and $\alpha_2+\alpha_3=\alpha$ (see also (8.26) below). The conclusions in part (i) follow for the quadratic components.

The conclusions for the cubic components follow by the same argument, using the assumption (7.12) instead of (7.11), and the formula (8.3). The contributions of the higher-order non-linearity $\mathcal{N}_{\geqslant 4}$ are estimated using directly the bootstrap hypothesis (8.4).

(ii) We assume that s is fixed and, for simplicity, drop it from the notation. In view of (8.4) and using interpolation, the functions $f^\mu:=\varepsilon_1^{-1}D^{\alpha_2}\Omega^{a_2}\mathcal{V}_\mu$ and $f^\nu:=\varepsilon_1^{-1}D^{\alpha_3}\Omega^{a_3}\mathcal{V}_\nu$ satisfy

$$\|f^\mu\|_{H^{N_0'}\cap Z_1\cap H_\Omega^{N_1'}} + \|f^\nu\|_{H^{N_0'}\cap Z_1\cap H_\Omega^{N_1'}} \lesssim 2^{\delta^2 m}, \tag{8.26}$$

where (compare with the notation in Theorem 1.1)

$$N_1' := \frac{N_1-N_4}{2} = \frac{1}{2\delta} \quad \text{and} \quad N_0' := \frac{N_0-N_3}{2} - N_4 = \frac{1}{\delta}. \tag{8.27}$$

In particular, the dispersive bounds (7.39)–(7.44) hold with $N=N_1'=1/2\delta$.

The contributions of the higher-order non-linearities $\mathcal{N}_3$ and $\mathcal{N}_{\geqslant 4}$ can all be estimated as part of the error term $P_k E_\sigma^{a,\alpha}$, so we focus on the quadratic non-linearity $\mathcal{N}_2$. Notice that

$$A_{k;k_1,j_1;k_2,j_2}^{a_1,\alpha_1;a_2,\alpha_2} = P_k I^{\sigma\mu\nu}(f_{j_1,k_1}^\mu, f_{j_2,k_2}^\nu).$$

Proof of property (1). In view of Lemma A.1 and (7.43), we have the general bound

$$\|A_{k;k_1,j_1;k_2,j_2}^{a_1,\alpha_1;a_2,\alpha_2}\|_{L^2} \lesssim 2^{k+\min(k_1,k_2)/2} 2^{-5m/6+5\delta^2 m} \min(2^{-(1/2-\delta)\max(j_1,j_2)}, 2^{-N_0'\max(k_1,k_2)}).$$

This bound suffices to prove the claims in (1). Indeed, if $k\geqslant m/N_0'$ or if $k\leqslant -\frac{3}{4}m+\mathcal{D}^2$, then the sum of all the terms can be bounded as in (8.9). Similarly, if $k\in[-\frac{3}{4}m+\mathcal{D}^2, m/N_0']$ then the sums of the L^2 norms corresponding to $\max(k_1,k_2)\geqslant m/N_0'$, or $\max(j_1,j_2)\geqslant 2m$, or $\min(k_1,k_2)\leqslant -\frac{3}{4}m+\mathcal{D}^2$, are all bounded by $2^{-3m/2}$ as desired. □

Proof of property (2). Assume now that

$$\min(k_1,k_2)\leqslant -\frac{2m}{N_0'}\quad\text{and}\quad j_2=\max(j_1,j_2)\geqslant (1-\delta^2)m-|k|.$$

Then, using the $L^2\times L^\infty$ estimate as before,

$$\|P_kI^{\sigma\mu\nu}[f^\mu_{j_1,k_1},A_{\leqslant 0,\gamma_1}f^\nu_{j_2,k_2}]\|_{L^2}\lesssim 2^{k+\min(k_1,k_2)/2}2^{-5m/6+5\delta^2m}2^{-j_2(1-50\delta)}\lesssim 2^{-3m/2}.$$

Moreover, we notice that, if $A_{\geqslant 1,\gamma_1}f^\nu_{j_2,k_2}$ is non-trivial, then $|k_2|\leqslant 10$ and $k_1\leqslant -2m/N_0'$, therefore

$$\|P_kI^{\sigma\mu\nu}[f^\mu_{j_1,k_1},A_{\geqslant 1,\gamma_1}f^\nu_{j_2,k_2}]\|_{L^2}\lesssim 2^{k+k_1/2}2^{-m+5\delta^2m}2^{-j_2(1/2-\delta)}\lesssim 2^{-3m/2+3\delta m}$$

if $j_1\leqslant(1-\delta^2)m$, using (7.41) if $k_1\geqslant -\frac{1}{2}m$ and (7.40) if $k_1\leqslant -\frac{1}{2}m$. On the other hand, if $j_1\geqslant(1-\delta^2)m$, then we use again the $L^2\times L^\infty$ estimate (placing $f^\mu_{j_1,k_1}$ in L^2) to conclude that

$$\|P_kI^{\sigma\mu\nu}[f^\mu_{j_1,k_1},A_{\geqslant 1,\gamma_1}f^\nu_{j_2,k_2}]\|_{L^2}\lesssim 2^{k+k_1/2}2^{-j_1+50\delta j_1}2^{-m+52\delta m}\lesssim 2^{-3m/2}.$$

The last three bounds show that

$$\|A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}\|_{L^2}\lesssim 2^{-3m/2+3\delta m},\quad \text{if } \max(j_1,j_2)\geqslant(1-\delta^2)m-|k|. \tag{8.28}$$

Assume now that

$$k_1=\min(k_1,k_2)\leqslant -\frac{2m}{N_0'}\quad\text{and}\quad \max(j_1,j_2)\leqslant(1-\delta^2)m-|k|.$$

If $k_2\geqslant k_1+20$, then $|\nabla_\eta\Phi(\xi,\eta)|\gtrsim 2^{|k_1|/2}$, and so $\|A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}\|_{L^2}\lesssim 2^{-3m}$ by Lemma 7.2 (i). On the other hand, if $k,k_2\leqslant k_1+30$ then, using again the $L^2\times L^\infty$ argument as before,

$$\|P_kI^{\sigma\mu\nu}[f^\mu_{j_1,k_1},f^\nu_{j_2,k_2}]\|_{L^2}\lesssim 2^{k+k_1}2^{-m+5\delta^2m}. \tag{8.29}$$

The L^2 bound in (8.9) follows if $k+k_1\leqslant -\frac{1}{2}m$. On the other hand, if $k+k_1\geqslant -\frac{1}{2}m$ and

$$\max(|k_1-k|,|k_2-k|)\geqslant 100\quad\text{or}\quad \mu=-\nu,$$

then $|\nabla_\eta\Phi(\xi,\eta)|\gtrsim 2^{k-\max(k_1,k_2)}$ in the support of the integral, in view of (10.18). Therefore, $\|A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}\|_{L^2}\lesssim 2^{-3m}$ in view of Lemma 7.2 (i). The inequalities in (8.12) follow. The bound (8.13) then follows from (8.29). □

Proof of property (3). Assume first that

$$\min(k_1,k_2)\geqslant -\frac{5m}{N_0'},\quad k\leqslant \min(k_1,k_2)-200,\quad \max(j_1,j_2)\geqslant (1-\delta^2)m-|k|-|k_2|. \tag{8.30}$$

We may assume that $j_2\geqslant j_1$. Using the $L^2\times L^\infty$ estimate and Lemma 7.5 (ii) as before,

$$\|P_kI^{\sigma\mu\nu}[f^{\mu}_{j_1,k_1},A^{(j_2)}_{n_2,\gamma_1}f^{\nu}_{j_2,k_2}]\|_{L^2}\lesssim 2^{k+k_1/2}2^{-5m/6+5\delta^2m}2^{-j_2(1-50\delta)}\lesssim 2^{-3m/2}$$

if $n_2\leqslant \mathcal{D}$. On the other hand, if $n_2\in[\mathcal{D},j_2]$, then

$$P_kI^{\sigma\mu\nu}[f^{\mu}_{j_1,k_1},A^{(j_2)}_{n_2,\gamma_1}f^{\nu}_{j_2,k_2}]=P_kI^{\sigma\mu\nu}[A_{\geqslant 1,\gamma_1}f^{\mu}_{j_1,k_1},A^{(j_2)}_{n_2,\gamma_1}f^{\nu}_{j_2,k_2}].$$

If $j_1\leqslant(1-\delta^2)m$, then we estimate

$$\begin{aligned}\|P_kI^{\sigma\mu\nu}[A_{\geqslant 1,\gamma_1}f^{\mu}_{j_1,k_1},A^{(j_2)}_{n_2,\gamma_1}f^{\nu}_{j_2,k_2}]\|_{L^2}&\lesssim 2^{k}2^{-m+5\delta^2m+2\delta m}2^{-j_2(1/2-\delta)}\\&\lesssim 2^{-3m/2+3\delta m+8\delta^2m}.\end{aligned}$$

Finally, if $j_2\geqslant j_1\geqslant(1-\delta^2)m$, then we use Schur's lemma in the Fourier space and estimate

$$\begin{aligned}&\|P_kI^{\sigma\mu\nu}[A^{(j_1)}_{n_1,\gamma_1}f^{\mu}_{j_1,k_1},A^{(j_2)}_{n_2,\gamma_1}f^{\nu}_{j_2,k_2}]\|_{L^2}\\&\quad\lesssim 2^k2^{-\max(n_1,n_2)/2}\|A^{(j_1)}_{n_1,\gamma_1}f^{\mu}_{j_1,k_1}\|_{L^2}\,\|A^{(j_2)}_{n_2,\gamma_1}f^{\nu}_{j_2,k_2}\|_{L^2}\\&\quad\lesssim 2^k2^{2\delta^2m}2^{-\max(n_1,n_2)/2}2^{-j_1(1-50\delta)}2^{(1/2-49\delta)n_1}2^{-j_2(1-50\delta)}2^{(1/2-49\delta)n_2}\\&\quad\lesssim 2^{2\delta^2m}2^{\min(n_1,n_2)/2}2^{-j_1(1-50\delta)}2^{-49\delta(n_1+n_2)}2^{-j_2(1-50\delta)}\\&\quad\lesssim 2^{2\delta^2m}2^{-(2-2\delta^2)(1-50\delta)m}2^{(1/2-98\delta)m}\end{aligned}\tag{8.31}$$

for any $n_1\in[1,j_1+1]$ and $n_2\in[1,j_2+1]$. Therefore, if (8.30) holds, then

$$\|A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}\|_{L^2}\lesssim 2^{-3m/2+4\delta m}. \tag{8.32}$$

Assume now that

$$\min(k_1,k_2)\geqslant -\frac{5m}{N_0'},\quad k\leqslant \min(k_1,k_2)-200,\quad \max(j_1,j_2)\leqslant (1-\delta^2)m-|k|-|k_2|. \tag{8.33}$$

If, in addition, $\max(|k_1|,|k_2|)\geqslant 11$ or $\mu=\nu$, then $|\nabla_\eta\Phi(\xi,\eta)|\gtrsim 2^{k-k_2}$ in the support of the integral. Indeed, this is a consequence of (10.18) if $k\leqslant -100$ and it follows easily from formula (10.22) if $k\geqslant -100$. Therefore, $\|A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}\|_{L^2}\lesssim 2^{-3m}$, using Lemma 7.2 (i). As a consequence, the functions $A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}$ can be absorbed into the error term $P_kE^{a,\alpha}_{\sigma}$, unless all the inequalities in (8.14) hold.

Assume now that (8.14) holds and we are looking to prove (8.15). It suffices to prove that

$$\|P_k I^{\sigma\mu\nu}[A_{\geqslant 1,\gamma_0} f^{\mu}_{j_1,k_1}, A_{\geqslant 1,\gamma_0} f^{\nu}_{j_2,k_2}]\|_{L^2} \lesssim 2^k 2^{-m+4\delta m}, \tag{8.34}$$

after using (7.41) and the $L^2\times L^\infty$ argument. We may assume that $\max(j_1,j_2)\leqslant\frac{1}{3}m$; otherwise, (8.34) follows from the $L^2\times L^\infty$ estimate. Using (7.37) and the more precise bound (7.42), we get

$$\|A_{p,\gamma_0} h\|_{L^2} \lesssim 2^{\delta^2 m} 2^{-p/2} \quad \text{and} \quad \|e^{-it\Lambda} A_{p,\gamma_0} h\|_{L^\infty} \lesssim 2^{-m+3\delta^2 m} \min(2^{p/2}, 2^{m/2-p}),$$

where $h\in\{f_{j_1,k_1}, g_{j_2,k_2}\}$ and $p\geqslant 1$. Therefore, using Lemma A.1,

$$\|P_k I^{\sigma\mu\nu}[A_{p_1,\gamma_0} f^{\mu}_{j_1,k_1}, A_{p_2,\gamma_0} f^{\nu}_{j_2,k_2}]\|_{L^2} \lesssim 2^k 2^{-m+5\delta^2 m} 2^{-\max(p_1,p_2)/2} 2^{\min(p_1,p_2)/2}.$$

The desired bound (8.34) follows, using also the simple estimate

$$\|P_k I^{\sigma\mu\nu}[A_{p_1,\gamma_0} f^{\mu}_{j_1,k_1}, A_{p_2,\gamma_0} f^{\nu}_{j_2,k_2}]\|_{L^2} \lesssim 2^k 2^{2\delta^2 m} 2^{-(p_1+p_2)/2}.$$

This completes the proof of (8.15). □

Proof of property (4). The same argument as in the proof of (8.32), using just $L^2\times L^\infty$ estimates, shows that $\|A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}\|_{L^2}\lesssim 2^{-3m/2+4\delta m}$ if either (8.16) or (8.18) do not hold. The bounds (8.20) follow in the same way. The same argument as in the proof of (8.34), together with $L^2\times L^\infty$ estimates using (7.43) and (7.39), gives (8.19). □

The proof of Lemma 8.1 is completed. □

In our second lemma we give a more precise description of the basic functions $A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}(s)$ in case $\min(k,k_1,k_2)\geqslant -6m/N_0'$.

LEMMA 8.2. *Assume that* $[(k_1,j_1),(k_2,j_2)]\in X_{m,k}$ *and* $k,k_1,k_2\in[-6m/N_0', m/N_0']$ (*as in Lemma* 8.1 (ii) (4)), *and recall the functions* $A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}(s)$ *defined in* (8.10).

(i) *We can decompose*

$$A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2} = \sum_{i=1}^{3} A^{a_1,\alpha_1;a_2,\alpha_2;[i]}_{k;k_1,j_1;k_2,j_2} = \sum_{i=1}^{3} G^{[i]}, \tag{8.35}$$

$$\begin{aligned} \mathcal{F}A^{a_1,\alpha_1;a_2,\alpha_2;[i]}_{k;k_1,j_1;k_2,j_2}(\xi,s) := \int_{\mathbb{R}^2} & e^{is\Phi(\xi,\eta)} \mathfrak{m}_{\sigma\mu\nu}(\xi,\eta)\varphi_k(\xi)\chi^{[i]}(\xi,\eta) \\ & \times \widehat{f^{\mu}_{j_1,k_1}}(\xi-\eta,s)\widehat{f^{\nu}_{j_2,k_2}}(\eta,s)\,d\eta, \end{aligned} \tag{8.36}$$

where $\chi^{[i]}$ *are defined as*

$$\begin{aligned} \chi^{[1]}(\xi,\eta) &= \varphi(2^{10\delta m}\Phi(\xi,\eta))\varphi(2^{30\delta m}\nabla_\eta\Phi(\xi,\eta))\mathbf{1}_{[0,5m/6]}(\max(j_1,j_2)), \\ \chi^{[2]}(\xi,\eta) &= \varphi_{\geqslant 1}(2^{10\delta m}\Phi(\xi,\eta))\varphi(2^{20\delta m}\Omega_\eta\Phi(\xi,\eta)), \\ \chi^{[3]} &= 1-\chi^{[1]}-\chi^{[2]}. \end{aligned}$$

The functions $A^{a_1,\alpha_1;a_2,\alpha_2;[1]}_{k;k_1,j_1;k_2,j_2}(s)$ *are non-trivial only when* $\max(|k|,|k_1|,|k_2|)\leqslant 10$. *Moreover*

$$\|G^{[1]}(s)\|_{L^2}\lesssim 2^{-m+4\delta m}2^{-(1-50\delta)\max(j_1,j_2)}, \tag{8.37}$$

$$\|G^{[2]}(s)\|_{L^2}\lesssim 2^k 2^{-m+4\delta m}, \tag{8.38}$$

$$\|G^{[3]}(s)\|_{L^2}\lesssim 2^{-3m/2+4\delta m}. \tag{8.39}$$

(ii) *We have*

$$\|\mathcal{F}\{A_{\leqslant\mathcal{D},2\gamma_0}A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}\}(s)\|_{L^\infty}\lesssim(2^{-k}+2^{3k})2^{-m+14\delta m}. \tag{8.40}$$

As a consequence, if $k\geqslant -6m/N_0'+\mathcal{D}$, *then we can decompose*

$$A_{\leqslant\mathcal{D}-10,2\gamma_0}\partial_t f^{\sigma}_{j,k}=h_2+h_\infty, \tag{8.41}$$

with

$$\|h_2(s)\|_{L^2}\lesssim 2^{-3m/2+5\delta m}\quad \textit{and}\quad \|\hat{h}_\infty(s)\|_{L^\infty}\lesssim(2^{-k}+2^{3k})2^{-m+15\delta m}.$$

(iii) *If* $j_1,j_2\leqslant\frac12 m+\delta m$, *then we can write*

$$\widehat{G^{[1]}}(\xi,s)=e^{is(\Lambda_\sigma(\xi)-2\Lambda_\sigma(\xi/2))}g^{[1]}(\xi,s)\varphi(2^{3\delta m}(|\xi|-\gamma_1))+h^{[1]}(\xi,s), \tag{8.42}$$

with

$$\begin{aligned}\|D^\alpha_\xi g^{[1]}(s)\|_{L^\infty}&\lesssim_\alpha 2^{-m+4\delta m}2^{|\alpha|(m/2+4\delta m)},\\ \|\partial_s g^{[1]}(s)\|_{L^\infty}&\lesssim 2^{-2m+18\delta m},\\ \|h^{[1]}(s)\|_{L^\infty}&\lesssim 2^{-4m}.\end{aligned} \tag{8.43}$$

Proof. (i) To prove the bounds (8.37)–(8.39), we decompose

$$A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}=\sum_{i=1}^5 A_i,\quad A_i:=P_kI_i[f^\mu_{j_1,k_1},f^\nu_{j_2,k_2}], \tag{8.44}$$

with

$$\mathcal{F}\{I_i[f,g]\}(\xi):=\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}m(\xi,\eta)\chi_i(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta, \tag{8.45}$$

where $m=\mathfrak{m}^{a_1}_{\sigma\mu\nu}$ and χ_i are defined as

$$\begin{aligned}
&\chi_1(\xi,\eta):=\varphi_{\geqslant 1}(2^{20\delta m}\Theta(\xi,\eta)),\\
&\chi_2(\xi,\eta):=\varphi_{\geqslant 1}(2^{10\delta m}\Phi(\xi,\eta))\varphi(2^{20\delta m}\Theta(\xi,\eta)),\\
&\chi_3(\xi,\eta):=\varphi(2^{10\delta m}\Phi(\xi,\eta))\varphi(2^{20\delta m}\Theta(\xi,\eta))\mathbf{1}_{(5m/6,\infty)}(\max(j_1,j_2)),\\
&\chi_4(\xi,\eta):=\varphi(2^{10\delta m}\Phi(\xi,\eta))\varphi(2^{20\delta m}\Theta(\xi,\eta))\varphi_{\geqslant 1}(2^{30\delta m}\Xi(\xi,\eta))\mathbf{1}_{[0,5m/6]}(\max(j_1,j_2)),\\
&\chi_5(\xi,\eta):=\varphi(2^{10\delta m}\Phi(\xi,\eta))\varphi(2^{20\delta m}\Theta(\xi,\eta))\varphi(2^{30\delta m}\Xi(\xi,\eta))\mathbf{1}_{[0,5m/6]}(\max(j_1,j_2)).
\end{aligned} \tag{8.46}$$

Notice that $A_2=G^{[2]}$, $A_5=G^{[1]}$, and $A_1+A_3+A_4=G^{[3]}$. We will show first that

$$\|A_1\|_{L^2}+\|A_3\|_{L^2}+\|A_4\|_{L^2}\lesssim 2^{-3m/2+4\delta m}. \tag{8.47}$$

It follows from Lemma 7.3 and (8.16)–(8.18) that $\|A_1\|_{L^2}\lesssim 2^{-2m}$, as desired. Also, $\|A_4\|_{L^2}\lesssim 2^{-4m}$, as a consequence of Lemma 7.2 (i). It remains to prove that

$$\|A_3\|_{L^2}\lesssim 2^{-3m/2+4\delta m}. \tag{8.48}$$

Assume that $j_2>\frac{5}{6}m$ (the proof of (8.48) when $j_1>\frac{5}{6}m$ is similar). We may assume that $|k_2|\leqslant 10$ (see (8.17)), and then $|k|,|k_1|\in[0,100]$ (due to the restrictions $|\Phi(\xi,\eta)|\lesssim 2^{-10\delta m}$ and $|\Theta(\xi,\eta)|\lesssim 2^{-20\delta m}$; see also (10.6)). We first show that

$$\|P_k I_3[f^{\mu}_{j_1,k_1}, A_{\leqslant 0,\gamma_1} f^{\nu}_{j_2,k_2}]\|_{L^2}\lesssim 2^{-3m/2+4\delta m}. \tag{8.49}$$

Indeed, we notice that, as a consequence of the $L^2\times L^\infty$ argument,

$$\|P_k I^{\sigma\mu\nu}[f^{\mu}_{j_1,k_1}, A_{\leqslant 0,\gamma_1} f^{\nu}_{j_2,k_2}]\|_{L^2}\lesssim 2^{-3m/2},$$

where $I^{\sigma\mu\nu}$ is defined as in (8.22). Let $I^{||}$ be defined by

$$\mathcal{F}\{I^{||}[f,g]\}(\xi):=\int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)}m(\xi,\eta)\varphi(2^{20\delta m}\Theta(\xi,\eta))\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta. \tag{8.50}$$

Using Lemma 7.3 and (8.18), it follows that

$$\|P_k I^{||}[f^{\mu}_{j_1,k_1}, A_{\leqslant 0,\gamma_1} f^{\nu}_{j_2,k_2}]\|_{L^2}\lesssim 2^{-3m/2}.$$

The same averaging argument as in the proof of Lemma 7.4 gives (8.49).

We show now that

$$\|P_k I_3[f^{\mu}_{j_1,k_1}, A_{\geqslant 1,\gamma_1} f^{\nu}_{j_2,k_2}]\|_{L^2}\lesssim 2^{-3m/2+4\delta m}. \tag{8.51}$$

Recall that $|k_2|\leqslant 10$ and $|k|,|k_1|\in[0,100]$. It follows that $|\nabla_\eta\Phi(\xi,\eta)|\geqslant 2^{-\mathcal{D}}$ in the support of the integral (otherwise $|\eta|$ would be close to $\frac{1}{2}\gamma_1$, as a consequence of Proposition 10.2 (iii), which is not the case). The bound (8.51) (in fact, rapid decay) follows using Lemma 7.2 (i), unless

$$j_2\geqslant(1-\delta^2)m. \tag{8.52}$$

Finally, assume that (8.52) holds. Notice that $P_k I_3[A_{\geqslant 1,\gamma_0} f^{\mu}_{j_1,k_1}, A_{\geqslant 1,\gamma_1} f^{\nu}_{j_2,k_2}]\equiv 0$. This is due to the fact that $|\lambda(\gamma_1)\pm\lambda(\gamma_0)\pm\lambda(\gamma_1\pm\gamma_0)|\gtrsim 1$; see Lemma 10.1 (iv). Moreover,

$$\|P_k I^{\sigma\mu\nu}[A_{\leqslant 0,\gamma_0} f^{\mu}_{j_1,k_1}, A_{\geqslant 1,\gamma_1} f^{\nu}_{j_2,k_2}]\|_{L^2}\lesssim 2^{-3m/2+3\delta m+6\delta^2 m},$$

as a consequence of the $L^2\times L^\infty$ argument and the bound (7.43). Therefore, using Lemma 7.3,

$$\|P_kI^{||}[A_{\leqslant 0,\gamma_0}f^{\mu}_{j_1,k_1},A_{\geqslant 1,\gamma_1}f^{\nu}_{j_2,k_2}]\|_{L^2}\lesssim 2^{-3m/2+3\delta m+6\delta^2 m}.$$

The same averaging argument as in the proof of Lemma 7.4 shows that

$$\|P_kI_3[A_{\leqslant 0,\gamma_0}f^{\mu}_{j_1,k_1},A_{\geqslant 1,\gamma_1}f^{\nu}_{j_2,k_2}]\|_{L^2}\lesssim 2^{-3m/2+3\delta m+6\delta^2 m},$$

and the desired bound (8.51) follows in this case as well. This completes the proof of (8.48).

We now prove the bounds (8.37). We notice that $|\eta|$ and $|\xi-\eta|$ are close to $\frac{1}{2}\gamma_1$ in the support of the integral, due to Proposition 10.2 (iii), so

$$\begin{aligned}\widehat{G^{[1]}}(\xi)=\int_{\mathbb{R}^2}& e^{is\Phi(\xi,\eta)}m(\xi,\eta)\varphi_k(\xi)\chi^{[1]}(\xi,\eta)\\ &\times\mathcal{F}\{A_{\geqslant 1,\gamma_1/2}f^{\mu}_{j_1,k_1}\}(\xi-\eta)\mathcal{F}\{A_{\geqslant 1,\gamma_1/2}f^{\nu}_{j_2,k_2}\}(\eta)\,d\eta.\end{aligned}$$

Then, we notice that the factor $\varphi(2^{30\delta m}\nabla_\eta\Phi(\xi,\eta))$ can be removed at the expense of negligible errors (due to Lemma 7.2 (i)). The bound follows using the $L^2\times L^\infty$ argument and Lemma 7.4.

The bound (8.38) follows using (8.19), (8.37), and (8.47).

(ii) The plan is to localize suitably in the Fourier space, both in the radial and the angular directions, and use (7.36) or (7.37). More precisely, let

$$\begin{aligned}B_{\varkappa_\theta,\varkappa_r}(\xi):=\int_{\mathbb{R}^2}& e^{is\Phi(\xi,\eta)}m(\xi,\eta)\varphi_k(\xi)\varphi(\varkappa_r^{-1}\Xi(\xi,\eta))\varphi(\varkappa_\theta^{-1}\Theta(\xi,\eta))\\ &\times\widehat{f^{\mu}_{j_1,k_1}}(\xi-\eta)\widehat{f^{\nu}_{j_2,k_2}}(\eta)\,d\eta,\end{aligned}\tag{8.53}$$

where $\varkappa_\theta$ and $\varkappa_r$ are to be fixed.

Let $\bar{j}:=\max(j_1,j_2)$. If

$$\min(k_1,k_2)\geqslant -\frac{2m}{N_0'}\quad\text{and}\quad \bar{j}\leqslant\frac{m}{2},$$

then we set $\varkappa_r=2^{2\delta m-m/2}$ (we do not localize in the angular variable in this case). Notice that

$$|\mathcal{F}\{A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}\}(\xi)-B_{\varkappa_\theta,\varkappa_r}(\xi)|\lesssim 2^{-4m},$$

in view of Lemma 7.2 (i). If $\big||\xi|-2\gamma_0\big|\geqslant 2^{-2\mathcal{D}}$, then we use Proposition 10.2 (ii) and conclude that the integration in η is over a ball of radius $\lesssim 2^{|k|}\varkappa_r$. Therefore,

$$\begin{aligned}|B_{\varkappa_\theta,\varkappa_r}(\xi)|&\lesssim 2^{k+\min(k_1,k_2)/2}(2^{|k|}\varkappa_r)^2\|\widehat{f^{\mu}_{j_1,k_1}}\|_{L^\infty}\|\widehat{f^{\nu}_{j_2,k_2}}\|_{L^\infty}\\ &\lesssim(2^{-k}+2^{3k})2^{-m+10\delta m}.\end{aligned}\tag{8.54}$$

If

$$\min(k_1,k_2)\geqslant -\frac{2m}{N_0'} \quad \text{and} \quad \bar{j}\in\left[\frac{m}{2}, m-10\delta m\right],$$

then we set $\varkappa_r=2^{2\delta m+\bar{j}-m}$ and $\varkappa_\theta=2^{3\delta m-m/2}$. Notice that

$$|\mathcal{F}\{A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}\}(\xi)-B_{\varkappa_\theta,\varkappa_r}(\xi)|\lesssim 2^{-2m}$$

in view of Lemma 7.2 (i) and Lemma 7.3. If $\big||\xi|-2\gamma_0\big|\geqslant 2^{-2\mathcal{D}}$, then we use Proposition 10.2 (ii) (notice that the hypothesis (10.16) holds in our case) to conclude that the integration in η in the integral defining $B_{\varkappa_\theta,\varkappa_r}(\xi)$ is over an $O(\varkappa\times\varrho)$ rectangle in the direction of the vector ξ, where $\varkappa:=2^{|k|}2^{\delta m}\varkappa_\theta$ and $\varrho:=2^{|k|}\varkappa_r$. Then, we use (7.36) for the function corresponding to the larger j and (7.37) to the other function to estimate

$$|B_{\varkappa_\theta,\varkappa_r}(\xi)|\lesssim 2^k\varkappa 2^{-\bar{j}+51\delta\bar{j}}\varrho^{49\delta}2^{2\delta\bar{j}}2^{2\delta m}\lesssim(2^{-k}+2^{3k})2^{-m+10\delta m}. \tag{8.55}$$

If

$$\min(k_1,k_2)\geqslant -\frac{2m}{N_0'} \quad \text{and} \quad \bar{j}\geqslant m-10\delta m,$$

then we have two subcases: if $\min(j_1,j_2)\leqslant m-10\delta m$, then we still localize in the angular direction (with $\varkappa_\theta=2^{3\delta m-m/2}$ as before) and do not localize in the radial direction. The same argument as above, with $\varrho\lesssim 2^{2\delta m}$, gives the same pointwise bound (8.55). On the other hand, if $\min(j_1,j_2)\geqslant m-10\delta m$, then the desired conclusion follows by Hölder's inequality. The bound (8.40) follows if $\min(k_1,k_2)\geqslant -2m/N_0'$.

On the other hand, if $\min(k_1,k_2)\leqslant -2m/N_0'$, then $2^k\approx 2^{k_1}\approx 2^{k_2}$ (due to (8.12)) and the bound (8.40) can be proved in a similar way. The decomposition (8.41) is a consequence of (8.40) and the L^2 bounds (8.9).

(iii) We now prove the decomposition (8.42). With $\varkappa:=2^{-m/2+\delta m+\delta^2 m}$, we define

$$\begin{aligned} g^{[1]}(\xi,s) &:= \int_{\mathbb{R}^2} e^{is\Phi'(\xi,\eta)}m(\xi,\eta)\varphi_k(\xi)\chi^{[1]}(\xi,\eta)\widehat{f^{\mu}_{j_1,k_1}}(\xi-\eta,s)\\ &\qquad\qquad\times\widehat{f^{\nu}_{j_2,k_2}}(\eta,s)\varphi(\varkappa^{-1}\Xi(\xi,\eta))\,d\eta,\\ h^{[1]}(\xi,s) &:= \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)}m(\xi,\eta)\varphi_k(\xi)\chi^{[1]}(\xi,\eta)\widehat{f^{\mu}_{j_1,k_1}}(\xi-\eta,s)\\ &\qquad\qquad\times\widehat{f^{\nu}_{j_2,k_2}}(\eta,s)\varphi_{\geqslant 1}(\varkappa^{-1}\Xi(\xi,\eta))\,d\eta, \end{aligned} \tag{8.56}$$

where $\Phi'(\xi,\eta)=\Phi_{\sigma\mu\nu}(\xi,\eta)-\Lambda_\sigma(\xi)+2\Lambda_\sigma\big(\tfrac12\xi\big)$. In view of Proposition 10.2 (iii) and the definition of $\chi^{[1]}$, the function $G^{[1]}$ is non-trivial only when $\mu=\nu=\sigma$, and it is supported in the set $\big\{\xi:\big||\xi|-\gamma_1\big|\lesssim 2^{-10\delta m}\big\}$. The conclusion $\|h^{[1]}(s)\|_{L^\infty}\lesssim 2^{-4m}$ in (8.43) follows from Lemma 7.2 (i) and the assumption $j_1,j_2\leqslant\frac12 m+\delta m$.

To prove the bounds on $g^{[1]}$, we notice that $\Phi'(\xi,\eta)=2\Lambda_\sigma\big(\frac{1}{2}\xi\big)-\Lambda_\sigma(\xi-\eta)-\Lambda_\sigma(\eta)$ and $\big|\eta-\frac{1}{2}\xi\big|\lesssim\varkappa$ (due to (10.21)). Therefore,

$$|\Phi'(\xi,\eta)|\lesssim\varkappa^2,\quad |(\nabla_\xi\Phi')(\xi,\eta)|\lesssim\varkappa,\quad\text{and}\quad |(D^\alpha_\xi\Phi')(\xi,\eta)|\lesssim_{|\alpha|}1$$

in the support of the integral. The bounds on $\|\mathcal{D}^\alpha_\xi g^{[1]}(s)\|_{L^\infty}$ in (8.43) follow using L^∞ bounds on $\widehat{f^\mu_{j_1,k_1}}(s)$ and $\widehat{f^\nu_{j_2,k_2}}(s)$. The bounds on $\|\partial_s g^{[1]}(s)\|_{L^\infty}$ follow in the same way, using also the decomposition (8.41) when the s-derivative hits either $\widehat{f^\mu_{j_1,k_1}}(s)$ or $\widehat{f^\nu_{j_2,k_2}}(s)$ (the contribution of the L^2 component is estimated using Hölder's inequality). This completes the proof. □

Our last lemma concerning $\partial_t\mathcal{V}$ is a refinement of Lemma 8.2 (ii). It is only used in the proof of the decomposition (5.29)–(5.30) in Lemma 5.4.

LEMMA 8.3. *For $s\in[2^m-1,2^{m+1}]$ and $k\in[-10,10]$ we can decompose*

$$\mathcal{F}\{P_kA_{\leqslant\mathcal{D},2\gamma_0}(D^\alpha\Omega^a\partial_t\mathcal{V}_\sigma)(s)\}(\xi)=g_d(\xi)+g_\infty(\xi)+g_2(\xi),\tag{8.57}$$

provided that $a\leqslant\frac{1}{2}N_1+20$ and $2a+|\alpha|\leqslant N_1+N_4$, where

$$\begin{aligned}\|g_2\|_{L^2}&\lesssim\varepsilon_1^2 2^{-3m/2+20\delta m},\\ \|g_\infty\|_{L^\infty}&\lesssim\varepsilon_1^2 2^{-m-4\delta m},\\ \sup_{|\varrho|\leqslant 2^{7m/9+4\delta m}}\|\mathcal{F}^{-1}\{e^{-i(s+\varrho)\Lambda_\sigma}g_d\}\|_{L^\infty}&\lesssim\varepsilon_1^2 2^{-16m/9-4\delta m}.\end{aligned}\tag{8.58}$$

Proof. Starting from Lemma 8.1 (ii), we notice that the error term $E^{a,\alpha}_\sigma$ can be placed in the L^2 component g_2 (due to (8.9)). It remains to decompose the functions $A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}$. We may assume that we are in case (4), $k_1,k_2\in[-2m/N_0',m/N_0']$. We define the functions $B_{\varkappa_\theta,\varkappa_r}$ as in (8.53). We notice that the argument in Lemma 8.2 (ii) already gives the desired conclusion if $\bar{j}=\max(j_1,j_2)\geqslant\frac{1}{2}m+20\delta m$ (without having to use the function g_d).

It remains to decompose the functions $A_{\leqslant\mathcal{D},2\gamma_0}A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}(s)$ when

$$\bar{j}=\max(j_1,j_2)\leqslant\tfrac{1}{2}m+20\delta m.\tag{8.59}$$

As in (8.53), let

$$B_{\varkappa_r}(\xi):=\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}m(\xi,\eta)\varphi_k(\xi)\varphi(\varkappa_r^{-1}\Xi(\xi,\eta))\widehat{f^\mu_{j_1,k_1}}(\xi-\eta)\widehat{f^\nu_{j_2,k_2}}(\eta)\,d\eta,\tag{8.60}$$

where $\varkappa_r:=2^{30\delta m-m/2}$ (we do not need angular localization here). In view of Lemma 7.2 (i), $|\mathcal{F}A^{a_1,\alpha_1;a_2,\alpha_2}_{k;k_1,j_1;k_2,j_2}(\xi)-B_{\varkappa_r}(\xi)|\lesssim 2^{-4m}$. It remains to prove that

$$\big\|\mathcal{F}^{-1}\big\{e^{-i(s+\varrho)\Lambda_\sigma(\xi)}\varphi_{\geqslant-\mathcal{D}}\big(2^{100}\big||\xi|-2\gamma_0\big|\big)B_{\varkappa_r}(\xi)\big\}\big\|_{L^\infty}\lesssim 2^{-16m/9-5\delta m}\tag{8.61}$$

for any k, j_1, k_1, j_2, k_2, and ϱ fixed, $|\varrho|\leqslant 2^{7m/9+4\delta m}$.

In proving (8.61), we may assume that $m\geqslant\mathcal{D}^2$. The condition $|\Xi(\xi,\eta)|\leqslant 2\varkappa_r$ shows that the variable η is localized to a small ball. More precisely, using Lemma 10.2, we have

$$|\eta-p(\xi)|\lesssim\varkappa_r \quad \text{for some } p(\xi)\in P_{\mu\nu}(\xi), \tag{8.62}$$

provided that $\big||\xi|-2\gamma_0\big|\gtrsim 1$. The sets $P_{\mu\nu}(\xi)$ are defined in (10.15) and contain two or three points. We parameterize these points by

$$p_\ell(\xi)=q_\ell(|\xi|)\frac{\xi}{|\xi|},$$

where

$$q_1(r)=\tfrac{1}{2}r, \quad q_2(r)=p_{++2}(r), \quad \text{and} \quad q_3(r)=r-p_{++2}(r),$$

if $\mu=\nu$, and

$$q_1(r)=p_{+-1}(r) \quad \text{and} \quad q_2(r)=r-p_{+-1}(r),$$

if $\mu=-\nu$. Then, we rewrite

$$B_{\varkappa_r}(\xi)=\sum_\ell e^{is\Lambda_\sigma(\xi)}e^{-is(\Lambda_\mu(\xi-p_\ell(\xi))+\Lambda_\nu(p_\ell(\xi)))}H_\ell(\xi), \tag{8.63}$$

where

$$\begin{aligned} H_\ell(\xi):=\int_{\mathbb{R}^2} & e^{is(\Phi(\xi,\eta)-\Phi(\xi,p_\ell(\xi))}m(\xi,\eta)\varphi_k(\xi)\varphi(\varkappa_r^{-1}\Xi(\xi,\eta)) \\ & \times\widehat{f^\mu_{j_1,k_1}}(\xi-\eta)\widehat{f^\nu_{j_2,k_2}}(\eta)\varphi(2^{m/2-31\delta m}(\eta-p_\ell(\xi))\,d\eta. \end{aligned} \tag{8.64}$$

Clearly,

$$|\Phi(\xi,\eta)-\Phi(\xi,p_\ell(\xi)|\lesssim|\eta-p_\ell(\xi)|^2 \quad \text{and} \quad |\nabla_\xi[\Phi(\xi,\eta)-\Phi(\xi,p_\ell(\xi)]|\lesssim|\eta-p_\ell(\xi)|.$$

Therefore,

$$|D^\beta H_\ell(\xi)|\lesssim_\beta 2^{-m+70\delta m}2^{|\beta|(m/2+35\delta m)}, \quad \text{if } \big||\xi|-2\gamma_0\big|\gtrsim 1. \tag{8.65}$$

We can now prove (8.61). Notice that the factor $e^{is\Lambda_\sigma(\xi)}$ simplifies and that the remaining phase $\xi\mapsto\Lambda_\mu(\xi-p_\ell(\xi))+\Lambda_\nu(p_\ell(\xi))$ is radial. Let $\Gamma_l=\Gamma_{l;\mu\nu}$ be defined such that $\Gamma_l(|\xi|)=\Lambda_\mu(\xi-p_\ell(\xi))+\Lambda_\nu(p_\ell(\xi))$. Standard stationary phase estimates, using also (8.65), show that (8.61) holds provided that

$$|\Gamma'_\ell(r)|\approx 1 \text{ and } |\Gamma''_\ell(r)|\approx 1, \quad \text{if } r\in[2^{-20},2^{20}] \text{ and } |r-2\gamma_0|\geqslant 2^{-3\mathcal{D}/2}. \tag{8.66}$$

To prove (8.66), assume first that $\mu=\nu$. If $\ell=1$, then $p_\ell(\xi)=\frac{1}{2}\xi$, and the desired conclusion is clear. If $\ell\in\{2,3\}$, then $\pm\Gamma_\ell(r)=\lambda(r-p_{++2}(r))+\lambda(p_{++2}(r))$. In view of Proposition 10.2 (i), $r-2\gamma_0\geqslant 2^{-2\mathcal{D}}$, $p_{++2}(r)\in(0,\gamma_0-2^{-2\mathcal{D}}]$, and $\lambda'(r-p_{++2}(r))=\lambda'(p_{++2}(r))$. Therefore,

$$|\Gamma_\ell'(r)|=\lambda'(r-p_{++2}(r)) \quad \text{and} \quad |\Gamma_\ell''(r)|=|\lambda''(r-p_{++2}(r))(1-p_{++2}'(r))|.$$

The desired conclusions in (8.66) follow, since $|1-p_{++2}'(r)|\approx 1$ in the domain of r (due to the identity $\lambda''(r-p_{++2}(r))(1-p_{++2}'(r))=\lambda''(p_{++2}(r))p_{++2}'(r)$).

The proof of (8.66) in the case $\mu=-\nu$ is similar. This completes the proof of the lemma. □

9. Dispersive analysis III: Proof of Proposition 7.1

9.1. Quadratic interactions

In this section we prove Proposition 7.1. We start with the quadratic component in the Duhamel formula (7.5) and show how to control its Z norm.

PROPOSITION 9.1. *With the hypothesis in Proposition 7.1, for any $t\in[0,T]$ we have*

$$\sup_{\substack{0\leqslant a\leqslant N_1/2+20\\ 2a+|\alpha|\leqslant N_1+N_4}} \|D^\alpha\Omega^a W_2(t)\|_{Z_1}\lesssim\varepsilon_1^2. \tag{9.1}$$

The rest of this section is concerned with the proof of this proposition. First notice that

$$\begin{aligned}\Omega_\xi^a\widehat{W_2}(\xi,t)=\sum_{\mu,\nu\in\{+,-\}}\sum_{a_1+a_2=a}\int_0^t\int_{\mathbb{R}^2} & e^{is\Phi_{+\mu\nu}(\xi,\eta)}\mathfrak{m}_{\mu\nu}(\xi,\eta)\\ & \times(\Omega^{a_1}\widehat{\mathcal{V}}_\mu)(\xi-\eta,s)(\Omega^{a_2}\widehat{\mathcal{V}}_\nu)(\eta,s)\,d\eta\,ds.\end{aligned} \tag{9.2}$$

Given $t\in[0,T]$, we fix a suitable decomposition of the function $\mathbf{1}_{[0,t]}$, i.e. we fix functions $q_0,...,q_{L+1}:\mathbb{R}\to[0,1]$, $|L-\log_2(2+t)|\leqslant 2$, as in (4.8). For $\mu,\nu\in\{+,-\}$ and $m\in[0,L+1]$ we define the operator $T_m^{\mu\nu}$ by

$$\mathcal{F}\{T_m^{\mu\nu}[f,g]\}(\xi):=\int_{\mathbb{R}}q_m(s)\int_{\mathbb{R}^2}e^{is\Phi_{+\mu\nu}(\xi,\eta)}\mathfrak{m}_{\mu\nu}(\xi,\eta)\hat{f}(\xi-\eta,s)\hat{g}(\eta,s)\,d\eta\,ds. \tag{9.3}$$

In view of Definition 2.5, Proposition 9.1 follows from Proposition 9.2 below.

PROPOSITION 9.2. *Assume that $t\in[0,T]$ is fixed and define the operators $T_m^{\mu\nu}$ as above. If $a_1+a_2=a$, $\alpha_1+\alpha_2=\alpha$, $\mu,\nu\in\{+,-\}$, $m\in[0,L+1]$, and $(k,j)\in\mathcal{J}$, then*

$$\sum_{k_1,k_2\in\mathbb{Z}}\|Q_{jk}T_m^{\mu\nu}[P_{k_1}D^{\alpha_1}\Omega^{a_1}\mathcal{V}_\mu,P_{k_2}D^{\alpha_2}\Omega^{a_2}\mathcal{V}_\nu]\|_{B_j}\lesssim 2^{-\delta^2 m}\varepsilon_1^2. \tag{9.4}$$

Assume that a_1, a_2, b, α_1, α_2, μ, and ν are fixed and let, for simplicity of notation,

$$f^\mu:=\varepsilon_1^{-1}D^{\alpha_1}\Omega^{a_1}\mathcal{V}_\mu,\quad f^\nu:=\varepsilon_1^{-1}D^{\alpha_2}\Omega^{a_2}\mathcal{V}_\nu,\quad \Phi:=\Phi_{+\mu\nu},\quad m_0:=\mathfrak{m}_{\mu\nu},\quad T_m:=T_m^{\mu\nu}. \tag{9.5}$$

The bootstrap assumption (7.15) gives, for any $s\in[0,t]$,

$$\|f^\mu(s)\|_{H^{N_0'}\cap Z_1\cap H_\Omega^{N_1'}}+\|f^\nu(s)\|_{H^{N_0'}\cap Z_1\cap H_\Omega^{N_1'}}\lesssim(1+s)^{\delta^2}. \tag{9.6}$$

We recall also the symbol-type bounds, which hold for any $k,k_1,k_2\in\mathbb{Z}$, $|\alpha|\geqslant 0$,

$$\begin{aligned}\|m_0^{k,k_1,k_2}\|_{S^\infty}&\lesssim 2^k2^{\min(k_1,k_2)/2},\\ \|D_\eta^\alpha m_0^{k,k_1,k_2}\|_{L^\infty}&\lesssim_{|\alpha|}2^{(|\alpha|+3/2)\max(|k_1|,|k_2|)},\\ \|D_\xi^\alpha m_0^{k,k_1,k_2}\|_{L^\infty}&\lesssim_{|\alpha|}2^{(|\alpha|+3/2)\max(|k_1|,|k_2|,|k|)},\end{aligned} \tag{9.7}$$

where $m_0^{k,k_1,k_2}(\xi,\eta)=m_0(\xi,\eta)\varphi_k(\xi)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta)$.

We first consider a few simple cases before moving to the main analysis in the next subsections. Recall (see (7.44)) that, for any $k\in\mathbb{Z}$, $m\in\{0,...,L+1\}$, and $s\in I_m:=\operatorname{supp} q_m$,

$$\begin{aligned}\|P_kf^\mu(s)\|_{L^2}+\|P_kf^\nu(s)\|_{L^2}&\lesssim 2^{\delta^2m}\min(2^{(1-50\delta)k},2^{-N_0'k}),\\ \|P_ke^{-is\Lambda_\mu}f^\mu(s)\|_{L^\infty}+\|P_ke^{-is\Lambda_\nu}f^\nu(s)\|_{L^\infty}&\lesssim 2^{3\delta^2m}\min(2^{(2-50\delta)k},2^{-5m/6}).\end{aligned} \tag{9.8}$$

LEMMA 9.3. *Assume that f^μ and f^ν are as in (9.5) and let $(k,j)\in\mathcal{J}$. Then,*

$$\sum_{\max(k_1,k_2)\geqslant 1.01(j+m)/N_0'-\mathcal{D}^2}\|Q_{jk}T_m[P_{k_1}f^\mu,P_{k_2}f^\nu]\|_{B_j}\lesssim 2^{-\delta^2m}, \tag{9.9}$$

$$\sum_{\min(k_1,k_2)\leqslant -(j+m)/2+\mathcal{D}^2}\|Q_{jk}T_m[P_{k_1}f^\mu,P_{k_2}f^\nu]\|_{B_j}\lesssim 2^{-\delta^2m}, \tag{9.10}$$

$$\sum_{k_1,k_2\in\mathbb{Z}}\|Q_{jk}T_m[P_{k_1}f^\mu,P_{k_2}f^\nu]\|_{B_j}\lesssim 2^{-\delta^2m},\quad \textit{if } 2k\leqslant -j-m+49\delta j-\delta m, \tag{9.11}$$

$$\sum_{-j\leqslant k_1,k_2\leqslant 2j/N_0'}\|Q_{jk}T_m[P_{k_1}f^\mu,P_{k_2}f^\nu]\|_{B_j}\lesssim 2^{-\delta^2m},\quad \textit{if } j\geqslant 2.1m. \tag{9.12}$$

Proof. Using (9.8), the left-hand side of (9.9) is dominated by

$$C\sum_{\max(k_1,k_2)\geqslant 1.01(m+j)/N_0'-\mathcal{D}^2}2^{j+m}2^{2k^+}2^{\min(k_1,k_2)/2}\sup_{s\in I_m}\|P_{k_1}f^\mu(s)\|_{L^2}\,\|P_{k_2}f^\nu(s)\|_{L^2}\lesssim 2^{-\delta m},$$

which is acceptable. Similarly, if $k_1\leqslant k_2$ and $k_1\leqslant\mathcal{D}^2$, then

$$\begin{aligned}2^j\|P_kT_m[P_{k_1}f^\mu,P_{k_2}f^\nu]\|_{L^2}&\lesssim 2^{j+m}2^{k+k_1/2}\sup_{s\in I_m}\|\widehat{P_{k_1}f^\mu}(s)\|_{L^1}\|P_{k_2}f^\nu(s)\|_{L^2}\\ &\lesssim 2^{j+m}2^{(5/2-50\delta)k_1}2^{-(N_0'-1)\max(k_2,0)},\end{aligned}$$

and the bound (9.10) follows by summation over $\min(k_1,k_2)\leqslant -\frac{1}{2}(j+m)+2\mathcal{D}^2$.

To prove (9.11), we may assume that

$$2k\leqslant -j-m+49\delta j-\delta m \quad \text{and} \quad -\frac{j+m}{2}\leqslant k_1,k_2\leqslant \frac{1.01(j+m)}{N_0'}. \tag{9.13}$$

Then

$$\begin{aligned}\|Q_{jk}T_m[P_{k_1}f^\mu,P_{k_2}f^\nu]\|_{B_j}&\\ &\lesssim 2^{j(1-50\delta)}\|P_kT_m[P_{k_1}f^\mu,P_{k_2}f^\nu]\|_{L^2}\\ &\lesssim 2^{j(1-50\delta)}2^m2^{k+\min(k_1,k_2)/2}2^k\sup_{s\in I_m}\|P_{k_1}f^\mu(s)\|_{L^2}\,\|P_{k_2}f^\nu(s)\|_{L^2}\\ &\lesssim 2^{-\delta(j+m)/3}.\end{aligned}$$

Summing in k_1 and k_2 as in (9.13), we obtain an acceptable contribution.

Finally, to prove (9.12), we may assume that

$$j\geqslant 2.1m,\quad j+k\geqslant \frac{j}{10}+\mathcal{D},\quad \text{and}\quad -j\leqslant k_1,k_2\leqslant \frac{2j}{N_0'},$$

and define

$$f^\mu_{j_1,k_1}:=P_{[k_1-2,k_1+2]}Q_{j_1k_1}f^\mu \quad \text{and} \quad f^\nu_{j_2,k_2}:=P_{[k_2-2,k_2+2]}Q_{j_2k_2}f^\nu. \tag{9.14}$$

If $\min\{j_1,j_2\}\geqslant \frac{99}{100}j-\mathcal{D}$ then, using also (7.36),

$$\begin{aligned}\|P_kT_m[f^\mu_{j_1,k_1},f^\nu_{j_2,k_2}]\|_{L^2}&\lesssim 2^m2^{k+\min(k_1,k_2)/2}\sup_{s\in I_m}\|\widehat{f^\mu_{j_1,k_1}}(s)\|_{L^1}\,\|f^\nu_{j_2,k_2}(s)\|_{L^2}\\ &\lesssim 2^m2^{k+3k_1/2}2^{-(1-\delta')j_1-(1/2-\delta)j_2}2^{4\delta^2m},\end{aligned}$$

and therefore

$$\sum_{-j\leqslant k_1,k_2\leqslant 2j/N_0'}\;\sum_{\min\{j_1,j_2\}\geqslant 99j/100-\mathcal{D}}\|Q_{jk}T_m[f^\mu_{j_1,k_1},f^\nu_{j_2,k_2}]\|_{B_j}\lesssim 2^{-\delta m}.$$

On the other hand, if $j_1\leqslant\frac{99}{100}j-\mathcal{D}$, then we rewrite

$$\begin{aligned}&Q_{jk}T_m[f^\mu_{j_1,k_1},f^\nu_{j_2,k_2}](x)\\ &\quad=C\widetilde{\varphi}^{(k)}_j(x)\int_{\mathbb{R}}q_m(s)\int_{\mathbb{R}^2}\bigg(\int_{\mathbb{R}^2}e^{i(s\Phi(\xi,\eta)+x\cdot\xi)}\varphi_k(\xi)m_0(\xi,\eta)\widehat{f^\mu_{j_1,k_1}}(\xi-\eta,s)\,d\xi\bigg)\\ &\qquad\times\widehat{f^\nu_{j_2,k_2}}(\eta,s)\,d\eta\,ds.\end{aligned} \tag{9.15}$$

In the support of integration, we have the lower bound $|\nabla_\xi[s\Phi(\xi,\eta)+x\cdot\xi]|\approx|x|\approx 2^j$. Integration by parts in ξ using Lemma 7.2 yields

$$|Q_{jk}T_m[f^\mu_{j_1,k_1},f^\nu_{j_2,k_2}](x)|\lesssim 2^{-10j}, \tag{9.16}$$

which gives an acceptable contribution. This finishes the proof. □

9.2. The main decomposition

We may assume that

$$k_1, k_2 \in \left[-\frac{j+m}{2}, \frac{1.01(j+m)}{N_0'}\right], \quad k \geqslant \frac{-j-m+49\delta j-\delta m}{2}, \quad j \leqslant 2.1m, \quad m \geqslant \frac{\mathcal{D}^2}{8}. \tag{9.17}$$

Recall the definition (2.9). We fix $l_- := \lfloor -(1-\frac{1}{2}\delta)m \rfloor$, and decompose

$$\begin{aligned} T_m[f,g] &= \sum_{l_- \leqslant l} T_{m,l}[f,g], \\ \widehat{T_{m,l}[f,g]}(\xi) &:= \int_{\mathbb{R}} q_m(s) \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} \varphi_l^{[l_-,m]}(\Phi(\xi,\eta)) m_0(\xi,\eta) \\ &\qquad\qquad \times \hat{f}(\xi-\eta,s)\hat{g}(\eta,s)\,d\eta\,ds. \end{aligned} \tag{9.18}$$

Assuming (9.17), we notice that $T_{m,l}[P_{k_1}f^\mu, P_{k_2}f^\nu] \equiv 0$ if $l \geqslant 10m/N_0'$. When $l > l_-$, we may integrate by parts in time to rewrite $T_{m,l}[P_{k_1}f^\mu, P_{k_2}f^\nu]$:

$$\begin{aligned} T_{m,l}[P_{k_1}f^\mu, P_{k_2}f^\nu] &= i\mathcal{A}_{m,l}[P_{k_1}f^\mu, P_{k_2}f^\nu] + i\mathcal{B}_{m,l}[P_{k_1}\partial_s f^\mu, P_{k_2}f^\nu] \\ &\qquad + i\mathcal{B}_{m,l}[P_{k_1}f^\mu, P_{k_2}\partial_s f^\nu], \\ \widehat{\mathcal{A}_{m,l}[f,g]}(\xi) &:= \int_{\mathbb{R}} q_m'(s) \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} 2^{-l}\widetilde{\varphi}_l(\Phi(\xi,\eta)) m_0(\xi,\eta) \\ &\qquad \times \hat{f}(\xi-\eta,s)\hat{g}(\eta,s)\,d\eta\,ds, \\ \widehat{\mathcal{B}_{m,l}[f,g]}(\xi) &:= \int_{\mathbb{R}} q_m(s) \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} 2^{-l}\widetilde{\varphi}_l(\Phi(\xi,\eta)) m_0(\xi,\eta) \\ &\qquad \times \hat{f}(\xi-\eta,s)\hat{g}(\eta,s)\,d\eta\,ds, \end{aligned} \tag{9.19}$$

where $\widetilde{\varphi}_l(x) := 2^l x^{-1}\varphi_l(x)$. For s fixed, let $\mathcal{I}_l$ denote the bilinear operator defined by

$$\widehat{\mathcal{I}_l[f,g]}(\xi) := \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} 2^{-l}\widetilde{\varphi}_l(\Phi(\xi,\eta)) m_0(\xi,\eta) \hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta. \tag{9.20}$$

It is easy to see that Proposition 9.2 follows from Lemma 9.3 and Lemmas 9.4–9.8 below.

