Acta Math., 216 (2016), 325–388
DOI: 10.1007/s11511-016-0140-6

Geometric measures in the dual Brunn–Minkowski theory and their associated Minkowski problems

by

Yong Huang

Hunan University
Changsha, China

Erwin Lutwak

New York University
Brooklyn, NY, U.S.A.

Deane Yang

New York University
Brooklyn, NY, U.S.A.

Gaoyong Zhang

New York University
Brooklyn, NY, U.S.A.

1. Introduction

The Brunn–Minkowski theory (or the theory of mixed volumes) of convex bodies, developed by Minkowski, Aleksandrov, Fenchel, et al., centers around the study of geometric functionals of convex bodies as well as the differentials of these functionals. The theory depends heavily on analytic tools such as the cosine transform on the unit sphere (a variant of the Fourier transform) and Monge–Ampère type equations. The fundamental geometric functionals in the Brunn–Minkowski theory are the quermassintegrals (which include volume and surface area as special cases). The differentials of volume, surface area and the other quermassintegrals are geometric measures called the area measures and (Federer's) curvature measures. These geometric measures are fundamental concepts in the Brunn–Minkowski theory.

A Minkowski problem is a characterization problem for a geometric measure generated by convex bodies: It asks for necessary and sufficient conditions in order that a given measure arises as the measure generated by a convex body. The solution of a Minkowski problem, in general, amounts to solving a degenerate fully non-linear partial differential equation. The study of Minkowski problems has a long history and strong influence on both the Brunn–Minkowski theory and fully non-linear partial differential equations, see [78] and [75]. Among the important Minkowski problems in the classical

Research of the first author supported, in part, by NSFC No.11371360; research of the other authors supported, in part, by NSF Grant DMS-1312181.

Brunn–Minkowski theory are the classical Minkowski problem itself, the Aleksandrov problem, the Christoffel problem, and the Minkowski–Christoffel problem.

There are two extensions of the Brunn–Minkowski theory: the dual Brunn–Minkowski theory, which emerged in the mid-1970s, and the L_p Brunn–Minkowski theory actively investigated since the 1990s but dating back to the 1950s. The important L_p surface area measure and its associated Minkowski problem in the L_p Brunn–Minkowski theory were introduced in [54]. The logarithmic Minkowski problem and the centro-affine Minkowski problem are unsolved singular cases, see [14] and [20]. The book [75] of Schneider presents a comprehensive account of the classical Brunn–Minkowski theory and its recent developments, see Chapters 8 and 9 for Minkowski problems.

For the dual Brunn–Minkowski theory, the situation is quite different. While, over the years, the "duals" of many concepts and problems of the classical Brunn–Minkowski theory have been discovered and studied, the duals of Federer's curvature measures and their associated Minkowski problems within the dual Brunn–Minkowski theory have remained elusive. Behind this lay our inability to calculate the differentials of the dual quermassintegrals. Since the revolutionary work of Aleksandrov in the 1930s, the non-linear partial differential equations that arise within the classical Brunn–Minkowski theory and within the L_p Brunn–Minkowski theory have done much to advance both theories. However, the intrinsic partial differential equations of the dual Brunn–Minkowski theory have had to wait a full 40 years after the birth of the dual theory to emerge. It was the elusive nature of the duals of Federer's curvature measures that kept these partial differential equations well hidden. As will be seen, the duals of Federer's curvature measures contain a number of surprises. Perhaps the biggest is that they connect known measures that were never imagined to be related. All this will be unveiled in the current work.

In the following, we first recall the important geometric measures and their associated Minkowski problems in the classical Brunn–Minkowski theory and the L_p Brunn–Minkowski theory. Then we explain how the missing geometric measures in the dual Brunn–Minkowski theory can be naturally discovered and how their associated Minkowski problems will be investigated.

As will be shown, the notion of *dual curvature measures* arises naturally from the fundamental geometric functionals (the dual quermassintegrals) in the dual Brunn–Minkowski theory. Their associated Minkowski problem will be called the *dual Minkowski problem*. Amazingly, both the logarithmic Minkowski problem as well as the Aleksandrov problem turn out to be special cases of the new dual Minkowski problem. Existence conditions for the solution of the dual Minkowski problem in the symmetric case will be given.

