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Global bifurcation of steady gravity water waves with critical layers

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1. Introduction

We construct large families of 2-dimensional periodic steady water waves propagating under the influence of gravity at the surface of a layer of an incompressible, inviscid and homogeneous fluid with a flat bed. In contrast to most previous mathematical studies, our waves can have internal stagnation points and critical layers. They can also have overturning profiles, that is, profiles that are not graphs. Such phenomena cannot occur for the much-studied irrotational flows. For an irrotational steady flow the wave profile is necessarily the graph of a single-valued function [32], [38], [43], [44], and there are no interior stagnation points or critical layers [1], [42].

In this paper we construct families of water waves on flows with non-vanishing constant vorticity. Vorticity is the hallmark of underlying non-uniform currents. Flows with constant vorticity are of interest because of their analytic tractability, being representative of a wide range of physical scenarios. For example, if the waves are long compared with the average water depth, then the existence of a non-zero mean vorticity is more important than its specific distribution [36]. In particular, constant vorticity gives a good description of tidal currents; that is, the alternating, horizontal movement of water associated with the rise and fall of the tide, with positive/negative vorticity being appropriate for the ebb and flood, respectively [10]. On areas of the continental shelf and in many coastal inlets these are the most significant currents; the fact that they are the most regular and predictable currents adds to their appeal. Note that tides refer to the vertical motion of water caused by the gravitational forces due to the relative motions of the Moon, the Sun and the Earth, whereas the tidal current flood/ebb is the horizontal movement of water (mostly 1-directional) associated with the rise and fall of the tide, respectively. A spectacular example of the effect of tidal currents on sea waves is encountered at the Columbia River entrance, where appreciable tidal currents make it one of the most hazardous navigational regions in the world because the wave height can easily double in just a few hours; cf. [25]. The presence of an underlying non-uniform current gives rise to new dynamically rich flow phenomena. In particular, critical layers have been observed in many numerical studies of water flows with an underlying current of constant vorticity, a typical feature of the flow dynamics being the "cat's eye" flow pattern of Thomson (Lord Kelvin) [37]. The analysis pursued for instance in [9] and [16] shows that no such patterns can occur in the flow beneath an irrotational periodic steady water wave. General overviews of the wave-current interaction theory can be found in [10], [13], [34] and [45], and of the numerics in [26], [27], [29], [36] and [41]. In the present notation, positive vorticity $\Upsilon>0$ is typical of pure ebb currents $(\Upsilon Y+U_0,0)$ whose surface horizontal velocity exceeds the horizontal current velocity U_0 on the bed $Y=0$. This is appropriate for the ebb current, while $\Upsilon<0$ captures flood currents [10]. For wave-current interactions in a flow with velocity $(\psi_Y,-\psi_X)$, where ψ is the stream function and X denotes the direction of wave propagation, positive vorticity $\Upsilon=\Delta\psi>0$ refers to a favorable underlying current, while negative vorticity $\Upsilon<0$ refers to an adverse underlying current.

The waves investigated in the present paper are constructed by means of bifurcation from laminar (flat) flows. Construction of small-amplitude waves via local bifurcation has already been accomplished in [49] and [18]. What is novel here is the construction of waves of *large amplitude* via global bifurcation. Now, global bifurcation has also been carried out for general vorticity under the limitation that there are no stagnation points or critical layers [15], that is, under the condition that $\psi_Y<0$, where the vector $(\psi_Y, -\psi_X)$ is the fluid velocity in the frame moving with the speed of the traveling wave. The reason for this limitation in past studies is because they rely upon the semi-Lagrangian transformation of Dubreil–Jacotin [21], which requires $\psi_Y<0$. The critical layers are comprised of the points where $\psi_Y=0$, while the stagnation points are those at which both $\psi_Y=0$ and $\psi_X=0$. On the other hand, a vertical scaling transformation was used in [49] to construct waves of small amplitude with critical layers. However, the intricate nature of the reformulated free-boundary problem precludes an attempt to develop an effective global bifurcation approach from that perspective. While for waves of small amplitude, the vertical scaling transformation induces tractable structural changes that can be captured by linear theory, as soon as the waves cease to represent small perturbations of a flat free surface, the inherent loss of structure cannot accommodate the pursuit of a physically tangible analysis. Two of the present authors introduced in [18] a new approach, based on a conformal transformation, for a different construction of waves with critical layers. This approach is further developed in the present paper to construct waves of *large amplitude*. We also prove that the constructed waves are critical points of a natural functional derived from the energy.

We begin §2 by introducing the governing equations of water waves. Then, as presented in [18], we use a conformal mapping from a fixed infinite strip in the plane onto the fluid domain with its free boundary (see Figure 1), to reformulate the problem as a non-linear pseudodifferential equation (2.20) on a horizontal line, where the unknown $x\mapsto v(x)$ is a real-valued function v of one real variable, expressing the elevation of the wave profile in a suitable parametrization. This equation involves the Hilbert transform for a strip as well as several parameters. For the sake of completeness, some of the technical background for the Hilbert transform on a strip is provided in Appendix A, including a representation formula as a singular integral that appears to be new. While a construction of waves of small amplitude by means of local bifurcation theory applied to (2.20) was carried out in [18], the applicability of *global bifurcation theory* requires certain compactness properties of the non-linear operators in question, which do not seem to be available for (2.20). Our first main result is to uncover a new reformulation of the problem, the quasilinear pseudodifferential system (2.22) for the same function v and the same parameters as before, involving the periodic Hilbert transform for a strip. The equivalence between (2.20) and (2.22) is proven by means of Riemann–Hilbert theory on the strip. It is somewhat surprising that the new formulation involves not only the functional equation (2.22a), but also the scalar constraint (2.22b). The system (2.22) is novel, even when restricted to irrotational flows (without vorticity) of finite depth, even though in that case it is an analogue of Babenko's formulation [2] for the irrotational water-wave problem of infinite depth. While Babenko's formulation has turned out to be instrumental in the recent theory of global and subharmonic bifurcation in the irrotational infinite-depth case [3], [4], [5], [31], [39], the finite depth and the vorticity make both the Riemann–Hilbert analysis and the subsequent global bifurcation theory for (2.22) substantially more involved.

Furthermore, we show in §2 that the equations in the new formulation are identical to the Euler–Lagrange equations of a certain functional Λ. As explained in Appendix B, the functional Λ comes naturally from the physical energy

$$E = \iint \left(\frac{1}{2}|\nabla\psi|^2 + \Upsilon\psi + \frac{Q}{2} - gY \right) dX\, dY,$$

where g is gravity, Υ is the vorticity ($\Upsilon = -\gamma$ in some references), and $Q/2g$ is the total head, namely, the greatest possible height of the wave, as discussed further at the beginning of §2. The variational structure of Babenko's equation is a key ingredient in the subharmonic bifurcation analysis in [3] and [4], and thus our formulation opens up the possibility of similar investigations for surface waves in water flows of finite depth and with constant non-zero vorticity. However, since in this case numerical studies [36], [41] indicate a much richer dynamics, in the present paper we content ourselves with establishing a global bifurcation theory, leaving the question of secondary and subharmonic bifurcation to remain the subject of subsequent investigations.

In §3 we construct a continuous solution curve $\mathcal{K}_{n,\pm}$, representing waves with spatial period $2\pi/n$, for every integer $n \geqslant 1$ and both choices of sign $\pm$. The solution curve $\mathcal{K}_{n,\pm}$ consists of triples (Q, m, v) belonging to a function space $\mathbb{R} \times \mathbb{R} \times C^{2,\alpha}_{2\pi}(\mathbb{R})$, where $0 < \alpha < 1$ is the Hölder-space index and 2π indicates the period. Each point on the solution curve $\mathcal{K}_{n,\pm}$ corresponds to a water wave, provided an associated curve in the plane is injective and contained in the upper half-plane. This property necessarily holds for solutions that are sufficiently close to the trivial one. The index $n=1$ stands for the bifurcation from the lowest eigenvalue and $n \geqslant 2$ for the higher modes; given $n \geqslant 1$, the dispersion relation has two roots which are distinguished by the notation $\pm$. While local bifurcation from a simple eigenvalue is a well-known technique, most often referred to by the names Crandall–Rabinowitz or Liapunov–Schmidt, the real-analytic structure of the operator in (2.22), together with suitable compactness properties that we establish for the solution set, enable us to go beyond local bifurcation and prove the existence of a global solution curve, by employing the “analytic theory of bifurcation” due to Dancer, Buffoni and Toland [6]. However, compared to the analysis in [6], the presence of vorticity and the finiteness of the depth introduce major additional difficulties. In particular, the proof of the required compactness property of the solution set is rather subtle and depends on commutator properties of the periodic Hilbert transform. We thus obtain the existence of solutions that are not merely small perturbations of a laminar flow (with a flat free surface).

The basic idea of continuation of the local bifurcation curve for abstract operator equations originates with the work of Rabinowitz [30], who used Leray–Schauder degree to extend the local bifurcation curve. Shortly thereafter Dancer [20] showed that, for equations with a real-analytic structure, analytic continuation can provide a similar construction. The two approaches provide somewhat different sets of solutions. The main advantage of the analytic theory is that it provides a continuous curve of solutions, whereas the degree-theoretic approach only ensures the existence of a global connected set of solutions that might lose its character as a curve beyond a small neighbourhood of the local bifurcation point. Our constructed solution curve has an analytic reparametrization locally around each point, even as it passes through its branch points. Theorem 5 provides three alternatives for how the curve “ends” in either direction. One is that the curve may be unbounded in $\mathbb{R}\times\mathbb{R}\times C^{2,\alpha}_{2\pi}(\mathbb{R})$. The second is that the solution curve $\mathcal{K}_{n,\pm}$ may approach a wave of greatest height $Q/2g$. The third alternative is that the curve is a closed loop that returns to the original laminar flow.

§4 is a key part of the analysis. We show that the third alternative cannot occur unless the curve of solutions contains a wave for which the free surface $\mathcal{S}$ intersects itself at a point directly above the trough. This intriguing possibility is supported by numerical computations [36], [41]. Moreover, we prove that the constructed waves have the property that, although their free surface is allowed to overturn (that is, Y could be a multivalued function of X), Y necessarily decreases from the crest to the trough. More precisely, this property holds either all along the global solution curve, or for all the solutions situated along the curve between the trivial one and the first limiting wave with a self-intersection on the vertical line above the trough, if such a solution ever occurs on the curve. These statements are proven by an intricate series of arguments based on repeated use of the strong maximum principle together with the Hopf and Serrin inequalities at the boundary. We conclude this section by mentioning some further results that will appear in a forthcoming paper (announced in [35]), as well as some conjectures that have gained some reasonable degree of credence due to a combination of analytical results and numerical simulations.

§5 highlights some detailed features of the small-amplitude waves that illustrate the profound effect of vorticity. In particular, in contrast to irrotational flows, there exist flows for which the streamlines are in the shape of “cat’s eyes”.

2. The free-boundary problem

The problem of periodic travelling gravity water waves in a flow of constant vorticity Υ over a flat bed can be formulated as the free-boundary problem of finding the following domain and function:

- a laterally unbounded domain Ω in the (X,Y)-plane, whose boundary consists of the real axis

$$\mathcal{B}=\{(X,0):X\in\mathbb{R}\} \tag{2.1a}$$

representing the flat impermeable water bed, and an a-priori unknown curve with a parametric representation

$$\mathcal{S}=\{(u(s),v(s)):s\in\mathbb{R}\}, \tag{2.1b}$$

where

$$\text{the mapping } s\mapsto(u(s)-s,v(s)) \text{ is periodic of period } L, \tag{2.1c}$$

representing the water’s free surface;

- a function $\psi(X,Y)$ that is L-periodic in X throughout Ω, called the *stream function*, providing the velocity field $(\psi_Y,-\psi_X)$ in a frame moving at the constant wave speed, which satisfies the following equations and boundary conditions:

$$\Delta\psi=\Upsilon \qquad \text{in } \Omega, \tag{2.2a}$$

$$\psi=0 \qquad \text{on } \mathcal{S}, \tag{2.2b}$$

$$\psi=-m \quad \text{on } \mathcal{B}, \tag{2.2c}$$

$$|\nabla\psi|^2+2gY=Q \qquad \text{on } \mathcal{S}. \tag{2.2d}$$

Here g is the gravitational constant of acceleration and the constant m is the relative mass flux. The constant $Q/2g$ is the total head, which is the greatest possible height of $\mathcal{S}$. The notion of relative mass flux captures the fact that in the moving frame the amount of water passing any vertical line is constant throughout the fluid domain, as

$$\int_0^{v(s)}\psi_Y(X,Y)\,dY=m, \quad s\in\mathbb{R}.$$

Neglecting friction and surface tension effects, the Bernoulli equation (2.2d) is a form of conservation of energy; cf. the discussion in [10]. Each term in the relation

$$\frac{|\nabla\psi|^2}{2g}+Y=\frac{Q}{2g} \quad \text{on } \mathcal{S},$$

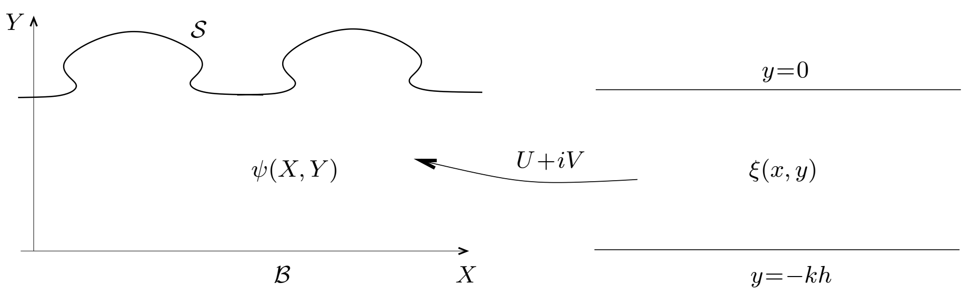

Figure 1. The conformal parametrization of the fluid domain.

obtained from (2.2d), has the dimension of length, the first being called the velocity head and representing the elevation needed for the fluid to reach the velocity $|\nabla\psi|$ during frictionless free fall, and the second term being the elevation head. The level sets of ψ are the streamlines. A point where the gradient of ψ vanishes is called a *stagnation point*. A *critical layer* is a curve along which $\psi_Y=0$. Such curves arise in the context of flow-reversal, when a fluid region where the flow is oriented towards the propagation direction of the wave is adjacent to a fluid region in which the flow is adverse to it.

For any integer $p\geqslant 0$ and $\alpha\in(0,1)$, we denote by $C^{p,\alpha}$ the space of functions whose partial derivatives up to order p are Hölder continuous with exponent α over their domain of definition. By $C^{p,\alpha}_{\mathrm{loc}}$ we denote the set of functions of class $C^{p,\alpha}$ over any compact subset of their domain of definition. Let $C^{p,\alpha}_L(\mathbb{R})$ be the space of functions of one real variable which are L-periodic and of Hölder class $C^{p,\alpha}$. Throughout this paper we are interested in solutions (Ω,ψ) of the water-wave problem (2.1)–(2.2) of class $C^{1,\alpha}$ for some $\alpha\in(0,1)$, in the sense that $\mathcal{S}$ has a parametrization (2.1b) with u,v functions of class $C^{1,\alpha}$, such that (2.1c) holds and

$$u'(s)^2+v'(s)^2\neq 0 \quad \text{for all } s\in\mathbb{R}, \tag{2.3}$$

while $\psi\in C^\infty(\Omega)\cap C^{1,\alpha}(\bar{\Omega})$.

2.1. The first reformulation via conformal mapping

In this subsection we present, following [18], the reformulation of the free-boundary problem problem (2.1)–(2.2) as the quasilinear pseudodifferential equation (2.18) for a periodic function of a single variable. This involves the periodic Dirichlet–Neumann operator $\mathcal{G}_{kh}$ and the periodic Hilbert transform $\mathcal{C}_{kh}$ for a strip, for whose definition and detailed properties we refer to Appendix A.

A domain Ω contained in the upper half of the (X,Y)-plane is called an *L-periodic strip-like domain* if its boundary consists of the real axis $\mathcal{B}$ and a curve $\mathcal{S}$ described in parametric form by (2.1b) such that (2.1c) holds. Given $L>0$, let $k>0$ be such that

$L=2\pi/k$ (so that, for a steady wave of spatial period L, k is the wave number). Also, for any $d>0$, let us denote by $\mathcal{R}_d$ the horizontal strip

$$\{(x,y)\in\mathbb{R}^2:\ -d<y<0\}.$$

Given any L-periodic strip-like domain Ω of class $C^{1,\alpha}$ for some $\alpha\in(0,1)$, the considerations in [18] show that there exists a unique constant $h>0$, called the *conformal mean depth* of Ω, and a conformal map $U+iV$ from $\mathcal{R}_{kh}$ onto Ω that admits an extension as a homeomorphism of class $C^{1,\alpha}$ between the closures of these domains (see Figure 1), such that

$$\begin{cases} U(x+2\pi,y)=U(x,y)+2\pi/k & \text{for } (x,y)\in\mathcal{R}_{kh}, \\ V(x+2\pi,y)=V(x,y) & \text{for } (x,y)\in\mathcal{R}_{kh}, \\ V(x,-kh)=0 & \text{for all } x\in\mathbb{R}, \end{cases} \tag{2.4}$$

and, moreover,

$$U_x^2(x,0)+V_x^2(x,0)\neq 0 \quad \text{for all } x\in\mathbb{R}, \tag{2.5}$$

$$\text{the mapping } x\mapsto (U(x,0),V(x,0)) \text{ is injective on } \mathbb{R}, \tag{2.6}$$

$$\mathcal{S}=\{(U(x,0),V(x,0)):x\in\mathbb{R}\} \text{ is the upper boundary of } \Omega. \tag{2.7}$$

We define

$$v(x)=V(x,0) \quad \text{for all } x\in\mathbb{R}. \tag{2.8}$$

It then follows that necessarily

$$[v]=h,$$

where, throughout this paper, $[f]$ denotes the mean over one period of a 2π-periodic function f. Note that V may be recovered uniquely from v as the solution of

$$\Delta V=0, \qquad \text{in } \mathcal{R}_{kh}, \tag{2.9a}$$

$$V(x,0)=v(x), \quad x\in\mathbb{R}, \tag{2.9b}$$

$$V(x,-kh)=0, \qquad x\in\mathbb{R}. \tag{2.9c}$$

Of course, U is a harmonic conjugate of $-V$ on $\mathcal{R}_{kh}$ and, moreover, by (A.5) with $d=kh$, we have that, up to an additive constant that can be neglected,

$$U(x,0)=\frac{x}{k}+(\mathcal{C}_{kh}(v-h))(x), \quad x\in\mathbb{R}. \tag{2.10}$$

Thus

$$U_x(x,0)=V_y(x,0)=\frac{1}{k}+(\mathcal{C}_{kh}(v'))(x), \quad x\in\mathbb{R}.$$

Let Ω be any L-periodic strip-like domain of class $C^{1,\alpha}$. Classical elliptic theory ensures that (2.2a)–(2.2c) has a unique solution. Thus, one may regard (2.1)–(2.2) as the problem of finding a domain Ω for which the corresponding solution of (2.2a)–(2.2c) satisfies also (2.2d). In what follows, we aim to recast this problem in terms of the function v of one variable defined above. If ψ is the unique solution of (2.2a)–(2.2c), we define $\xi\colon \mathcal{R}_{kh}\to\mathbb{R}$ by

$$\xi(x,y)=\psi(U(x,y),V(x,y)),\quad (x,y)\in\mathcal{R}_{kh}, \tag{2.11}$$

and $\zeta\colon \mathcal{R}_{kh}\to\mathbb{R}$ by

$$\zeta=\xi+m-\tfrac{1}{2}\Upsilon V^2. \tag{2.12}$$

Then we may recast (2.2a)–(2.2c) as

$$\Delta\zeta=0 \qquad \text{in } \mathcal{R}_{kh}, \tag{2.13a}$$

$$\zeta(x,0)=m-\tfrac{1}{2}\Upsilon v^2(x) \qquad \text{for all } x\in\mathbb{R}, \tag{2.13b}$$

$$\zeta(x,-kh)=0 \qquad \text{for all } x\in\mathbb{R}. \tag{2.13c}$$

Moreover, by calculating

$$|\nabla\xi|^2=(Q-2gY)|\nabla V|^2,$$

we see that (2.2d) is satisfied if and only if

$$(\zeta_y+\Upsilon VV_y)^2=(Q-2gV)(V_x^2+V_y^2) \text{ at } (x,0) \text{ for all } x\in\mathbb{R}. \tag{2.14}$$

From (2.13) and the definition (A.1) of the Dirichlet–Neumann operator $\mathcal{G}_{kh}$ we infer that

$$\zeta_y=\frac{m}{kh}-\frac{\Upsilon}{2}\mathcal{G}_{kh}(v^2) \quad \text{on} \quad y=0, \tag{2.15}$$

while clearly

$$V_y=\mathcal{G}_{kh}(v) \quad \text{on} \quad y=0. \tag{2.16}$$

Therefore (2.14) may now be rewritten as

$$\left(\frac{m}{kh}-\Upsilon\left(\mathcal{G}_{kh}\left(\frac{v^2}{2}\right)-v\mathcal{G}_{kh}(v)\right)\right)^2=(Q-2gv)((v')^2+(\mathcal{G}_{kh}(v))^2), \tag{2.17}$$

where $[v]=h$. Equation (2.17) was first derived in [18], with $\Upsilon=-\gamma$ (due to a change of sign of the 2-dimensional vorticity).

The above considerations show that the free-boundary problem (2.1)–(2.2) leads to the problem of finding a positive number h and a function $v\in C^{1,\alpha}_{2\pi}(\mathbb{R})$ which satisfy

$$\left(\frac{m}{kh}-\Upsilon\left(\mathcal{G}_{kh}\left(\frac{v^2}{2}\right)-v\mathcal{G}_{kh}(v)\right)\right)^2=(Q-2gv)((v')^2+(\mathcal{G}_{kh}(v))^2), \tag{2.18a}$$

$$[v] = h, \tag{2.18b}$$

$$v(x) > 0 \quad \text{for all } x \in \mathbb{R}, \tag{2.18c}$$

$$\text{the mapping } x \mapsto (x/k + \mathcal{C}_{kh}(v-h)(x), v(x)) \text{ is injective on } \mathbb{R}, \tag{2.18d}$$

$$(v'(x))^2 + (\mathcal{G}_{kh}(v)(x))^2 \neq 0 \quad \text{for all } x \in \mathbb{R}. \tag{2.18e}$$

Indeed, (2.18a) is precisely (2.17), condition (2.18b) comes from the definition of h as the conformal mean depth of Ω, and (2.18c) is ensured by the assumption that $\mathcal{S}$ lies in the upper half-plane. As for (2.18d) and (2.18e), they are obtained from (2.6) and (2.5), respectively, in view of (2.16), (2.8), (2.10), and the Cauchy–Riemann equations for the analytic function $U+iV$. Note also that the relations (2.15)-(2.16), in combination with (A.4), may be written as

$$\begin{cases} V_y = \dfrac{1}{k} + \mathcal{C}_{kh}(v'), \\ \zeta_y = \dfrac{m}{kh} - \dfrac{\Upsilon}{2kh}\,[v^2] - \Upsilon\,\mathcal{C}_{kh}(vv'), \end{cases} \qquad \text{on } y=0, \tag{2.19}$$

since $[v]=h$. Thus (2.18a) may be equivalently rewritten as

$$\begin{aligned} \left(\frac{m}{kh} - \frac{\Upsilon}{2kh}[v^2] - \Upsilon\mathcal{C}_{kh}(vv') + \Upsilon v\left(\frac{1}{k} + \mathcal{C}_{kh}(v')\right)\right)^2 & \\ -(Q-2gv)\left((v')^2 + \left(\frac{1}{k} + \mathcal{C}_{kh}(v')\right)^2\right) &= 0. \end{aligned} \tag{2.20}$$

Conversely, given a positive number h and a function $v \in C^{1,\alpha}_{2\pi}$ satisfying (2.18) with $k=2\pi/L$, one can construct a solution of (2.1)–(2.2) by reversing the process that has just been presented. Indeed, let V be the unique solution of (2.9). If $U: \mathcal{R}_{kh} \to \mathbb{R}$ is a harmonic function such that $U+iV$ is holomorphic, then $U, V \in C^{1,\alpha}(\overline{\mathcal{R}}_{kh})$; cf. [18]. Condition (2.18b) ensures that the first two relations in (2.4) hold. Using (2.18d) and (2.18c), we infer that the curve $\mathcal{S}$ defined by (2.7) is non-self-intersecting and contained in the upper half-plane. Moreover, cf. [18], the map $U+iV$ is a conformal mapping from $\mathcal{R}_{kh}$ onto an L-periodic strip-like domain Ω of conformal depth h whose upper boundary is $\mathcal{S}$, and admits an extension as a homeomorphism between the closures of these domains, with $\{(x,0): x \in \mathbb{R}\}$ being mapped onto $\mathcal{S}$ and $\{(x,-kh): x \in \mathbb{R}\}$ onto $\mathcal{B}$. Due to (2.18e), $\mathcal{S}$ is a $C^{1,\alpha}$ curve. If ζ is the unique solution of (2.13), then $\zeta \in C^{1,\alpha}(\overline{\mathcal{R}}_{kh}) \cap C^\infty(\mathcal{R}_{kh})$. Defining ξ by (2.12), and then ψ by (2.11), we see that $\psi \in C^{1,\alpha}(\overline{\Omega}) \cap C^\infty(\Omega)$ satisfies (2.2a)-(2.2c). Finally, since (2.18a) holds, we obtain that ψ satisfies (2.2d).

Note that, at the location $(X,Y) = (U(x,y), V(x,y)) \in \Omega$, where $(x,y) \in \mathcal{R}_{kh}$, the fluid velocity is given by

$$(\psi_Y, -\psi_X) = \left(\frac{V_x\zeta_x + V_y\zeta_y}{V_x^2 + V_y^2} + \Upsilon V, \frac{V_x\zeta_y - V_y\zeta_x}{V_x^2 + V_y^2}\right) \tag{2.21}$$

in terms of $\zeta(x,y)$ and of the conformal map $U+iV$ from $\mathcal{R}_{kh}$ to Ω. The formula (2.21) is obtained by differentiating (2.11), solving the resulting linear system for ψ_X and ψ_Y and substituting (2.12).