Lemma 9.4. *Assume that* (9.17) *holds and, in addition,*

$$j \geqslant m + 2\mathcal{D} + \tfrac{1}{2}\max(|k|,|k_1|,|k_2|). \tag{9.21}$$

Then, for $l_- \leqslant l \leqslant 10m/N_0'$,

$$2^{(1-50\delta)j} \|Q_{jk} T_{m,l}[P_{k_1}f^\mu, P_{k_2}f^\nu]\|_{L^2} \lesssim 2^{-2\delta^2 m}.$$

Notice that the assumptions (9.17) and $j\leqslant m+2\mathcal{D}+\frac{1}{2}\max(|k|,|k_1|,|k_2|)$ show that

$$k,k_1,k_2\in\Big[-\frac{4m}{3}-2\mathcal{D},\frac{3.2m}{N_0'}\Big]\quad\text{and}\quad m\geqslant\frac{\mathcal{D}^2}{8}. \tag{9.22}$$

LEMMA 9.5. *Assume that* (9.22) *holds and, in addition,*

$$j\leqslant m+2\mathcal{D}+\frac{\max(|k|,|k_1|,|k_2|)}{2}\quad\textit{and}\quad\min(k,k_1,k_2)\leqslant-\frac{3.5m}{N_0'}. \tag{9.23}$$

Then, for $l_-\leqslant l\leqslant 10m/N_0'$,

$$2^{(1-50\delta)j}\|Q_{jk}T_{m,l}[P_{k_1}f^\mu,P_{k_2}f^\nu]\|_{L^2}\lesssim 2^{-2\delta^2m}.$$

LEMMA 9.6. *Assume that* (9.22) *holds and, in addition,*

$$j\leqslant m+2\mathcal{D}+\frac{\max(|k|,|k_1|,|k_2|)}{2}\quad\textit{and}\quad\min(k,k_1,k_2)\geqslant-\frac{3.5m}{N_0'}. \tag{9.24}$$

Then, for $l_-<l\leqslant 10m/N_0'$,

$$\|Q_{jk}T_{m,l_-}[P_{k_1}f^\mu,P_{k_2}f^\nu]\|_{B_j}+\|Q_{jk}\mathcal{A}_{m,l}[P_{k_1}f^\mu,P_{k_2}f^\nu]\|_{B_j}\lesssim 2^{-2\delta^2m}.$$

LEMMA 9.7. *Assume that* (9.22) *holds and, in addition,*

$$j\leqslant m+2\mathcal{D}+\frac{\max(|k|,|k_1|,|k_2|)}{2},\quad\min(k,k_1,k_2)\geqslant-\frac{3.5m}{N_0'},\quad\textit{and}\quad l\geqslant-\frac{m}{14}. \tag{9.25}$$

Then,

$$2^{(1-50\delta)j}\|Q_{jk}\mathcal{B}_{m,l}[P_{k_1}f^\mu,P_{k_2}\partial_sf^\nu]\|_{L^2}\lesssim 2^{-2\delta^2m}.$$

LEMMA 9.8. *Assume that* (9.22) *holds and, in addition,*

$$j\leqslant m+2\mathcal{D}+\frac{\max(|k|,|k_1|,|k_2|)}{2},\quad\min(k,k_1,k_2)\geqslant-\frac{3.5m}{N_0'},\quad\textit{and}\quad l_-<l\leqslant-\frac{m}{14}. \tag{9.26}$$

Then,

$$\|Q_{jk}T_{m,l}[P_{k_1}f^\mu,P_{k_2}f^\nu]\|_{B_j}\lesssim 2^{-2\delta^2m}.$$

We prove these lemmas in the following five subsections. Lemma 9.4 takes advantage of the approximate finite speed of propagation. Lemma 9.5 uses the null structure at low frequencies. Lemma 9.6 controls interactions that lead to the creation of a space-time resonance. Lemmas 9.7 and 9.8 correspond to interactions that are particularly difficult to control in dimension 2, and contain the main novelty of our analysis (see also [30]). They rely on all the estimates in Lemmas 8.1 and 8.2, and on the "slow propagation of iterated resonances" properties in Lemma 10.6.

We will use repeatedly the symbol bounds (9.7) and the main assumption (9.6).

9.3. Approximate finite speed of propagation

In this subsection we prove Lemma 9.4. We define the functions $f^{\mu}_{j_1,k_1}$ and $f^{\nu}_{j_2,k_2}$ as before (see (9.14)), and further decompose

$$f^{\mu}_{j_1,k_1}=\sum_{n_1=0}^{j_1+1} f^{\mu}_{j_1,k_1,n_1} \quad \text{and} \quad f^{\nu}_{j_2,k_2}=\sum_{n_2=0}^{j_2+1} f^{\nu}_{j_2,k_2,n_2}, \tag{9.27}$$

as in (7.33). If $\min\{j_1,j_2\}\leqslant j-\delta m$, then the same argument as in the proof of (9.12) leads to rapid decay, as in (9.16). To bound the sum over $\min\{j_1,j_2\}\geqslant j-\delta m$, we consider several cases.

Case 1. Assume first that

$$\min(k,k_1,k_2)\leqslant -\tfrac{1}{2}m. \tag{9.28}$$

Then we notice that

$$\begin{aligned}\|\mathcal{F}\{P_kT_{m,l}[f^{\mu}_{j_1,k_1},f^{\nu}_{j_2,k_2}]\}\|_{L^\infty} &\lesssim 2^m2^{k+\min(k_1,k_2)/2}\sup_{s\in I_m}(\|\widehat{f^{\mu}_{j_1,k_1}}(s)\|_{L^2}\,\|\widehat{f^{\nu}_{j_2,k_2}}(s)\|_{L^2})\\ &\lesssim 2^m2^{2\delta^2m}2^k2^{-(1/2-\delta)(j_1+j_2)}.\end{aligned}$$

Therefore, the sum over j_1 and j_2, with $\min(j_1,j_2)\geqslant j-\delta m$, is controlled as claimed, provided $k\leqslant-\frac{1}{2}m$. On the other hand, if $k_1=\min(k_1,k_2)\leqslant-\frac{1}{2}m$, then we estimate

$$\begin{aligned}&\|P_kT_{m,l}[f^{\mu}_{j_1,k_1},f^{\nu}_{j_2,k_2}]\|_{L^2}\\ &\quad\lesssim 2^m2^{k+k_1/2}\sup_{s\in I_m}(\|\widehat{f^{\mu}_{j_1,k_1}}(s)\|_{L^1}\|\widehat{f^{\nu}_{j_2,k_2}}(s)\|_{L^2})\\ &\quad\lesssim 2^m2^{2\delta^2m}2^{k+k_1/2}2^{k_1}2^{-(1-50\delta)j_1}2^{-(1/2-\delta)j_2}2^{-4\max(k_2,0)}.\end{aligned} \tag{9.29}$$

The sum over j_1 and j_2, with $\min(j_1,j_2)\geqslant j-\delta m$, is controlled as claimed in this case as well.

Case 2. Assume now that

$$\min(k,k_1,k_2)\geqslant-\tfrac{1}{2}m \quad \text{and} \quad l\leqslant\tfrac{1}{2}\min(k,k_1,k_2,0)-\tfrac{1}{5}m. \tag{9.30}$$

We use Lemma 10.5: we may assume that $\min(k,k_1,k_2)+\max(k,k_1,k_2)\geqslant-100$ and estimate

$$\begin{aligned}&\|P_kT_{m,l}[f^{\mu}_{j_1,k_1,n_1},f^{\nu}_{j_2,k_2,n_2}]\|_{L^2}\\ &\quad\lesssim 2^m2^{k+\min(k_1,k_2)/2}2^{5\max(k_1,k_2,0)}2^{l/2-n_1/2-n_2/2}\\ &\qquad\times\sup_{s\in I_m}\Big(\Big\|\sup_{\theta}|\widehat{f^{\mu}_{j_1,k_1,n_1}}(r\theta,s)|\Big\|_{L^2(r\,dr)}\Big\|\sup_{\theta}|\widehat{f^{\nu}_{j_2,k_2,n_2}}(r\theta,s)|\Big\|_{L^2(r\,dr)}\Big).\end{aligned}$$

Using (7.35) and (9.6), and summing over n_1 and n_2, we have

$$2^{(1-50\delta)j}\|P_kT_{m,l}[f^{\mu}_{j_1,k_1},f^{\nu}_{j_2,k_2}]\|_{L^2}\lesssim 2^{7\max(k_1,k_2,0)}2^m2^{2\delta^2m}2^{(1-50\delta)j}2^{l/2}2^{-(1-\delta')(j_1+j_2)}.$$

The sum over j_1 and j_2, with $\min(j_1,j_2)\geqslant j-\delta m$, is controlled as claimed.

Case 3. Finally, assume that

$$\min(k,k_1,k_2)\geqslant -\tfrac{1}{2}m \quad\text{and}\quad l\geqslant \tfrac{1}{2}\min(k,k_1,k_2,0)-\tfrac{1}{5}m. \tag{9.31}$$

We use formula (9.19). The contribution of $\mathcal{A}_{m,l}$ can be estimated as in (9.29), with 2^m replaced by 2^{-l}, and we focus on the contribution of $\mathcal{B}_{m,l}[P_{k_1}f^{\mu},P_{k_2}\partial_sf^{\nu}]$. We decompose $\partial_sf^{\nu}(s)$, according to (8.8). The contribution of $P_{k_2}E^{a_2,\alpha_2}_{\nu}$ can be estimated easily:

$$\begin{aligned}
&\|P_k\mathcal{B}_{m,l}[f^{\mu}_{j_1,k_1},P_{k_2}E^{a_2,\alpha_2}_{\nu}]\|_{L^2}\\
&\quad\lesssim 2^m2^{-l}2^{k+\min(k_1,k_2)/2}\sup_{s\in I_m}(\|\widehat{f^{\mu}_{j_1,k_1}}(s)\|_{L^1}\,\|P_{k_2}E^{a_2,\alpha_2}_{\nu}(s)\|_{L^2})\\
&\quad\lesssim 2^m2^{2\delta^2m}2^{m/5-\min(k,k_1,k_2,0)/2}2^{k+k_2/2}2^{k_1}2^{-(1-51\delta)j_1}2^{-3m/2+5\delta m}\\
&\quad\lesssim 2^{-(1-51\delta)j_1}2^{-m/4},
\end{aligned}\tag{9.32}$$

and the sum over $j_1\geqslant j-\delta m$ of

$$2^{(1-50\delta)j}\|P_k\mathcal{B}_{m,l}[f^{\mu}_{j_1,k_1},P_{k_2}E^{a_2,\alpha_2}_{\nu}]\|_{L^2}$$

is suitably bounded.

We consider now the terms $A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3,k_4,j_4}(s)$ in (8.8), $[(k_3,j_3),(k_4,j_4)]\in X_{m,k_2}$, with $\alpha_3+\alpha_4=\alpha_2$ and $a_3+a_4\leqslant a_2$. In view of (8.12), (8.14), and (8.20),

$$\|A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3,k_4,j_4}(s)\|_{L^2}\lesssim 2^{-4m/3+4\delta m}$$

if

$$\max(j_3,j_4)\geqslant(1-\delta^2)m-|k_2|\quad\text{or}\quad |k_2|+\tfrac{1}{2}\mathcal{D}\leqslant\min(|k_3|,|k_4|).$$

The contributions of these terms can be estimated as in (9.32). On the other hand, to control the contribution of $Q_{jk}\mathcal{B}_{m,l}[f^{\mu}_{j_1,k_1},A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3,k_4,j_4}]$ when

$$\max(j_3,j_4)\leqslant(1-\delta^2)m-|k_2|\quad\text{and}\quad |k_2|+\tfrac{1}{2}\mathcal{D}\geqslant|k_3|,$$

we simply rewrite this in the form

$$\begin{aligned}
c\widetilde{\varphi}^{(k)}_j(x)\int_{\mathbb{R}}q_m(s)\int_{\mathbb{R}^2}\widehat{f^{\mu}_{j_1,k_1}}(\eta,s)\biggl(\int_{\mathbb{R}^2\times\mathbb{R}^2}&e^{i[x\cdot\xi+s\tilde{\Phi}'(\xi,\eta,\sigma)]}2^{-l}\widetilde{\varphi}_l(\Phi_{\sigma\mu\nu}(\xi,\xi-\eta))\\
&\times\varphi_k(\xi)\varphi_{k_2}(\xi-\eta)\mathfrak{m}_{\mu\nu}(\xi,\xi-\eta)\mathfrak{m}_{\nu\beta\gamma}(\xi-\eta,\sigma)\\
&\times\widehat{f^{\beta}_{j_3,k_3}}(\xi-\eta-\sigma,s)\widehat{f^{\gamma}_{j_4,k_4}}(\sigma,s)\,d\xi\,d\sigma\biggr)\,d\eta\,ds,
\end{aligned}\tag{9.33}$$

where $\tilde{\Phi}'(\xi,\eta,\sigma):=\Lambda(\xi)-\Lambda_\mu(\eta)-\Lambda_\beta(\xi-\eta-\sigma)-\Lambda_\gamma(\sigma)$. Notice that

$$|\nabla_\xi(x\cdot\xi+s\Lambda(\xi)-s\Lambda_\mu(\eta)-s\Lambda_\beta(\xi-\eta-\sigma)-s\Lambda_\gamma(\sigma))|\approx|x|\approx 2^j. \tag{9.34}$$

We can integrate by parts in ξ using Lemma 7.2 (i) to conclude that these are negligible contributions, pointwise bounded by $C2^{-5m}$. This completes the proof of the lemma.

9.4. The case of small frequencies

In this subsection we prove Lemma 9.5. The main point is that, if

$$\underline{k}:=\min(k,k_1,k_2)\leqslant-\frac{3.5m}{N_0'},$$

then $|\Phi(\xi,\eta)|\gtrsim 2^{\underline{k}/2}$ for any $(\xi,\eta)\in\mathcal{D}_{k,k_1,k_2}$, as a consequence of (10.6) and (9.22). Therefore, the operators $T_{m,l}$ are non-trivial only if

$$l\geqslant\tfrac{1}{2}\underline{k}-\mathcal{D}. \tag{9.35}$$

Step 1. We consider first the operators $\mathcal{A}_{m,l}$. Since $l\geqslant-\frac{2}{3}m-2\mathcal{D}$, it suffices to prove that

$$2^{(1-50\delta)(m-\underline{k}/2)}\|P_k\mathcal{I}_l[f^\mu_{j_1,k_1}(s),f^\nu_{j_2,k_2}(s)]\|_{L^2}\lesssim 2^{-3\delta^2 m}, \tag{9.36}$$

for any $s\in I_m$ and j_1,j_2, where $\mathcal{I}_l$ are the operators defined in (9.20), and $f^\mu_{j_1,k_1}$ and $f^\nu_{j_2,k_2}$ are as in (9.14). We may assume $k_1\leqslant k_2$ and consider two cases.

Case 1. If $\underline{k}=k_1$ then we estimate first the left-hand side of (9.36) by

$$C2^{(1-50\delta)(m-\underline{k}/2)}2^{k+\underline{k}/2}2^{-l}\Big(\sup_{s,t\approx 2^m}\|e^{-it\Lambda_\mu}f^\mu_{j_1,k_1}(s)\|_{L^\infty}\|f^\nu_{j_2,k_2}(s)\|_{L^2}+2^{-8m}\Big)$$
$$\lesssim 2^{(1-50\delta)(m-\underline{k}/2)}2^k2^{6\delta^2 m}(2^{\underline{k}}2^{-m+50\delta j_1}2^{-4k^+}+2^{-8m}),$$

using Lemma 7.4 and (7.40). This suffices to prove (9.36) if $j_1\leqslant\frac{9}{10}m$. On the other hand, if $j_1\geqslant\frac{9}{10}m$, then we estimate the left-hand side of (9.36) by

$$C2^{(1-50\delta)(m-\underline{k}/2)}2^{k+\underline{k}/2}2^{-l}\Big(\sup_{s,t\approx 2^m}\|f^\mu_{j_1,k_1}(s)\|_{L^2}\,\|e^{-it\Lambda_\nu}f^\nu_{j_2,k_2}(s)\|_{L^\infty}+2^{-8m}\Big)$$
$$\lesssim 2^{(1-50\delta)(m-\underline{k}/2)}2^k2^{6\delta^2 m}(2^{-(1-50\delta)j_1}2^{-5m/6}2^{-2k^+}+2^{-8m}),$$

using Lemma 7.4 and (7.44). This suffices to prove the desired bound (9.36).

Case 2. If $\underline{k}=k$, then (9.36) follows using the $L^2\times L^\infty$ estimate, as in Case 1, unless

$$\max(|k_1|,|k_2|)\leqslant 20\quad\text{and}\quad\max(j_1,j_2)\leqslant\tfrac{1}{3}m. \tag{9.37}$$

On the other hand, if (9.37) holds, then it suffices to prove that, for $|\varrho|\leqslant 2^{m-\mathcal{D}}$,

$$2^{(1-50\delta)(m-k/2)}2^{-k/2}\|P_kI_0[f^{\mu}_{j_1,k_1}(s),f^{\nu}_{j_2,k_2}(s)]\|_{L^2}\lesssim 2^{-3\delta^2 m},\\ \widehat{I_0[f,g]}(\xi):=\int_{\mathbb{R}^2}e^{i(s+\varrho)\Phi(\xi,\eta)}m_0(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta. \tag{9.38}$$

Indeed, (9.36) would follow from (9.38) and the inequality $l\geqslant\frac{1}{2}\underline{k}-\mathcal{D}\geqslant-\frac{2}{3}m-2\mathcal{D}$ (see (9.22)–(9.35)), using the superposition argument in Lemma 7.4. On the other hand, the proof of (9.38) is similar to the proof of (8.15) in Lemma 8.1.

Step 2. We consider now the operators $\mathcal{B}_{m,l}$. In some cases, we prove the stronger bound

$$2^{(1-50\delta)(m-\underline{k}/2)}2^m\|P_k\mathcal{I}_l[f^{\mu}_{j_1,k_1}(s),P_{k_2}\partial_sf^{\nu}(s)]\|_{L^2}\lesssim 2^{-3\delta^2 m}, \tag{9.39}$$

for any $s\in I_m$ and j_1. We consider three cases.

Case 1. If $\underline{k}=k_1$, then we use the bounds

$$\|P_{k_2}\partial_sf^{\nu}(s)\|_{L^2}\lesssim 2^{-m+5\delta m}(2^{k_2}+2^{-m/2}),\\ \|e^{-is\Lambda_\nu}P_{k_2}\partial_sf^{\nu}(s)\|_{L^\infty}\lesssim 2^{-5m/3+6\delta^2 m}; \tag{9.40}$$

see (8.21) and (8.7). We also record the bound, which can be easily verified using integration by parts and Plancherel for any $\varrho\in\mathbb{R}$ and $k'\in\mathbb{Z}$,

$$\|e^{-i\varrho\Lambda}P_{k'}\|_{L^\infty\to L^\infty}\lesssim\|\mathcal{F}^{-1}\{e^{-i\varrho\Lambda(\xi)}\varphi_{k'}(\xi)\}\|_{L^1}\lesssim 1+2^{k'/2}2^{k'_+}|\varrho|. \tag{9.41}$$

If

$$k_1\geqslant-\tfrac{1}{4}m\quad\text{and}\quad j_1\leqslant(1-\delta^2)m, \tag{9.42}$$

then we use (7.43), (9.40), and Lemma 7.4 to estimate the left-hand side of (9.39) by

$$C2^{k+k_1/2}2^{(1-50\delta)(m-\underline{k}/2)}2^m\\ \times\Big(2^{-l}\sup_{|\varrho|\leqslant 2^{m/2}}\|e^{-i(s+\varrho)\Lambda_\mu}f^{\mu}_{j_1,k_1}(s)\|_{L^\infty}\|P_{k_2}\partial_sf^{\nu}(s)\|_{L^2}+2^{-8m}\Big)\\ \lesssim 2^{6k^+}2^{k_1/2}2^{-40\delta m}.$$

This suffices to prove (9.39) when (9.42) holds (recall the choice of δ, N_0, and N_1 in Definition 2.5). On the other hand, if

$$k_1\geqslant-\tfrac{1}{4}m\quad\text{and}\quad j_1\geqslant(1-\delta^2)m, \tag{9.43}$$

then we use (9.41), (7.39), (9.40), and Lemma 7.4 to estimate the left-hand side of (9.39) by

$$C2^{k+k_1/2}2^{(1-50\delta)(m-\underline{k}/2)}2^m \times\Big(2^{-l}\|f^{\mu}_{j_1,k_1}(s)\|_{L^2}\sup_{|\varrho|\leqslant 2^{-l+4\delta^2m}}\|e^{-i(s+\varrho)\Lambda_\nu}P_{k_2}\partial_s f^{\nu}(s)\|_{L^\infty}+2^{-8m}\Big) \lesssim 2^{10k^+}2^{-2m/3+10\delta m}2^{-2l}.$$

This suffices to prove (9.39), provided that (9.43) holds.

Finally, if $k_1\leqslant -\frac{1}{4}m$, then we use the bound

$$\sup_{|\varrho|\leqslant 2^{m-\mathcal{D}}}\|e^{-i(s+\varrho)\Lambda_\mu}f^{\mu}_{j_1,k_1}(s)\|_{L^\infty}\lesssim 2^{(3/2-25\delta)k_1}2^{-m+50\delta m}2^{\delta^2 m},$$

which follows from (7.39)–(7.40).Then, we estimate the left-hand side of (9.39) by

$$C2^{2k^++k_1/2}2^{(1-50\delta)(m-\underline{k}/2)}2^m2^{-l}2^{(3/2-25\delta)k_1}2^{-m+51\delta m}2^{-m+5\delta m}\lesssim 2^{6k^+}2^{10\delta m}2^{k_1}.$$

The desired bound (9.39) follows, provided that $k_1\leqslant -\frac{1}{4}m$.

Case 2. If $\underline{k}=k$, then (9.39) follows using $L^2\times L^\infty$ estimates, as in Case 1, unless

$$\max(|k_1|,|k_2|)\leqslant 20. \tag{9.44}$$

Assuming (9.44), we notice that

$$\begin{aligned} &\sup_{|\varrho|\leqslant 2^{m-\mathcal{D}}}\|e^{-i(s+\varrho)\Lambda_\mu}A_{\leqslant 0,\gamma_0}f^{\mu}_{j_1,k_1}(s)\|_{L^\infty}\lesssim 2^{-m+3\delta m}, && \text{if } j_1\leqslant(1-\delta^2)m,\\ &\sup_{|\varrho|\leqslant 2^{m-\mathcal{D}}}\|e^{-i(s+\varrho)\Lambda_\mu}A_{\geqslant 1,\gamma_0}f^{\mu}_{j_1,k_1}(s)\|_{L^\infty}\lesssim 2^{-m}, && \text{if } \tfrac{1}{2}m\leqslant j_1\leqslant(1-\delta^2)m, \end{aligned} \tag{9.45}$$

as a consequence of (7.43). Therefore, using the $L^2\times L^\infty$ estimate and (9.40), as before,

$$2^{(1-50\delta)(m-k/2)}2^m\|P_k\mathcal{I}_l[A_{\leqslant 0,\gamma_0}f^{\mu}_{j_1,k_1}(s),P_{k_2}\partial_s f^{\nu}(s)]\|_{L^2}\lesssim 2^{-3\delta^2 m}, \tag{9.46}$$

if $j_1\leqslant(1-\delta^2)m$, and

$$2^{(1-50\delta)(m-k/2)}2^m\|P_k\mathcal{I}_l[A_{\geqslant 1,\gamma_0}f^{\mu}_{j_1,k_1}(s),P_{k_2}\partial_s f^{\nu}(s)]\|_{L^2}\lesssim 2^{-3\delta^2 m}, \tag{9.47}$$

if $\frac{1}{2}m\leqslant j_1\leqslant(1-\delta^2)m$.

On the other hand, if $j_1\geqslant(1-\delta^2)m$, then we can use the L^∞ bound

$$\|e^{-is\Lambda_\nu}P_{k_2}\partial_s f^{\nu}(s)\|_{L^\infty}\lesssim 2^{-5m/3+6\delta^2 m}$$

in (9.40), together with the general bound (9.41). As in (9.27), we decompose

$$f^{\mu}_{j_1,k_1}=\sum_{n_1=0}^{j_1} f^{\mu}_{j_1,k_1,n_1},$$

and record the bound $\|f^{\mu}_{j_1,k_1,n_1}(s)\|_{L^2}\lesssim 2^{-j_1+50\delta j_1}2^{n_1/2-49\delta n_1}2^{\delta^2 m}$. Let

$$X:=2^{(1-50\delta)(m-\underline{k}/2)}2^m\|P_k\mathcal{I}_l[f^{\mu}_{j_1,k_1,n_1}(s),P_{k_2}\partial_s f^{\nu}(s)]\|_{L^2}.$$

Using Lemma 7.4, it follows that

$$\begin{aligned}X&\lesssim 2^{(1-50\delta)(m-k/2)}2^m\\&\quad\times\Big(2^k2^{-l}\|f^{\mu}_{j_1,k_1,n_1}(s)\|_{L^2}\sup_{|\varrho|\leqslant 2^{-l+2\delta^2 m}}\|e^{-i(s+\varrho)\Lambda_\nu}P_{k_2}\partial_s f^{\nu}(s)\|_{L^\infty}+2^{-8m}\Big)\\&\lesssim 2^{-k/2}2^{-2m/3}2^{n_1/2-49\delta n_1}2^{4\delta m}.\end{aligned}$$

Using only L^2 bounds (see (9.40)) and Cauchy–Schwarz inequality, we also have

$$X\lesssim 2^{(1-50\delta)(m-k/2)}2^m2^{2k}2^{-l}\|f^{\mu}_{j_1,k_1,n_1}(s)\|_{L^2}\,\|P_{k_2}\partial_s f^{\nu}(s)\|_{L^2}\lesssim 2^k2^{n_1/2-49\delta n_1}2^{6\delta m}.$$

Finally, using (7.36), we have

$$X\lesssim 2^{(1-50\delta)(m-k/2)}2^m2^k2^{-l}\|\widehat{f^{\mu}_{j_1,k_1,n_1}}(s)\|_{L^1}\,\|P_{k_2}\partial_s f^{\nu}(s)\|_{L^2}\lesssim 2^{-49\delta n_1}2^{7\delta m}.$$

We can combine the last three estimates (using the last one for $n_1\geqslant\frac14 m$ and the first two for $n_1\leqslant\frac14 m$) to conclude that, if $j_1\geqslant(1-\delta^2)m$, then

$$2^{(1-50\delta)(m-k/2)}2^m\|P_k\mathcal{I}_l[f^{\mu}_{j_1,k_1}(s),P_{k_2}\partial_s f^{\nu}(s)]\|_{L^2}\lesssim 2^{-3\delta^2 m}. \tag{9.48}$$

In view of (9.46)–(9.48), it remains to prove that, for $j_1\leqslant\frac12 m$,

$$2^{(1-50\delta)(m-k/2)}2^m\|P_k\mathcal{I}_l[A_{\geqslant 1,\gamma_0}f^{\mu}_{j_1,k_1}(s),P_{k_2}\partial_s f^{\nu}(s)]\|_{L^2}\lesssim 2^{-3\delta^2 m}. \tag{9.49}$$

To prove (9.49), we decompose $P_{k_2}\partial_s f^{\nu}(s)$ as in (8.8). The terms that are bounded in L^2 by $2^{-4m/3+4\delta m}$ lead to acceptable contributions, using the $L^2\times L^\infty$ argument with Lemma 7.4 and (7.44). It remains to consider the terms $A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3,k_4,j_4}(s)$ when $\max(j_3,j_4)\leqslant(1-\delta^2)m$ and $k_3,k_4\in[-2m/N_0',300]$. For these terms, it suffices to prove that

$$\|P_k\mathcal{I}_l[A_{\geqslant 1,\gamma_0}f^{\mu}_{j_1,k_1}(s),A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3,k_4,j_4}(s)]\|_{L^2}\lesssim 2^{-4m}. \tag{9.50}$$

Notice that $A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3,k_4,j_4}(s)$ is given by an expression similar to (8.10). Therefore,

$$\begin{aligned}\mathcal{F}\{P_k\mathcal{I}_l[A_{\geqslant 1,\gamma_0}f^{\mu}_{j_1,k_1}(s),A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3,k_4,j_4}(s)]\}(\xi)\\ =\int_{\mathbb{R}^2\times\mathbb{R}^2}e^{is\tilde{\Phi}(\xi,\eta,\sigma)}\widehat{f^{\mu}_{j_1,k_1}}(\xi-\eta,s)\varphi_{\leqslant -101}(|\xi-\eta|-\gamma_0)2^{-l}\widetilde{\varphi}_l(\Phi_{+\mu\nu}(\xi,\eta))\varphi_k(\xi)\\ \times\varphi_{k_2}(\eta)\mathfrak{m}_{\mu\nu}(\xi,\eta)\mathfrak{m}_{\nu\beta\gamma}(\eta,\sigma)\widehat{f^{\beta}_{j_3,k_3}}(\eta-\sigma,s)\widehat{f^{\gamma}_{j_4,k_4}}(\sigma,s)\,d\sigma\,d\eta,\end{aligned}\tag{9.51}$$

where

$$\tilde{\Phi}(\xi,\eta,\sigma)=\Lambda(\xi)-\Lambda_\mu(\xi-\eta)-\Lambda_\beta(\eta-\sigma)-\Lambda_\gamma(\sigma).$$

The main observation is that either

$$|\nabla_\eta\tilde{\Phi}(\xi,\eta,\sigma)|=|\nabla\Lambda_\mu(\xi-\eta)-\nabla\Lambda_\beta(\eta-\sigma)|\gtrsim 1,\tag{9.52}$$

or

$$|\nabla_\sigma\tilde{\Phi}(\xi,\eta,\sigma)|=|\nabla\Lambda_\beta(\eta-\sigma)-\nabla\Lambda_\gamma(\sigma)|\gtrsim 1,\tag{9.53}$$

in the support of the integral. Indeed, $\big||\eta|-\gamma_0\big|\leqslant 2^{-95}$ in view of the cutoffs on the variables ξ and $\xi-\eta$. If $|\nabla_\sigma\tilde{\Phi}(\xi,\eta,\sigma)|\leqslant 2^{-\mathcal{D}}$, then $\max(|k_3|,|k_4|)\leqslant 300$ and, using Proposition 10.2 (ii) (in particular (10.17)), it follows that $|\eta-\sigma|$ is close to either $\frac12\gamma_0$, or $p_{+-1}(\gamma_0)\geqslant 1.1\gamma_0$, or $p_{+-1}(\gamma_0)-\gamma_0\leqslant 0.9\gamma_0$. In these cases, the lower bound (9.52) follows. The desired bound (9.50) then follows using Lemma 7.2 (i).

Case 3. If $\underline{k}=k_2$, then we do not prove the stronger estimate (9.39). In this case, the desired bound follows from Lemma 9.9 below.

LEMMA 9.9. *Assume that* (9.22) *holds and, in addition,*

$$j\leqslant m+2\mathcal{D}+\tfrac12\max(|k|,|k_1|,|k_2|),\quad k_2\leqslant -2\mathcal{D},\quad \textit{and}\quad 2^{-l}\leqslant 2^{10\delta m}+2^{-k_2/2+\mathcal{D}}.\tag{9.54}$$

Then, for any j_1,

$$2^{(1-50\delta)j}\|Q_{jk}\mathcal{B}_{m,l}[f^{\mu}_{j_1,k_1},P_{k_2}\partial_s f^{\nu}]\|_{L^2}\lesssim 2^{-3\delta^2 m}.\tag{9.55}$$

Proof. We record the bounds

$$\begin{aligned}\|P_{k_2}\partial_s f^{\nu}(s)\|_{L^2}\lesssim 2^{-m+5\delta m}(2^{k_2}+2^{-m/2}),\\ \sup_{|\varrho|\leqslant 2^{-l+2\delta^2 m}}\|e^{-i(s+\varrho)\Lambda_\nu}P_{k_2}\partial_s f^{\nu}(s)\|_{L^\infty}\lesssim 2^{-5m/3+10\delta^2 m}(2^{k_2/2+10\delta m}+1);\end{aligned}\tag{9.56}$$

see (8.7), (8.21), and (9.41). We will prove that, for any $s\in\mathcal{I}_m$,

$$2^{(1-50\delta)j}2^m\|Q_{jk}\mathcal{I}_l[f^{\mu}_{j_1,k_1}(s),P_{k_2}\partial_s f^{\nu}(s)]\|_{L^2}\lesssim 2^{-3\delta^2 m}.\tag{9.57}$$

Step 1. We notice the identity

$$
\begin{aligned}
&Q_{jk}\mathcal{I}_l[f^{\mu}_{j_1,k_1}(s), P_{k_2}\partial_s f^{\nu}(s)](x)\\
&\quad = C\widetilde{\varphi}^{(k)}_j(x)\int_{\mathbb{R}^2\times\mathbb{R}^2} e^{i(s\Phi(\xi,\eta)+x\cdot\xi)}2^{-l}\widetilde{\varphi}_l(\Phi(\xi,\eta))\varphi_k(\xi)m_0(\xi,\eta)\\
&\qquad\qquad\qquad\qquad \times\widehat{f^{\mu}_{j_1,k_1}}(\xi-\eta,s)\widehat{P_{k_2}\partial_s f^{\nu}}(\eta,s)\,d\xi\,d\eta.
\end{aligned}
$$

Therefore, $\|Q_{jk}\mathcal{I}_l[f^{\mu}_{j_1,k_1}(s), P_{k_2}\partial_s f^{\nu}(s)]\|_{L^2}\lesssim 2^{-4m}$, using integration by parts in ξ and Lemma 7.2 (i), unless

$$
2^j\leqslant \max(2^{j_1+\delta m}, 2^{m+\max(|k|,|k_1|)/2+\mathcal{D}}). \tag{9.58}
$$

On the other hand, assuming (9.58), $L^2\times L^\infty$ bounds using Lemma 7.4, the bounds (9.56), and Lemma 7.5 show that (9.57) holds in the following cases:

$$
\begin{aligned}
\text{either}\quad &(k_1\leqslant -10 \ \text{ and } \ j_1\leqslant m-\delta m),\\
\text{or}\quad &(k_1\leqslant -10 \ \text{ and } \ j_1\geqslant m-\delta m),\\
\text{or}\quad &\big(k_1\geqslant 10 \quad \text{and } \ j_1\leqslant \tfrac{2}{3}m\big),\\
\text{or}\quad &\big(k_1\geqslant 10 \quad \text{and } \ j_1\geqslant \tfrac{2}{3}m\big).
\end{aligned} \tag{9.59}
$$

See the similar estimates in the proof of Lemma 9.5, in particular those in Cases 1 and 2 of Step 2. In each case, we estimate $e^{-i(s+\varrho)\Lambda_\mu}f^{\mu}_{j_1,k_1}(s)$ in L^∞ and $e^{-i(s+\varrho)\Lambda_\nu}P_{k_2}\partial_s f^{\nu}(s)$ in L^2 when j_1 is small, and we estimate $e^{-i(s+\varrho)\Lambda_\mu}f^{\mu}_{j_1,k_1}(s)$ in L^2 and $e^{-i(s+\varrho)\Lambda_\nu}P_{k_2}\partial_s f^{\nu}(s)$ in L^∞ when j_1 is large. We estimate the contribution of the symbol m_0 by $2^{(k+k_1+k_2)/2}$ in all cases.

It remains to prove the desired bound (9.57) when $k,k_1\in[-20,20]$. We can still prove this, when $f^{\mu}_{j_1,k_1}(s)$ is replaced by $A_{\leqslant 0,\gamma_0}f^{\mu}_{j_1,k_1}(s)$, or when $j_1\geqslant\frac{1}{3}m-\delta m$, or when $k_2\leqslant -\frac{1}{3}m+\delta m$, using $L^2\times L^\infty$ estimates as before.

Step 2. To deal with the remaining cases, we use the decomposition (8.8). The contribution of the error component $P_{k_2}E^{a_2,\alpha_2}_{\nu}$ can also be estimated in the same way when $j_1\leqslant\frac{1}{3}m-\delta m$. After these reductions, we may assume that

$$
\begin{gathered}
k,k_1\in[-20,20],\quad j_1\leqslant\tfrac{1}{3}m-\delta m,\quad j\leqslant m+2\mathcal{D},\quad k_2\in\big[-\tfrac{1}{3}m+\delta m,-2\mathcal{D}\big],\\
2^{-l}\lesssim 2^{10\delta m}+2^{-k_2/2}.
\end{gathered} \tag{9.60}
$$

It remains to prove that, for any $[(k_3,j_3),(k_4,j_4)]\in X_{m,k_2}$,

$$
2^{(1-50\delta)j}2^m\|Q_{jk}\mathcal{I}_l[A_{\geqslant 1,\gamma_0}f^{\mu}_{j_1,k_1}, A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3;k_4,j_4}]\|_{L^2}\lesssim 2^{-4\delta^2 m}. \tag{9.61}
$$

The $L^2\times L^\infty$ argument still works to prove (9.61) if

$$\|A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3;k_4,j_4}(s)\|_{L^2}\lesssim 2^{-7m/6+25\delta m}. \tag{9.62}$$

We notice that this bound holds if $\max(j_3,j_4)\geqslant\frac{1}{3}m-\delta m$. Indeed, since $k_2\leqslant -2\mathcal{D}$, we have

$$P_{k_2}I^{\nu\beta\gamma}[A_{\geqslant 1,\gamma_1}f^{\beta}_{j_3,k_3}(s),A_{\geqslant 1,\gamma_0}f^{\gamma}_{j_4,k_4}(s)]\equiv 0,$$

and the bound (9.62) follows by $L^2\times L^\infty$ arguments as in the proof of Lemma 8.1.

Thus, we may assume that $j_3,j_4\leqslant\frac{1}{3}m-\delta m$. We examine the explicit formula (9.51). We claim that

$$|\mathcal{F}\{P_k\mathcal{I}_l[A_{\geqslant 1,\gamma_0}f^{\mu}_{j_1,k_1}(s),A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3;k_4,j_4}(s)]\}(\xi)|\lesssim 2^{-10m},\quad \text{if } |k_3|\geqslant 100.$$

Indeed, in this case the η derivative of the phase $\tilde{\Phi}$ is $\gtrsim 2^{|k_3|/2}$ in the support of the integral (recall that $|k_1|\leqslant 20$). Integration by parts in η, using Lemma 7.2 (i), shows that the resulting integral is negligible, as desired.

In view of Lemma 8.1 (ii) (3), it remains to prove (9.61) when, in addition to (9.60),

$$k_3,k_4\in[-100,100],\quad j_3,j_4\leqslant\tfrac{1}{3}m-\delta m,\quad\text{and}\quad\beta=-\gamma. \tag{9.63}$$

We examine again formula (9.51) and notice that the (η,σ) derivative of the phase $\tilde{\Phi}$ is $\gtrsim 1$ unless $\big||\eta-\sigma|-\gamma_0\big|\leqslant 2^{-98}$ and $\big||\sigma|-\gamma_0\big|\leqslant 2^{-98}$. Therefore, we may replace $f^{\beta}_{j_3,k_3}$ by $A_{\geqslant -5,\gamma_0}f^{\beta}_{j_3,k_3}$, and $f^{\gamma}_{j_4,k_4}$ by $A_{\geqslant -5,\gamma_0}f^{\gamma}_{j_4,k_4}$, at the expense of negligible errors. Finally, we may assume that $l\geqslant-\mathcal{D}$ if $\mu=-$, and that $j\leqslant m+k_2+\mathcal{D}$ if $\mu=+$ (otherwise, the approximate-finite-speed-of-propagation argument used in the proof of (9.12) and Lemma 9.4, which relies on integration by parts in ξ, gives rapid decay). Therefore, in proving (9.61), we may assume that

$$2^{-l}2^{(1-50\delta)j}\lesssim 2^{m-50\delta m}(1+2^{k_2/2+10\delta m}). \tag{9.64}$$

Let $\varkappa_r:=2^{\delta^2 m}2^{k_2/2-m/2}$. We now observe that, if $\big||\eta-\sigma|-\gamma_0\big|+\big||\sigma|-\gamma_0\big|\leqslant 2^{-90}$ and $|\Xi_{\beta\gamma}(\eta,\sigma)|=|(\nabla_\sigma\Phi_{\nu\beta\gamma})(\eta,\sigma)|\leqslant 2\varkappa_r$, then

$$\big||\sigma|-\gamma_0\big|\geqslant 2^{k_2-10}\quad\text{and}\quad\big||\eta-\sigma|-\gamma_0\big|\geqslant 2^{k_2-10}. \tag{9.65}$$

Indeed, we may assume that $\sigma=(\sigma_1,0)$, $\eta=(\eta_1,\eta_2)$, $|\sigma_1-\gamma_0|\leqslant 2^{-90}$, $|\eta|\in[2^{k_2-2},2^{k_2+2}]$. Recalling that $\beta=-\gamma$ and using formula (10.22), the condition $|\Xi_{\beta\gamma}(\eta,\sigma)|\leqslant 2\varkappa_r$ gives

$$\left|\lambda'(\sigma_1)-\frac{\sigma_1-\eta_1}{|\sigma-\eta|}\lambda'(|\sigma-\eta|)\right|\leqslant 2\varkappa_r\quad\text{and}\quad\frac{|\eta_2|}{|\sigma-\eta|}\lambda'(|\sigma-\eta|)\leqslant 2\varkappa_r.$$

Since $k_2\in\left[-\frac{1}{3}m+\delta m,-2\mathcal{D}\right]$ and $\varkappa_r=2^{\delta^2 m+k_2/2-m/2}$, it follows that $|\eta_2|\leqslant\varkappa_r 2^{\mathcal{D}}\leqslant 2^{k_2-\mathcal{D}}$, $|\eta_1|\in[2^{k_2-3},2^{k_2+3}]$, and $|\lambda'(\sigma_1)-\lambda'(\sigma_1-\eta_1)|\leqslant 4\varkappa_r$. On the other hand, if $|\sigma_1-\gamma_0|\leqslant 2^{k_2-10}$ and $|\eta_1|\in[2^{k_2-3},2^{k_2+3}]$, then $|\lambda'(\sigma_1)-\lambda'(\sigma_1-\eta_1)|\gtrsim 2^{2k_2}$ (as $\lambda''(\gamma_0)=0$ and $\lambda'''(\gamma_0)\approx 1$), which gives a contradiction. The claims in (9.65) follow.

We now examine formula (9.51) and recall (9.63) and (9.65). Using Lemma 7.2 (i) and integration by parts in σ, we notice that we may insert the factor $\varphi(\varkappa_r^{-1}\Xi_{\beta\gamma}(\eta,\sigma))$, at the expense of a negligible error. It remains to prove that

$$2^{(1-50\delta)j}2^m\|H\|_{L^2}\lesssim 2^{-4\delta^2 m}, \tag{9.66}$$

where, with

$$g_1:=A_{\geqslant 1,\gamma_0}f^{\mu}_{j_1,k_1}(s),\quad g_3:=A_{[-20,20-k_2],\gamma_0}f^{\beta}_{j_3,k_3}(s),\quad\text{and}\quad g_4:=A_{[-20,20-k_2],\gamma_0}f^{\gamma}_{j_4,k_4}(s),$$

we have

$$\widehat{H}(\xi):=\varphi_k(\xi)\int_{\mathbb{R}^2}e^{is(\Lambda(\xi)-\Lambda_\mu(\xi-\eta)-\Lambda_\nu(\eta))}\hat{g}_1(\xi-\eta)2^{-l}\widetilde{\varphi}_l(\Phi_{+\mu\nu}(\xi,\eta))\mathfrak{m}_{\mu\nu}(\xi,\eta)\widehat{G}_2(\eta)\,d\eta,$$
$$\widehat{G}_2(\eta):=\varphi_{k_2}(\eta)\int_{\mathbb{R}^2}e^{is(\Lambda_\nu(\eta)-\Lambda_\beta(\eta-\sigma)-\Lambda_\gamma(\sigma))}\mathfrak{m}_{\nu\beta\gamma}(\eta,\sigma)\varphi(\varkappa_r^{-1}\Xi_{\beta\gamma}(\eta,\sigma))\hat{g}_3(\eta-\sigma)\hat{g}_4(\sigma)\,d\sigma.$$

We use now the more precise bound (7.42) to see that

$$\|e^{-is\Lambda_\beta}g_3\|_{L^\infty}+\|e^{-is\Lambda_\gamma}g_4\|_{L^\infty}\lesssim 2^{-m+4\delta^2 m}2^{-k_2/2}.$$

This bound is the main reason for proving (9.65). After removing the factor

$$\varphi(\varkappa_r^{-1}\Xi_{\beta\gamma}(\eta,\sigma))$$

at the expense of a small error, and using also (A.2) and (9.41), it follows that

$$\|e^{-i(s+\varrho)\Lambda_\nu}G_2\|_{L^\infty}\lesssim(1+|\varrho|2^{k_2/2})2^{k_2}$$

for any $\varrho\in\mathbb{R}$. We now use the $L^2\times L^\infty$ argument, together with Lemma 7.4, to estimate

$$\|H\|_{L^2}\lesssim 2^{k_2/2}2^{-l}(1+2^{-l}2^{k_2/2})2^{-2m+12\delta^2 m}\lesssim 2^{-2m+12\delta^2 m}2^{k_2/2}2^{-l}(1+2^{10\delta m+k_2/2}).$$

The desired bound (9.66) follows using also (9.64). □

9.5. The case of strongly resonant interactions I

In this subsection we prove Lemma 9.6. This is where we need the localization operators $A^{(j)}_{n,\gamma_1}$ to control the output. It is an instantaneous estimate, in the sense that the time evolution will play no role. Hence, it suffices to show the following: let $\chi\in C_c^\infty(\mathbb{R}^2)$ be supported in $[-1,1]$ and assume that j, l, s, and m satisfy

$$-m+\frac{\delta m}{2}\leqslant l\leqslant\frac{10m}{N_0'}\quad\text{and}\quad 2^{m-4}\leqslant s\leqslant 2^{m+4}. \tag{9.67}$$

Assume that

$$\|f\|_{H^{N_0'}\cap H_\Omega^{N_1'}\cap Z_1}+\|g\|_{H^{N_0'}\cap H_\Omega^{N_1'}\cap Z_1}\leqslant 1, \tag{9.68}$$

and define, with $\chi_l(x)=\chi(2^{-l}x)$,

$$\widehat{I[f,g]}(\xi):=\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}\chi_l(\Phi(\xi,\eta))m_0(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta.$$

Assume also that k, k_1, k_2, j, and m satisfy (9.22) and (9.24). Then,

$$2^{\delta m/2}2^{-l}\|Q_{jk}I[P_{k_1}f,P_{k_2}g]\|_{B_j}\lesssim 2^{-5\delta^2 m}. \tag{9.69}$$

To prove (9.69), we define f_{j_1,k_1}, g_{j_2,k_2}, f_{j_1,k_1,n_1}, and g_{j_2,k_2,n_2}, for $(k_1,j_1),(k_2,j_2)\in\mathcal{J}$, $n_1\in[0,j_1+1]$, and $n_2\in[0,j_2+1]$, as in (7.33). We will analyze several cases depending on the relative sizes of the main parameters m, l, k, j, k_1, j_1, k_2, and j_2. In many cases, we will prove the stronger bound

$$2^{\delta m/2}2^{-l}2^{(1-50\delta)j}\|Q_{jk}I[f_{j_1,k_1},g_{j_2,k_2}]\|_{L^2}\lesssim 2^{-6\delta^2 m}. \tag{9.70}$$

However, in the main case (9.72), we can only prove the weaker bound

$$2^{\delta m/2}2^{-l}\|Q_{jk}I[f_{j_1,k_1},g_{j_2,k_2}]\|_{B_j}\lesssim 2^{-6\delta^2 m}. \tag{9.71}$$

These bounds clearly suffice to prove (9.69).

Case 1. We first prove the bound (9.71) under the assumption

$$\max(j_1,j_2)\leqslant\tfrac{9}{10}m\quad\text{and}\quad 2l\leqslant\min(k,k_1,k_2,0)-\mathcal{D}. \tag{9.72}$$

We may assume $j_1\leqslant j_2$. With

$$\varkappa_\theta:=2^{-m/2+\delta^2 m}\quad\text{and}\quad\varkappa_r:=2^{\delta^2 m}(2^{-m/2+3\max(|k|,|k_1|,|k_2|)/4}+2^{j_2-m}),$$

we decompose

$$\mathcal{F}I[f_{j_1,k_1},g_{j_2,k_2}]=\mathcal{R}_1+\mathcal{R}_2+\mathcal{NR},$$

where

$$\mathcal{R}_1(\xi) := \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)}\chi_l(\Phi(\xi,\eta))m_0(\xi,\eta)\varphi(\varkappa_r^{-1}\Xi(\xi,\eta))\varphi(\varkappa_\theta^{-1}\Theta(\xi,\eta)) \times \hat{f}_{j_1,k_1}(\xi-\eta)\hat{g}_{j_2,k_2}(\eta)\,d\eta,$$

$$\mathcal{R}_2(\xi) := \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)}\chi_l(\Phi(\xi,\eta))m_0(\xi,\eta)\varphi(\varkappa_r^{-1}\Xi(\xi,\eta))\varphi_{\geqslant 1}(\varkappa_\theta^{-1}\Theta(\xi,\eta)) \times \hat{f}_{j_1,k_1}(\xi-\eta)\hat{g}_{j_2,k_2}(\eta)\,d\eta,$$

$$\mathcal{NR}(\xi) := \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)}\chi_l(\Phi(\xi,\eta))m_0(\xi,\eta)\varphi_{\geqslant 1}(\varkappa_r^{-1}\Xi(\xi,\eta)) \times \hat{f}_{j_1,k_1}(\xi-\eta)\hat{g}_{j_2,k_2}(\eta)\,d\eta.$$

With $\psi_1 := \varphi_{\leqslant(1-\delta/4)m}$ and $\psi_2 := \varphi_{>(1-\delta/4)m}$, we rewrite

$$\mathcal{NR}(\xi) = C2^l(\mathcal{NR}_1(\xi)+\mathcal{NR}_2(\xi)),$$

$$\mathcal{NR}_i(\xi) := \int_{\mathbb{R}}\int_{\mathbb{R}^2} e^{i(s+\lambda)\Phi(\xi,\eta)}\widehat{\chi}(2^l\lambda)\psi_i(\lambda)m_0(\xi,\eta)\varphi_{\geqslant 1}(\varkappa_r^{-1}\Xi(\xi,\eta)) \times \hat{f}_{j_1,k_1}(\xi-\eta)\hat{g}_{j_2,k_2}(\eta)\,d\eta\,d\lambda.$$

Since $\widehat{\chi}$ is rapidly decreasing, we have $\|\varphi_k\mathcal{NR}_2\|_{L^\infty}\lesssim 2^{-4m}$, which gives an acceptable contribution. On the other hand, in the support of the integral defining $\mathcal{NR}_1$, we have that $|s+\lambda|\approx 2^m$, and integration by parts in η (using Lemma 7.2 (i)) gives

$$\|\varphi_k\mathcal{NR}_1\|_{L^\infty}\lesssim 2^{-4m}.$$

The contribution of $\mathcal{R}=\mathcal{R}_1+\mathcal{R}_2$ is only present if we have a space-time resonance. In particular, in view of Proposition 10.2 (iii) (notice that the assumption (10.20) is satisfied, due to (9.72)), we may assume that

$$-10\leqslant k,k_1,k_2\leqslant 10,\ \ \pm(\sigma,\mu,\nu)=(+,+,+),\ \ \text{and}\ \ \big||\xi|-\gamma_1|+|\eta-\tfrac{1}{2}\xi\big|\leqslant 2^{-\mathcal{D}}. \tag{9.73}$$

Notice that, if $\mathcal{R}(\xi)\neq 0$, then

$$\big||\xi|-\gamma_1\big|\lesssim\big|\Phi\big(\xi,\tfrac{1}{2}\xi\big)\big|\lesssim|\Phi(\xi,\eta)|+\big|\Phi(\xi,\eta)-\Phi\big(\xi,\tfrac{1}{2}\xi\big)\big|\lesssim 2^l+\varkappa_r^2. \tag{9.74}$$

Integration by parts using Lemma 7.3 shows that $\|\varphi_k\mathcal{R}_2\|_{L^\infty}\lesssim 2^{-5m/2}$, which gives an acceptable contribution. To bound the contribution of $\mathcal{R}_1$, we will show that

$$2^{\delta m/2}2^{-l}\sup_{|\xi|\approx 1}\Big|\big(1+2^m\big||\xi|-\gamma_1\big|\big)\mathcal{R}_1(\xi)\Big|\lesssim 2^{9\delta m/10}, \tag{9.75}$$

which is stronger than the bound we need in (9.71). Indeed, for j fixed, we estimate

$$\begin{aligned}\sup_{0\leqslant n\leqslant j} & 2^{(1-50\delta)j}2^{-n/2+49\delta n}\|A^{(j)}_{n,\gamma_1}Q_{jk}\mathcal{F}^{-1}\mathcal{R}_1\|_{L^2}\\ &\lesssim \sup_{0\leqslant n\leqslant j} 2^{(1-50\delta)j}2^{-n/2+49\delta n}\big\|\varphi^{[-j,0]}_{-n}\big(2^{100}\big||\xi|-\gamma_1\big|\big)\mathcal{R}_1(\xi)\big\|_{L^2_\xi}\\ &\lesssim \sum_{n\geqslant 0} 2^{(1-50\delta)j}2^{-n/2-(1/2-49\delta)\min(n,j)}\big\|\varphi^{(-\infty,0]}_{-n}\big(2^{100}\big||\xi|-\gamma_1\big|\big)\mathcal{R}_1(\xi)\big\|_{L^\infty_\xi},\end{aligned} \tag{9.76}$$

and notice that (9.71) would follow from (9.75) and the assumption $j\leqslant m+3\mathcal{D}$.