1.1. Geometric measures and their associated Minkowski problems in the Brunn–Minkowski theory

The fundamental geometric functional for convex bodies in Euclidean n-space, $\mathbb{R}^n$, is volume (Lebesgue measure), denoted by V. The support function $h_K\colon S^{n-1}\to\mathbb{R}$ of a compact convex set $K\subset\mathbb{R}^n$, is defined, for v in the unit sphere S^{n-1}, by

$$h_K(v)=\max\{v\cdot x : x\in K\},$$

where $v\cdot x$ is the inner product of v and x in $\mathbb{R}^n$. For a continuous $f\colon S^{n-1}\to\mathbb{R}$, some small $\delta=\delta_K>0$, and $t\in(-\delta,\delta)$, define the t-perturbation of K by f by

$$[K,f]_t=\{x\in\mathbb{R}^n : x\cdot v\leqslant h_K(v)+tf(v) \text{ for all } v\in S^{n-1}\}.$$

This convex body is called the *Wulff shape* of (K,f) with parameter t.

Surface area measure, area measures, and curvature measures. Aleksandrov established the following variational formula,

$$\frac{d}{dt}V([K,f]_t)\Big|_{t=0}=\int_{S^{n-1}} f(v)\,dS(K,v), \tag{1.1}$$

where $S(K,\cdot)$ is the Borel measure on S^{n-1} known as the *surface area measure of* K. This formula suggests that the surface area measure can be viewed as the differential of the volume functional. The total measure $S(K)=|S(K,\cdot)|$ of the surface area measure is the ordinary surface area of K. Aleksandrov's proof of (1.1) makes critical use of the Minkowski mixed-volume inequality—an inequality that is an extension of the classical isoperimetric inequality, see Schneider [75, Lemma 7.5.3]. In this paper, we shall present the first proof of (1.1) that makes no use of mixed-volume inequalities.

The surface area measure of a convex body can be defined directly, for each Borel set $\eta\subset S^{n-1}$, by

$$S(K,\eta)=\mathcal{H}^{n-1}(\nu_K^{-1}(\eta)), \tag{1.2}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. Here the Gauss map

$$\nu_K\colon\partial' K\longrightarrow S^{n-1}$$

is defined on the subset $\partial' K$ of those points of ∂K that have a unique outer unit normal and is hence defined $\mathcal{H}^{n-1}$-a.e. on ∂K (see (2.11) for a precise definition). If one views the reciprocal Gauss curvature of a smooth convex body as a function of the outer unit normals of the body, then surface area measure is the extension to arbitrary convex bodies (that are not necessarily smooth) of the reciprocal Gauss curvature. In fact, if ∂K is of class C^2 and has everywhere positive curvature, then the surface area measure has a positive density,

$$\frac{dS(K,v)}{dv} = \det(h_{ij}(v)+h_K(v)\delta_{ij}), \tag{1.3}$$

where $(h_{ij})_{i,j}$ is the Hessian matrix of h_K with respect to an orthonormal frame on S^{n-1}, δ_{ij} is the Kronecker delta, the determinant is precisely the reciprocal Gauss curvature of ∂K at the point of ∂K whose outer unit normal is v, and where the Radon–Nikodym derivative is with respect to spherical Lebesgue measure.

We recall that the *quermassintegrals* are the principal geometric functionals in the Brunn–Minkowski theory. These are the elementary mixed volumes which include volume, surface area, and mean width. In differential geometry, the quermassintegrals are the integrals of intermediate mean curvatures of closed smooth convex hypersurfaces. In integral geometry, the quermassintegrals are the means of the projection areas of convex bodies:

$$W_{n-i}(K) = \frac{\omega_n}{\omega_i}\int_{G(n,i)} \mathrm{vol}_i(K|\xi)\,d\xi, \quad i=1,\dots,n, \tag{1.4}$$

where $G(n,i)$ is the Grassmann manifold of i-dimensional subspaces in $\mathbb{R}^n$, $K|\xi$ is the image of the orthogonal projection of K onto ξ, where vol_i is just $\mathcal{H}^i$ (or Lebesgue measure in ξ), and ω_i is the i-dimensional volume of the i-dimensional unit ball. The integration here is with respect to the rotation-invariant probability measure on $G(n,i)$.