2.2. The new reformulation of Riemann–Hilbert type

In the theory of irrotational waves of infinite depth, it is known [31] that the equation corresponding to (2.20) admits an equivalent reformulation via Riemann–Hilbert theory. We now derive an analogous reformulation in the present setting as well. The argument is, however, substantially more intricate, and the new equation is, somewhat surprisingly, coupled to a scalar constraint. The new formulation is the system

$$\begin{aligned}\mathcal{C}_{kh}((Q-2gv-\Upsilon^2v^2)v')+(Q-2gv-\Upsilon^2v^2)\left(\frac{1}{k}+\mathcal{C}_{kh}(v')\right)\\ -2\Upsilon v\left(\frac{m}{kh}-\frac{\Upsilon}{2kh}[v^2]-\Upsilon\mathcal{C}_{kh}(vv')\right)-\frac{Q-2\Upsilon m-2gh}{k}+2g[v\mathcal{C}_{kh}(v')]=0,\end{aligned} \tag{2.22a}$$

$$\begin{aligned}\left[\left(\frac{m}{kh}-\frac{\Upsilon}{2kh}[v^2]-\Upsilon\mathcal{C}_{kh}(vv')+\Upsilon v\left(\frac{1}{k}+\mathcal{C}_{kh}(v')\right)\right)^2\right]\\ -\left[(Q-2gv)\left((v')^2+\left(\frac{1}{k}+\mathcal{C}_{kh}(v')\right)^2\right)\right]=0,\end{aligned} \tag{2.22b}$$

for unknowns $(m,Q,v)\in\mathbb{R}\times\mathbb{R}\times C^{1,\alpha}_{2\pi}(\mathbb{R})$. Recall that $m\in\mathbb{R}$ is the relative mass flux, $h=[v]>0$ is the conformal mean depth of the fluid domain Ω, $k>0$ is the wave number corresponding to the wave period $L=2\pi/k$, and Q is the total head. The square bracket $[\cdot]$ means the average over a period, so (2.22b) is a scalar equation.

It is convenient to introduce the following class of functions.

Definition. A 2π-periodic function $z+iw:\mathbb{R}\to\mathbb{C}$ is said to belong to the class $A^{p,\alpha}_d$ for some $d>0$, $\alpha\in(0,1)$ and some integer $p\geqslant 0$, if $[w]=0$ and there exists a holomorphic function $Z+iW:\mathcal{R}_d\to\mathbb{C}$ such that $Z,W\in C^{p,\alpha}_{2\pi}(\overline{\mathcal{R}}_d)$, W satisfies (A.2), and $Z(x,0)=z(x)$ for all $x\in\mathbb{R}$.

The discussion in Appendix A shows that, given a 2π-periodic function $z+iw:\mathbb{R}\to\mathbb{C}$ with $[w]=0$, we have that

$$z+iw\in A^{p,\alpha}_d \quad \text{if and only if} \quad z=[z]+\mathcal{C}_d(w). \tag{2.23}$$

Note also that $A^{p,\alpha}_d$ is an algebra. Indeed, let $z_j+iw_j\in A^{p,\alpha}_d$ for $j=1,2$, and let Z_j+iW_j be the corresponding holomorphic functions in $\mathcal{R}_d$. Then the product

$$Z+iW=(Z_1+iW_1)(Z_2+iW_2)$$

is also a holomorphic function in $\mathcal{R}_d$, whose boundary values on the real axis are given by $z+iw=(z_1+iw_1)(z_2+iw_2)$. The Cauchy–Riemann equation $Z_x=W_y$ implies that $[W(\cdot,y)]$ is independent of y. Since $W(\cdot,-d)\equiv 0$, it follows that $[w]=0$. Thus $A_d^{p,\alpha}$ is an algebra.

Theorem 1. (Equivalence of formulations) *Let* $(m,Q,v)\in\mathbb{R}\times\mathbb{R}\times C_{2\pi}^{1,\alpha}(\mathbb{R})$, *with* $[v]=h$. *Then, we have the following*:

(i) *If* (2.22) *holds, then* (2.20) *holds.*

(ii) *If* (2.20) *holds and, in addition,*

$$V_x^2+V_y^2\neq 0 \quad \textit{in } \overline{\mathcal{R}}_{kh}, \tag{2.24}$$

where V *is the solution of* (2.9), *then* (2.22) *holds.*

Proof. For any $(m,Q,v)\in\mathbb{R}\times\mathbb{R}\times C_{2\pi}^{1,\alpha}(\mathbb{R})$ with $[v]=h$, let V be the solution of (2.9) and ζ the solution of (2.13). In particular,

$$\zeta_x(x,0)=-\Upsilon V(x,0)V_x(x,0) \quad \text{for all } x\in\mathbb{R}, \tag{2.25}$$

with V_y and ζ_y given by (2.19). Also, recall from the previous section that (2.20) is merely (2.14). As a consequence (2.22b) is equivalently expressed as

$$[(Q-2gV(\cdot,0))(V_x^2(\cdot,0)+V_y^2(\cdot,0))-(\zeta_y(\cdot,0)+\Upsilon V(\cdot,0)V_y(\cdot,0))^2]=0. \tag{2.26}$$

Notice that

$$V_y(\cdot,0)+iV_x(\cdot,0)\in A_{kh}^{0,\alpha}, \tag{2.27}$$

$$\zeta_y(\cdot,0)+i\zeta_x(\cdot,0)\in A_{kh}^{0,\alpha}, \tag{2.28}$$

$$(\zeta_y^2-\zeta_x^2+2i\zeta_y\zeta_x)|_{(\cdot,0)}\in A_{kh}^{0,\alpha}. \tag{2.29}$$

We now claim that (2.22a) may be expressed in an equivalent way as

$$(x\mapsto((Q-2gV-\Upsilon^2V^2)(V_y-iV_x)-2\Upsilon\zeta_yV)|_{(x,0)})\in A_{kh}^{0,\alpha}. \tag{2.30}$$

To justify this claim, let us consider the function in (2.30) in more detail. Its imaginary part is

$$w=-(Q-2gv-\Upsilon^2v^2)v', \tag{2.31}$$

which satisfies $[w]=0$. Because of (2.19), its real part is

$$z=(Q-2gv-\Upsilon^2v^2)\left(\frac{1}{k}+\mathcal{C}_{kh}(v')\right)-2\Upsilon v\left(\frac{m}{kh}-\frac{\Upsilon}{2kh}[v^2]-\Upsilon\,\mathcal{C}_{kh}(vv')\right), \tag{2.32}$$

which has the average

$$[z]=\frac{Q-2gh-2\Upsilon m}{k}-2g[v\mathcal{C}_{kh}(v')]-\Upsilon^2[v^2\,\mathcal{C}_{kh}(v')]+2\Upsilon^2[v\,\mathcal{C}_{kh}(vv')].$$

But $f\mapsto\mathcal{C}_{kh}(f')$ being a self-adjoint linear operator, we have

$$[v^2\mathcal{C}_{kh}(v')]=\frac{1}{2\pi}\int_{-\pi}^{\pi}v^2\mathcal{C}_{kh}(v')\,dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}v\mathcal{C}_{kh}((v^2)')\,dx=2[v\mathcal{C}_{kh}(vv')], \tag{2.33}$$

so that

$$[z]=\frac{Q-2gh-2\Upsilon m}{k}-2g[v\mathcal{C}_{kh}(v')]. \tag{2.34}$$

According to (2.23), the statement (2.30) is equivalent to

$$z=[z]+\mathcal{C}_d(w). \tag{2.35}$$

We express z by (2.32), $[z]$ by (2.34) and w by (2.31). Then all the terms in the equation that result from (2.35) are compared with the terms in (2.22a). This leads to the conclusion that (2.30) is equivalent to (2.22a), as we claimed.

(i) Suppose that (2.22a) and (2.22b) hold. Then, as we have just pointed out, (2.30) and (2.26) hold. Since $A^{0,\alpha}_{kh}$ is an algebra, it follows from (2.30) and (2.27) that

$$(x\mapsto((Q-2gV-\Upsilon^2V^2)(V_x^2+V_y^2)-2\Upsilon\zeta_yV(V_y+iV_x))|_{(x,0)})\in A^{0,\alpha}_{kh}. \tag{2.36}$$

Upon using (2.25), this may be rewritten as

$$(x\mapsto((Q-2gV-\Upsilon^2V^2)(V_x^2+V_y^2)-2\Upsilon\zeta_yVV_y+2i\zeta_x\zeta_y)|_{(x,0)})\in A^{0,\alpha}_{kh}. \tag{2.37}$$

It now follows, taking (2.29) into account, that

$$(x\mapsto((Q-2gV-\Upsilon^2V^2)(V_x^2+V_y^2)-2\Upsilon\zeta_yVV_y-(\zeta_y^2-\zeta_x^2))|_{(x,0)})\in A^{0,\alpha}_{kh}. \tag{2.38}$$

Since this function is real-valued, it follows from (2.23) that it must be a constant, a fact which may be written, upon using (2.25), as

$$((Q-2gV)(V_x^2+V_y^2)-(\zeta_y+\Upsilon VV_y)^2)|_{(\cdot,0)}\equiv C_0. \tag{2.39}$$

It now follows from (2.26) that $C_0=0$, so that (2.14), and therefore (2.20) holds.

(ii) Conversely, suppose that (2.20) holds. Then (2.22b) is obtained by taking averages. Also (2.20) is exactly the same as (2.39) with $C_0=0$. This implies that (2.38) holds, then, in view of (2.29), that (2.37) holds, and then, upon using (2.25), that (2.36) holds. It is a consequence of (2.24) that

$$\left(x\mapsto\frac{1}{V_y(x,0)+iV_x(x,0)}\right)\in A^{0,\alpha}_{kh}.$$

Again, using the fact that $A^{0,\alpha}_{kh}$ is an algebra, we deduce from (2.36) that (2.30) holds, which means, as proved earlier, that (2.22a) holds. □

Remark 2. Let us make an observation about the statement (2.30), which is at the heart of the proof of Theorem 1. If we denote, for all $x\in\mathbb{R}$,

$$\begin{aligned} F(x) &= V_y(x,0)+iV_x(x,0),\\ a(x) &= Q-2gV(x,0)-\Upsilon^2V^2(x,0),\\ b(x) &= -2\Upsilon V(x,0)\zeta_y(x,0),\end{aligned}$$

then a and b are real-valued functions and $F\in A^{0,\alpha}_{kh}$. Moreover, if we denote, for all $x\in\mathbb{R}$,

$$G(x)=a(x)\overline{F}(x)+b(x), \tag{2.40}$$

where $\overline{F}$ denotes the complex conjugate of F, then (2.30) states that $G\in A^{0,\alpha}_{kh}$. We recall that a *Riemann–Hilbert problem* is an equation of the type (2.40), but where the coefficients a and b are given, and the functions $F,G\in A^{0,\alpha}_{kh}$ are to be determined. In our problem, however, a, b, F and G all depend on the unknown function v. Also, our proof of Theorem 1 did not make use of any general result of Riemann–Hilbert theory.

2.3. Variational structure of the new formulation

Given $v\in C^{2,\alpha}_{2\pi}(\mathbb{R})$, we separate the average h of v by setting

$$v=w+h \quad \text{with } [w]=0. \tag{2.41}$$

With this notation, we introduce the functional

$$\begin{aligned}\Lambda(w,h) = \int_{-\pi}^{\pi}&\left(Qv-gv^2-\frac{\Upsilon^2}{3}v^3\right)\left(\frac{1}{k}+\mathcal{C}_{kh}(v')\right)dx\\ &+\int_{-\pi}^{\pi}\left(m-\frac{\Upsilon}{2}v^2\right)\left(\frac{m}{kh}-\frac{\Upsilon}{2kh}[v^2]-\Upsilon\mathcal{C}_{kh}(vv')\right)dx,\end{aligned} \tag{2.42}$$

having as domain of definition the space $\mathcal{S}$ of pairs (w,h) with $h>0$ and $w\in C^{2,\alpha}_{2\pi,\circ}(\mathbb{R})$, where the subscript $\circ$ indicates mean zero. Taking variations of Λ with respect to the function w and with respect to the scalar h, we will obtain the reformulation of the governing equations (2.22). While the particular form of the functional Λ in (2.42) may appear at this point unmotivated, the reason for considering it is that, as we show in full detail in Appendix B.2, this form is obtained by transforming via conformal mapping the natural energy associated with the flow in the physical plane. Here we content ourselves with merely deriving the equation for the critical points of the functional Λ.

THEOREM 3. *Any critical point v of Λ in the space $\mathcal{S}$ satisfies the equation* (2.22a) *as well as the constraint* (2.22b).

Proof. We compute the first variation of Λ at v by considering in turn the variations of Λ with respect to w and h. First, for φ smooth, 2π-periodic and with $[\varphi]=0$, we compute from (2.42) the variation

$$\frac{\delta\Lambda}{\delta w}(w,h)\varphi=\lim_{\varepsilon\to 0}\frac{\Lambda(w+\varepsilon\varphi,h)-\Lambda(w,h)}{\varepsilon}$$

as

$$\begin{aligned}\frac{\delta\Lambda}{\delta w}(w,h)\varphi=&\int_{-\pi}^{\pi}\left(Qv-gv^2-\frac{\Upsilon^2}{3}v^3\right)\mathcal{C}_{kh}(\varphi')\,dx\\&+\int_{-\pi}^{\pi}\left(Q-2gv-\Upsilon^2v^2\right)\varphi\left(\frac{1}{k}+\mathcal{C}_{kh}(v')\right)dx\\&+\int_{-\pi}^{\pi}\left(m-\frac{\Upsilon}{2}v^2\right)\left(-\frac{\Upsilon}{kh}[v\varphi]-\Upsilon\mathcal{C}_{kh}((v\varphi)')\right)dx\\&-\Upsilon\int_{-\pi}^{\pi}v\varphi\left(\frac{m}{kh}-\frac{\Upsilon}{2kh}[v^2]-\Upsilon\mathcal{C}_{kh}(vv')\right)dx.\end{aligned}$$

Using the fact that $f\mapsto\mathcal{C}_{kh}(f')$ is a self-adjoint linear operator, this may be written as

$$\begin{aligned}\frac{\delta\Lambda}{\delta w}(w,h)\varphi=&\int_{-\pi}^{\pi}\mathcal{C}_{kh}((Q-2gv-\Upsilon^2v^2)v')\varphi\,dx\\&+\int_{-\pi}^{\pi}(Q-2gv-\Upsilon^2v^2)\left(\frac{1}{k}+\mathcal{C}_{kh}(v')\right)\varphi\,dx\\&-\frac{\Upsilon}{kh}\left(m-\frac{\Upsilon}{2}[v^2]\right)\int_{-\pi}^{\pi}v\varphi\,dx+\Upsilon^2\int_{-\pi}^{\pi}v\mathcal{C}_{kh}(vv')\varphi\,dx\\&-\Upsilon\int_{-\pi}^{\pi}v\left(\frac{m}{kh}-\frac{\Upsilon}{2kh}[v^2]-\Upsilon\mathcal{C}_{kh}(vv')\right)\varphi\,dx.\end{aligned}$$

Therefore

$$\frac{\delta\Lambda}{\delta w}(w,h)\varphi=\int_{-\pi}^{\pi}\eta\varphi\,dx,\tag{2.43}$$

where

$$\begin{aligned}\eta=\mathcal{C}_{kh}((Q-2gv-\Upsilon^2v^2)v')+(Q-2gv-\Upsilon^2v^2)\left(\frac{1}{k}+\mathcal{C}_{kh}(v')\right)&\\-2\Upsilon v\left(\frac{m}{kh}-\frac{\Upsilon}{2kh}[v^2]-\Upsilon\mathcal{C}_{kh}(vv')\right).&\end{aligned}\tag{2.44}$$

Now, at a critical point $v=w+h$ of $\Lambda(w,h)$, we of course have

$$\frac{\delta\Lambda}{\delta w}(w,h)\varphi=0$$

for all smooth 2π-periodic functions φ with $[\varphi]=0$, so that (2.43) implies that η is a constant.

Next, we compute the variation of Λ with respect to h. For this purpose, we notice that for any function $f\in C^{2,\alpha}_{2\pi}(\mathbb{R})$ we have

$$\frac{d}{dh}(\mathcal{C}_{kh}(f'))=-kf''-k\mathcal{C}_{kh}^2(f''). \tag{2.45}$$

Indeed, writing

$$f(x)=[f]+\sum_{n=1}^{\infty}(a_n\cos(nx)+b_n\sin(nx)),$$

we have

$$\mathcal{C}_{kh}\,(f')=\sum_{n=1}^{\infty}n\coth(nkh)(a_n\cos(nx)+b_n\sin(nx)),$$

so that

$$\frac{d}{dh}(\mathcal{C}_{kh}\,(f'))=\sum_{n=1}^{\infty}kn^2(1-\coth^2(nkh))(a_n\cos(nx)+b_n\sin(nx))=-kf''-k\mathcal{C}_{kh}^2(f'').$$

Expressing Λ explicitly in terms of w and h, the functional $\Lambda(w,h)$ equals

$$\begin{aligned}&\int_{-\pi}^{\pi}\left(Qw+Qh-gw^2-2gwh-gh^2-\frac{\Upsilon^2}{3}w^3-\Upsilon^2w^2h-\Upsilon^2wh^2-\frac{\Upsilon^2}{3}h^3\right)\left(\frac{1}{k}+\mathcal{C}_{kh}(w')\right)dx\\&\quad+\int_{-\pi}^{\pi}\left(m-\frac{\Upsilon}{2}w^2-\Upsilon wh-\frac{\Upsilon}{2}h^2\right)\left(\frac{m}{kh}-\frac{\Upsilon}{2kh}[w^2]-\frac{\Upsilon h}{2k}-\Upsilon\mathcal{C}_{kh}(ww')-\Upsilon h\mathcal{C}_{kh}(w')\right)dx.\end{aligned}$$

Using (2.45) we now compute

$$\begin{aligned}\frac{\delta\Lambda}{\delta h}(w,h)&=\int_{-\pi}^{\pi}(Q-2gv-\Upsilon^2v^2)\left(\frac{1}{k}+\mathcal{C}_{kh}(v')\right)dx\\&\quad-\int_{-\pi}^{\pi}\left(Qv-gv^2-\frac{\Upsilon^2}{3}v^3\right)\left(kv''+k\mathcal{C}_{kh}^2(v'')\right)dx\\&\quad-\Upsilon\int_{-\pi}^{\pi}v\left(\frac{m}{kh}-\frac{\Upsilon}{2kh}[v^2]-\Upsilon\mathcal{C}_{kh}(vv')\right)dx\\&\quad+\int_{-\pi}^{\pi}\left(m-\frac{\Upsilon}{2}v^2\right)\\&\qquad\times\left(-\frac{m}{kh^2}+\frac{\Upsilon}{2kh^2}[v^2]-\frac{\Upsilon}{k}-\Upsilon\mathcal{C}_{kh}(v')+\Upsilon k(vv')'+\Upsilon k\mathcal{C}_{kh}^2((vv')')\right)dx,\end{aligned} \tag{2.46}$$

since the terms $\pm\Upsilon hk(1+\mathcal{C}_{kh}^2)v''$ have canceled each other. In the second and fourth integral in (2.46) we integrate once by parts the terms involving v'' and $(vv')'$, and for the terms involving $\mathcal{C}_{kh}^2(v'')$ and $\mathcal{C}_{kh}^2\big((vv')'\big)$ we use the fact that $f\mapsto\mathcal{C}_{kh}(f')$ is self-adjoint, to express (2.46) as

$$\begin{aligned}\frac{\delta\Lambda}{\delta h}(w,h)=\int_{-\pi}^{\pi}&(Q-2gv-\Upsilon^2v^2)\left(\frac{1}{k}+\mathcal{C}_{kh}(v')\right)dx\\&+k\int_{-\pi}^{\pi}(Q-2gv-\Upsilon^2v^2)(v')^2\,dx\\&-k\int_{-\pi}^{\pi}\mathcal{C}_{kh}((Q-2gv-\Upsilon^2v^2)v')\mathcal{C}_{kh}(v')\,dx\\&-\Upsilon\int_{-\pi}^{\pi}v\left(\frac{m}{kh}-\frac{\Upsilon}{2kh}[v^2]-\Upsilon\mathcal{C}_{kh}(vv')\right)dx\\&+\int_{-\pi}^{\pi}\left(m-\frac{\Upsilon}{2}v^2\right)\left(-\frac{m}{kh^2}+\frac{\Upsilon}{2kh^2}[v^2]-\frac{\Upsilon}{k}-\Upsilon\mathcal{C}_{kh}(v')\right)dx\\&+k\Upsilon^2\int_{-\pi}^{\pi}v^2(v')^2\,dx-k\Upsilon^2\int_{-\pi}^{\pi}(\mathcal{C}_{kh}(vv'))^2\,dx.\end{aligned}\tag{2.47}$$

We further use (2.44) to substitute the third integral in (2.47) by

$$\begin{aligned}-k\int_{-\pi}^{\pi}\eta\mathcal{C}_{kh}(v')\,dx+k\int_{-\pi}^{\pi}(Q-2gv-\Upsilon^2v^2)\left(\frac{1}{k}+\mathcal{C}_{kh}(v')\right)\mathcal{C}_{kh}(v')\,dx\\-2k\Upsilon\int_{-\pi}^{\pi}v\left(\frac{m}{kh}-\frac{\Upsilon}{2kh}[v^2]-\Upsilon\mathcal{C}_{kh}(vv')\right)\mathcal{C}_{kh}(v')\,dx.\end{aligned}$$

Consequently, (2.47) takes the form

$$\begin{aligned}\frac{\delta\Lambda}{\delta h}(w,h)=-k\int_{-\pi}^{\pi}&\eta\mathcal{C}_{kh}(v')\,dx+k\int_{-\pi}^{\pi}(Q-2gv-\Upsilon^2v^2)\left((v')^2+\left(\frac{1}{k}+\mathcal{C}_{kh}(v')\right)^2\right)dx\\&-2k\Upsilon\int_{-\pi}^{\pi}v\left(\frac{m}{kh}-\frac{\Upsilon}{2kh}[v^2]-\Upsilon\mathcal{C}_{kh}(vv')\right)\mathcal{C}_{kh}(v')\,dx\\&-\Upsilon\int_{-\pi}^{\pi}v\left(\frac{m}{kh}-\frac{\Upsilon}{2kh}[v^2]-\Upsilon\mathcal{C}_{kh}(vv')\right)dx\\&+\int_{-\pi}^{\pi}\left(m-\frac{\Upsilon}{2}v^2\right)\left(-\frac{m}{kh^2}+\frac{\Upsilon}{2kh^2}[v^2]-\frac{\Upsilon}{k}\right)dx\\&+\frac{\Upsilon^2}{2}\int_{-\pi}^{\pi}v^2\mathcal{C}_{kh}(v')\,dx+k\Upsilon^2\int_{-\pi}^{\pi}v^2(v')^2\,dx-k\Upsilon^2\int_{-\pi}^{\pi}(\mathcal{C}_{kh}(vv'))^2\,dx.\end{aligned}$$

One can easily check that the fifth integral term in the above expression is precisely

$$-k\int_{-\pi}^{\pi}\left(\frac{m}{kh}-\frac{\Upsilon}{2kh}[v^2]\right)^2dx-\Upsilon\int_{-\pi}^{\pi}v\left(\frac{m}{kh}-\frac{\Upsilon}{2kh}[v^2]\right)dx,$$

while the self-adjointness of $f\mapsto\mathcal{C}_{kh}(f')$ yields

$$\frac{\Upsilon^2}{2}\int_{-\pi}^{\pi}v^2\mathcal{C}_{kh}(v')\,dx=\Upsilon^2\int_{-\pi}^{\pi}v\mathcal{C}_{kh}(vv')\,dx.$$

Since $[\mathcal{C}_{kh}(vv')]=0$, we get

$$\begin{aligned}\frac{\delta\Lambda}{\delta h}(w,h)=-k\int_{-\pi}^{\pi}\eta\mathcal{C}_{kh}(v')\,dx+k\int_{-\pi}^{\pi}(Q-2gv-\Upsilon^2v^2)\bigg((v')^2+\bigg(\frac{1}{k}+\mathcal{C}_{kh}(v')\bigg)^2\bigg)\,dx\\ -k\int_{-\pi}^{\pi}\bigg(\frac{m}{kh}-\frac{\Upsilon}{2kh}[v^2]-\Upsilon\mathcal{C}_{kh}(vv')\bigg)^2\,dx+k\Upsilon^2\int_{-\pi}^{\pi}v^2(v')^2\,dx\\ -2k\Upsilon\int_{-\pi}^{\pi}v\bigg(\frac{m}{kh}-\frac{\Upsilon}{2kh}[v^2]-\Upsilon\mathcal{C}_{kh}(vv')\bigg)\bigg(\frac{1}{k}+\mathcal{C}_{kh}(v')\bigg)\,dx.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{\delta\Lambda}{\delta h}(w,h)=-k\int_{-\pi}^{\pi}\eta\mathcal{C}_{kh}(v')\,dx+k\int_{-\pi}^{\pi}(Q-2gv)\bigg((v')^2+\bigg(\frac{1}{k}+\mathcal{C}_{kh}(v')\bigg)^2\bigg)\,dx\\ -k\int_{-\pi}^{\pi}\bigg(\frac{m}{kh}-\frac{\Upsilon}{2kh}[v^2]-\Upsilon\mathcal{C}_{kh}(vv')+\Upsilon v\bigg(\frac{1}{k}+\mathcal{C}_{kh}(v')\bigg)\bigg)^2\,dx,\end{aligned}\tag{2.48}$$

where η is given by (2.44).

At a critical point $v=w+h$ of $\Lambda(w,h)$ we also have

$$\frac{\delta\Lambda}{\delta h}(w,h)=0.$$

We have already shown that η is a constant. Its exact value is computed by taking averages in (2.44). Using the identity (2.33), we obtain precisely (2.22a). Due to the fact that $[\mathcal{C}_{kh}(v')]=0$, the condition

$$\frac{\delta\Lambda}{\delta h}(w,h)=0$$

is easily seen to be equivalent to (2.22b). This completes the proof of Theorem 3. □

Remark 4. Let us now discuss the special case $\Upsilon=0$ (irrotational flow). For $k=1$ and $w=v-h$, (2.22a) reduces to

$$\mu\mathcal{C}_h(w')=w+w\mathcal{C}_h(w')+\mathcal{C}_h(ww')-[w\mathcal{C}_h(w')]\tag{2.49}$$

with $\mu=(Q-2gh)/g$. Note that for $\Upsilon=0$ the constant m disappears from (2.22a). In this case, the constraint (2.22b) merely specifies the value of m. Setting $\beta=[w\mathcal{C}_hw']$, $\tilde{v}=w-\beta$, we transform (2.49) into

$$\tilde{\mu}\mathcal{C}_h(\tilde{v}')=\tilde{v}+\tilde{v}\mathcal{C}_h(\tilde{v}')+\mathcal{C}_h(\tilde{v}\tilde{v}')\tag{2.50}$$

with $\tilde{\mu}=\mu-2\beta$. On the other hand, the equation for irrotational steady waves in water of infinite depth, mentioned in the introduction, is

$$\tilde{\mu}\mathcal{C}(\tilde{v}')=\tilde{v}+\tilde{v}\mathcal{C}(\tilde{v}')+\mathcal{C}(\tilde{v}\tilde{v}'),\tag{2.51}$$

where $\mathcal{C}$ is the standard Hilbert transform and $\tilde{\mu}>0$ is a constant (see [3, equation (1.8)]). Note the direct analogy between (2.50) and (2.51).