Recall from Lemma 7.5 and (9.73) (note that we may assume that $f_{j_1,k_1}=f_{j_1,k_1,0}$ and $g_{j_2,k_2}=g_{j_2,k_2,0}$) that

$$\begin{aligned}2^{(1/2-\delta')j_1}\|\hat f_{j_1,k_1}\|_{L^\infty}+2^{(1-\delta')j_1}\sup_{\theta\in\mathbb{S}^1}\|\hat f_{j_1,k_1}(r\theta)\|_{L^2(r\,dr)}\lesssim 1,\\ 2^{(1/2-\delta')j_2}\|\hat g_{j_2,k_2}\|_{L^\infty}+2^{(1-\delta')j_2}\sup_{\theta\in\mathbb{S}^1}\|\hat g_{j_2,k_2}(r\theta)\|_{L^2(r\,dr)}\lesssim 1.\end{aligned} \tag{9.77}$$

We first ignore the factor $\chi_l(\Phi(\xi,\eta))$. In view of Proposition 10.2 (ii), the η integration in the definition of $\mathcal{R}_1(\xi)$ takes place essentially over a $\varkappa_\theta\times\varkappa_r$ box in the neighborhood of $\frac12\xi$. Using (9.74) and (9.77), and estimating $\|\hat f_{j_1,k_1}\|_{L^\infty}\lesssim 1$, we have, if $j_2\geqslant\frac12 m$,

$$\big|\big(1+2^m\big||\xi|-\gamma_1\big|\big)\mathcal{R}_1(\xi)\big|\lesssim 2^m(2^l+\varkappa_r^2)2^{-j_2+\delta' j_2}\varkappa_\theta\varkappa_r^{1/2}\lesssim(2^l+\varkappa_r^2)2^{-j_2(1/2-\delta')}2^{2\delta^2 m}.$$

On the other hand, if $j_2\leqslant\frac12 m$, we estimate $\|\hat f_{j_1,k_1}\|_{L^\infty}+\|\hat f_{j_2,k_2}\|_{L^\infty}\lesssim 1$ and conclude that

$$\big|\big(1+2^m\big||\xi|-\gamma_1\big|\big)\mathcal{R}_1(\xi)\big|\lesssim 2^{m+l}\varkappa_\theta\varkappa_r\lesssim 2^l2^{2\delta^2 m}.$$

The desired bound (9.75) follows if $\varkappa_r^2 2^{-l}\leqslant 2^{j_2/4}$.

Assume now $\varkappa_r^2\geqslant 2^l2^{j_2/4}$ (in particular $j_2\geqslant\frac{11}{20}m$). In this case, the restriction

$$|\Phi(\xi,\eta)|\leqslant 2^l$$

is stronger, and we have to use it. We decompose, with $p_-:=\lfloor\log_2(2^{l/2}\varkappa_r^{-1})+\mathcal{D}\rfloor$,

$$\begin{aligned}\mathcal{R}_1(\xi)&=\sum_{p\in[p_-,0]}\mathcal{R}_1^p(\xi),\\ \mathcal{R}_1^p(\xi)&:=\int_{\mathbb{R}^2}e^{is\Phi(\xi,\eta)}\chi_l(\Phi(\xi,\eta))m_0(\xi,\eta)\varphi_p^{[p_-,1]}(\varkappa_r^{-1}\Xi(\xi,\eta))\\ &\qquad\qquad\times\varphi(\varkappa_\theta^{-1}\Theta(\xi,\eta))\hat f_{j_1,k_1}(\xi-\eta)\hat g_{j_2,k_2}(\eta)\,d\eta.\end{aligned}$$

As in (9.74), notice that, if $\mathcal{R}_1^p(\xi)\neq 0$, then $\big||\xi|-\gamma_1\big|\lesssim 2^{2p}\varkappa_r^2$. The term $\mathcal{R}_1^{p_-}(\xi)$ can be bounded as before. Moreover, using formula (10.46), it is easy to see that, if $\xi=(s,0)$ is fixed, then the set of points η that satisfy the three restrictions

$$|\Phi(\xi,\eta)|\lesssim 2^l,\quad |\nabla_\eta\Phi(\xi,\eta)|\approx 2^p\varkappa_r,\quad\text{and}\quad |\xi\cdot\eta^\perp|\lesssim\varkappa_\theta$$

is essentially contained in a union of two $\varkappa_\theta \times 2^l 2^{-p}\varkappa_r^{-1}$ boxes. Using (9.77), and estimating $\|\hat{f}_{j_1,k_1}\|_{L^\infty} \lesssim 1$, we have

$$\Big|\big(1+2^m\big||\xi|-\gamma_1\big|\big)\mathcal{R}_1^p(\xi)\Big| \lesssim 2^{m+2p}\varkappa_r^2 2^{-j_2+\delta' j_2}\varkappa_\theta(2^l 2^{-p}\varkappa_r^{-1})^{1/2}$$
$$\lesssim 2^{3p/2}2^{-m+4\delta^2 m}2^{l/2}2^{j_2/2+\delta' j_2}.$$

This suffices to prove (9.75), since $2^p \leqslant 1$, $2^{-l/2}\leqslant 2^{m/2}$, and $2^{j_2}\leqslant 2^{9m/10}$; see (9.72).

Case 2. We now assume that

$$2l \geqslant \min(k, k_1, k_2, 0) - \mathcal{D}. \tag{9.78}$$

In this case, we prove the stronger bound (9.70). We can still use the standard $L^2\times L^\infty$ argument, with Lemmas 7.4 and 7.5, to bound the contributions away from γ_0. For (9.70), it remains to prove that

$$2^{-l}2^{(1-50\delta)(m+|k|/2)}\|P_k I[A_{\geqslant 1,\gamma_0}f_{j_1,k_1}, A_{\geqslant 1,\gamma_0}g_{j_2,k_2}]\|_{L^2} \lesssim 2^{-\delta m}. \tag{9.79}$$

The bound (9.79) follows if $\max(j_1,j_2)\geqslant \frac{1}{3}m$, using the same $L^2\times L^\infty$ argument. On the other hand, if $j_1, j_2 \leqslant \frac{1}{3}m$, then we use (7.37) and the more precise bound (7.42) to see that

$$\|A_{p,\gamma_0}h\|_{L^2} \lesssim 2^{-p/2} \quad \text{and} \quad \|e^{-it\Lambda}A_{p,\gamma_0}h\|_{L^\infty} \lesssim 2^{-m+2\delta^2 m}\min(2^{p/2}, 2^{m/2-p}),$$

where $h\in\{f_{j_1,k_1}, g_{j_2,k_2}\}$, $p\geqslant 1$, and $t\approx 2^m$. Therefore, using Lemma 7.4,

$$\|P_k I[A_{p_1,\gamma_0}f_{j_1,k_1}, A_{p_2,\gamma_0}g_{j_2,k_2}]\|_{L^2} \lesssim 2^k 2^{-\max(p_1,p_2)/2}2^{-m+2\delta^2 m}2^{\min(p_1,p_2)/2}.$$

The desired bound (9.79) follows, using also the simple estimate

$$\|P_k I[A_{p_1,\gamma_0}f_{j_1,k_1}, A_{p_2,\gamma_0}g_{j_2,k_2}]\|_{L^2} \lesssim 2^k 2^{-(p_1+p_2)/2}.$$

Case 3. Assume now that

$$\max(j_1,j_2)\geqslant \tfrac{9}{10}m, \quad j\leqslant \min(j_1,j_2)+\tfrac{1}{4}m, \quad \text{and} \quad 2l\leqslant \min(k,k_1,k_2,0)-\mathcal{D}.$$

Using Lemma 10.5 and (7.35), we estimate

$$\begin{aligned}\|P_k I[f_{j_1,k_1,n_1}, g_{j_2,k_2,n_2}]\|_{L^2} &\\ \lesssim 2^{k/2}2^{30\delta m}2^{l/2-n_1/2-n_2/2}&\Big\|\sup_{\theta\in\mathbb{S}^1}|\hat{f}_{j_1,k_1,n_1}(r\theta)|\Big\|_{L^2(r\,dr)}\Big\|\sup_{\theta\in\mathbb{S}^1}|\hat{g}_{j_2,k_2,n_2}(r\theta)|\Big\|_{L^2(r\,dr)}\\ \lesssim 2^{k/2}2^{l/2}2^{-j_1+\delta' j_1}2^{-j_2+\delta' j_2}2^{30\delta m},&\end{aligned} \tag{9.80}$$

and the desired bound (9.70) follows.

Case 4. Finally, assume that

$$j_2 \geqslant \tfrac{9}{10}m, \quad j \geqslant j_1+\tfrac{1}{4}m, \quad \text{and} \quad 2l \leqslant \min(k,k_1,k_2,0)-\mathcal{D}. \tag{9.81}$$

In particular, $j_1 \leqslant \frac{7}{8}m$. We decompose, with $\varkappa_\theta = 2^{-2m/5}$,

$$\begin{aligned} I[f_{j_1,k_1}, g_{j_2,k_2}] &= I_{||}[f_{j_1,k_1}, g_{j_2,k_2}] + I_{\perp}[f_{j_1,k_1}, g_{j_2,k_2}], \\ \widehat{I_{||}[f,g]}(\xi) &= \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} \chi_l(\Phi(\xi,\eta)) \varphi(\varkappa_\theta^{-1}\Omega_\eta \Phi(\xi,\eta)) \hat{f}(\xi-\eta)\hat{g}(\eta)\, d\eta, \\ \widehat{I_{\perp}[f,g]}(\xi) &= \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} \chi_l(\Phi(\xi,\eta))(1-\varphi(\varkappa_\theta^{-1}\Omega_\eta \Phi(\xi,\eta))) \hat{f}(\xi-\eta)\hat{g}(\eta)\, d\eta. \end{aligned} \tag{9.82}$$

Integration by parts using Lemma 7.3 shows that $\|\mathcal{F}P_k I_{\perp}[f_{j_1,k_1}, g_{j_2,k_2}]\|_{L^\infty} \lesssim 2^{-5m/2}$. In addition, using Schur's test and Proposition 10.4 (i), (iii),

$$\begin{aligned} \|P_k I_{||}[f_{j_1,k_1}, g_{j_2,k_2,n_2}]\|_{L^2} &\lesssim 2^{80\delta m} 2^l \varkappa_\theta^{1/2} \|\hat{f}_{j_1,k_1}\|_{L^\infty} \|\hat{g}_{j_2,k_2,n_2}\|_{L^2} \\ &\lesssim 2^{95\delta m} 2^{l-m/5} 2^{-(1-50\delta)j_2} 2^{n_2/2}, \end{aligned}$$

which gives an acceptable contribution if $n_2 \leqslant \mathcal{D}$.

It remains to estimate the contribution of $I_{||}[f_{j_1,k_1}, g_{j_2,k_2,n_2}]$ for $n_2 \geqslant \mathcal{D}$. Since $|\eta|$ is close to γ_1 and $|\Phi(\xi,\eta)|$ is sufficiently small (see (9.81)), it follows from (10.6) that $\min(k,k_1,k_2) \geqslant -40$; moreover, the vectors ξ and η are almost aligned and $|\Phi(\xi,\eta)|$ is small, so we may also assume that $\max(k,k_1,k_2) \leqslant 100$. Moreover, $|\nabla_\eta \Phi(\xi,\eta)| \gtrsim 1$ in the support of integration of $I_{||}[f_{j_1,k_1}, g_{j_2,k_2,n_2}]$, in view of Proposition 10.2 (iii). Integration by parts in η using Lemma 7.2 (i) then gives an acceptable contribution, unless $j_2 \geqslant (1-\delta^2)m$. We may also reset $\varkappa_\theta = 2^{\delta^2 m - m/2}$, up to small errors, using Lemma 7.3.

To summarize, we may assume that

$$\begin{gathered} j_2 \geqslant (1-\delta^2)m, \quad j \geqslant j_1 + \tfrac{1}{4}m, \quad k,k_1,k_2 \in [-100,100], \\ n_2 \geqslant \mathcal{D}, \quad \text{and} \quad \varkappa_\theta = 2^{\delta^2 m - m/2}. \end{gathered} \tag{9.83}$$

We decompose, with $p_- := \lfloor \frac{1}{2} l \rfloor$,

$$\begin{aligned} I_{||}[f_{j_1,k_1}, g_{j_2,k_2,n_2}] &= \sum_{p_- \leqslant p \leqslant \mathcal{D}} I_{||}^p[f_{j_1,k_1}, g_{j_2,k_2,n_2}], \\ \widehat{I_{||}^p[f,g]}(\xi) &:= \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} \chi_l(\Phi(\xi,\eta)) \varphi(\varkappa_\theta^{-1}\Theta(\xi,\eta)) \varphi_p^{[p_-,\mathcal{D}]}(\nabla_\xi \Phi(\xi,\eta)) \\ &\qquad \times \hat{f}(\xi-\eta)\hat{g}(\eta)\, d\eta. \end{aligned}$$

It suffices to prove that, for any p,

$$2^{-l}2^{(1-50\delta)j}\|Q_{jk}I^p_{||}[f_{j_1,k_1},g_{j_2,k_2,n_2}]\|_{L^2}\lesssim 2^{-\delta m}. \tag{9.84}$$

As a consequence of Proposition 10.4 (iii), under our assumptions in (9.83) and recalling that $|\nabla_\eta\Phi(\xi,\eta)|\gtrsim 1$ in the support of the integral,

$$\sup_\xi\int_{\mathbb{R}^2}|\chi_l(\Phi(\xi,\eta))|\varphi(\varkappa_\theta^{-1}\Theta(\xi,\eta))\varphi_{\leqslant -\mathcal{D}/2}(|\eta|-\gamma_1)\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\eta)\,d\eta\lesssim 2^{\delta^2 m}2^l\varkappa_\theta,$$

and, for any $p\geqslant p_-$,

$$\begin{aligned}\sup_\eta\int_{\mathbb{R}^2}|\chi_l(\Phi(\xi,\eta))|\varphi(\varkappa_\theta^{-1}\Theta(\xi,\eta))\varphi_p(\nabla_\xi\Phi(\xi,\eta))\varphi_{\leqslant -\mathcal{D}/2}(|\eta|-\gamma_1)\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\eta)\,d\xi\\ \lesssim 2^{\delta^2 m}2^{l-p}\varkappa_\theta.\end{aligned}$$

Using Schur's test, we can then estimate, for $p\geqslant p_-$,

$$\begin{aligned}\|P_kI^p_{||}[f_{j_1,k_1},g_{j_2,k_2,n_2}]\|_{L^2}&\lesssim 2^{-p/2}2^l2^{-m/2+4\delta^2 m}\|\hat f_{j_1,k_1}\|_{L^\infty}\|g_{j_2,k_2,n_2}\|_{L^2}\\ &\lesssim 2^{-p/2}2^l2^{-m+5\delta m}.\end{aligned}$$

The desired bound (9.83) follows if $j\leqslant m+p+4\delta m$. On the other hand, if

$$j\geqslant m+p+4\delta m,$$

then we use the approximate-finite-speed-of-propagation argument to show that

$$\|Q_{jk}I^p_{||}[f_{j_1,k_1},g_{j_2,k_2,n_2}]\|_{L^2}\lesssim 2^{-3m}. \tag{9.85}$$

Indeed we write, as in Lemma 7.4,

$$\chi_l(\Phi(\xi,\eta))=c2^l\int_{\mathbb{R}}\widehat{\chi}(2^l\varrho)e^{i\varrho\Phi(\xi,\eta)}\,d\varrho,$$

and notice that $|\nabla_\xi(x\cdot\xi+(s+\varrho)\Phi(\xi,\eta))|\approx 2^j$ in the support of the integral, provided that $|x|\approx 2^j$ and $|\varrho|\leqslant 2^m$. Then, we recall that $j\geqslant j_1+\frac14 m$ (see (9.83)) and use Lemma 7.2 (i) to prove (9.85). This completes the proof of Lemma 9.6.

9.6. The case of weakly resonant interactions

In this subsection we prove Lemma 9.7. We decompose $P_{k_2}\partial_s f^\nu$ as in (8.8) and notice that the contribution of the error term can be estimated using the $L^2\times L^\infty$ argument as before.

To estimate the contributions of the terms $A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3;k_4,j_4}$, we need more careful analysis of trilinear operators. With $\tilde\Phi(\xi,\eta,\sigma)=\Lambda(\xi)-\Lambda_\mu(\xi-\eta)-\Lambda_\beta(\eta-\sigma)-\Lambda_\gamma(\sigma)$ and $p\in\mathbb{Z}$, we define the trilinear operators $\mathcal{J}_{l,p}$ by

$$\begin{aligned}\widehat{\mathcal{J}_{l,p}[f,g,h]}(\xi,s):=\int_{\mathbb{R}^2\times\mathbb{R}^2} & e^{is\tilde\Phi(\xi,\eta,\sigma)}\hat f(\xi-\eta)2^{-l}\widetilde\varphi_l(\Phi_{+\mu\nu}(\xi,\eta))\varphi_p(\tilde\Phi(\xi,\eta,\sigma))\\ &\times\varphi_{k_2}(\eta)\mathfrak{m}_{\mu\nu}(\xi,\eta)\mathfrak{m}_{\nu\beta\gamma}(\eta,\sigma)\hat g(\eta-\sigma)\hat h(\sigma)\,d\sigma\,d\eta.\end{aligned}\tag{9.86}$$

Let $\mathcal{J}_{l,\leqslant p}=\sum_{q\leqslant p}\mathcal{J}_{l,q}$ and $\mathcal{J}_l=\sum_{q\in\mathbb{Z}}\mathcal{J}_{l,q}$. Let

$$\mathcal{C}_{l,p}[f,g,h]:=\int_{\mathbb{R}}q_m(s)\mathcal{J}_{l,p}[f,g,h](s)\,ds,\quad \mathcal{C}_{l,\leqslant p}:=\sum_{q\leqslant p}\mathcal{C}_{l,q},\quad \mathcal{C}_l=\sum_{q\in\mathbb{Z}}\mathcal{C}_{l,q}.\tag{9.87}$$

Notice that

$$\mathcal{B}_{m,l}[f^\mu_{j_1,k_1},A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3;k_4,j_4}]=\mathcal{C}_l[f^\mu_{j_1,k_1},f^\beta_{j_3,k_3},f^\gamma_{j_4,k_4}].\tag{9.88}$$

To prove the lemma, it suffices to show that

$$2^{(1-50\delta)j}\|Q_{jk}\mathcal{C}_l[f^\mu_{j_1,k_1},f^\beta_{j_3,k_3},f^\gamma_{j_4,k_4}]\|_{L^2}\lesssim 2^{-3\delta^2m},\tag{9.89}$$

provided that

$$\begin{gathered}k,k_1,k_2\in\Big[-\frac{3.5m}{N_0'},\frac{3.2m}{N_0'}\Big],\quad j\leqslant m+2\mathcal{D}+\frac{\max(|k|,|k_1|,|k_2|)}{2},\\ l\geqslant-\frac{m}{14},\quad m\geqslant\frac{\mathcal{D}^2}{8},\quad k_2,k_3,k_4\leqslant\frac{m}{N_0'},\quad [(k_3,j_3),(k_4,j_4)]\in X_{m,k_2}.\end{gathered}\tag{9.90}$$

The bound (9.41) and the same argument as in the proof of Lemma 7.4 show that

$$\begin{aligned}&\|P_k\mathcal{J}_{l,\leqslant p}[f,g,h](s)\|_{L^2}\\ &\quad\lesssim 2^{(k+k_1+k_2)/2}2^{(k_2+k_3+k_4)/2}2^{-l}\\ &\qquad\times\min(|f|_\infty|g|_2\,|h|_\infty,|f|_\infty\,|g|_\infty\,|h|_2,(1+2^{-l+2\delta^2m+3\max(k_2,0)/2})|f|_2\,|g|_\infty\,|h|_\infty)\\ &\qquad+2^{-10m}|f|_2\,|g|_2\,|h|_2,\end{aligned}\tag{9.91}$$

provided that $s\in I_m$, $2^{-p}+2^{-l}\leqslant 2^{m-2\delta^2m}$,

$$f=P_{[k_1-8,k_1+8]}f,\quad g=P_{[k_3-8,k_3+8]}g,\quad h=P_{[k_4-8,k_4+8]}h,$$

and, for $F\in\{f,g,h\}$,

$$|F|_q:=\sup_{|t|\in[2^{m-4},2^{m+4}]}\|e^{it\Lambda}F\|_{L^q}.\tag{9.92}$$

In particular, the bounds (9.91) and (7.43) show that

$$2^{(1-50\delta)j}\|Q_{jk}\mathcal{C}_l[f^{\mu}_{j_1,k_1},f^{\beta}_{j_3,k_3},f^{\gamma}_{j_4,k_4}]\|_{L^2}\lesssim 2^{-\delta m},$$

provided that $\max(j_1,j_3,j_4)\geqslant\frac{20}{21}m$. Therefore, it remains to prove (9.89) when

$$\max(j_1,j_3,j_4)\leqslant\tfrac{20}{21}m. \tag{9.93}$$

Step 1. We first consider the contributions of $\mathcal{C}_{l,p}[f^{\mu}_{j_1,k_1},f^{\beta}_{j_3,k_3},f^{\gamma}_{j_4,k_4}]$ for $p\geqslant-\frac{11}{21}m$. In this case, we integrate by parts in s and rewrite

$$\begin{aligned}\mathcal{C}_{l,p}&[f^{\mu}_{j_1,k_1},f^{\beta}_{j_3,k_3},f^{\gamma}_{j_4,k_4}]\\&=i2^{-p}\biggl(\int_{\mathbb{R}}q_m'(s)\widetilde{\mathcal{J}}_{l,p}[f^{\mu}_{j_1,k_1},f^{\beta}_{j_3,k_3},f^{\gamma}_{j_4,k_4}](s)\,ds+\tilde{\mathcal{C}}_{l,p}[\partial_sf^{\mu}_{j_1,k_1},f^{\beta}_{j_3,k_3},f^{\gamma}_{j_4,k_4}]\\&\qquad+\tilde{\mathcal{C}}_{l,p}[f^{\mu}_{j_1,k_1},\partial_sf^{\beta}_{j_3,k_3},f^{\gamma}_{j_4,k_4}]+\tilde{\mathcal{C}}_{l,p}[f^{\mu}_{j_1,k_1},f^{\beta}_{j_3,k_3},\partial_sf^{\gamma}_{j_4,k_4}]\biggr),\end{aligned}$$

where the operators $\widetilde{\mathcal{J}}_{l,p}$ and $\tilde{\mathcal{C}}_{l,p}$ are defined in the same way as the operators $\mathcal{J}_{l,p}$ and $\mathcal{C}_{l,p}$, but with $\varphi_p(\tilde{\Phi}(\xi,\eta,\sigma))$ replaced by $\widetilde{\varphi}_p(\tilde{\Phi}(\xi,\eta,\sigma))$, with $\widetilde{\varphi}_p(x)=2^px^{-1}\varphi_p(x)$ (see formula (9.86)). The operator $\widetilde{\mathcal{J}}_{l,p}$ also satisfies the L^2 bound (9.91). Recall the L^2 bounds (8.21) on $\partial_sP_{k'}f_\sigma$. Using (9.91) (with $\partial_sP_{k'}f_\sigma$ always placed in L^2, notice that $2^{-2l}\leqslant 2^{m/7}$), it follows that

$$\sum_{p\geqslant-11m/21}2^{(1-50\delta)j}\|P_k\mathcal{C}_{l,p}[f^{\mu}_{j_1,k_1},f^{\beta}_{j_3,k_3},f^{\gamma}_{j_4,k_4}]\|_{L^2}\lesssim 2^{-3\delta^2m}.$$

Step 2. For (9.89) it remains to prove that

$$2^{(1-50\delta)j}\|Q_{jk}\mathcal{C}_{l,\leqslant-11m/21}[f^{\mu}_{j_1,k_1},f^{\beta}_{j_3,k_3},f^{\gamma}_{j_4,k_4}]\|_{L^2}\lesssim 2^{-3\delta^2m}. \tag{9.94}$$

Since $\max(j_1,j_3,j_4)\leqslant\frac{20}{21}m$, see (9.93), we have the pointwise approximate identity

$$\begin{aligned}P_k\mathcal{C}_{l,\leqslant-11m/21}&[f^{\mu}_{j_1,k_1},f^{\beta}_{j_3,k_3},f^{\gamma}_{j_4,k_4}]\\&=P_k\mathcal{C}_{l,\leqslant-11m/21}[A_{\geqslant\mathcal{D}_1,\gamma_0}f^{\mu}_{j_1,k_1},A_{\geqslant\mathcal{D}_1-10,\gamma_0}f^{\beta}_{j_3,k_3},A_{\geqslant\mathcal{D}_1-20,\gamma_0}f^{\gamma}_{j_4,k_4}]\\&\quad+P_k\mathcal{C}_{l,\leqslant-11m/21}[A_{<\mathcal{D}_1,\gamma_0}f^{\mu}_{j_1,k_1},A_{\leqslant\mathcal{D}_1+10,\gamma_0}f^{\beta}_{j_3,k_3},A_{\leqslant\mathcal{D}_1+20,\gamma_0}f^{\gamma}_{j_4,k_4}]\\&\quad+O(2^{-4m}),\end{aligned} \tag{9.95}$$

where $\mathcal{D}_1$ is the large constant used in §10. This is a consequence of Lemma 7.2 (i) and the observation that $|\nabla_{\eta,\sigma}\tilde{\Phi}(\xi,\eta,\sigma)|\gtrsim 1$ in the other cases. Letting

$$\begin{aligned}g_1&=A_{\geqslant\mathcal{D}_1,\gamma_0}f^{\mu}_{j_1,k_1}, & g_3&=A_{\geqslant\mathcal{D}_1-10,\gamma_0}f^{\beta}_{j_3,k_3}, & g_4&=A_{\geqslant\mathcal{D}_1-20,\gamma_0}f^{\gamma}_{j_4,k_4},\\h_1&=A_{<\mathcal{D}_1,\gamma_0}f^{\mu}_{j_1,k_1}, & h_3&=A_{<\mathcal{D}_1+10,\gamma_0}f^{\beta}_{j_3,k_3}, & h_4&=A_{<\mathcal{D}_1+20,\gamma_0}f^{\gamma}_{j_4,k_4},\end{aligned}$$

it remains to prove that

$$2^{(1-50\delta)j}\|Q_{jk}\mathcal{C}_{l,\leqslant -11m/21}[g_1,g_3,g_4]\|_{L^2}\lesssim 2^{-3\delta^2 m} \tag{9.96}$$

and

$$2^{(1-50\delta)j}\|Q_{jk}\mathcal{C}_{l,\leqslant -11m/21}[h_1,h_3,h_4]\|_{L^2}\lesssim 2^{-3\delta^2 m}. \tag{9.97}$$

Proof of (9.96). We use Lemma 10.6 (i). If $l\leqslant -4m/N_0'$, then $|\nabla_{\eta,\sigma}\tilde{\Phi}(\xi,\eta,\sigma)|\gtrsim 1$ in the support of the integral (due to (10.66)) and the contribution is negligible (due to Lemma 7.2 (i) and (9.93)). On the other hand, if

$$l\geqslant -\frac{4m}{N_0'}\quad\text{and}\quad j\leqslant \frac{2m}{3}+\max(j_1,j_3,j_4), \tag{9.98}$$

then we apply (9.91). The left-hand side of (9.96) is dominated by

$$C2^{(1-50\delta)j}2^m(1+2^{-2l})2^{-5m/3+8\delta^2 m}2^{-\max(j_1,j_3,j_4)(1-50\delta)}\lesssim 2^{-10\delta},$$

as we notice that $\max(k,k_1,k_2,k_3,k_4)\leqslant 20$. This suffices to prove (9.96) in this case.

Finally, if

$$l\geqslant -\frac{4m}{N_0'}\quad\text{and}\quad j\geqslant \frac{2m}{3}+\max(j_1,j_3,j_4), \tag{9.99}$$

then $\max(j_1,j_3,j_4)\leqslant \frac{1}{3}m+10\delta m$ and $j\geqslant\frac{2}{3}m$. We define the localized trilinear operators

$$\begin{aligned}&\mathcal{F}\{\mathcal{J}_{l,\leqslant p,\varkappa}[f,g,h]\}(\xi,s)\\ &\quad:=\int_{\mathbb{R}^2\times\mathbb{R}^2} e^{is\tilde{\Phi}(\xi,\eta,\sigma)}\hat{f}(\xi-\eta)2^{-l}\widetilde{\varphi}_l(\Phi_{+\mu\nu}(\xi,\eta))\varphi_{\leqslant p}(\tilde{\Phi}(\xi,\eta,\sigma))\\ &\qquad\quad\times\varphi(\varkappa^{-1}\nabla_{\eta,\sigma}\tilde{\Phi}(\xi,\eta,\sigma))\varphi_{k_2}(\eta)\mathfrak{m}_{\mu\nu}(\xi,\eta)\mathfrak{m}_{\nu\beta\gamma}(\eta,\sigma)\hat{g}(\eta-\sigma)\hat{h}(\sigma)\,d\sigma\,d\eta,\end{aligned} \tag{9.100}$$

which are similar to the trilinear operators defined in (9.86), with the additional cutoff factor in $\nabla_{\eta,\sigma}\tilde{\Phi}(\xi,\eta,\sigma)$ and $p=-\frac{11}{21}m$. Set $\varkappa:=2^{-m/2+\delta^2 m}$, and notice that

$$\|\mathcal{F}\{\mathcal{J}_{l,\leqslant -11m/21}[g_1,g_3,g_4]-\mathcal{J}_{l,\leqslant -11m/21,\varkappa}[g_1,g_3,g_4]\}\|_{L^\infty}\lesssim 2^{-6m},$$

as a consequence of Lemma 7.2 (i). Also, $|\nabla_\xi\tilde{\Phi}(\xi,\eta,\sigma)|\lesssim 2^{2p/3}\approx 2^{-22m/63}$ in the support of the integral defining $\mathcal{J}_{l,\leqslant -11m/21,\varkappa}[g_1,g_3,g_4]$, due to Lemma 10.6 (i). Thus, using the approximate-finite-speed-of-propagation argument (integration by parts in ξ),

$$\|Q_{jk}\mathcal{J}_{l,\leqslant -11m/21,\varkappa}[g_1,g_3,g_4]\|_{L^\infty}\lesssim 2^{-6m}.$$

The desired bound (9.96) follows in this case as well (in fact, one has rapid decay if (9.99) holds). □

Proof of (9.97). The desired estimate follows from (9.91) and the dispersive bounds (7.41)–(7.42) if $\max(j_1,j_3,j_4)\geqslant\frac{1}{3}m$ or if $j\leqslant\frac{2}{3}m$ or if $l\geqslant -10\delta m$. Assume that

$$\max(j_1,j_3,j_4)\leqslant\tfrac{1}{3}m,\quad j\geqslant\tfrac{2}{3}m,\quad\text{and}\quad l\leqslant -10\delta m. \tag{9.101}$$

As before, we may replace $\mathcal{J}_{l,\leqslant -11m/21}[h_1,h_3,h_4]$ by $\mathcal{J}_{l,\leqslant -11m/21,\varkappa}[h_1,h_3,h_4]$, at the expense of a small error, where $\varkappa=2^{-m/2+20\delta m}$. Moreover, $|\nabla_\xi\tilde{\Phi}(\xi,\eta,\sigma)|\lesssim\varkappa$ in the support of the integral defining $\mathcal{J}_{l,\leqslant -11m/21,\varkappa}[h_1,h_3,h_4]$, due to Lemma 10.6 (ii). The approximate-finite-speed-of-propagation argument (integration by parts in ξ) then gives rapid decay in the case when (9.101) holds. This completes the proof. □

9.7. The case of strongly resonant interactions II

In this subsection we prove Lemma 9.8. Let $\bar{k}:=\max(k,k_1,k_2,0)$. It suffices to prove the lemma in the case

$$k,k_1,k_2\in[-\bar{k}-20,\bar{k}],\quad j\leqslant m+3\mathcal{D}+\frac{\bar{k}}{2},\quad \bar{k}\leqslant\frac{7m}{6N_0'},\quad\text{and}\quad l_-<l\leqslant -\frac{m}{14}. \tag{9.102}$$

Indeed, we may assume that $k,k_1,k_2\geqslant -\bar{k}-20$, since otherwise the operator is trivial (due to (10.6)). Moreover, if $\max(k_1,k_2)\geqslant 7m/6N_0'-10$, then the $L^2\times L^\infty$ argument (with Lemma 7.4) easily gives the desired conclusion due to the assumption (9.6).

We define (compare with the definition of the operators $T_{m,l}$ in (9.18))

$$\widehat{T^{\|}_{m,l}[f,g]}(\xi) = \int_{\mathbb{R}} q_m(s)\int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)}\varphi(\varkappa_\theta^{-1}\Theta(\xi,\eta))\varphi_l(\Phi(\xi,\eta))m_0(\xi,\eta)\hat{f}(\xi-\eta,s)\hat{g}(\eta,s)\,d\eta\,ds,$$

where $\varkappa_\theta:=2^{-m/2+6\bar{k}+\delta^2 m}$. Let $T^{\perp}_{m,l}=T_{m,l}-T^{\|}_{m,l}$, and define $\mathcal{A}^{\|}_{m,l}$ and $\mathcal{B}^{\|}_{m,l}$ similarly, by inserting the factor $\varphi(\varkappa_\theta^{-1}\Theta(\xi,\eta))$ in the integrals in (9.19). We notice that

$$\begin{aligned}T^{\|}_{m,l}[P_{k_1}f^\mu,P_{k_2}f^\nu]&=i\mathcal{A}^{\|}_{m,l}[P_{k_1}f^\mu,P_{k_2}f^\nu]+i\mathcal{B}^{\|}_{m,l}[P_{k_1}\partial_s f^\mu,P_{k_2}f^\nu]\\&\quad+i\mathcal{B}^{\|}_{m,l}[P_{k_1}f^\mu,P_{k_2}\partial_s f^\nu].\end{aligned}$$

It remains to prove that, for any j_1 and j_2,

$$2^{(1-50\delta)j}\|Q_{jk}T^{\perp}_{m,l}[f^\mu_{j_1,k_1},f^\nu_{j_2,k_2}]\|_{L^2}\lesssim 2^{-3\delta^2 m}, \tag{9.103}$$

$$\|Q_{jk}\mathcal{A}^{\|}_{m,l}[f^\mu_{j_1,k_1},f^\nu_{j_2,k_2}]\|_{B_j}\lesssim 2^{-3\delta^2 m}, \tag{9.104}$$

and

$$\|Q_{jk}\mathcal{B}^{\|}_{m,l}[f^\mu_{j_1,k_1},\partial_s P_{k_2}f^\nu]\|_{B_j}\lesssim 2^{-3\delta^2 m}. \tag{9.105}$$

Proof of (9.103). We may assume that $\min(j_1,j_2)\geqslant m-2\bar{k}-\delta^2 m$, otherwise the conclusion follows from Lemma 7.3. We decompose

$$f^{\mu}_{j_1,k_1}=\sum_{n_1=0}^{j_1+1} f_{j_1,k_1,n_1} \quad \text{and} \quad f^{\nu}_{j_2,k_2}=\sum_{n_2=0}^{j_j+1} f_{j_2,k_2,n_2},$$

and estimate, using Lemma 10.5 and (7.35),

$$\begin{aligned}
&\|P_k T^{\perp}_{m,l}[f_{j_1,k_1,n_1}, f_{j_2,k_2,n_2}]\|_{L^2}\\
&\quad\lesssim 2^{2\bar{k}}2^m 2^{l/2-n_1/2-n_2/2}\Big\|\sup_{\theta\in\mathbb{S}^1}|\hat{f}_{j_1,k_1,n_1}(r\theta)|\Big\|_{L^2(r\,dr)}\Big\|\sup_{\theta\in\mathbb{S}^1}|\hat{f}_{j_2,k_2,n_2}(r\theta)|\Big\|_{L^2(r\,dr)}\\
&\quad\lesssim 2^{2\bar{k}}2^m 2^{l/2}2^{6\delta^2 m}2^{-j_1+51\delta j_1}2^{-j_2+51\delta j_2}.
\end{aligned}$$

Therefore, using also (9.102), the left-hand side of (9.103) is dominated by

$$2^{(1-50\delta)j}2^{6\delta^2 m}2^{2\bar{k}}2^m 2^{l/2}2^{-j_1+51\delta j_1}2^{-j_2+51\delta j_2}\lesssim 2^{8\bar{k}}2^{l/2}2^{54\delta m}.$$

This suffices to prove the desired bound, since

$$2^{l/2}\lesssim 2^{-m/28} \quad \text{and} \quad 2^{8\bar{k}}2^{54\delta m}\lesssim 2^{64\delta m}\lesssim 2^{m/30}. \qquad \square$$

Proof of (9.104). In view of Lemma 9.6, it suffices to prove that

$$2^{(1-50\delta)j}\|Q_{jk}\mathcal{A}^{\perp}_{m,l}[f^{\mu}_{j_1,k_1}, f^{\nu}_{j_2,k_2}]\|_{L^2}\lesssim 2^{-3\delta^2 m}.$$

This is similar to the proof of (9.103) above, using Lemma 10.5 and (7.35). $\square$

Proof of (9.105). This is the more difficult estimate, where we need to use the more precise information in Lemma 8.2. We may assume $j_1\leqslant 3m$, since in the case $j_1\geqslant 3m$ we can simply estimate $\|\widehat{f^{\mu}_{j_1,k_1}}\|_{L^1}\lesssim 2^{-j_1+51\delta j_1}$ (see (7.36)) and the desired estimate easily follows. We decompose $\partial_s P_{k_2} f^{\nu}$ as in (8.8), and then we decompose

$$A^{a_3,\alpha_3;a_4,\alpha_4}_{k_2;k_3,j_3;k_4,j_4}=\sum_{i=1}^{3} A^{a_3,\alpha_3;a_4,\alpha_4;[i]}_{k_2;k_3,j_3;k_4,j_4}$$

as in (8.35). Note that, as $k_2\geqslant -3m/2N_0'$ (see (9.102)), it follows from Lemma 8.1 (ii) (2) that $\min(k_2,k_3,k_4)\geqslant -2m/N_0'$, so Lemma 8.2 applies. It remains to prove that

$$\|Q_{jk}\mathcal{B}^{\|}_{m,l}[f^{\mu}_{j_1,k_1}, P_{k_2}E^{a_2,\alpha_2}_{\nu}]\|_{B_j}\lesssim 2^{-4\delta^2 m} \tag{9.106}$$

and, for any $[(k_3,j_3),(k_4,j_4)]\in X_{m,k_2}$, $i\in\{1,2,3\}$,

$$\|Q_{jk}\mathcal{B}^{\|}_{m,l}[f^{\mu}_{j_1,k_1}, A^{a_3,\alpha_3;a_4,\alpha_4;[i]}_{k_2;k_3,j_3;k_4,j_4}]\|_{B_j}\lesssim 2^{-4\delta^2 m}. \tag{9.107}$$

These bounds follow from Lemmas 9.10–9.12 below. Recall the definition

$$\begin{aligned}
\widehat{\mathcal{B}^{\|}_{m,l}[f,g]}(\xi)=\int_{\mathbb{R}} q_m(s)\int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)}\varphi(\varkappa_\theta^{-1}\Theta(\xi,\eta))2^{-l}& \qquad (9.108)\\
\times\widetilde{\varphi}_l(\Phi(\xi,\eta))m_0(\xi,\eta)\hat{f}(\xi-\eta,s)\hat{g}(\eta,s)\,d\eta\,ds.& \qquad \square
\end{aligned}$$

LEMMA 9.10. *Assume that* (9.102) *holds and* $\varkappa_\theta = 2^{-m/2+6\bar{k}+\delta^2 m}$. *Then,*

$$\|Q_{jk}\mathcal{B}^{\|}_{m,l}[f^{\mu}_{j_1,k_1}, h]\|_{B_j} \lesssim 2^{-4\delta^2 m}, \tag{9.109}$$

provided that, for any $s \in I_m$,

$$h(s) = P_{[k_2-2,k_2+2]}h(s), \quad \textit{with } \|h(s)\|_{L^2} \lesssim 2^{-3m/2+35\delta m-22\bar{k}}. \tag{9.110}$$

Proof. The lemma is slightly stronger (with a weaker assumption on h) than we need to prove (9.106), since we intend to apply it in some cases in the proof of (9.107) as well. We would like to use Schur's lemma and Proposition 10.4 (iii). For this, we need to further decompose the operator $\mathcal{B}^{\|}_{m,l}$. For $p, q \in \mathbb{Z}$ we define the operators $\mathcal{B}'_{p,q}$ by

$$\begin{aligned}\widehat{\mathcal{B}'_{p,q}[f,g]}(\xi) := \int_{\mathbb{R}} q_m(s) \int_{\mathbb{R}^2} e^{is\Phi(\xi,\eta)} \varphi(\varkappa_\theta^{-1}\Theta(\xi,\eta)) 2^{-l} \widetilde{\varphi}_l(\Phi(\xi,\eta)) \\ \times \varphi_p(\nabla_\xi \Phi(\xi,\eta)) \varphi_q(\nabla_\eta \Phi(\xi,\eta)) m_0(\xi,\eta) \hat{f}(\xi-\eta,s)\hat{g}(\eta,s)\, d\eta\, ds.\end{aligned} \tag{9.111}$$

Let $H_{p,q} := P_k \mathcal{B}'_{p,q}[f^{\mu}_{j_1,k_1}, h]$. Using the bounds $\|\widehat{f^{\mu}_{j_1,k_1}}\|_{L^\infty} \lesssim 2^{2\delta j_1} 2^{5\delta^2 m} 2^{51\delta\bar{k}} \lesssim 2^{7\delta m}$ (see (7.37)), Proposition 10.4 (iii), and (9.110), we estimate

$$\begin{aligned}\|H_{p,q}\|_{L^2} &\lesssim 2^{2\bar{k}} 2^m (2^{10\bar{k}} 2^l \varkappa_\theta 2^{-p_-/2} 2^{-q_-/2} 2^{\delta^2 m}) 2^{-l} \sup_{s\in I_m} \|\widehat{f^{\mu}_{j_1,k_1}}(s)\|_{L^\infty} \|h(s)\|_{L^2} \\ &\lesssim 2^{-4\bar{k}} 2^{-p_-/2} 2^{-q_-/2} 2^{-m+43\delta m},\end{aligned} \tag{9.112}$$

where $x_- = \min(x, 0)$. In particular,

$$\sum_{\substack{p \geqslant -4\delta m \\ q \geqslant -4\delta m}} 2^{j-50\delta j} \|P_k \mathcal{B}'_{p,q}[f^{\mu}_{j_1,k_1}, h]\|_{L^2} \lesssim 2^{-\delta m}. \tag{9.113}$$

We now show that

$$\sum_{\substack{p \leqslant -4\delta m \\ q \in \mathbb{Z}}} 2^{j-50\delta j} \|P_k \mathcal{B}'_{p,q}[f^{\mu}_{j_1,k_1}, h]\|_{L^2} \lesssim 2^{-\delta m}. \tag{9.114}$$

For this, we now notice that, if $p \leqslant -4\delta m$, then $P_k \mathcal{B}'_{p,q}[f^{\mu}_{j_1,k_1}, h]$ is non-trivial only when $|\eta|$ is close to γ_1, and $|\xi|$ and $|\xi-\eta|$ are close to $\frac{1}{2}\gamma_1$ (as a consequence of Proposition 10.2 (iii)). In particular, $2^{\bar{k}} \lesssim 1$, $2^q \approx 1$, and $|\widehat{f^{\mu}_{j_1,k_1}}(\xi-\eta, s)| \lesssim 2^{2\delta^2 m} 2^{-j_1/2+51\delta j_1}$ in the support of the integral. Therefore, using also (10.44), we have the stronger estimate (compare with (9.112))

$$\begin{aligned}\|H_{p,q}\|_{L^2} &\lesssim 2^{m-l} 2^l \varkappa_\theta \min(2^{-p/2}, 2^{p/2-l/2}) 2^{\delta^2 m} \sup_{s\in I_m} \|\widehat{f^{\mu}_{j_1,k_1}}(s)\|_{L^\infty} \|h(s)\|_{L^2} \\ &\lesssim 2^{-j_1/2+51\delta j_1} \min(2^{-p/2}, 2^{p/2-l/2}) 2^{-m+36\delta m}.\end{aligned} \tag{9.115}$$

The desired bound (9.114) follows if $j_1\geqslant j-\delta m$ or if $j\leqslant\frac{3}{4}3m-5\delta m$, since

$$\min(2^{-p/2},2^{p/2-l/2})\lesssim 2^{-l/4}\lesssim 2^{m/4}.$$

On the other hand, if

$$j_1\leqslant j-\delta m \quad\text{and}\quad j\geqslant \tfrac{3}{4}m-5\delta m,$$

then the sum over $p\geqslant(j-m)-10\delta m$ in (9.114) can also be estimated using (9.115). The remaining sum over $p\leqslant(j-m)-10\delta m$ is negligible using the approximate-finite-speed-of-propagation argument (integration by parts in ξ). This completes the proof of (9.114).

Finally, we show that

$$\sum_{\substack{p\in\mathbb{Z}\\ q\leqslant -4\delta m}} \|Q_{jk}\mathcal{B}'_{p,q}[f^{\mu}_{j_1,k_1},h]\|_{B_j}\lesssim 2^{-\delta m}. \tag{9.116}$$

As before, we now notice that, if $q\leqslant-4\delta m$, then $P_k\mathcal{B}'_{p,q}[f^{\mu}_{j_1,k_1},h]$ is non-trivial only when $|\xi|$ is close to γ_1, and $|\eta|$ and $|\xi-\eta|$ are close to $\frac{1}{2}\gamma_1$ (as a consequence of Proposition 10.2 (iii)). In particular $2^{\bar{k}}\lesssim 1$, $2^p\approx 1$, and we have the stronger estimate (compare with (9.115))

$$\|H_{p,q}\|_{L^2}\lesssim 2^{-j_1/2+51\delta j_1}\min(2^{-q/2},2^{q/2-l/2})2^{-m+36\delta m}\lesssim \frac{2^{q/2}}{2^q+2^{l/2}}2^{-m+36\delta m}. \tag{9.117}$$

Moreover, since $|\Phi(\xi,\eta)|\lesssim 2^l$ and $|\nabla_\eta\Phi(\xi,\eta)|\lesssim 2^q$, the function $\widehat{H}_{p,q}$ is supported in the set $\{\xi:\big||\xi|-\gamma_1\big|\lesssim 2^l+2^{2q}\}$ (see (10.21)). The main observation is that the B_j norm for functions supported in such a set carries an additional small factor. More precisely, after localization to a 2^j-ball in the physical space, the function $\mathcal{F}\{Q_{jk}\mathcal{B}'_{p,q}[f^{\mu}_{j_1,k_1},h]\}(\xi)$ is supported in the set $\{\xi:\big||\xi|-\gamma_1\big|\lesssim 2^l+2^{2q}+2^{-j+2\delta m}\}$, up to a negligible error. Therefore, using (9.117),

$$\begin{aligned}\|Q_{jk}\mathcal{B}'_{p,q}[f^{\mu}_{j_1,k_1},P_{k_2}E^{a_3}_{\nu}]\|_{B_j}&\lesssim 2^{j-50\delta j}(2^l+2^{2q}+2^{-j+2\delta m})^{1/2-49\delta}\|H_{p,q}\|_{L^2}\\ &\lesssim 2^{j-50\delta j}2^{-m+36\delta m}(2^{l/2}+2^q+2^{-j/2+\delta m})\frac{2^{q/2-100\delta q}}{2^q+2^{l/2}}\\ &\lesssim 2^{q/8}2^{-4\delta m}.\end{aligned}$$

So, (9.116) follows. The bound (9.109) follows from (9.113), (9.114), and (9.116). □

Lemma 9.11. *Assume that* (9.102) *holds and* $\varkappa_\theta=2^{-m/2+6\bar{k}+\delta^2 m}$. *Then,*

$$\|Q_{jk}\mathcal{B}^{\|}_{m,l}[f^{\mu}_{j_1,k_1},A^{a_3,\alpha_3;a_4,\alpha_4;[1]}_{k_2;k_3,j_3;k_4,j_4}]\|_{B_j}\lesssim 2^{-4\delta^2 m}. \tag{9.118}$$

Proof. Notice that $A^{a_3,\alpha_3;a_4,\alpha_4;[1]}_{k_2;k_3,j_3;k_4,j_4}$ is supported in the set $\big||\eta|-\gamma_1\big|\leqslant 2^{-\mathcal{D}}$. Using also the conditions $\Phi(\xi,\eta)\lesssim 2^l$ and $\Theta(\xi,\eta)\lesssim\varkappa_\theta$, we have

$$\big||\eta|-\gamma_1\big|\leqslant 2^{-\mathcal{D}},\quad |\xi|,|\xi-\eta|\in[2^{-50},2^{50}],\quad \min\big(\big||\xi|-\gamma_1\big|,\big||\xi-\eta|-\gamma_1\big|\big)\geqslant 2^{-50}\tag{9.119}$$

in the support of the integral defining $\mathcal{F}\{P_k\mathcal{B}^{\|}_{m,l}[f^{\mu}_{j_1,k_1},G^{[1]}](\xi)\}$, where

$$G^{[1]}=A^{a_3,\alpha_3;a_4,\alpha_4;[1]}_{k_2;k_3,j_3;k_4,j_4}.$$

Case 1. Assume first that

$$\max(j_3,j_4)\geqslant \tfrac{1}{2}m.\tag{9.120}$$

In this case,

$$\|G^{[1]}\|_{L^2}\lesssim 2^{-3m/2+30\delta m}$$

(see (8.37)), and the conclusion follows from Lemma 9.10.

Case 2. Assume now that

$$\max(j_3,j_4)\leqslant \tfrac{1}{2}m\quad\text{and}\quad j_1\geqslant\tfrac{1}{2}m.\tag{9.121}$$

The bound (9.118) again follows by the same argument as in the proof of (9.109) above. In this case, $\|\widehat{G^{[1]}}(s)\|_{L^\infty}\lesssim 2^{-m+4\delta m}$ (due to (8.42) and (8.43)) and

$$\|\mathcal{F}\{A_{\leqslant 0,\gamma_1}f^{\mu}_{j_1,k_1}\}(s)\|_{L^2}\lesssim 2^{2\delta^2 m}2^{-j_1+50\delta j_1}$$

(see (7.37)). We make the change of variables $\eta\mapsto\xi-\eta$, define $\Phi'(\xi,\eta)=\Phi(\xi,\xi-\eta)$, and define the operators $\mathcal{B}''_{p,q}$ as in (9.111), by inserting cutoff factors $\varphi_p((\nabla_\xi\Phi')(\xi,\eta))$ and $\varphi_q((\nabla_\eta\Phi')(\xi,\eta))$. In this case, we notice that we may assume both $p\geqslant-\mathcal{D}$ and $q\geqslant-\mathcal{D}$. Indeed, we have $|\Phi'(\xi,\eta)|\leqslant 2^{-\mathcal{D}}$ and $\big||\xi-\eta|-\gamma_1\big|\leqslant 2^{-\mathcal{D}}$, so

$$|(\nabla_\xi\Phi')(\xi,\eta)|\gtrsim 1\quad\text{and}\quad|(\nabla_\eta\Phi')(\xi,\eta)|\gtrsim 1$$

in the support of the integral (in view of Proposition 10.2 (iii)). Then we estimate, using (10.42),

$$\|P_k\mathcal{B}''_{p,q}[A_{\leqslant 0,\gamma_1}f^{\mu}_{j_1,k_1},G^{[1]}]\|_{L^2}\lesssim 2^{-j_1+50\delta j_1}2^{-m/2+5\delta m}.$$

The bound (9.118) follows by summation over p and q.

Case 3. Assume now that

$$\max(j_1,j_3,j_4)\leqslant\tfrac{1}{2}m\quad\text{and}\quad j\leqslant\tfrac{1}{2}m+10\delta m.\tag{9.122}$$

We use the bounds $\|\widehat{G^{[1]}}(s)\|_{L^\infty} \lesssim 2^{-m+4\delta m}$ (see (8.42)–(8.43)) and $\|\widehat{f^{\mu}_{j_1,k_1}}(s)\|_{L^\infty} \lesssim 2^{3\delta m}$. Moreover, $|\nabla_\eta \Phi(\xi,\eta)| \gtrsim 1$ in the support of the integral. Therefore, using the first bound in (10.42),

$$\begin{aligned}\|\mathcal{F}\{P_k\mathcal{B}^{\|}_{m,l}[f^{\mu}_{j_1,k_1}, G^{[1]}]\}\|_{L^\infty} &\lesssim 2^{m-l}\varkappa_\theta 2^l 2^{\delta^2 m} \sup_{s\in I_m} \|\widehat{G^{[1]}}(s)\|_{L^\infty}\, \|\widehat{f^{\mu}_{j_1,k_1}}(s)\|_{L^\infty}\\ &\lesssim 2^{-m/2+8\delta m}.\end{aligned}$$

The desired bound (9.118) follows when $j \leqslant \frac{1}{2}m + 10\delta m$.

Case 4. Finally, assume that

$$\max(j_1, j_3, j_4) \leqslant \tfrac{1}{2}m \quad \text{and} \quad j \geqslant \tfrac{1}{2}m + 10\delta m. \tag{9.123}$$

We examine formula (9.108), decompose $G^{[1]}$ as in (8.42) and notice that the contribution of the error term is easy to estimate. To estimate the main term, we define the modified phase

$$\mathfrak{p}(\xi,\eta) := \Phi_{+\mu\nu}(\xi,\eta) + \Lambda_\nu(\eta) - 2\Lambda_\nu\big(\tfrac{1}{2}\eta\big) = \Lambda(\xi) - \Lambda_\mu(\xi-\eta) - 2\Lambda_\nu\big(\tfrac{1}{2}\eta\big). \tag{9.124}$$

For $r\in\mathbb{Z}$ we define the functions $\mathcal{G}_r = \mathcal{G}_{r,m,l,j,j_1}$ by

$$\begin{aligned}\widehat{\mathcal{G}_r}(\xi) := \int_{\mathbb{R}} q_m(s) \int_{\mathbb{R}^2} & e^{is\mathfrak{p}(\xi,\eta)} \varphi(\varkappa_\theta^{-1}\Theta(\xi,\eta)) 2^{-l} \widetilde{\varphi}_l(\Phi(\xi,\eta)) m_0(\xi,\eta)\\ &\times \varphi_r(\nabla_\eta \mathfrak{p}(\xi,\eta)) \widehat{f^{\mu}_{j_1,k_1}}(\xi-\eta,s) g^{[1]}(\eta,s) \varphi(2^{3\delta m}(|\eta|-\gamma_1))\, d\eta\, ds.\end{aligned} \tag{9.125}$$

Notice that the functions $\mathcal{G}_r$ are negligible for, say, $r \leqslant -10m$. It suffices to prove that

$$2^{j-50\delta j} \|Q_{jk}\mathcal{G}_r\|_{L^2} \lesssim 2^{-5\delta^2 m} \quad \text{for any } r\in\mathbb{Z}. \tag{9.126}$$

We first notice that $\|P_k\mathcal{G}_r\|_{L^2} \lesssim 2^{-4m}$ if $r \geqslant \max\big(\delta^2 m - l - m, 6\delta m - \frac{1}{2}m\big)$, in view of Lemma 7.2 (i). In particular, we may assume that $r \leqslant -\mathcal{D}$. In this case, the functions $\mathcal{G}_r$ are non-trivial only when $-\mu = \nu = +$ and ξ is close to $\frac{1}{2}\eta$. Therefore,

$$\mathfrak{p}(\xi,\eta) = \Lambda(\xi) + \Lambda(\eta-\xi) - 2\Lambda\big(\tfrac{1}{2}\eta\big),$$

and we have, in the support of the integral defining $\widehat{\mathcal{G}_r}(\xi)$,

$$\begin{aligned}|\nabla_\eta \mathfrak{p}(\xi,\eta)| &\approx \big|\xi - \tfrac{1}{2}\eta\big| \approx |\nabla_\xi \mathfrak{p}(\xi,\eta)| \approx |\nabla_\xi \Phi(\xi,\eta)| \approx 2^r,\\ |\mathfrak{p}(\xi,\eta)| &\approx \big|\xi - \tfrac{1}{2}\eta\big|^2 \approx 2^{2r},\\ \big||\eta| - \gamma_1\big| &\approx \big|\Lambda(\eta) - 2\Lambda\big(\tfrac{1}{2}\eta\big)\big| \lesssim |\Phi(\xi,\eta)| + |\mathfrak{p}(\xi,\eta)| \lesssim 2^l + 2^{2r},\\ \big||\xi| - \tfrac{1}{2}\gamma_1\big| &\lesssim 2^l + 2^r.\end{aligned} \tag{9.127}$$

The finite-speed-of-propagation argument (integration by parts in ξ) shows that

$$\|Q_{jk}\mathcal{G}_r\|_{L^2}\lesssim 2^{-4m}$$

if $j\geqslant 3\delta^2 m+\max(m+r,-r)$. To summarize, it remains to prove that

$$(2^{m+r}+2^{-r})^{1-50\delta}\|P_k\mathcal{G}_r\|_{L^2}\lesssim 2^{-\delta m},\quad \text{if } r\leqslant\max\big(\delta^2 m-l-m, 6\delta m-\tfrac{1}{2}m\big). \tag{9.128}$$

For ξ fixed, the variable η satisfies three restrictions:

$$|\eta\cdot\xi^\perp|\lesssim\varkappa_\theta,\quad |\Phi(\xi,\eta)|\lesssim 2^l,\quad\text{and}\quad |\eta-2\xi|\lesssim 2^r.$$

Therefore, using also (8.42) and (8.43), we have the pointwise bound

$$\begin{aligned}|\widehat{\mathcal{G}}_r(\xi)|&\lesssim 2^{5\delta^2 m}2^{m-l}\min(2^r,2^{-m/2})\min(2^r,2^l)\sup_{s\in I_m}\|\widehat{f^{\mu}_{j_1,k_1}}(s)\|_{L^\infty}\|g^{[1]}(s)\|_{L^\infty}\\&\lesssim 2^{8\delta m}\min(2^r,2^{-m/2})\min(2^{r-l},1).\end{aligned}\tag{9.129}$$

The desired bound (9.128) follows, using also the support assumption $\big||\xi|-\frac{1}{2}\gamma_1\big|\lesssim 2^l+2^r$ in (9.127), if $r\leqslant-\frac{1}{2}m$ or if $r\in\big[-\frac{1}{2}m,-\frac{1}{3}m\big]$.

It remains to prove (9.128) when $-\frac{1}{3}m\leqslant r\leqslant -l-m+\delta^2 m$. The main observation in this case is that $|\mathfrak{p}(\xi,\eta)|\approx 2^{2r}$ is large enough to be able to integrate by parts in s. It follows that

$$\begin{aligned}|\widehat{\mathcal{G}}_r(\xi)|\lesssim\int_{\mathbb{R}}\int_{\mathbb{R}^2}2^{-2r}|\varphi(\varkappa_\theta^{-1}\Theta(\xi,\eta))2^{-l}\widetilde{\varphi}_l(\Phi(\xi,\eta))\varphi_r(\nabla_\eta\mathfrak{p}(\xi,\eta))\varphi(2^{3\delta m}(|\eta|-\gamma_1))|\\ \times|\partial_s(\widehat{f^{\mu}_{j_1,k_1}}(\xi-\eta,s)g^{[1]}(\eta,s)q_m(s))|\,d\eta\,ds.\end{aligned}$$

For ξ fixed, the integral is supported in an $O(\varkappa_\theta\times 2^l)$ rectangle centered at $\eta=2\xi$. In this support, we have the bounds see Lemma 8.2 (ii) and (iii),

$$\begin{gathered}\|\widehat{f^{\mu}_{j_1,k_1}}(s)\|_{L^\infty}\lesssim 2^{\delta^2 m},\qquad \|g^{[1]}(s)\|_{L^\infty}\lesssim 2^{-m+4\delta m},\qquad \|\partial_s g^{[1]}(s)\|_{L^\infty}\lesssim 2^{-2m+18\delta m},\\ \partial_s f^{\mu}_{j_1,k_1}=h_2+h_\infty,\qquad \|h_2(s)\|_{L^2}\lesssim 2^{-3m/2+5\delta m},\qquad \|\hat{h}_\infty(s)\|_{L^\infty}\lesssim 2^{-m+15\delta m}.\end{gathered}$$

The integrals that do not contain the function h_2 can all be estimated pointwise, as in (9.129) by $C2^{-2r}2^{-l}2^{-m+20\delta m}(2^l\varkappa_\theta)\lesssim 2^{-2r}2^{-3m/2+21\delta m}$. The integral that contains the function h_2 can be estimated pointwise, using Hölder's inequality, by

$$C2^{-2r}2^{-l}2^{-3m/2+10\delta m}(2^l\varkappa_\theta)^{1/2}\lesssim 2^{-2r}2^{-l/2}2^{-7m/4+11\delta m}\lesssim 2^{-2r}2^{-5m/4+11\delta m}.$$

Therefore, using also the support assumption $\big||\xi|-\frac{1}{2}\gamma_1\big|\lesssim 2^r$ in (9.127), and recalling that $r\geqslant-\frac{1}{3}m$ and $l\leqslant-\frac{1}{2}m$, we have

$$2^{m+r}\|P_k\mathcal{G}_r\|_{L^2}\lesssim 2^{-r/2}2^{-m/4+11\delta m}.$$

This suffices to prove (9.128), which completes the proof of the lemma. □

LEMMA 9.12. *With the same notation as in Lemma 9.11, and assuming (9.102), we have*

$$\|Q_{jk}\mathcal{B}^{\|}_{m,l}[f^{\mu}_{j_1,k_1},A^{a_3,\alpha_3;a_4,\alpha_4;[2]}_{k_2;k_3,j_3;k_4,j_4}]\|_{B_j}\lesssim 2^{-4\delta^2 m}. \tag{9.130}$$

Proof. The main observation here is that, since

$$|\Phi_{+\mu\nu}(\xi,\eta)|\lesssim 2^{l} \quad \text{and} \quad |\Phi_{\nu\beta\gamma}(\eta,\sigma)|\gtrsim 2^{-10\delta m},$$

we have $|\tilde{\Phi}(\xi,\eta,\sigma)|\gtrsim 2^{-10\delta m}$, and thus we can integrate by parts in s once more. Before this, however, we notice that we may assume that

$$k_3,k_4\in\Big[-\frac{2m}{N_0'},\frac{m}{N_0'}\Big] \quad \text{and} \quad \min(j_3,j_4)\leqslant m-4\delta m. \tag{9.131}$$

Indeed, we first use Lemma 8.1 (ii) (2), (3). Moreover, if

$$\min(j_3,j_4)\geqslant m-4\delta m \quad \text{or} \quad \max(k_3,k_4)\geqslant\frac{m}{N_0'},$$

then we would have

$$\|A^{a_3,\alpha_3;a_4,\alpha_4;[2]}_{k_2;k_3,j_3;k_4,j_4}\|_{L^2}\lesssim 2^{-3m/2+8\delta m}$$

(by the same argument as in the proof of (8.31) or an $L^2\times L^\infty$ estimate), and the desired bound would follow from Lemma 9.10.

Step 1. For $r\in\mathbb{Z}$ we define (compare with (9.86)) the trilinear operators $\mathcal{J}^{[2]}_{l,r}$ by

$$\begin{aligned}\mathcal{F}\{&\mathcal{J}^{[2]}_{l,r}[f,g,h]\}(\xi,s)\\ &:=\int_{\mathbb{R}^2\times\mathbb{R}^2}e^{is\tilde{\Phi}(\xi,\eta,\sigma)}\hat{f}(\xi-\eta)\varphi(\varkappa_\theta^{-1}\Theta(\xi,\eta))2^{-l}\widetilde{\varphi}_l(\Phi_{+\mu\nu}(\xi,\eta))\varphi_r(\tilde{\Phi}(\xi,\eta,\sigma))\\ &\qquad\times\chi^{[2]}(\eta,\sigma)\varphi_{k_2}(\eta)\mathfrak{m}_{\mu\nu}(\xi,\eta)\mathfrak{m}_{\nu\beta\gamma}(\eta,\sigma)\hat{g}(\eta-\sigma)\hat{h}(\sigma)\,d\sigma\,d\eta.\end{aligned} \tag{9.132}$$

Let

$$\mathcal{C}^{[2]}_{l,r}[f,g,h]:=\int_{\mathbb{R}}q_m(s)\mathcal{J}^{[2]}_{l,r}[f,g,h](s)\,ds, \tag{9.133}$$

and notice that

$$\mathcal{B}^{\|}_{m,l}[f^{\mu}_{j_1,k_1},A^{b_1,b_2,b_3,[2]}_{k_2;k_3,j_3;k_4,j_4}]=\sum_{r\geqslant-11\delta m}\mathcal{C}^{[2]}_{l,r}[f^{\mu}_{j_1,k_1},f^{\beta}_{j_3,k_3},f^{\gamma}_{j_4,k_4}].$$

We integrate by parts in s to rewrite

$$\begin{aligned}&\mathcal{C}^{[2]}_{l,r}[f^{\mu}_{j_1,k_1},f^{\beta}_{j_3,k_3},f^{\gamma}_{j_4,k_4}]\\ &\quad=i2^{-r}\bigg(\int_{\mathbb{R}}q_m'(s)\widetilde{\mathcal{J}^{[2]}_{l,r}}[f^{\mu}_{j_1,k_1},f^{\beta}_{j_3,k_3},f^{\gamma}_{j_4,k_4}](s)\,ds+\widetilde{\mathcal{C}^{[2]}_{l,r}}[\partial_s f^{\mu}_{j_1,k_1},f^{\beta}_{j_3,k_3},f^{\gamma}_{j_4,k_4}]\\ &\qquad\qquad+\widetilde{\mathcal{C}^{[2]}_{l,r}}[f^{\mu}_{j_1,k_1},\partial_s f^{\beta}_{j_3,k_3},f^{\gamma}_{j_4,k_4}]+\widetilde{\mathcal{C}^{[2]}_{l,r}}[f^{\mu}_{j_1,k_1},f^{\beta}_{j_3,k_3},\partial_s f^{\gamma}_{j_4,k_4}]\bigg),\end{aligned}$$

where the operators $\widetilde{\mathcal{J}_{l,r}^{[2]}}$ and $\widetilde{\mathcal{C}_{l,r}^{[2]}}$ are defined in the same way as the operators $\mathcal{J}_{l,r}^{[2]}$ and $\mathcal{C}_{l,r}^{[2]}$, but with $\varphi_p(\tilde{\Phi}(\xi,\eta,\sigma))$ replaced by $\widetilde{\varphi}_p(\tilde{\Phi}(\xi,\eta,\sigma))$, where $\widetilde{\varphi}_p(x)=2^p x^{-1}\varphi_p(x)$ (see formula (9.132)). It suffices to prove that, for any $s\in I_m$ and $r\geqslant -11\delta m$,

$$2^{j-50\delta j}\|Q_{jk}\widetilde{\mathcal{J}_{l,r}^{[2]}}[f,g,h]\|_{L^2}\lesssim 2^{-12\delta m}, \tag{9.134}$$

where one of the following hold:

$$\begin{aligned}
[f,g,h]&=[f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}](s),\\
[f,g,h]&=[2^m\partial_s f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}](s),\\
[f,g,h]&=[f^{\mu}_{j_1,k_1}, 2^m\partial_s f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}](s),\\
[f,g,h]&=[f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, 2^m\partial_s f^{\gamma}_{j_4,k_4}](s).
\end{aligned}$$

Step 2. As in the proof of Lemma 7.4, the function $\widetilde{\varphi}_r(\tilde{\Phi}(\xi,\eta,\sigma))$ can be incorporated with the phase $e^{is\tilde{\Phi}(\xi,\eta,\sigma)}$, using formula (7.30) and the fact that $2^{-r}\leqslant 2^{11\delta m}$. Then, we integrate in the variable σ and denote by H_1, H_2, and H_3 the resulting functions:

$$\begin{aligned}
H_1&:=I^{[2]}[f^{\beta}_{j_3,k_3}(s), f^{\gamma}_{j_4,k_4}(s)],\\
H_2&:=I^{[2]}[\partial_s f^{\beta}_{j_3,k_3}(s), f^{\gamma}_{j_4,k_4}(s)],\\
H_3&:=I^{[2]}[f^{\beta}_{j_3,k_3}(s), \partial_s f^{\gamma}_{j_4,k_4}(s)],\\
\mathcal{F}\{I^{[2]}[g,h]\}(\eta)&:=\int_{\mathbb{R}^2} e^{i(s+\lambda)\Phi_{\nu\beta\gamma}(\eta,\sigma)}\chi^{[2]}(\eta,\sigma)\varphi_{k_2}(\eta)\mathfrak{m}_{\nu\beta\gamma}(\eta,\sigma)\hat{g}(\eta-\sigma)\hat{h}(\sigma)\,d\sigma.
\end{aligned}$$

We claim that, for $|\lambda|\leqslant 2^{m-100}$,

$$\|H_1\|_{L^2}+2^m\|H_2\|_{L^2}+2^m\|H_3\|_{L^2}\lesssim 2^{-5m/6+10\delta m}. \tag{9.135}$$

Notice that the bound on H_1 is already proved (in a stronger form) in the proof of (8.38) and (8.39). The bounds on H_2 and H_3 follow in the same way from the $L^2\times L^\infty$ argument: indeed, we have $\|\partial_s f^{\beta}_{j_3,k_3}(s)\|_{L^2}+\|\partial_s f^{\gamma}_{j_4,k_4}(s)\|_{L^2}\lesssim 2^{-m+7\delta m}$ (due to (8.21)). Then, we notice that we can remove the factor $\varphi(2^{20\delta m}\Theta_\beta(\eta,\sigma))$ from the multiplier $\chi^{[2]}(\eta,\sigma)$, at the expenses of a small error (due to Lemma 7.3 and (9.131)). The desired bounds in (9.135) follow using the $L^2\times L^\infty$ argument with Lemma 7.4.

Step 3. We now prove (9.134) for $[f,g,h]=[f^{\mu}_{j_1,k_1}, f^{\beta}_{j_3,k_3}, f^{\gamma}_{j_4,k_4}](s)$. It suffices to show that

$$2^{4\bar{k}}2^{m-30\delta m}\|S[f^{\mu}_{j_1,k_1}(s),H_1]\|_{L^2}\lesssim 1 \tag{9.136}$$

for any $s\in I_m$, where

$$\begin{aligned}\mathcal{F}\{S[f,g]\}(\xi):=|\varphi_k(\xi)|\int_{\mathbb{R}^2}|\hat f(\xi-\eta)\varphi(\varkappa_\theta^{-1}\Theta(\xi,\eta))2^{-l}\widetilde\varphi_l(\Phi(\xi,\eta))\\ \times\varphi_{[k_2-2,k+2]}(\eta)\hat g(\eta)|\,d\eta.\end{aligned}\tag{9.137}$$

This follows using Schur's lemma, the bound (9.135), and Proposition 10.4 (iii). Indeed, we have $|\nabla_\eta\Phi(\xi,\eta)|+|\nabla_\xi\Phi(\xi,\eta)|\gtrsim 2^{-4\delta m}$ in the support of the integral (due to the location of space-time resonances), therefore the left-hand side of (9.136) is dominated by

$$C2^{4\bar k}2^{m-30\delta m}2^{-l}(2^{10\bar k}\varkappa_\theta 2^{3l/4}2^{4\delta m})\|\widehat{f^\mu_{j_1,k_1}}(s)\|_{L^\infty}\|\widehat H_1\|_{L^2}\lesssim 2^{30\bar k}2^{-l/4}2^{-m/3}.$$

This suffices to prove (9.136), since $2^{-l}\leqslant 2^m$. Moreover, (9.134) follows in the same way for $[f,g,h]=[f^\mu_{j_1,k_1},2^m\partial_s f^\beta_{j_3,k_3},f^\gamma_{j_4,k_4}](s)$ or $[f,g,h]=[f^\mu_{j_1,k_1},f^\beta_{j_3,k_3},2^m\partial_s f^\gamma_{j_4,k_4}](s)$, as the L^2 bounds on 2^mH_2 and 2^mH_3 are the same as for H_1.

It remains to prove (9.134) for $[f,g,h]=[2^m\partial_s f^\mu_{j_1,k_1},f^\beta_{j_3,k_3},f^\gamma_{j_4,k_4}](s)$. It suffices to prove that

$$2^{4\bar k}2^{m-30\delta m}\|S[2^m\partial_s f^\mu_{j_1,k_1}(s),H_1]\|_{L^2}\lesssim 1\quad\text{for any } s\in I_m.\tag{9.138}$$

Let $f=2^m\partial_s f^\mu_{j_1,k_1}(s)$ and $f_{2\gamma_0}:=A_{\geqslant\mathcal{D}-11,2\gamma_0}f$. We decompose, using (8.41),

$$f=f_{2\gamma_0}+f_2+f_\infty,$$

with

$$\|f_{2\gamma_0}\|_{L^2}\lesssim 2^{7\delta m},\quad \|f_2\|_{L^2}\lesssim 2^{-m/2+5\delta m},\quad\text{and}\quad \|\hat f_\infty\|_{L^\infty}\lesssim 2^{3\bar k+15\delta m}.$$

The contribution of f_∞ can be estimated as before, using Schur's lemma, (9.135), and Proposition 10.4 (iii). To estimate the other contributions, we also use the bound (see (8.40))

$$\|\widehat H_{1,\infty}\|_{L^\infty}\lesssim 2^{3\bar k}2^{-m+14\delta m}$$

where

$$H_1=H_{1,2\gamma_0}+H_{1,\infty}=A_{\geqslant\mathcal{D}+1,2\gamma_0}H_1+A_{\leqslant\mathcal{D},2\gamma_0}H_1.$$

As before, we use Schur's test and Proposition 10.4 (iii), together with the fact that space-time resonances are possible only when $|\xi|$, $|\eta|$, and $|\xi-\eta|$ are all close to either γ_1 or $\frac12\gamma_1$. We estimate

$$\begin{aligned}\|S[f_2,H_{1,\infty}]\|_{L^2}&\lesssim 2^{-l}(2^{12\bar k}\varkappa_\theta 2^{3l/4}2^{4\delta m})\|\hat f_2\|_{L^2}\|\widehat H_{1,\infty}\|_{L^\infty}\lesssim 2^{20\bar k}2^{-l/4}2^{-2m+40\delta m},\\ \|S[f_{2\gamma_0},H_{1,\infty}]\|_{L^2}&\lesssim 2^{-l}(2^{12\bar k}\varkappa_\theta 2^{3l/4}2^{4\delta m})\|\hat f_{2\gamma_0}\|_{L^2}\|\widehat H_{1,\infty}\|_{L^\infty}\lesssim 2^{20\bar k}2^{-l/4}2^{-3m/2+40\delta m},\\ \|S[f_2,H_{1,2\gamma_0}]\|_{L^2}&\lesssim 2^{-l}(2^{10\bar k}\varkappa_\theta 2^l2^{4\delta m})^{1/2}\|\hat f_2\|_{L^2}\|\widehat H_{1,2\gamma_0}\|_{L^2}\lesssim 2^{15\bar k}2^{-l/2}2^{-19m/12+20\delta m},\\ S[f_{2\gamma_0},H_{1,2\gamma_0}]&\equiv 0.\end{aligned}$$

These bounds suffice to prove (9.138), which completes the proof of the lemma. □

9.8. Higher-order terms

In this subsection we consider the higher-order components in the Duhamel formula (7.5) and show how to control their Z norms.

PROPOSITION 9.13. *With the hypothesis in Proposition 7.1, for any $t\in[0,T]$ we have*

$$\|W_3(t)\|_Z+\bigg\|\int_0^t e^{is\Lambda}\mathcal{N}_{\geqslant 4}(s)\,ds\bigg\|_Z\lesssim \varepsilon_1^2. \tag{9.139}$$

The rest of this section is concerned with the proof of Proposition 9.13. The bound on $\mathcal{N}_{\geqslant 4}$ follows directly from the hypothesis $\|e^{is\Lambda}\mathcal{N}_{\geqslant 4}(s)\|_Z\leqslant\varepsilon_1^2(1+s)^{-1-\delta^2}$; see (7.15). To prove the bound on W_3 we start from the formula

$$\begin{aligned}\Omega_\xi^a\widehat{W}_3(\xi,t)=\sum_{\substack{\mu,\nu,\beta\in\{+,-\}\\ a_1+a_2+a_3=a}}\int_0^t\int_{\mathbb{R}^2\times\mathbb{R}^2} & e^{is\tilde{\Phi}_{+\mu\nu\beta}(\xi,\eta,\sigma)}\mathfrak{n}_{\mu\nu\beta}(\xi,\eta,\sigma)(\Omega^{a_1}\widehat{\mathcal{V}}_\mu)(\xi-\eta,s)\\ &\times(\Omega^{a_2}\widehat{\mathcal{V}}_\nu)(\eta-\sigma,s)(\Omega^{a_3}\widehat{\mathcal{V}}_\beta)(\sigma,s)\,d\eta\,d\sigma\,ds.\end{aligned} \tag{9.140}$$

We define the functions q_m as in (4.8) and the trilinear operators $C_m=C_{m,b}^{\mu\nu\beta}$ by

$$\begin{aligned}\mathcal{F}\{C_m[f,g,h]\}(\xi):=\int_{\mathbb{R}}q_m(s)\int_{\mathbb{R}^2\times\mathbb{R}^2} & e^{is\tilde{\Phi}(\xi,\eta,\sigma)}n_0(\xi,\eta,\sigma)\\ &\times\hat{f}(\xi-\eta,s)\hat{g}(\eta-\sigma,s)\hat{h}(\sigma,s)\,d\eta\,d\sigma\,ds,\end{aligned} \tag{9.141}$$

where $\tilde{\Phi}:=\tilde{\Phi}_{+\mu\nu\beta}$ and $n_0:=\mathfrak{n}_{\mu\nu\beta}$. It remains to prove that, for any $(k,j)\in\mathcal{J}$ and any $m\in[0,L+1]$,

$$\sum_{k_1,k_2,k_3\in\mathbb{Z}}2^{j-50\delta j}\|Q_{jk}C_m[P_{k_1}D^{\alpha_1}\Omega^{a_1}\mathcal{V}_\mu,P_{k_2}D^{\alpha_2}\Omega^{a_2}\mathcal{V}_\nu,P_{k_3}D^{\alpha_3}\Omega^{a_3}\mathcal{V}_\beta]\|_{L^2}\lesssim 2^{-\delta^2 m}\varepsilon_1^3 \tag{9.142}$$

for any $\mu,\nu,\beta\in\{+,-\}$, provided that $a_1+a_2+a_3=a$ and $\alpha_1+\alpha_2+\alpha_3=\alpha$. Let

$$f^\mu:=\varepsilon^{-1}D^{\alpha_1}\Omega^{a_1}\mathcal{V}_\mu,\quad f^\nu:=\varepsilon^{-1}D^{\alpha_2}\Omega^{a_2}\mathcal{V}_\nu,\quad\text{and}\quad f^\beta:=\varepsilon^{-1}D^{\alpha_3}\Omega^{a_3}\mathcal{V}_\beta. \tag{9.143}$$

The bootstrap assumption (7.15) gives, for any $s\in[0,t]$ and $\gamma\in\{\mu,\nu,\beta\}$,

$$\|f^\gamma(s)\|_{H^{N_0'}\cap Z_1\cap H_\Omega^{N_1'}}\lesssim(1+s)^{\delta^2}. \tag{9.144}$$

Simple estimates, as in the proof of Lemma 9.3, show that the parts of the sum in (9.142) over $\max(k_1,k_2,k_3)\geqslant 2(j+m)/N_0'-\mathcal{D}^2$ or over $\min(k_1,k_2,k_3)\leqslant-\frac{1}{2}(j+m)$ are bounded as claimed. For (9.142) it remains to prove that

$$2^{j-50\delta j}\|Q_{jk}C_m[P_{k_1}f^\mu,P_{k_2}f^\nu,P_{k_3}f^\beta]\|_{L^2}\lesssim 2^{-2\delta^2 m-\delta^2 j} \tag{9.145}$$

for any fixed $m\in[0,L+1]$, $(k,j)\in\mathcal{J}$, and $k_1,k_2,k_3\in\mathbb{Z}$ satisfying

$$k_1,k_2,k_3\in\Big[-\frac{j+m}{2},\frac{2(j+m)}{N_0'}-\mathcal{D}^2\Big]. \tag{9.146}$$

Let $\bar{k}:=\max(k,k_1,k_2,k_3,0)$, $\underline{k}:=\min(k,k_1,k_2,k_3)$, and $[k]:=\max(|k|,|k_1|,|k_2|,|k_3|)$. The S^∞ bound in (7.12) and Lemma A.1 (ii) show that

$$\begin{aligned}\|C_m[P_{k_1}f^\mu,P_{k_2}f^\nu,P_{k_3}f^\beta]\|_{L^2}\\ \lesssim 2^{\underline{k}/2}2^{3\bar{k}}2^m\sup_{s\in I_m}\|e^{-is\Lambda_\mu}P_{k_1}f^\mu\|_{L^{p_1}}\,\|e^{-is\Lambda_\nu}P_{k_2}f^\nu\|_{L^{p_2}}\,\|e^{-is\Lambda_\beta}P_{k_3}f^\beta\|_{L^{p_3}}\end{aligned} \tag{9.147}$$

if $p_1,p_2,p_3\in\{2,\infty\}$ and $1/p_1+1/p_2+1/p_3=\frac{1}{2}$. The desired bound (9.145) follows unless

$$j\geqslant\tfrac{2}{3}m+\tfrac{1}{2}[k]+\mathcal{D}^2, \tag{9.148}$$

using the pointwise bounds in (7.44). Also, by estimating $\|P_kH\|_{L^2}\lesssim 2^k\|P_kH\|_{L^1}$, and using a bound similar to (9.147), the desired bound (9.145) follows unless

$$k\geqslant-\tfrac{2}{3}\big(j+\tfrac{1}{6}m+\delta m\big). \tag{9.149}$$

Next, we notice that, if $j\geqslant m+\frac{1}{2}\mathcal{D}+[k]$ and (9.149) holds, then the desired bound (9.145) follows. Indeed, we use the approximate-finite-speed-of-propagation argument as in the proof of (9.12). First, we define $f^\mu_{j_1,k_1}$, $f^\nu_{j_2,k_2}$, and $f^\beta_{j_3,k_3}$ as in (9.14). Then, we notice that the contribution in the case $\min(j_1,j_2,j_3)\geqslant\frac{9}{10}j$ is suitably controlled, due to (9.147). On the other and, if

$$\min(j_1,j_2,j_3)\leqslant\tfrac{9}{10}j,$$

then we may assume that $j_1\leqslant\frac{9}{10}j$ (using changes of variables), and it follows that the contribution is negligible, using integration by parts in ξ as before. To summarize, in proving (9.145) we may assume that

$$\frac{2m}{3}+\frac{[k]}{2}+\mathcal{D}^2\leqslant j\leqslant m+\mathcal{D}+\frac{[k]}{2},\quad \max(j,[k])\leqslant 2m+2\mathcal{D},\quad\text{and}\quad \bar{k}\leqslant\frac{6m}{N_0'}. \tag{9.150}$$

We define now the functions $f^\mu_{j_1,k_1}$, $f^\nu_{j_2,k_2}$, and $f^\beta_{j_3,k_3}$ as in (9.14). The contribution in the case $\max(j_1,j_2,j_3)\geqslant\frac{2}{3}m$ can be bounded using (9.147). On the other hand, if $\max(j_1,j_2,j_3)\leqslant\frac{2}{3}m$, then we can argue as in the proof of Lemma 9.7 when $2^l\approx 1$. More precisely, we define

$$g_1:=A_{\geqslant\mathcal{D}_1,\gamma_0}f^\mu_{j_1,k_1},\quad g_2:=A_{\geqslant\mathcal{D}_1-10,\gamma_0}f^\nu_{j_2,k_2},\quad\text{and}\quad A_{\geqslant\mathcal{D}_1-20,\gamma_0}f^\beta_{j_3,k_3}. \tag{9.151}$$

As in the proof of Lemma 9.7 (see (9.95)–(9.97)), and after inserting cutoff functions of the form $\varphi_{\leqslant l}(\eta)$ and $\varphi_{>l}(\eta)$, $l=m-\delta m$, to bound the other terms, for (9.145) it suffices to prove that

$$2^{j-50\delta j}\|Q_{jk}C_m[g_1,g_2,g_3]\|_{L^2}\lesssim 2^{-\delta m}. \tag{9.152}$$

In proving (9.152), we may assume that $\max(j_1,j_2,j_3)\leqslant\frac{1}{3}m$ and $m\leqslant L$ (otherwise we could use directly (9.147)) and that $k\geqslant-100$ (otherwise the contribution is negligible, by integrating by parts in η and σ). Therefore, using (9.150), we may assume that

$$[k]\leqslant 100,\quad m\leqslant L,\quad \tfrac{2}{3}m+\mathcal{D}^2\leqslant j\leqslant m+2\mathcal{D},\quad\text{and}\quad j_1,j_2,j_3\in\left[0,\tfrac{1}{3}m\right]. \tag{9.153}$$

As in the proof of Lemma 9.7, we decompose the operator C_m in dyadic pieces depending on the size of the modulation. More precisely, let

$$\widehat{\mathcal{J}_p[f,g,h]}(\xi,s):=\int_{\mathbb{R}^2\times\mathbb{R}^2}e^{is\tilde{\Phi}(\xi,\eta,\sigma)}\varphi_p(\tilde{\Phi}(\xi,\eta,\sigma))n_0(\xi,\eta,\sigma)\\ \times\hat{f}(\xi-\eta,s)\hat{g}(\eta-\sigma,s)\hat{h}(\sigma,s)\,d\sigma\,d\eta.$$

Let $\mathcal{J}_{\leqslant p}=\sum_{q\leqslant p}\mathcal{J}_q$ and

$$C_{m,p}[f,g,h]:=\int_{\mathbb{R}}q_m(s)\mathcal{J}_{l,p}[f,g,h](s)\,ds.$$

For $p\geqslant-\frac{2}{3}m$ we integrate by parts in s. As in Step 1 in the proof of Lemma 9.7, using also the L^2 bound (8.21), it easily follows that

$$2^{j-50\delta j}\sum_{p\geqslant-2m/3}\|P_kC_{m,p}[g_1,g_2,g_3]\|_{L^2}\lesssim 2^{-\delta m}.$$

To complete the proof of (9.152), it suffices to show that

$$2^{j-50\delta j}2^m\sup_{s\in I_m}\|Q_{jk}\mathcal{J}_{\leqslant -m/2}[g_1,g_2,g_3](s)\|_{L^2}\lesssim 2^{-\delta m}. \tag{9.154}$$

Let $\varkappa=2^{-m/3}$ and define the operators $\mathcal{J}_{\leqslant -m/2,\leqslant 0}$ and $\mathcal{J}_{\leqslant -m/2,l}$ by inserting the factors $\varphi(\varkappa^{-1}\nabla_{\eta,\sigma}\tilde{\Phi}(\xi,\eta,\sigma))$ and $\varphi_l(\varkappa^{-1}\nabla_{\eta,\sigma}\tilde{\Phi}(\xi,\eta,\sigma))$, $l\geqslant1$, in the definition of the operators $\mathcal{J}_p$ above. The point is to observe that $|\nabla_\xi\tilde{\Phi}(\xi,\eta,\sigma)|\leqslant 2^{-m/3+\mathcal{D}}$ in the support of the integral defining the operator $\mathcal{J}_{\leqslant -m/2,\geqslant 0}$, due to Lemma 10.6 (i). Since $j\geqslant\frac{2}{3}m+\mathcal{D}^2$ (see (9.153)), the contribution of this operator is negligible, using integration by parts in ξ.

To estimate the operators $\mathcal{J}_{\leqslant -m/2,l}$, we may insert a factor $\varphi(2^{2m/3+l-\delta m}\eta)$, at the expense of a negligible error (due to Lemma 7.2 (i)). To summarize, we define

$$\widehat{\mathcal{J}'_{\leqslant -m/2,l}[f,g,h]}(\xi,s)\\ :=\int_{\mathbb{R}^2\times\mathbb{R}^2}e^{is\tilde{\Phi}(\xi,\eta,\sigma)}\varphi_l(\varkappa^{-1}\nabla_{\eta,\sigma}\tilde{\Phi}(\xi,\eta,\sigma))\varphi_{\leqslant -m/2}(\tilde{\Phi}(\xi,\eta,\sigma))\\ \times\varphi(2^{2m/3+l-\delta m}\eta)n_0(\xi,\eta,\sigma)\hat{f}(\xi-\eta,s)\hat{g}(\eta-\sigma,s)\hat{h}(\sigma,s)\,d\sigma\,d\eta,$$

and it remains to show that, for $l\geqslant 1$ and $s\in I_m$,

$$2^{j-50\delta j}2^m\|Q_{jk}\mathcal{J}'_{\leqslant -m/2,l}[g_1,g_2,g_3](s)\|_{L^2}\lesssim 2^{-2\delta m}. \tag{9.155}$$

Using L^∞ estimates in the Fourier space, (9.155) follows when $l\geqslant\frac{1}{3}m-\delta m$, since $2^j\lesssim 2^m$ (see (9.153)). On the other hand, if $l\leqslant\frac{1}{3}m-\delta m$, then the operator is non-trivial only if

$$\tilde{\Phi}(\xi,\eta,\sigma)=\Lambda(\xi)-\Lambda(\xi-\eta)-\Lambda_\nu(\eta-\sigma)+\Lambda_\nu(\sigma),\quad \nu\in\{+,-\},$$

due to the smallness of $|\eta|$, $|\nabla_\sigma\tilde{\Phi}(\xi,\eta,\sigma)|$, and $|\tilde{\Phi}(\xi,\eta,\sigma)|$ (recall the support restrictions in (9.151)). In this case, $|\nabla_\xi\tilde{\Phi}(\xi,\eta,\sigma)|\leqslant 2^{-m/2}$ in the support of the integral, and the contribution is again negligible using integration by parts in ξ. This completes the proof of Proposition 9.13.

10. Analysis of phase functions

In this section we collect and prove some important facts about the phase functions Φ.

10.1. Basic properties

Recall that

$$\begin{gathered}\Phi(\xi,\eta)=\Phi_{\sigma\mu\nu}(\xi,\eta)=\Lambda_\sigma(\xi)-\Lambda_\mu(\xi-\eta)-\Lambda_\nu(\eta),\quad \sigma,\mu,\nu\in\{+,-\},\\ \Lambda_\varkappa(\xi)=\lambda_\varkappa(|\xi|)=\varkappa\lambda(|\xi|)=\varkappa\sqrt{|\xi|+|\xi|^3}.\end{gathered} \tag{10.1}$$

We have

$$\lambda'(x)=\frac{1+3x^2}{2\sqrt{x+x^3}},\quad \lambda''(x)=\frac{3x^4+6x^2-1}{4(x+x^3)^{3/2}},\quad\text{and}\quad \lambda'''(x)=\frac{3(1+5x^2-5x^4-x^6)}{8(x+x^3)^{5/2}}. \tag{10.2}$$

Therefore,

$$\lambda''(x)\geqslant 0\ \text{ if } x\geqslant\gamma_0,\quad \lambda''(x)\leqslant 0\ \text{ if } x\in[0,\gamma_0],\quad\text{and}\quad \gamma_0:=\sqrt{\frac{2\sqrt{3}-3}{3}}\approx 0.393. \tag{10.3}$$

It follows that

$$\lambda(\gamma_0)\approx 0.674,\quad \lambda'(\gamma_0)\approx 1.086,\quad \lambda'''(\gamma_0)\approx 4.452,\quad\text{and}\quad \lambda''''(\gamma_0)\approx -28.701. \tag{10.4}$$

Let $\gamma_1:=\sqrt{2}\approx 1.414$ denote the radius of the space-time resonant sphere, and notice that

$$\lambda(\gamma_1)=\sqrt{3\sqrt{2}}\approx 2.060,\quad \lambda'(\gamma_1)=\frac{7}{2\sqrt{3\sqrt{2}}}\approx 1.699,\quad\text{and}\quad \lambda''(\gamma_1)=\frac{23}{4\sqrt{54\sqrt{2}}}\approx 0.658. \tag{10.5}$$

The following simple observation will be used many times: if $U_2\geqslant 1$, $\xi,\eta\in\mathbb{R}^2$,

$$\max(|\xi|,|\eta|,|\xi-\eta|)\leqslant U_2, \quad \text{and} \quad \min(|\xi|,|\eta|,|\xi-\eta|)=a\leqslant 2^{-10}U_2^{-1},$$

then

$$|\Phi(\xi,\eta)|\geqslant\lambda(a)-\sup_{b\in[a,U_2]}(\lambda(a+b)-\lambda(b))\geqslant\lambda(a)-a\max\{\lambda'(a),\lambda'(U_2+1)\}\geqslant\tfrac{1}{4}\lambda(a). \tag{10.6}$$

LEMMA 10.1. (i) *The function* λ' *is strictly decreasing on the interval* $(0,\gamma_0]$ *and strictly increasing on the interval* $[\gamma_0,\infty)$, *and*

$$\lim_{x\to\infty}\left(\lambda'(x)-\frac{3\sqrt{x}}{2}\right)=0 \quad \text{and} \quad \lim_{x\to 0}\left(\lambda'(x)-\frac{1}{2\sqrt{x}}\right)=0. \tag{10.7}$$

The function λ' *is concave up on the interval* $(0,1]$ *and concave down on the interval* $[1,\infty)$. *For every* $y>\lambda'(\gamma_0)$ *the equation* $\lambda'(r)=y$ *has two solutions* $r_1(y)\in(0,\gamma_0)$ *and* $r_2(y)\in(\gamma_0,\infty)$.

(ii) *If* $a\neq b\in(0,\infty)$, *then*

$$\lambda'(a)=\lambda'(b) \quad \text{if and only if} \quad (a-b)^2=\frac{(3ab+1)(3a^2b^2+6ab-1)}{1-9ab}. \tag{10.8}$$

In particular, if $a\neq b\in(0,\infty)$ *and* $\lambda'(a)=\lambda'(b)$, *then* $ab\in\left(\frac{1}{9},\gamma_0^2\right]$.

(iii) *Let* $b\colon[\gamma_0,\infty)\to(0,\gamma_0]$ *be the implicit function defined by* $\lambda'(a)=\lambda'(b(a))$. *Then,* b *is a smooth decreasing function and*(6)

$$\begin{gathered} b'(a)\in\left[-1,-\frac{b(a)}{a}\right], \quad a+b(a) \text{ is increasing on } [\gamma_0,\infty), \\ b(a)\approx\frac{1}{a}, \quad -b'(a)\approx\frac{1}{a^2}, \quad b'(a)+1\approx\frac{a-\gamma_0}{a}. \end{gathered} \tag{10.9}$$

In particular,

$$a+b(a)-2\gamma_0\approx\frac{(a-\gamma_0)^2}{a}. \tag{10.10}$$

Moreover,

$$-(\lambda''(b(a))+\lambda''(a))\approx a^{-1/2}(a-\gamma_0)^2. \tag{10.11}$$

(iv) *If* $a,b\in(0,\infty)$, *then*

$$\lambda(a+b)=\lambda(a)+\lambda(b) \quad \text{if and only if} \quad (a-b)^2=\frac{4+8ab-32a^2b^2}{9ab-4}. \tag{10.12}$$

In particular, if $a,b\in(0,\infty)$ *and* $\lambda(a+b)=\lambda(a)+\lambda(b)$, *then* $ab\in\left[\frac{4}{9},\frac{1}{2}\right]$. *Moreover,*

$$\begin{gathered} \text{if } ab>\tfrac{1}{2}, \text{ then } \lambda(a+b)-\lambda(a)-\lambda(b)>0, \\ \text{if } ab<\tfrac{4}{9}, \text{ then } \lambda(a+b)-\lambda(a)-\lambda(b)<0. \end{gathered} \tag{10.13}$$

(6) In a neighborhood of γ_0, $\lambda'(x)$ behaves like $A+B(x-\gamma_0)^2-C(x-\gamma_0)^3$, where $A,B,C>0$. The asymptotics described in (10.9)–(10.11) are consistent with this behaviour.

Proof. The conclusions (i) and (ii) follow from (10.2)–(10.4) by elementary arguments. For part (iii) we notice that, with $Y=ab$,

$$(a+b(a))^2=F(Y):=\frac{-9Y^3-21Y^2-3Y+1}{9Y-1}+4Y=\frac{32/81}{9Y-1}-Y^2+\frac{14Y}{9}-\frac{49}{81},$$

as a consequence of (10.8). Taking the derivative with respect to a, it follows that

$$2(a+b(a))(1+b'(a))=(ab'(a)+b(a))F'(Y). \tag{10.14}$$

Since $F'(Y)\leqslant -\frac{1}{10}$ for all $Y\in\left(\frac{1}{9},\gamma_0^2\right]$, it follows that $b'(a)\in[-1,-b(a)/a]$ for all $a\in[\gamma_0,\infty)$. The claims in the first line of (10.9) follow.

The claim $-b'(a)\approx 1/a^2$ follows from the identity $\lambda''(a)-\lambda''(b(a))b'(a)=0$. The last claim in (10.9) is clear if $a-\gamma_0\gtrsim 1$; on the other hand, if $a-\gamma_0=\varrho\ll 1$, then (10.14) gives

$$-\frac{1+b'(a)}{b'(a)+b(a)/a}\approx 1 \quad \text{and} \quad \gamma_0-b(a)\approx\varrho.$$

In particular, $1-b(a)/a\approx\varrho$, and the last conclusion in (10.9) follows.

The claim in (10.10) follows by integrating the approximate identity

$$b'(x)+1\approx\frac{x-\gamma_0}{x}$$

between γ_0 and a. To prove (10.11), we recall that $\lambda''(a)-\lambda''(b(a))b'(a)=0$. Therefore,

$$-(\lambda''(b(a))+\lambda''(a))=-\lambda''(b(a))(1+b'(a))=\lambda''(a)\frac{1+b'(a)}{-b'(a)},$$

and the desired conclusion follows using also (10.9).

To prove (iv), note that (10.12) and the claim $ab\in\left[\frac{4}{9},\frac{1}{2}\right]$ follow from (10.2)–(10.4) by elementary arguments. To prove (10.13), let $G(x):=\lambda(a+x)-\lambda(a)-\lambda(x)$. For $a\in(0,\infty)$ fixed, we notice that $G(x)>0$ if x is sufficiently large, and $G(x)<0$ if $x>0$ is sufficiently small. The desired conclusion follows from the continuity of G. □

10.2. Resonant sets

We now prove an important proposition describing the geometry of resonant sets.

PROPOSITION 10.2. (Structure of resonance sets) *The following claims hold*:

(i) *There are functions* $p_{++1}=p_{--1}\colon(0,\infty)\to(0,\infty)$, $p_{++2}=p_{--2}\colon[2\gamma_0,\infty)\to(0,\gamma_0]$, $p_{+-1}=p_{-+1}\colon(0,\infty)\to(\gamma_0,\infty)$ *such that, if* $\sigma,\mu,\nu\in\{+,-\}$ *and* $\xi\neq 0$, *then*

$$(\nabla_\eta\Phi_{\sigma\mu\nu})(\xi,\eta)=0 \quad \textit{if and only if} \quad \eta\in P_{\mu\nu}(\xi), \tag{10.15}$$

where (*the* $p_{\mu\nu 2}$ *parts are absent when* $\mu\neq\nu$)

$$P_{\mu\nu}(\xi):=\left\{p_{\mu\nu 1}(|\xi|)\frac{\xi}{|\xi|},p_{\mu\nu 2}(|\xi|)\frac{\xi}{|\xi|},\xi-p_{\mu\nu 1}(|\xi|)\frac{\xi}{|\xi|},\xi-p_{\mu\nu 2}(|\xi|)\frac{\xi}{|\xi|}\right\}.$$

(ii) (Space resonances) *With* $\mathcal{D}_{k,k_1,k_2}$ *as in* (2.11), *assume that*

$$(\xi,\eta)\in\mathcal{D}_{k,k_1,k_2}\quad \text{and}\quad |(\nabla_\eta\Phi_{\sigma\mu\nu})(\xi,\eta)|\leqslant\varepsilon_2\leqslant 2^{-\mathcal{D}_1}2^{k-\max(k_1,k_2)} \tag{10.16}$$

for some constant $\mathcal{D}_1$ *sufficiently large. So,* $\big||k_1|-|k_2|\big|\leqslant 20$ *and, for some* $p\in P_{\mu\nu}(\xi)$,(7) *the following conditions are satisfied*:

- *if* $|k|\leqslant 100$, *then* $\max(|k_1|,|k_2|)\leqslant 200$ *and*

$$\begin{aligned}&\text{either } (\mu=-\nu \text{ and } |\eta-p|\lesssim\varepsilon_2),\\ &\text{or } \left(\mu=\nu,\ \left|\frac{(\eta-p)\cdot\xi^\perp}{|\xi|}\right|\lesssim\varepsilon_2,\ \text{and } \left|\frac{(\eta-p)\cdot\xi}{|\xi|}\right|\lesssim\frac{\varepsilon_2}{\varepsilon_2^{2/3}+\big||\xi|-2\gamma_0\big|}\right);\end{aligned} \tag{10.17}$$

- *if* $k\leqslant -100$, *then*

$$\begin{aligned}&\text{either } (\mu=-\nu,\ k_1,k_2\in[-10,10],\ \text{and } |\eta-p|\lesssim\varepsilon_2 2^{|k|}),\\ &\text{or } \big(\mu=\nu,\ k_1,k_2\in[k-10,k+10],\ \text{and } \big|\eta-\tfrac12\xi\big|\lesssim 2^{-3|k|/2}\varepsilon_2\big);\end{aligned} \tag{10.18}$$

- *if* $k\geqslant 100$, *then*

$$|\eta-p|\lesssim\varepsilon_2 2^{k/2}. \tag{10.19}$$

(iii) (Space-time resonances) *Assume that* $(\xi,\eta)\in\mathcal{D}_{k,k_1,k_2}$,

$$\begin{aligned}|\Phi_{\sigma\mu\nu}(\xi,\eta)|&\leqslant\varepsilon_1\leqslant 2^{-\mathcal{D}_1}2^{\min(k,k_1,k_2,0)/2},\\ |(\nabla_\eta\Phi_{\sigma\mu\nu})(\xi,\eta)|&\leqslant\varepsilon_2\leqslant 2^{-\mathcal{D}_1}2^{k-\max(k_1,k_2)}2^{-2k^+}.\end{aligned} \tag{10.20}$$

Then, with $\gamma_1:=\sqrt{2}$,

$$\pm(\sigma,\mu,\nu)=(+,+,+),\quad |\eta-p_{++1}(\xi)|=\big|\eta-\tfrac12\xi\big|\lesssim\varepsilon_2,\quad \big||\xi|-\gamma_1\big|\lesssim\varepsilon_1+\varepsilon_2^2. \tag{10.21}$$

Proof. (i) We have

$$(\nabla_\eta\Phi_{\sigma\mu\nu})(\xi,\eta)=\mu\lambda'(|\xi-\eta|)\frac{\xi-\eta}{|\xi-\eta|}-\nu\lambda'(|\eta|)\frac{\eta}{|\eta|}. \tag{10.22}$$

(7) The set $P_{\mu\nu}(\xi)$ contains two points if $(\mu,\nu)\in\{(+,-),(-,+)\}$ and at most three points if $(\mu,\nu)\in\{(+,+),(-,-)\}$.

Assume that $\xi=\alpha e$ for some $\alpha\in(0,\infty)$ and $e\in\mathbb{S}^1$. In view of (10.22), $(\nabla_\eta\Phi_{\sigma\mu\nu})(\xi,\eta)=0$ if and only if

$$\eta=\beta e,\quad \beta\in\mathbb{R}\setminus\{0,\alpha\},\quad\text{and}\quad \mu\lambda'(|\alpha-\beta|)\operatorname{sgn}(\alpha-\beta)=\nu\lambda'(|\beta|)\operatorname{sgn}(\beta). \tag{10.23}$$

We observe that it suffices to define the functions p_{++1}, p_{++2}, and p_{+-1} satisfying (10.15), since clearly $p_{--1}=p_{++1}$, $p_{--2}=p_{++2}$, and $p_{-+1}=p_{+-1}$.

If $(\mu,\nu)=(+,+)$ then, as a consequence of (10.23), $\beta\in(0,\alpha)$ and $\lambda'(\alpha-\beta)=\lambda'(\beta)$. Therefore, according to Lemma 10.1 (i)–(iii), there are two possible solutions:

$$\begin{aligned}&\beta=p_{++1}(\alpha):=\tfrac{1}{2}\alpha,\\&\beta=p_{++2}(\alpha),\ \text{ uniquely determined by } \lambda'(\beta)=\lambda'(\alpha-\beta) \text{ and } \beta\in(0,\gamma_0].\end{aligned} \tag{10.24}$$

The uniqueness of the point $p_{++2}(\alpha)$ is due to the fact that the function $x\mapsto x+b(x)$ is increasing in $[\gamma_0,\infty)$; see (10.9). On the other hand, if $(\mu,\nu)=(+,-)$ then, as a consequence of (10.23), either $\beta<0$, or $\beta>\alpha$ and $\lambda'(|\alpha-\beta|)=\lambda'(|\beta|)$. Therefore, according to Lemma 10.1, there is only one solution $\beta\geqslant\gamma_0$:

$$\beta=p_{+-1}(\alpha),\ \text{ uniquely determined by } \lambda'(\beta)=\lambda'(\beta-\alpha) \text{ and } \beta\in[\max(\alpha,\gamma_0),\alpha+\gamma_0]. \tag{10.25}$$

The conclusions in part (i) follow.

(ii) Assume that (10.16) holds and that $(\mu,\nu)\in\{(+,+),(+,-)\}$. Let $\xi=\alpha e$, $|e|=1$, $\alpha\in[2^{k-4},2^{k+4}]$, $\eta=\beta e+v$, $v\cdot e=0$, and $(\beta^2+|v|^2)^{1/2}\in[2^{k_2-4},2^{k_2+4}]$. The condition

$$|(\nabla_\eta\Phi_{\sigma\mu\nu})(\xi,\eta)|\leqslant\varepsilon_2$$

gives, using (10.22), $\big||k_1|-|k_2|\big|\leqslant 20$,

$$\left|\mu\lambda'(|\xi-\eta|)\frac{(\alpha-\beta)}{|\xi-\eta|}-\nu\lambda'(|\eta|)\frac{\beta}{|\eta|}\right|\leqslant\varepsilon_2,\quad\text{and}\quad\left|-\mu\frac{\lambda'(|\xi-\eta|)}{|\xi-\eta|}-\nu\frac{\lambda'(|\eta|)}{|\eta|}\right||v|\leqslant\varepsilon_2. \tag{10.26}$$

Since $\alpha\gtrsim 2^k$ and $|\xi-\eta|^{-1}\lambda'(|\xi-\eta|)\gtrsim 2^{|k_1|/2-k_1}$, the first inequality in (10.26) shows that

$$\left|\mu\lambda'(|\xi-\eta|)\frac{-\beta}{|\xi-\eta|}-\nu\lambda'(|\eta|)\frac{\beta}{|\eta|}\right|\gtrsim 2^{k+|k_1|/2-k_1}.$$

Since $1/|\beta|\geqslant 2^{-k_2-4}$, using also the second inequality in (10.26), it follows that

$$|v|\lesssim\varepsilon_2 2^{-k-|k_1|/2+k_1+k_2} \tag{10.27}$$

and

$$\left|-\mu\frac{\lambda'(|\xi-\eta|)}{|\xi-\eta|}-\nu\frac{\lambda'(|\eta|)}{|\eta|}\right|\gtrsim 2^{k+|k_1|/2-k_1-k_2}.$$

In particular, $|v|\leqslant 2^{-20}2^{\min(k_1,k_2)}$,

$$\big||\eta|-|\beta|\big|\lesssim\varepsilon_2^2 2^{-2k-|k_1|+2k_1+k_2},\quad\text{and}\quad \big||\xi-\eta|-|\alpha-\beta|\big|\lesssim\varepsilon_2^2 2^{-2k-|k_1|+k_1+2k_2}. \tag{10.28}$$

Using the first inequality in (10.26), it follows that

$$\big|\mu\lambda'(|\alpha-\beta|)\operatorname{sgn}(\alpha-\beta)-\nu\lambda'(|\beta|)\operatorname{sgn}(\beta)\big|\leqslant\varepsilon_2+C\varepsilon_2^2 2^{-2k-|k_1|/2+2\max(k_1,k_2)}. \tag{10.29}$$

Proof of (10.17). Assume first that $|k|\leqslant 100$. Then, $\max(|k_1|,|k_2|)\leqslant 200$, since otherwise (10.29) cannot hold (so there are no points (ξ,η) satisfying (10.16)). The conclusion $\big|(\eta-p)\cdot\xi^\perp/|\xi|\big|\lesssim\varepsilon_2$ in (10.17) follows from (10.27).

Case 1. If $(\mu,\nu)=(+,-)$ then (10.29) gives

$$\big|\lambda'(|\alpha-\beta|)-\lambda'(|\beta|)\big|\leqslant 2\varepsilon_2\quad\text{and}\quad\operatorname{sgn}(\alpha-\beta)+\operatorname{sgn}(\beta)=0.$$

Therefore, either $\beta>\alpha$ and $|\lambda'(\beta-\alpha)-\lambda'(\beta)|\leqslant 2\varepsilon_2$, in which case $\beta-\alpha<\gamma_0$, $\beta>\gamma_0$, and $|\beta-p_{+-1}(\alpha)|\lesssim\varepsilon_2$, or $\beta<0$ and $|\lambda'(\alpha-\beta)-\lambda'(-\beta)|\leqslant 2\varepsilon_2$, in which case $\alpha-\beta>\gamma_0$, $-\beta<\gamma_0$, and $|\alpha-\beta-p_{+-1}(\alpha)|\lesssim\varepsilon_2$. The desired conclusion follows in the stronger form $|\eta-p|\lesssim\varepsilon_2$.

Case 2. If $(\mu,\nu)=(+,+)$, then (10.29) gives

$$\big|\lambda'(|\alpha-\beta|)-\lambda'(|\beta|)\big|\leqslant 2\varepsilon_2\quad\text{and}\quad\operatorname{sgn}(\alpha-\beta)=\operatorname{sgn}(\beta).$$

Therefore,

$$\beta\in(0,\alpha)\quad\text{and}\quad|\lambda'(\alpha-\beta)-\lambda'(\beta)|\leqslant 2\varepsilon_2. \tag{10.30}$$

Assume α fixed and let $G(\beta):=\lambda'(\beta)-\lambda'(\alpha-\beta)$. The function G vanishes when $\beta=\frac12\alpha$ or $\beta\in\{p_{++2}(\alpha),\alpha-p_{++2}(\alpha)\}$ (if $\alpha\geqslant 2\gamma_0$).

Assume that $\alpha=2\gamma_0+\varrho\geqslant 2\gamma_0$, $\varrho\in[0,2^{110}]$. Then, using Lemma 10.1 (iii),

$$p_{++2}(\alpha)\leqslant\gamma_0\leqslant\tfrac12\alpha\leqslant\alpha-p_{++2}(\alpha),\quad \tfrac12\alpha-\gamma_0=\tfrac12\varrho,\quad \gamma_0-p_{++2}(\alpha)\approx\sqrt{\varrho}, \tag{10.31}$$

where the last conclusion follows from (10.10) with $a=\alpha-p_{++2}(\alpha)$ and $b(a)=p_{++2}(\alpha)$. Moreover, $|G'(\beta)|=|\lambda''(\beta)+\lambda''(\alpha-\beta)|\approx\varrho$ if $\beta\in\{\frac12\alpha,p_{++2}(\alpha),\alpha-p_{++2}(\alpha)\}$, using (10.11) and (10.31). Also, $|G''(\beta)|=|\lambda'''(\beta)-\lambda'''(\alpha-\beta)|\lesssim\sqrt{\varrho}$ if $\big|\beta-\frac12\alpha\big|\lesssim\sqrt{\varrho}$, therefore

$$|G'(\beta)|\approx\varrho,\quad\text{if }\beta\in I_\alpha, \tag{10.32}$$

where

$$I_\alpha:=\bigg\{x:\min\bigg(\bigg|x-\frac{\alpha}{2}\bigg|,|x-p_{++2}(\alpha)|,|x-\alpha+p_{++2}(\alpha)|\bigg)\leqslant\frac{\sqrt{\varrho}}{C_0}\bigg\},$$

for some large constant C_0.