3. Existence theory

A powerful approach for establishing the existence of travelling water waves relies on bifurcation theory, tailored for the study in-the-large of parameter-dependent families of solutions. The existence of waves of *small* amplitude is addressed by means of local bifurcation theory: identifying suitable parameters that, by crossing through certain thresholds, lead to sudden changes of the corresponding flat-surface flows into genuine waves. Global bifurcation theory uses topological methods to show that these families of perturbations of simple solutions belong to connected sets of solutions of *global* extent. Since global bifurcation is not a perturbative approach, exploiting instead the topological structure of the solution set, this global continuum provides wave patterns that are not small disturbances of flows with a flat-free surface.

In this section we study the existence of solutions (m, Q, v) of (2.22) in the space $\mathbb{R}\times\mathbb{R}\times C^{2,\alpha}_{2\pi,e}(\mathbb{R})$ and such that $[v]=h$, where

$$C^{2,\alpha}_{2\pi,e}(\mathbb{R}) = \{f\in C^{2,\alpha}_{2\pi}(\mathbb{R}) : f(x)=f(-x) \text{ for all } x\in\mathbb{R}\},$$

for some constant $\alpha\in(0,1)$. Our construction provides parametrized families of solutions, which reduce to $v\equiv h$ when the parameter vanishes. The requirement that v is an even function reflects the symmetry of the corresponding wave profile about the crest located at $x=0$. Symmetric travelling periodic waves are ubiquitous in nature (see e.g. the photographs in [10]). Moreover, in the absence of stagnation points in the flow and for surface waves that are represented by graphs of functions, one can show that a wave profile that is monotonic between crests and troughs has to be symmetric cf. [12], [11]. In view of Theorem 1, any solution of (2.22) satisfies also (2.18a)–(2.18b). Whether or not it will give rise to solutions of the free-boundary problem (2.1)–(2.2) depends on whether or not it also satisfies conditions (2.18c)–(2.18e). The existence of solutions of (2.22) will be proved in this section, while the extent to which the constructed solutions satisfy also (2.18c)–(2.18e) will be investigated in §4. Some further qualitative properties of the constructed solutions will also be studied there.

Note first that (2.22) has a family of trivial solutions, for which $v\equiv h$, while Q and m are related by

$$Q = 2gh + \left(\frac{m}{h}+\frac{\Upsilon h}{2}\right)^2, \tag{3.1}$$

where $m\in\mathbb{R}$ is arbitrary. This family represents a curve

$$\mathcal{K}_{\text{triv}} = \left\{\left(m, 2gh+\left(\frac{m}{h}+\frac{\Upsilon h}{2}\right)^2, h\right) : m\in\mathbb{R}\right\}$$

in the space $\mathbb{R}\times\mathbb{R}\times C^{2,\alpha}_{2\pi,e}(\mathbb{R})$. These solutions correspond to laminar flows in the fluid domain bounded below by the rigid bed $\mathcal{B}$ and above by the free surface $Y=h$, with stream function

$$\psi(X,Y)=\frac{\Upsilon}{2}Y^2+\left(\frac{m}{h}-\frac{\Upsilon h}{2}\right)Y-m,\quad X\in\mathbb{R},\ 0\leqslant Y\leqslant h,$$

velocity field

$$(\psi_Y,-\psi_X)=\left(\Upsilon Y+\frac{m}{h}-\frac{\Upsilon h}{2},0\right),\quad X\in\mathbb{R},\ 0\leqslant Y\leqslant h, \tag{3.2}$$

and period $L=2\pi/k$.

THEOREM 5. (Global bifurcation) *Let $h,k>0$ and $\Upsilon\in\mathbb{R}$ be given. For each $n\in\mathbb{N}$, let*

$$m^*_{n,\pm}=-\frac{\Upsilon h^2}{2}+\frac{\Upsilon h\tanh(nkh)}{2nk}\pm h\sqrt{\frac{\gamma^2\tanh^2(nkh)}{4n^2k^2}+g\frac{\tanh(nkh)}{nk}} \tag{3.3}$$

and

$$Q^*_{n,\pm}=2gh+\left(\frac{m^*_{n,\pm}}{h}+\frac{\Upsilon h}{2}\right)^2. \tag{3.4}$$

First, for any $m\in\mathbb{R}\setminus\{m^*_{n,\pm}:n\in\mathbb{N}\}$, there exists a neighbourhood in $\mathbb{R}\times\mathbb{R}\times C^{2,\alpha}_{2\pi,e}(\mathbb{R})$ of the point (m,Q,h) on $\mathcal{K}_{\mathrm{triv}}$, where Q is related to m by (3.1), in which the only solutions of (2.22) are those on $\mathcal{K}_{\mathrm{triv}}$. Secondly, consider the points $m^*_{n,\pm}$. For each integer $n\geqslant 1$ and each choice of sign $\pm$, there exists in the space $\mathbb{R}\times\mathbb{R}\times C^{2,\alpha}_{2\pi,e}(\mathbb{R})$ a continuous curve

$$\mathcal{K}_{n,\pm}=\{(m(s),Q(s),v_s):s\in\mathbb{R}\} \tag{3.5}$$

of solutions of (2.22) such that the following properties hold:

(i) *$(m(0),Q(0),v_0)=(m^*_{n,\pm},Q^*_{n,\pm},h)$, where $m^*_{n,\pm}$ and $Q^*_{n,\pm}$ are given by (3.3) and (3.4);*

(ii) *$v_s(x)=h+s\cos(nx)+o(s)$ in $C^{2,\alpha}_{2\pi,e}(\mathbb{R})$ if $0<|s|<\varepsilon$, for some $\varepsilon>0$ sufficiently small;*

(iii) *there exist a neighbourhood $\mathcal{W}_{n,\pm}$ of $(m^*_{n,\pm},Q^*_{n,\pm},h)$ in $\mathbb{R}\times\mathbb{R}\times C^{2,\alpha}_{2\pi,e}(\mathbb{R})$ and $\varepsilon>0$ sufficiently small such that*

$$\{(m,Q,v)\in\mathcal{W}_{n,\pm}:v\not\equiv h \text{ and } (2.22) \text{ holds}\}=\{(m(s),Q(s),v_s):0<|s|<\varepsilon\};$$

(iv) *$Q(s)-2gv_s(x)>0$ for all $s,x\in\mathbb{R}$;*

(v) *$\mathcal{K}_{n,\pm}$ has a real-analytic reparametrization locally around each of its points;*

(vi) *one of the following alternatives occurs*:

(α) *either*

$$\min\left\{\frac{1}{1+\|(m(s),Q(s),v_s)\|_{\mathbb{R}\times\mathbb{R}\times C^{2,\alpha}_{2\pi,e}(\mathbb{R})}},\min_{x\in\mathbb{R}}(Q(s)-2gv_s(x))\right\}\to 0 \qquad (3.6)$$

as $s\to\pm\infty$;

(β) *or there exists* $T>0$ *such that*

$$(m(s+T),Q(s+T),v_{s+T})=(m(s),Q(s),v_s) \quad \textit{for all } s\in\mathbb{R}.$$

Moreover, for each integer $n\geqslant 1$ *and both choices of sign* $\pm$, *such a curve of solutions of* (2.22) *with the properties* (i)–(vi) *is unique* (*up to reparametrization*).

In this theorem, (3.3) and (3.4) identify the local bifurcation points along the trivial solution curve $\mathcal{K}_{\text{triv}}$, (i) states where the curve of non-trivial solutions begins, (ii) and (iii) describe the local behavior of the curve, and (iv)–(vi) describe the global behavior. The alternative (α) means that either the curve is unbounded in the function space or it approaches a wave of greatest height, while the alternative (β) is that the curve forms a loop.

Our main tool in the proof of Theorem 5 is the following version of the global bifurcation theorem for real-analytic operators due to Dancer [20] and improved by Buffoni and Toland [6, Theorem 9.1.1]. That result is however slightly inaccurate as stated there, and therefore we provide a corrected version, slightly modified to better suit our purposes. For a linear operator $\mathcal{L}$ between two Banach spaces, let us denote by $\mathcal{N}(\mathcal{L})$ its null space and by $\mathcal{R}(\mathcal{L})$ its range.

THEOREM 6. (Analytic bifurcation theory) *Let* X *and* Y *be Banach spaces*, $\mathcal{O}$ *be an open subset of* $\mathbb{R}\times X$ *and* $F:\mathcal{O}\to Y$ *be a real-analytic function. Suppose that*

(H_1) $(\lambda,0)\in\mathcal{O}$ *and* $F(\lambda,0)=0$ *for all* $\lambda\in\mathbb{R}$;

(H_2) *for some* $\lambda^*\in\mathbb{R}$, $\mathcal{N}(\partial_u F(\lambda^*,0))$ *and* $Y/\mathcal{R}(\partial_u F(\lambda^*,0))$ *are* 1-*dimensional*, *with the nullspace generated by* u^*, *and the transversality condition*

$$\partial^2_{\lambda,u}F(\lambda^*,0)(1,u^*)\notin\mathcal{R}(\partial_u F(\lambda^*,0))$$

holds;

(H_3) $\partial_u F(\lambda,u)$ *is a Fredholm operator of index zero for any* $(\lambda,u)\in\mathcal{O}$ *such that* $F(\lambda,u)=0$;

(H_4) *for some sequence* $(\mathcal{Q}_j)_{j\in\mathbb{N}}$ *of bounded closed subsets of* $\mathcal{O}$ *with* $\mathcal{O}=\bigcup_{j\in\mathbb{N}}\mathcal{Q}_j$, *the set* $\{(\lambda,u)\in\mathcal{O}:F(\lambda,u)=0\}\cap\mathcal{Q}_j$ *is compact for each* $j\in\mathbb{N}$.

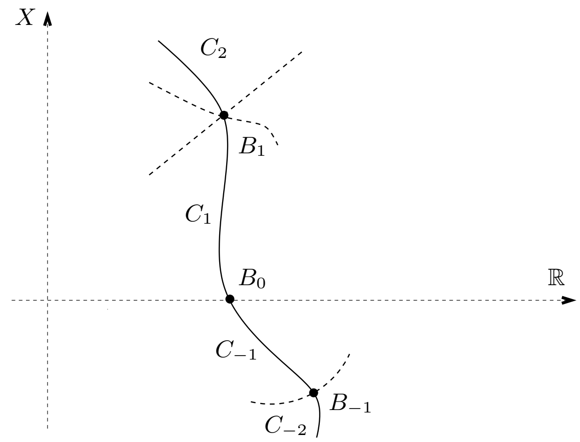

Figure 2. The global bifurcation curve $\mathcal{K}\subset\mathbb{R}\times X$ consists of distinguished real-analytic open arcs $\mathcal{C}_j$ that end in the branch points B_{j-1} and B_j if $j>0$, or in the branch points B_j and B_{j+1} if $j<0$, with $B_0=(\lambda^*,0)$ and $\mathcal{C}_1\cup\{B_0\}\cup\mathcal{C}_{-1}$ being the local bifurcation curve $\mathcal{K}_{\text{loc}}$. A point on $\mathcal{C}_j$ corresponds to a non-singular solution (near which the implicit function theorem applies), while each B_i arises as the unique intersection point of the closures of a finite even number of open 1-dimensional real-analytic varieties. A distinguished arc $\mathcal{C}_j$ can be uniquely continued across B_j by choosing an outgoing branch $\mathcal{C}_{j+1}$ if $j>0$, or $\mathcal{C}_{j-1}$ if $j<0$, so that each curve $\mathcal{C}_j\cup\{B_j\}\cup\mathcal{C}_{j+1}$ if $j>0$, or $\mathcal{C}_j\cup\{B_j\}\cup\mathcal{C}_{j-1}$ if $j<0$, admits a local uniformizing real-analytic parametrization near the branch point B_j.

Then there exists in $\mathcal{O}$ *a continuous curve* $\mathcal{K}=\{(\lambda(s),u(s)):s\in\mathbb{R}\}$ *of solutions to* $F(\lambda,u)=0$ *such that:*

(C_1) $(\lambda(0),u(0))=(\lambda^*,0)$;

(C_2) $u(s)=su^*+o(s)$ *in* X, $|s|<\varepsilon$ *as* $s\to0$;

(C_3) *there exist a neighbourhood* $\mathcal{W}$ *of* $(\lambda^*,0)$ *and* $\varepsilon>0$ *sufficiently small such that*

$$\{(\lambda,u)\in\mathcal{W}: u\neq0 \text{ and } F(\lambda,u)=0\}=\{(\lambda(s),u(s)):0<|s|<\varepsilon\};$$

(C_4) $\mathcal{K}$ *has a real-analytic reparametrization locally around each of its points*;

(C_5) *one of the following alternatives occurs*:

(α) *for every* $j\in\mathbb{N}$, *there exists* $s_j>0$ *such that* $(\lambda(s),u(s))\notin\mathcal{Q}_j$ *for all* $s\in\mathbb{R}$ *with* $|s|>s_j$;

(β) *there exists* $T>0$ *such that* $(\lambda(s+T),u(s+T))=(\lambda(s),u(s))$ *for all* $s\in\mathbb{R}$.

Moreover, such a curve of solutions to $F(\lambda,u)=0$ *having the properties* (C_1)–(C_5) *is unique* (*up to reparametrization*).

Remark 7. The local version of Theorem 6, in which assumptions $(H_1)-(H_2)$ imply the existence of a real-analytic local bifurcating curve

$$\mathcal{K}_{\text{loc}}=\{(\lambda(s),u(s)):s\in(-\varepsilon,\varepsilon)\}$$

of solutions to $F(\lambda,u)=0$ with the properties $(C_1)-(C_3)$ is the real-analytic version of the standard Crandall–Rabinowitz local bifurcation theorem [19]. The curve $\mathcal{K}$ exhausts all possibilities of adding real-analytic arcs to the local bifurcation curve $\mathcal{K}_{\text{loc}}$ in such a way that $\mathcal{K}$ has a real-analytic parametrization around each of its points (see Figure 2), but is not necessarily a maximal connected subset of the solution set.

Remark 8. We now discuss how Theorem 6 relates to [6, Theorem 9.1.1]. In [6], assumption (H_4) is replaced by the slightly stronger assumption that all bounded and closed subsets (in $\mathbb{R}\times X$) of $\{(\lambda,u)\in\mathcal{O}:F(\lambda,u)=0\}$ are compact. In [6] it is proven that there does not exist any sequence $\{s_k\}$, with $s_k\to\infty$, such that the sequence $\{(\lambda(s_k),u(s_k))\}_{k\geqslant 1}$ is *both* bounded *and* bounded away from the boundary of $\mathcal{O}$. However, [6] incorrectly claims from there that, if $(C_5)(\beta)$ does not hold, then

$(C_5)(\alpha')$ either $\|(\lambda(s),u(s))\|_{\mathbb{R}\times X}\to\infty$ or $\operatorname{dist}((\lambda(s),u(s)),\partial\mathcal{O})\to 0$, as $s\to\infty$,

which is a strictly stronger statement than the one proved. Instead, the correct conclusion for [6, Theorem 9.1.1] should be that either $(C_5)(\beta)$ holds or the following is true: for every bounded and closed subset $\mathcal{Q}$ of $\mathcal{O}$, the curve $(\lambda(s),u(s))$ eventually leaves $\mathcal{Q}$ as $s\to\infty$.

Moreover, what is needed in the proof of the above result for any given bounded and closed subset $\mathcal{Q}$ of $\mathcal{O}$ is that $\mathcal{Q}\cap\{(\lambda,u)\in\mathcal{O}:F(\lambda,u)=0\}$ is compact. Thus Theorem 6 can be proved by means of exactly the same arguments as in the proof of [6, Theorem 9.1.1].

For the purpose of applying Theorem 6 to problem (2.22), it is necessary to make some simple changes of variables. Since one must necessarily have $[v]=h$, it is natural to work with the function

$$w=v-h, \tag{3.7}$$

for which $[w]=0$. Then w satisfies

$$\begin{aligned}2(Q-2gh)&\mathcal{C}_{kh}(w')-2g(\mathcal{C}_{kh}(ww')+w\mathcal{C}_{kh}(w'))\\&-\Upsilon^2(\mathcal{C}_{kh}(w^2w')+w^2\mathcal{C}_{kh}(w')-2w\mathcal{C}_{kh}(ww'))\\&+\frac{\Upsilon^2}{kh}w[w^2]-\frac{\Upsilon^2}{k}w^2+\frac{\Upsilon^2}{k}[w^2]+2g[w\mathcal{C}_{kh}(w')]-\frac{2g}{k}w-\frac{2\Upsilon}{k}\left(\frac{m}{h}+\frac{\Upsilon h}{2}\right)w=0,\end{aligned} \tag{3.8a}$$

and

$$\begin{aligned}\left[\left(\frac{1}{k}\left(\frac{m}{h}+\frac{\Upsilon h}{2}\right)-\Upsilon\left(\frac{[w^2]}{2kh}-\frac{w}{k}+\mathcal{C}_{kh}(ww')-w\mathcal{C}_{kh}(w')\right)\right)^2\right]&\\=\left[(Q-2gh-2gw)\left((w')^2+\left(\frac{1}{k}+\mathcal{C}_{kh}(w')\right)^2\right)\right].&\end{aligned} \tag{3.8b}$$

We observe that, although Q and m are related by (3.1) for trivial solutions, this need not be the case in general. This observation suggests the introduction of a new parameter

$$\mu = Q - 2gh - \left(\frac{m}{h} + \frac{\Upsilon h}{2}\right)^2. \tag{3.9}$$

Note that, for the laminar flows given by (3.2), the horizontal velocity at the free surface is $m/h + \Upsilon h/2$. Since this expression occurs naturally in (3.8), while in the fluid dynamics literature it is customary to identify the laminar flows at which non-linear small-amplitude waves bifurcate using the speed of their particles at the free surface (rather than the value of their flux), we introduce another parameter

$$\lambda = \frac{m}{h} + \frac{\Upsilon h}{2}. \tag{3.10}$$

The change of parameters $(m, Q) \mapsto (\lambda, \mu)$ given by (3.9)–(3.10) is a bijection from $\mathbb{R}^2$ onto itself. In terms of the new parameters, the system (3.8) may be written as

$$F(\lambda, (\mu, w)) = 0, \tag{3.11}$$

where $F: \mathbb{R} \times X \to Y$, with

$$X = \mathbb{R} \times C^{2,\alpha}_{2\pi,\circ,e}(\mathbb{R}) \quad \text{and} \quad Y = C^{1,\alpha}_{2\pi,\circ,e}(\mathbb{R}) \times \mathbb{R}, \tag{3.12}$$

where the subscripts indicate period 2π, zero average and evenness, respectively, and where $F = (F_1, F_2)$ is given by

$$\begin{aligned} F_1(\lambda, (\mu, w)) = {} & 2(\mu + \lambda^2)\mathcal{C}_{kh}(w') - 2g(\mathcal{C}_{kh}(ww') + w\mathcal{C}_{kh}(w')) \\ & - \Upsilon^2(\mathcal{C}_{kh}(w^2 w') + w^2 \mathcal{C}_{kh}(w') - 2w\mathcal{C}_{kh}(ww')) \\ & + \frac{\Upsilon^2}{kh} w[w^2] - \frac{\Upsilon^2}{k} w^2 + \frac{\Upsilon^2}{k}[w^2] + 2g[w\mathcal{C}_{kh}(w')] - \frac{2g}{k} w - \frac{2\lambda\Upsilon}{k} w, \end{aligned} \tag{3.13a}$$

and

$$\begin{aligned} F_2(\lambda, (\mu, w)) = \Upsilon^2 & \left[\left(\mathcal{C}_{kh}(ww') - w\mathcal{C}_{kh}(w') - \frac{1}{k} w + \frac{[w^2]}{2kh} \right)^2 \right] \\ & + \frac{2(2g + \lambda\Upsilon)}{k} [w\mathcal{C}_{kh}(w')] + 2g[w(w')^2] + 2g[w(\mathcal{C}_{kh}(w'))^2] \\ & - \frac{\lambda\Upsilon}{k^2 h}[w^2] - (\mu + \lambda^2)([(\mathcal{C}_{kh}(w'))^2] + [(w')^2]) - \frac{\mu}{k^2}. \end{aligned} \tag{3.13b}$$

Note that $[F_1] = 0$ because $[w] = 0$ and $\mathcal{C}_{kh}$ maps into functions of zero average. Among the twelve terms in $[F_1]$, the third and tenth cancel, and the fifth and sixth cancel as well.

Proof of Theorem 5. We are going to apply Theorem 6 in the setting (3.11)–(3.13), after which we shall transfer in a rather straightforward manner the results obtained to the corresponding results for (2.22). It is obvious that the mapping F is *real-analytic* on $\mathbb{R}\times X$. Let

$$\mathcal{O}=\{(\lambda,(\mu,w))\in\mathbb{R}\times X:\mu+\lambda^2-2gw(x)>0 \text{ for all } x\in\mathbb{R}\}, \tag{3.14}$$

which is an open set in $\mathbb{R}\times X$. We now check the validity of the assumptions (H_1)–(H_4).

It is obvious that (H_1) holds. As for (H_2), for $(\nu,\varphi)\in X$ we easily compute

$$\partial_{(\mu,w)}F(\lambda,(0,0))(\nu,\varphi)=\left(2\lambda^2\mathcal{C}_{kh}(\varphi')-\frac{2g}{k}\varphi-\frac{2\lambda\Upsilon}{k}\varphi,-\frac{\nu}{k^2}\right). \tag{3.15}$$

Expanding the function φ, which is even, has period 2π, and has zero average, in a Fourier series $\varphi(x)=\sum_{n=1}^{\infty}a_n\cos(nx)$, we obtain for $(\nu,\varphi)\in X$ the representation

$$\begin{aligned}\partial_{(\mu,w)}&F(\lambda,(0,0))(\nu,\varphi)\\&=\left(\sum_{n=1}^{\infty}a_n\left(2\lambda^2 n\coth(nkh)-\frac{2g}{k}-\frac{2\lambda\Upsilon}{k}\right)\cos(nx),-\frac{\nu}{k^2}\right).\end{aligned} \tag{3.16}$$

It follows from (3.16) that the bounded linear operator $\partial_{(\mu,w)}F(\lambda,(0,0)):X\to Y$ is invertible whenever λ is *not* a solution of

$$\lambda^2 nk\coth(nkh)=g+\lambda\Upsilon, \tag{3.17}$$

for any integer $n\geqslant 1$. Hence by the implicit function theorem these points are not bifurcation points. The solutions of (3.17) are, for any integer $n\geqslant 1$, given by

$$\lambda_{n,\pm}^*=\frac{\Upsilon\tanh(nkh)}{2nk}\pm\sqrt{\frac{\Upsilon^2\tanh^2(nkh)}{4n^2k^2}+g\frac{\tanh(nkh)}{nk}}. \tag{3.18}$$

Observe that all of these values are distinct and none of them vanishes. We claim that (H_2) holds for every $\lambda^*\in\{\lambda_{n,\pm}^*:n\in\mathbb{N}\}$. Indeed, consider any such λ^*. It follows easily from (3.16) that $\mathcal{N}(\partial_{(\mu,w)}F(\lambda^*,(0,0)))$ is 1-dimensional and generated by $(0,w^*)\in X$, where

$$w^*(x)=\cos(nx)\quad\text{for all } x\in\mathbb{R},$$

while $\mathcal{R}(\partial_{(\mu,w)}F(\lambda^*,(0,0)))$ is the closed subspace of Y formed by the elements $(f,c)\in Y$ where $c\in\mathbb{R}$ is arbitrary and f satisfies

$$\int_{-\pi}^{\pi}f(x)\cos(nx)\,dx=0.$$

From (3.16), we now compute

$$\partial^2_{\lambda,(\mu,w)}F(\lambda^*,(0,0))(1,(0,w^*)) = \left(\left(4\lambda^* n\coth(nkh) - \frac{2\Upsilon}{k}\right)w^*, 0\right)$$
$$\notin \mathcal{R}(\partial_{(\mu,w)}F(\lambda^*,(0,0)))$$

since, using (3.17), we have

$$4\lambda^* n\coth(nkh) - \frac{2\Upsilon}{k} = 2\lambda^*\left(n\coth(nkh) + \frac{g}{k(\lambda^*)^2}\right) \neq 0.$$

This shows that (H_2) holds for every $\lambda^* \in \{\lambda^*_{n,\pm} : n \in \mathbb{N}\}$.