If $\varrho\leqslant C_0^4\varepsilon_2^{2/3}$, then the points $\frac{1}{2}\alpha$, $p_{++2}(\alpha)$, and $\alpha-p_{++2}(\alpha)$ are at distance $\leqslant C_0^4\varepsilon_2^{1/3}$. In this case, it suffices to prove that $|G(\beta)|\geqslant 3\varepsilon_2$ if $\left|\beta-\frac{1}{2}\alpha\right|\geqslant 2C_0^4\varepsilon_2^{1/3}$. Assume, by contradiction, that this is not true, so there is $\beta\leqslant\gamma_0-C_0^4\varepsilon_2^{1/3}$ such that

$$|\lambda'(\beta)-\lambda'(\alpha-\beta)|\leqslant 3\varepsilon_2.$$

So, there is x close to β, say $|x-\beta|\lesssim\varepsilon_2^{2/3}$, such that $\lambda'(x)=\lambda'(\alpha-\beta)$. In particular, using (10.10) with $a=\alpha-\beta$ and $b(a)=x$, we have $\alpha-\beta+x-2\gamma_0\geqslant C_0^7\varepsilon_2^{2/3}$. Therefore, $\alpha-2\gamma_0\geqslant C_0^6\varepsilon_2^{2/3}$, in contradiction with the assumption $\alpha-2\gamma_0=\varrho\leqslant C_0^4\varepsilon_2^{2/3}$.

Assume now $\varrho\geqslant C_0^4\varepsilon_2^{2/3}$. In view of (10.32), it suffices to prove that, if $\beta\notin I_\alpha$, then $|G(\beta)|\geqslant 3\varepsilon_2$. Assume, by contradiction, that this is not true, so there is $\beta\in\left(0,\frac{1}{2}\alpha\right]\setminus I_\alpha$ such that $|\lambda'(\beta)-\lambda'(\alpha-\beta)|\leqslant 3\varepsilon_2$. Since $\beta\leqslant\frac{1}{2}\alpha-\sqrt{\varrho}/C_0$, we may in fact assume that $\beta\leqslant\gamma_0-\sqrt{\varrho}/2C_0$, provided that the constant $\mathcal{D}_1$ in (10.16) is sufficiently large. So, there is x close to β, say $|x-\beta|\lesssim\varepsilon_2 C_0/\sqrt{\varrho}$, such that $\lambda'(x)=\lambda'(\alpha-\beta)$. Using (10.9), it follows that there is a point y close to x, say $|y-x|\lesssim\varepsilon_2 C_0^2/\varrho$, such that $\lambda'(y)=\lambda'(\alpha-y)$. Therefore, $y=p_{++2}(\alpha)$. In particular, $|\beta-p_{++2}(\alpha)|\lesssim\varepsilon_2 C_0^2/\varrho$, in contradiction with the assumption $\beta\notin I_\alpha$, so $|\beta-p_{++2}(\alpha)|\geqslant\sqrt{\varrho}/C_0$ (recall that $\varrho\geqslant C_0^4\varepsilon_2^{2/3}$).

The case $\alpha=2\gamma_0-\varrho\leqslant 2\gamma_0$ is easier, since there is only one point to consider, namely $\frac{1}{2}\alpha$. As in (10.32), $|G'(\beta)|\approx\varrho$ if $\left|\beta-\frac{1}{2}\alpha\right|\leqslant\sqrt{\varrho}/C_0$. The proof then proceeds as before, by considering the two cases $\varrho\leqslant C_0^4\varepsilon_2^{2/3}$ and $\varrho\geqslant C_0^4\varepsilon_2^{2/3}$. □

Proof of (10.18). Assume now that $k\leqslant -100$, so $|k_1-k_2|\leqslant 20$, and consider two cases.

Case 1. Assume first that $(\mu,\nu)=(+,-)$. In view of (10.22), we have

$$\left|\lambda'(|\eta|)\frac{\eta}{|\eta|}-\lambda'(|w|)\frac{w}{|w|}\right|\leqslant\varepsilon_2, \quad \text{where } w=\eta-\xi. \tag{10.33}$$

If $\max(|\eta|,|w|)\leqslant\gamma_0-2^{-10}$ or $\min(|\eta|,|w|)\geqslant\gamma_0+2^{-10}$, then it follows from (10.33) that $\left|\lambda'(|\eta|)-\lambda'(|w|)\right|\leqslant\varepsilon_2$, and thus $\left||\eta|-|w|\right|\lesssim\varepsilon_2 2^{-|k_1|/2+k_1}$. Therefore,

$$\left|\frac{\eta}{|\eta|}-\frac{w}{|w|}\right|\lesssim\varepsilon_2 2^{-|k_1|/2} \quad\text{and}\quad \left|\frac{1}{|\eta|}-\frac{1}{|w|}\right|\lesssim\varepsilon_2 2^{-|k_1|/2-k_1}.$$

As a consequence, $|\eta-w|\lesssim\varepsilon_2 2^{-|k_1|/2+k_1}$. On the other hand, $|\eta-w|=|\xi|\gtrsim 2^k$, in contradiction with the assumption $\varepsilon_2\leqslant 2^{-\mathcal{D}_1}2^{k-k_1}$.

Since $|\eta-w|\leqslant 2^{-90}$, it remains to consider the case

$$|\eta|,|\eta-\xi|\in[\gamma_0-2^{-9},\gamma_0+2^{-9}]. \tag{10.34}$$

In particular, $k_1,k_2\in[-10,10]$ as claimed. Moreover, $|v|\lesssim\varepsilon_2 2^{|k|}$ as desired, in view of (10.27). The condition (10.29) gives

$$\big|\lambda'(|\alpha-\beta|)-\lambda'(|\beta|)\big|\leqslant\varepsilon_2+C\varepsilon_2^2 2^{-2k}\quad\text{and}\quad\operatorname{sgn}(\alpha-\beta)+\operatorname{sgn}(\beta)=0.$$

Without loss of generality, we may assume that

$$\beta>\alpha\quad\text{and}\quad|\lambda'(\beta-\alpha)-\lambda'(\beta)|\leqslant\varepsilon_2+C\varepsilon_2^2 2^{-2k}. \tag{10.35}$$

Notice that $p_{+-1}(\alpha)\in(\gamma_0,\alpha+\gamma_0)$. We have two cases: if $\varepsilon_2\geqslant 2^{-\mathcal{D}_1}2^{2k}$, then we need to prove that $|\beta-\gamma_0|\leqslant 2^{4\mathcal{D}_1}\varepsilon_2 2^{|k|}$. This follows from (10.33): otherwise, if $|\beta-\gamma_0|=d\geqslant 2^{4\mathcal{D}_1}\varepsilon_2 2^{|k|}\geqslant 2^{3\mathcal{D}_1}2^k$, then $\big||\eta|-\gamma_0\big|\approx d$ and $\big||w|-\gamma_0\big|\approx d$, using also (10.27). As a consequence of (10.33), we have $\big||\eta|-|w|\big|\lesssim\varepsilon_2 d^{-1}$, so

$$\left|\frac{\eta}{|\eta|}-\frac{w}{|w|}\right|\lesssim\varepsilon_2\quad\text{and}\quad\left|\frac{1}{|\eta|}-\frac{1}{|w|}\right|\lesssim\varepsilon_2 d^{-1}.$$

Thus, $|\eta-w|\lesssim\varepsilon_2+\varepsilon_2 d^{-1}\lesssim\varepsilon_2+2^{k-4\mathcal{D}_1}$, in contradiction with $|\eta-w|=|\xi|\gtrsim 2^k$.

On the other hand, if $\varepsilon_2\leqslant 2^{-\mathcal{D}_1}2^{2k}$, then (10.35) gives $|\lambda'(\beta-\alpha)-\lambda'(\beta)|\leqslant 2\varepsilon_2$ and $\beta\in(\gamma_0,\gamma_0+\alpha)$. Let $H(\beta):=\lambda'(\beta)-\lambda'(\beta-\alpha)$, and notice that

$$|H'(\beta)|\gtrsim|\beta-\gamma_0|+|\beta-\alpha-\gamma_0|\gtrsim 2^k$$

if β is in this set. The desired conclusion follows, since $H(p_{+-1}(\alpha))=0$.

Case 2. If $(\mu,\nu)=(+,+)$, then (10.29) gives

$$|\lambda'(\alpha-\beta)-\lambda'(\beta)|\leqslant\varepsilon_2+C\varepsilon_2^2 2^{-2k-|k_1|/2+2\max(k_1,k_2)},\quad\beta\in(0,\alpha).$$

This easily shows that $k_1,k_2\in[k-10,k+10]$ and $|\alpha-2\beta|\lesssim 2^{-3|k|/2}\varepsilon_2$. The desired conclusion follows using also (10.27). □

Proof of (10.19). Assume now that $k\geqslant 100$ and consider two cases.

Case 1. If $(\mu,\nu)=(+,-)$, then (10.29) gives

$$|\lambda'(|\alpha-\beta|)-\lambda'(|\beta|)|\leqslant\varepsilon_2+C\varepsilon_2^2 2^{-2k-|k_1|/2+2\max(k_1,k_2)}\quad\text{and}\quad\operatorname{sgn}(\alpha-\beta)+\operatorname{sgn}(\beta)=0.$$

We may assume $\beta>\alpha$, $|\max(k_1,k_2)-k|\leqslant 20$, and $|\lambda'(\beta-\alpha)-\lambda'(\beta)|\leqslant 2\varepsilon_2$. In particular, $\beta\in(\alpha,\alpha+\gamma_0)$. Let $H(\beta):=\lambda'(\beta)-\lambda'(\beta-\alpha)$ as before, and notice that $|H'(\beta)|\gtrsim 2^{3k/2}$ in this set. The desired conclusion follows, since $H(p_{+-1}(\alpha))=0$, using also (10.27).

Case 2. If $(\mu,\nu)=(+,+)$, then (10.29) gives

$$|\lambda'(\alpha-\beta)-\lambda'(\beta)|\leqslant\varepsilon_2+C\varepsilon_2^2 2^{-2k-|k_1|/2+2\max(k_1,k_2)},\quad\beta\in(0,\alpha). \tag{10.36}$$

If both β and $\alpha-\beta$ are in $[\gamma_0,\infty)$, then (10.36) gives $\big|\beta-\frac{1}{2}\alpha\big|\lesssim\varepsilon_2 2^{k/2}$, which suffices (using also (10.27)). Otherwise, assuming for example that $\beta\in(0,\gamma_0)$, (10.36) implies that $\beta\leqslant 2^{-k+20}$. Let, as before, $G(\beta):=\lambda'(\beta)-\lambda'(\alpha-\beta)$, and notice that $|G'(\beta)|\gtrsim 2^{3k/2}$ if $\beta\in(0,2^{-k+20}]$. The desired conclusion follows, since $G(p_{++2}(\alpha))=0$, using also (10.27). □

(iii) If $k\leqslant -100$, then $\Phi_{\sigma\mu\nu}(\xi,\eta)\gtrsim 2^{k/2}$, in view of (10.6) and (10.18), which is not allowed by the condition on ε_1.

If $k\geqslant 100$ and $(\mu,\nu)=(+,-)$, then $p_{+-1}(\alpha)-\alpha\leqslant 2^{-k+10}\leqslant 2^{k-10}\leqslant\alpha$ and

$$|\Phi(\xi,\eta)|\geqslant|\pm\lambda(\alpha)-\lambda(p_{+-1}(\alpha))+\lambda(p_{+-1}(\alpha)-\alpha)|-C\varepsilon_2 2^k,$$

for some constant C sufficiently large. Moreover, in view of Lemma 10.1 (i),

$$\alpha(p_{+-1}(\alpha)-\alpha)\leqslant\gamma_0^2\leqslant 0.2.$$

In particular, using also Lemma 10.1 (iv), $|\Phi(\xi,\eta)|\gtrsim 2^{-k/2}$, which is impossible in view of the assumption on ε_1. A similar argument works also in the case $k\geqslant 100$ and $(\mu,\nu)=(+,+)$ to show that there are no points (ξ,η) satisfying (10.20).

Finally, assume that $|k|\leqslant 100$, so $|k_1|,|k_2|\in[0,200]$. If $(\mu,\nu)=(+,-)$, then there are still no solutions (ξ,η) of (10.20), using the same argument as before: in view of Lemma 10.1 (i),

$$\alpha(p_{+-1}(\alpha)-\alpha)\leqslant\gamma_0^2\leqslant 0.2,$$

so $|\Phi(\xi,\eta)|\gtrsim 1$, as a consequence of Lemma 10.1 (iv).

On the other hand, if $(\mu,\nu)=(+,+)$, then we may also assume that $\sigma=+$. If β is close to $p_{++2}(\alpha)$ or to $\alpha-p_{++2}(\alpha)$, then $\Phi(\xi,\eta)\gtrsim 1$, for the same reason as before. We are left with the case $\left|\beta-\frac{1}{2}\alpha\right|\lesssim\varepsilon_2$ and $\alpha\geqslant 1$. Therefore, $\left|\eta-\frac{1}{2}\xi\right|\lesssim\varepsilon_2$. We now notice that the equation $\lambda(x)-2\lambda\left(\frac{1}{2}x\right)=0$ has the unique solution $x=\sqrt{2}=:\gamma_1$, and the desired bound on $\big||\xi|-\gamma_1\big|$ follows, since

$$\big||\xi|-\gamma_1\big|\lesssim\left|\Phi_{\sigma\mu\nu}\left(\xi,\tfrac{1}{2}\xi\right)\right|\lesssim|\Phi_{\sigma\mu\nu}(\xi,\eta)|+\left|\Phi_{\sigma\mu\nu}\left(\xi,\tfrac{1}{2}\xi\right)-\Phi_{\sigma\mu\nu}(\xi,\eta)\right|\lesssim\varepsilon_1+\varepsilon_2^2.$$

This completes the proof of the proposition. □

10.3. Bounds on sublevel sets

In this subsection we analyze the sublevel sets of the phase functions Φ, and the interaction of these sublevel sets with several other structures. We start with a general bound on the size of sublevel sets of functions; see [30, Lemma 8.5] for the proof.

LEMMA 10.3. *Let $L,R,M\in\mathbb{R}$, with $M\geqslant\max(1,L,L/R)$, and let $Y:B_R\to\mathbb{R}$, with $B_R:=\{x\in\mathbb{R}^n:|x|<R\}$, be a function satisfying $\|\nabla Y\|_{C^l(B_R)}\leqslant M$ for some $l\geqslant 1$. Then, for any $\varepsilon>0$,*

$$\left|\left\{x\in B_R:|Y(x)|\leqslant\varepsilon \text{ and } \sum_{|\alpha|\leqslant l}|\partial_x^\alpha Y(x)|\geqslant L\right\}\right|\lesssim R^n M L^{-1-1/l}\varepsilon^{1/l}. \tag{10.37}$$

Moreover, if $n=l=1$, K is a union of at most A intervals, and $|Y'(x)|\geqslant L$ on K, then

$$\big|\{x\in K:|Y(x)|\leqslant\varepsilon\}\big|\lesssim AL^{-1}\varepsilon. \tag{10.38}$$

We now prove several important bounds on the sets of time resonances. Assume that $\Phi=\Phi_{\sigma\mu\nu}$ for some choice of $\sigma,\mu,\nu\in\{+,-\}$, and let $\mathcal{D}_1$ be the large constant fixed in Proposition 10.2.

PROPOSITION 10.4. (Volume bounds of sublevel sets) *Assume that $k,k_1,k_2\in\mathbb{Z}$, define $\mathcal{D}_{k,k_1,k_2}$ as in (2.11), let $\bar{k}:=\max(k,k_1,k_2)$, and assume that*

$$\min(k,k_1,k_2)+\max(k,k_1,k_2)\geqslant -100. \tag{10.39}$$

(i) *Let*

$$E_{k,k_1,k_2;\varepsilon}:=\{(\xi,\eta)\in\mathcal{D}_{k,k_1,k_2}:|\Phi(\xi,\eta)|\leqslant\varepsilon\}.$$

Then,

$$\begin{aligned}\sup_\xi\int_{\mathbb{R}^2}\mathbf{1}_{E_{k,k_1,k_2;\varepsilon}}(\xi,\eta)\,d\eta&\lesssim 2^{-\bar{k}/2}\varepsilon\log\Big(2+\frac{1}{\varepsilon}\Big)2^{4\min(k_1^+,k_2^+)},\\ \sup_\eta\int_{\mathbb{R}^2}\mathbf{1}_{E_{k,k_1,k_2;\varepsilon}}(\xi,\eta)\,d\xi&\lesssim 2^{-\bar{k}/2}\varepsilon\log\Big(2+\frac{1}{\varepsilon}\Big)2^{4\min(k_1^+,k^+)}.\end{aligned} \tag{10.40}$$

(ii) *Assume that $r_0\in[2^{-\mathcal{D}_1},2^{\mathcal{D}_1}]$, $\varepsilon\leqslant 2^{\min(k,k_1,k_2,0)/2-\mathcal{D}_1}$, and $\varepsilon'\leqslant 1$, and let*

$$E'_{k,k_1,k_2;\varepsilon,\varepsilon'}=\big\{(\xi,\eta)\in\mathcal{D}_{k,k_1,k_2}:|\Phi(\xi,\eta)|\leqslant\varepsilon \text{ and } \big||\xi-\eta|-r_0\big|\leqslant\varepsilon'\big\}.$$

Then, we can write $E'_{k,k_1,k_2;\varepsilon,\varepsilon'}=E'_1\cup E'_2$, with

$$\sup_\xi\int_{\mathbb{R}^2}\mathbf{1}_{E'_1}(\xi,\eta)\,d\eta+\sup_\eta\int_{\mathbb{R}^2}\mathbf{1}_{E'_2}(\xi,\eta)\,d\xi\lesssim\varepsilon\Big(\log\frac{1}{\varepsilon}\Big)2^{2\bar{k}}(\varepsilon')^{1/2}. \tag{10.41}$$

(iii) *Assume that $\varepsilon\leqslant 2^{\min(k,k_1,k_2,0)/2-\mathcal{D}_1}$, $\varkappa\leqslant 1$, and $p,q\leqslant 0$, and let*

$$E''_{k,k_1,k_2;\varepsilon,\varkappa}=\{(\xi,\eta)\in\mathcal{D}_{k,k_1,k_2}:|\Phi(\xi,\eta)|\leqslant\varepsilon \text{ and } |(\Omega_\eta\Phi)(\xi,\eta)|\leqslant\varkappa\}.$$

Then,

$$\begin{aligned}\sup_\xi\int_{\mathbb{R}^2}\mathbf{1}_{E''_{k,k_1,k_2;\varepsilon,\varkappa}}(\xi,\eta)\varphi_{\geqslant q}(\nabla_\eta\Phi(\xi,\eta))\,d\eta&\lesssim 2^{8\min(|k_1|,|k_2|)}\varepsilon\Big(\log\frac{1}{\varepsilon}\Big)\varkappa 2^{-q}2^{2\bar{k}},\\ \sup_\eta\int_{\mathbb{R}^2}\mathbf{1}_{E''_{k,k_1,k_2;\varepsilon,\varkappa}}(\xi,\eta)\varphi_{\geqslant p}(\nabla_\xi\Phi(\xi,\eta))\,d\xi&\lesssim 2^{8\min(|k_1|,|k|)}\varepsilon\Big(\log\frac{1}{\varepsilon}\Big)\varkappa 2^{-p}2^{2\bar{k}}.\end{aligned} \tag{10.42}$$

As a consequence, we can write $E''_{k,k_1,k_2;\varepsilon,\varkappa}=E''_1\cup E''_2$, with

$$\sup_\xi\int_{\mathbb{R}^2}\mathbf{1}_{E''_1}(\xi,\eta)\,d\eta+\sup_\eta\int_{\mathbb{R}^2}\mathbf{1}_{E''_2}(\xi,\eta)\,d\xi\lesssim\varepsilon\Big(\log\frac{1}{\varepsilon}\Big)\varkappa 2^{12\bar{k}}. \tag{10.43}$$

Moreover, if $\varkappa \leqslant 2^{-8\max(k,k_1,k_2)-\mathcal{D}_1}$, *then*

$$\begin{aligned}&\sup_{\xi}\int_{\mathbb{R}^2}\mathbf{1}_{E''_{k,k_1,k_2;\varepsilon,\varkappa}}(\xi,\eta)\varphi_{\leqslant q}(\nabla_\eta\Phi(\xi,\eta))\,d\eta\lesssim\varkappa 2^q2^{8\bar{k}},\\&\sup_{\eta}\int_{\mathbb{R}^2}\mathbf{1}_{E''_{k,k_1,k_2;\varepsilon,\varkappa}}(\xi,\eta)\varphi_{\leqslant p}(\nabla_\xi\Phi(\xi,\eta))\,d\xi\lesssim\varkappa 2^p2^{8\bar{k}}.\end{aligned}\tag{10.44}$$

Proof. The condition (10.39) is natural, due to (10.6), otherwise

$$|\Phi(\xi,\eta)|\gtrsim 2^{\min(k,k_1,k_2)/2}\quad\text{in } \mathcal{D}_{k,k_1,k_2}.$$

Compare also with the condition $\varepsilon\leqslant 2^{\min(k,k_1,k_2,0)/2-\mathcal{D}_1}$ in (ii) and (iii).

(i) By symmetry, it suffices to prove the inequality in the first line of (10.40). We may assume that $k_2\leqslant k_1$, so, using (10.39),

$$k_1,\max(k,k_2)\in[\bar{k}-10,\bar{k}]\quad\text{and}\quad k,k_2\geqslant-\bar{k}-100.\tag{10.45}$$

Assume that $\xi=(s,0)$ and $\eta=(r\cos\theta,r\sin\theta)$, so

$$-\Phi(\xi,\eta)=-\sigma\lambda(s)+\nu\lambda(r)+\mu\lambda((s^2+r^2-2sr\cos\theta)^{1/2})=:Z(r,\theta).\tag{10.46}$$

We may assume that $\varepsilon\leqslant 2^{\min(k,k_2)}2^{\bar{k}/2-\mathcal{D}_1}$. Notice that

$$\left|\frac{d}{d\theta}Z(r,\theta)\right|=\left|\lambda'((s^2+r^2-2sr\cos\theta)^{1/2})\frac{sr\sin\theta}{(s^2+r^2-2sr\cos\theta)^{1/2}}\right|.\tag{10.47}$$

Assume that $|s-r|\geqslant 2^{\bar{k}-100}$, $s\in[2^{k-4},2^{k+4}]$, and $r\in[2^{k_2-4},2^{k_2+4}]$. Then, for r and s fixed,

$$\big|\{\theta\in[0,2\pi]:|Z(r,\theta)|\leqslant\varepsilon\}\big|\lesssim\sum_{b\in\{0,1\}}\frac{\varepsilon}{\sqrt{2^{\bar{k}/2}2^{\min(k,k_2)}(\varepsilon+Z(r,b\pi))}}.\tag{10.48}$$

Indeed, this follows from (10.47), since in this case $|\partial_\theta Z(r,\theta)|\approx 2^{\min(k,k_2)}2^{\bar{k}/2}|\sin\theta|$ for all $\theta\in[0,2\pi]$. Next, we observe that

$$\big|\{r\in[2^{k_2-4},2^{k_2+4}]:|s-r|\geqslant 2^{\bar{k}-100}\text{ and }|Z(r,b\pi)|\leqslant\varkappa 2^{\min(k,k_2)}2^{\bar{k}/2}\}\big|\lesssim\varkappa 2^{k_2},\tag{10.49}$$

provided that $\bar{k}\geqslant 200$ and $b\in\{0,1\}$. Indeed, in proving (10.49), we may assume that $\varkappa\leqslant 2^{-\mathcal{D}_1}$. Then, we notice that the set in the left-hand side of (10.49) is non-trivial only if either

$$\pm Z(r,b\pi)=\lambda(s)-\lambda(s\pm r)\pm\lambda(r)\quad\text{and}\quad s\in[2^{\bar{k}-10},2^{\bar{k}+10}],\ r\in[2^{-\bar{k}-10},2^{-\bar{k}+10}],$$

or

$$\pm Z(r,b\pi)=\lambda(r)-\lambda(r\pm s)\pm\lambda(s) \quad \text{and} \quad r\in[2^{\bar{k}-10},2^{\bar{k}+10}],\ s\in[2^{-\bar{k}-10},2^{-\bar{k}+10}].$$

In all cases, the desired conclusion (10.49) easily follows, since $|\partial_r Z(r,b\pi)|$ is suitably bounded away from zero. Using also (10.48) it follows that

$$\Big|\big\{\eta:|\eta|\in[2^{k_2-4},2^{k_2+4}],\ \big||\xi|-|\eta|\big|\geqslant 2^{\bar{k}-100},\ \text{and } |\Phi(\xi,\eta)|\leqslant\varepsilon\big\}\Big|\lesssim\varepsilon 2^{-\bar{k}/2}2^{4k_2^+}, \tag{10.50}$$

provided that $|\xi|\in[2^{k-4},2^{k+4}]$, $\bar{k}\geqslant 200$, and (10.45) holds.

The case $\bar{k}\leqslant 200$ is easier. In this case we have $2^k, 2^{k_1}, 2^{k_2}\approx 1$, due to (10.45). In view of Proposition 10.2 (iii), if $|Z(r,b\pi)|\leqslant\varkappa\leqslant 2^{-2\mathcal{D}_1}$ and $|\partial_r Z(r,b\pi)|\leqslant 2^{-2\mathcal{D}_1}$, then s is close to γ_1, r is close to $\frac{1}{2}\gamma_1$, and $b=0$. As a consequence, $|\partial_r^2 Z(r,b\pi)|\gtrsim 1$. It follows from Lemma 10.3 that

$$\big|\{r\in[2^{k_2-4},2^{k_2+4}]:|s-r|\geqslant 2^{\bar{k}-100} \text{ and } |Z(r,b\pi)|\leqslant\varkappa\}\big|\lesssim\varkappa^{1/2},$$

provided that $\bar{k}\leqslant 200$ and $\varkappa>0$. Using (10.48) again, it follows that

$$\Big|\big\{\eta:|\eta|\in[2^{k_2-4},2^{k_2+4}],\ \big||\xi|-|\eta|\big|\geqslant 2^{\bar{k}-100},\ \text{and } |\Phi(\xi,\eta)|\leqslant\varepsilon\big\}\Big|\lesssim\varepsilon\log\Big(2+\frac{1}{\varepsilon}\Big), \tag{10.51}$$

provided that $|\xi|\in[2^{k-4},2^{k+4}]$ and $\bar{k}\leqslant 200$.

Finally, we estimate the contribution of the set where $\big||\xi|-|\eta|\big|\leqslant 2^{\bar{k}-100}$. In this case, we may assume that $k,k_1,k_2\geqslant\bar{k}-20$. We replace (10.48) by

$$\big|\{\theta\in[2^{-\mathcal{D}_1},2\pi-2^{-\mathcal{D}_1}]:|Z(r,\theta)|\leqslant\varepsilon\}\big|\lesssim\frac{\varepsilon}{\sqrt{2^{3\bar{k}/2}(\varepsilon+Z(r,\pi))}}, \tag{10.52}$$

which follows from (10.47) (since $|\partial_\theta Z(r,\theta)|\approx 2^{3\bar{k}/2}|\sin\theta|$ for all $\theta\in[2^{-\mathcal{D}_1},2\pi-2^{-\mathcal{D}_1}]$). The proof proceeds as before, by analyzing the vanishing of the function $r\mapsto Z(r,\pi)$ (it is in fact slightly easier, since $|Z(r,\pi)|\gtrsim 2^{3\bar{k}/2}$ if $\bar{k}\geqslant 200$). It follows that

$$\Big|\big\{\eta:|\eta|\in[2^{k_2-4},2^{k_2+4}],\ \big||\xi|-|\eta|\big|\leqslant 2^{\bar{k}-100},\ \text{and } |\Phi(\xi,\eta)|\leqslant\varepsilon\big\}\Big|\lesssim\varepsilon\Big(\log\Big(2+\frac{1}{\varepsilon}\Big)\Big)2^{\bar{k}/2}.$$

The desired bound in the first line of (10.40) follows using also (10.50)–(10.51).

(ii) We may assume that $\min(k,k_2)\geqslant -2\mathcal{D}_1$ and that $\varepsilon'\leqslant 2^{-\mathcal{D}_1^2}$. Define

$$\begin{aligned} E_1' &:= \{(\xi,\eta)\in E'_{k,k_1,k_2;\varepsilon,\varepsilon'}:|\nabla_\eta\Phi(\xi,\eta)|\geqslant 2^{-20\mathcal{D}_1}\},\\ E_2' &:= \{(\xi,\eta)\in E'_{k,k_1,k_2;\varepsilon,\varepsilon'}:|\nabla_\xi\Phi(\xi,\eta)|\geqslant 2^{-20\mathcal{D}_1}\}. \end{aligned} \tag{10.53}$$

It is easy to see that

$$E'_{k,k_1,k_2;\varepsilon,\varepsilon'}=E'_1\cup E'_2,$$

using Proposition 10.2 (ii). By symmetry, it suffices to prove (10.41) for the first term in the left-hand side. Let $\xi=(s,0)$, $\eta=(r\cos\theta, r\sin\theta)$, and

$$\begin{aligned} E'_{1,\xi,1}&:=\{\eta:(\xi,\eta)\in E'_1 \text{ and } |\sin\theta|\leqslant(\varepsilon')^{1/2}2^{-2k_2}\},\\ E'_{1,\xi,2}&:=\{\eta:(\xi,\eta)\in E'_1 \text{ and } |\sin\theta|\geqslant(\varepsilon')^{1/2}2^{-2k_2}\}. \end{aligned}\tag{10.54}$$

It follows from Lemma 10.3 that $|E'_{1,\xi,1}|\lesssim\varepsilon\cdot(\varepsilon')^{1/2}$. Indeed, as $|\nabla_\eta\Phi(\xi,\eta)|\geqslant 2^{-20\mathcal{D}_1}$ and $|\sin\theta|\leqslant(\varepsilon')^{1/2}2^{-2k_2}$, it follows from formula (10.46) that $|\partial_r(\Phi(\xi,\eta))|\geqslant 2^{-21\mathcal{D}_1}$ in $E'_{1,\xi,1}$. The desired conclusion follows by applying Lemma 10.3 for every suitable angle θ.

To estimate $|E'_{1,\xi,2}|$, we use formula (10.46). It follows from the definitions that

$$E'_{1,\xi,2}\subseteq\{\eta:r\in[2^{k_2-4},2^{k_2+4}],\ \lambda(r)\in K_{s,r_0},\ |\sin\theta|\geqslant(\varepsilon')^{1/2}2^{-2k_2},\ \text{and } |\Phi(\xi,\eta)|\leqslant\varepsilon\},$$

where K_{s,r_0} is an interval of length $\lesssim\varepsilon'$ and $k_2\geqslant-2\mathcal{D}_1$. Therefore, using formula (10.46) as before, $|E'_{1,\xi,2}|\lesssim 2^{2k_2}\varepsilon(\varepsilon')^{1/2}$, as desired.

(iii) For (10.42) it suffices to prove the inequality in the first line. We may also assume that (10.39) holds, and that $\varkappa\leqslant 2^{q-2\max(k,k_1,k_2)-\mathcal{D}_1}$. Assume, as before, that $\xi=(s,0)$ and $\eta=(r\cos\theta, r\sin\theta)$. Since

$$|(\Omega_\eta\Phi)(\xi,\eta)|=\frac{\lambda'(|\xi-\eta|)}{|\xi-\eta|}|(\xi\cdot\eta^\perp)|,$$

the condition $|(\Omega_\eta\Phi)(\xi,\eta)|\leqslant\varkappa$ gives

$$|\sin\theta|\lesssim\varkappa 2^{k_1-k-k_2-|k_1|/2}\tag{10.55}$$

in the support of the integral. Formula (10.46) shows that

$$r^{-1}|\partial_\theta\Phi(\xi,\eta)|=\frac{\lambda'(|\xi-\eta|)}{|\xi-\eta|}s|\sin\theta|\lesssim\varkappa 2^{-k_2}$$

in the support of the integral. Therefore, $|\partial_r\Phi(\xi,\eta)|\geqslant 2^{q-4}$ in the support of the integral.

We now assume that θ is fixed satisfying (10.55). If $\big||k_2|-|k_1|\big|\geqslant 100$, then

$$|\partial_r\Phi(\xi,\eta)|\gtrsim 2^{|k_1|/2}+2^{|k_2|/2}\quad\text{for all } (\xi,\eta)\in\mathcal{D}_{k,k_1,k_2},$$

and the desired bound follows from (10.37), with $l=1$ and $n=1$. If $\big||k_2|-|k_1|\big|\leqslant 100$, then we still use (10.37) to conclude that the integral is dominated by

$$C\varepsilon 2^{-2q}2^{5|k_1|/2}\varkappa 2^{k_1-k-|k_1|/2}\lesssim\varepsilon\varkappa 2^{-2q}2^{4|k_1|}.$$

This suffices to prove (10.42) if $2^q \geqslant 2^{-6\max(k,k_1,k_2)-\mathcal{D}_1}$. Finally, if

$$\big||k_2|-|k_1|\big| \leqslant 100, \quad 2^q \leqslant 2^{-6\max(k,k_1,k_2)-\mathcal{D}_1}, \quad \text{and} \quad \varkappa \leqslant 2^{q-2\max(k,k_1,k_2)-\mathcal{D}_1},$$

then we would like to apply (10.38). For this, it suffices to verify that, for any θ fixed satisfying (10.55), the number of intervals (in the variable r) where $|\partial_r\Phi(\xi,\eta)| \leqslant 2^{q-4}$ is uniformly bounded. In view of Proposition 10.2 (iii), these intervals are present only when $k, k_1, k_2 \in [-10, 10]$, $|s-\gamma_1| \ll 1$, $\big|r-\tfrac{1}{2}\gamma_1\big| \ll 1$, and

$$\Phi(\xi,\eta) = \pm(\lambda(s)-\lambda(r)-\lambda((s^2+r^2-2sr\cos\theta)^{1/2})).$$

In this case, however, $|\partial_r^2\Phi(\xi,\eta)| \gtrsim 1$. As a consequence, for any s and θ there is at most one interval in r where $|\partial_r\Phi(\xi,\eta)| \leqslant 2^{q-4}$, and the desired bound follows from (10.38).

The decomposition (10.43) follows from (10.42) and Proposition 10.2 (iii), by setting $2^p = 2^q = 2^{-2\mathcal{D}_1} 2^{-2\max(k,k_1,k_2)}$.

To prove the first inequality in (10.44), we may assume $q \leqslant -5\max(k,k_1,k_2) - \mathcal{D}_1$ (due to (10.55)). By Proposition 10.2 (iii), we may assume $k, k_1, k_2 \in [-10, 10]$, $|s-\gamma_1| \ll 1$, $\big|r-\tfrac{1}{2}\gamma_1\big| \ll 1$, and

$$\Phi(\xi,\eta) = \pm(\lambda(s)-\lambda(r)-\lambda((s^2+r^2-2sr\cos\theta)^{1/2})).$$

As before, $|\partial_r^2\Phi(\xi,\eta)| \gtrsim 1$ in this case. As a consequence, for any s and θ fixed, the measure of the set of numbers r for which $|\partial_r\Phi(\xi,\eta)| \lesssim 2^q$ is bounded by $C2^q$, and the desired bound follows. □

We will also need a variant of Schur's lemma for suitably localized kernels.

LEMMA 10.5. *Assume that* $n, p \leqslant -\frac{1}{10}\mathcal{D}$, $k, k_1, k_2 \in \mathbb{Z}$, $l \leqslant \frac{1}{2}\min(k, k_1, k_2, 0) - \frac{1}{10}\mathcal{D}$, *and* $\varrho_1, \varrho_2 \in \{\gamma_0, \gamma_1\}$. *Then, with* $\mathcal{D}_{k,k_1,k_2}$ *as in* (2.11), *and assuming that*

$$\Big\| \sup_{\omega\in\mathbb{S}^1} |\hat{f}(r\omega)| \Big\|_{L^2(r\,dr)} \leqslant 1,$$

we have

$$\bigg\| \int_{\mathbb{R}^2} \mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\eta)\varphi_l(\Phi(\xi,\eta))\varphi_n(|\xi-\eta|-\varrho_1)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta \bigg\|_{L^2_\xi} \lesssim 2^{(l+n)/2}\|g\|_{L^2}, \quad (10.56)$$

$$\bigg\| \int_{\mathbb{R}^2} \mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\eta)\varphi_l(\Phi(\xi,\eta))\varphi_n(|\xi-\eta|-\varrho_1)\varphi_p(|\eta|-\varrho_2)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta \bigg\|_{L^2_\xi} \lesssim \min(2^{l/2}, 2^{p/2}) 2^{(l+n)/2}\|g\|_{L^2}, \quad (10.57)$$

and

$$\bigg\| \int_{\mathbb{R}^2} \mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\eta)\varphi_l(\Phi(\xi,\eta))\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta \bigg\|_{L^2_\xi} \lesssim 2^{5|k_1|} 2^{3l/4}(1+|l|)\|g\|_{L^2}. \quad (10.58)$$

Proof. By (10.6), we may assume that $\min(k,k_1,k_2)+\bar{k}\geqslant -100$, where

$$\bar{k}=\max(k,k_1,k_2).$$

We start with (10.56). We may assume that $\min(k,k_1,k_2)\geqslant -200$. By Schur's test, it suffices to show that

$$\begin{aligned}\sup_\xi \int_{\mathbb{R}^2} \mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\eta)\varphi_l(\Phi(\xi,\eta))\varphi_n(|\xi-\eta|-\varrho_1)|\hat{f}(\xi-\eta)|\,d\eta \lesssim 2^{(l+n)/2},\\ \sup_\eta \int_{\mathbb{R}^2} \mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\eta)\varphi_l(\Phi(\xi,\eta))\varphi_n(|\xi-\eta|-\varrho_1)|\hat{f}(\xi-\eta)|\,d\xi \lesssim 2^{(l+n)/2}.\end{aligned} \tag{10.59}$$

We focus on the first inequality. Fix $\xi\in\mathbb{R}^2$ and introduce polar coordinates, $\eta=\xi-r\omega$, $r\in(0,\infty)$, $\omega\in\mathbb{S}^1$. The left-hand side is dominated by

$$C\int_{\omega\in\mathbb{S}^1}\int_{2^{k_1-4}}^{2^{k_1+4}} \mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\xi-r\omega)\varphi_l(\Phi(\xi,\xi-r\omega))\varphi_n(r-\varrho_1)|\hat{f}(r\omega)|r\,dr\,d\omega,$$

for a constant C sufficiently large. Therefore, it suffices to show that

$$\sup_{r,\xi}\int_{\omega\in\mathbb{S}^1} \mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\xi-r\omega)\varphi_l(\Phi(\xi,\xi-r\omega))\,d\omega \lesssim 2^{l/2}2^{|k_1|/2}, \tag{10.60}$$

which is easily verified as in Proposition 10.4, using the identity (10.46). Indeed, for ξ and r fixed, and letting $\omega=(\cos\theta,\sin\theta)$, the absolute value of the $d/d\theta$ derivative of the function $\theta\mapsto\Phi(\xi,\xi-r(\cos\theta,\sin\theta))$ is bounded from below by

$$c|\sin\theta|2^{k+k_1-k_2}2^{|k_2|/2}\gtrsim|\sin\theta|2^{-|k_1|/2}.$$

The bound (10.60) follows using also (10.38). The second inequality in (10.59) follows similarly.

We now prove (10.57). We may assume that $k,k_1,k_2\in[-80,80]$, and it suffices to show that

$$\begin{aligned}\sup_\xi \int_{\mathbb{R}^2} \mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\eta)\varphi_l(\Phi(\xi,\eta))\varphi_n(|\xi-\eta|-\varrho_1)\varphi_p(|\eta|-\varrho_2)|\hat{f}(\xi-\eta)|\,d\eta \lesssim 2^{n/2}\min(2^l,2^p),\\ \sup_\eta \int_{\mathbb{R}^2} \mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\eta)\varphi_l(\Phi(\xi,\eta))\varphi_n(|\xi-\eta|-\varrho_1)\varphi_p(|\eta|-\varrho_2)|\hat{f}(\xi-\eta)|\,d\xi \lesssim 2^{l+n/2}.\end{aligned}$$

We proceed as for (10.59), but replace (10.60) by

$$\begin{aligned}\sup_{|\xi|\approx 1}\sup_r\int_{\omega\in\mathbb{S}^1}\varphi_l(\Phi(\xi,\xi-r\omega))\varphi_n(r-\varrho_1)\varphi_p(|\xi-r\omega|-\varrho_2)\,d\omega\lesssim\min\{2^l,2^p\},\\ \sup_\eta\sup_r\int_{\omega\in\mathbb{S}^1}\varphi_l(\Phi(\eta+r\omega,\eta))\varphi_n(r-\varrho_1)\varphi_p(|\eta|-\varrho_2)\varphi_{\geqslant -90}(\eta+r\omega)\,d\omega\lesssim 2^l.\end{aligned} \tag{10.61}$$

The bounds (10.61) easily follow, using also formula (10.46) to prove the 2^l bounds, once we notice that $|\sin\theta|\gtrsim 1$ in the support of the integrals. For this, we only need to verify that the points ξ and η cannot be almost aligned; more precisely, we need to verify that, if ξ and η are aligned, then $|\Phi(\xi,\xi-\eta)|+\big||\xi-\eta|-\varrho_2\big|+\big||\eta|-\varrho_1\big|\gtrsim 1$. For this, it suffices to notice that

$$\big|\pm\lambda(|\xi|)\pm\lambda(\varrho_1)\pm\lambda(\varrho_2)\big|\gtrsim 1,\quad \text{if } |\xi|\gtrsim 1 \text{ and } \pm|\xi|\pm\varrho_1\pm\varrho_2=0.$$

Recalling that $\varrho_1,\varrho_2\in\{\gamma_0,\gamma_1\}$, it suffices to verify $\lambda(2\gamma_0)-2\lambda(\gamma_0)\neq 0$, $\lambda(2\gamma_1)-2\lambda(\gamma_1)\neq 0$, $\lambda(\gamma_0+\gamma_1)-\lambda(\gamma_0)-\lambda(\gamma_1)\neq 0$, and $\lambda(-\gamma_0+\gamma_1)+\lambda(\gamma_0)-\lambda(\gamma_1)\neq 0$. These claims follow from Lemma 10.1 (iv), since the numbers γ_0^2, γ_1^2, $\gamma_0\gamma_1$, and $\gamma_0(\gamma_1-\gamma_0)$ are not in the interval $\big[\frac{4}{9},\frac{1}{2}\big]$.

We now turn to (10.58). By Schur's lemma, it suffices to show that

$$\begin{aligned}&\sup_\xi\int_{\mathbb{R}^2}\varphi_l(\Phi(\xi,\eta))\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\eta)|\hat f(\xi-\eta)|\,d\eta\lesssim 2^{5|k_1|}2^{3l/4}(1+|l|),\\ &\sup_\eta\int_{\mathbb{R}^2}\varphi_l(\Phi(\xi,\eta))\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\eta)|\hat f(\xi-\eta)|\,d\xi\lesssim 2^{5|k_1|}2^{3l/4}(1+|l|).\end{aligned}\tag{10.62}$$

We show the first inequality. Introducing polar coordinates, as before, we estimate

$$\begin{aligned}&\int_{\mathbb{R}^2}\varphi_l(\Phi(\xi,\xi-r\omega))\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\xi-r\omega)|\hat f(r\omega)|r\,dr\,d\omega\\ &\lesssim\Big\|\sup_\omega|\hat f(r\omega)|\Big\|_{L^2(r\,dr)}\Big\|\int_{\mathbb{S}^1}\varphi_l(\Phi(\xi,\xi-r\omega))\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\xi-r\omega)\,d\omega\Big\|_{L^2(r\,dr)}\\ &\lesssim\|\varphi_{\leqslant l+2}(\Phi(\xi,\xi-\eta))\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\xi-\eta)\|_{L^2_\eta}\,\|\varphi_{\leqslant l+2}(\Phi(\xi,\xi-r\omega))\mathbf{1}_{\mathcal{D}_{k,k_1,k_2}}(\xi,\xi-r\omega)\|_{L^\infty_r L^2_\omega}\\ &\lesssim 2^{5|k_1|}2^{3l/4}(1+|l|),\end{aligned}$$

using Proposition 10.4 (i) and the bound (10.60). The second inequality in (10.62) follows similarly. □

10.4. Iterated resonances

In this subsection we prove a lemma concerning some properties of the cubic phases

$$\tilde\Phi(\xi,\eta,\sigma)=\tilde\Phi_{+\mu\beta\gamma}(\xi,\eta,\sigma)=\Lambda(\xi)-\Lambda_\mu(\xi-\eta)-\Lambda_\beta(\eta-\sigma)-\Lambda_\gamma(\sigma).\tag{10.63}$$

These properties are used only in the proofs of Lemmas 9.7 and 9.8.

LEMMA 10.6. (i) *Assume that $\xi,\eta,\sigma\in\mathbb{R}^2$ satisfy*

$$\max\big(\big||\xi-\eta|-\gamma_0\big|,\big||\eta-\sigma|-\gamma_0\big|,\big||\sigma|-\gamma_0\big|\big)\leqslant 2^{-\mathcal{D}_1/2},\tag{10.64}$$

and

$$|\nabla_{\eta,\sigma}\tilde{\Phi}(\xi,\eta,\sigma)| \leqslant \varkappa_1 \leqslant 2^{-4\mathcal{D}_1}. \tag{10.65}$$

Then, for $\nu\in\{+,-\}$,

$$\Lambda(\xi)-\Lambda_\mu(\xi-\eta)-\Lambda_\nu(\eta) \gtrsim |\eta|. \tag{10.66}$$

Moreover,

$$\text{if } |\nabla_\xi\tilde{\Phi}(\xi,\eta,\sigma)| \geqslant \varkappa_2 \geqslant 2^{\mathcal{D}_1}\varkappa_1, \text{ then } |\tilde{\Phi}(\xi,\eta,\sigma)| \gtrsim \varkappa_2^{3/2}. \tag{10.67}$$

(ii) *Assume that* $\xi,\eta,\sigma\in\mathbb{R}^2$ *satisfy* $|\xi-\eta|$, $|\eta-\sigma|$, $|\sigma|\in[2^{-10},2^{10}]$ *and*

$$\begin{aligned} |\Phi_{+\mu\nu}(\xi,\eta)| &= |\Lambda(\xi)-\Lambda_\mu(\xi-\eta)-\Lambda_\nu(\eta)| \leqslant 2^{-2\mathcal{D}_1},\\ |\Phi_{\nu\beta\gamma}(\eta,\sigma)| &= |\Lambda_\nu(\eta)-\Lambda_\beta(\eta-\sigma)-\Lambda_\gamma(\sigma)| \leqslant 2^{-2\mathcal{D}_1}. \end{aligned} \tag{10.68}$$

If

$$|\nabla_{\eta,\sigma}\tilde{\Phi}(\xi,\eta,\sigma)| \leqslant \varkappa \leqslant 2^{-4\mathcal{D}_1} \tag{10.69}$$

then

$$\mu=-, \quad \nu=\beta=\gamma=+, \quad |\eta-2\sigma|+|\xi-\sigma| \lesssim \varkappa, \quad \text{and} \quad |\nabla_\xi\tilde{\Phi}(\xi,\eta,\sigma)| \lesssim \varkappa. \tag{10.70}$$

Proof. (i) If (10.64) and (10.65) hold, then the vectors $\xi-\eta$, $\eta-\sigma$, and σ are almost aligned. Thus, either $|\eta|\leqslant 2^{-\mathcal{D}_1/2+10}$ or $\big||\eta|-2\gamma_0\big|\leqslant 2^{-\mathcal{D}_1/2+10}$. We will assume that we are in the second case, $\big||\eta|-2\gamma_0\big|\leqslant 2^{-\mathcal{D}_1/2+10}$ (the other case is similar, in fact slightly easier because the inequality (10.66) is a direct consequence of (10.6)). Therefore, either $\big||\xi|-3\gamma_0\big|\leqslant 2^{-\mathcal{D}_1/2+20}$, and the desired conclusions are trivial, or $\big||\xi|-\gamma_0\big|\leqslant 2^{-\mathcal{D}_1/2+20}$. In the latter case, (10.66) follows since $|\lambda(\gamma_0)\pm\lambda(\gamma_0)\pm\lambda(2\gamma_0)|\gtrsim 1$; it remains to prove (10.67) in the case $\mu=-$, $\beta=\gamma=+$,

$$\begin{gathered} \tilde{\Phi}(\xi,\eta,\sigma) = \Lambda(\xi)+\Lambda(\xi-\eta)-\Lambda(\eta-\sigma)-\Lambda(\sigma),\\ \big||\eta|-2\gamma_0\big| \leqslant 2^{-\mathcal{D}_1/2+20}, \quad \big||\xi|-\gamma_0\big| \leqslant 2^{-\mathcal{D}_1/2+20}. \end{gathered} \tag{10.71}$$

In view of (10.65), the angle between any two of the vectors $\xi-\eta$, $\eta-\sigma$, and σ is either $O(\varkappa_1)$ or $\pi+O(\varkappa_1)$. Given $\sigma=ze$ for some $e\in\mathbb{S}^1$, we write $\eta=ye+\eta'$, $\xi=xe+\xi'$, with $e\cdot\eta'=e\cdot\xi'=0$, and $|\eta'|+|\xi'|\lesssim\varkappa_1$. Notice that $|\tilde{\Phi}(\xi,\eta,\sigma)-\tilde{\Phi}(xe,ye,ze)|\lesssim\varkappa_1^2$. Therefore, we may assume that

$$\begin{aligned} |x-\gamma_0|+|y-2\gamma_0|+|z-\gamma_0| &\leqslant 2^{-\mathcal{D}_1/2+30},\\ |\lambda'(y-z)-\lambda'(z)| &\leqslant 2\varkappa_1,\\ |\lambda'(y-x)-\lambda'(y-z)| &\leqslant 2\varkappa_1,\\ |\lambda'(x)-\lambda'(y-x)| &\geqslant \tfrac{1}{2}\varkappa_2, \end{aligned} \tag{10.72}$$

and it remains to prove that

$$|\tilde{\Phi}(xe, ye, ze)| = |\lambda(x)+\lambda(y-x)-\lambda(y-z)-\lambda(z)| \gtrsim \varkappa_2^{3/2}. \tag{10.73}$$

Let $z'\neq z$ denote the unique solution to the equation $\lambda'(z')=\lambda'(z)$, and let $d:=|z-\gamma_0|$. Then $|z'-\gamma_0|\approx d$, in view of (10.10). Moreover, $d\geqslant\sqrt{\varkappa_1}$; otherwise $|y-z-\gamma_0|\lesssim\sqrt{\varkappa_1}$ and $|y-x-\gamma_0|\lesssim\sqrt{\varkappa_1}$, so $|x-\gamma_0|\lesssim\sqrt{\varkappa_1}$, in contradiction with the assumption

$$|\lambda'(x)-\lambda'(y-x)| \geqslant \tfrac{1}{2}\varkappa_2.$$

Moreover,

$$\text{there are } \sigma_1, \sigma_2 \in \{z, z'\} \text{ such that } |y-z-\sigma_1|+|y-x-\sigma_2| \lesssim \frac{\varkappa_1}{d}. \tag{10.74}$$

In fact, we may assume $d\geqslant 2^{-\mathcal{D}_1/4}\varkappa_2^{1/2}$, since otherwise $|x-\gamma_0|+|y-x-\gamma_0|\lesssim d$, and hence $|\lambda'(x)-\lambda'(y-x)|\lesssim d^2$, which contradicts (10.65).

Now, we must have $\sigma_1=z$; in fact, if $\sigma_1=z'$, then $x=z+z'-\sigma_2+O(\varkappa_1/d)$, and thus

$$|\lambda'(x)-\lambda'(\sigma_2)| \lesssim \varkappa_1,$$

which again contradicts (10.72). Similarly, $\sigma_2=z'$. Therefore,

$$y=2z+O\Big(\frac{\varkappa_1}{d}\Big), \quad x=2z-z'+O\Big(\frac{\varkappa_1}{d}\Big), \quad y-x=z'+O\Big(\frac{\varkappa_1}{d}\Big). \tag{10.75}$$

We expand the function λ at γ_0 in its Taylor series:

$$\lambda(v)=\lambda(\gamma_0)+c_1(v-\gamma_0)+c_3(v-\gamma_0)^3+O(v-\gamma_0)^4,$$

where $c_1, c_3\neq 0$. Using (10.75), we have

$$\begin{aligned}\tilde{\Phi}(xe, ye, ze) &= c_3((x-\gamma_0)^3+(y-x-\gamma_0)^3-(z-\gamma_0)^3-(y-z-\gamma_0)^3)+O(d^4)\\ &= c_3((2z-z'-\gamma_0)^3+(z'-\gamma_0)^3-2(z-\gamma_0)^3)+O(d^4+\varkappa_1 d).\end{aligned}$$

In view of (10.10), $z+z'-2\gamma_0=O(d^2)$. Therefore, $\tilde{\Phi}(xe, ye, ze)=24(z-\gamma_0)^3+O(d^4+\varkappa_1 d)$, which shows that $|\tilde{\Phi}(xe, ye, ze)|\gtrsim d^3$. The desired conclusion (10.73) follows.

(ii) The conditions $|\Phi_{\nu\beta\gamma}(\eta,\sigma)|\leqslant 2^{-2\mathcal{D}_1}$ and $|(\nabla_\sigma\Phi_{\nu\beta\gamma})(\eta,\sigma)|\leqslant\varkappa$ show that η corresponds to a space-time resonance output. It follows from Lemma 10.2 (iii) that

$$|\eta-ye|+\big|\sigma-\tfrac{1}{2}ye\big| \lesssim \varkappa, \quad |y-\gamma_1| \lesssim 2^{-2\mathcal{D}_1}, \quad \text{and} \quad \nu=\beta=\gamma, \tag{10.76}$$

for some $e\in\mathbb{S}^1$. Let $b\approx 0.207$ denote the unique non-negative number $b\neq\frac{1}{2}\gamma_1$ with the property that $\lambda'(b)=\lambda'\big(\frac{1}{2}\gamma_1\big)$. The condition $|\nabla_\eta\tilde{\Phi}(\xi,\eta,\sigma)|\leqslant\varkappa$ shows that $\xi-\eta$ is close to one of the vectors $\big(\frac{1}{2}\gamma_1\big)e$, $-\big(\frac{1}{2}\gamma_1\big)e$, be, and $-be$. However, $\lambda(b)\approx 0.465$, $\lambda(\gamma_1+b)\approx 2.462$, $\lambda(\gamma_1-b)\approx 1.722$, and $\lambda(\gamma_1)\approx 2.060$. Therefore, the condition $|\Phi_{+\mu\nu}(\xi,\eta)|\leqslant 2^{-2\mathcal{D}_1}$ prevents $\xi-\eta$ from being close to one of the vectors be or $-be$. Similarly, $\xi-\eta$ cannot be close to the vector $\big(\frac{1}{2}\gamma_1\big)e$, since $\lambda\big(\frac{1}{2}\gamma_1\big)\approx 1.030$, $\lambda\big(\frac{3}{2}\gamma_1\big)\approx 3.416$. It follows that

$$\big|(\xi-\eta)+\big(\tfrac{1}{2}\gamma_1\big)e\big|\lesssim 2^{-2\mathcal{D}_1},\quad \big||\xi|-\tfrac{1}{2}\gamma_1\big|\lesssim 2^{-2\mathcal{D}_1},\quad \mu=-,\quad \text{and}\quad \nu=+.$$

The condition $|\nabla_\eta\tilde{\Phi}(\xi,\eta,\sigma)|\leqslant\varkappa$ then gives

$$|(\eta-\xi)-(\eta-\sigma)|\lesssim\varkappa,$$

and remaining bounds in (10.70) follow using also (10.76). □

11. The functions Υ

The analysis in the proofs of the crucial L^2 lemmas in §6 depends on understanding the properties of the functions $\Upsilon\colon\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$,

$$\Upsilon(\xi,\eta):=(\nabla^2_{\xi,\eta}\Phi)(\xi,\eta)[(\nabla^\perp_\xi\Phi)(\xi,\eta),(\nabla^\perp_\eta\Phi)(\xi,\eta)]. \tag{11.1}$$

We calculate

$$\begin{aligned}(\nabla_\eta\Phi)(\xi,\eta)&=-\lambda'_\nu(|\eta|)\frac{\eta}{|\eta|}+\lambda'_\mu(|\xi-\eta|)\frac{\xi-\eta}{|\xi-\eta|},\\ (\nabla_\xi\Phi)(\xi,\eta)&=\lambda'_\sigma(|\xi|)\frac{\xi}{|\xi|}-\lambda'_\mu(|\xi-\eta|)\frac{\xi-\eta}{|\xi-\eta|},\end{aligned} \tag{11.2}$$

and

$$\begin{aligned}(\nabla^2_{\xi,\eta}\Phi)(\xi,\eta)[\partial_i,\partial_j]&=\lambda''_\mu(|\xi-\eta|)\frac{(\xi_i-\eta_i)(\xi_j-\eta_j)}{|\xi-\eta|^2}\\ &\quad+\lambda'_\mu(|\xi-\eta|)\frac{\delta_{ij}|\xi-\eta|^2-(\xi_i-\eta_i)(\xi_j-\eta_j)}{|\xi-\eta|^3}.\end{aligned} \tag{11.3}$$

Using these formulas and the identity $(v\cdot w^\perp)^2+(v\cdot w)^2=|v|^2|w|^2$, we calculate

$$\begin{aligned}-\Upsilon(\xi,\eta)&=\frac{\lambda''_\mu(|z|)}{|z|^2}\frac{\lambda'_\sigma(|\xi|)}{|\xi|}\frac{\lambda'_\nu(|\eta|)}{|\eta|}(\eta\cdot\xi^\perp)^2\\ &\quad+\frac{\lambda'_\mu(|z|)}{|z|^3}\bigg(\lambda'_\mu(|z|)|z|-\frac{\lambda'_\sigma(|\xi|)}{|\xi|}\xi\cdot z\bigg)\bigg(\lambda'_\mu(|z|)|z|-\frac{\lambda'_\nu(|\eta|)}{|\eta|}\eta\cdot z\bigg),\end{aligned} \tag{11.4}$$

where $z:=\xi-\eta$. We also define the normalized function

$$\widehat{\Upsilon}(\xi,\eta):=\frac{\Upsilon(\xi,\eta)}{|(\nabla_\xi\Phi)(\xi,\eta)|\,|(\nabla_\eta\Phi)(\xi,\eta)|}. \tag{11.5}$$

We first consider the case of large frequencies.

LEMMA 11.1. *Assume that* $\sigma=\nu=+$, $k\geqslant\mathcal{D}_1$, *and* $p-\frac{1}{2}k\leqslant-\mathcal{D}_1$.

(i) *Assume that*

$$|\Phi(\xi,\eta)|\leqslant 2^p,\quad |\xi|,|\eta|\in[2^{k-2},2^{k+2}],\quad \textit{and}\quad 2^{-20}\leqslant|\xi-\eta|\leqslant 2^{20}. \tag{11.6}$$

Let $z:=\xi-\eta$. *Then, with* $p^+=\max(p,0)$,

$$\frac{|\xi\cdot\eta^\perp|}{|\xi|\,|\eta|}\approx 2^{-k}\quad \textit{and}\quad \frac{|\xi\cdot z|}{|\xi|\,|z|}+\frac{|\eta\cdot z|}{|\eta|\,|z|}\lesssim 2^{p^+-k/2}. \tag{11.7}$$

Moreover, we can write

$$\begin{gathered}-\mu\Upsilon(\xi,\eta)=\lambda''(|z|)A(\xi,\eta)+B(\xi,z)B(\eta,z),\\ |A(\xi,\eta)|\gtrsim 2^k,\quad \|D^\alpha A\|_{L^\infty}\lesssim_\alpha 2^k,\quad \|B\|_{L^\infty}\lesssim 2^{p^+},\quad \|D^\alpha B\|_{L^\infty}\lesssim_\alpha 2^{k/2}.\end{gathered} \tag{11.8}$$

(ii) *Assume that* $z=(\varrho\cos\theta,\varrho\sin\theta)$, $|\varrho|\in[2^{-20},2^{20}]$. *There exist functions* $\theta^1=\theta^1_{|\xi|,\mu}$ *and* $\theta^2=\theta^2_{|\eta|,\mu}$ *such that*

$$\begin{aligned}&\textit{if}\ \ 2^{k-2}\leqslant|\xi|\leqslant 2^{k+2}\ \ \textit{and}\ \ |\Phi(\xi,\xi-z)|\leqslant 2^p,\ \textit{then}\ \min\nolimits_{\mp}|\theta-\arg(\xi)\mp\theta^1(\varrho)|\lesssim 2^{p-k/2},\\ &\textit{if}\ \ 2^{k-2}\leqslant|\eta|\leqslant 2^{k+2}\ \ \textit{and}\ \ |\Phi(\eta+z,\eta)|\leqslant 2^p,\ \textit{then}\ \min\nolimits_{\mp}|\theta-\arg(\eta)\mp\theta^2(\varrho)|\lesssim 2^{p-k/2}.\end{aligned} \tag{11.9}$$

Moreover,

$$\bigl|\theta^1(\varrho)-\tfrac{1}{2}\pi\bigr|+\bigl|\theta^2(\varrho)-\tfrac{1}{2}\pi\bigr|\lesssim 2^{-k/2}\quad \textit{and}\quad |\partial_\varrho\theta^1|+|\partial_\varrho\theta^2|\lesssim 2^{-k/2}. \tag{11.10}$$

(iii) *Assume that* $|\xi|,|\eta|\in[2^{k-2},2^{k+2}]$. *For* $0<\varkappa\leqslant 2^{-\mathcal{D}_1}$ *and integers* r *and* q *such that* $q\leqslant-\mathcal{D}_1$ *and* $|\varkappa r|\in\bigl[\frac{1}{4},4\bigr]$, *define*

$$\begin{aligned}\mathcal{S}^{1,\mp}_{p,q,r}(\xi):=\{z:|z|=\varrho\in[2^{-15},2^{15}],\ &|\Phi(\xi,\xi-z)|\leqslant 2^p,\\ &|\arg(z)-\arg(\xi)\mp\theta^1(\varrho)|\leqslant 2^{-\mathcal{D}_1/2},\\ &\textit{and}\ |\widehat{\Upsilon}(\xi,\xi-z)-\varkappa r2^q|\leqslant 10\varkappa 2^q\},\end{aligned} \tag{11.11}$$

and

$$\begin{aligned}\mathcal{S}^{2,\mp}_{p,q,r}(\eta):=\{z:|z|=\varrho\in[2^{-15},2^{15}],\ &|\Phi(\eta+z,\eta)|\leqslant 2^p,\\ &|\arg(z)-\arg(\eta)\mp\theta^1(\varrho)|\leqslant 2^{-\mathcal{D}_1/2},\\ &\textit{and}\ |\widehat{\Upsilon}(\eta+z,\eta)-\varkappa r2^q|\leqslant 10\varkappa 2^q\}.\end{aligned} \tag{11.12}$$

Then, for any $\iota\in\{+,-\}$,

$$\begin{gathered}|\mathcal{S}^{1,\iota}_{p,q,r}(\xi)|+|\mathcal{S}^{2,\iota}_{p,q,r}(\eta)|\lesssim 2^{q+p-k/2},\\ \operatorname{diam}(\mathcal{S}^{1,\iota}_{p,q,r}(\xi))+\operatorname{diam}(\mathcal{S}^{2,\iota}_{p,q,r}(\eta))\lesssim 2^{p-k/2}+\varkappa 2^q.\end{gathered} \tag{11.13}$$

Moreover, if $2^{p-k/2} \ll \varkappa 2^q$, *then there exist intervals* $I^1_{p,q,r}$ *and* $I^2_{p,q,r}$ *such that*

$$\begin{aligned}\mathcal{S}^{1,\mp}_{p,q,r}(\xi) &\subseteq \{(\varrho\cos\theta, \varrho\sin\theta) : \varrho \in I^1_{p,q,r}, |\theta - \arg(\xi) \mp \theta^1(\varrho)| \lesssim 2^{p-k/2}\}, \quad |I^1_{p,q,r}| \lesssim \varkappa 2^q,\\ \mathcal{S}^{2,\mp}_{p,q,r}(\eta) &\subseteq \{(\varrho\cos\theta, \varrho\sin\theta) : \varrho \in I^2_{p,q,r}, |\theta - \arg(\eta) \mp \theta^2(\varrho)| \lesssim 2^{p-k/2}\}, \quad |I^2_{p,q,r}| \lesssim \varkappa 2^q.\end{aligned} \tag{11.14}$$

Proof. (i) Notice that, if $|\xi|=s$, $|\eta|=r$, and $z=\xi-\eta=(\varrho\cos\theta, \varrho\sin\theta)$, then

$$\begin{gathered}2\xi\cdot\eta = r^2+s^2-\varrho^2, \quad 2z\cdot\xi = \varrho^2+s^2-r^2, \quad 2z\cdot\eta = s^2-r^2-\varrho^2,\\ (2\eta\cdot\xi^\perp)^2 = 4r^2s^2-(r^2+s^2-\varrho^2)^2.\end{gathered} \tag{11.15}$$

Under the assumptions (11.6), we see that $|\lambda(r)-\lambda(s)| \lesssim 2^{p^+}$, therefore $|r-s| \lesssim 2^{-k/2}2^{p^+}$. The bounds (11.7) follow using also (11.15). The decomposition (11.8) follows from (11.4), with

$$A(x,y) := \frac{\lambda'(|x|)}{|x|}\frac{\lambda'(|y|)}{|y|}\frac{(x\cdot y^\perp)^2}{|x-y|^2}, \quad B(w,z) := \frac{\sqrt{\lambda'(|z|)}}{|z|^{3/2}}\biggl(|z|\lambda'(|z|) - \frac{\lambda'(|w|)}{|w|}(w\cdot z)\biggr).$$

The bounds in the second line of (11.8) follow from this definition and (11.7).

(ii) We will show the estimates for fixed ξ, since the estimates for fixed η are similar. We may assume that $\xi=(s,0)$, so

$$\Phi(\xi,\xi-z) = \lambda(s) - \lambda_\mu(\varrho) - \lambda(\sqrt{s^2+\varrho^2-2s\varrho\cos\theta}\,). \tag{11.16}$$

Let

$$f(\theta) := -\lambda(s)+\lambda_\mu(\varrho)+\lambda(\sqrt{s^2+\varrho^2-2s\varrho\cos\theta}\,).$$

We notice that $-f(0) \gtrsim 2^{k/2}$, $f(\pi) \gtrsim 2^{k/2}$, and $f'(\theta) \approx 2^{k/2}\sin\theta$ for $\theta\in[0,\pi]$. Therefore, f is increasing on the interval $[0,\pi]$ and vanishes at a unique point $\theta^1(\varrho)=\theta^1_{s,\mu}(\varrho)$. Moreover, it is easy to see that $|\cos(\theta^1(\varrho))| \lesssim 2^{-k/2}$, and therefore $\bigl|\theta^1(\varrho)-\frac{1}{2}\pi\bigr| \lesssim 2^{-k/2}$. The remaining conclusions in (11.9)–(11.10) easily follow.

(iii) We will only prove the estimates for the sets $\mathcal{S}^{1,-}_{p,q,r}(\xi)$, since the others are similar. With $z=(\varrho\cos\theta, \varrho\sin\theta)$ and $\xi=(s,0)$, we define

$$F(\varrho,\theta) := \Phi(\xi,\xi-z) \quad \text{and} \quad G(\varrho,\theta) := \widehat{\Upsilon}(\xi,\xi-z).$$

The condition $|\widehat{\Upsilon}(\xi,\xi-z)| \lesssim 2^{-\mathcal{D}_1}$ shows that $|\Upsilon(\xi,\xi-z)| \lesssim 2^{k-\mathcal{D}_1}$, thus $|\varrho-\gamma_0| \leqslant 2^{-\mathcal{D}_1/2}$ (see (11.8)). Moreover, $\bigl|\theta-\frac{1}{2}\pi\bigr| \lesssim 2^{-\mathcal{D}_1/2}$, in view of (11.9) and (11.10). Using (11.16),

$$|\partial_\theta F(\varrho,\theta)| \approx 2^{k/2} \quad \text{and} \quad |\partial_\varrho F(\varrho,\theta)| \lesssim 2^{k/2-\mathcal{D}_1/2}$$

in the set $\{(\varrho,\theta):|\varrho-\gamma_0|\leqslant 2^{-\mathcal{D}_1/2}$ and $|\theta-\frac{1}{2}\pi|\lesssim 2^{-\mathcal{D}_1/2}\}$. In addition, using (11.8), we have

$$-\mu\partial_\varrho G(\varrho,\theta)=\lambda'''(\varrho)\frac{A(\xi,\xi-z)}{|\Lambda'(\xi)|\,|\Lambda'(\xi-z)|}+O(2^{-\mathcal{D}_1/2})\quad\text{and}\quad|\partial_\theta G(\varrho,\theta)|=O(2^{-\mathcal{D}_1/2}).$$

Therefore, the mapping $(\varrho,\theta)\mapsto(2^{-k/2}F(\varrho,\theta),G(\varrho,\theta))$ is a regular change of variables for (ϱ,θ) satisfying $|\varrho-\gamma_0|\leqslant 2^{-\mathcal{D}_1/2}$ and $|\theta-\frac{1}{2}\pi|\lesssim 2^{-\mathcal{D}_1/2}$. The conclusions follow. □

It follows from (11.4) and (11.15) that, if $|\xi|=s$, $|\eta|=r$, and $|\xi-\eta|=\varrho$, then

$$-4\Upsilon(\xi,\eta)\frac{\varrho^3}{\lambda'_\mu(\varrho)}\frac{s}{\lambda'_\sigma(s)}\frac{r}{\lambda'_\nu(r)}=G(s,r,\varrho),\tag{11.17}$$

where

$$\begin{aligned}G(s,r,\varrho):=&\frac{\varrho\lambda''(\varrho)}{\lambda'(\varrho)}(4r^2s^2-(r^2+s^2-\varrho^2)^2)\\&+\left(2\varrho s\frac{\lambda'(\varrho)}{\lambda'(s)}-\varrho^2-s^2+r^2\right)\left(2\varrho r\frac{\lambda'(\varrho)}{\lambda'(r)}+\varrho^2+r^2-s^2\right).\end{aligned}\tag{11.18}$$

We now assume that $|\xi-\eta|$ is close to γ_0, and consider the case of bounded frequencies.

LEMMA 11.2. *If* $|\xi|=s$, $|\eta|=r$, $|\xi-\eta|=\varrho$, $|\varrho-\gamma_0|\leqslant 2^{-8\mathcal{D}_1}$, *and* $2^{-200}\leqslant r,s\leqslant 2^{2\mathcal{D}_1}$, *then*

$$|\Phi(\xi,\eta)|+|\Upsilon(\xi,\eta)|\gtrsim 1.\tag{11.19}$$

Proof. Case 1: $(\sigma,\mu,\nu)=(+,+,+)$. Notice first that the function

$$f(r):=\lambda(r)+\lambda(\gamma_0)-\lambda(r+\gamma_0)$$

is concave down for $r\in[0,\gamma_0]$ (in view of (10.3)) and satisfies $f(0)=0$ and $f(\gamma_0)\geqslant 0.1$. Therefore, $f(r)\gtrsim 1$ if $r\in[2^{-200},\gamma_0]$, so

$$|\Phi(\xi,\eta)|\gtrsim 1,\quad\text{if } r\leqslant\gamma_0 \text{ or } s\leqslant 2\gamma_0.\tag{11.20}$$

Assume, by contradiction, that (11.19) fails. In view of (11.17), $|\Phi(\xi,\eta)|\ll 1$ and

$$\left|\left(2\varrho r\frac{\lambda'(\varrho)}{\lambda'(r)}+(\varrho^2+r^2-s^2)\right)\left(2\varrho s\frac{\lambda'(\varrho)}{\lambda'(s)}-(\varrho^2+s^2-r^2)\right)\right|\ll 1+s+r.\tag{11.21}$$

It is easy to see that, if $|\Phi(\xi,\eta)|=|\lambda(s)-\lambda(\varrho)-\lambda(r)|\ll 1$, $r\geqslant 100$, and $|\varrho-\gamma_0|\leqslant 2^{-8\mathcal{D}_1}$, then

$$r\leqslant s-\frac{\lambda(\varrho)-0.1}{\lambda'(s)}\quad\text{and}\quad s\geqslant r+\frac{\lambda(\varrho)-0.1}{\lambda'(r)}.$$

Therefore, using (10.2)–(10.4), if $r \geqslant 100$, then

$$-2\varrho s\frac{\lambda'(\varrho)}{\lambda'(s)}+\varrho^2+s^2-r^2 \geqslant \frac{2s}{\lambda'(s)}(\lambda(\varrho)-0.1-\varrho\lambda'(\varrho)) \gtrsim \sqrt{s}$$
$$-2\varrho r\frac{\lambda'(\varrho)}{\lambda'(r)}-\varrho^2-r^2+s^2 \geqslant \frac{2r}{\lambda'(r)}(\lambda(\varrho)-0.1-\varrho\lambda'(\varrho))-\varrho^2 \gtrsim \sqrt{r}.$$

In particular, (11.21) cannot hold if $r \geqslant 100$.

For $y \in [0,\infty)$, the equation $\lambda(x)=y$ admits a unique solution $x \in [0,\infty)$:

$$x=-\frac{1}{Y(y)}+\frac{Y(y)}{3}, \quad Y(y):=\left(\frac{27y^2+\sqrt{27}\sqrt{27y^4+4}}{2}\right)^{1/3}. \tag{11.22}$$

Assuming $|\varrho-\gamma_0| \leqslant 2^{-8\mathcal{D}_1}$, $2\gamma_0 \leqslant s \leqslant 110$, and $|\lambda(s)-\lambda(r)-\lambda(\varrho)| \ll 1$, we now show that $G(s,r,\varrho) \gtrsim 1$, where G is as in (11.18). Indeed, we solve the equation $\lambda(r(s))=\lambda(s)-\lambda(\gamma_0)$ according to (11.22), and define the function $G_0(s):=G(s,r(s),\gamma_0)$. A simple `Mathematica` program shows that $G_0(s) \gtrsim 1$ if $2\gamma_0 \leqslant s \leqslant 110$. This completes the proof of (11.19) when $(\sigma,\mu,\nu)=(+,+,+)$.

Case 2: the other triplets. Notice that, if $(\sigma,\mu,\nu)=(+,-,+)$, then

$$\Phi_{+-+}(\xi,\eta)=-\Phi_{+++}(\eta,\xi) \quad \text{and} \quad \Upsilon_{+-+}(\xi,\eta)=-\Upsilon_{+++}(\eta,\xi). \tag{11.23}$$

The desired bound in this case follows from the case $(\sigma,\mu,\nu)=(+,+,+)$ analyzed earlier.

On the other hand, if $(\sigma,\mu,\nu)=(+,-,-)$, then $\Phi(\xi,\eta)=\lambda(s)+\lambda(r)+\lambda(\varrho) \gtrsim 1$, so (11.19) is clearly verified. Finally, if $(\sigma,\mu,\nu)=(+,+,-)$, then $\Phi(\xi,\eta)=\lambda(s)+\lambda(r)-\lambda(\varrho)$ and we estimate, assuming $2^{-200} \leqslant r \leqslant \frac{1}{2}\varrho$,

$$\lambda(s)+\lambda(r)-\lambda(\varrho) \geqslant \lambda(r)+\lambda(\varrho-r)-\lambda(\varrho)=\int_0^r (\lambda'(x)-\lambda'(x+\varrho-r))\,dx \gtrsim 1.$$

A similar estimate holds if $2^{-200} \leqslant s \leqslant \frac{1}{2}\varrho$, or if $s,r \geqslant \frac{1}{2}\varrho$. Therefore, $\Phi(\xi,\eta) \gtrsim 1$ in this case.

The cases corresponding to $\sigma=-$ are similar, by replacing Φ by $-\Phi$ and Υ by $-\Upsilon$. This completes the proof of the lemma. □

Finally, we consider the case when $|\xi-\eta|$ is close to γ_1.

LEMMA 11.3. *If* $|\xi|=s$, $|\eta|=r$, $|\xi-\eta|=\varrho$, $|\varrho-\gamma_1|\leqslant 2^{-\mathcal{D}_1}$, *and* $2^{-200}\leqslant r,s$, *then*

$$\begin{aligned}|\Phi(\xi,\eta)|+\frac{|\Upsilon(\xi,\eta)|}{|\xi|+|\eta|}+\frac{|(\nabla_\eta\Upsilon)(\xi,\eta)\cdot(\nabla_\eta^\perp\Phi)(\xi,\eta)|}{(|\xi|+|\eta|)^6}&\gtrsim 1,\\ |\Phi(\xi,\eta)|+\frac{|\Upsilon(\xi,\eta)|}{|\xi|+|\eta|}+\frac{|(\nabla_\xi\Upsilon)(\xi,\eta)\cdot(\nabla_\xi^\perp\Phi)(\xi,\eta)|}{(|\xi|+|\eta|)^6}&\gtrsim 1,\end{aligned}\tag{11.24}$$

and

$$\begin{aligned}|\Phi(\xi,\eta)|+\frac{|\Upsilon(\xi,\eta)|}{|\xi|+|\eta|}+\frac{|(\xi-\eta)\cdot(\nabla_\eta^\perp\Phi)(\xi,\eta)|}{(|\xi|+|\eta|)^6}&\gtrsim 1,\\ |\Phi(\xi,\eta)|+\frac{|\Upsilon(\xi,\eta)|}{|\xi|+|\eta|}+\frac{|(\xi-\eta)\cdot(\nabla_\xi^\perp\Phi)(\xi,\eta)|}{(|\xi|+|\eta|)^6}&\gtrsim 1.\end{aligned}\tag{11.25}$$

Proof. *Case* 1: $(\sigma,\mu,\nu)=(+,+,+)$. Notice first that the function

$$f(r):=\lambda(r)+\lambda(\gamma_1)-\lambda(r+\gamma_1)$$

is concave down for $r\in[0,0.3]$ (in view of (10.3)) and satisfies $f(0)=0$ and $f(0.3)\geqslant 0.02$. Therefore, $f(r)\gtrsim 1$ if $r\in[2^{-200},0.3]$, so

$$|\Phi(\xi,\eta)|\gtrsim 1,\quad \text{if } r\leqslant 0.3 \text{ or } s\leqslant\gamma_1+0.3.\tag{11.26}$$

On the other hand, if $|\Phi(\xi,\eta)|\ll 1$, $r\geqslant 1000$, and $|\varrho-\gamma_1|\leqslant 2^{-\mathcal{D}_1}$, then

$$s\leqslant r+\frac{\lambda(\varrho)+0.2}{\lambda'(r)}\quad\text{and}\quad r\geqslant s-\frac{\lambda(\varrho)+0.2}{\lambda'(s)}.$$

Therefore, using also (10.5), if $r\geqslant 1000$ then

$$\begin{aligned}2\varrho r\frac{\lambda'(\varrho)}{\lambda'(r)}+\varrho^2+r^2-s^2&\geqslant\frac{2r}{\lambda'(r)}(\varrho\lambda'(\varrho)-\lambda(\varrho)-0.2)\gtrsim\sqrt{r},\\ 2\varrho s\frac{\lambda'(\varrho)}{\lambda'(s)}-\varrho^2-s^2+r^2&\geqslant\frac{2s}{\lambda'(s)}(\varrho\lambda'(\varrho)-\lambda(\varrho)-0.2)-\varrho^2\gtrsim\sqrt{s},\\ \frac{\varrho\lambda''(\varrho)}{\lambda'(\varrho)}(4r^2s^2-(r^2+s^2-\varrho^2)^2)&\gtrsim r^2.\end{aligned}$$

Using the formula (11.17) and assuming $|\varrho-\gamma_1|\leqslant 2^{-\mathcal{D}_1}$, it follows that

$$\text{if }\ |\Phi(\xi,\eta)|\ll 1\ \text{ and }\ r\geqslant 1000\ \text{ then }\ -\Upsilon(\xi,\eta)\gtrsim r.\tag{11.27}$$

Therefore both (11.24) and (11.25) follow if $r\geqslant 1000$.

It remains to consider the case $\gamma_1+0.3\leqslant s\leqslant 1010$. We show first that

$$\text{if } 3\leqslant s\leqslant 1010 \text{ and } |\lambda(s)-\lambda(r)-\lambda(\varrho)|\ll 1, \text{ then } -\Upsilon(\xi,\eta)\gtrsim 1.\tag{11.28}$$

Indeed, we solve the equation $\lambda(r(s))=\lambda(s)-\lambda(\gamma_1)$ according to (11.22), and define the function $G_1(s):=G(s,r(s),\gamma_1)$, see (11.17)–(11.18). A simple `Mathematica` program shows that $G_1(s)\gtrsim 1$ if $3\leqslant s\leqslant 1010$. The bound (11.28) follows, so both (11.24) and (11.25) follow if $3\leqslant s\leqslant 1010$.

On the other hand, the function $G_1(s)$ does vanish for some $s\in[\gamma_1+0.3,3]$ (more precisely at $s\approx 1.94$). In this range we can only prove the weaker estimates in the lemma. Notice that

$$\Upsilon(\xi,\eta)=\widetilde{\Upsilon}(|\xi|,|\eta|,|\xi-\eta|)\quad\text{and}\quad \widetilde{\Upsilon}(s,r,\varrho):=-\frac{1}{4}G(s,r,\varrho)\frac{\lambda'(\varrho)}{\varrho^3}\frac{\lambda'(s)}{s}\frac{\lambda'(r)}{r}.$$

Then, using also (11.2), we have

$$\begin{aligned}(\nabla_\eta\Upsilon)(\xi,\eta)\cdot(\nabla_\eta^\perp\Phi)(\xi,\eta)&=(r\varrho)^{-1}(\eta\cdot\xi^\perp)((\partial_r\widetilde{\Upsilon})(s,r,\varrho)\lambda'(\varrho)-(\partial_\varrho\widetilde{\Upsilon})(s,r,\varrho)\lambda'(r)),\\(\nabla_\xi\Upsilon)(\xi,\eta)\cdot(\nabla_\xi^\perp\Phi)(\xi,\eta)&=(s\varrho)^{-1}(\xi\cdot\eta^\perp)((\partial_s\widetilde{\Upsilon})(s,r,\varrho)\lambda'(\varrho)+(\partial_\varrho\widetilde{\Upsilon})(s,r,\varrho)\lambda'(s)).\end{aligned}\tag{11.29}$$

It is easy to see, using formulas (11.15) and (11.17), that

$$|\Phi(\xi,\eta)|+|\Upsilon(\xi,\eta)|+|\xi\cdot\eta^\perp|\gtrsim 1\tag{11.30}$$

if $s\in[\gamma_1+0.3,3]$. Moreover, let

$$\begin{aligned}G_{11}(s)&:=(\partial_r\widetilde{\Upsilon})(s,r(s),\gamma_1)\lambda'(\gamma_1)-(\partial_\varrho\widetilde{\Upsilon})(s,r(s),\gamma_1)\lambda'(r(s)),\\G_{12}(s)&:=(\partial_s\widetilde{\Upsilon})(s,r(s),\gamma_1)\lambda'(\gamma_1)+(\partial_\varrho\widetilde{\Upsilon})(s,r(s),\gamma_1)\lambda'(s),\end{aligned}$$

where, as before, $r(s)$ is the unique solution of the equation $\lambda(r(s))=\lambda(s)-\lambda(\gamma_1)$, according to (11.22). A simple `Mathematica` program shows that $G_1(s)+G_{11}(s)\gtrsim 1$ and $G_1(s)+G_{12}(s)\gtrsim 1$ if $s\in[\gamma_1+0.3,3]$. Using also (11.29) and (11.30), it follows that

$$\begin{aligned}|\Upsilon(\xi,\eta)|+|(\nabla_\eta\Upsilon)(\xi,\eta)\cdot(\nabla_\eta^\perp\Phi)(\xi,\eta)|&\gtrsim 1,\\|\Upsilon(\xi,\eta)|+|(\nabla_\xi\Upsilon)(\xi,\eta)\cdot(\nabla_\xi^\perp\Phi)(\xi,\eta)|&\gtrsim 1,\end{aligned}\tag{11.31}$$

if $s\in[\gamma_1+0.3,3]$, $|\Phi(\xi,\eta)|\ll 1$, and $|\varrho-\gamma_0|\leqslant 2^{-\mathcal{D}_1}$. The bounds in (11.24) follow from (11.26)–(11.28) and (11.31). Those in (11.25) follow from (11.26)–(11.28) and (11.30).

Case 2: the other triplets. The desired bounds in case $(\sigma,\mu,\nu)=(+,-,+)$ follow from the corresponding bounds in case $(\sigma,\mu,\nu)=(+,+,+)$ and (11.23). Moreover, if $(\sigma,\mu,\nu)=(+,-,-)$, then $\Phi(\xi,\eta)=\lambda(s)+\lambda(r)+\lambda(\varrho)\gtrsim 1$, so (11.24)–(11.25) are clearly verified.

Finally, if $(\sigma,\mu,\nu)=(+,+,-)$, then $\Phi(\xi,\eta)=\lambda(s)+\lambda(r)-\lambda(\varrho)$. We may assume that $s,r\in[2^{-20},\gamma_1]$. In this case, we prove the stronger bound

$$|\Phi(\xi,\eta)|+|\Upsilon(\xi,\eta)|\gtrsim 1. \tag{11.32}$$

Indeed, for this, it suffices to notice that the function $x\mapsto\lambda(x)+\lambda(\gamma_1-x)-\lambda(\gamma_1)$ is non-negative for $x\in[0,\gamma_1]$ and vanishes only when $x\in\{0,\frac{1}{2}\gamma_1,\gamma_1\}$. Moreover,

$$\Upsilon\big((\tfrac{1}{2}\gamma_1)e,-(\tfrac{1}{2}\gamma_1)e\big)\neq 0$$

if $|e|=1$ (using (11.4)), and the lower bound (11.32) follows.

The cases corresponding to $\sigma=-$ are similar, by replacing Φ by $-\Phi$ and Υ by $-\Upsilon$. This completes the proof of the lemma. □

Appendix A. Paradifferential calculus

The paradifferential calculus allows us to understand the high-frequency structure of our system. In this section we record the definitions, and state and prove several useful lemmas.

A.1. Operator bounds

In this subsection we define our main objects, and prove several basic non-linear bounds.

A.1.1. Fourier multipliers

We will mostly work with bilinear and trilinear multipliers. Many of the simpler estimates follow from the following basic result (see [44, Lemma 5.2] for the proof).

LEMMA A.1. (i) *Assume $l\geqslant 2$, $f_1,...,f_l,f_{l+1}\in L^2(\mathbb{R}^2)$, and let $m:(\mathbb{R}^2)^l\to\mathbb{C}$ be a continuous compactly supported function. Then,*

$$\begin{aligned}\bigg|\int_{(\mathbb{R}^2)^l} m(\xi_1,...,\xi_l)\hat{f}_1(\xi_1)\,...\,\hat{f}_l(\xi_l)\hat{f}_{l+1}(-\xi_1-...-\xi_l)\,d\xi_1\,...\,d\xi_l\bigg|&\\ \lesssim\|\mathcal{F}^{-1}(m)\|_{L^1}\,\|f_1\|_{L^{p_1}}\,...\,\|f_{l+1}\|_{L^{p_{l+1}}},&\end{aligned} \tag{A.1}$$

for any exponents $p_1,...\,p_{l+1}\in[1,\infty]$ satisfying $1/p_1+...+1/p_{l+1}=1$.

(ii) *Assume $l\geqslant 2$ and let L_m be the multilinear operator defined by*

$$\mathcal{F}\{L_m[f_1,...,f_l]\}(\xi)=\int_{(\mathbb{R}^2)^{l-1}} m(\xi,\eta_2,...,\eta_l)\hat{f}_1(\xi-\eta_2)\,...\,\hat{f}_{l-1}(\eta_{l-1}-\eta_l)\hat{f}_l(\eta_l)\,d\eta_2\,...\,d\eta_l.$$

Then, for any exponents $p, q_1, \dots q_l \in [1,\infty]$ *satisfying* $1/q_1+\dots+1/q_l=1/p$, *we have*

$$\|L_m[f_1,\dots,f_l]\|_{L^p} \lesssim \|\mathcal{F}^{-1}(m)\|_{L^1}\,\|f_1\|_{L^{q_1}}\dots\|f_l\|_{L^{q_l}}. \tag{A.2}$$

Given a multiplier $m\colon(\mathbb{R}^2)^2\to\mathbb{C}$, we define the bilinear operator M by the formula

$$\mathcal{F}(M[f,g])(\xi)=\frac{1}{4\pi^2}\int_{\mathbb{R}^2} m(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta. \tag{A.3}$$

With $\Omega=x_1\partial_2-x_2\partial_1$, we notice the formula

$$\Omega M[f,g]=M[\Omega f,g]+M[f,\Omega g]+\widetilde{M}[f,g], \tag{A.4}$$

where $\widetilde{M}$ is the bilinear operator defined by the multiplier $\widetilde{m}(\xi,\eta)=(\Omega_\xi+\Omega_\eta)m(\xi,\eta)$.

For simplicity of notation, we define the following classes of bilinear multipliers:

$$\begin{aligned} S^\infty &:= \{m\colon(\mathbb{R}^2)^n\to\mathbb{C}\text{ continuous}: \|m\|_{S^\infty}:=\|\mathcal{F}^{-1}m\|_{L^1}<\infty\},\\ S^\infty_\Omega &:= \Big\{m:(\mathbb{R}^2)^2\to\mathbb{C}\text{ continuous}: \|m\|_{S^\infty_\Omega}:=\sup_{l\leqslant N_1}\|(\Omega_\xi+\Omega_\eta)^l m\|_{S^\infty}<\infty\Big\}. \end{aligned} \tag{A.5}$$

We will often need to analyze bilinear operators more carefully, by localizing in the frequency space. We therefore define, for any symbol m,

$$m^{k,k_1,k_2}(\xi,\eta):=\varphi_k(\xi)\varphi_{k_1}(\xi-\eta)\varphi_{k_2}(\eta)m(\xi,\eta). \tag{A.6}$$

For any $t\in[0,T]$, $p\geqslant -N_3$, and $m\geqslant 1$ let $\langle t\rangle=1+t$ and let $\mathcal{O}_{m,p}=\mathcal{O}_{m,p}(t)$ denote the Banach spaces of functions $f\in L^2$ defined by the norms

$$\|f\|_{\mathcal{O}_{m,p}}:=\langle t\rangle^{(m-1)(5/6-20\delta^2)-\delta^2}\big(\|f\|_{H^{N_0+p}}+\|f\|_{H^{N_1,N_3+p}_\Omega}+\langle t\rangle^{5/6-2\delta^2}\|f\|_{\widetilde{W}^{N_1/2,N_2+p}_\Omega}\big). \tag{A.7}$$

This is similar to the definition of the spaces $O_{m,p}$ in Definition 2.4, except for the supremum over $t\in[0,T]$. We first show that these spaces are compatible with S^∞_Ω multipliers.

Lemma A.2. *Let M be a bilinear operator with symbol m satisfying*

$$\|m^{k,k_1,k_2}\|_{S^\infty_\Omega}\leqslant 1 \quad \textit{for any } k,k_1,k_2\in\mathbb{Z}.$$

Then, if $p\in[-N_3,10]$, $t\in[0,T]$, *and* $m,n\geqslant 1$,

$$\langle t\rangle^{12\delta^2}\|M[f,g]\|_{\mathcal{O}_{m+n,p}}\lesssim\|f\|_{\mathcal{O}_{m,p}}\,\|g\|_{\mathcal{O}_{n,p}}. \tag{A.8}$$

Proof. By definition, we may assume that $m=n=1$ and $\|f\|_{\mathcal{O}_{m,p}}=\|g\|_{\mathcal{O}_{n,p}}=1$. Thus, we may assume that

$$\|h\|_{H^{N_0+p}}+\sup_{j\leqslant N_1}\|\Omega^j h\|_{H^{N_3+p}}\leqslant\langle t\rangle^{\delta^2}\quad\text{and}\quad\sup_{j\leqslant N_1/2}\|\Omega^j h\|_{\widetilde{W}^{N_2+p}}\leqslant\langle t\rangle^{3\delta^2-5/6},\tag{A.9}$$

where $h\in\{f(t),g(t)\}$. With $F:=M[f(t),g(t)]$, it suffices to prove that

$$\begin{aligned}\|F\|_{H^{N_0+p}}+\sup_{j\leqslant N_1}\|\Omega^j F\|_{H^{N_3+p}}&\lesssim\langle t\rangle^{6\delta^2-5/6},\\ \sup_{j\leqslant N_1/2}\|\Omega^j P_kF\|_{\widetilde{W}^{N_2+p}}&\lesssim\langle t\rangle^{8\delta^2-5/3}.\end{aligned}\tag{A.10}$$

For $k,k_1,k_2\in\mathbb{Z}$ let

$$F_k:=P_kM[f(t),g(t)]\quad\text{and}\quad F_{k,k_1,k_2}:=P_kM[P_{k_1}f(t),P_{k_2}g(t)].$$

For $k\in\mathbb{Z}$ let

$$\begin{aligned}\mathcal{X}_k^1&:=\{(k_1,k_2)\in\mathbb{Z}\times\mathbb{Z}:k_1\leqslant k-8\text{ and }|k_2-k|\leqslant 4\},\\ \mathcal{X}_k^2&:=\{(k_1,k_2)\in\mathbb{Z}\times\mathbb{Z}:k_2\leqslant k-8\text{ and }|k_1-k|\leqslant 4\},\\ \mathcal{X}_k^3&:=\{(k_1,k_2)\in\mathbb{Z}\times\mathbb{Z}:\min(k_1,k_2)\geqslant k-7\text{ and }|k_1-k_2|\leqslant 20\},\end{aligned}$$

and let $\mathcal{X}_k:=\mathcal{X}_k^1\cup\mathcal{X}_k^2\cup\mathcal{X}_k^3$. Let

$$\begin{aligned}a_k&:=\|P_kh\|_{H^{N_0+p}},\quad b_k:=\sup_{0\leqslant j\leqslant N_1}\|\Omega^jP_kh\|_{H^{N_3+p}},\quad c_k:=\sup_{0\leqslant j\leqslant N_1/2}\|\Omega^jP_kh\|_{\widetilde{W}^{N_2+p}},\\ \tilde{a}_k&:=\sum_{m\in\mathbb{Z}}a_{k+m}2^{-|m|/100},\quad \tilde{b}_k:=\sum_{m\in\mathbb{Z}}b_{k+m}2^{-|m|/100},\quad \tilde{c}_k:=\sum_{m\in\mathbb{Z}}c_{k+m}2^{-|m|/100}.\end{aligned}\tag{A.11}$$

We can now prove (A.10). Assuming $k\in\mathbb{Z}$ fixed, we estimate, using Lemma A.1 (ii),

$$\begin{aligned}\|F_{k,k_1,k_2}\|_{H^{N_0+p}}&\lesssim a_{k_1}(2^{-4\max(k_2,0)}c_{k_2}),\quad\text{if }(k_1,k_2)\in\mathcal{X}_k^2,\\ \|F_{k,k_1,k_2}\|_{H^{N_0+p}}&\lesssim a_{k_2}(2^{-4\max(k_1,0)}c_{k_1}),\quad\text{if }(k_1,k_2)\in\mathcal{X}_k^1\cup\mathcal{X}_k^3.\end{aligned}\tag{A.12}$$

Since $\sum_l c_l\leqslant\langle t\rangle^{3\delta^2-5/6}$, it follows that

$$\sum_{(k_1,k_2)\in\mathcal{X}_k}\|F_{k,k_1,k_2}\|_{H^{N_0+p}}\lesssim\langle t\rangle^{3\delta^2-5/6}\biggl(\tilde{a}_k+\sum_{l\geqslant k}\tilde{a}_l2^{-4l_+}\biggr).\tag{A.13}$$

Therefore, since $\sum_{k\in\mathbb{Z}}\tilde{a}_k^2\lesssim\langle t\rangle^{2\delta^2}$, it follows that

$$\biggl(\sum_{2^k\geqslant(1+t)^{-10}}\|F_k\|^2_{H^{N_0+p}}\biggr)^{1/2}\lesssim\langle t\rangle^{6\delta^2-5/6}.\tag{A.14}$$

To bound the contribution of small frequencies, $2^k\leqslant\langle t\rangle^{-10}$, we also use the bound

$$\|F_{k,k_1,k_2}\|_{L^2}\lesssim 2^k\|F_{k,k_1,k_2}\|_{L^1}\lesssim 2^k a_{k_1}a_{k_2}, \tag{A.15}$$

when $(k_1,k_2)\in\mathcal{X}_k^3$, in addition to the bounds (A.12). Therefore,

$$\sum_{(k_1,k_2)\in\mathcal{X}_k}\|F_{k,k_1,k_2}\|_{H^{N_0+p}}\lesssim\langle t\rangle^{3\delta^2-5/6}\tilde{a}_k+2^k\sum_{l\in\mathbb{Z}}a_l^2, \tag{A.16}$$

if $2^k\leqslant\langle t\rangle^{-10}$. It follows that

$$\biggl(\sum_{2^k\leqslant\langle t\rangle^{-10}}\|F_k\|_{H^{N_0+p}}^2\biggr)^{1/2}\lesssim\langle t\rangle^{6\delta^2-5/6}, \tag{A.17}$$

and the desired bound $\|F\|_{H^{N_0+p}}\lesssim(1+t)^{6\delta^2-5/6}$ in (A.10) follows.

The proof of the second bound in (A.10) is similar. We start by estimating, as in (A.12),

$$\begin{aligned}&\|\Omega^jF_{k,k_1,k_2}\|_{H^{N_3+p}}\\&\quad\lesssim 2^{(N_3+p)k^+}(b_{k_1}2^{-(N_3+p)k_1^+}c_{k_2}2^{-(N_2+p)k_2^+}+b_{k_2}2^{-(N_3+p)k_2^+}c_{k_1}2^{-(N_2+p)k_1^+})\end{aligned}$$

for any $j\in[0,N_1]$. We remark that this is weaker than (A.12), since the Ω-derivatives can distribute on either $P_{k_1}f(t)$ or $P_{k_2}(t)$, and we are forced to estimate the factor with more than $\frac{1}{2}N_1$ Ω-derivatives in L^2. To bound the contributions of small frequencies, we also estimate

$$\|\Omega^jF_{k,k_1,k_2}\|_{H^{N_3+p}}\lesssim 2^{\min(k,k_1,k_2)}b_{k_1}b_{k_2},$$

as in (A.15). Recall that $N_2-N_3\geqslant 5$. We combine these two bounds to estimate

$$\sum_{(k_1,k_2)\in\mathcal{X}_k}\|\Omega^jF_{k,k_1,k_2}\|_{H^{N_3+p}}\lesssim\langle t\rangle^{3\delta^2-5/6}\biggl(\tilde{b}_k+\sum_{l\geqslant k}\tilde{b}_l2^{-4l^+}\biggr)+\langle t\rangle^{2\delta^2}2^{-(N_2-N_3)k^+}\tilde{c}_k.$$

When $2^k\leqslant(1+t)^{-10}$, this does not suffice; we have instead the bound

$$\sum_{(k_1,k_2)\in\mathcal{X}_k}\|\Omega^jF_{k,k_1,k_2}\|_{H^{N_3+p}}\lesssim\langle t\rangle^{3\delta^2-5/6}\tilde{b}_k+2^k\sum_{l\in\mathbb{Z}}b_l^2+\langle t\rangle^{2\delta^2}2^{-(N_2-N_3)k^+}\tilde{c}_k.$$

The desired estimate $\|\Omega^jF\|_{H^{N_3+p}}\lesssim\langle t\rangle^{6\delta^2-5/6}$ in (A.10) follows.

For the last bound in (A.10), we estimate as before, for any $j\in\bigl[0,\frac{1}{2}N_1\bigr]$,

$$\begin{aligned}\|\Omega^jF_{k,k_1,k_2}\|_{\widetilde{W}^{N_2+p}}&\lesssim 2^{(N_2+p)k^+}c_{k_1}2^{-(N_2+p)k_1^+}c_{k_2}2^{-(N_2+p)k_2^+},\\\|\Omega^jF_{k,k_1,k_2}\|_{\widetilde{W}^{N_2+p}}&\lesssim 2^{2k}b_{k_1}b_{k_2},\end{aligned}$$

where the last estimate holds only for $k\leqslant 0$. The desired bound follows as before. $\square$

A.1.2. Paradifferential operators

We first recall the definition of paradifferential operators (see (2.22)): given a symbol $a=a(x,\zeta)\colon\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{C}$, we define the operator T_a by

$$\mathcal{F}\{T_a f\}(\xi)=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\chi\biggl(\frac{|\xi-\eta|}{|\xi+\eta|}\biggr)\tilde{a}\biggl(\xi-\eta,\frac{\xi+\eta}{2}\biggr)\hat{f}(\eta)\,d\eta, \tag{A.18}$$

where $\tilde{a}$ denotes the partial Fourier transform of a in the first coordinate, and $\chi=\varphi_{-20}$. We define the Poisson bracket between two symbols a and b by

$$\{a,b\}:=\nabla_x a\cdot\nabla_\zeta b-\nabla_\zeta a\cdot\nabla_x b. \tag{A.19}$$

We will use several norms to estimate symbols of degree zero. For $q\in\{2,\infty\}$ and $r\in\mathbb{Z}_+$ let

$$\|a\|_{\mathcal{M}_{r,q}}:=\sup_{\zeta}\bigl\||a|_r(\,\cdot\,,\zeta)\bigr\|_{L^q_x},\quad \text{where } |a|_r(x,\zeta):=\sum_{|\alpha|+|\beta|\leqslant r}|\zeta|^{|\beta|}\,|\partial_\zeta^\beta\partial_x^\alpha a(x,\zeta)|. \tag{A.20}$$

At later stages, we will use more complicated norms, which also keep track of multiplicity and degree. For now, we record a few simple properties, which directly follow from the definitions:

$$\begin{aligned}\|ab\|_{\mathcal{M}_{r,q}}+\bigl\||\zeta|\{a,b\}\bigr\|_{\mathcal{M}_{r-2,q}}&\lesssim\|a\|_{\mathcal{M}_{r,q_1}}\,\|b\|_{\mathcal{M}_{r,q_2}},\quad \{\infty,q\}=\{q_1,q_2\},\\ \|P_k a\|_{\mathcal{M}_{r,q}}&\lesssim 2^{-sk}\|P_k a\|_{\mathcal{M}_{r+s,q}},\quad q\in\{2,\infty\},\ k\in\mathbb{Z},\ s\in\mathbb{Z}_+.\end{aligned} \tag{A.21}$$

We start with some simple properties.

LEMMA A.3. (i) *Let a be a symbol and $1\leqslant q\leqslant\infty$. Then,*

$$\|P_kT_af\|_{L^q}\lesssim\|a\|_{\mathcal{M}_{8,\infty}}\,\|P_{[k-2,k+2]}f\|_{L^q} \tag{A.22}$$

and

$$\|P_kT_af\|_{L^2}\lesssim\|a\|_{\mathcal{M}_{8,2}}\,\|P_{[k-2,k+2]}f\|_{L^\infty}. \tag{A.23}$$

(ii) *If $a\in\mathcal{M}_{8,\infty}$ is real valued, then T_a is a bounded self-adjoint operator on L^2.*

(iii) *We have*

$$\overline{T_af}=T_{a'}\bar{f},\quad \textit{where } a'(y,\zeta):=\overline{a(y,-\zeta)} \tag{A.24}$$

and

$$\Omega(T_af)=T_a(\Omega f)+T_{a''}f,\quad \textit{where } a''(y,\zeta)=(\Omega_ya)(y,\zeta)+(\Omega_\zeta a)(y,\zeta). \tag{A.25}$$

Proof. (i) Inspecting the Fourier transform, we directly see that

$$P_k T_a f = P_k T_a P_{[k-2,k+2]} f.$$

By rescaling, we may assume that $k=0$ and write

$$\langle P_0 T_a h, g\rangle = C\int_{\mathbb{R}^4} \bar{g}(x) h(y) I(x,y)\,dx\,dy$$

and

$$\begin{aligned} I(x,y) &= \int_{\mathbb{R}^6} a\left(z, \frac{\xi+\eta}{2}\right) e^{i\xi\cdot(x-z)} e^{i\eta\cdot(z-y)} \chi\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right)\varphi_0(\xi)\,d\eta\,d\xi\,dz \\ &= \int_{\mathbb{R}^6} a\left(z, \xi+\frac{\theta}{2}\right) e^{i\theta\cdot(z-y)} e^{i\xi\cdot(x-y)} \chi\left(\frac{|\theta|}{|2\xi+\theta|}\right)\varphi_0(\xi)\,d\xi\,d\theta\,dz. \end{aligned}$$

We observe that

$$\begin{aligned} (1+|x-y|^2)^2 I(x,y) = \int_{\mathbb{R}^6} & \frac{a(z,\xi+\theta/2)}{(1+|z-y|^2)^2} \chi\left(\frac{|\theta|}{|2\xi+\theta|}\right)\varphi_0(\xi) \\ & \times (1-\Delta_\theta)^2 (1-\Delta_\xi)^2 (e^{i\theta\cdot(z-y)} e^{i\xi\cdot(x-y)})\,d\xi d\theta\,dz. \end{aligned}$$

By integration by parts in ξ and θ, it follows that

$$(1+|x-y|^2)^2\,|I(x,y)| \lesssim \int_{\mathbb{R}^6} \frac{|a|_8(z,\xi+\theta/2)}{(1+|z-y|^2)^2} \varphi_{[-4,4]}(\xi)\varphi_{\leqslant -10}(\theta)\,d\xi\,d\theta\,dz, \qquad \text{(A.26)}$$

where $|a|_8$ is defined as in (A.20).

The bounds (A.22) and (A.23) now easily follow. Indeed, it follows from (A.26) that

$$(1+|x-y|^2)^2\,|I(x,y)| \lesssim \|a\|_{\mathcal{M}_{8,\infty}}.$$

Therefore, $|\langle P_0 T_a h, g\rangle| \lesssim \|a\|_{\mathcal{M}_{8,\infty}} \|h\|_{L^q} \|g\|_{L^{q'}}$. This gives (A.22), and (A.23) follows similarly.

Part (ii) and (A.24) follow directly from definitions. To prove (A.25), we start from the formula

$$\mathcal{F}\{\Omega T_a f\}(\xi) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} (\Omega_\xi+\Omega_\eta)\left(\chi\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right)\tilde{a}\left(\xi-\eta, \frac{\xi+\eta}{2}\right)\hat{f}(\eta)\right), d\eta,$$

and notice that

$$(\Omega_\xi+\Omega_\eta)\left(\chi\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right)\right) \equiv 0.$$

Formula (A.25) follows. □

The paradifferential calculus is useful to linearize products and compositions. More precisely, we have the following.

LEMMA A.4. (i) *If* $f, g \in L^2$, *then*

$$fg = T_f g + T_g f + \mathcal{H}(f,g),$$

where $\mathcal{H}$ *is smoothing in the sense that*

$$\|P_k \mathcal{H}(f,g)\|_{L^q} \lesssim \sum_{\substack{k',k''\geqslant k-40\\ |k'-k''|\leqslant 40}} \min(\|P_{k'}f\|_{L^q}\,\|P_{k''}g\|_{L^\infty}, \|P_{k'}f\|_{L^\infty}\,\|P_{k''}g\|_{L^q}).$$

As a consequence, if $f \in \mathcal{O}_{m,-5}$ *and* $g \in \mathcal{O}_{n,-5}$, *then*

$$\langle t\rangle^{12\delta^2}\|\mathcal{H}(f,g)\|_{\mathcal{O}_{m+n,5}} \lesssim \|f\|_{\mathcal{O}_{m,-5}}\|g\|_{\mathcal{O}_{n,-5}}. \tag{A.27}$$

(ii) *Let* $F(z) = z + h(z)$, *where* h *is analytic for* $|z| < \frac{1}{2}$ *and satisfies* $|h(z)| \lesssim |z|^3$. *If* $\|u\|_{L^\infty} \leqslant \frac{1}{100}$ *and* $N \geqslant 10$, *then*

$$\begin{aligned} F(u) &= T_{F'(u)}u + E(u),\\ \langle t\rangle^{12\delta^2}\|E(u)\|_{\mathcal{O}_{3,5}} &\lesssim \|u\|^3_{\mathcal{O}_{1,-5}} \quad \text{if } \|u\|_{\mathcal{O}_{1,-5}} \leqslant 1. \end{aligned} \tag{A.28}$$

Proof. (i) This follows easily by defining $\mathcal{H}(f,g) = fg - T_f g - T_g f$ and observing that

$$P_k \mathcal{H}(P_{k'}f, P_{k''}g) \equiv 0 \quad \text{unless } k', k'' \geqslant k-40, \text{ with } |k'-k''| \leqslant 40.$$

The bound (A.27) follows as in the proof of Lemma A.2 (the remaining bilinear interactions correspond essentially to the set $\mathcal{X}_k^3$).

(ii) Since F is analytic, it suffices to show this for $F(x) = x^n$, $n \geqslant 3$. This follows, however, as in part (i), using the Littlewood–Paley decomposition for u. □

We show now that compositions of paradifferential operators can be approximated well by paradifferential operators with suitable symbols. More precisely, we have the following.

PROPOSITION A.5. *Let* $1 \leqslant q \leqslant \infty$. *Given symbols* a *and* b, *we may decompose*

$$T_a T_b = T_{ab} + \tfrac{1}{2} i T_{\{a,b\}} + E(a,b), \tag{A.29}$$

where $\{a,b\}$ *denotes the Poisson bracket as defined in* (A.19). *The error* E *obeys the following bounds*: *assuming* $k \geqslant -100$,

$$\|P_k E(a,b) f\|_{L^q} \lesssim 2^{-2k}\|a\|_{\mathcal{M}_{16,\infty}}\,\|b\|_{\mathcal{M}_{16,\infty}}\,\|P_{[k-5,k+5]}f\|_{L^q} \quad \text{for } q \in \{2,\infty\}, \tag{A.30}$$

and

$$\begin{aligned} \|P_k E(a,b) f\|_{L^2} &\lesssim 2^{-2k}\|a\|_{\mathcal{M}_{16,2}}\,\|b\|_{\mathcal{M}_{16,\infty}}\,\|P_{[k-5,k+5]}f\|_{L^\infty},\\ \|P_k E(a,b) f\|_{L^2} &\lesssim 2^{-2k}\|a\|_{\mathcal{M}_{16,\infty}}\,\|b\|_{\mathcal{M}_{16,2}}\,\|P_{[k-5,k+5]}f\|_{L^\infty}. \end{aligned} \tag{A.31}$$

Moreover, $E(a,b) = 0$ *if both* a *and* b *are independent of* x.