To check the validity of (H_3)-(H_4), we rewrite F_1 in (3.13a) in the following form

$$\begin{aligned} F_1(\lambda,(\mu,w)) &= 2(\mu+\lambda^2-2gw)\mathcal{C}_{kh}(w') + 4g[w\mathcal{C}_{kh}(w')] - 2g(\mathcal{C}_{kh}(ww') - w\mathcal{C}_{kh}(w')) \\ &\quad - \Upsilon^2(\mathcal{C}_{kh}(w^2w') + w^2\mathcal{C}_{kh}(w') - 2w\mathcal{C}_{kh}(ww')) \\ &\quad + \frac{\Upsilon^2}{kh}w[w^2] - \frac{\Upsilon^2}{k}w^2 + \frac{\Upsilon^2}{k}[w^2] - 2g[w\mathcal{C}_{kh}(w')] - \frac{2g}{k}w - \frac{2\lambda\Upsilon}{k}w \\ &= 2(\mu+\lambda^2-2gw)\mathcal{C}_{kh}(w') + 4g[w\mathcal{C}_{kh}(w')] + J(\lambda,w). \end{aligned} \tag{3.19}$$

Here

$$J(\lambda,w) = J_1(w) + J_2(w) + J_3(\lambda,w),$$

with

$$\begin{aligned} J_1(w) &= -2g(\mathcal{C}_{kh}(ww') - w\mathcal{C}_{kh}(w')), \\ J_2(w) &= -\Upsilon^2(\mathcal{C}_{kh}(w^2w') + w^2\mathcal{C}_{kh}(w') - 2w\mathcal{C}_{kh}(ww')), \end{aligned}$$

and $J_3(\lambda,w)$ gathers all the remaining terms in (3.19). Since J_2 may also be rewritten as

$$J_2(w) = -\Upsilon^2(\mathcal{C}_{kh}(w(ww')) - w\mathcal{C}_{kh}(ww') + w(w\mathcal{C}_{kh}(w') - \mathcal{C}_{kh}(ww'))),$$

it is a consequence of Lemma 17 in Appendix A that the continuous non-linear mappings

$$J_1, J_2 : C^{2,\alpha}_{2\pi,\circ}(\mathbb{R}) \longrightarrow C^{1,\alpha}_{2\pi}(\mathbb{R})$$

map bounded sets of $C^{2,\alpha}_{2\pi,\circ}(\mathbb{R})$ into bounded sets of $C^{2,\alpha/2}_{2\pi}(\mathbb{R})$, and thus into relatively compact subsets of $C^{1,\alpha}_{2\pi}(\mathbb{R})$. We also rewrite (3.13b) as

$$F_2(\lambda,(\mu,w)) = -\frac{\mu}{k^2} + K(\lambda,(\mu,w)). \tag{3.20}$$

It follows that the non-linear mapping from $\mathbb{R}\times X$ into Y

$$(\lambda,(\mu,w)) \longmapsto (J(\lambda,w), K(\lambda,(\mu,w))) \tag{3.21}$$

maps bounded sets of $\mathbb{R}\times X$ into bounded sets of $C^{2,\alpha/2}_{2\pi}\times\mathbb{R}$, and thus into relatively compact subsets of Y, and is therefore a non-linear compact operator. It then follows (see [6, Lemma 3.1.12]) that any of its partial derivatives is a linear compact operator. Note that, for any $(\lambda,(\nu,w))\in\mathcal{O}$, one may write

$$\begin{aligned}\partial_{(\mu,w)}F_1(\lambda,(\mu,w))(\nu,\varphi) = {} & 2(\mu+\lambda^2-2gw)\mathcal{C}_{kh}(\varphi')+4g[w\mathcal{C}_{kh}(\varphi')] \\ & -4g\mathcal{C}_{kh}(w')\varphi+4g[\mathcal{C}_{kh}(w')\varphi] \\ & +2\mathcal{C}_{kh}(w')\nu+\partial_{(\mu,w)}J(\lambda,(\mu,w))(\nu,\varphi), \\ \partial_{(\mu,w)}F_2(\lambda,(\mu,w))(\nu,\varphi) = {} & -\frac{\nu}{k^2}+\partial_{(\mu,w)}K(\lambda,(\mu,w))(\nu,\varphi).\end{aligned}$$

Since the condition $\mu+\lambda^2-2gw(x)>0$ for all $x\in\mathbb{R}$, guaranteed by the definition of $\mathcal{O}$, ensures that the bounded linear operator from X to Y given by

$$(\nu,\varphi)\mapsto\left(2(\mu+\lambda^2-2gw)\mathcal{C}_{kh}(\varphi')+4g[w\mathcal{C}_{kh}(\varphi')],-\frac{\nu}{k^2}\right)$$

is invertible, it follows that $\partial_{(\mu,w)}F(\lambda,(\mu,w))\colon X\to Y$ is, for any $(\lambda,(\mu,w))\in\mathcal{O}$, the sum of an invertible linear operator and a compact linear operator. It is therefore a Fredholm operator of index zero (see [6, Theorem 2.7.6]). Thus assumption (H_3) is indeed satisfied.

We now verify assumption (H_4) for the sequence $\{\mathcal{Q}_j\}_{j\geqslant 1}$ given by

$$\mathcal{Q}_j=\left\{(\lambda,(\mu,w))\in\mathcal{O}:\|(\lambda,(\mu,w))\|\leqslant j \text{ and } \mu+\lambda^2-2gw(x)\geqslant\frac{1}{j} \text{ for all } x\in\mathbb{R}\right\}. \quad (3.22)$$

It is obvious that each $\mathcal{Q}_j$ is bounded and closed, with $\bigcup_{j\in J}\mathcal{Q}_j=\mathcal{O}$. Let $j\in\mathbb{N}$ be arbitrary and consider $(\lambda,(\mu,w))\in\mathcal{Q}_j$ such that $F(\lambda,(\mu,w))=0$. Then we have in particular that

$$2(\mu+\lambda^2-2gw)\mathcal{C}_{kh}(w')+4[w\mathcal{C}_{kh}(w')]+J(\lambda,w)=0,$$

using the notation introduced earlier in the proof. Since

$$\mu+\lambda^2-2gw\geqslant\frac{1}{j}>0,$$

one may invert the linear operator $w\mapsto\mathcal{C}_{kh}(w')$ in this equation to get

$$w=-(\mathcal{C}_{kh}\partial_x)^{-1}\left(\frac{1}{\mu+\lambda^2-2gw}(J(\lambda,w)+4[w\mathcal{C}_{kh}(w')])\right).$$

Combining the bounds ensured by the definition of $\mathcal{Q}_j$ with the commutator estimates satisfied by J_1 and J_2 that we have used above, we obtain a uniform upper bound for w in the space $C^{3,\alpha/2}_{2\pi}(\mathbb{R})$. This implies that $\{(\lambda,(\mu,w))\in\mathcal{O}:F(\lambda,(\mu,w))=0\}\cap\mathcal{Q}_j$ is bounded in $\mathbb{R}\times\mathbb{R}\times C^{3,\alpha/2}_{2\pi}(\mathbb{R})$, and therefore compact in $\mathbb{R}\times X$. Thus assumption (H_4) is satisfied by the sequence of sets given by (3.22).

We have thus checked that the assumptions (H_1)–(H_4) are satisfied in the setting (3.11)–(3.13). In view of the definition of the set $\mathcal{O}$ in (3.14), the relation between (λ,μ,w) and (m,Q,v) expressed by (3.7), (3.9) and (3.10), and the definition of the sets $\mathcal{Q}_j$ in (3.22), the conclusion of Theorem 6 can be rephrased in a straightforward way to yield the result claimed by Theorem 5. □

Remark 9. By bootstrapping the improvement of the regularity that is based on Lemma 17 from Appendix A in the above proof, one can easily show in addition that, for any solution (m,Q,v) of (2.22), the function v is necessarily of class C^∞ on $\mathbb{R}$.

4. Nodal analysis

In this section we investigate qualitative features of the wave profiles corresponding to points lying on the solution curves $\mathcal{K}_{n,\pm}$, while also addressing the question of the validity of (2.18c)–(2.18e).

Firstly, the waves constructed are symmetric, since the function space in Theorem 3 is $C^{2,\alpha}_{2\pi,e}(\mathbb{R})$. Furthermore, for any wave in $\mathcal{K}_{n,\pm}$ except for the laminar flow, the elevation of the free surface is strictly decreasing between any wave crest and the successive wave trough. For waves that are close to the bifurcating laminar flow, this is a direct consequence of local bifurcation, while away from the bifurcating flow with a flat free surface this will be proved in what follows by means of a continuation argument.

LEMMA 10. (Periodicity) *Let $h,k>0$ and $\Upsilon\in\mathbb{R}$ be given. For each integer $n\geqslant 1$ and both choices of sign $\pm$, denote by*

$$\mathcal{K}_{n,\pm}=\{(m(s),Q(s),v_s):s\in\mathbb{R}\} \tag{4.1}$$

the continuous curve of solutions of (2.22) in the space $\mathbb{R}\times\mathbb{R}\times C^{2,\alpha}_{2\pi,e}(\mathbb{R})$ given by Theorem 5. Then the following additional properties hold along $\mathcal{K}_{n,\pm}$:

(i) *v_s is periodic of period $2\pi/n$, for each $s\in\mathbb{R}$;*

(ii) *$m(-s)=m(s)$, $Q(-s)=Q(s)$, and $v_{-s}(x)=v_s(x+\pi/n)$ for all $x\in\mathbb{R}$, for each $s\in\mathbb{R}$.*

Proof. (i) Fix some arbitrary integer $n\geqslant 2$. Consider again equations (3.11)–(3.13), but this time in the setting of even functions of period $2\pi/n$, that is, with the spaces X and Y being replaced by $\widetilde{X}$ and $\widetilde{Y}$ given by

$$\widetilde{X}=\mathbb{R}\times C^{2,\alpha}_{2\pi/n,\circ,e}(\mathbb{R}) \quad\text{and}\quad \widetilde{Y}=C^{1,\alpha}_{2\pi/n,\circ,e}(\mathbb{R})\times\mathbb{R}, \tag{4.2}$$

where the notation has the obvious meaning. Then it is immediate to check that Theorem 6 is applicable in the new setting, and global real-analytic bifurcation takes place exactly from the local bifurcation points $\{\lambda^*_{np,\pm}\}_{p\geqslant 1}$ given by (3.18). This leads to the existence of global bifurcation curves $\widetilde{\mathcal{K}}_{p,\pm}$ for each $p\geqslant 1$ of solutions to (2.22) with the properties as in Theorem 5. Since $\widetilde{X}\subset X$ and $\widetilde{Y}\subset Y$, it follows that $\widetilde{\mathcal{K}}_{1,\pm}$ are curves of solutions of (2.22) in the space X too, with properties analogous to (i)–(vi) as satisfied by $\mathcal{K}_{n,\pm}$. The uniqueness claim in Theorem 5 then ensures that $\widetilde{\mathcal{K}}_{1,\pm}=\mathcal{K}_{n,\pm}$, up to reparametrization. It therefore follows that $\mathcal{K}_{n,\pm}\subset\widetilde{X}$, which is the required result.

(ii) Fix some arbitrary integer $n\geqslant 1$. It is easy to check that

$$\mathbb{R}\ni s\longmapsto (m(-s),Q(-s),v_{-s}(\cdot+\pi/n))$$

is a curve of solutions of (2.22) satisfying the properties (i)–(vi) in Theorem 5. The required result is a consequence of the uniqueness claim in Theorem 5. □

For simplicity, we will concentrate in what follows on discussing qualitative properties of the solutions on the curves $\mathcal{K}_{1,\pm}$. Similar properties (with obvious modifications) can be proven for solutions on the curves $\mathcal{K}_{n,\pm}$ for any integer $n\geqslant 2$ if one works in the space of functions of period $2\pi/n$ (this choice being justified by Lemma 10 (i)). Also denoting, for both choices of sign $\pm$ and for $n=1$,

$$\mathcal{K}_{1,\pm}=\mathcal{K}^<_\pm\cup\{(m^*_{1,\pm},Q^*_{1,\pm},h)\}\cup\mathcal{K}^>_\pm, \tag{4.3}$$

where

$$\begin{aligned}\mathcal{K}^<_\pm&=\{(m(s),Q(s),v_s):s\in(-\infty,0)\},\\ \mathcal{K}^>_\pm&=\{(m(s),Q(s),v_s):s\in(0,\infty)\},\end{aligned} \tag{4.4}$$

it suffices, because of Lemma 10(ii), to study the properties of the solutions on $\mathcal{K}^>_\pm$.

For any function $v\in C^{2,\alpha}_{2\pi,e}$ with $[v]=h$, we consider in $\overline{\mathcal{R}}_{kh}$ the functions U and V as defined in §2. Then the evenness of v implies that

$$x\longmapsto V(x,y) \text{ is an even function, for each } y\in[-kh,0] \tag{4.5}$$

and that the arbitrary additive constant in the definition of U may be chosen so that

$$x\longmapsto U(x,y) \text{ is an odd function, for each } y\in[-kh,0]. \tag{4.6}$$

This ensures that (2.10) holds. It follows in particular that

$$U(0,y)=0 \quad \text{for all } y\in[-kh,0], \tag{4.7}$$

and as a consequence of (2.4) that

$$U(m\pi,y)=\frac{m\pi}{k} \quad \text{for all } y\in[-kh,0] \text{ and } m\in\mathbb{Z}. \tag{4.8}$$

Lemma 11. (Injectivity) *If*

$$V_x(x,0) \neq 0 \quad \textit{for all } x \in (0,\pi), \tag{4.9}$$

then the injectivity condition (2.6) *holds if and only if*

$$0 < U(x,0) < \frac{\pi}{k} \quad \textit{for all } x \in (0,\pi). \tag{4.10}$$

Proof. Suppose first that condition (2.6) holds. If there exists $\tilde{x}_0 \in (0,\pi)$ such that $U(\tilde{x}_0,0) \leqslant 0$, then the continuity of $x \mapsto U(x,0)$ implies the existence of $x_0 \in [\tilde{x}_0,\pi)$ such that $U(x_0,0)=0$. It then follows that

$$(U(x_0,0),V(x_0,0)) = (0,V(x_0,0)) = (U(-x_0,0),V(-x_0,0)),$$

which contradicts condition (2.6). If, on the other hand, there exists $\tilde{x}_0 \in (0,\pi)$ such that $U(\tilde{x}_0,0) \geqslant \pi/k$, then the continuity of $x \mapsto U(x,0)$ implies the existence of $x_0 \in (0,\tilde{x}_0]$ such that $U(x_0,0)=\pi/k$. It then follows, because of (4.5), (4.6) and (2.4), that

$$(U(x_0,0),V(x_0,0)) = \left(U(-x_0,0) + \frac{2\pi}{k}, V(-x_0,0) \right) = (U(2\pi - x_0,0), V(2\pi - x_0,0)),$$

which also contradicts (2.6). In conclusion, (4.10) is valid.

For the converse, suppose that (4.10) holds, and let x_1 and x_2 be any real numbers such that $(U(x_1,0),V(x_1,0))=(U(x_2,0),V(x_2,0))$. It is a consequence of (4.10), (4.6) and (2.4) that, for any $n \in \mathbb{Z}$,

$$U(x,0) \in \left[\frac{n\pi}{k}, \frac{(n+1)\pi}{k} \right) \quad \text{for all } x \in [n\pi,(n+1)\pi).$$

As $U(x_1,0)=U(x_2,0)$, it follows that there exists $m \in \mathbb{Z}$ such that $x_1,x_2 \in [m\pi,(m+1)\pi)$. Now, using the fact that $V(x_1,0)=V(x_2,0)$ and assumption (4.9), which implies that $V_x(x,0) \neq 0$ for any $x \in (m\pi,(m+1)\pi)$, it follows that necessarily $x_1=x_2$. This shows that the injectivity condition (2.6) holds, as required. □

We shall see in Lemma 12 below that (4.9) and (4.10) are satisfied by the solutions on the bifurcation curve $\mathcal{K}_{1,\pm}$ that are close enough to the trivial solution $v \equiv h$. Of course, in order for v to give rise to a water wave it is also necessary that

$$V(x,0) > 0 \quad \text{for all } x \in \mathbb{R}. \tag{4.11}$$

The previous discussion leads us to consider the following seven properties of a pair $(m,v) \in \mathbb{R} \times C^{2,\alpha}_{2\pi,e}$:

$$v(x) > 0 \text{ for all } x \in \mathbb{R}, \tag{4.12}$$

$$v \not\equiv h, \tag{4.13}$$

$$v'(x) < 0 \quad \text{for all } x \in (0,\pi), \tag{4.14}$$

$$v''(0) < 0 \quad \text{and} \quad v''(\pi) > 0, \tag{4.15}$$

$$0 < \frac{x}{k} + (\mathcal{C}_{kh}(v-[v]))(x) < \frac{\pi}{k} \quad \text{for all } x \in (0,\pi), \tag{4.16}$$

$$\frac{1}{k} + (\mathcal{C}_{kh}(v'))(0) > 0 \quad \text{and} \quad \frac{1}{k} + (\mathcal{C}_{kh}(v'))(\pi) > 0, \tag{4.17}$$

$$\pm\left(\frac{m}{kh} - \frac{\Upsilon}{2kh}[v^2] - \Upsilon\mathcal{C}_{kh}(vv') + \Upsilon v\left(\frac{1}{k} + \mathcal{C}_{kh}(v')\right)\right) > 0 \quad \text{for all } x \in \mathbb{R}. \tag{4.18}$$

Note that, for solutions (m,Q,v) of (2.20), condition (4.18) is equivalent to

$$Q - 2gv(x) \neq 0 \quad \text{and} \quad (v'(x))^2 + \left(\frac{1}{k} + (\mathcal{C}_{kh}(v'))(x)\right)^2 \neq 0 \quad \text{for all } x \in \mathbb{R}.$$

Because of Lemma 11 and the discussion in §2, any solution in $\mathcal{K}_\pm^>$ that satisfies (4.12)–(4.18) corresponds to a water wave.

We will study to what extent these properties are satisfied along $\mathcal{K}_\pm^>$. To that aim, it is convenient to define the sets

$$\mathcal{V}_\pm = \{(m,Q,v) \in \mathbb{R} \times \mathbb{R} \times C^{2,\alpha}_{2\pi,e}(\mathbb{R}) : (4.12)\text{–}(4.18) \text{ hold}\} \tag{4.19}$$

the choice of sign in $\mathcal{V}_\pm$ being the same as that in (4.18). Note that $\mathcal{V}_\pm$ are *open sets* in $\mathbb{R} \times \mathbb{R} \times C^{2,\alpha}_{2\pi,e}(\mathbb{R})$. (It may be worth pointing out here that, while condition (4.14) does *not* define an open set in $C^{2,\alpha}_{2\pi,e}(\mathbb{R})$, the combination of conditions (4.14)) and (4.15) defines an open set; a similar statement holds for conditions (4.16) and (4.17).) The next lemma deals with solutions on $\mathcal{K}_\pm^>$ which are close to the trivial one.

LEMMA 12. (Local properties) *For either choice of sign* $\pm$, *let*

$$\mathcal{K}_\pm^> = \{(m(s), Q(s), v_s) : s \in (0,\infty)\}$$

be the curve of solutions of (2.22) *given by* (4.4). *Then there exists* $\varepsilon > 0$ *sufficiently small such that*

$$\{(m(s), Q(s), v_s) : s \in (0,\varepsilon)\} \subset \mathcal{V}_\pm. \tag{4.20}$$

Proof. Since the set of (m,v) defined by the conditions (4.12), (4.16), (4.17), and (4.18) is open in $\mathbb{R} \times C^{2,\alpha}_{2\pi,e}(\mathbb{R})$, and the mapping $s \mapsto (m(s), v_s)$ is continuous from $\mathbb{R}$ into $\mathbb{R} \times C^{2,\alpha}_{2\pi,e}(\mathbb{R})$, the fact that those conditions are satisfied at $s=0$ implies that they are satisfied on $(-\varepsilon_1, \varepsilon_1)$, for some $\varepsilon_1 > 0$ sufficiently small. Similarly, note that the set

$$\mathcal{Y} = \{u \in C^{2,\alpha}_{2\pi,e}(\mathbb{R}) : u \not\equiv 0, u'(x) < 0 \text{ for all } x \in (0,\pi), u''(0) < 0 \text{ and } u''(\pi) > 0\}$$

is open in $C^{2,\alpha}_{2\pi,e}(\mathbb{R})$ and the mapping

$$s \mapsto u_s = \begin{cases} \dfrac{v_s - h}{s} & \text{for } s \neq 0, \\ (x \mapsto \cos x) & \text{for } s = 0, \end{cases}$$

is continuous from $\mathbb{R}$ into $C^{2,\alpha}_{2\pi,e}(\mathbb{R})$. Thus, since Theorem 3 ensures $u_0 \in \mathcal{Y}$, we deduce that $u_s \in \mathcal{Y}$ for all $s \in (-\varepsilon_2, \varepsilon_2)$, for some $\varepsilon_2 > 0$ sufficiently small. It follows that v_s satisfies (4.13)–(4.15) for all $s \in (0, \varepsilon_2)$. Setting $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\} > 0$ yields the claimed result. □

Fix some $\varepsilon > 0$ given by Lemma 12, and let us define

$$\mathcal{K}^{>}_{\pm,\mathrm{loc}} = \{(m(s), Q(s), v_s) : s \in (0, \varepsilon)\}.$$

Let us reparametrize $\mathcal{K}^{<}_{\pm,\mathrm{loc}}$ such that Proposition 10 (ii) holds, and set

$$\mathcal{K}^{<}_{\pm,\mathrm{loc}} = \{(m(s), Q(s), v_s) : s \in (-\varepsilon, 0)\}.$$

The main result of this section is the following theorem.

Theorem 13. (Global continuation) *For either choice of sign* $\pm$, *let*

$$\mathcal{K}^{>}_{\pm} = \{(m(s), Q(s), v_s) : s \in (0, \infty)\}$$

be the curve of solution of (2.22) *given by* (4.4). *Then one of the following alternatives occurs*:

(A_1) $\mathcal{K}^{>}_{\pm} \subset \mathcal{V}_{\pm}$, *in which case alternative* (α) *in Theorem* 5 *occurs*;

(A_2) *there exists some* $s^* \in (0, \infty)$ *such that* $\{(m(s), Q(s), v_s) : s \in (0, s^*)\} \subset \mathcal{V}_{\pm}$, *while* $(m(s^*), Q(s^*), v_{s^*})$ *satisfies* (4.12)–(4.15), (4.17) *and* (4.18), *and instead of* (4.16) *it satisfies*

$$0 < \frac{x}{k} + (\mathcal{C}_{kh}(v-h))(x) \leqslant \frac{\pi}{k} \quad \textit{for all } x \in (0, \pi), \tag{4.21a}$$

$$\frac{x_0}{k} + (\mathcal{C}_{kh}(v-h))(x_0) = \frac{\pi}{k} \quad \textit{for some } x_0 \in (0, \pi). \tag{4.21b}$$

Remark 14. (i) Roughly speaking, alternative (A_1) means that all the solutions on $\mathcal{K}^{>}_{\pm}$ correspond to physical water waves that are symmetric and whose vertical coordinate strictly decreases between each of its consecutive global maxima and minima, which are unique per minimal period. However, since we make no claim that the horizontal coordinate is strictly monotone, such waves could have overhanging profiles. The existence of overhanging waves is strongly suggested by numerical simulations and remains an important open problem. More generally, the behaviour of the solutions on $\mathcal{K}_\pm^>$ as $s\to\infty$ remains another important open problem.

(ii) Alternative (A_2) means that solutions on $\mathcal{K}_\pm^>$ that correspond to physical water waves with qualitative properties as described above do exist until a limiting configuration with a profile that self-intersects on the line strictly above the trough is reached at $s=s^*$. Indeed, let us denote $u(x)=U(x,0)$ for $x\in\mathbb{R}$. By (2.10), (4.21b) and (4.8),

$$u(x_0)=\frac{x_0}{k}+(\mathcal{C}_{hk}(v-h))(x_0)=\frac{\pi}{k}=u(\pi).$$

This implies that the physical point $(u(x_0),v(x_0))\in\mathcal{S}$ lies directly above the trough (which is at $(u(\pi),v(\pi))$). By (2.4) and (4.6),

$$u(2\pi-x_0)=u(-x_0)+\frac{2\pi}{k}=\frac{2\pi}{k}-u(x_0)=u(x_0),$$

while (2.4) and (4.5) yield $v(2\pi-x_0)=v(-x_0)=v(x_0)$. Thus the curve $\mathcal{S}$ intersects itself at the physical point $(u(x_0),v(x_0))$, which lies directly above the trough.

Proof of Theorem 13. Fix a choice of sign $\pm$. In what follows we choose the $+$ sign merely for definiteness. All the arguments below can be straightforwardly adapted to the choice of the $-$ sign. For convenience, define

$$I=\{s\in(0,\infty):(m(s),Q(s),v_s)\in\mathcal{V}_+\}.$$

Since $\mathcal{V}_+$ is an open set in $\mathbb{R}\times\mathbb{R}\times C^{2,\alpha}_{2\pi,e}(\mathbb{R})$, it follows that I is an open subinterval of $(0,\infty)$. In case I equals all of $(0,\infty)$, the definition of $\mathcal{V}_+$ implies $\mathcal{K}_+^>\subset\mathcal{V}_+$, so that (β) in Theorem 5 is excluded and therefore (α) is valid.

Hence we are left with the case that the open interval I is not the whole of $(0,\infty)$. With $\varepsilon>0$ given by Lemma 12, we have $(0,\varepsilon)\subset I$. Let s^* be the upper end-point of the largest interval that contains $(0,\varepsilon)$ and is contained in I. Then $(0,s^*)\subset I$ and $s^*\notin I$. In what follows we shall investigate the properties of the solution $(m(s^*),Q(s^*),v_{s^*})$.

We first claim that necessarily $v_{s^*}\not\equiv h$. Suppose on the contrary that $v_{s^*}\equiv h$. Then the point $(m(s^*),Q(s^*),h)$ belongs to $\mathcal{K}_{\text{triv}}$ and is a limit of a sequence of non-trivial solutions. It follows from Theorem 5 that necessarily there exist an integer $n\geqslant1$ and a choice of sign $\pm$ such that $m(s^*)=m^*_{n,\pm}$ and $Q(s^*)=Q^*_{n,\pm}$, where $m^*_{n,\pm}$ and $Q^*_{n,\pm}$ are given by (3.3) and (3.4). At this point note that it is a consequence of Theorem 5(iv) and the periodicity in Lemma 10 (i) that, for any $n\geqslant2$, all non-trivial solutions in a neighbourhood of $(m^*_{n,\pm},Q^*_{n,\pm},h)$ are periodic of period $2\pi/n$. By (4.14), $v'_{s^*}(x)\leqslant0$ for $0\leqslant x\leqslant\pi$. Thus $n=1$. It follows that either $(m(s^*),Q(s^*))=(m^*_{1,+},Q^*_{1,+})$ or $(m(s^*),Q(s^*))=(m^*_{1,-},Q^*_{1,-})$. However the possibility that $(m(s^*),Q(s^*))=(m^*_{1,-},Q^*_{1,-})$ is ruled out by the fact proved

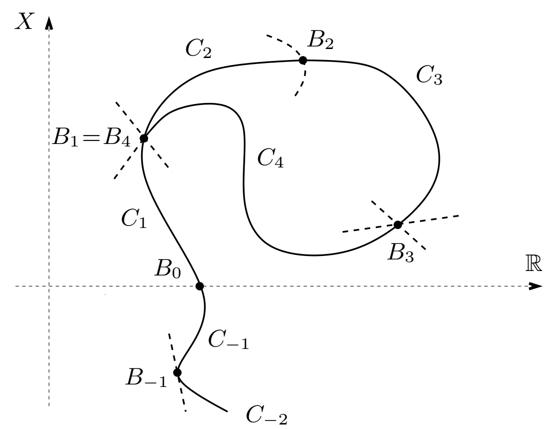

Figure 3. The possibility of a loop is eliminated by a nodal pattern analysis.

in Lemma 12 that all non-trivial solutions in a neighbourhood of $(m^*_{1,-}, Q^*_{1,-}, h)$ satisfy (4.18) with the minus sign, combined with the fact that all solutions on $\mathcal{K}^>_+$ satisfy (4.18) with the plus sign. Thus the only remaining possibility is that $(m(s^*), Q(s^*))=(m^*_{1,+}, Q^*_{1,+})$.