Proof. We may assume that $a=P_{\leqslant k-100}a$ and $b=P_{\leqslant k-100}$, since the other contributions can also be estimated using Lemma A.3 (i) and (A.21). In this case, we write

$$\begin{aligned}16\pi^4\mathcal{F}\{P_k(T_aT_b-T_{ab})f\}(\xi)\\&=\varphi_k(\xi)\int_{\mathbb{R}^4}\hat{f}(\eta)\varphi_{\leqslant k-100}(\xi-\theta)\varphi_{\leqslant k-100}(\theta-\eta)\\&\quad\times\bigg(\tilde{a}\Big(\xi-\theta,\frac{\xi+\theta}{2}\Big)\tilde{b}\Big(\theta-\eta,\frac{\eta+\theta}{2}\Big)-\tilde{a}\Big(\xi-\theta,\frac{\xi+\eta}{2}\Big)\tilde{b}\Big(\theta-\eta,\frac{\xi+\eta}{2}\Big)\bigg)\,d\eta\,d\theta.\end{aligned}$$

Moreover, using the definition,

$$\begin{aligned}16\pi^4\mathcal{F}\Big\{P_k\Big(\frac{i}{2}\Big)T_{\{a,b\}}f\Big\}(\xi)=\varphi_k(\xi)\int_{\mathbb{R}^4}\hat{f}(\eta)\varphi_{\leqslant k-100}(\xi-\theta)\varphi_{\leqslant k-100}(\theta-\eta)\\\times\bigg(\frac{\theta-\eta}{2}(\nabla_\zeta\tilde{a})\Big(\xi-\theta,\frac{\xi+\eta}{2}\Big)\tilde{b}\Big(\theta-\eta,\frac{\xi+\eta}{2}\Big)\\-\tilde{a}\Big(\xi-\theta,\frac{\xi+\eta}{2}\Big)\frac{\xi-\theta}{2}(\nabla_\zeta\tilde{b})\Big(\theta-\eta,\frac{\xi+\eta}{2}\Big)\bigg)\,d\eta\,d\theta.\end{aligned}$$

Therefore,

$$\begin{aligned}&16\pi^4P_kE(a,b)f=U^1f+U^2f+U^3f,\\&\mathcal{F}(U^jf)(\xi)=\varphi_k(\xi)\int_{\mathbb{R}^4}\hat{f}(\eta)\varphi_{\leqslant k-100}(\xi-\theta)\varphi_{\leqslant k-100}(\theta-\eta)m^j(\xi,\eta,\theta)\,d\eta\,d\theta,\end{aligned}\tag{A.32}$$

where

$$\begin{aligned}m^1(\xi,\eta,\theta):=\tilde{a}\Big(\xi-\theta,\frac{\xi+\eta}{2}\Big)\tilde{b}\Big(\theta-\eta,\frac{\eta+\theta}{2}\Big)-\tilde{a}\Big(\xi-\theta,\frac{\xi+\eta}{2}\Big)\tilde{b}\Big(\theta-\eta,\frac{\xi+\eta}{2}\Big)\\-\tilde{a}\Big(\xi-\theta,\frac{\xi+\eta}{2}\Big)\frac{\theta-\xi}{2}(\nabla_\zeta\tilde{b})\Big(\theta-\eta,\frac{\xi+\eta}{2}\Big),\end{aligned}\tag{A.33}$$

$$\begin{aligned}m^2(\xi,\eta,\theta):=\tilde{a}\Big(\xi-\theta,\frac{\xi+\theta}{2}\Big)\tilde{b}\Big(\theta-\eta,\frac{\eta+\theta}{2}\Big)-\tilde{a}\Big(\xi-\theta,\frac{\xi+\eta}{2}\Big)\tilde{b}\Big(\theta-\eta,\frac{\eta+\theta}{2}\Big)\\-\frac{\theta-\eta}{2}(\nabla_\zeta\tilde{a})\Big(\xi-\theta,\frac{\xi+\eta}{2}\Big)\tilde{b}\Big(\theta-\eta,\frac{\eta+\theta}{2}\Big),\end{aligned}\tag{A.34}$$

and

$$m^3(\xi,\eta,\theta):=\frac{\theta-\eta}{2}(\nabla_\zeta\tilde{a})\Big(\xi-\theta,\frac{\xi+\eta}{2}\Big)\Big(\tilde{b}\Big(\theta-\eta,\frac{\eta+\theta}{2}\Big)-\tilde{b}\Big(\theta-\eta,\frac{\xi+\eta}{2}\Big)\Big).\tag{A.35}$$

It remains to prove the bounds (A.30) and (A.31) for the operators U^j, $j\in\{1,2,3\}$. The operators U^j are similar, so we will only provide the details for the operator U^1. We rewrite

$$m^1(\xi,\eta,\theta)=\int_0^1\tilde{a}\Big(\xi-\theta,\frac{\xi+\eta}{2}\Big)\frac{(\theta-\xi)_j(\theta-\xi)_k}{4}(\partial_{\zeta_j}\partial_{\zeta_k}\tilde{b})\Big(\theta-\eta,\frac{\xi+\eta}{2}+s\frac{\theta-\xi}{2}\Big)(1-s)\,ds.\tag{A.36}$$

Therefore,

$$U^1 f(x) = \int_{\mathbb{R}^2} f(y) K^1(x,y)\, dy, \tag{A.37}$$

where

$$K^1(x,y) := C\int_{\mathbb{R}^6} e^{-iy\cdot\eta} e^{ix\cdot\xi} \varphi_k(\xi)\varphi_{\leqslant k-100}(\xi-\theta)\varphi_{\leqslant k-100}(\theta-\eta) m^1(\xi,\eta,\theta)\, d\eta\, d\theta\, d\xi.$$

We use formula (A.36) and make changes of variables to rewrite

$$\begin{aligned} K^1(x,y) = C\int_0^1 ds\,(1-s)\int_{\mathbb{R}^{10}} & e^{-iy\cdot(\xi+\mu+\nu)} e^{ix\cdot\xi} e^{iz\cdot\mu} e^{iw\cdot\nu} \varphi_k(\xi)\varphi_{\leqslant k-100}(\mu)\varphi_{\leqslant k-100}(\nu) \\ & \times (\partial_{x_j}\partial_{x_k} a)\Big(z, \xi+\frac{\mu}{2}+\frac{\nu}{2}\Big)(\partial_{\zeta_j}\partial_{\zeta_k} b)\Big(w, \xi+\frac{\mu}{2}+\frac{\nu}{2}+\frac{s\mu}{2}\Big)\, d\mu\, d\nu\, d\xi\, dz\, dw. \end{aligned}$$

We integrate by parts in ξ, μ, and ν, using the operators $(2^{-2k}-\Delta_\xi)^2$, $(2^{-2k}-\Delta_\mu)^2$, and $(2^{-2k}-\Delta_\nu)^2$. It follows that

$$|K^1(x,y)| \lesssim \int_{\mathbb{R}^{10}} \frac{2^{-2k}}{(2^{-2k}+|x-y|^2)^2}\frac{2^{-2k}}{(2^{-2k}+|z-y|^2)^2}\frac{2^{-2k}}{(2^{-2k}+|w-y|^2)^2} F_{a,b}(z,w)\, dz\, dw, \tag{A.38}$$

where, with $\underline{\varphi}(X,Y,Z) := \varphi_0(X)\varphi_{\leqslant -100}(Y)\varphi_{\leqslant -100}(Z)$,

$$\begin{aligned} & F_{a,b}(z,w) \\ & := 2^{6k}\int_0^1 ds\int_{\mathbb{R}^6}\Big|(2^{-2k}-\Delta_\xi)^2(2^{-2k}-\Delta_\mu)^2(2^{-2k}-\Delta_\nu)^2\Big\{\underline{\varphi}(2^{-k}\xi, 2^{-k}\mu, 2^{-k}\nu) \\ & \qquad \times (\partial_{x_j}\partial_{x_k} a)\Big(z,\xi+\frac{\mu}{2}+\frac{\nu}{2}\Big)(\partial_{\zeta_j}\partial_{\zeta_k} b)\Big(w,\xi+\frac{\mu}{2}+\frac{\nu}{2}+\frac{s\mu}{2}\Big)\Big\}\Big|\, d\xi\, d\mu\, d\nu. \end{aligned}$$

With $|a|_{16}$ and $|b|_{16}$ defined as in (A.20), it follows that

$$\begin{aligned} |F_{a,b}(z,w)| \lesssim 2^{-2k}\int_0^1 ds\int_{\mathbb{R}^6} & |a|_{16}\Big(z,\xi+\frac{\mu}{2}+\frac{\nu}{2}\Big)|b|_{16}\Big(w,\xi+\frac{\mu}{2}+\frac{\nu}{2}+\frac{s\mu}{2}\Big) \\ & \times \varphi_{[-4,4]}(2^{-k}\xi)\varphi_{\leqslant -10}(2^{-k}\mu)\varphi_{\leqslant -10}(2^{-k}\nu)\,\frac{d\xi\, d\mu\, d\nu}{2^{6k}}. \end{aligned}$$

The desired bounds (A.30) and (A.31) for U^1 follow using also (A.37) and (A.38). □

We also make the following observation: if $a=a(\zeta)$ is a Fourier multiplier, b is a symbol, and f is a function, then

$$\begin{aligned} \widehat{E(a,b)f}(\xi) &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\chi\Big(\frac{|\xi-\eta|}{|\xi+\eta|}\Big)\Big(a(\xi)-a\Big(\frac{\xi+\eta}{2}\Big)-\frac{\xi-\eta}{2}\cdot\nabla a\Big(\frac{\xi+\eta}{2}\Big)\Big) \\ & \qquad \times \tilde{b}\Big(\xi-\eta,\frac{\xi+\eta}{2}\Big)\hat{f}(\eta)\, d\eta, \\ \widehat{E(b,a)f}(\xi) &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\chi\Big(\frac{|\xi-\eta|}{|\xi+\eta|}\Big)\Big(a(\eta)-a\Big(\frac{\xi+\eta}{2}\Big)-\frac{\eta-\xi}{2}\cdot\nabla a\Big(\frac{\xi+\eta}{2}\Big)\Big) \\ & \qquad \times \tilde{b}\Big(\xi-\eta,\frac{\xi+\eta}{2}\Big)\hat{f}(\eta)\, d\eta. \end{aligned} \tag{A.39}$$

A.2. Decorated norms and estimates

In the previous subsection we proved bounds on paraproduct operators. In our study of the problem, we need to keep track of several parameters, such as order, decay, and vector fields. It is convenient to use two compatible hierarchies of bounds, one for functions and one for symbols of operators.

A.2.1. Decorated norms

Recall the spaces $\mathcal{O}_{m,p}$ defined in (A.7). We define now the norms we will use to measure symbols.

Definition A.6. For $l\in[-10,10]$, $r\in\mathbb{Z}_+$, $m\in\{1,2,3,4\}$, $t\in[0,T]$, and $q\in\{2,\infty\}$, we define classes of symbols $\mathcal{M}^{l,m}_{r,q}=\mathcal{M}^{l,m}_{r,q}(t)\subseteq C(\mathbb{R}^2\times\mathbb{R}^2:\mathbb{C})$ by the norms

$$\|a\|_{\mathcal{M}^{l,m}_{r,\infty}} := \sup_{j\leqslant N_1/2}\ \sup_{|\alpha|+|\beta|\leqslant r}\ \sup_{\zeta\in\mathbb{R}^2} \langle t\rangle^{m(5/6-20\delta^2)+16\delta^2}\langle\zeta\rangle^{-l}\big\||\zeta|^{|\beta|}\partial_\zeta^\beta\partial_x^\alpha\Omega^j_{x,\zeta}a\big\|_{L^\infty_x}, \quad (A.40)$$

$$\|a\|_{\mathcal{M}^{l,m}_{r,2}} := \sup_{j\leqslant N_1}\ \sup_{|\alpha|+|\beta|\leqslant r}\ \sup_{\zeta\in\mathbb{R}^2} \langle t\rangle^{(m-1)(5/6-20\delta^2)-2\delta^2}\langle\zeta\rangle^{-l}\big\||\zeta|^{|\beta|}\partial_\zeta^\beta\partial_x^\alpha\Omega^j_{x,\zeta}a\big\|_{L^2_x}. \quad (A.41)$$

Here,

$$\Omega_{x,\zeta}a := \Omega_x a+\Omega_\zeta a = (x_1\partial_{x_2}-x_2\partial_{x_1}+\zeta_1\partial_{\zeta_2}-\zeta_2\partial_{\zeta_1})a;$$

see (A.25). We also define

$$\|a\|_{\mathcal{M}^{l,m}_r} := \|a\|_{\mathcal{M}^{l,m}_{r,\infty}}+\|a\|_{\mathcal{M}^{l,m}_{r,2}}, \quad m\geqslant 1. \quad (A.42)$$

Note that this hierarchy is naturally related to the hierarchy in terms of $\mathcal{O}_{m,p}$. In this definition the parameters m (the "multiplicity" of a, related to the decay rate) and l (the "order") will play an important role. Observe that for a function $f=f(x)$, and $m\in[1,4]$,

$$\|f\|_{\mathcal{M}^{0,m}_{N_3+p}} \lesssim \|f\|_{\mathcal{O}_{m,p}}. \quad (A.43)$$

Note also that we have the simple linear rule

$$\|P_k a\|_{\mathcal{M}^{l,m}_{r,q}} \lesssim 2^{-sk}\|P_k a\|_{\mathcal{M}^{l,m}_{r+s,q}}, \quad k\in\mathbb{Z},\ s\geqslant 0,\ q\in\{2,\infty\}, \quad (A.44)$$

and the basic multiplication rules

$$\langle t\rangle^{2\delta^2}\big(\|ab\|_{\mathcal{M}^{l_1+l_2,m_1+m_2}_r}+\|\zeta\{a,b\}\|_{\mathcal{M}^{l_1+l_2,m_1+m_2}_{r-2}}\big) \lesssim \|a\|_{\mathcal{M}^{l_1,m_1}_r}\|b\|_{\mathcal{M}^{l_2,m_2}_r}. \quad (A.45)$$

A.2.2. Bounds on operators

We may now pass the bounds proved in §A.1 to decorated norms. We consider the action of paradifferential operators on the classes $\mathcal{O}_{k,p}$. We will often use the following simple facts. Paradifferential operators preserve frequency localizations:

$$P_kT_af=P_kT_aP_{[k-4,k+4]}f=P_kT_{a(x,\zeta)\varphi_{\leqslant k+4}(\zeta)}f. \tag{A.46}$$

The rotation vector field Ω acts nicely on such operators (see (A.25)):

$$\Omega(T_af)=T_{\Omega_{x,\zeta}a}f+T_a(\Omega f). \tag{A.47}$$

The following relations between basic and decorated norms for symbols hold:

$$\begin{aligned}\|\Omega^j_{x,\zeta}a(x,\zeta)\varphi_{\leqslant k}(\zeta)\|_{\mathcal{M}_{r,\infty}}&\lesssim 2^{lk^+}\|a\|_{\mathcal{M}^{l,m}_{r,\infty}}\langle t\rangle^{-m(5/6-20\delta^2)-16\delta^2}, && 0\leqslant j\leqslant \tfrac12 N_1,\\ \|\Omega^j_{x,\zeta}a(x,\zeta)\varphi_{\leqslant k}(\zeta)\|_{\mathcal{M}_{r,2}}&\lesssim 2^{lk^+}\|a\|_{\mathcal{M}^{l,m}_{r,2}}\langle t\rangle^{-(m-1)(5/6-20\delta^2)+2\delta^2}, && 0\leqslant j\leqslant N_1.\end{aligned} \tag{A.48}$$

A simple application of the above remarks and Lemma A.3 (i) gives the bound

$$\|T_\sigma f\|_{H^s}\lesssim \langle t\rangle^{-m(5/6-20\delta^2)-16\delta^2}\|\sigma\|_{\mathcal{M}^{l,m}_8}\|f\|_{H^{s+l}}. \tag{A.49}$$

We now prove two useful lemmas.

LEMMA A.7. *If $q,q-l\in[-N_3,10]$ and $m,m_1\geqslant 1$, then*

$$\langle t\rangle^{12\delta^2}T_a\mathcal{O}_{m,q}\subseteq\mathcal{O}_{m+m_1,q-l}\quad \textit{for } a\in\mathcal{M}^{l,m_1}_{10}, \tag{A.50}$$

In particular, using also (A.43),

$$\langle t\rangle^{12\delta^2}T_{\mathcal{O}_{m_1,-10}}\mathcal{O}_{m,q}\subseteq\mathcal{O}_{m+m_1,q}. \tag{A.51}$$

Proof. The estimate (A.50) follows using the definitions and the linear estimates (A.22) and (A.23) in Lemma A.3. We may assume that $m=m_1=1$. Using (A.22) and (A.48), we estimate

$$\begin{aligned}2^{(N_0+q-l)k^+}\|P_kT_af\|_{L^2}&\lesssim\|a\|_{\mathcal{M}_{8,\infty}}2^{(N_0+q-l)k^+}\|P_{[k-2,k+2]}f\|_{L^2}\\&\lesssim\langle t\rangle^{-5/6+4\delta^2}\|a\|_{\mathcal{M}^{l,1}_{8,\infty}}2^{(N_0+q)k^+}\|P_{[k-2,k+2]}f\|_{L^2}\end{aligned}$$

for any $f\in\mathcal{O}_{1,q}$. By orthogonality, we deduce the desired bound on the H^{N_0} norm.

To bound the weighted norm, we use (A.22), (A.23), and (A.48) to estimate

$$\begin{aligned}
&2^{(N_3+q-l)k^+}\|\Omega^j P_k T_a f\|_{L^2}\\
&\quad\lesssim \sum_{n\leqslant j/2} 2^{(N_3+q-l)k^+}\big(\|P_k T_{\Omega^n_{x,\zeta}a}\Omega^{j-n}f\|_{L^2}+\|P_k T_{\Omega^{j-n}_{x,\zeta}a}\Omega^n f\|_{L^2}\big)\\
&\quad\lesssim \sum_{n\leqslant j/2} 2^{(N_3+q-l)k^+}\big(\|\Omega^n_{x,\zeta}a\|_{\mathcal{M}_{8,\infty}}\|P_{[k-2,k+2]}\Omega^{j-n}f\|_{L^2}\\
&\qquad\qquad\qquad\qquad\qquad+\|\Omega^{j-n}_{x,\zeta}a\|_{\mathcal{M}_{8,2}}\|P_{[k-2,k+2]}\Omega^n f\|_{L^\infty}\big)\\
&\quad\lesssim \sum_{n\leqslant j/2} 2^{(N_3+q)k^+}\|a\|_{\mathcal{M}^{l,1}_8}\big(\langle t\rangle^{-5/6+4\delta^2}\|P_{[k-2,k+2]}\Omega^{j-n}f\|_{L^2}\\
&\qquad\qquad\qquad\qquad\qquad+\langle t\rangle^{2\delta^2}\|P_{[k-2,k+2]}\Omega^n f\|_{L^\infty}\big)
\end{aligned}$$

for every $j\in[0,N_1]$. The desired weighted L^2 bound follows since

$$\begin{aligned}
&\Big(\sum_{k\in\mathbb{Z}} 2^{2(N_3+q)k^+}\|P_{[k-2,k+2]}\Omega^{j-n}f\|^2_{L^2}\Big)^{1/2}\\
&\qquad+\langle t\rangle^{5/6-2\delta^2}\Big(\sum_{k\in\mathbb{Z}} 2^{2(N_3+q)k^+}\|P_{[k-2,k+2]}\Omega^n f\|^2_{L^\infty}\Big)^{1/2}\lesssim \langle t\rangle^{2\delta^2}\|f\|_{\mathcal{O}_{1,q}}.
\end{aligned}$$

Finally, for the L^∞ bound, we use (A.22) to estimate

$$\begin{aligned}
2^{(N_2+q-l)k^+}\|\Omega^j P_k T_a f\|_{L^\infty} &\lesssim \sum_{j_1,j_2\leqslant N_1/2} 2^{(N_2+q-l)k^+}\|\Omega^{j_1}_{x,\zeta}a\|_{\mathcal{M}_{8,\infty}}\|P_{[k-2,k+2]}\Omega^{j_2}f\|_{L^\infty}\\
&\lesssim \langle t\rangle^{-5/6+4\delta^2}\|a\|_{\mathcal{M}^{l,1}_{8,\infty}}\sum_{j_2\leqslant N_1/2} 2^{(N_2+q)k^+}\|P_{[k-2,k+2]}\Omega^{j_2}f\|_{L^\infty}
\end{aligned}$$

for any $j\in\big[0,\tfrac12 N_1\big]$. The desired bound follows by summation over k. □

LEMMA A.8. *Let E be defined as in Proposition A.5. Assume that $m,m_1,m_2\geqslant 1$, $q,q-l_1,q-l_2,q-l_1-l_2\in[-N_3,10]$, and consider $a\in\mathcal{M}^{l_1,m_1}_{20}$ and $b\in\mathcal{M}^{l_2,m_2}_{20}$. Then,*

$$\begin{aligned}
\langle t\rangle^{12\delta^2}P_{\geqslant -100}E(a,b)\mathcal{O}_{m,q}&\subset\mathcal{O}_{m+m_1+m_2,q-l_1-l_2+2},\\
\langle t\rangle^{12\delta^2}P_{\geqslant -100}(T_aT_b+T_bT_a-2T_{ab})\mathcal{O}_{m,q}&\subset\mathcal{O}_{m+m_1+m_2,q-l_1-l_2+2}.
\end{aligned}\tag{A.52}$$

In addition,

$$\begin{aligned}
\langle t\rangle^{12\delta^2}[T_a,T_b]\mathcal{O}_{m,q}&\subseteq\mathcal{O}_{m+m_1+m_2,q-l_1-l_2+1},\\
\langle t\rangle^{12\delta^2}(T_aT_b-T_{ab})\mathcal{O}_{m,q}&\subseteq\mathcal{O}_{m+m_1+m_2,q-l_1-l_2+1}.
\end{aligned}\tag{A.53}$$

Moreover, if $a\in\mathcal{M}^{0,m_1}_{20}$ and $b\in\mathcal{M}^{0,m_2}_{20}$ are functions, then

$$\langle t\rangle^{12\delta^2}(T_aT_b-T_{ab})\mathcal{O}_{m,-5}\subseteq\mathcal{O}_{m+m_1+m_2,5}.\tag{A.54}$$

Proof. We record the formulas

$$\Omega_{x,\zeta}(ab)=(\Omega_{x,\zeta}a)b+a(\Omega_{x,\zeta}b) \quad \text{and} \quad \Omega_{x,\zeta}(\{a,b\})=\{\Omega_{x,\zeta}a,b\}+\{a,\Omega_{x,\zeta}b\}. \tag{A.55}$$

Therefore, letting $U(a,b):=T_aT_b-T_{ab}$, we have

$$\begin{gathered}[T_a,T_b]=U(a,b)-U(b,a), \quad E(a,b)=U(a,b)-\tfrac{1}{2}iT_{\{a,b\}},\\ T_aT_b+T_bT_a-2T_{ab}=E(a,b)+E(b,a),\end{gathered} \tag{A.56}$$

and

$$\begin{aligned}\Omega(U(a,b)f)&=U(\Omega_{x,\zeta}a,b)f+U(a,\Omega_{x,\zeta}b)f+U(a,b)\Omega f,\\ \Omega(T_{\{a,b\}}f)&=T_{\{\Omega_{x,\zeta}a,b\}}f+T_{\{a,\Omega_{x,\zeta}b\}}f+T_{\{a,b\}}\Omega f,\\ \Omega(E(a,b)f)&=E(\Omega_{x,\zeta}a,b)f+E(a,\Omega_{x,\zeta}b)f+E(a,b)\Omega f.\end{aligned} \tag{A.57}$$

The bound (A.54) follows as in the proof of Lemma A.2, once we notice that

$$P_k[(T_aT_b-T_{ab})f]=\sum_{\max(k_1,k_2)\geqslant k-40}P_k((T_{P_{k_1}a}T_{P_{k_2}b}-T_{P_{k_1}aP_{k_2}b})f).$$

The bounds (A.52) follow from (A.30)–(A.31) and (A.48), in the same way the bound (A.50) in Lemma A.7 follows from (A.22)–(A.23). Moreover, using (A.45),

$$\langle t\rangle^{12\delta^2}\|\{a,b\}(x,\zeta)\varphi_{\geqslant -200}(\zeta)\|_{\mathcal{M}_{18}^{l_1+l_2-1,m_1+m_2}}\lesssim\|a\|_{\mathcal{M}_{20}^{l_1,m_1}}\|b\|_{\mathcal{M}_{20}^{l_2,m_2}}.$$

Therefore, using (A.50) and frequency localization,

$$\langle t\rangle^{12\delta^2}P_{\geqslant -100}T_{\{a,b\}}\mathcal{O}_{m,q}\subseteq\mathcal{O}_{m+m_1+m_2,q-l_1-l_2+1}. \tag{A.58}$$

Therefore, using (A.56) and (A.52),

$$\langle t\rangle^{12\delta^2}P_{\geqslant -100}U(a,b)\mathcal{O}_{m,q}\subseteq\mathcal{O}_{m+m_1+m_2,q-l_1-l_2+1}.$$

For (A.53), it remains to prove that

$$\langle t\rangle^{12\delta^2}P_{\leqslant 0}U(a,b)\mathcal{O}_{m,q}\subseteq\mathcal{O}_{m+m_1+m_2,q-l_1-l_2+1}. \tag{A.59}$$

However, using (A.50) and (A.45), we have

$$\langle t\rangle^{12\delta^2}T_aT_b\mathcal{O}_{m,q}\subseteq\mathcal{O}_{m+m_1+m_2,q-l_1-l_2} \quad \text{and} \quad \langle t\rangle^{12\delta^2}T_{ab}\mathcal{O}_{m,q}\subseteq\mathcal{O}_{m+m_1+m_2,q-l_1-l_2},$$

and (A.59) follows. This completes the proof of (A.53). ∎

Appendix B. The Dirichlet–Neumann operator

Let (h,ϕ) be as in Proposition 2.2 and let $\Omega:=\{(x,z)\in\mathbb{R}^3: z\leqslant h(x)\}$. Let Φ denote the (unique in a suitable space, see Lemma B.4) harmonic function in Ω satisfying $\Phi(x,h(x))=\phi(x)$. We define the *Dirichlet–Neumann*([8]) *map* as

$$G(h)\phi=\sqrt{1+|\nabla h|^2}(\nu\cdot\nabla\Phi), \tag{B.1}$$

where ν denotes the outward-pointing unit normal to the domain Ω. The main result of this section is the following paralinearization of the Dirichlet–Neumann map.

PROPOSITION B.1. *Assume that $t\in[0,T]$ is fixed and let (h,ϕ) satisfy*

$$\|\langle\nabla\rangle h\|_{\mathcal{O}_{1,0}}+\big\||\nabla|^{1/2}\phi\big\|_{\mathcal{O}_{1,0}}\lesssim\varepsilon_1. \tag{B.2}$$

Define

$$B:=\frac{G(h)\phi+\nabla_x h\cdot\nabla_x\phi}{1+|\nabla h|^2},\quad V:=\nabla_x\phi-B\nabla_x h,\quad \textit{and}\quad \omega:=\phi-T_B h. \tag{B.3}$$

Then, we can paralinearize the Dirichlet–Neumann operator as

$$G(h)\phi=T_{\lambda_{DN}}\omega-\operatorname{div}(T_V h)+G_2+\varepsilon_1^3\mathcal{O}_{3,3/2} \tag{B.4}$$

(*recall definition* (A.7)), *where*

$$\begin{aligned}\lambda_{DN}&:=\lambda^{(1)}+\lambda^{(0)},\\ \lambda^{(1)}(x,\zeta)&:=\sqrt{(1+|\nabla h|^2)|\zeta|^2-(\zeta\cdot\nabla h)^2},\\ \lambda^{(0)}(x,\zeta)&:=\left(\frac{(1+|\nabla h|^2)^2}{2\lambda^{(1)}}\left\{\frac{\lambda^{(1)}}{1+|\nabla h|^2},\frac{\zeta\cdot\nabla h}{1+|\nabla h|^2}\right\}+\frac{1}{2}\Delta h\right)\varphi_{\geqslant 0}(\zeta).\end{aligned} \tag{B.5}$$

The quadratic terms are given by

$$G_2=G_2(h,|\nabla|^{1/2}\omega)\in\varepsilon_1^2\mathcal{O}_{2,5/2},\quad \widehat{G}_2(\xi)=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}g_2(\xi,\eta)\hat{h}(\xi-\eta)|\eta|^{1/2}\widehat{\omega}(\eta)\,d\eta, \tag{B.6}$$

where g_2 is a symbol satisfying (*see the definition of the class S^∞_Ω in* (A.5))

$$\|g_2^{k,k_1,k_2}(\xi,\eta)\|_{S^\infty_\Omega}\lesssim 2^k 2^{\min(k_1,k_2)/2}\left(\frac{1+2^{\min(k_1,k_2)}}{1+2^{\max(k_1,k_2)}}\right)^{7/2}. \tag{B.7}$$

([8]) To be precise, this is $\sqrt{1+|\nabla h|^2}$ times the standard Dirichlet–Neumann operator, but we will slightly abuse notation and call $G(h)\phi$ the Dirichlet–Neumann operator.

Remark B.2. Using (B.5), Definition A.6, and (A.43)–(A.45) we see that, for any $t\in[0,T]$,

$$\lambda^{(1)}=|\zeta|(1+\varepsilon_1^2\mathcal{M}^{0,2}_{N_3-1}) \quad\text{and}\quad \lambda^{(0)}\in\varepsilon_1\mathcal{M}^{0,1}_{N_3-2}. \tag{B.8}$$

For later use, we further decompose $\lambda^{(0)}$ into its linear and higher-order parts:

$$\lambda^{(0)}=\lambda_1^{(0)}+\lambda_2^{(0)},\quad \lambda_1^{(0)}:=\Big[\frac{1}{2}\Delta h-\frac{1}{2}\frac{\zeta_j\zeta_k\partial_j\partial_k h}{|\zeta|^2}\Big]\varphi_{\geqslant 0}(\zeta),\quad \lambda_2^{(0)}\in\varepsilon_1^3\mathcal{M}^{0,3}_{N_3-2}. \tag{B.9}$$

According to the formulas in (B.5) and (B.9), we have

$$\lambda_{DN}-|\zeta|-\lambda_1^{(0)}\in\varepsilon_1^2\mathcal{M}^{1,2}_{N_3-2}\quad\text{and}\quad \lambda_{DN}-\lambda^{(1)}-\lambda_1^{(0)}\in\varepsilon_1^3\mathcal{M}^{0,3}_{N_3-2}. \tag{B.10}$$

The proof of Proposition B.1 relies on several results and is given at the end of the section.

B.1. Linearization

We start with a result that identifies the linear and quadratic part of the Dirichlet–Neumann operator.

We first use a change of variable to flatten the surface. We thus define

$$\begin{aligned} &u(x,y):=\Phi(x,h(x)+y),\quad (x,y)\in\mathbb{R}^2\times(-\infty,0],\\ &\Phi(x,z)=u(x,z-h(x)). \end{aligned} \tag{B.11}$$

In particular, $u|_{y=0}=\phi$ and $\partial_y u|_{y=0}=B$, and the Dirichlet–Neumann operator is given by

$$G(h)\phi=(1+|\nabla h|^2)\partial_y u|_{y=0}-\nabla_x h\cdot\nabla_x u|_{y=0}. \tag{B.12}$$

A simple computation yields

$$0=\Delta_{x,z}\Phi=(1+|\nabla_x h|^2)\partial_y^2 u+\Delta_x u-2\partial_y\nabla_x u\cdot\nabla_x h-\partial_y u\Delta_x h. \tag{B.13}$$

Since we will also need to study the linearized operator, it is convenient to also allow for error terms and consider the equation

$$(1+|\nabla_x h|^2)\partial_y^2 u+\Delta_x u-2\partial_y\nabla_x u\cdot\nabla_x h-\partial_y u\Delta_x h=\partial_y\mathfrak{e}_a+|\nabla|\mathfrak{e}_b. \tag{B.14}$$

With $\mathcal{R}:=|\nabla|^{-1}\nabla$ (the Riesz transform), this can be rewritten in the form

$$\begin{aligned} &(\partial_y^2-|\nabla|^2)u=\partial_y Q_a+|\nabla|Q_b,\\ &Q_a:=\nabla u\cdot\nabla h-|\nabla h|^2\partial_y u+\mathfrak{e}_a,\quad Q_b:=\mathcal{R}(\partial_y u\nabla h)+\mathfrak{e}_b. \end{aligned} \tag{B.15}$$

To study the solution u, we will need an additional class of Banach spaces, to measure functions that depend on $y\in(-\infty,0]$ and $x\in\mathbb{R}^2$. These spaces are only used in this section.

Definition B.3. For $t\in[0,T]$, $p\geqslant -10$, and $m\geqslant 1$ let $\mathcal{L}_{m,p}=\mathcal{L}_{m,p}(t)$ denote the Banach space of functions $g\in C((-\infty,0]:\dot H^{1/2,1/2})$ defined by the norm

$$\|g\|_{\mathcal{L}_{m,p}} := \big\||\nabla|g\big\|_{L^2_y\mathcal{O}_{m,p}}+\|\partial_y g\|_{L^2_y\mathcal{O}_{m,p}}+\big\||\nabla|^{1/2}g\big\|_{L^\infty_y\mathcal{O}_{m,p}}. \tag{B.16}$$

The point of these spaces is to estimate solutions of equations of the form

$$(\partial_y-|\nabla|)u=\mathcal{N},$$

in terms of the initial data $u(0)=\psi$. It is easy to see that, if $|\nabla|^{1/2}\psi\in\mathcal{O}_{m,p}$, then

$$\|e^{y|\nabla|}\psi\|_{\mathcal{L}_{m,p}}\lesssim \big\||\nabla|^{1/2}\psi\big\|_{\mathcal{O}_{m,p}}. \tag{B.17}$$

To see this estimate for the $L^2_y\widetilde{W}^{N_1/2,N_2+p}_\Omega$ component, we use the bound

$$\|c\|_{L^2_y\ell^1_k}\lesssim\|c\|_{\ell^1_kL^2_y}$$

for any $c:\mathbb{Z}\times(-\infty,0]\to\mathbb{C}$. Moreover, if $Q\in L^2_y\mathcal{O}_{m,p}$, then

$$\bigg\||\nabla|^{1/2}\int_{-\infty}^0 e^{-|y-s|\,|\nabla|}\mathbf{1}_\pm(y-s)Q(s)\,ds\bigg\|_{L^\infty_y\mathcal{O}_{m,p}}\lesssim\langle t\rangle^{\delta^2/2}\|Q\|_{L^2_y\mathcal{O}_{m,p}} \tag{B.18}$$

and

$$\bigg\||\nabla|\int_{-\infty}^0 e^{-|y-s|\,|\nabla|}\mathbf{1}_\pm(y-s)Q(s)\,ds\bigg\|_{L^2_y\mathcal{O}_{m,p}}\lesssim\langle t\rangle^{\delta^2/2}\|Q\|_{L^2_y\mathcal{O}_{m,p}}. \tag{B.19}$$

Indeed, these bounds follow directly from the definitions for the L^2-based components of the space $\mathcal{O}_{m,p}$, which are H^{N_0+p} and H^{N_1,N_3+p}_Ω. For the remaining component, one can control uniformly the $\widetilde{W}^{N_1/2,N_2+p}_\Omega$ norm of the function localized at every single dyadic frequency, without the factor $\langle t\rangle^{\delta^2/2}$ in the right-hand side. The full bounds follow, once we notice that only the frequencies satisfying $2^k\in[\langle t\rangle^{-8},\langle t\rangle^{8}]$ are relevant in the $\widetilde{W}^{N_1/2,N_2+p}_\Omega$ component of the space $\mathcal{O}_{1,p}$; the other frequencies are already accounted by the stronger Sobolev norms.

Our first result is the following.

LEMMA B.4. (i) *Assume that $t\in[0,T]$ is fixed, $\|\langle\nabla\rangle h\|_{\mathcal{O}_{1,0}}\lesssim\varepsilon_1$, as in* (B.2), *and*

$$\big\||\nabla|^{1/2}\psi\big\|_{\mathcal{O}_{1,p}}\leqslant A<\infty\quad\textit{and}\quad\|\mathfrak{e}_a\|_{L^2_y\mathcal{O}_{1,p}}+\|\mathfrak{e}_b\|_{L^2_y\mathcal{O}_{1,p}}\leqslant A\varepsilon_1\langle t\rangle^{-12\delta^2} \tag{B.20}$$

for some $p\in[-10,0]$. Then, there is a unique solution $u\in\mathcal{L}_{1,p}$ of the equation

$$\begin{aligned}u(y)=e^{y|\nabla|}\bigg(\psi-\frac12\int_{-\infty}^0 e^{s|\nabla|}(Q_a(s)-Q_b(s))\,ds\bigg)\\+\frac12\int_{-\infty}^0 e^{-|y-s|\,|\nabla|}(\operatorname{sgn}(y-s)Q_a(s)-Q_b(s))\,ds,\end{aligned} \tag{B.21}$$

where Q_a *and* Q_b *are as in* (B.15). *Moreover, u is a solution of the equation*

$$(\partial_y^2-|\nabla|^2)u=\partial_y Q_a+|\nabla|Q_b$$

in (B.15) (*and therefore a solution of* (B.14) *in* $\mathbb{R}^2\times(-\infty,0]$), *and*

$$\|u\|_{\mathcal{L}_{1,p}}=\big\||\nabla|u\big\|_{L^2_y\mathcal{O}_{1,p}}+\|\partial_y u\|_{L^2_y\mathcal{O}_{1,p}}+\big\||\nabla|^{1/2}u\big\|_{L^\infty_y\mathcal{O}_{1,p}}\lesssim A. \tag{B.22}$$

(ii) *Assume that we make the stronger assumptions* (*compare with* (B.20))

$$\big\||\nabla|^{1/2}\psi\big\|_{\mathcal{O}_{1,p}}\leqslant A<0 \quad \textit{and} \quad \|\partial_y^j\mathfrak{e}\|_{L^2_y\mathcal{O}_{2,p-j}}+\|\partial_y^j\mathfrak{e}\|_{L^\infty_y\mathcal{O}_{2,p-1/2-j}}\leqslant A\varepsilon_1\langle t\rangle^{-12\delta^2} \tag{B.23}$$

for $\mathfrak{e}\in\{\mathfrak{e}_a,\mathfrak{e}_b\}$ *and* $j\in\{0,1,2\}$. *Then,*

$$\big\|\partial_y^j(\partial_y u-|\nabla|u)\big\|_{L^2_y\mathcal{O}_{2,p-j}}+\big\|\partial_y^j(\partial_y u-|\nabla|u)\big\|_{L^\infty_y\mathcal{O}_{2,p-1/2-j}}\lesssim A\varepsilon_1. \tag{B.24}$$

Proof. (i) We use a fixed-point argument in a ball of radius $\approx A$ in $\mathcal{L}_{1,p}$ for the functional

$$\begin{aligned}\Phi(u):=e^{y|\nabla|}\bigg(\psi-\frac{1}{2}\int_{-\infty}^0 e^{s|\nabla|}(Q_a(s)-Q_b(s))\,ds\bigg)&\\ +\frac{1}{2}\int_{-\infty}^0 e^{-|y-s|\,|\nabla|}(\operatorname{sign}(y-s)Q_a(s)-Q_b(s))\,ds.&\end{aligned} \tag{B.25}$$

Notice that, using Lemma A.2 and (B.20), if $\|u\|_{\mathcal{L}_{1,p}}\lesssim 1$, then

$$\|Q_a\|_{L^2_y\mathcal{O}_{1,p}}+\|Q_b\|_{L^2_y\mathcal{O}_{1,p}}\lesssim A\varepsilon_1\langle t\rangle^{-12\delta^2}.$$

Therefore, using (B.17)–(B.19), $\|\Phi(u)-e^{y|\nabla|}\psi\|_{\mathcal{L}_{1,p}}\lesssim A\varepsilon_1$. Similarly, one can show that $\|\Phi(u)-\Phi(v)\|_{\mathcal{L}_{1,p}}\lesssim\varepsilon_1\|u-v\|_{\mathcal{L}_{1,p}}$, and the desired conclusion follows.

(ii) The identity (B.21) shows that

$$\partial_y u(y)-|\nabla|u(y)=Q_a(y)+\int_{-\infty}^y|\nabla|e^{-|s-y|\,|\nabla|}(Q_b(s)-Q_a(s))\,ds. \tag{B.26}$$

Given (B.22), definition (B.15), and the stronger assumptions in (B.23), we have

$$\|Q\|_{L^2_y\mathcal{O}_{2,p}}+\|Q\|_{L^\infty_y\mathcal{O}_{2,p-1/2}}\lesssim A\varepsilon_1\langle t\rangle^{-12\delta^2} \tag{B.27}$$

for $Q\in\{Q_a,Q_b\}$. Using estimates similar to (B.18) and (B.19), it follows that

$$\big\|\partial_y u-|\nabla|u\big\|_{L^2_y\mathcal{O}_{2,p}}+\big\|\partial_y u-|\nabla|u\big\|_{L^\infty_y\mathcal{O}_{2,p-1/2}}\lesssim A\varepsilon_1. \tag{B.28}$$

To prove (B.24) for $j\in\{1,2\}$, we observe that, as a consequence of (B.14),

$$\partial_y^2 u-|\nabla|^2u=(1+|\nabla_x h|^2)^{-1}(-|\nabla|^2u|\nabla_x h|^2+2\partial_y\nabla_x u\cdot\nabla_x h+\partial_y u\Delta_x h+\partial_y\mathfrak{e}_a+|\nabla|\mathfrak{e}_b). \tag{B.29}$$

Using (B.22) and (B.28), together with Lemma A.2, it follows that

$$\big\|\partial_y^2 u-|\nabla|^2u\big\|_{L^2_y\mathcal{O}_{2,p-1}}+\big\|\partial_y^2u-|\nabla|^2u\big\|_{L^\infty_y\mathcal{O}_{2,p-3/2}}\lesssim A\varepsilon_1.$$

The desired bound (B.24) for $j=1$ follows using also (B.28). The bound for $j=2$ then follows by differentiating (B.29) with respect to y. This completes the proof of the lemma. □

B.2. Paralinearization

The previous analysis allows us to isolate the linear (and the higher-order) components of the Dirichlet–Neumann operator. However, this is insufficient for our purpose, because we also need to avoid losses of derivatives in the equation. To deal with this, we follow the approach of Alazard–Metivier [5], Alazard–Burq–Zuily [1], [2] and Alazard–Delort [3] using paradifferential calculus. Our choice is to work with the (somewhat unusual) Weyl quantization, instead of the standard one used by the cited authors. We refer to Appendix A for a review of the paradifferential calculus using the Weyl quantization.

For simplicity of notation, we set $\alpha=|\nabla h|^2$ and let

$$\omega := u - T_{\partial_y u} h. \tag{B.30}$$

Notice that ω is naturally extended to the fluid domain; compare with the definition (B.3). We will also assume (B.2) and use Lemma B.4. Using (A.51) in Lemma A.7 and (B.24), we see that

$$\|\omega - u\|_{L^2_y \mathcal{O}_{2,1} \cap L^\infty_y \mathcal{O}_{2,1}} \lesssim \varepsilon_1^2. \tag{B.31}$$

Using Lemma A.4 to paralinearize products, we may rewrite the equation (B.13) as

$$T_{1+\alpha}\partial_y^2\omega + \Delta\omega - 2T_{\nabla h}\nabla\partial_y\omega - T_{\Delta h}\partial_y\omega = \mathcal{Q}+\mathcal{C}, \tag{B.32}$$

where

$$\begin{aligned} -\mathcal{Q} &= -2\mathcal{H}(\nabla h, \nabla\partial_y u) - \mathcal{H}(\Delta h, \partial_y u), \\ -\mathcal{C} &= \partial_y(T_{1+\alpha}T_{\partial_y^2 u} + T_{\Delta u} - 2T_{\nabla h}T_{\nabla\partial_y u} - T_{\Delta h}T_{\partial_y u})h + 2(T_{\partial_y^2 u}T_{\nabla h} - T_{\nabla h}T_{\partial_y^2 u})\nabla h \\ &\qquad + T_{\partial_y^2 u}\mathcal{H}(\nabla h, \nabla h) + \mathcal{H}(\alpha, \partial_y^2 u). \end{aligned} \tag{B.33}$$

Notice that the error terms are quadratic and cubic strongly semilinear. More precisely, using Lemmas A.4 and A.8, and equation (B.13), we see that

$$\mathcal{Q} \in \varepsilon_1^2[L^2_y\mathcal{O}_{2,4} \cap L^\infty_y \mathcal{O}_{2,4}] \quad \text{and} \quad \mathcal{C} \in \varepsilon_1^3 \langle t\rangle^{-11\delta^2}[L^2_y\mathcal{O}_{3,4} \cap L^\infty_y\mathcal{O}_{3,4}]. \tag{B.34}$$

We now look for a factorization of the main elliptic equation into

$$\begin{aligned} &T_{1+\alpha}\partial_y^2 + \Delta - 2T_{\nabla h}\nabla\partial_y - T_{\Delta h}\partial_y \\ &\qquad = (T_{\sqrt{1+\alpha}}\partial_y - A + B)(T_{\sqrt{1+\alpha}}\partial_y - A - B) + \mathcal{E} \\ &\qquad = T^2_{\sqrt{1+\alpha}}\partial_y^2 - ((AT_{\sqrt{1+\alpha}} + T_{\sqrt{1+\alpha}}A) + [T_{\sqrt{1+\alpha}}, B])\partial_y + A^2 - B^2 + [A, B] + \mathcal{E}, \end{aligned}$$

where the error term is acceptable (in a suitable sense to be made precise later), and $[A, \partial_y]=0$ and $[B, \partial_y]=0$. Identifying the terms, this leads to the system

$$\begin{aligned} T_{\sqrt{1+\alpha}}A + AT_{\sqrt{1+\alpha}} + [T_{\sqrt{1+\alpha}}, B] &= 2T_{i\zeta\cdot\nabla h} + \mathcal{E}, \\ A^2 - B^2 + [A, B] &= \Delta + \mathcal{E}. \end{aligned}$$

We may now look for paraproduct solutions in the form

$$A = iT_a, \quad a = a^{(1)} + a^{(0)}, \quad B = T_b, \quad b = b^{(1)} + b^{(0)},$$

where both a and b are real and are a sum of two symbols of order 1 and 0, respectively. Therefore, A corresponds to the skew-symmetric part of the system, while B corresponds to the symmetric part. Using Proposition A.5, and formally identifying the symbols, we obtain the system

$$\begin{aligned} 2ia\sqrt{1+\alpha} + i\{\sqrt{1+\alpha}, b\} &= 2i\zeta\cdot\nabla h + \varepsilon_1^2 \mathcal{M}^{-1,2}_{N_3-2}, \\ a^2 + b^2 + \{a, b\} &= |\zeta|^2 + \varepsilon_1^2 \mathcal{M}^{0,2}_{N_3-2}. \end{aligned}$$

We can solve this by letting

$$\begin{aligned} a^{(1)} &:= \frac{\zeta\cdot\nabla h}{\sqrt{1+\alpha}}, & a^{(0)} &:= -\frac{1}{2\sqrt{1+\alpha}}\{\sqrt{1+\alpha}, b^{(1)}\}\varphi_{\geqslant 0}(\zeta), \\ b^{(1)} &= \sqrt{|\zeta|^2 - (a^{(1)})^2}, & b^{(0)} &= \frac{1}{2b^{(1)}}(-2a^{(1)}a^{(0)} - \{a^{(1)}, b^{(1)}\}\varphi_{\geqslant 0}(\zeta)). \end{aligned}$$

This gives us the following formulas:

$$a^{(1)} = \frac{1}{\sqrt{1+|\nabla h|^2}}(\zeta\cdot\nabla h) = (\zeta\cdot\nabla h)(1 + \varepsilon_1^2 \mathcal{M}^{0,2}_{N_3}), \tag{B.35}$$

$$b^{(1)} = \sqrt{\frac{(1+|\nabla h|^2)|\zeta|^2 - (\zeta\cdot\nabla h)^2}{1+|\nabla h|^2}} = |\zeta|(1 + \varepsilon_1^2 \mathcal{M}^{0,2}_{N_3}), \tag{B.36}$$

$$a^{(0)} = -\frac{\{\sqrt{1+|\nabla h|^2}, b^{(1)}\}}{2\sqrt{1+|\nabla h|^2}}\,\varphi_{\geqslant 0}(\zeta) = \varphi_{\geqslant 0}(\zeta)\varepsilon_1^2 \mathcal{M}^{0,2}_{N_3-1}, \tag{B.37}$$

$$b^{(0)} = -\frac{\sqrt{1+|\nabla h|^2}}{2b^{(1)}}\left\{\frac{\zeta\cdot\nabla h}{1+|\nabla h|^2}, b^{(1)}\right\}\varphi_{\geqslant 0}(\zeta) = \varphi_{\geqslant 0}(\zeta)\left(-\frac{\zeta_j\zeta_k\partial_j\partial_k h}{2|\zeta|^2} + \varepsilon_1^3 \mathcal{M}^{0,3}_{N_3-1}\right). \tag{B.38}$$

We now verify that

$$\begin{aligned} &(T_{\sqrt{1+\alpha}}\partial_y - iT_a + T_b)(T_{\sqrt{1+\alpha}}\partial_y - iT_a - T_b) \\ &\quad = T_{1+\alpha}\partial_y^2 - (2T_{a\sqrt{1+\alpha}} + T_{\{\sqrt{1+\alpha}, b^{(1)}\}})i\partial_y - T_{a^2} - T_{b^2} - T_{\{a^{(1)}, b^{(1)}\}\varphi_{\geqslant 0}(\zeta)} + \mathcal{E}, \end{aligned} \tag{B.39}$$

where

$$\begin{aligned} \mathcal{E} := {} & (T_{\sqrt{1+\alpha}}T_{\sqrt{1+\alpha}} - T_{1+\alpha})\partial_y^2 - (T_aT_{\sqrt{1+\alpha}} + T_{\sqrt{1+\alpha}}T_a - 2T_{a\sqrt{1+\alpha}})i\partial_y \\ & - [T_{\sqrt{1+\alpha}}, T_{b^{(0)}}]\partial_y - ([T_{\sqrt{1+\alpha}}, T_{b^{(1)}}] - iT_{\{\sqrt{1+\alpha}, b^{(1)}\}})\partial_y + (T_{a^2} - T_a^2) \\ & + (T_{b^2} - T_b^2) + i[T_a, T_b] + T_{\{a^{(1)}, b^{(1)}\}\varphi_{\geqslant 0}(\zeta)}. \end{aligned}$$

We also verify that

$$\begin{aligned} 2a\sqrt{1+\alpha} + \{\sqrt{1+\alpha}, b^{(1)}\} &= 2\zeta\cdot\nabla h + \{\sqrt{1+\alpha}, b^{(1)}\}\varphi_{\leqslant -1}(\zeta), \\ a^2 + b^2 + \{a^{(1)}, b^{(1)}\}\varphi_{\geqslant 0}(\zeta) &= |\zeta|^2 + (a^{(0)})^2 + (b^{(0)})^2. \end{aligned}$$

LEMMA B.5. *With the definitions above, we have*

$$(T_{\sqrt{1+\alpha}}\partial_y - iT_a + T_b)(T_{\sqrt{1+\alpha}}\partial_y - iT_a - T_b)\omega = Q_0 + \tilde{\mathcal{C}}, \tag{B.40}$$

where

$$\begin{aligned} &\tilde{\mathcal{C}} \in \varepsilon_1^3 \langle t\rangle^{-11\delta^2}[L^\infty_y \mathcal{O}_{3,1/2} \cap L^2_y \mathcal{O}_{3,1}], \quad Q_0 \in \varepsilon_1^2[L^\infty_y \mathcal{O}_{2,3/2} \cap L^2_y \mathcal{O}_{2,2}], \\ &\widehat{Q}_0(\xi, y) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} q_0(\xi,\eta)\hat{h}(\xi-\eta)\hat{u}(\eta,y)\,d\eta, \end{aligned} \tag{B.41}$$

and

$$\begin{aligned} q_0(\xi,\eta) := {}& \chi\Big(\frac{|\xi-\eta|}{|\xi+\eta|}\Big)\frac{(|\xi|-|\eta|)^2(|\xi|+|\eta|)}{2}\Big(\frac{2\xi\cdot\eta - 2|\xi|\,|\eta|}{|\xi+\eta|^2}\varphi_{\geqslant 0}\Big(\frac{\xi+\eta}{2}\Big) + \varphi_{\leqslant -1}\Big(\frac{\xi+\eta}{2}\Big)\Big) \\ &+ \Big(1 - \chi\Big(\frac{|\xi-\eta|}{|\xi+\eta|}\Big) - \chi\Big(\frac{|\eta|}{|2\xi-\eta|}\Big)\Big)(|\eta|^2 - |\xi|^2)|\eta|. \end{aligned} \tag{B.42}$$

Notice that (*see* (A.6) *for the definition*)

$$\begin{aligned} &\|q_0^{k,k_1,k_2}\|_{S^\infty_\Omega} \lesssim 2^{k_2}2^{2k_1}[2^{-(2k_2-2k_1)}\mathbf{1}_{[-40,\infty)}(k_2-k_1) + \mathbf{1}_{(-\infty,4]}(k_2)], \\ &(\Omega_\xi + \Omega_\eta)q_0 = 0. \end{aligned} \tag{B.43}$$

Proof. Using (B.32) and (B.39), we have

$$\begin{aligned} &(T_{\sqrt{1+\alpha}}\partial_y - iT_a + T_b)(T_{\sqrt{1+\alpha}}\partial_y - iT_a - T_b)\omega \\ &\qquad = \mathcal{Q} + \mathcal{C} + \mathcal{E}\omega - T_{(a^{(0)})^2 + (b^{(0)})^2}\omega - T_{\{\sqrt{1+\alpha}, b^{(1)}\}\varphi_{\leqslant -1}(\zeta)} i\partial_y \omega. \end{aligned}$$

The terms $\mathcal{C}$, $T_{(a^0)^2+(b^{(0)})^2}\omega$, and $T_{\{\sqrt{1+\alpha},b^{(1)}\}\varphi_{\leqslant -1}(\zeta)} i\partial_y\omega$ are in

$$\varepsilon_1^3(1+t)^{-11\delta^2}[L^\infty_y\mathcal{O}_{3,1/2} \cap L^2_y \mathcal{O}_{3,1}].$$

Moreover, using Lemmsa B.4 and A.8, and (B.35)–(B.38), we can verify that

$$\mathcal{E}\omega - (T_{2|\zeta|b_1^{(0)}} - T_{|\zeta|}T_{b_1^{(0)}} - T_{b_1^{(0)}}T_{|\zeta|})\omega - (i[T_{\zeta\cdot\nabla h}, T_{|\zeta|}] + T_{\{\zeta\cdot\nabla h, |\zeta|\}\varphi_{\geqslant 0}(\zeta)})\omega$$

is an acceptable cubic error, where

$$b_1^{(0)} := -\varphi_{\geqslant 0}(\zeta)\frac{\zeta_j\zeta_k\partial_j\partial_k h}{2|\zeta|^2}.$$

Indeed, most of the terms in $\mathcal{E}$ are already acceptable cubic errors; the last three terms become acceptable cubic errors after removing the quadratic components corresponding to the symbols $\zeta\cdot\nabla h$ in $a^{(1)}$, $|\zeta|$ in $b^{(1)}$, and $b_1^{(0)}$ in $b^{(0)}$. As a consequence, we have that $\mathcal{E}\omega-Q_0'\in\varepsilon_1^3\langle t\rangle^{-11\delta^2}[L^\infty_y\mathcal{O}_{3,1/2}\cap L^2_y\mathcal{O}_{3,1}]$, where

$$\widehat{Q_0'}(\xi,y):=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\chi\Big(\frac{|\xi-\eta|}{|\xi+\eta|}\Big)q_0'(\xi,\eta)\hat{h}(\xi-\eta)\widehat{\omega}(\eta,y)\,d\eta,$$

$$q_0'(\xi,\eta):=\frac{(|\xi|-|\eta|)^2(|\xi|+|\eta|)(\xi\cdot\eta-|\xi|\,|\eta|)}{|\xi+\eta|^2}\varphi_{\geqslant 0}\Big(\frac{\xi+\eta}{2}\Big)+\frac{(|\xi|-|\eta|)^2(|\xi|+|\eta|)}{2}\varphi_{\leqslant -1}\Big(\frac{\xi+\eta}{2}\Big).$$

The desired conclusions follow, using also the formula $\mathcal{Q}=2\mathcal{H}(\nabla h,\nabla\partial_y u)+\mathcal{H}(\Delta h,\partial_y u)$ in (B.33), and the approximations $\partial_y u\approx|\nabla|u$ and $\omega\approx u$, up to suitable quadratic errors. □

In order to continue, we want to invert the first operator in (B.40), which is elliptic in the domain under consideration.

LEMMA B.6. *Let* $U:=(T_{\sqrt{1+\alpha}}\partial_y-iT_a-T_b)\omega\in\varepsilon_1[L^\infty_y\mathcal{O}_{1,-1/2}\cap L^2_y\mathcal{O}_{1,0}]$, *so*

$$(T_{\sqrt{1+\alpha}}\partial_y-iT_a+T_b)U=Q_0+\tilde{\mathcal{C}}. \tag{B.44}$$

Define

$$\widehat{M_0[f,g]}(\xi)=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}m_0(\xi,\eta)\hat{f}(\xi-\eta)\hat{g}(\eta)\,d\eta,\quad m_0(\xi,\eta):=\frac{q_0(\xi,\eta)}{|\xi|+|\eta|}. \tag{B.45}$$

Then, recalling notation (A.7), *and letting* $U_0:=U|_{y=0}$ *and* $u_0:=u|_{y=0}=\phi$, *we have*

$$P_{\geqslant -10}(U_0-M_0[h,u_0])\in\varepsilon_1^3\langle t\rangle^{-\delta^2}\mathcal{O}_{3,3/2}. \tag{B.46}$$

Proof. Set

$$\widetilde{U}:=T_{(1+\alpha)^{1/4}}U\in\varepsilon_1[L^\infty_y\mathcal{O}_{1,-1/2}\cap L^2_y\mathcal{O}_{1,0}],\quad \sigma:=\frac{b-ia}{\sqrt{1+\alpha}}=|\zeta|(1+\varepsilon_1\mathcal{M}^{0,1}_{N_3-1}). \tag{B.47}$$

Using (B.44) and Lemma A.8, and letting $f:=(1+\alpha)^{1/4}-1\in\varepsilon_1^2\mathcal{O}_{2,0}$, we calculate

$$T_{(1+\alpha)^{1/4}}(\partial_y+T_\sigma)\widetilde{U}=Q_0+\mathcal{C}_1$$

and

$$\mathcal{C}_1:=\tilde{\mathcal{C}}+(T_f^2-T_{f^2})\partial_yU+(T_{f+1}T_\sigma T_{f+1}-T_{(f+1)^2\sigma})U\in\varepsilon_1^3\langle t\rangle^{-11\delta^2}[L^\infty_y\mathcal{O}_{3,1/2}\cap L^2_y\mathcal{O}_{3,1}].$$

Let $g=(1+f)^{-1}-1\in\varepsilon_1^2\mathcal{O}_{2,0}$, and apply the operator T_{1+g} to the identity above. Using Lemma A.8, it follows that

$$(\partial_y+T_\sigma)\widetilde{U}=Q_0+\mathcal{C}_2,\quad \mathcal{C}_2\in\varepsilon_1^3\langle t\rangle^{-11\delta^2}[L^\infty_y\mathcal{O}_{3,1/2}\cap L^2_y\mathcal{O}_{3,1}]. \tag{B.48}$$

Notice that, using Lemma B.4, (B.43), (B.45), and Lemma A.2,

$$M_0[h,u]\in\varepsilon_1^2[L^\infty_y\mathcal{O}_{2,5/2}\cap L^2_y\mathcal{O}_{2,3}]\quad\text{and}\quad M_0[h,\partial_y u]\in\varepsilon_1^2[L^\infty_y\mathcal{O}_{2,3/2}\cap L^2_y\mathcal{O}_{2,2}].\tag{B.49}$$

We define $V:=\widetilde{U}-M_0[h,u]$. Since

$$V=T_{(1+\alpha)^{1/4}}U-M_0[h,u]=T_{(1+\alpha)^{1/4}}(U-M_0[h,u])+\mathcal{C}'\ \text{ and }\ \mathcal{C}'\in\varepsilon_1^3\langle t\rangle^{-11\delta^2}L^\infty_y\mathcal{O}_{3,3/2},$$

for (B.46) it suffices to prove that

$$P_{\geqslant -20}V(y)\in\varepsilon_1^3\langle t\rangle^{-\delta^2}\mathcal{O}_{3,3/2}\quad\text{for any } y\in(-\infty,0].\tag{B.50}$$

Using also (B.24), we verify that

$$\begin{aligned}(\partial_y+T_\sigma)V&=(\partial_y+T_\sigma)\widetilde{U}-(\partial_y+|\nabla|)M_0[h,u]-T_{(\sigma-|\zeta|)}M_0[h,u]\\&=\mathcal{C}_2+M_0[h,|\nabla|u-\partial_y u]-T_{(\sigma-|\zeta|)}M_0[h,u]\\&=\mathcal{C}_3\in\varepsilon_1^3\langle t\rangle^{-11\delta^2}[L^\infty_y\mathcal{O}_{3,1/2}\cap L^2_y\mathcal{O}_{3,1}].\end{aligned}\tag{B.51}$$

Letting $\sigma':=\sigma-|\zeta|$ and $V_k:=P_kV$, $k\in\mathbb{Z}$, we calculate

$$(\partial_y+T_{|\zeta|})V_k=P_k\mathcal{C}_3-P_kT_{\sigma'}V.$$

We can rewrite this equation in integral form,

$$V_k(y)=\int_{-\infty}^y e^{(s-y)|\nabla|}(P_k\mathcal{C}_3(s)-P_kT_{\sigma'}V(s))\,ds.\tag{B.52}$$

To prove the desired bound for the high Sobolev norm, let, for $k\in\mathbb{Z}$,

$$X_k:=\sup_{y\leqslant 0}2^{(N_0+3/2)k}\|V_k(y)\|_{L^2}.$$

Since $\sigma'/|\zeta|\in\varepsilon_1\mathcal{M}^{0,1}_{N_3-1}$, it follows from Lemma A.7 that, for any $y\leqslant 0$,

$$\begin{aligned}&2^{(N_0+3/2)k}\int_{-\infty}^y\|e^{(s-y)|\nabla|}P_kT_{\sigma'}V(s)\|_{L^2}\,ds\\&\quad\lesssim 2^{(N_0+3/2)k}\varepsilon_1\sum_{|k'-k|\leqslant 4}\int_{-\infty}^y e^{(s-y)2^{k-4}}2^k\|P_{k'}V(s)\|_{L^2}\,ds\lesssim\varepsilon_1\sum_{|k'-k|\leqslant 4}X_{k'}.\end{aligned}$$

It follows from (B.52) that, for any $k\in\mathbb{Z}$

$$\begin{aligned}X_k&\lesssim\varepsilon_1\sum_{|k'-k|\leqslant 4}X_{k'}+\sup_{y\leqslant 0}2^{(N_0+3/2)k}\int_{-\infty}^y e^{(s-y)2^{k-4}}\|P_k\mathcal{C}_3(s)\|_{L^2}\,ds\\&\lesssim\varepsilon_1\sum_{|k'-k|\leqslant 4}X_{k'}+2^{(N_0+1)k}\bigg(\int_{-\infty}^0\|P_k\mathcal{C}_3(s)\|_{L^2}^2\,ds\bigg)^{1/2}.\end{aligned}$$

We take l^2 summation in k, and absorb the first term in the right-hand side([9]) into the left-hand side, to conclude that

$$\Big(\sum_{k\in\mathbb{Z}} X_k^2\Big)^{1/2} \lesssim \Big(\sum_{k\in\mathbb{Z}} 2^{2(N_0+1)k}\int_{-\infty}^0 \|P_k\mathcal{C}_3(s)\|_{L^2}^2\,ds\Big)^{1/2} \lesssim \varepsilon_1^3\langle t\rangle^{-11\delta^2}\langle t\rangle^{-2(5/6-20\delta^2)+\delta^2}, \tag{B.53}$$

where the last inequality in this estimate is a consequence of $\mathcal{C}_3\in\varepsilon_1^3\langle t\rangle^{-11\delta^2}L_y^2\mathcal{O}_{3,1}$. The desired bound $\|P_{\geqslant -20}V(y)\|_{H^{N_0+3/2}}\lesssim\varepsilon_1^3\langle t\rangle^{-11\delta^2}\langle t\rangle^{-2(5/6-20\delta^2)+\delta^2}$ in (B.50) follows.

The proof of the bound for the weighted norms is similar. For $k\in\mathbb{Z}$ let

$$Y_k := \sup_{y\leqslant 0} 2^{(N_3+3/2)k}\sum_{j\leqslant N_1}\|\Omega^j V_k(y)\|_{L^2}.$$

As before, we have the bounds

$$2^{(N_3+3/2)k}\int_{-\infty}^y \|e^{(s-y)|\nabla|}\Omega^j P_k T_{\sigma'}V(s)\|_{L^2}\,ds \lesssim \varepsilon_1\sum_{|k'-k|\leqslant 4}[Y_{k'}+\langle t\rangle^{6\delta^2}X_{k'}]$$

for any $y\in(-\infty,0]$ and $j\leqslant N_1$, and therefore, using (B.52),

$$Y_k\lesssim\varepsilon_1\sum_{|k'-k|\leqslant 4}Y_{k'}+\varepsilon_1\langle t\rangle^{6\delta^2}\sum_{|k'-k|\leqslant 4}X_{k'}+\sum_{j\leqslant N_1}2^{(N_3+1)k}\Big(\int_{-\infty}^0\|\Omega^j P_k\mathcal{C}_3(s)\|_{L^2}^2\,ds\Big)^{1/2}.$$

As before, we take the l^2 sum in k, and use (B.53) and the hypothesis

$$\mathcal{C}_3\in\varepsilon_1^3\langle t\rangle^{-11\delta^2}L_y^2\mathcal{O}_{3,1}.$$

The desired bound

$$\|P_{\geqslant -20}V(y)\|_{H_\Omega^{N_1,N_3+3/2}}\lesssim\varepsilon_1^3\langle t\rangle^{-4\delta^2}\langle t\rangle^{-2(5/6-20\delta^2)+\delta^2}$$

in (B.50) follows.

Finally, for the L^∞ bound, we let, for $k\in\mathbb{Z}$,

$$Z_k := \sup_{y\leqslant 0} 2^{(N_2+3/2)k}\sum_{j\leqslant N_1/2}\|\Omega^j V_k(y)\|_{L^\infty}.$$

([9]) To make this step rigorous, one can modify the definition of X_k to

$$X_k' := \sup_{y\leqslant 0} 2^{(N_0+3/2)\min(k,K)}\|V_k(y)\|_{L^2},$$

in order to make sure that $\sum_k (X_k')^2<\infty$, and then prove uniform estimates in K and finally let $K\to\infty$.

As before, using (B.52), it follows that

$$Z_k \lesssim \varepsilon_1 \sum_{|k'-k|\leqslant 4} Z_{k'} + \sum_{j\leqslant N_1/2} 2^{(N_2+1)k}\biggl(\int_{-\infty}^{0} \|\Omega^j P_k \mathcal{C}_3(s)\|_{L^\infty}^2\, ds\biggr)^{1/2}.$$

After taking l^2 summation in k, it follows that

$$\begin{aligned}\biggl(\sum_{k\in\mathbb{Z}} Z_k^2\biggr)^{1/2} &\lesssim \sum_{j\leqslant N_1/2}\biggl(\sum_{k\in\mathbb{Z}} 2^{2(N_2+1)k}\int_{-\infty}^{0} \|\Omega^j P_k \mathcal{C}_3(s)\|_{L^\infty}^2\, ds\biggr)^{1/2}\\ &\lesssim \varepsilon_1^3 \langle t\rangle^{-11\delta^2}\langle t\rangle^{-5/2+45\delta^2},\end{aligned}$$

where the last inequality is a consequence of $\mathcal{C}_3\in\varepsilon_1^3\langle t\rangle^{-11\delta^2}L_y^2\mathcal{O}_{3,1}$. The desired bound on $\|P_{\geqslant -20}V(y)\|_{\widetilde{W}_\Omega^{N_1/2,N_2+3/2}}$ in (B.50) follows, once we recall that only the sum over $2^{|k|}\leqslant\langle t\rangle^8$ is relevant when estimating the $\widetilde{W}_\Omega^{N_1/2,N_2+3/2}$ norm; the remaining frequencies are already accounted for by the stronger Sobolev norms. □

We are now ready to obtain the paralinearization of the Dirichlet–Neumann operator.

Proof of Proposition B.1. Recall that $G(h)\phi=(1+|\nabla h|^2)\partial_y u|_{y=0}-\nabla h\cdot\nabla u|_{y=0}$; see (B.12), and $B=\partial_y u|_{y=0}$. All the calculations below are done on the interface, at $y=0$. We observe that, using Corollary C.1,

$$\begin{aligned}&P_{\leqslant 6}((1+|\nabla h|^2)\partial_y u-\nabla h\cdot\nabla u)\\ &\quad=P_{\leqslant 6}(\partial_y u-\nabla h\cdot\nabla u)+\varepsilon_1^3\mathcal{O}_{3,3/2}\\ &\quad=P_{\leqslant 6}(|\nabla|\omega-\operatorname{div}(T_V h))+P_{\leqslant 6}(\operatorname{div}(T_V h)+|\nabla|T_{|\nabla|\omega}h+N_2[h,\omega])+\varepsilon_1^3\mathcal{O}_{3,3/2}.\end{aligned}\tag{B.54}$$

Thus, low frequencies give acceptable contributions. To estimate high frequencies, we compute

$$\begin{aligned}&(1+|\nabla h|^2)\partial_y u-\nabla h\cdot\nabla u\\ &\quad=T_{1+\alpha}\partial_y u-T_{\nabla h}\nabla u-T_{\nabla u}\nabla h+T_{\partial_y u}\alpha+\mathcal{H}(\alpha,\partial_y u)-\mathcal{H}(\nabla h,\nabla u)\\ &\quad=T_{1+\alpha}\partial_y\omega-T_{\nabla h}\nabla\omega-T_{\nabla u}\nabla h+T_{\nabla h}T_{\partial_y u}\nabla h+(T_{\partial_y u}\alpha-2T_{\nabla h}T_{\partial_y u}\nabla h)\\ &\qquad+T_{1+\alpha}T_{\partial_y^2 u}h-T_{\nabla h}T_{\nabla\partial_y u}h+\mathcal{H}(\alpha,\partial_y u)-\mathcal{H}(\nabla h,\nabla u).\end{aligned}$$

Using Lemma B.6 with $U=(T_{\sqrt{1+\alpha}}\partial_y-iT_a-T_b)\omega$, (B.49) and Lemmas A.7 and A.8, we find that

$$\begin{aligned}T_{1+\alpha}\partial_y\omega&=T_{\sqrt{1+\alpha}}(iT_a\omega+T_b\omega+M_0[h,u]+\mathcal{C}')+(T_{1+\alpha}-T^2_{\sqrt{1+\alpha}})\partial_y\omega\\ &=T_{\sqrt{1+\alpha}}(T_b+iT_a)\omega+M_0[h,u]+\mathcal{C}'',\end{aligned}$$

where $\mathcal{C}''$ satisties $P_{\geqslant -6}\mathcal{C}''\in\varepsilon_1^3\mathcal{O}_{3,3/2}$. Therefore, with $V=\nabla u-\partial_y u\nabla h$,

$$(1+|\nabla h|^2)\partial_y u-\nabla h\cdot\nabla u=T_{\sqrt{1+\alpha}}(T_b+iT_a)\omega+M_0[h,u]+\mathcal{C}'' \\ -T_{\nabla h}\nabla\omega-\operatorname{div}(T_V h)+\mathcal{C}_1+\mathcal{C}_2-\mathcal{H}(\nabla h,\nabla u), \tag{B.55}$$

with cubic terms $\mathcal{C}_1$ and $\mathcal{C}_2$ given explicitly by

$$\mathcal{C}_1=(T_{\partial_y u}\alpha-2T_{\nabla h}T_{\partial_y u}\nabla h)+\mathcal{H}(\alpha,\partial_y u),$$

$$\mathcal{C}_2=(T_{\operatorname{div} V}+T_{1+\alpha}T_{\partial_y^2 u}-T_{\nabla h}T_{\nabla\partial_y u})h+(T_{\nabla h}T_{\partial_y u}-T_{\partial_y u\nabla h})\nabla h.$$

Notice that $\operatorname{div} V+(1+\alpha)\partial_y^2 u-\nabla h\nabla\partial_y u=0$, as a consequence of (B.13). Using also Lemma A.8, it follows that $\mathcal{C}_1,\mathcal{C}_2\in\varepsilon_1^3\mathcal{O}_{3,3/2}$.

Moreover, using formulas (B.36) and (B.38), Lemmas A.5 and A.8, we see that

$$T_{\sqrt{1+\alpha}}T_b\omega=T_{b\sqrt{1+\alpha}}\omega+\tfrac{1}{2}iT_{\{\sqrt{1+\alpha},b\}}\omega+E(\sqrt{1+\alpha}-1,b)\omega \\ =T_{\lambda^{(1)}}\omega+T_{b^{(0)}\sqrt{1+\alpha}}\omega+\tfrac{1}{2}iT_{\{\sqrt{1+\alpha},b^{(1)}\}}\omega+\varepsilon_1^3\mathcal{O}_{3,3/2},$$

where $\lambda^{(1)}$ is the principal symbol in (B.5). Similarly, using (B.35)and (B.37),

$$iT_{\sqrt{1+\alpha}}T_a\omega-T_{\nabla h}\nabla\omega \\ =T_{i\zeta\cdot\nabla h}\omega-T_{\nabla h}\nabla\omega+iT_{a^{(0)}\sqrt{1+\alpha}}\omega-\tfrac{1}{2}T_{\{\sqrt{1+\alpha},a\}}\omega+iE(\sqrt{1+\alpha}-1,a)\omega \\ =\tfrac{1}{2}T_{\Delta h}\omega+iT_{a^{(0)}\sqrt{1+\alpha}}\omega-\tfrac{1}{2}T_{\{\sqrt{1+\alpha},a^{(1)}\}}\omega+\varepsilon_1^3\mathcal{O}_{3,3/2}.$$

Summing these last two identities and using (B.35)–(B.38), we see that

$$T_{\sqrt{1+\alpha}}T_b\omega+iT_{\sqrt{1+\alpha}}T_a\omega-T_{\nabla h}\nabla\omega=T_{\lambda^{(1)}}\omega+T_m\omega+\varepsilon_1^3\mathcal{O}_{3,3/2}, \tag{B.56}$$

where

$$m:=b^{(0)}\sqrt{1+\alpha}-\frac{1}{2}\{\sqrt{1+\alpha},a^{(1)}\}+\frac{1}{2}\Delta h \\ =\frac{(1+\alpha)^{3/2}}{2\lambda^{(1)}}\left\{\frac{\lambda^{(1)}}{\sqrt{1+\alpha}},\frac{\zeta\cdot\nabla h}{1+\alpha}\right\}\varphi_{\geqslant 0}(\zeta)-\frac{1}{2}\left\{\sqrt{1+\alpha},\frac{\zeta\cdot\nabla h}{\sqrt{1+\alpha}}\right\}+\frac{1}{2}\Delta h \\ =\lambda^{(0)}-\frac{1}{2}\left\{\sqrt{1+\alpha},\frac{\zeta\cdot\nabla h}{\sqrt{1+\alpha}}\right\}\varphi_{\leqslant -1}(\zeta)+\frac{\Delta h}{2}\varphi_{\leqslant -1}(\zeta). \tag{B.57}$$

We conclude from (B.55) and (B.56) that

$$P_{\geqslant 7}((1+|\nabla h|^2)\partial_y u-\nabla h\nabla u) \\ =P_{\geqslant 7}(T_{\lambda_{DN}}\omega-\operatorname{div}(T_V h)+M_0[h,u]-\mathcal{H}(\nabla h,\nabla u)+\varepsilon_1^3\widetilde{\mathcal{O}}_{3,3/2}).$$

Moreover, the symbol of the bilinear operator $M_0[h,u]-\mathcal{H}(\nabla h,\nabla u)$ is

$$\frac{q_0(\xi,\eta)}{|\xi|+|\eta|}+\left(1-\chi\left(\frac{|\xi-\eta|}{|\xi+\eta|}\right)-\chi\left(\frac{|\eta|}{|2\xi-\eta|}\right)\right)(\xi-\eta)\cdot\eta,$$

where q_0 is defined in (B.42). The symbol bounds (B.7) follow. Combining this with (B.54), we finish the proof. □

Appendix C. Taylor expansion of the Dirichlet–Neumann operator

C.1. A simple expansion

We start with a simple expansion the Dirichlet–Neumann operator, using only the $O_{m,p}$ hierarchy, which suffices in many cases.

Corollary C.1. (i) *Assume that* $\|\langle\nabla\rangle h\|_{\mathcal{O}_{1,0}}+\big\||\nabla|^{1/2}\psi\big\|_{\mathcal{O}_{1,0}}\lesssim\varepsilon_1$ *and* $\mathfrak{c}_a=\mathfrak{c}_b=0$, *and define* u *as in Lemma B.4. Then, we have an expansion*

$$\partial_y u=|\nabla|u+\nabla h\cdot\nabla u+N_2[h,u]+\mathcal{E}^{(3)},\quad \|\mathcal{E}^{(3)}\|_{L^2_y\mathcal{O}_{3,0}\cap L^\infty_y\mathcal{O}_{3,-1/2}}\lesssim\varepsilon_1^3\langle t\rangle^{-11\delta^2},\tag{C.1}$$

where

$$\mathcal{F}\{N_2[h,\phi]\}(\xi)=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}n_2(\xi,\eta)\hat{h}(\xi-\eta)\hat{\phi}(\eta)\,d\eta,\quad n_2(\xi,\eta):=\xi\cdot\eta-|\xi|\,|\eta|.\tag{C.2}$$

In particular,

$$\big\|G(h)\psi-|\nabla|\psi-N_2[h,\psi]\big\|_{\mathcal{O}_{3,-1/2}}\lesssim\varepsilon_1^3\langle t\rangle^{-11\delta^2}.\tag{C.3}$$

Moreover,

$$\|n_2^{k,k_1,k_2}\|_{S^\infty_\Omega}\lesssim 2^{\min\{k,k_1\}}2^{k_2},\quad (\Omega_\xi+\Omega_\eta)n_2\equiv 0.\tag{C.4}$$

(ii) *As in Proposition 2.2, assume that* $(h,\phi)\in C([0,T]\colon H^{N_0+1}\times\dot{H}^{N_0+1/2,1/2})$ *is a solution of the system (2.1) with* $g=1$ *and* $\sigma=1$, $t\in[0,T]$ *is fixed, and (B.2) holds. Then,*

$$\big\|\partial_t(G(h)\phi)-|\nabla|\partial_t\phi\big\|_{\mathcal{O}_{2,-2}}\lesssim\varepsilon_1^2.\tag{C.5}$$

Proof. (i) Let $u^{(1)}:=e^{y|\nabla|}\psi$, $Q_a^{(1)}:=\nabla u^{(1)}\cdot\nabla h$, and $Q_b^{(1)}:=\mathcal{R}(\partial_y u^{(1)}\nabla h)$. It follows from (B.18)–(B.19) and Lemma B.4 (more precisely, from (B.22), (B.24), and (B.27)) that

$$\begin{aligned}\big\||\nabla|^{1/2}(u-u^{(1)})\big\|_{L^\infty_y\mathcal{O}_{2,0}}&+\big\||\nabla|(u-u^{(1)})\big\|_{L^2_y\mathcal{O}_{2,0}}\\&+\|\partial_y(u-u^{(1)})\|_{L^\infty_y\mathcal{O}_{2,-1/2}}+\|\partial_y(u-u^{(1)})\|_{L^2_y\mathcal{O}_{2,0}}\lesssim\varepsilon_1^2.\end{aligned}\tag{C.6}$$

Therefore, using Lemma A.2, for $d\in\{a,b\}$,

$$\|Q_d-Q_d^{(1)}\|_{L^\infty_y\mathcal{O}_{3,-1/2}}+\|Q_d-Q_d^{(1)}\|_{L^2_y\mathcal{O}_{3,0}}\lesssim\varepsilon_1^3\langle t\rangle^{-12\delta^2}.\tag{C.7}$$

Therefore, using (B.18)–(B.19) and (B.26),

$$\begin{aligned}\bigg\|\partial_y u-|\nabla|u-\nabla h\cdot\nabla u-\int_{-\infty}^y|\nabla|e^{-|s-y|\,|\nabla|}(Q_b^{(1)}(s)-Q_a^{(1)}(s))\,ds\bigg\|_{L^2_y\mathcal{O}_{3,0}\cap L^\infty_y\mathcal{O}_{3,-1/2}}&\\ \lesssim\varepsilon_1^3\langle t\rangle^{-11\delta^2}.&\end{aligned}$$

Since

$$\mathcal{F}\{Q_b^{(1)}(s)-Q_a^{(1)}(s)\}(\xi)=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\Big(\eta\cdot(\xi-\eta)-\frac{\xi\cdot(\xi-\eta)}{|\xi|}|\eta|\Big)\hat{h}(\xi-\eta)e^{s|\eta|}\hat{\psi}(\eta)\,d\eta,$$

we have

$$\begin{aligned}&\mathcal{F}\Big\{\int_{-\infty}^{y}|\nabla|e^{-|s-y|\,|\nabla|}(Q_b^{(1)}(s)-Q_a^{(1)}(s))\,ds\Big\}(\xi)\\&\quad=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\Big(\eta\cdot(\xi-\eta)-\frac{\xi\cdot(\xi-\eta)}{|\xi|}|\eta|\Big)\frac{|\xi|}{|\xi|+|\eta|}\hat{h}(\xi-\eta)e^{y|\eta|}\hat{\psi}(\eta)\,d\eta=\mathcal{F}\{N_2[h,u^{(1)}]\}(\xi).\end{aligned}$$

Moreover, using the assumption $\|\langle\nabla\rangle h\|_{\mathcal{O}_{1,0}}\lesssim\varepsilon_1$ and the bounds (C.6), we have

$$\|N_2[h,u-u^{(1)}]\|_{L^2_y\mathcal{O}_{3,0}\cap L^\infty_y\mathcal{O}_{3,-1/2}}\lesssim\varepsilon_1^3\langle t\rangle^{-11\delta^2},$$

as a consequence of Lemma A.2. The desired identity (C.1) follows. The bound (C.3) follows using also identity (B.12).

(ii) We define $u=u(x,y,t)$ as in (B.11), let $v=\partial_t u$, differentiate (B.13) with respect to t, and find that v satisfies (B.14) with

$$\mathfrak{e}_a=\nabla_x u\cdot\nabla_x\partial_t h-2\partial_y u\nabla_x h\cdot\nabla_x\partial_t h\quad\text{and}\quad\mathfrak{e}_b=\mathcal{R}(\partial_y u\nabla_x\partial_t h).$$

In view of (C.3),

$$\|\partial_t h\|_{\mathcal{O}_{1,-1/2}}+\|\partial_t\phi\|_{\mathcal{O}_{1,-1}}\lesssim\varepsilon_1.$$

Therefore, the triplet $(\partial_t\phi,\mathfrak{e}_a,\mathfrak{e}_b)$ satisfies (B.23) with $p=-\frac{3}{2}$. Therefore, using (B.24),

$$\big\|\partial_y v-|\nabla|v\big\|_{L^\infty_y\mathcal{O}_{2,-2}}\lesssim\varepsilon_1^2,$$

and the desired bound (C.5) follows using also (B.12). □

C.2. Proof of Proposition 2.3

We now show that Proposition 2.3 follows from Proposition 7.1. The starting point is the system (2.1). We need to verify that it can be rewritten in the form stated in Proposition 7.1. For this, we need to expand the Dirichlet–Neumann operator

$$G(h)\phi=|\nabla|\phi+N_2[h,\phi]+N_3[h,h,\phi]+\text{quartic remainder},$$

and then prove the required claims. To justify this rigorously and estimate the remainder, the main issue is to prove space localization. We prefer not to work with the Z norm itself, which is too complicated, but define instead certain auxiliary spaces which are used only in this section.

Step 1. We assume that the bootstrap assumption (2.6) holds. Notice first that

$$\sup_{\substack{2a+|\alpha|\leqslant N_1+N_4\\ a\leqslant N_1/2+20}} \sum_{(k,j)\in\mathcal{J}} 2^{\theta j}2^{-\theta|k|/2}\|Q_{jk}D^\alpha\Omega^a\mathcal{U}(t)\|_{L^2}\lesssim\varepsilon_1(1+t)^{\theta+6\delta^2}, \tag{C.8}$$

$$\sup_{\substack{2a+|\alpha|\leqslant N_1+N_4\\ a\leqslant N_1/2+20}} \sum_{(k,j)\in\mathcal{J}} 2^{\theta j}2^{-\theta|k|/2}\|Q_{jk}D^\alpha\Omega^a\mathcal{U}(t)\|_{L^\infty}\lesssim\varepsilon_1(1+t)^{-5/6+\theta+6\delta^2}, \tag{C.9}$$

for $\theta\in\big[0,\frac{1}{3}\big]$, where the operators Q_{jk} are defined as in (2.10). Indeed, let

$$f=e^{it\Lambda}\Omega^a D^\alpha\mathcal{U}(t),$$

and assume that $t\in[2^m-1,2^{m+1}]$, $m\geqslant 0$. We have

$$\|f\|_{H^{N_0'}\cap H_\Omega^{N_1'}}+\|f\|_{Z_1}\lesssim\varepsilon_1 2^{\delta^2 m}, \tag{C.10}$$

as a consequence of (2.6), where, as in (8.27),

$$N_1':=\frac{N_1-N_4}{2}=\frac{1}{2\delta}\quad\text{and}\quad N_0':=\frac{N_0-N_3}{2}-N_4=\frac{1}{\delta}.$$

To prove (C.8), we need to show that

$$\sum_{(k,j)\in\mathcal{J}} 2^{\theta j}2^{-\theta|k|/2}\|Q_{jk}e^{-it\Lambda}f\|_{L^2}\lesssim\varepsilon_1 2^{\theta m+6\delta^2 m}. \tag{C.11}$$

The sum over $j\leqslant m+\delta^2 m+\frac{1}{2}|k|$ or over $j\leqslant |k|+\mathcal{D}$ is easy to control. On the other hand, if $j\geqslant\max\big(m+\delta^2 m+\frac{1}{2}|k|,|k|+\mathcal{D}\big)$, then we decompose $f=\sum_{(k',j')\in\mathcal{J}} f_{j',k'}$ as in (7.33). We may assume that $|k'-k|\leqslant 10$; the contribution of $j'\leqslant j-\delta^2 j$ is negligible, using integration by parts, while for $j'\geqslant j-\delta^2 j-10$ we have

$$\|Q_{jk}e^{-it\Lambda}f_{j',k'}\|_{L^2}\lesssim\varepsilon_1 2^{\delta^2 m}\min(2^{-2j'/5},2^{-N_0'k^+}).$$

The desired bound (C.11) follows, which completes the proof of (C.8). The proof of (C.9) is similar, using also the decay bound (7.44). As a consequence, it follows that

$$\begin{gathered}\sum_{(k,j)\in\mathcal{J}} 2^{\theta j}2^{-\theta|k|/2}\|Q_{jk}g(t)\|_{L^2}\lesssim\varepsilon_1 2^{\theta m+6\delta^2 m},\\ \sum_{(k,j)\in\mathcal{J}} 2^{\theta j}2^{-\theta|k|/2}\|Q_{jk}g(t)\|_{L^\infty}\lesssim\varepsilon_1 2^{-5m/6+\theta m+6\delta^2 m},\end{gathered} \tag{C.12}$$

for $g\in\big\{D^\alpha\Omega^a\langle\nabla\rangle h, D^\alpha\Omega^a|\nabla|^{1/2}\phi: 2a+|\alpha|\leqslant N_1+N_4, a\leqslant\frac{1}{2}N_1+20\big\}$ and $\theta\in\big[0,\frac{1}{3}\big]$.

Step 2. We now need to define certain norms that allow us to extend our estimates to the region $\{y\leqslant 0\}$; compare with the analysis in §B.1.

LEMMA C.2. *For $q\geqslant 0$, $\theta\in[0,1]$, and $p,r\in[1,\infty]$, define the norms*

$$\|f\|_{Y^p_{\theta,q}(\mathbb{R}^2)} := \sum_{(k,j)\in\mathcal{J}} 2^{\theta j}2^{qk^+}\|Q_{jk}f\|_{L^p},$$

$$\|f\|_{L^r_yY^p_{\theta,q}(\mathbb{R}^2\times(-\infty,0])} := \sum_{(k,j)\in\mathcal{J}} 2^{\theta j}2^{qk^+}\|Q_{jk}f\|_{L^r_yL^p_x}.$$

(i) *Then, for any $p\in[2,\infty]$ and $\theta\in[0,1]$,*

$$\|e^{y|\nabla|}f\|_{L^\infty_yY^p_{\theta,q}} + \big\||\nabla|^{1/2}e^{y|\nabla|}f\big\|_{L^2_yY^p_{\theta,q}} \lesssim \|f\|_{Y^p_{\theta,q}} \tag{C.13}$$

and

$$\begin{aligned}&\Big\|\int_{-\infty}^0 |\nabla|^{1/2}e^{-|s-y|\,|\nabla|}\mathbf{1}_\pm(y-s)f(s)\,ds\Big\|_{L^\infty_yY^2_{\theta,q}}\\ &\qquad+\Big\|\int_{-\infty}^0 |\nabla|e^{-|s-y|\,|\nabla|}\mathbf{1}_\pm(y-s)f(s)\,ds\Big\|_{L^2_yY^2_{\theta,q}} \lesssim \|f\|_{L^2_yY^2_{\theta,q}}.\end{aligned} \tag{C.14}$$

(ii) *If $p_1,p_2,p,r_1,r_2,r\in\{2,\infty\}$, $1/p=1/p_1+1/p_2$, $1/r=1/r_1+1/r_2$, then*

$$\|fg\|_{L^r_yY^p_{\theta_1+\theta_2-\delta^2,q-\delta^2}} \lesssim \|f\|_{L^{r_1}_yY^{p_1}_{\theta_1,q}}\,\|g\|_{L^{r_2}_yY^{p_2}_{\theta_2,q}}, \tag{C.15}$$

provided that $\theta_1,\theta_2\in[0,1]$, $\theta_1+\theta_2\in[\delta^2,1]$, and $q\geqslant\delta^2$. Moreover,

$$\|fg\|_{L^2_yY^2_{\theta_1-\delta^2,q-\delta^2}} \lesssim \|f\|_{L^\infty_yY^\infty_{\theta_1,q}}\,\|g\|_{L^2_yH^q_x}. \tag{C.16}$$

Proof. The linear bounds in part (i) follow by parabolic estimates, once we notice that the kernel of the operator $e^{y|\nabla|}P_k$ is essentially localized in a ball of radius $\lesssim 2^{-k}$ and is bounded by $C2^{2k}(1+2^k|y|)^{-4}$.

The bilinear estimates in part (ii) follow by unfolding the definitions. The implicit factors $2^{-\delta^2j}2^{-\delta^2k^+}$ in the left-hand side allow one to prove the estimate for (k,j) fixed. Then, one can decompose $f=\sum f_{j_1,k_1}$ and $g=\sum g_{j_2,k_2}$ as in (7.33) and estimate $\|Q_{jk}(f_{j_1,k_1}g_{j_2,k_2})\|_{L^r_yL^p_x}$ using simple product estimates. The case $j=-k\gg\min(j_1,j_2)$ requires some additional attention; in this case, one can use first Sobolev imbedding and the hypothesis $\theta_1+\theta_2\leqslant 1$. □

Step 3. Recall now formula (B.21):

$$\begin{aligned}u&=e^{y|\nabla|}\phi+L(u),\\ L(u)&:=-\frac{1}{2}e^{y|\nabla|}\int_{-\infty}^0 e^{s|\nabla|}(Q_a(s)-Q_b(s))\,ds\\ &\qquad+\frac{1}{2}\int_{-\infty}^0 e^{-|y-s|\,|\nabla|}(\operatorname{sgn}(y-s)Q_a(s)-Q_b(s))\,ds,\end{aligned}$$

where $Q_a[u]=\nabla u\cdot\nabla h-|\nabla h|^2\partial_y u$ and $Q_b[u]=\mathcal{R}(\partial_y u\nabla h)$. Let, as in Corollary C.1,

$$u^{(1)}=e^{y|\nabla|}\phi \quad \text{and} \quad u^{(n+1)}=e^{y|\nabla|}\phi+L(u^{(n)}),\ \ n\geqslant 1. \tag{C.17}$$

We can now prove a precise asymptotic expansion on the Dirichlet–Neumann operator.

LEMMA C.3. *We have*

$$G(h)\phi=|\nabla|\phi+N_2[h,\phi]+N_3[h,\phi]+|\nabla|^{1/2}N_4[h,\phi], \tag{C.18}$$

where N_2 *is as in* (C.2),

$$\begin{aligned}\mathcal{F}\{N_3[h,\phi]\}(\xi)&=\frac{1}{(4\pi^2)^2}\int_{(\mathbb{R}^2)^2} n_3(\xi,\eta,\sigma)\hat{h}(\xi-\eta)\hat{h}(\eta-\sigma)\hat{\phi}(\sigma)\,d\eta\,d\sigma,\\ n_3(\xi,\eta,\sigma)&:=\frac{|\xi|\,|\sigma|}{|\xi|+|\sigma|}((|\xi|-|\eta|)(|\eta|-|\sigma|)-(\xi-\eta)\cdot(\eta-\sigma)),\end{aligned} \tag{C.19}$$

and, for $\theta\in\big[\delta^2,\frac{1}{3}\big]$ *and* $V\in\big\{D^\alpha\Omega^a:a\leqslant\frac{1}{2}N_1+20$ *and* $2a+|\alpha|\leqslant N_1+N_4-2\big\}$,

$$\|VN_4[h,\phi]\|_{Y^2_{3\theta-3\delta^2,1-3\delta^2}}\lesssim\varepsilon_1^4 2^{3\theta m-5m/2+24\delta^2 m}. \tag{C.20}$$

Proof. Recall that h is constant in y. In view of (C.12) we have, for $t\in[2^m-1,2^{m+1}]$,

$$\big\||\nabla|^{1/6}\langle\nabla\rangle^{5/6}Vh(t)\big\|_{L^\infty_y Y^2_{\theta,1}}\lesssim\varepsilon_1 2^{\theta m+6\delta^2 m},\quad \theta\in\big[0,\tfrac{1}{3}\big], \tag{C.21}$$

and

$$\big\||\nabla|^{1/6}\langle\nabla\rangle^{5/6}Vh(t)\big\|_{L^\infty_y Y^\infty_{\theta,1}}\lesssim\varepsilon_1 2^{\theta m-5m/6+6\delta^2 m},\quad \theta\in\big[0,\tfrac{1}{3}\big], \tag{C.22}$$

for $V\in\big\{D^\alpha\Omega^a:a\leqslant\frac{1}{2}N_1+20$ and $2a+|\alpha|\leqslant N_1+N_4-2\big\}$. Moreover, using also (B.22),

$$\big\||\nabla|Vu(t)\big\|_{L^2_yH^1_x}+\|(\partial_yVu)(t)\|_{L^2_yH^1_x}\lesssim\varepsilon_1 2^{6\delta^2 m} \tag{C.23}$$

for operators V as before. Therefore, using (C.16),

$$\|V[Q[u]]\|_{L^2_yY^2_{\theta-\delta^2,1-\delta^2}}\lesssim\varepsilon_1^2 2^{\theta m-5m/6+12\delta^2 m}$$

for $Q\in\{Q_a,Q_b\}$ and $\theta\in\big[\delta^2,\frac{1}{3}\big]$. Therefore,

$$\big\||\nabla|VL(u)\big\|_{L^2_yY^2_{\theta-\delta^2,1-\delta^2}}+\|\partial_yVL(u)\|_{L^2_yY^2_{\theta-\delta^2,1-\delta^2}}\lesssim\varepsilon_1^2 2^{\theta m-5m/6+12\delta^2 m}, \tag{C.24}$$

using (C.13)–(C.14). Thus, using the definition,

$$\big\||\nabla|V[u-u^{(1)}]\big\|_{L^2_yY^2_{\theta-\delta^2,1-\delta^2}}+\|\partial_yV[u-u^{(1)}]\|_{L^2_yY^2_{\theta-\delta^2,1-\delta^2}}\lesssim\varepsilon_1^2 2^{\theta m-5m/6+12\delta^2 m}. \tag{C.25}$$

Since $u-u^{(2)}=L(u-u^{(1)})$, we can repeat this argument to prove that, for $\theta\in\big[\delta^2,\frac{1}{3}\big]$ and $V\in\{D^\alpha\Omega^a:a\leqslant\frac{1}{2}N_1+20 \text{ and } 2a+|\alpha|\leqslant N_1+N_4-2\}$,

$$\big\||\nabla|V[u-u^{(2)}]\big\|_{L^2_yY^2_{2\theta-2\delta^2,1-2\delta^2}}+\|\partial_yV[u-u^{(2)}]\|_{L^2_yY^2_{2\theta-2\delta^2,1-2\delta^2}}\lesssim\varepsilon_1^3 2^{2\theta m-5m/3+18\delta^2m}. \tag{C.26}$$

To prove the decomposition (C.18), we start from the identities (B.26) and (B.12), which gives $G(h)\phi=\partial_y u-Q_a$. Letting $Q_a^{(n)}=Q_a[u^{(n)}]$ and $Q_b^{(n)}=Q_b[u^{(n)}]$, $n\in\{1,2\}$, it follows that

$$\begin{aligned} G(h)\phi&=|\nabla|\phi+\int_{-\infty}^0|\nabla|e^{-|s|\,|\nabla|}(Q_b^{(2)}(s)-Q_a^{(2)}(s))\,ds+N_{4,1},\\ N_{4,1}&:=\int_{-\infty}^0|\nabla|e^{-|s|\,|\nabla|}((Q_b-Q_b^{(2)})(s)-(Q_a-Q_a^{(2)})(s))\,ds. \end{aligned} \tag{C.27}$$

In view of (C.26), (C.22), and the algebra rule (C.16), we have

$$\|V(Q-Q^{(2)})\|_{L^2_yY^2_{3\theta-3\delta^2,1-3\delta^2}}\lesssim\varepsilon_1^4 2^{3\theta m-5m/2+24\delta^2m}$$

for $Q\in\{Q_a,Q_b\}$. Therefore, using (C.14), $|\nabla|^{-1/2}N_{4,1}$ satisfies the desired bound (C.20).

It remains to calculate the integral in the first line of (C.27). Letting $\alpha=|\nabla h|^2$, we have

$$\begin{aligned} &\mathcal{F}\{u^{(1)}\}(\xi,y)=e^{y|\xi|}\hat\phi(\xi),\\ &\mathcal{F}\{Q_a^{(1)}\}(\xi,y)=-\frac{1}{4\pi^2}\int_{\mathbb{R}^2}(\xi-\eta)\cdot\eta e^{y|\eta|}\hat h(\xi-\eta)\hat\phi(\eta)\,d\eta-\frac{1}{4\pi^2}\int_{\mathbb{R}^2}|\eta|e^{y|\eta|}\widehat\alpha(\xi-\eta)\hat\phi(\eta)\,d\eta,\\ &\mathcal{F}\{Q_b^{(1)}\}(\xi,y)=-\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\frac{(\xi-\eta)\cdot\xi}{|\xi|}|\eta|e^{y|\eta|}\hat h(\xi-\eta)\hat\phi(\eta)\,d\eta. \end{aligned} \tag{C.28}$$

Therefore,

$$\begin{aligned} &\mathcal{F}\{L(u^{(1)})\}(\xi,y)\\ &\quad=\frac{1}{8\pi^2}\int_{\mathbb{R}^2}(e^{y|\xi|}-e^{y|\eta|})\bigg(\frac{(\xi-\eta)\cdot\eta}{|\xi|+|\eta|}-\frac{|\eta|(\xi-\eta)\cdot\xi}{|\xi|(|\xi|+|\eta|)}\bigg)\hat h(\xi-\eta)\hat\phi(\eta)\,d\eta\\ &\qquad+\frac{1}{8\pi^2}\int_{\mathbb{R}^2}(e^{y|\xi|}-e^{y|\eta|})\bigg(\frac{(\xi-\eta)\cdot\eta}{-|\xi|+|\eta|}+\frac{|\eta|(\xi-\eta)\cdot\xi}{|\xi|(-|\xi|+|\eta|)}\bigg)\hat h(\xi-\eta)\hat\phi(\eta)\,d\eta+\widehat{E}_1(\xi,y), \end{aligned}$$

where

$$\big\||\nabla|VE_1\big\|_{L^2_yY^2_{2\theta-2\delta^2,1-2\delta^2}}+\|\partial_yVE_1\|_{L^2_yY^2_{2\theta-2\delta^2,1-2\delta^2}}\lesssim\varepsilon_1^3 2^{2\theta m-5m/3+18\delta^2m}.$$

After algebraic simplifications, this gives

$$\mathcal{F}\{L(u^{(1)})\}(\xi,y)=-\frac{1}{4\pi^2}\int_{\mathbb{R}^2}(e^{y|\xi|}-e^{y|\eta|})|\eta|\hat{h}(\xi-\eta)\hat{\phi}(\eta)\,d\eta+\widehat{E}_1(\xi,y).$$

Since $u^{(2)}-u^{(1)}=L(u^{(1)})$, we calculate

$$\begin{aligned}&\mathcal{F}\{Q_a^{(2)}-Q_a^{(1)}\}(\xi,y)\\&\quad=\frac{1}{16\pi^4}\int_{(\mathbb{R}^2)^2}|\sigma|(\xi-\eta)\cdot\eta(e^{y|\eta|}-e^{y|\sigma|})\hat{h}(\xi-\eta)\hat{h}(\eta-\sigma)\hat{\phi}(\sigma)\,d\eta\,d\sigma+\widehat{E}_2(\xi,y)\end{aligned}\tag{C.29}$$

and

$$\begin{aligned}&\mathcal{F}\{Q_b^{(2)}-Q_b^{(1)}\}(\xi,y)\\&\quad=\frac{1}{16\pi^4}\int_{(\mathbb{R}^2)^2}|\sigma|\frac{(\xi-\eta)\cdot\xi}{|\xi|}(|\eta|e^{y|\eta|}-|\sigma|e^{y|\sigma|})\hat{h}(\xi-\eta)\hat{h}(\eta-\sigma)\hat{\phi}(\sigma)\,d\eta\,d\sigma+\widehat{E}_3(\xi,y),\end{aligned}\tag{C.30}$$

where

$$\|VE_2\|_{L^2_yY^2_{3\theta-3\delta^2,1-3\delta^2}}+\|VE_3\|_{L^2_yY^2_{3\theta-3\delta^2,1-3\delta^2}}\lesssim\varepsilon_1^4 2^{3\theta m-5m/2+24\delta^2 m}.$$

We now examine the formula in the first line of (C.27). The contributions of E_2 and E_3 can be estimated as part of the quartic error term, using also (C.14). The main contributions can be divided into quadratic terms (coming from $Q_a^{(1)}$ and $Q_b^{(1)}$ in (C.28)), and cubic terms coming from (C.29)–(C.30) and the cubic term in $Q_a^{(1)}$. The conclusion of the lemma follows. □

Step 4. Finally, we can prove the desired expansion of the water-wave system.

LEMMA C.4. *Assume that (h,ϕ) satisfy (2.1) and (2.6). Then,*

$$(\partial_t+i\Lambda)\mathcal{U}=\mathcal{N}_2+\mathcal{N}_3+\mathcal{N}_{\geqslant 4},\tag{C.31}$$

where $\mathcal{U}=\langle\nabla\rangle h+i|\nabla|^{1/2}\phi$ and $\mathcal{N}_2$, $\mathcal{N}_3$, $\mathcal{N}_{\geqslant 4}$ are as in §7.1.

Proof. We rewrite (2.1) in the form

$$\partial_t\mathcal{U}=\langle\nabla\rangle G(h)\phi+i|\nabla|^{1/2}\Big(-h+\operatorname{div}\Big(\frac{\nabla h}{(1+|\nabla h|^2)^{1/2}}\Big)-\frac{1}{2}|\nabla\phi|^2+\frac{(G(h)\phi+\nabla h\cdot\nabla\phi)^2}{2(1+|\nabla h|^2)}\Big).\tag{C.32}$$

We now use formula (C.18) to extract the linear, the quadratic, and the cubic terms in the right-hand side of this formula. More precisely, we set

$$\begin{aligned}\mathcal{N}_1&:=\langle\nabla\rangle|\nabla|\phi+i|\nabla|^{1/2}(-h+\Delta h)=-i\Lambda\mathcal{U},\\\mathcal{N}_2&:=\langle\nabla\rangle N_2[h,\phi]+i|\nabla|^{1/2}\big(-\tfrac{1}{2}|\nabla\phi|^2+\tfrac{1}{2}(|\nabla|\phi)^2\big),\\\mathcal{N}_3&:=\langle\nabla\rangle N_3[h,h,\phi]+i|\nabla|^{1/2}\big(-\tfrac{1}{2}\operatorname{div}(\nabla h|\nabla h|^2)+|\nabla|\phi\cdot(N_2[h,\phi]+\nabla h\cdot\nabla\phi)\big).\end{aligned}\tag{C.33}$$

Then, we substitute

$$h=\langle\nabla\rangle^{-1}\frac{\mathcal{U}+\bar{\mathcal{U}}}{2}\quad\text{and}\quad|\nabla|^{1/2}\phi=\frac{\mathcal{U}-\bar{\mathcal{U}}}{2i}.$$

The symbols that define the quadratic component $\mathcal{N}_2$ are linear combinations of the symbols

$$n_{2,1}(\xi,\eta)=\sqrt{1+|\xi|^2}\,\frac{\xi\cdot\eta-|\xi|\,|\eta|}{|\eta|^{1/2}\sqrt{1+|\xi-\eta|^2}}\quad\text{and}\quad n_{2,2}(\xi,\eta)=|\xi|^{1/2}\frac{(\xi-\eta)\cdot\eta+|\xi-\eta|\,|\eta|}{|\xi-\eta|^{1/2}|\eta|^{1/2}}.$$

It is easy to see that these symbols verify the properties (7.11). A slightly non-trivial argument is needed for $n_{2,1}$ in the case $k_1=\min(k,k_1,k_2)\ll k$.

The cubic terms in $\mathcal{N}_3$ in (C.33) are defined by finite linear combinations of the symbols

$$\begin{aligned}
n_{3,1}(\xi,\eta,\sigma)&=\sqrt{\frac{1+|\xi|^2}{(1+|\xi-\eta|^2)(1+|\eta-\sigma|^2)}}\,\frac{|\xi|\,|\sigma|^{1/2}}{|\xi|+|\sigma|}((|\xi|-|\eta|)(|\eta|-|\sigma|)-(\xi-\eta)\cdot(\eta-\sigma)),\\
n_{3,2}(\xi,\eta,\sigma)&=|\xi|^{1/2}\frac{(\xi\cdot(\xi-\eta))((\eta-\sigma)\cdot\sigma)}{\sqrt{(1+|\xi-\eta|^2)(1+|\eta-\sigma|^2)(1+|\sigma|^2)}},\\
n_{3,3}(\xi,\eta,\sigma)&=|\xi|^{1/2}|\xi-\eta|^{1/2}|\sigma|^{1/2}\frac{|\sigma-\eta|}{\sqrt{1+|\eta-\sigma|^2}}.
\end{aligned}$$

It is easy to verify the properties (7.12) for these explicit symbols.

The higher-order remainder in the right-hand side of (C.32) can be written in the form

$$\mathcal{N}_{\geqslant 4}=|\nabla|^{1/2}N_4',\qquad \sup_{\substack{a\leqslant N_1/2+20\\ 2a+|\alpha|\leqslant N_1+N_4-4}}\|D^\alpha\Omega^a N_4'\|_{Y^2_{1-\delta,1-\delta}}\lesssim\varepsilon_1^4 2^{-3m/2+\delta m},\tag{C.34}$$

using (C.20), (C.12), and the algebra property (C.15). Moreover, using only the $\mathcal{O}$ hierarchy as in the proof of Corollary C.1, we have $\|\mathcal{N}_{\geqslant 4}\|_{\mathcal{O}_{4,-4}}\lesssim\varepsilon_1^4$, i.e.

$$\|\mathcal{N}_{\geqslant 4}\|_{H^{N_0-4}}+\|\mathcal{N}_{\geqslant 4}\|_{H^{N_1,N_3-4}}\lesssim\varepsilon_1^4 2^{-5m/2+\delta m}.\tag{C.35}$$

These two bounds suffice to prove the desired claims on $\mathcal{N}_{\geqslant 4}$ in (7.15). Indeed, the L^2 bound follows directly from (C.35). For the Z norm bound, it suffices to prove that, for any $(k,j)\in\mathcal{J}$,

$$\sup_{\substack{a\leqslant N_1/2+20\\ 2a+|\alpha|\leqslant N_1+N_4}}2^{j(1-50\delta)}\|Q_{jk}e^{it\Lambda}D^\alpha\Omega^a\mathcal{N}_{\geqslant 4}\|_{L^2}\lesssim\varepsilon_1^4 2^{-m-\delta m}.\tag{C.36}$$

This follows easily from (C.35) and (C.34), unless

$$j \geqslant \tfrac{3}{2}m + \tfrac{1}{4}N_0 k^+ + \mathcal{D} \quad \text{and} \quad j \geqslant \tfrac{3}{2}m - \tfrac{1}{2}k + \mathcal{D}.$$

On the other hand, if these inequalities hold, then let

$$f = D^\alpha \Omega^a \mathcal{N}_{\geqslant 4}, \quad a \leqslant \tfrac{1}{2}N_1 + 20, \quad 2a + |\alpha| \leqslant N_1 + N_4,$$

and decompose

$$f = \sum_{(k',j') \in \mathcal{J}} f_{j',k'}$$

as in (7.33). The bound (C.34) shows that

$$\sum_{(k',j') \in \mathcal{J}} 2^{-4\max(k',0)} 2^{j'(1-\delta)} \|f_{j',k'}\|_{L^2} \lesssim \varepsilon_1^4 2^{-3m/2+\delta m}. \tag{C.37}$$

The desired bound (C.35) follows by the usual approximate-finite-speed-of-propagation argument: we may assume that $|k'-k| \leqslant 4$, and consider the cases $j' \leqslant j - \delta j$ (which gives negligible contributions) and $j' \geqslant j - \delta j$ (in which case (C.37) suffices). This completes the proof. □

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