Now all the non-trivial solutions in a neighbourhood of $(m^*_{1,+}, Q^*_{1,+}, h)$ belong to $\mathcal{K}^>_+ \cup \mathcal{K}^<_+$. By Lemma 10 (ii), the solutions on $\mathcal{K}^<_{+,\mathrm{loc}}$ satisfy the opposite of (4.14), so that $\mathcal{K}^<_+$ is excluded. Since $v_0 = v_{s^*} \equiv h$ by Theorem 5, there exist $\delta_1, \delta_2 > 0$ sufficiently small such that $\{(m(s), Q(s), v_s): s \in (0, \delta_1)\} = \{(m(s), Q(s), v_s): s \in (s^* - \delta_2, s^*)\}$. However, this possibility is ruled out by (C_4) in Theorem 6, that is, by the construction of the real-analytic global bifurcation curve in [6]; see Figure 3. We have thus proven that $v_{s^*} \equiv h$ is not possible, which proves the claim.

For notational simplicity, throughout the remainder of the proof we shall denote $(m(s^*), Q(s^*), v_{s^*})$ by (m, Q, v). This is a limit of solutions satisfying (4.12) to (4.18). The definition of I implies that the following inequalities hold:

$$v(x) \geqslant 0 \text{ for all } x \in \mathbb{R}, \quad \text{and } v \text{ is even and of period } 2\pi, \tag{4.22}$$

$$v'(x) \leqslant 0 \text{ for all } x \in [0, \pi], \text{ so that } v' \geqslant 0 \text{ in } [\pi, 2\pi], \tag{4.23}$$

$$0 \leqslant \frac{x}{k} + (\mathcal{C}_{kh}(v-h))(x) \leqslant \frac{\pi}{k} \quad \text{for all } x \in [0, \pi], \tag{4.24}$$

$$\frac{m}{kh} - \frac{\Upsilon}{2kh}[v^2] - \Upsilon \mathcal{C}_{kh}(vv') + \Upsilon v \left(\frac{1}{k} + \mathcal{C}_{kh}(v') \right) \geqslant 0 \quad \text{on } \mathbb{R}. \tag{4.25}$$

By (2.10), (4.24) may be rewritten as

$$0 = U(0,0) \leqslant U(x,0) \leqslant U(\pi,0) = \frac{\pi}{k} \quad \text{for all } x \in [0, \pi]. \tag{4.26}$$

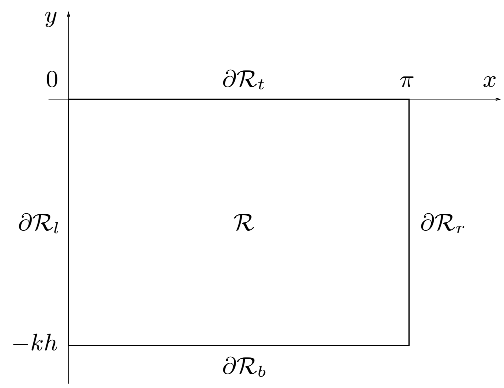

Figure 4. The rectangular domain $\mathcal{R}$, the conformal image of which is the fluid domain between the wave crest and a successive wave trough.

Also, by (2.19) with ζ defined as in §2, (4.25) may be rewritten as

$$\zeta_y(x,0)+\Upsilon V(x,0)V_y(x,0)\geqslant 0 \quad \text{for all } x\in\mathbb{R}. \tag{4.27}$$

By Theorem 1, v also satisfies (2.20), which is equivalent to (2.14).

We are now going to show that (m,Q,v) satisfies the remaining properties claimed in the statement of alternative (A_2) of Theorem 13. The proof will be based on sharp forms of maximum principles in the infinite strip $\mathcal{R}_{kh}$, as well as in the rectangular domain

$$\mathcal{R}=\{(x,y):0<x<\pi \text{ and } -kh<y<0\}, \tag{4.28}$$

whose boundary $\partial\mathcal{R}$ is the union of the four line segments (see Figure 4)

$$\begin{aligned}\partial\mathcal{R}_t&=\{(x,0):0\leqslant x\leqslant\pi\}, & \partial\mathcal{R}_b&=\{(x,-kh):0\leqslant x\leqslant\pi\},\\ \partial\mathcal{R}_l&=\{(0,y):-kh\leqslant y\leqslant 0\}, & \partial\mathcal{R}_r&=\{(\pi,y):-kh\leqslant y\leqslant 0\}.\end{aligned}$$

Recall at this point that, as noted in the last remark of §3, any solution v of (2.22) is of class C^∞ on $\mathbb{R}$.

Suppose on the contrary that (4.12) fails. Then (4.22) and (4.23) imply that $v(\pi)=0$. Thus the harmonic function V in $\mathcal{R}_{kh}$ has a global minimum in $\overline{\mathcal{R}}_{kh}$ at $(\pi,0)$, so that, by the Hopf boundary-point lemma, $V_y(\pi,0)<0$. On the other hand, it is a consequence of (4.26) and the Cauchy–Riemann equations that $V_y(\pi,0)=U_x(\pi,0)\geqslant 0$, which is a contradiction. It follows that (4.12) does hold.

We now claim the strict inequality (4.18) with the + sign; that is,

$$(\zeta_y+\Upsilon VV_y)|_{y=0}=\frac{m}{kh}-\frac{\Upsilon}{2kh}[v^2]-\Upsilon\mathcal{C}_{kh}(vv')+\Upsilon v\left(\frac{1}{k}+\mathcal{C}_{kh}(v')\right)>0 \quad \text{on } \mathbb{R}. \tag{4.29}$$

Recalling from Theorem 5 (iv) that

$$Q-2gv(x)>0 \quad \text{for all } x\in\mathbb{R}, \tag{4.30}$$

and taking (2.14) into account, the validity of (4.29) is equivalent to

$$(\zeta_y+\Upsilon VV_y)^2=(Q-2gV)(V_x^2+V_y^2)>0 \quad \text{at } y=0,$$

or, equivalently

$$v'(x)^2+\left(\frac{1}{k}+(\mathcal{C}_{kh}(v'))(x)\right)^2>0 \quad \text{for all } x\in\mathbb{R}, \tag{4.31}$$

or, also equivalently

$$V_x^2(x,0)+V_y^2(x,0)>0 \quad \text{for all } x\in\mathbb{R}. \tag{4.32}$$

We will prove (4.32) by contradiction. Suppose that there exists a point $x_0\in\mathbb{R}$ such that $V_x^2(x_0,0)+V_y^2(x_0,0)=0$. It follows from (2.14) that

$$\zeta_y(x_0,0)+\Upsilon V(x_0,0)V_y(x_0,0)=0.$$

Combined with (4.27), this ensures that $\zeta_y+\Upsilon VV_y=O((x-x_0)^2)$ as $x\to x_0$ at $y=0$. By (2.14) and (4.30), this implies that $V_x^2(x,0)+V_y^2(x,0)=O((x-x_0)^4)$ as $x\to x_0$. Examining the second derivative of the expression in the right-hand side of (2.14), we see that

$$V_{xx}(x_0,0)=V_{xy}(x_0,0)=0. \tag{4.33}$$

Because of the evenness and periodicity of v, we may assume with no loss of generality that $0\leqslant x_0\leqslant\pi$.

If $x_0=0$, it follows from (4.23) that the harmonic function V in $\mathcal{R}_{kh}$ has at $(0,0)$ a global maximum in $\overline{\mathcal{R}}_{kh}$. By the Hopf boundary-point lemma, $V_y(0,0)>0$. Hence

$$\frac{1}{k}+(\mathcal{C}_{kh}(v'))(0)=V_y(0,0)>0, \tag{4.34}$$

a contradiction. So $x_0\neq0$. Now suppose $0<x_0<\pi$. The harmonic function V_x has its global maximum in $\overline{\mathcal{R}}$ at $(x_0,0)$ because $V_x=0$ on three sides (by oddness and the bottom boundary condition) while $V_x\leqslant0$ on the top $\partial\mathcal{R}_t$ by (4.23). By the Hopf boundary-point lemma, $V_{xy}(x_0,0)>0$, which contradicts (4.33).

It remains to consider the case $x_0=\pi$. Since $x\mapsto V(x,y)$ is even about π, for each $y\in[-kh,0]$, it follows that

$$V_{xxx}(\pi,0)=V_{xyy}(\pi,0)=0. \tag{4.35}$$

The harmonic function V_x in the rectangle $\mathcal{R}$ has its global maximum in $\overline{\mathcal{R}}$ at $(\pi,0)$, as above. It follows from the Serrin corner-point lemma, by taking into account (4.33) and (4.35), that $V_{xxy}(\pi,0)<0$. By the Cauchy–Riemann equations, $U_{xxx}(\pi,0)<0$. Similarly, we also deduce that $U_x(\pi,0)=U_{xx}(\pi,0)=0$, since $V_y(\pi,0)=0$ by assumption and $V_{xy}(\pi,0)=0$ by (4.33). These properties of the derivatives of the function $U(\cdot,0)$ at $(\pi,0)$ imply that

$$U(x,0)-U(\pi,0)=\tfrac{1}{6}U_{xxx}(\pi,0)(x-\pi)^3+O((x-\pi)^4)>0 \quad \text{as } x\nearrow\pi.$$

This contradicts (4.26).

Thus x_0 does not exist and (4.32) is true. As $V_x(\pi,0)=0$, we must have $V_y(\pi,0)\neq 0$. As a consequence of (4.26) and the Cauchy–Riemann equations, $V_y(\pi,0)=U_x(\pi,0)\geqslant 0$. It follows that

$$\frac{1}{k}+(\mathcal{C}_{kh}(v'))(\pi)=V_y(\pi,0)>0. \tag{4.36}$$

We have therefore proved not only (4.18), but also (4.17) and (4.31). It remains to prove (4.14), (4.15) and (4.21).

Recall at this point that the solution $(m(s^*),Q(s^*),v_{s^*})$ is the limit of the sequence

$$(m^j,Q^j,v^j)=(m(s_j),Q(s_j),v_{s_j})$$

in $\mathbb{R}\times\mathbb{R}\times C^{2,\alpha}_{2\pi,e}(\mathbb{R})$, where $s_j\in I$ for all $j\in\mathbb{N}$ and $s_j\nearrow s^*$ as $j\to\infty$. For each $j\in\mathbb{N}$, let V^j and U^j be the associated harmonic functions in $\mathcal{R}_{kh}$. By the definition of I, for each $j\in\mathbb{N}$, we have that $(m^j,Q^j,v^j)\in\mathcal{V}_+$, which implies by Lemma 11 that the mapping $x\mapsto(U^j(x,0),V^j(x,0))$ is injective on $\mathbb{R}$. As noted in [44] and [18], a suitable application of the Darboux–Picard theorem [7, p. 3, Corollary 9.16] yields that U^j+iV^j is a conformal mapping between the strip $\mathcal{R}_{kh}$ and the domain Ω whose boundary consists of the curve

$$\mathcal{S}^j=\{(U^j(x,0),V^j(x,0)):x\in\mathbb{R}\}$$

and the real axis $\mathcal{B}$, that extends continuously from $\overline{\mathcal{R}}_{kh}$ to $\overline{\Omega}$.

As a consequence,

$$(V^j_x)^2+(V^j_y)^2>0 \quad \text{in } \mathcal{R}_{kh}\cup\{(x,-kh):x\in\mathbb{R}\}.$$

On $\{(x,-kh):x\in\mathbb{R}\}$ we have that $V^j=V^j_x=0$ and $V^j_y\neq 0$. Using the maximum principle and the fact that $U+iV$ is not constant, we may pass to the limit as $j\to\infty$ to obtain

$$V_x^2+V_y^2>0 \quad \text{in } \mathcal{R}_{kh}\cup\{(x,-kh):x\in\mathbb{R}\}.$$

By taking (4.32) into account, we therefore deduce that

$$V_x^2+V_y^2>0 \quad \text{in } \overline{\mathcal{R}}_{kh}. \tag{4.37}$$

At this point we state the following lemma, whose proof is somewhat technical. In order not to disrupt the flow of the argument, we choose to take its validity for granted for the moment, and defer proving it until after the proof of Theorem 13.

LEMMA 15. *Let* $(m,Q,v)\in\mathbb{R}\times\mathbb{R}\times C^3_{2\pi,e}(\mathbb{R})$, *with* $[v]=h$ *and* $v\not\equiv h$, *be a solution of* (2.22) *which satisfies* (4.30), (4.37) *and* (4.23). *Then* v *also satisfies* (4.14) *and* (4.15).

As noted earlier in the proof, any solution v of class $C^{2,\alpha}$ of (2.22) is necessarily of class C^∞, and therefore Lemma 15 is applicable. As a consequence of Lemma 15, our solution (m,Q,v), which is in fact $(m(s^*),Q(s^*),v_{s^*})$, satisfies (4.14) and (4.15). It finally remains to prove (4.21). Recall that (4.24) is indeed valid. Seeking a contradiction, let us suppose that (4.21a) fails, so that there exists $x_0\in(0,\pi)$ such that

$$0=\frac{x_0}{k}+\mathcal{C}_{kh}(v-h)(x_0)=U(x_0,0).$$

Then the harmonic function U in $\mathcal{R}$ has at $(x_0,0)$ a global minimum in $\overline{\mathcal{R}}$. Hence it follows from the Hopf lemma that $U_y(x_0,0)<0$. By the Cauchy–Riemann equations, $V_x(x_0,0)>0$, which contradicts (4.23). This proves that (4.21a) is indeed valid. On the other hand, (4.21b) must necessarily also be true, since otherwise, taking into account everything we have proved so far, it would follow that $(m(s^*),Q(s^*),v_{s*})\in\mathcal{V}_+$, a contradiction to the definition of s^*. This completes the proof of Theorem 13, modulo Lemma 15. □

Proof of Lemma 15. By Theorem 1, equation (2.20) is also satisfied. Let ζ be the solution of (2.13) and V the solution of (2.9). Then (2.14) also holds due to the discussion after (2.20). The assumptions (4.30) and (4.37) ensure that

$$\zeta_y=-\Upsilon VV_y\pm(Q-2gV)^{1/2}(V_x^2+V_y^2)^{1/2} \quad \text{at } (x,0) \text{ for all } x\in\mathbb{R}, \tag{4.38}$$

the choice of sign $\pm$ above being the same for all $x\in\mathbb{R}$. The required result will be obtained by applying maximum-principle-type arguments to the function $f:\overline{\mathcal{R}}_{kh}\to\mathbb{R}$ given by

$$f=\frac{V_x\zeta_y-V_y\zeta_x}{V_x^2+V_y^2}. \tag{4.39}$$

(The choice of f is motivated by the formula (2.21) for the velocity of the fluid in a steady water wave. However, here we are not assuming (2.6), so that our solution need not correspond to a water wave.) It is easy to see that f is a harmonic function in

$\mathcal{R}_{kh}$ because it is the imaginary part of σ/θ, where $\sigma=-(\zeta_y+i\zeta_x)$ and $\theta=V_y+iV_x$ are holomorphic functions. Note that $f=0$ on $\partial\mathcal{R}_l\cup\partial\mathcal{R}_b\cup\partial\mathcal{R}_r$, while

$$f=\frac{V_x(\zeta_y+\Upsilon VV_y)}{V_x^2+V_y^2}=\pm\frac{V_x(Q-2gV)^{1/2}}{(V_x^2+V_y^2)^{1/2}}\quad\text{on }\partial\mathcal{R}_t, \tag{4.40}$$

as a consequence of (2.25) and then (4.38). Therefore, in view of the assumptions (4.30) and (4.37), f has a constant sign on $\partial\mathcal{R}_t$ (depending on the choice of sign $\pm$), and $f\not\equiv0$ on $\partial\mathcal{R}_t$. By the maximum principle it follows that f has a strict sign in $\mathcal{R}$.

We will first prove that $v'(x)<0$ for all $x\in(0,\pi)$. Suppose on the contrary that $v'(x_0)=0$ for some $x_0\in(0,\pi)$. Since x_0 is a global maximum for v' on $(0,\pi)$, it follows that $v''(x_0)=0$. Also, it follows from (4.40) that $f(x_0,0)=0$, and thus the harmonic function f in $\mathcal{R}$ has at $(x_0,0)$ a global extremum in $\mathcal{R}\cup\partial\mathcal{R}$. It follows from the Hopf boundary-point lemma that $f_y(x_0,0)\neq0$. We shall now prove, by direct calculation, that $f_y(x_0,0)=0$, thus obtaining a contradiction. To simplify the following calculations, it is convenient to write

$$f=\frac{p}{q},\quad\text{where } p=V_x\zeta_y-V_y\zeta_x \text{ and } q=V_x^2+V_y^2.$$

Then, at every point in $\overline{\mathcal{R}}_{kh}$ we have

$$f_y=\frac{p_yq-pq_y}{q^2}. \tag{4.41}$$

This implies in particular that, at the point $(x_0,0)$, at which $p=0$, $V_x=0$, and $V_{xx}=0$, we have

$$f_y=\frac{p_y}{q}=\frac{V_{xy}\zeta_y-V_y\zeta_{xy}}{V_y^2}. \tag{4.42}$$

We now differentiate equation (4.38) with respect to x to obtain

$$\zeta_{xy}=-\Upsilon(V_xV_y+VV_{xy})\pm\left(\frac{-gV_x(V_x^2+V_y^2)^{1/2}}{(Q-2gV)^{1/2}}+\frac{(Q-2gV)^{1/2}(V_xV_{xx}+V_yV_{xy})}{(V_x^2+V_y^2)^{1/2}}\right) \tag{4.43}$$

at $(x,0)$ for all $x\in\mathbb{R}$. It follows from (4.38) and (4.43) that, at the point $(x_0,0)$, at which we know that $V_x=0$ and $V_{xx}=0$, we have

$$\zeta_y=-\Upsilon VV_y\pm(Q-2gV)^{1/2}|V_y|, \tag{4.44}$$

$$\zeta_{xy}=-\Upsilon VV_{xy}\pm\frac{(Q-2gV)^{1/2}V_yV_{xy}}{|V_y|}, \tag{4.45}$$

from which one can see that $V_{xy}\zeta_y=V_y\zeta_{xy}$ and therefore, by (4.42), that $f_y=0$, thus obtaining a contradiction. This proves that indeed $v'(x)<0$ for all $x\in(0,\pi)$.

We will now prove that $v''(0)<0$ and $v''(\pi)>0$. Since $v'(0)=v'(\pi)=0$ and $v'\leqslant 0$ on $[0,\pi]$, it follows that $v''(0)\leqslant 0$ and $v''(\pi)\geqslant 0$, so we just need to prove that the two inequalities are strict. Suppose on the contrary that $v''(x_0)=0$ for some $x_0\in\{0,\pi\}$. Then $V_x=0$ and $V_{xx}=0$ at $(x_0,0)$. Since the harmonic function f in $\mathcal{R}$ has at $(x_0,0)$ a global extremum in $\mathcal{R}\cup\partial\mathcal{R}$, the Serrin corner-point lemma [10] ensures that not all first and second order derivatives of f can vanish at $(x_0,0)$. We now show however by direct calculation that under the present assumptions all first and second order derivatives of f do vanish at $(x_0,0)$, which will constitute a contradiction. Indeed, because $f=0$ on $\partial\mathcal{R}_l\cup\partial\mathcal{R}_r$, it follows that $f_y=0$ and $f_{yy}=0$ and also that $f_{xx}=0$ on $\partial\mathcal{R}_l\cup\partial\mathcal{R}_r$ since f is harmonic. Thus we can see from (4.40) that each term in the expression for the derivative with respect to x of f at $(x_0,0)$ contains as a factor either V_x or V_{xx}, so that $f_x(x_0,0)=0$. It thus remains to examine the mixed derivative $f_{xy}(x_0,0)$. To accomplish this task, we calculate from (4.41) that at every point in $\overline{\mathcal{R}}_{kh}$ we have

$$f_{xy}=f_{yx}=\frac{(p_{xy}q+p_yq_x-p_xq_y-pq_{xy})q^2-(p_yq-pq_y)2qq_x}{q^4}. \tag{4.46}$$

This implies in particular that at the point $(x_0,0)$, at which

$$\zeta_x=V_x=\zeta_{xy}=V_{xy}=V_{xx}=V_{yy}=p=p_x=p_y=0,$$

we have

$$f_{xy}=\frac{p_{xy}}{q}. \tag{4.47}$$

It is easy to see, when calculating p_{xy} by differentiation in the formula $p=V_x\zeta_y-V_y\zeta_x$, that at the point $(x_0,0)$ six out the eight terms are zero and we are left with

$$p_{xy}=V_{xxy}\zeta_y-V_y\zeta_{xxy}. \tag{4.48}$$

We now calculate $\zeta_{xxy}(x_0,0)$ by differentiating with respect to x in (4.43) and, taking into account that at $(x_0,0)$ we have $V_x=V_{xx}=V_{xy}=0$, we obtain

$$\zeta_{xxy}=-\Upsilon VV_{xxy}\pm\frac{(Q-2gV)^{1/2}V_yV_{xxy}}{|V_y|}. \tag{4.49}$$

Because (4.44) is also valid at $(x_0,0)$, we see that $V_{xxy}\zeta_y=V_y\zeta_{xxy}$. Therefore by (4.47) and (4.48) we have $f_{xy}(x_0,0)=0$. Thus all first and second derivatives of f vanish at $(x_0,0)$, a contradiction. As explained earlier, this implies that indeed $v''(0)<0$ and $v''(\pi)>0$. □

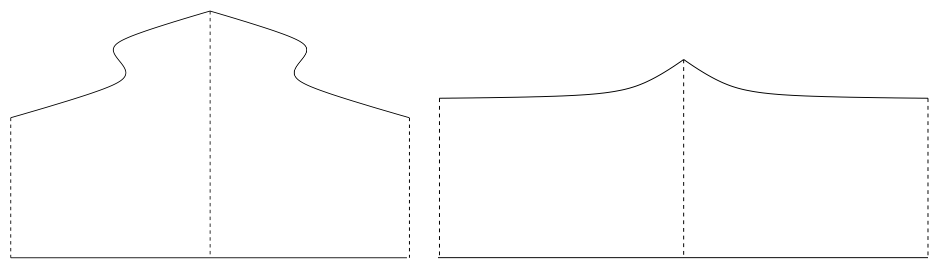

Figure 5. Waves with stagnation points and corners of 120° at their crests: overhanging profiles (on the left) and profiles that are graphs (on the right).

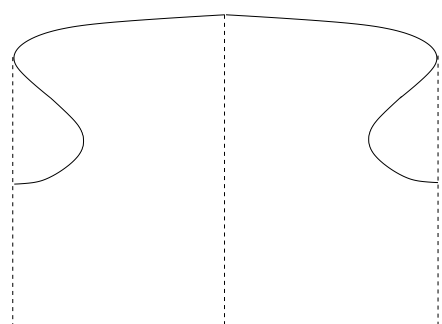

Figure 6. Overhanging wave with self-intersections on the trough line.

We conclude this section by pointing out that many questions about the global behaviour of the solution curves $\mathcal{K}_{n,\pm}$ remain to be elucidated. In a sequel to this paper, we will prove the following results:

- If $\mathcal{K}_{\pm}^{>}$ is unbounded in $\mathbb{R}\times\mathbb{R}\times C_{2\pi}^{2,\alpha}(\mathbb{R})$, then v' is unbounded in $L_{2\pi}^{2}(\mathbb{R})$;
- For all waves in $\mathcal{K}_{n,\pm}$, $n\in\mathbb{N}$, the free surface $\mathcal{S}$ is a real-analytic curve.

Furthermore, we have a conjecture about the furthest boundary of the global curve $\mathcal{K}_{\pm}^{>}$ that is more specific than Theorem 5. Our conjecture is that at this boundary we reach

- either a wave with a stagnation point and a corner of 120° at its crest, whose surface may be overhanging or a graph (see Figure 5),
- or a wave that has no stagnation point but its surface is overhanging with self-intersections on the trough line (see Figure 6).

This conjecture is supported by some analysis and by the numerical simulations in [36] and [41].

5. The flow beneath a wave of small amplitude

In this section we point out some features of the small amplitude waves whose existence was established in §3. As before, we only discuss the bifurcation from the two eigenvalues that are closest to zero on either side, so that throughout this section $n=1$. For simplicity, we refrain from using the subscript 1 when referring to $\lambda^*_{1,\pm}$ or to $\mathcal{K}_{1,\pm}$.

By restricting the local bifurcation curve to a sufficiently small neighborhood of $(\lambda^*_\pm, 0)\in\mathbb{R}\times X$, the condition $v'\neq 0$ in $(0,\pi)$ will hold everywhere on it, except for the bifurcating laminar flow (with a flat surface profile) corresponding to $s=0$. For $s\neq 0$ the wave profiles $\mathcal{S}$ obtained thereby are graphs that are strictly monotone between crests and troughs and are symmetric about the crest line. For $s>0$ the crest is located at $x=0$, while for $s<0$ the trough is at $x=0$. As for the velocity field in the fluid, for a given fluid domain Ω beneath the surface $\mathcal{S}$ and above the flat bed $Y=0$, the boundary-value problem (2.2a)–(2.2c) with L-periodicity in the X-variable has a unique solution. The symmetry of $\mathcal{S}$ implies that the stream function ψ is even in the X-variable.

Recall that by a *critical point* we mean a point where $\psi_Y=0$, while a critical layer is a curve of critical points, and by a *stagnation point* we mean a point where $\psi_X=\psi_Y=0$. We will state the results for the local bifurcating curve $\mathcal{K}_{-,\mathrm{loc}}$, corresponding to the choice of the minus sign in (3.18), the case of $\mathcal{K}_{+,\mathrm{loc}}$ being obtained by interchanging $\pm$ in $\mathcal{K}_\pm$ and in the sign of Υ.

THEOREM 16. (The local curve $\mathcal{K}_{-,\mathrm{loc}}$) *The following statements hold*:

(i) *A laminar flow on $\mathcal{K}_-$ that is a bifurcation point admits critical points if and only if $\Upsilon<0$ and*

$$\frac{\tanh(kh)}{kh}\leqslant\frac{\Upsilon^2 h}{g+\Upsilon^2 h}. \tag{5.1}$$

If they exist, all critical points are located on a unique horizontal line beneath the flat free surface.

(ii) *If either $\Upsilon>0$, or $\Upsilon<0$ is such that* (5.1) *fails, then the flows on the local bifurcating curve $\mathcal{K}_{-,\mathrm{loc}}$ that are sufficiently close to the bifurcating laminar flow have no critical points and the streamlines foliate the fluid domain* (*see Figure* 7).

(iii) *If $\Upsilon<0$ and if the laminar flow at the bifurcation point has a critical line* strictly above *the flat bed, then the nearby waves on $\mathcal{K}_{-,\mathrm{loc}}$ have a cat's eye structure with critical points* (*see Figure* 8).

(iv) *If $\Upsilon<0$ and if the flat bed is itself a critical line of the laminar flow at the bifurcation point, then the nearby waves on $\mathcal{K}_{-,\mathrm{loc}}$ have an isolated region of flow reversal near the bed, delimited by a critical layer* (*see Figure* 9).

Proof. (i) The stream function $\psi(X,Y)$ of the *laminar flow* corresponding to the bifurcation point $(\lambda_-^*, 0, 0)$ solves (2.2) with $\psi_X=0$ and $m=m_-^*=\lambda_-^* h-\frac{1}{2}\Upsilon h^2$ throughout $\Omega=\{(X,Y):0\leqslant Y\leqslant h\}$. We have explicitly

$$\psi(X,Y)=\tfrac{1}{2}\Upsilon Y^2+(\lambda_-^*-\Upsilon h)Y-\lambda_-^* h+\tfrac{1}{2}\Upsilon h^2, \quad 0\leqslant Y\leqslant h, \tag{5.2}$$

the corresponding velocity field being

$$(\psi_Y, -\psi_X)=(\Upsilon Y+\lambda_-^*-\Upsilon h, 0), \quad 0\leqslant Y\leqslant h. \tag{5.3}$$

Since $\lambda_-^*<0$ by (3.18), there are no critical points in the irrotational case $\Upsilon=0$. On the other hand, for $\Upsilon\neq 0$, it is easily seen that critical points exist in the bifurcating laminar flow corresponding to λ_-^* if and only if

$$h\geqslant\frac{\lambda_-^*}{\Upsilon}\geqslant 0. \tag{5.4}$$

If (5.4) is satisfied, then necessarily $\Upsilon<0$, and the critical points lie on the critical line

$$Y=h-\frac{\lambda_-^*}{\Upsilon} \tag{5.5}$$

that is located *beneath* the flat free surface $Y=h$. Straightforward manipulations starting from (3.18) show that (5.4) is satisfied if and only if $\Upsilon<0$ and (5.1) holds.

If $\Upsilon<0$, we remark that since the function $s\mapsto\tanh(s)/s$ is a strictly decreasing bijection from $(0,\infty)$ onto $(0,1)$, critical layers in the bifurcating laminar flow occur if and only if k exceeds a certain critical value; since $L=2\pi/k$, this means that critical layers occur whenever the wavelength is sufficiently small.

(ii) If $\Upsilon>0$, then (5.3) shows that the maximum of ψ_Y throughout the laminar flow at the bifurcation point is attained at the flat free surface $Y=h$, where $\psi_Y(h)=\lambda_-^*<0$, so that $\psi_Y<0$ throughout the flow. If $\Upsilon<0$ is such that (5.1) fails, then (5.3) with the choice λ_-^* shows that the maximum of ψ_Y throughout the laminar flow at the bifurcation point is attained on the flat bed $Y=0$, with the inequality opposite to (5.1) ensuring that $\psi_Y<0$ there. Consequently, we have again that $\psi_Y<0$ throughout the laminar flow at the bifurcation point. Using (2.21), we infer that this inequality will persist along the curve $\mathcal{K}_-$ of non-trivial solutions, provided that these are sufficiently close to the bifurcating laminar flow. All these waves will therefore not present critical points in the flow, with the streamlines in this case providing a foliation of the fluid domain. Typical streamline patterns for the laminar flows with bifurcation parameter λ_-^* are depicted in Figure 7.

On the other hand, for a bifurcating laminar flow with critical points in the case $\Upsilon<0$, the streamline pattern for a wave corresponding to a non-trivial solution on $\mathcal{K}_{-,\mathrm{loc}}$

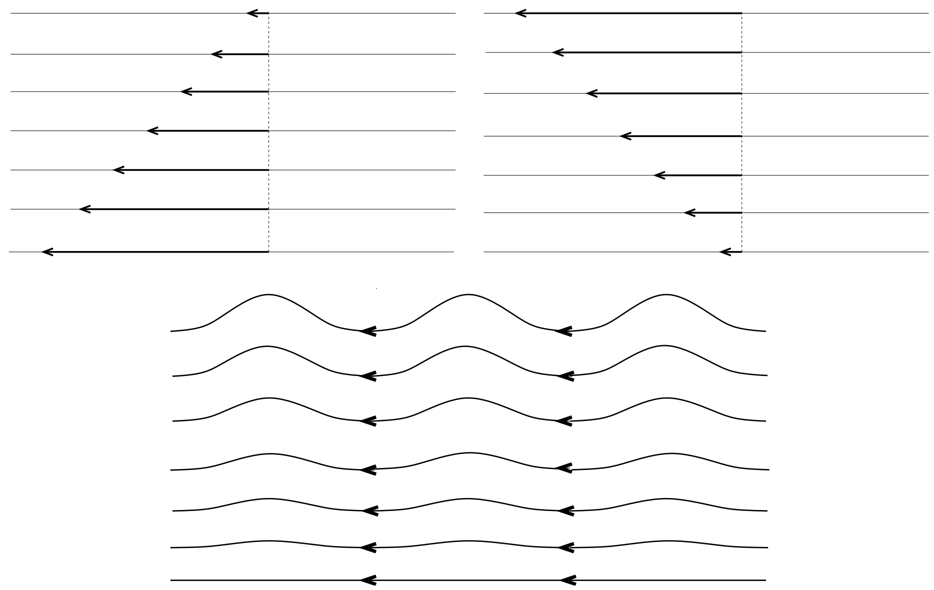

Figure 7. Streamlines of a flow of constant vorticity Υ, in the absence of critical points: On the top left laminar flows for $\Upsilon>0$, on the top right laminar flows for $\Upsilon<0$, and below nearby waves of small amplitude, lying in function space on the local bifurcation curve. In both settings the horizontal fluid velocity is negative throughout the flow.

is in marked contrast to that familiar from flows without critical points. Indeed, the previous considerations show that the bifurcation parameter λ_-^* is negative. Moreover, from (5.3) we get that the horizontal fluid velocity ψ_Y of this laminar flow takes on the value $\psi_Y(h)=\lambda_-^*<0$ at the surface $Y=h$, with $\psi_Y(0)=\lambda_-^*-\Upsilon h\geqslant 0$ on the bed $Y=0$, the latter inequality being a restatement of (5.4).

In the specific scenario (iii) the stream function ψ of the laminar bifurcating flow is such that $\psi_{YY}<0$ throughout the fluid, with $\psi_Y<0$ near the surface and $\psi_Y>0$ near the bed. All these inequalities will persist for the stream functions associated with the flows along the curve $\mathcal{K}_-$ of non-trivial solutions, provided that these flows are sufficiently close to the bifurcating laminar flow. Let $Y=\eta(X)$ be the free surface of such a flow, with the wave crest located at $X=0$ and the wave troughs at $X=\pm\frac{1}{2}L$. (Such a wave corresponds to a positive value of the parameter s along $\mathcal{K}_-$; in view of Lemma 10 (ii), a similar analysis can be carried out for waves corresponding to negative values of s.) At every fixed X, the corresponding real-analytic stream function ψ is such that the function $Y\mapsto\psi(X,Y)$ is strictly concave on $[0,\eta(X)]$, with $\psi_Y(X,0)>0$ and $\psi_Y(X,\eta(X))<0$. Consequently, there is a unique maximum point $Y_0(X)\in(0,\eta(X))$ of this function, with $\psi_Y(X,Y_0(X))=0$, and $\psi_Y(X,Y)>0$ for $0\leqslant Y<Y_0(X)$, while $\psi_Y(X,Y)<0$ for $Y\in(Y_0(X),\eta(X)]$. Since $\psi_{YY}<0$ throughout the flow, the implicit function theorem [6] ensures that the critical layer

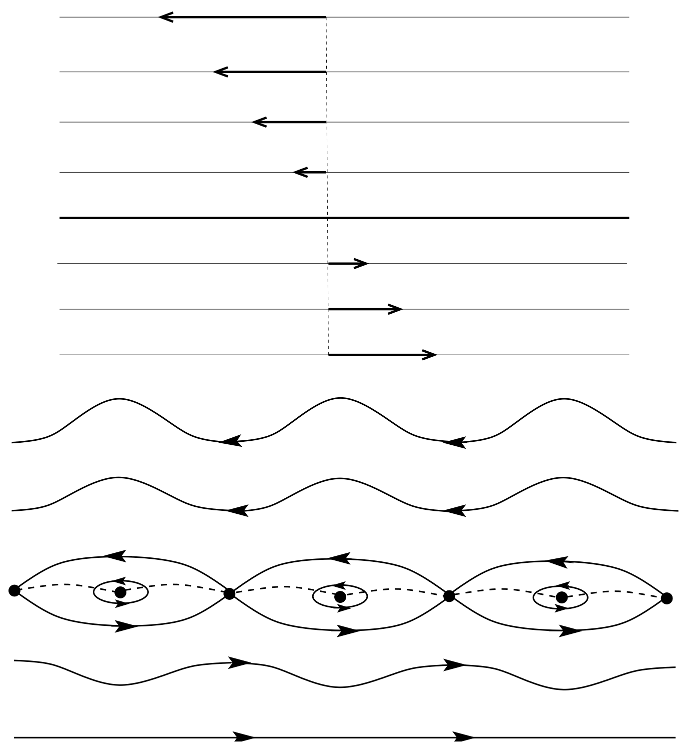

Figure 8. Flows with constant vorticity $\Upsilon<0$, with critical points above the flat bed: On top, for a laminar flow the horizontal critical layer consists of stagnation points that mark the flow reversal, while below, for a nearby wave of small amplitude (lying in function space on the local bifurcation curve) only the points of the critical layer that are beneath the crest or the trough are stagnation points. In the moving frame the stagnation point beneath the crest is surrounded by closed streamlines (Kelvin's "cat's eye" flow pattern). The critical layer, depicted by the dashed curve, delimits an upper region of negative horizontal fluid velocity, and beneath it the flow direction is reversed.

$\{(X,Y)\in\Omega:\psi_Y(X,Y)=0\}$ coincides with the graph of the real-analytic map $X\mapsto Y_0(X)$.

On the other hand, by (2.2b) we have $\psi(X,\eta(X))=0$ for all $X\in\mathbb{R}$. Differentiating this relation with respect to X and taking into account the fact that $\psi_Y(X,\eta(X))<0$ while $\eta'(X)<0$ for $X\in\bigl(0,\frac{1}{2}L\bigr)$ in view of (4.14), we deduce that $\psi_X(X,\eta(X))<0$ for all $X\in\bigl(0,\frac{1}{2}L\bigr)$. The boundary condition (2.2c) and the fact that ψ is even and L-periodic in the X-variable yield $\psi_X(0,Y)=0$ for $Y\in[0,\eta(0)]$, $\psi_X(X,0)=0$ for $X\in\bigl[0,\frac{1}{2}L\bigr]$, and $\psi_X\bigl(\frac{1}{2}L,Y\bigr)=0$ for $Y\in\bigl[0,\eta\bigl(\frac{1}{2}L\bigr)\bigr]$. Applying the strong maximum principle to the harmonic function ψ_X in the domain

$$\Omega_+=\bigl\{(X,Y):0<X<\tfrac{1}{2}L \text{ and } 0<Y<\eta(X)\bigr\},$$

we conclude that $\psi_X(X,Y)<0$ in Ω_+. The fact that ψ is even in the X-variable yields

$\psi_X(X,Y)>0$ in

$$\Omega_- = \left\{(X,Y) : -\tfrac{1}{2}L < X < 0 \text{ and } 0 < Y < \eta(X)\right\}.$$

These considerations show that the only stagnation points of this flow are the points on the critical layer that lie beneath the wave troughs and crests.

To describe qualitatively the streamline pattern of the flow, notice first that

$$\partial_X(\psi(X,Y_0(X))) = \psi_X(X,Y_0(X))$$

is positive in Ω_- and negative in Ω_+. Consequently the quantity $M(X)=\psi(X,Y_0(X))$, representing the maximum of the function $Y\mapsto\psi(X,Y)$ on $[0,\eta(X)]$, attained uniquely on the critical layer, is strictly increasing as X runs from $-\frac{1}{2}L$ to 0 and strictly decreasing as X runs from 0 to $\frac{1}{2}L$. Set $M_-=M\left(-\frac{1}{2}L\right)=M\left(\frac{1}{2}L\right)$ and $M_+=M(0)$. We know that $\psi_Y<0$ in the region

$$\Omega^+ = \left\{(X,Y) : -\tfrac{1}{2}L < X < \tfrac{1}{2}L \text{ and } Y_0(X) < Y < \eta(X)\right\}$$

above the critical layer, while $\psi_Y>0$ in the region

$$\Omega^- = \left\{(X,Y) : -\tfrac{1}{2}L < X < \tfrac{1}{2}L \text{ and } 0 < Y < Y_0(X)\right\}$$

beneath the critical layer. Since

$$\max\{0,-m\} < M_- < M(X) = M(-X) < M_+$$

for every $X\in\left(-\frac{1}{2}L,0\right)$, the implicit function theorem yields that in $\Omega_-^+=\Omega_-\cap\Omega^+$ the level set $\{\psi=M_-\}$ consists of a curve C_+, which is the graph of a smooth strictly increasing function, joining the points $\left(-\frac{1}{2}L, Y_0\left(-\frac{1}{2}L\right)\right)$ and $(0,Y_+)$, where $Y_+\in(Y_0(0),\eta(0))$ is the unique solution to $\psi(0,Y)=M_-$ in this interval. Similarly, in $\Omega_-^-=\Omega_-\cap\Omega^-$ the level set $\{\psi=M_-\}$ consists of a curve C_-, which is the graph of a smooth strictly decreasing function, that joins $\left(-\frac{1}{2}L, Y_0\left(-\frac{1}{2}L\right)\right)$ and $(0,Y_-)$, where $Y_-\in(0,Y_0(0))$ is the unique solution to $\psi(0,Y)=M_-$ in this interval. The level set $\{\psi=M_-\}$ in Ω_+ is the mirror image of the union of these two curves. In the closure of $\Omega_-\cup\Omega_+$, the level set $\{\psi=M_+\}$ consists of the stagnation point $(0,Y_0(0))$, while each level set $\{\psi=\gamma\}$ with $\gamma\in(M_-,M_+)$ is a closed curve encircling $(0,Y_0(0))$. For $\gamma\in(0,M_-)$, the implicit function theorem yields that the level set $\{\psi=\gamma\}$ in Ω_-^+ is the graph of a strictly increasing function, located between C_+ and the free surface, $\{\psi=\gamma\}$ in Ω_-^- is the graph of a strictly decreasing function, located between C_- and the flat bed, while $\{\psi=\gamma\}$ in Ω_+ is the mirror image

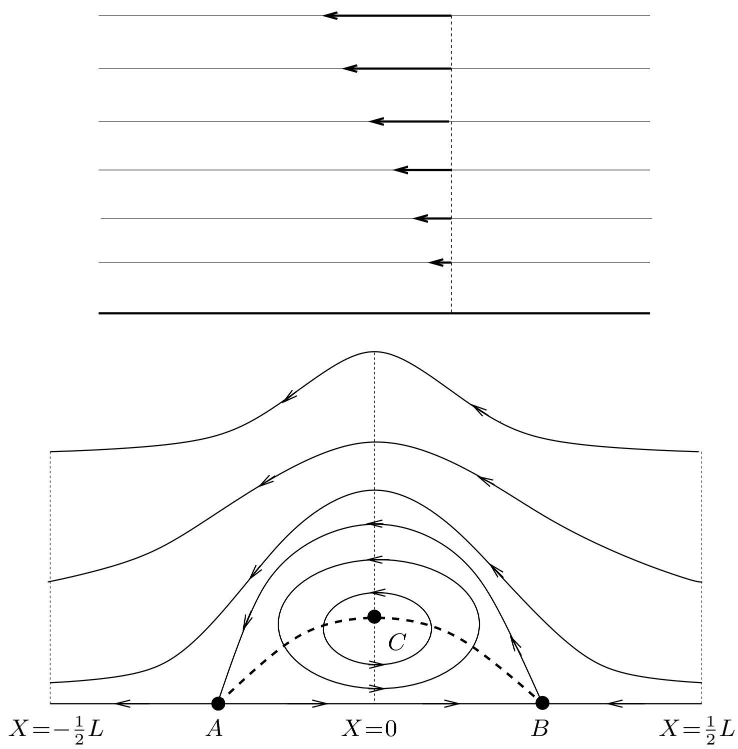

Figure 9. Flows with constant vorticity $\Upsilon<0$, admitting in the moving frame stagnation points on the flat bed: On top, a laminar flow for which all points on the bed are stagnation points, and below, a nearby wave of small amplitude (lying in function space on the local bifurcation curve) presents in a periodicity window only three stagnation points (identified as A, B and C in the figure), connected by the critical layer curve (depicted by the bold dashed curve) that encloses a near-bed region of flow-reversal. The stagnation point located beneath the wave crest is surrounded by closed streamlines.

of the union of these two curves. The cat's eye type flow pattern depicted in Figure 8 emerges.

(iv) We finally consider the case $\Upsilon<0$ in which the critical points of the bifurcating laminar flow (with a flat free surface) are confined to the flat bed, that is, $\lambda_-^*-\Upsilon h=0$. The proof of Theorem 3 shows that along the solution curve $\mathcal{K}_-$ the non-trivial solutions sufficiently close to the bifurcating laminar flow correspond to $v_s(x)=h+s\cos(x)+o(s)$ in $C^{2,\alpha}_{2\pi,e}(\mathbb{R})$ and $m(s)=m_-^*+o(s)$ for $s>0$ small enough, where $m_-^*=\frac{1}{2}\Upsilon h^2$ due to (3.18), (3.3) and $\lambda_-^*=\Upsilon h$. From (A.3) we deduce that

$$u_s(x)=\frac{x}{k}+s\coth(kh)\sin(x)+o(s) \text{ in } C^{2,\alpha}_{2\pi}(\mathbb{R}),$$

where $u_s(x)=x/k+(\mathcal{C}_{kh}(v_s-h))(x)$ for all $x\in\mathbb{R}$. The corresponding solution of (2.9) is given by

$$V^s(x,y)=\frac{y+kh}{k}+s\frac{\sinh(y+kh)}{\sinh(kh)}\cos(x)+o(s) \quad \text{in } C^{2,\alpha}(\overline{\mathcal{R}}_{kh}), \tag{5.6}$$

so that, in view of (2.10), the harmonic conjugate of $-V^s$ on $\mathcal{R}_{kh}$ is

$$U^s(x,y)=\frac{x}{k}+s\frac{\cosh(y+kh)}{\sinh(kh)}\sin(x)+o(s) \quad \text{in } C^{2,\alpha}(\overline{\mathcal{R}}_{kh}).$$

Furthermore, we have that the solution to (2.13) is given by

$$\zeta^s(x,y)=-\Upsilon sh\frac{\sinh(y+kh)}{\sinh(kh)}\cos(x)+o(s) \quad \text{in } C^{2,\alpha}(\overline{\mathcal{R}}_{kh}). \tag{5.7}$$

For $s>0$ small enough, the expansion

$$(u_s'(x),v_s'(x))=\left(\frac{1}{k}+s\coth(kh)\cos(x)+o(s), -s\sin(x)+o(s)\right)$$

in $C^{1,\alpha}_{2\pi}(\mathbb{R})$ ensures that the free surface is the graph of a function $Y=\eta_s(X)$ with $\eta_s'(X)<0$ for $X\in\left(0,\frac{1}{2}L\right)$. Throughout the bifurcating laminar flow we have that $\psi_{YY}=\Upsilon<0$, while $\psi_Y(h)=\lambda_-^*<0$, so that for nearby waves we will have $\psi^s_{YY}<0$ throughout the flow and $\psi^s_Y<0$ near the free surface $Y=\eta_s(X)$. Recalling that $\eta_s'(X)<0$ for $X\in\left(0,\frac{1}{2}L\right)$, by differentiating the relation $\psi^s(X,\eta_s(X))=0$, valid due to (2.2b), we deduce that $\psi^s_X(X,\eta_s(X))<0$ for $X\in\left(0,\frac{1}{2}L\right)$. On the other hand, $\psi^s_X(X,0)=0$ for all $X\in\mathbb{R}$ due to (2.2c), while the symmetry properties ensure $\psi^s_X(0,Y)=0$ for $0\leqslant Y\leqslant\eta_s(0)$ and $\psi^s_X\left(\frac{1}{2}L,Y\right)=0$ for $0\leqslant Y\leqslant\eta_s\left(\frac{1}{2}L\right)$. The function ψ^s_X being harmonic in the domain

$$\left\{(X,Y): 0<X<\tfrac{1}{2}L \text{ and } 0<Y<\eta_s(X)\right\}$$

by (2.2a), the maximum principle permits us to deduce that $\psi^s_X(X,Y)<0$ throughout the domain, while Hopf's boundary-point lemma yields

$$\psi^s_{XY}(X,0)<0, \quad X\in\left(0,\tfrac{1}{2}L\right). \tag{5.8}$$

To elucidate the behaviour of ψ^s_Y in the closure of the domain

$$\left\{(X,Y): -\tfrac{1}{2}L<X<\tfrac{1}{2}L \text{ and } 0<Y<\eta_s(X)\right\},$$

note that (2.21) yields

$$\psi^s_Y(U^s(x,-kh),0)=-\Upsilon s\frac{kh}{\sinh(kh)}\cos(x)+o(s) \quad \text{in } C^{1,\alpha}_{2\pi}(\mathbb{R}), \tag{5.9}$$

since $V^s(x,-kh)=0$ by (2.9c), while by (5.6) and (5.7),

$$V_Y^s(x,-kh)=\frac{1}{k}+s\frac{1}{\sinh(kh)}\cos(x)+o(s) \quad \text{in } C_{2\pi}^{1,\alpha}(\mathbb{R}),$$

and

$$\zeta_Y^s(x,-kh)=-\Upsilon s\frac{h}{\sinh(kh)}\cos(x)+o(s) \quad \text{in } C_{2\pi}^{1,\alpha}(\mathbb{R}).$$

For $x=0$ and $x=\pm\pi$ in (5.9) we get $\psi_Y^s(0,0)>0$ and $\psi_Y^s\big(\pm\frac{1}{2}L,0\big)<0$, respectively, for all $s>0$ sufficiently small. Taking into account (5.8) and the fact that $X\mapsto\psi^s(X,0)$ is even, we deduce the existence of some $X_0\in\big(0,\frac{1}{2}L\big)$ with $\psi_Y^s(\pm X_0,0)=0$ and

$$\psi_Y^s(X,0)\begin{cases} >0 & \text{for } X\in(-X_0,X_0),\\ <0 & \text{for } X\in\big[-\frac{1}{2}L,-X_0\big)\cup\big(X_0,\frac{1}{2}L\big].\end{cases}$$

Denote the points $(-X_0,0)$ and $(X_0,0)$ by A and B, respectively. Along the vertical segment $\{(X,Y):0\leqslant Y\leqslant\eta_s(X)\}$ we know that $\psi_{YY}^s(X,Y)<0$, with $\psi_Y^s(X,\eta_s(X))<0$. Thus $\psi_Y^s(X,Y)<0$ for all $X\in\big[-\frac{1}{2}L,-X_0\big]\cup\big[X_0,\frac{1}{2}L\big]$ and $Y\in(0,\eta_s(X)]$. On the other hand, for all $X\in(-X_0,X_0)$, there is a unique $Y_0^s(X)\in(0,\eta_s(X))$ such that $\psi_Y^s(X,Y_0^s(X))=0$, with

$$\psi_Y^s(X,Y)\begin{cases} >0 & \text{for } Y\in[0,Y_0^s(X)),\\ <0 & \text{for } Y\in(Y_0^s(X),\eta_s(X)].\end{cases}$$

Denote the stagnation point $(0,Y_0^s(0))$ by C. The curve $X\mapsto Y_0^s(X)$ with $X\in[-X_0,X_0]$ is the critical layer and A, B and C are the stagnation points of the flow in the closure of the domain $\big\{(X,Y):-\frac{1}{2}L<X<\frac{1}{2}L \text{ and } 0<Y<\eta_s(X)\big\}$. Recall from (2.2) that $\psi^s(X,\eta_s(X))=0$ while $\psi^s(X,0)=-m(s)>0$. For a fixed $X\in\big[-\frac{1}{2}L,-X_0\big]\cup\big[X_0,\frac{1}{2}L\big]$, the function $Y\mapsto\psi^s(X,Y)$ is strictly decreasing on $[0,\eta_s(X)]$, while for $X\in(-X_0,X_0)$, the function $Y\mapsto\psi^s(X,Y)$ attains its maximum $M^s(X)>-m(s)$ on $[0,\eta_s(X)]$ at $y=Y_0^s(X)$, being strictly monotone on either side of $Y_0^s(X)$. These considerations suffice to infer the full qualitative flow pattern (see Figure 9). In particular, the stagnation point C is surrounded by closed streamlines. □

It would be interesting to extend the above analysis of the nature of the flow beneath the wave profile from the case of small-amplitude waves to larger waves.

Appendix A. The periodic Hilbert transform $\mathcal{C}_d$ on a strip

We discuss the conjugation and Dirichlet–Neumann operators acting on periodic functions on a strip. The Dirichlet–Neumann operator $\mathcal{G}_d$ for the strip $\mathcal{R}_d$ is defined, for $w\in C_{2\pi}^{p,\alpha}(\mathbb{R})$ with $p\geqslant 1$ an integer and $\alpha\in(0,1)$, by

$$(\mathcal{G}_d(w))(x)=W_y(x,0), \quad x\in\mathbb{R}, \tag{A.1}$$

where $W\in C^{p,\alpha}_{2\pi}(\overline{\mathcal{R}}_d)$ is the unique solution to the boundary-value problem

$$\begin{cases} \Delta W=0, & \text{in } \mathcal{R}_d, \\ W(x,0)=w(x), & x\in\mathbb{R}, \\ W(x,-d)=0, & x\in\mathbb{R}. \end{cases} \tag{A.2}$$

$\mathcal{G}_d$ is a bounded linear operator from $C^{p,\alpha}_{2\pi}(\mathbb{R})$ to $C^{p-1,\alpha}_{2\pi}(\mathbb{R})$, given by

$$(\mathcal{G}_d(w))(x)=\frac{[w]}{d}+\sum_{n=1}^{\infty} na_n\coth(nd)\cos(nx)+\sum_{n=1}^{\infty} nb_n\coth(nd)\sin(nx),$$

for $x\in\mathbb{R}$, where $w\in C^{p,\alpha}_{2\pi}(\mathbb{R})$ has the Fourier series expansion

$$w(x)=[w]+\sum_{n=1}^{\infty} a_n\cos(nx)+\sum_{n=1}^{\infty} b_n\sin(nx),\quad x\in\mathbb{R}.$$

The conjugation operator $\mathcal{C}_d$ is defined for 2π-periodic functions $w\in C^{p,\alpha}_{2\pi,\circ}(\mathbb{R})$ of zero mean, $[w]=0$, having the Fourier series expansion

$$w(x)=\sum_{n=1}^{\infty} a_n\cos(nx)+\sum_{n=1}^{\infty} b_n\sin(nx),\quad x\in\mathbb{R},$$

by

$$(\mathcal{C}_d(w))(x)=\sum_{n=1}^{\infty} a_n\coth(nd)\sin(nx)-\sum_{n=1}^{\infty} b_n\coth(nd)\cos(nx), \tag{A.3}$$

for $x\in\mathbb{R}$. For any integer $p\geqslant 0$ and any $\alpha\in(0,1)$, the operator $\mathcal{C}_d$ is a bounded invertible linear operator from $C^{p,\alpha}_{2\pi,\circ}(\mathbb{R})$ into itself. The two operators $\mathcal{G}_d$ and $\mathcal{C}_d$ are related by means of the identity

$$\mathcal{G}_d(w)=\frac{[w]}{d}+(\mathcal{C}_d(w-[w]))'=\frac{[w]}{d}+\mathcal{C}_d(w'),\quad w\in C^{p,\alpha}_{2\pi}(\mathbb{R}), \tag{A.4}$$

that holds for all integers $p\geqslant 1$ and all $\alpha\in(0,1)$.

Given W as in (A.2), let $Z\in C^{p,\alpha}_{2\pi}(\overline{\mathcal{R}}_d)$ be the harmonic function in $\mathcal{R}_d$, uniquely determined up to a constant, such that $Z+iW$ is holomorphic in $\mathcal{R}_d$. Then

$$Z(x,y)=\frac{[w]}{d}x+Z_0(x,y)$$

throughout $\mathcal{R}_d$, for a harmonic function $(x,y)\mapsto Z_0(x,y)$ that is 2π-periodic in the x-variable (see [18]). The function Z_0 being unique up to an additive constant, we normalize it by requiring that $x\mapsto Z_0(x,0)$ has zero mean over one period. Then $x\mapsto Z_0(x,y)$ has zero mean for every $y\in[-d,0]$. The restriction of this particular harmonic conjugate of $-W$ to $y=0$ is given by

$$Z(x,0)=\frac{[w]}{d}x+(\mathcal{C}_d(w-[w]))(x),\quad x\in\mathbb{R}. \tag{A.5}$$

Let $L^2_{2\pi,\circ}(\mathbb{R})$ be the space of 2π-periodic locally square integrable functions of one real variable, with zero mean over one period. The operator $\mathcal{C}_d$ can be extended by complex-linearity to complex-valued functions in $L^2_{2\pi,\circ}(\mathbb{R})$, being characterized by its action on the trigonometric system $\{e^{int}\}_{n\in\mathbb{Z}\setminus\{0\}}$ as

$$\mathcal{C}_d(e^{int})=-i\coth(nd)e^{int},\quad n\in\mathbb{Z}\setminus\{0\}. \tag{A.6}$$

It is a skew-adjoint linear operator. Let $\mathcal{C}$ denote the standard periodic Hilbert transform [6], [40], defined by

$$\mathcal{C}(e^{int})=-i\,\mathrm{sgn}(n)e^{int},\quad n\in\mathbb{Z}\setminus\{0\}. \tag{A.7}$$

It is well known (see [6]) that $\mathcal{C}$ has a pointwise almost everywhere representation as a singular integral

$$\big(\mathcal{C}(w)\big)(x)=\frac{1}{2\pi}\,\mathrm{PV}\int_{-\pi}^{\pi}\cot\left(\frac{x-s}{2}\right)w(s)\,ds,\quad x\in\mathbb{R}, \tag{A.8}$$

where PV denotes a principal value integral [33], which is instrumental in the investigation of the structural properties of the operator $\mathcal{C}$. Writing

$$\mathcal{C}_d=\mathcal{C}+\mathcal{K}_d, \tag{A.9}$$

we see that the operator $\mathcal{K}_d$ corresponds to the Fourier multiplier operator on $L^2_{2\pi,\circ}(\mathbb{R})$ given by

$$w=\sum_{n\in\mathbb{Z}\setminus\{0\}}c_ne^{int}\longmapsto\sum_{n\in\mathbb{Z}\setminus\{0\}}-i\,\mathrm{sgn}(n)\lambda_nc_ne^{int}, \tag{A.10}$$

with $\lambda_n=2/(e^{2|n|d}-1)$ for $|n|\geqslant1$. Since $\sum_{n\in\mathbb{Z}\setminus\{0\}}|n|^{2p}\lambda_n^2<\infty$ for every integer $p\geqslant0$, the function $\varkappa_d\in L^2_{2\pi,\circ}(\mathbb{R})$ given by

$$\varkappa_d(t)=\sum_{n\in\mathbb{Z}\setminus\{0\}}-i\,\mathrm{sgn}(n)\lambda_ne^{int}=\sum_{n=1}^{\infty}2\lambda_n\sin(nt),\quad t\in\mathbb{R}, \tag{A.11}$$

is of class C^∞ (see [22]). From (A.10) we infer that $\mathcal{K}_d(w)$ is the convolution of w with the smooth function $\varkappa_d$, that is,

$$(\mathcal{K}_d(w))(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\varkappa_d(x-s)w(s)\,ds,\quad x\in\mathbb{R}. \tag{A.12}$$

This representation is useful in establishing the following commutator estimate (see [18] for the proof).

Lemma 17. *If* $f\in C^{j,\alpha}_{2\pi,\circ}(\mathbb{R})$ *and* $g\in C^{j-1,\alpha}_{2\pi,\circ}(\mathbb{R})$, *with* $j\in\mathbb{N}$ *and* $\alpha\in(0,1)$, *are such that* $[fg]=0$, *then* $f\mathcal{C}_d(g)-\mathcal{C}_d(fg)\in C^{j,\delta}_{2\pi}(\mathbb{R})$ *for all* $\delta\in(0,\alpha)$ (*with the inequality* $\delta<\alpha$ *being sharp*), *and there exists a constant* $C=C(j,\alpha,\delta)$ *such that*

$$\|f\mathcal{C}_d(g)-\mathcal{C}_d(fg)\|_{j,\delta}\leqslant C\|f\|_{j,\alpha}\|g\|_{j-1,\alpha}.$$

In order to complete the description of the periodic Hilbert transform in a strip, we now derive the analogue of the fundamental formula (A.8).

Theorem 18. *For any* $v\in C^{1,\alpha}_{2\pi,\circ}(\mathbb{R})$, *we have*

$$(\mathcal{C}_d(v))(x)=\frac{1}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi}\beta_d(x-s)v(s)\,ds,\quad x\in\mathbb{R}, \tag{A.13}$$

where the function $\beta_d:\mathbb{R}\setminus 2\pi\mathbb{Z}\to\mathbb{R}$ *is given by*

$$\beta_d(s)=-\frac{s}{d}+\frac{\pi}{d}\coth\Big(\frac{\pi s}{2d}\Big)+\frac{\pi}{d}\sum_{k\in\mathbb{N}}\frac{2\sinh(\pi s/d)}{\cosh(\pi s/d)-\cosh(2\pi^2k/d)},\quad s\in\mathbb{R}\setminus 2\pi\mathbb{Z}. \tag{A.14}$$

The function β_d *is* 2π*-periodic, odd, and continuous on* $\mathbb{R}\setminus 2\pi\mathbb{Z}$, *while*

$$s\longmapsto\Big(\beta_d(s)-\frac{\pi}{d}\coth\Big(\frac{\pi s}{2d}\Big)\Big)$$

is continuous at $s=0$.

Remark 19. Note that, for any $s\in\mathbb{R}\setminus 2\pi\mathbb{Z}$,

$$\lim_{d\to\infty}\beta_d(s)=\frac{2}{s}+\sum_{k\in\mathbb{N}}\frac{4s}{s^2-(2\pi k)^2}=\cot\Big(\frac{s}{2}\Big),$$

which shows that, at least formally, the limit as $d\to\infty$ of the family of linear operators $\mathcal{C}_d$ is the usual Hilbert transform $\mathcal{C}$.

Also, observe that term-by-term differentiation in (A.14) yields that β_d is decreasing on $(0,2\pi)$, a property that is not immediately obtainable from (A.11).

Proof. Let $U+iV$ be a holomorphic function in the strip $\mathcal{R}_d$, which is 2π-periodic in x and has for some $\alpha\in(0,1)$ a $C^{1,\alpha}_{\mathrm{loc}}$ extension to $\overline{\mathcal{R}}_d$, with $V=0$ on $y=-d$. Let $u+iv$ denote its boundary values on the horizontal line $y=0$, and assume that $[u]=[v]=0$. Then $u=\mathcal{C}_d(v)$, and we need to show that u and v are related by (A.13). Observe that the reflection principle [7] permits the analytic continuation of $U+iV$ to $\mathcal{R}_{2d}$ by setting

$$(U+iV)(x+iy)=\overline{(U+iV)(x-2id-iy)},\quad x\in\mathbb{R}\text{ and }-2d<y<-d. \tag{A.15}$$

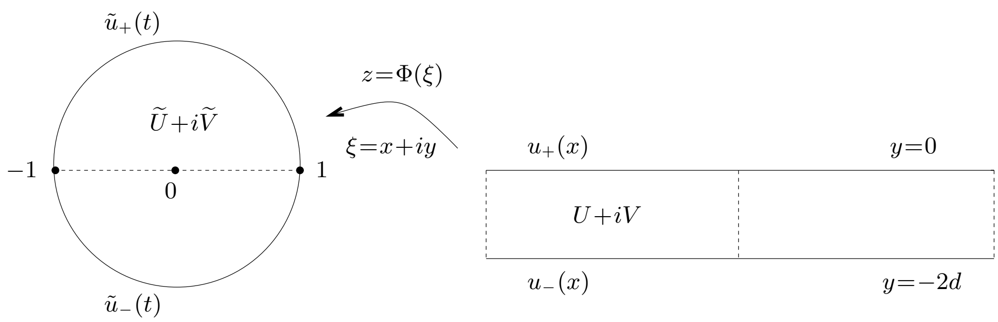

Figure 10. The correspondence between boundary values of harmonic functions on a strip and on the unit disk.

The function so obtained has a $C^{1,\alpha}_{\mathrm{loc}}$ extension to $\overline{\mathcal{R}}_{2d}$, and

$$U(x-2id)=u(x) \text{ and } V(x-2id)=-v(x), \quad \text{for all } x\in\mathbb{R}.$$

This suggests working with a holomorphic function in the strip $\mathcal{R}_{2d}$. Moreover, we adopt the approach of initially disregarding its 2π-periodicity, as well as the validity of (A.15), and take these conditions into consideration only later on.

Given $d>0$, let us therefore consider a holomorphic function $U+iV$ in $\mathcal{R}_{2d}$, having *bounded imaginary part* and admitting a $C^{1,\alpha}_{\mathrm{loc}}(\overline{\mathcal{R}}_{2d})$ extension. We denote by $u_\pm+iv_\pm$ its boundary values on the horizontal lines $y=0$ and $y=-2d$, respectively. Note that the function

$$\Phi(\xi)=\frac{e^{\pi(\xi+id)/2d}-1}{e^{\pi(\xi+id)/2d}+1}=\tanh\Big(\frac{\pi}{4d}(\xi+id)\Big),$$

with $\xi=x+iy$, maps the horizontal strip $\mathcal{R}_{2d}$ conformally onto the unit disc

$$\mathbb{D}=\{z\in\mathbb{C}:|z|<1\};$$

see Figure 10. For $z=\Phi(\xi)$, with $\xi\in\mathcal{R}_{2d}$, let $(\widetilde{U}+i\widetilde{V})(z)=(U+iV)(\xi)$ be the corresponding holomorphic function in $\mathbb{D}$, whose boundary values at $z=e^{it}$ on the upper/lower semicircles that make up the boundary of $\mathbb{D}$ are denoted by $\tilde{u}_\pm(t)+i\tilde{v}_\pm(t)$.

Now the holomorphic function $-i(\widetilde{U}+i\widetilde{V})=\widetilde{V}-i\widetilde{U}$ has bounded real part in $\mathbb{D}$, and therefore can be represented by Poisson's formula

$$\widetilde{V}(z)-i\widetilde{U}(z)=\frac{1}{2\pi}\int_0^\pi \frac{e^{it}+z}{e^{it}-z}\tilde{v}_+(t)\,dt+\frac{1}{2\pi}\int_{-\pi}^0 \frac{e^{it}+z}{e^{it}-z}\tilde{v}_-(t)\,dt-iC, \quad z\in\mathbb{D}, \tag{A.16}$$

where C is a real constant (more precisely, $C=\widetilde{U}(0)$). For $t\in(0,\pi)$, if $e^{it}=\Phi(s)$ with $s\in\mathbb{R}$, then the identity

$$e^{it}=\tanh\Big(\frac{\pi}{4d}(s+id)\Big)=\frac{\sinh(\pi s/2d)+i}{\cosh(\pi s/2d)}$$

yields $\cos(t)=\tanh(\pi s/2d)$ and $\sin(t)=1/\cosh(\pi s/2d)$, so that $t=2\arctan(e^{-\pi s/2d})$. On the other hand, for $t\in(-\pi,0)$, if $e^{it}=\Phi(s-2id)$ with $s\in\mathbb{R}$, then the identity

$$e^{it}=\tanh\left(\frac{\pi}{4d}(s-id)\right)=\frac{\sinh(\pi s/2d)-i}{\cosh(\pi s/2d)}$$

yields $\cos(t)=\tanh(\pi s/2d)$ and $\sin(t)=-1/\cosh(\pi s/2d)$, so that $t=-2\arctan(e^{-\pi s/2d})$. Thus the change of variables $t=2\arctan(e^{-\pi s/2d})$ for $t\in(0,\pi)$ and $t=-2\arctan(e^{-\pi s/2d})$ for $t\in(-\pi,0)$ transforms the right-hand side of (A.16) into

$$\begin{aligned}
&\frac{1}{4d}\int_{\mathbb{R}}\frac{\tanh(\pi(s+id)/4d)+\tanh(\pi(x+iy+id)/4d)}{\tanh(\pi(s+id)/4d)-\tanh(\pi(x+iy+id)/4d)}\frac{v_+(s)}{\cosh(\pi s/2d)}\,ds\\
&\quad+\frac{1}{4d}\int_{\mathbb{R}}\frac{\tanh(\pi(s-id)/4d)+\tanh(\pi(x+iy+id)/4d)}{\tanh(\pi(s-id)/4d)-\tanh(\pi(x+iy+id)/4d)}\frac{v_-(s)}{\cosh(\pi s/2d)}\,ds\\
&\quad=\frac{1}{4d}\int_{\mathbb{R}}\frac{\sinh(\pi(s+x+2id+iy)/4d)}{\sinh(\pi(s-x-iy)/4d)}\frac{v_+(s)}{\cosh(\pi s/2d)}\,ds\\
&\qquad+\frac{1}{4d}\int_{\mathbb{R}}\frac{\sinh(\pi(s+x+iy)/4d)}{\sinh(\pi(s-x-2id-iy)/4d)}\frac{v_-(s)}{\cosh(\pi s/2d)}\,ds
\end{aligned}\tag{A.17}$$

Using the identities

$$\begin{cases}2\sinh(\xi_1)\sinh(\xi_2)=\cosh(\xi_1+\xi_2)-\cosh(\xi_1-\xi_2),\\ \cosh(a+ib)=\cosh(a)\cos(b)+i\sinh(a)\sin(b),\end{cases}$$

for $\xi_1,\xi_2\in\mathbb{C}$ and $a,b\in\mathbb{R}$, respectively, we get

$$\begin{aligned}
&\frac{\sinh(\pi(s+x+2id+iy)/4d)}{\sinh(\pi(s-x-iy)/4d)}\\
&\quad=\frac{\sinh(\pi(s+x+2id+iy)/4d)\sinh(\pi(s-x+iy)/4d)}{\sinh(\pi(s-x-iy)/4d)\sinh(\pi(s-x+iy)/4d)}\\
&\quad=\frac{\cosh(\pi(s+id+iy)/2d)-\cosh(\pi(x+id)/2d)}{\cosh(\pi(s-x)/2d)-\cosh(\pi iy/2d)}\\
&\quad=-\frac{\cosh(\pi s/2d)\sin(\pi y/2d)}{\cosh(\pi(x-s)/2d)-\cos(\pi y/2d)}+i\frac{\sinh(\pi s/2d)\cos(\pi y/2d)-\sinh(\pi x/2d)}{\cosh(\pi(x-s)/2d)-\cos(\pi y/2d)},
\end{aligned}\tag{A.18}$$

and, similarly,

$$\begin{aligned}
\frac{\sinh(\pi(s+x+iy)/4d)}{\sinh(\pi(s-x-2id-iy)/4d)}=&-\frac{\cosh(\pi s/2d)\sin(\pi y/2d)}{\cosh(\pi(x-s)/2d)+\cos(\pi y/2d)}\\
&+i\frac{\sinh(\pi s/2d)\cos(\pi y/2d)+\sinh(\pi x/2d)}{\cosh(\pi(x-s)/2d)+\cos(\pi y/2d)}.
\end{aligned}\tag{A.19}$$

Taking (A.18) and (A.19) into account, from (A.16) and (A.17) we deduce that, for any $(x,y)\in\mathcal{R}_{2d}$, we have

$$\begin{aligned}
&V(x,y)-iU(x,y)\\
&\quad=-\frac{1}{4d}\int_{\mathbb{R}}\frac{\sin(\pi y/2d)}{\cosh(\pi(x-s)/2d)-\cos(\pi y/2d)}v_+(s)\,ds\\
&\qquad+\frac{i}{4d}\int_{\mathbb{R}}\frac{\tanh(\pi s/2d)\cos(\pi y/2d)-\sinh(\pi x/2d)/\cosh(\pi s/2d)}{\cosh(\pi(x-s)/2d)-\cos(\pi y/2d)}v_+(s)\,ds\\
&\qquad-\frac{1}{4d}\int_{\mathbb{R}}\frac{\sin(\pi y/2d)}{\cosh(\pi(x-s)/2d)+\cos(\pi y/2d)}v_-(s)\,ds\\
&\qquad+\frac{i}{4d}\int_{\mathbb{R}}\frac{\tanh(\pi s/2d)\cos(\pi y/2d)+\sinh(\pi x/2d)/\cosh(\pi s/2d)}{\cosh(\pi(x-s)/2d)+\cos(\pi y/2d)}v_-(s)\,ds-iC.
\end{aligned}\tag{A.20}$$

Using the identity

$$\sinh\Big(\frac{\pi x}{2d}\Big)=\sinh\Big(\frac{\pi(x-s)}{2d}\Big)\cosh\Big(\frac{\pi s}{2d}\Big)+\cosh\Big(\frac{\pi(x-s)}{2d}\Big)\sinh\Big(\frac{\pi s}{2d}\Big),$$

valid for all $x,s\in\mathbb{R}$, we obtain, for any $(x,y)\in\mathcal{R}_{2d}$, that

$$\begin{aligned}
U(x,y)=C+\frac{1}{4d}\int_{\mathbb{R}}\Big(\tanh\Big(\frac{\pi s}{2d}\Big)+\frac{\sinh(\pi(x-s)/2d)}{\cosh(\pi(x-s)/2d)-\cos(\pi y/2d)}\Big)v_+(s)\,ds\\
-\frac{1}{4d}\int_{\mathbb{R}}\Big(\tanh\Big(\frac{\pi s}{2d}\Big)+\frac{\sinh(\pi(x-s)/2d)}{\cosh(\pi(x-s)/2d)+\cos(\pi y/2d)}\Big)v_-(s)\,ds.
\end{aligned}\tag{A.21}$$

The above formula is valid also on $y=0$ and on $y=-2d$, with the proviso that in these cases one of the two integrals must be considered as a principal value, due to the singularity of the integrand at $s=x$.

Let us assume now, in addition, that V is 2π-periodic in the x-variable throughout $\overline{\mathcal{R}}_{2d}$. Due to the periodicity of $v_\pm$, the representation formula (A.21) may be written as

$$\begin{aligned}
&U(x,y)\\
&\quad=C+\frac{1}{4d}\int_{-\pi}^{\pi}\sum_{k\in\mathbb{Z}}\Big(\tanh\Big(\frac{\pi(s+2\pi k)}{2d}\Big)+\frac{\sinh(\pi(x-s-2\pi k)/2d)}{\cosh(\pi(x-s-2\pi k)/2d)-\cos(\pi y/2d)}\Big)v_+(s)\,ds\\
&\qquad-\frac{1}{4d}\int_{-\pi}^{\pi}\sum_{k\in\mathbb{Z}}\Big(\tanh\Big(\frac{\pi(s+2\pi k)}{2d}\Big)+\frac{\sinh(\pi(x-s-2\pi k)/2d)}{\cosh(\pi(x-s-2\pi k)/2d)+\cos(\pi y/2d)}\Big)v_-(s)\,ds
\end{aligned}\tag{A.22}$$

for $(x,y)\in\overline{\mathcal{R}}_{2d}$. We define the function $\hat{g}_d:\mathbb{R}\to\mathbb{R}$ by

$$\hat{g}_d(s)=\frac{\pi}{2d}\sum_{k\in\mathbb{Z}}\Big(\tanh\Big(\frac{\pi(s+2\pi k)}{2d}\Big)-\operatorname{sgn}(k)\Big),\quad s\in\mathbb{R},$$

and, for any choice of sign $\mp$, the functions $\hat{\beta}_{d,y,\mp}: D \to \mathbb{R}$ with

$$D = \begin{cases} \mathbb{R}, & \text{if } y \in (-2d, 0), \\ \mathbb{R} \setminus 2\pi\mathbb{Z}, & \text{if } y = 0 \text{ or } y = -2d, \end{cases}$$

by

$$\hat{\beta}_{d,y,\mp}(s) = \frac{\pi}{2d} \sum_{k \in \mathbb{Z}} \left(\frac{\sinh(\pi(s-2\pi k)/2d)}{\cosh(\pi(s-2\pi k)/2d) \mp \cos(\pi y/2d)} + \operatorname{sgn}(k) \right), \quad s \in D,$$

where

$$\operatorname{sgn}(k) = \begin{cases} 1, & \text{if } k > 0, \\ 0, & \text{if } k = 0, \\ -1, & \text{if } k < 0. \end{cases}$$

Let M be an arbitrary positive constant. For any $y \in [-2d, 0]$, the validity, for every $s \in (-M, M)$ and $k \in \mathbb{Z}$ with $2\pi|k| > M$, of the estimates

$$\begin{aligned} \left| \tanh\left(\frac{\pi(s+2\pi k)}{2d} \right) - \operatorname{sgn}(k) \right| &= \left| \tanh\left(\frac{\pi|2\pi k+s|}{2d} \right) - 1 \right| \\ &= \frac{2}{1+e^{\pi|2\pi k+s|/d}} \leqslant \frac{2}{1+e^{\pi(2\pi|k|-M)/d}}, \\ \left| \frac{\sinh(\pi(s-2\pi k)/2d)}{\cosh(\pi(s-2\pi k)/2d) \mp \cos(\pi y/2d)} + \operatorname{sgn}(k) \right| &= \left| 1 - \frac{\sinh(\pi|2\pi k-s|/2d)}{\cosh(\pi|2\pi k-s|/2d) \mp \cos(\pi y/2d)} \right| \\ &\leqslant \frac{2}{\cosh(\pi|2\pi k-s|/2d) \mp \cos(\pi y/2d)} \\ &\leqslant \frac{2}{\cosh(\pi(2\pi|k|-M)/2d) \mp \cos(\pi y/2d)} \end{aligned}$$

ensures the continuity on the interval $(-M, M)$ of the functions

$$s \longmapsto \sum_{\substack{k \in \mathbb{Z} \\ 2\pi|k| > M}} \left(\frac{\sinh(\pi(s-2\pi k)/2d)}{\cosh(\pi(s-2\pi k)/2d) \mp \cos(\pi y/2d)} + \operatorname{sgn}(k) \right).$$

On the other hand, the functions

$$s \longmapsto \sum_{\substack{k \in \mathbb{Z} \\ 2\pi|k| \leqslant M}} \left(\frac{\sinh(\pi(s-2\pi k)/2d)}{\cosh(\pi(s-2\pi k)/2d) \mp \cos(\pi y/2d)} + \operatorname{sgn}(k) \right)$$

are continuous on $(-M, M)$ if $y \in (-2d, 0)$, and continuous on $(-M, M) \setminus 2\pi\mathbb{Z}$ if $y=0$ or $y=-2d$. Since $M>0$ was arbitrary, it follows that $\hat{g}_d$ is continuous on $\mathbb{R}$ and the functions $\hat{\beta}_{d,y,\mp}$ are continuous on D.

Now observe that, for any $s\in\mathbb{R}$, we have

$$\hat{g}_d(s+2\pi)-\hat{g}_d(s)=\frac{\pi}{2d}\sum_{k\in\mathbb{Z}}(\operatorname{sgn}(k+1)-\operatorname{sgn}(k))=\frac{\pi}{d},$$
$$\hat{\beta}_{d,y,\mp}(s+2\pi)-\hat{\beta}_d(s)=\frac{\pi}{2d}\sum_{k\in\mathbb{Z}}(\operatorname{sgn}(k)-\operatorname{sgn}(k-1))=\frac{\pi}{d}.$$

Let $g_d:\mathbb{R}\to\mathbb{R}$ and $\beta_{d,y,\mp}:D\to\mathbb{R}$ be given by

$$g_d(s)=\hat{g}_d(s)-\frac{s}{2d},\quad s\in\mathbb{R},$$

and

$$\beta_{d,y,\mp}(s)=\hat{\beta}_{d,y,\mp}(s)-\frac{s}{2d},\quad s\in D.$$

Then, the preceding considerations show that each of these functions is 2π-periodic, and that g_d is continuous on $\mathbb{R}$ and $\beta_{d,y,\mp}$ are continuous on D. Moreover, we have

$$\begin{aligned}\frac{\pi}{2d}\sum_{k\in\mathbb{Z}}\biggl(\tanh\biggl(\frac{\pi(s+2\pi k)}{2d}\biggr)+\frac{\sinh(\pi(x-s-2\pi k)/2d)}{\cosh(\pi(x-s-2\pi k)/2d)\mp\cos(\pi y/2d)}\biggr)\\ =\frac{x}{2d}+g_d(s)+\beta_{d,y,\mp}(x-s)\end{aligned}\tag{A.23}$$

for all $x,s\in\mathbb{R}$ (with $x\neq s$ if $y=0$ or $y=-2d$). We may therefore write (A.22) in the form

$$\begin{aligned}U(x,y)=C+\frac{[v_+]-[v_-]}{2d}\,x+\frac{1}{2\pi}\int_{-\pi}^{\pi}g_d(s)(v_+(s)-v_-(s))\,ds\\ +\frac{1}{2\pi}\int_{-\pi}^{\pi}\beta_{d,y,-}(x-s)v_+(s)\,ds-\frac{1}{2\pi}\int_{-\pi}^{\pi}\beta_{d,y,+}(x-s)v_-(s)\,ds,\end{aligned}\tag{A.24}$$

for any $(x,y)\in\overline{\mathcal{R}}_{2d}$. Since $\beta_{d,y,\mp}$ are 2π-periodic functions on $\mathbb{R}$ for $y\in(-2d,0)$, it follows that

$$U(x+2\pi,y)-U(x,y)=\frac{\pi}{d}([v_+]-[v_-])\quad\text{for all } (x,y)\in\mathcal{R}_{2d}.$$

This implies, in particular, that U is 2π-periodic in x if and only if $[v_+]=[v_-]$.

Let us now further assume that $V=0$ on $y=d$ and that (A.15) holds. In this case $v_-(x)=-v_+(x)$ for all $x\in\mathbb{R}$, which implies, in view of the preceding considerations, that U is 2π-periodic in x if and only if $[v_+]=0$. We assume that this is the case, and seek to determine a formula for $u_+=\mathcal{C}_d(v_+)$. Under the present assumptions, (A.24) may be rewritten as

$$U(x,y)=C+\frac{1}{2\pi}\int_{-\pi}^{\pi}2g_d(s)v_+(s)\,ds+\frac{1}{2\pi}\int_{-\pi}^{\pi}(\beta_{d,y,-}(x-s)+\beta_{d,y,+}(x-s))v_+(s)\,ds,\tag{A.25}$$

for any $(x,y)\in\overline{\mathcal{R}}_d$. Using the Cauchy–Riemann equations and the 2π-periodicity in the x variable of V, we have

$$\frac{d}{dy}\int_{-\pi}^{\pi} U(t,y)\,dt = \int_{-\pi}^{\pi} U_y(t,y)\,dt = -\int_{-\pi}^{\pi} V_x(t,y)\,dt = V(-\pi,0)-V(\pi,0)=0,$$

and therefore the function U has the same mean on any horizontal line segment of length 2π that is contained in $\mathcal{R}_d$. To determine the Hilbert transform of v_+, the constant C in the definition of U has to be chosen so that the above mean is zero. Since, for $y\in[-d,0)$, the functions $\beta_{d,y,\mp}$ are odd (and therefore have zero mean), their convolution with any 2π-periodic function also has zero mean. It follows that the harmonic conjugate U of $-V$ in $\mathcal{R}_d$, whose boundary values on $y=0$ yield $\mathcal{C}_d(v_+)$, is

$$U(x,y)=\frac{1}{2\pi}\int_{-\pi}^{\pi}(\beta_{d,y,-}(x-s)+\beta_{d,y,+}(x-s))v_+(s)\,ds, \quad (x,y)\in\overline{\mathcal{R}}_d. \tag{A.26}$$

In particular, for $y=0$, we have

$$u_+(x)=\frac{1}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi}(\beta_{d,0,-}(x-s)+\beta_{d,0,+}(x-s))v_+(s)\,ds, \quad x\in\mathbb{R}, \tag{A.27}$$

the principal value being due to the singularity of the integrand at $s=x$. Now observe that, for any $s\in\mathbb{R}\setminus 2\pi\mathbb{Z}$, we have

$$\begin{aligned}
&\beta_{d,0,-}(s)+\beta_{d,0,+}(s)\\
&\quad=-\frac{s}{d}+\frac{\pi}{d}\sum_{k\in\mathbb{Z}}\left(\coth\left(\frac{\pi(s-2\pi k)}{2d}\right)+\operatorname{sgn}(k)\right)\\
&\quad=-\frac{s}{d}+\frac{\pi}{d}\coth\left(\frac{\pi s}{2d}\right)+\frac{\pi}{d}\sum_{k\in\mathbb{N}}\left(\coth\left(\frac{\pi(s-2\pi k)}{2d}\right)+\coth\left(\frac{\pi(s+2\pi k)}{2d}\right)\right),\\
&\quad=-\frac{s}{d}+\frac{\pi}{d}\coth\left(\frac{\pi s}{2d}\right)+\frac{\pi}{d}\sum_{k\in\mathbb{N}}\frac{2\sinh(\pi s/d)}{\cosh(\pi s/d)-\cosh(2\pi^2k/d)}\\
&\quad=\beta_d(s),
\end{aligned}$$

where β_d is the function defined in (A.14). Therefore, (A.27) can be written as

$$u_+(x)=\frac{1}{2\pi}\,\mathrm{PV}\!\int_{-\pi}^{\pi}\beta_d(x-s)v_+(s)\,ds, \quad x\in\mathbb{R},$$

which is the required formula (A.13). It is obvious that β_d is 2π-periodic and odd, while the nature of its singularity at $s=0$ is also plain. □

Appendix B. Some variational considerations

We present two new variational formulations for the problem of travelling periodic gravity water waves with constant non-zero vorticity over a flat bed. We make no assumptions on the shape of the wave profile, thus allowing profiles that are not graphs.

The much-studied irrotational flows (that is, flows without vorticity) are models for swell entering a region of still water, in which case there is no current, or else models for swell entering a region with currents that are uniform with depth. The earliest variational formulation dates back to Friedrichs [23]. It was followed by the work of Luke [28], Zakharov [50], Babenko [2] and others, all of which expressed the Lagrangian in terms of a velocity potential. See [14] for further references.

While there is no velocity potential in the presence of vorticity, there is a stream function. The first formulation, valid for general vorticity distributions, recasts the steady water waves as extremals of a suitably defined Lagrangian functional that is roughly the total energy of the wave. Lagrangians of a similar type, expressed in terms of a hodograph transform involving the stream function, were considered in [14] under the assumption that there are no stagnation points in the flow and that the free surface is the graph of a function (no overturning). A related approach was pursued in the paper [8], which considers a water wave beneath an elastic sheet obtained by minimization in a class of rearrangements. The novelty of our formulation is that it allows rotational waves, overturning free surface profiles, stagnation points and critical layers in the flow. Another advantage is that the Lagrangian is expressed directly in terms of the physical variables instead of depending on a specific choice of coordinates. A key aspect of our formulation is that the Lagrangian involves a variable domain of integration. Although we are not aware of an earlier deduction of this formulation in this generality, we refer to it as "the standard variational formulation" due to its rather classical form.

The second variational formulation is specific to waves with constant vorticity and it turns out to be more useful than the first formulation because it involves just a single function of a single variable. Its essential advantage is that it reduces the governing equations to one pseudodifferential equation for a function of one variable, namely, the elevation of the free surface when the fluid domain is regarded as the conformal image of a strip. Somewhat unexpectedly, this equation for the elevation is coupled to a scalar constraint. The corresponding Lagrangian is essentially obtained by composing the first Lagrangian, suitably restricted, with a conformal mapping from a strip.

B.1. The standard variational formulation

In this subsection we construct a functional on a certain function space, critical points of which are solutions of the travelling water-wave problem (2.1)–(2.2). Since we are dealing with a free-boundary problem, in which the domain Ω is unknown, the function space, to be denoted by $\mathcal{A}$, will consist of pairs (Ω,ψ), where ψ is a function on Ω. Some of the conditions expressed by (2.1)–(2.2) will be required to hold for every element of $\mathcal{A}$, while the remaining ones will emerge from the condition of criticality.

We consider the space $\mathcal{A}$ of pairs (Ω,ψ), where Ω is an L-periodic strip-like domain of class $C^{2,\alpha}$, and $\psi\in C^{2,\alpha}_L(\mathbb{R}^2;\mathbb{R})$ satisfies (2.2b)–(2.2c). The subscript "L" is used to indicate periodicity in the X-variable, with period $L>0$, while $\alpha\in(0,1)$ is a Hölder exponent. The behaviour of ψ outside $\bar{\Omega}$ is of no importance. The only restriction we impose on the geometry of the free surface $\mathcal{S}$ is the requirement that the curve is not self-intersecting. No restrictions are imposed on the pattern of the streamlines, so that we can handle overhanging profiles and critical layers.

For any pair (Ω,ψ) in $\mathcal{A}$, the periodicity in the X-variable permits us to restrict attention to a cell $\Omega^\dagger$ bounded below by the real axis $\mathcal{B}$, above by the free surface $\mathcal{S}$ and laterally by two vertical lines situated at horizontal distance L. For definiteness, and to ensure that $\Omega^\dagger$ is a connected set, one may choose one of the vertical lines in the definition of $\Omega^\dagger$ (and hence the other one too) so as to pass through the lowest point of $\mathcal{S}$. Consider on the space $\mathcal{A}$ the functional

$$\mathcal{L}(\Omega,\psi)=\iint_{\Omega^\dagger}(|\nabla\psi|^2+2\Upsilon\psi-2gY+Q)\,d\mathbb{X}, \tag{B.1}$$

where $\mathbb{X}=(X,Y)$. Variations of $\mathcal{L}$ with respect to Ω are defined by means of changes of independent variables, as described below.

THEOREM 20. *Any critical point (Ω,ψ) of the functional $\mathcal{L}$ over the space $\mathcal{A}$ is a solution to the governing equations* (2.1)–(2.2).

Proof. We only need to show that a critical point (Ω,ψ) of $\mathcal{L}$ on $\mathcal{A}$ satisfies (2.2a) and (2.2d). We first investigate, for a fixed domain Ω the rate of change of $\mathcal{L}$ with respect to variations of the *dependent* variable ψ which do not change the boundary values on $\mathcal{S}$ and $\mathcal{B}$. For smooth functions $\varphi\colon\Omega\to\mathbb{R}$ that are L-periodic in the horizontal variable and vanish in a neighbourhood of $\mathcal{S}$ and of $\mathcal{B}$, the first variation $\delta\mathcal{L}(\Omega,\psi;\varphi)$ of $\mathcal{L}$ at (Ω,ψ) in the direction of φ is defined by

$$\delta\mathcal{L}(\Omega,\psi;\varphi)=\frac{d}{d\varepsilon}\mathcal{L}(\Omega,\psi+\varepsilon\varphi)\bigg|_{\varepsilon=0};$$

cf. [24, §2.1]. A *weak extremal* ψ of $\mathcal{L}$ is a solution of the Euler–Lagrange equation $\delta\mathcal{L}(\Omega,\psi;\varphi)=0$ for all φ of the type described above, and a straight-forward computation shows that the weak extremals $\psi\in C_L^{2,\alpha}(\Omega;\mathbb{R})$ are precisely the solutions to the *Euler–Lagrange equation*

$$\Delta\psi=\Upsilon \quad \text{in } \Omega. \tag{B.2}$$

We now consider variations $\mathbb{X}\mapsto\mathfrak{D}_\varepsilon(\mathbb{X})$ of the *independent* variables $\mathbb{X}$, allowing modifications of the domain Ω, but such that the modified domains are still L-periodic strip-like domains. This leads to the notion of *strong inner extremals* that will not only satisfy an equation of Euler–Lagrange-type but also a free boundary condition, cf. [24, §3.2]. Given a vector field $\mathfrak{f}\in C_L^{2,\alpha}(\mathbb{R}^2;\mathbb{R}^2)$, the support of which is contained in the upper half-plane $\{\mathbb{X}:Y>0\}$, consider the parameter-dependent family of mappings $\mathfrak{D}_\varepsilon:\mathbb{R}^2\to\mathbb{R}^2$ defined by

$$\mathfrak{D}_\varepsilon(\mathbb{X})=\mathbb{X}+\varepsilon\mathfrak{f}(\mathbb{X}).$$

Let $\Omega_\varepsilon=\mathfrak{D}_\varepsilon(\Omega)$. For $|\varepsilon|<\varepsilon_0$ with $\varepsilon_0>0$ sufficiently small, $\mathfrak{D}_\varepsilon$ is a diffeomorphism from $\bar{\Omega}$ onto $\bar{\Omega}_\varepsilon$. For a given $\psi\in C_L^{2,\alpha}(\bar{\Omega};\mathbb{R})$ we can now define a *strong inner variation in the direction of* $\mathfrak{f}$ by

$$\psi_\varepsilon(\mathbb{X})=\psi(\mathfrak{D}_\varepsilon^{-1}(\mathbb{X})) \quad \text{for } \mathbb{X}\in\bar{\Omega}_\varepsilon \text{ and } \varepsilon\in(-\varepsilon_0,\varepsilon_0).$$

Note that, for any $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$, the pair $(\Omega_\varepsilon,\psi_\varepsilon)$ still belongs to $\mathcal{A}$. Correspondingly, the expression

$$\partial\mathcal{L}(\Omega,\psi;\mathfrak{f})=\frac{d}{d\varepsilon}\iint_{(\Omega_\varepsilon)^\dagger}F(\mathbb{X},\psi_\varepsilon(\mathbb{X}),\nabla\psi_\varepsilon(\mathbb{X}))\,d\mathbb{X}\bigg|_{\varepsilon=0},$$

where

$$F(X,Y,z,p_1,p_2)=p_1^2+p_2^2+2\Upsilon z-2gY+Q \tag{B.3}$$

is the integrand in the functional $\mathcal{L}$, is called the *inner variation* of the functional $\mathcal{L}$ at ψ in the direction of the vector field $\mathfrak{f}$. A mapping $\psi\in C_L^{2,\alpha}(\bar{\Omega};\mathbb{R})$ is said to be a *strong inner extremal* of $\mathcal{L}$ if $\partial\mathcal{L}(\Omega,\psi;\mathfrak{f})=0$ holds for all such vector fields $\mathfrak{f}$. Also, a mapping $\psi\in C_L^{2,\alpha}(\bar{\Omega};\mathbb{R})$ is said to be an *inner extremal* of $\mathcal{L}$ if $\partial\mathcal{F}(\Omega,\psi;\mathfrak{f})=0$ holds for the subclass of vector fields $\mathfrak{f}$ that vanish in a neighbourhood of $\mathcal{S}$ and of $\mathcal{B}$. Any inner extremal $\psi\in C_L^{2,\alpha}(\bar{\Omega};\mathbb{R})$ will satisfy the *Noether equation*

$$(\Delta\psi-\Upsilon)\nabla\psi=0 \quad \text{in } \Omega; \tag{B.4}$$

cf. [24, §3.1 and Proposition 1, §3.2]. Note the equivalence of (B.2) and (B.4) if stagnation points do not occur. However, in our setting stagnation points are permissible.

Moreover, a strong inner extremal satisfies (B.4) as well as the boundary conditions

$$\begin{cases} \nu_1\left(p_1\dfrac{\partial F}{\partial p_1}-F\right)+\nu_2 p_1\dfrac{\partial F}{\partial p_2}=0, \\ \nu_1 p_2\dfrac{\partial F}{\partial p_1}+\nu_2\left(p_2\dfrac{\partial F}{\partial p_2}-F\right)=0, \end{cases} \quad \text{on } \mathcal{S}, \tag{B.5}$$

where $\nu=(\nu_1,\nu_2)$ is the outer unit normal to $\mathcal{S}$ and F is defined in (B.3). The condition $\psi=0$ on $\mathcal{S}$, which must hold since $(\Omega,\psi)\in\mathcal{A}$, ensures that ν is collinear with $\nabla_{\mathbb{X}}\psi$ at all points $\mathbb{X}\in\mathcal{S}$ where $\nabla_{\mathbb{X}}\psi(\mathbb{X})\neq(0,0)$. At every such point, a mere substitution into (B.5) in combination with (2.2b) leads us to (2.2d). On the other hand, at points $\mathbb{X}\in\mathcal{S}$ where $\nabla_{\mathbb{X}}\psi(\mathbb{X})=(0,0)$, inspection of (B.5) yields $F=0$, which in this case is the same as (2.2d). This completes the proof of Theorem 20. □

Remark 21. While Theorem 20 is specifically formulated for flows with constant vorticity, there is a counterpart for travelling waves on flows with *general vorticity distributions*, in which case equation (2.2a) is replaced by

$$\Delta\psi=\Upsilon(\psi),$$

where now Υ is an arbitrary function of one variable. In this case, we would work on the same space $\mathcal{A}$, but with the functional $\mathcal{L}$ in (B.1) replaced by

$$\mathcal{L}(\Omega,\psi)=\iint_{\Omega^\dagger}(|\nabla\psi|^2+2\Xi(\psi)-2gY+Q)\,d\mathbb{X},$$

and, correspondingly, the integrand F in (B.3) being replaced by

$$F(X,Y,z,p_1,p_2)=p_1^2+p_2^2+2\Xi(z)-2gY+Q,$$

where $\Xi(z)=\int_0^z\Upsilon(s)\,ds$. Even discontinuous vorticity functions Υ are permissible if one lowers the regularity requirements and uses a suitable weak formulation (see [17] and [48]). The setting of weak solutions is well-suited for the use of geometric methods to investigate the behaviour of the surface wave profile near stagnation points; cf. [46], [47].

B.2. An alternative variational formulation

Since one of the great advantages of Lagrangian dynamics is the freedom it allows in the choice of coordinates and since variational formulations involving as few dependent and independent variables as possible are preferable, in what follows we express the functional $\mathcal{L}$, with a suitably restricted domain of definition, in terms of the function v associated as in §2.1 with each strip-like domain Ω. We obtain in this way exactly the functional Λ given by (2.42), the critical points of which have been shown in §2.3 to correspond to the single pseudodifferential equation (2.22a) coupled with the scalar constraint (2.22b). We establish therefore a precise correspondence between the functional Λ acting on functions of one variable and the energy-type functional $\mathcal{L}$ in the physical variables.

Let Ω be an arbitrary $2\pi/k$-periodic strip-like domain of class $C^{1,\alpha}$, for some $\alpha\in(0,1)$, and let ψ^Ω be the unique solution of (2.2a)–(2.2c). Let $h>0$ denote the conformal mean depth of Ω, let $U+iV$ be a conformal mapping from $\mathcal{R}_{kh}$ to Ω having the properties (2.4)–(2.7), and let v be given by (2.8). Let $\Omega^\dagger$ be defined as in the previous subsection, so that each of its lateral boundaries passes through a lowest point (trough) of $\mathcal{S}$, and let us denote by $\mathcal{S}^\dagger$ and $\mathcal{B}^\dagger$ the top and bottom boundaries of $\Omega^\dagger$. We choose for convenience the conformal mapping $U+iV$ so that $\mathcal{S}^\dagger$ is the image of the horizontal line segment $\{(x,0):x\in[-\pi,\pi]\}$. (This can be achieved by a suitable horizontal translation.)

THEOREM 22. *Let $\mathcal{L}$ be the functional given by* (B.1). *Then*

$$\begin{aligned}\mathcal{L}(\Omega,\psi^\Omega)=&\int_{-\pi}^{\pi}\left(Qv-gv^2-\frac{\Upsilon^2}{3}v^3\right)\left(\frac{1}{k}+\mathcal{C}_{kh}(v')\right)dx\\&+\int_{-\pi}^{\pi}\left(m-\frac{\Upsilon}{2}v^2\right)\left(\frac{m}{kh}-\frac{\Upsilon}{2kh}[v^2]-\Upsilon\mathcal{C}_{kh}(vv')\right)dx.\end{aligned}\tag{B.6}$$

Therefore, if we write

$$v=h+w,$$

where w is a 2π-periodic function of zero mean, we have

$$\mathcal{L}(\Omega,\psi^\Omega)=\Lambda(w,h),$$

where the functional Λ is given by (2.42).

Proof. For notational convenience, we use throughout the proof the notation ψ instead of ψ^Ω. (It should be kept in mind, however, that ψ satisfies (2.2a)–(2.2c), and is not an arbitrary function on Ω.) Let $\beta\in\mathbb{R}$ be such that $\mathcal{B}^\dagger$ is be the conformal image of a segment $\{(x,-kh):x\in[\beta-\pi,\beta+\pi]\}$.

Firstly, denoting by ν the outward unit normal at the boundary $\partial\Omega$ of Ω, Green's formula yields

$$\iint_{\Omega^\dagger}|\nabla\psi|^2\,d\mathbb{X}=-\Upsilon\iint_{\Omega^\dagger}\psi\,d\mathbb{X}-m\int_{\mathcal{B}^\dagger}\frac{\partial\psi}{\partial\nu}\,d\sigma,$$

by periodicity in the X-variable and by taking (2.2a)–(2.2c) into account. The outer normal on $\mathcal{B}$ is $\nu=(0,-1)$, so that

$$\frac{\partial\psi}{\partial\nu}(X,0)=-\psi_Y(X,0)\quad\text{on }\mathcal{B}.$$

Recall that the conformal mapping is $X=U(x,y)$, $Y=V(x,y)$. Since $\psi_X(X,0)=0$ by (2.2c), from (2.11) we infer that

$$\xi_y(x,-kh)=\psi_Y(X,0)U_x(x,-kh),\quad x\in\mathbb{R},$$

as $V_y=U_x$ by the Cauchy–Riemann equations. Therefore, using the periodicity of ξ,

$$\int_{\mathcal{B}^\dagger} \frac{\partial\psi}{\partial\nu}\,d\sigma = -\int_{\beta-\pi}^{\beta+\pi} \xi_y(x,-kh)\,dx = -\int_{-\pi}^{\pi} \xi_y(x,-kh)\,dx,$$

so that

$$\iint_{\Omega^\dagger} |\nabla\psi|^2\,d\mathbb{X} = -\Upsilon \iint_{\Omega^\dagger} \psi\,d\mathbb{X} + m\int_{-\pi}^{\pi} \xi_y(x,-kh)\,dx. \tag{B.7}$$

Similarly,

$$\begin{aligned}\iint_{\Omega^\dagger} \psi\,d\mathbb{X} &= \iint_{\Omega^\dagger} \left(\Delta\frac{Y^2}{2}\right)\psi\,d\mathbb{X}\\ &= \iint_{\Omega^\dagger} \frac{Y^2}{2}\Delta\psi\,d\mathbb{X} + \int_{\partial\Omega^\dagger} \psi\frac{\partial}{\partial\nu}\left(\frac{Y^2}{2}\right)d\sigma - \int_{\partial\Omega^\dagger} \frac{Y^2}{2}\frac{\partial\psi}{\partial\nu}\,d\sigma\\ &= \frac{\Upsilon}{2}\iint_{\Omega^\dagger} Y^2\,d\mathbb{X} - \int_{\mathcal{S}^\dagger} \frac{Y^2}{2}\frac{\partial\psi}{\partial\nu}\,d\sigma.\end{aligned}$$

But

$$\nu = \frac{(-V_x, U_x)}{\sqrt{U_x^2+V_x^2}}$$

is the outer normal on $\mathcal{S}$, so that on $\mathcal{S}$ we have

$$\frac{\partial\psi}{\partial\nu} = (\psi_X,\psi_Y)\cdot\nu = \frac{\psi_X U_y + \psi_Y V_y}{\sqrt{U_x^2+V_x^2}}\bigg|_{(x,0)} = \frac{\xi_y}{\sqrt{U_x^2+V_x^2}}\bigg|_{(x,0)},$$

in view of (2.11) and the Cauchy–Riemann equations

$$U_x = V_y \quad \text{and} \quad U_y = -V_x.$$

Since $d\sigma = \sqrt{U_x^2+V_x^2}\,|_{(x,0)}\,dx$, we obtain

$$\int_{\mathcal{S}^\dagger} \frac{Y^2}{2}\frac{\partial\psi}{\partial\nu}\,d\sigma = \int_{-\pi}^{\pi} \frac{V^2}{2}\xi_y\bigg|_{(x,0)}\,dx.$$

Moreover, using the Cauchy–Riemann equations for $U+iV$, the divergence theorem, the periodicity in the X-variable and the fact that $V(x,-kh)=0$ from (2.4), we get

$$\begin{aligned}\iint_{\Omega^\dagger} Y^2 d\mathbb{X} &= \iint_{\Omega^\dagger} \nabla_{\mathbb{X}}\cdot\left(0,\frac{Y^3}{3}\right)d\mathbb{X} = \frac{1}{3}\int_{-\pi}^{\pi} V^3(x,0)V_y(x,0)\,dx\\ &= \frac{1}{3}\int_{-\pi}^{\pi} V^3(x,0)U_x(x,0)\,dx.\end{aligned}$$

Combining the last four displayed equations, we obtain

$$\iint_{\Omega^\dagger} \psi\,d\mathbb{X} = \frac{\Upsilon}{6}\int_{-\pi}^{\pi} V^3(x,0)U_x(x,0)\,dx - \frac{1}{2}\int_{-\pi}^{\pi} V^2(x,0)\xi_y(x,0)\,dx.$$

Proceeding as above, we also get

$$\iint_{\Omega^\dagger} 1\,d\mathbb{X} = \int_{-\pi}^{\pi} V(x,0)U_x(x,0)\,dx \quad \text{and} \quad \iint_{\Omega^\dagger} Y\,d\mathbb{X} = \frac{1}{2}\int_{-\pi}^{\pi} V^2(x,0)U_x(x,0)\,dx.$$

Substituting the last three relations and (B.7) into (B.1) yields

$$\begin{aligned}\mathcal{L}(\Omega,\psi) &= m\int_{-\pi}^{\pi} \xi_y(x,-kh)\,dx - \frac{\Upsilon}{2}\int_{-\pi}^{\pi} V^2(x,0)\xi_y(x,0)\,dx \\ &\quad + \int_{-\pi}^{\pi}\left(\frac{\Upsilon^2}{6}V^3(x,0) - V^2(x,0) + QV(x,0)\right)U_x(x,0)\,dx.\end{aligned}$$

Taking (2.12) into account to express

$$\xi_y = \Upsilon V V_y + \zeta_y = \Upsilon V U_x + \zeta_y$$

and using $V=0$ on $y=-hk$, we get

$$\begin{aligned}\mathcal{L}(\Omega,\psi) &= m\int_{-\pi}^{\pi} \zeta_y(x,-kh)\,dx - \frac{\Upsilon}{2}\int_{-\pi}^{\pi} V^2(x,0)\zeta_y(x,0)\,dx \\ &\quad + \int_{-\pi}^{\pi}\left(-\frac{\Upsilon^2}{3}V^3(x,0) - gV^2(x,0) + QV(x,0)\right)U_x(x,0)\,dx.\end{aligned}$$

However, the harmonicity (2.13a) of ζ in the strip $\mathcal{R}_{kh}$ yields

$$\int_{-\pi}^{\pi} \zeta_y(x,-kh)\,dx = \int_{-\pi}^{\pi} \zeta_y(x,0)\,dx,$$

so that

$$\begin{aligned}\mathcal{L}(\Omega,\psi) &= \int_{-\pi}^{\pi}\left(QV(x,0) - gV^2(x,0) - \frac{\Upsilon^2}{3}V^3(x,0)\right)U_x(x,0)\,dx \\ &\quad + \int_{-\pi}^{\pi}\left(m - \frac{\Upsilon}{2}V^2(x,0)\right)\zeta_y(x,0)\,dx.\end{aligned}$$

Using the relations (2.19), the fact that

$$U_x(x,0) = V_y(x,0) = \frac{1}{k} + (\mathcal{C}_{kh}(v'))(x), \quad x\in\mathbb{R},$$

and recalling that ψ stands for ψ^Ω, we obtain (B.6). □

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