Since $V=W_0$ and with $S(K,\cdot)=S_{n-1}(K,\cdot)$, it would be desirable if Aleksandrov's variational formula (1.1) could be extended to quermassintegrals; i.e., if it were the case that

$$\frac{d}{dt}W_{n-j-1}([K,f]_t)\bigg|_{t=0} = \int_{S^{n-1}} f(v)\,dS_j(K,v), \quad j=0,\dots,n-1, \tag{1.5}$$

for each continuous $f\colon S^{n-1}\to\mathbb{R}$. In the special case where K is sufficiently smooth and has positive curvature everywhere, formula (1.5) can be easily verified. Unfortunately, in general (1.5) is only known for the very special case where f is a support function and where the derivative is a right derivative. The measures defined by (1.5) (for the case where f is a support function and the derivative is a right derivative) are called the *area measures* and were introduced by Fenchel–Jessen and Aleksandrov (see Schneider [75, p. 214]). The proof of the variational formula (1.5), for the case where f is a support function and the derivative is a right derivative, depends on the Steiner formula for mixed volumes. But the special cases in which (1.5) are known to hold are of little use in the study of the "Minkowski problems" for area measures. The lack of knowledge concerning the left derivative for the quermassintegrals in (1.5) is one of the obstacles to tackling the partial differential equations associated with the area measures. That (1.5) does not hold for arbitrary convex bodies was already known by the middle of the last century; see e.g [42] for recent work.

In addition to the area measures of Aleksandrov and Fenchel–Jessen, there exists another set of measures $\mathcal{C}_0(K,\cdot),...,\mathcal{C}_{n-1}(K,\cdot)$ called *curvature measures*, which were introduced by Federer [23] for sets of positive reach, and are also closely related to the quermassintegrals. A direct treatment of curvature measures for convex bodies was given by Schneider [73], [74]; see also [75, p. 214]. If K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, then each ray emanating from the origin intersects a unique point on ∂K and a unique point on the unit sphere S^{n-1}. This fact induces a bi-Lipschitz map $r_K: S^{n-1}\to\partial K$. The pull-back of the curvature measure $\mathcal{C}_j(K,\cdot)$ on ∂K via r_K is the measure $C_j(K,\cdot)$ on the unit sphere S^{n-1}, which is called the *j-th curvature measure* of K. The measure $C_0(K,\cdot)$ was first defined by Aleksandrov, who called it the *integral curvature of K*; see [3]. The total measures of both area measures and curvature measures give the quermassintegrals:

$$S_j(K,S^{n-1}) = C_j(K,S^{n-1}) = nW_{n-j}(K),$$

for $j=0,1,...,n-1$; see Schneider [75, p. 213].

Minkowski problems in the Brunn–Minkowski theory. One of the main problems in the Brunn–Minkowski theory is characterizing the area and curvature measures. The well-known classical Minkowski problem is: *given a finite Borel measure μ on S^{n-1}, what are the necessary and sufficient conditions on μ so that μ is the surface area measure $S(K,\cdot)$ of a convex body K in $\mathbb{R}^n$?* The Minkowski problem was first studied by Minkowski [63], [64], who demonstrated both existence and uniqueness of solutions for the problem when the given measure is either discrete or has a continuous density. Aleksandrov [1], [2] and Fenchel–Jessen [24] independently solved the problem in 1938 for arbitrary measures. Their methods are variational and (1.1) is crucial for transforming the Minkowski problem into an optimization problem. Analytically, the Minkowski problem is equivalent to solving a degenerate Monge–Ampère equation. Establishing the regularity of the solution to the Minkowski problem is difficult and has led to a long series of influential works (see, for example, Nirenberg [67], Cheng–Yau [19], Pogorelov [71], Caffarelli [17]).

After solving the Minkowski problem, Aleksandrov went on to characterize his integral curvature $C_0(K,\cdot)$, which is called the *Aleksandrov problem.* He was able to solve it completely by using his *mapping lemma*; see [3]. Further work on the Aleksandrov problem from the partial differential equation and the mass transport viewpoints is due to Guan–Li [34] and Oliker [68].

Finding necessary and sufficient conditions so that a given measure is the area measure $S_1(K,\cdot)$ of a convex body K is the *Christoffel problem*. Firey [25] and Berg [11] solved the problem independently. See Pogorelov [70] for a partial result in the smooth case, Schneider [72] for a more explicit solution in the polytope case, Grinberg–Zhang [30] for an abbreviated approach to Firey's and Berg's solution, Goodey–Yaskin–Yaskina [29], and Schneider [75, §8.3.2], for a Fourier transform approach. In general, characterizing the area measure $S_j(K,\cdot)$ is called the *Minkowski–Christoffel problem*: *given an integer* $1\leqslant j\leqslant n-1$ *and a finite Borel measure* μ *on* S^{n-1}, *what are the necessary and sufficient conditions so that* μ *is the area measure* $S_j(K,\cdot)$ *of a convex body* K *in* $\mathbb{R}^n$. The case where $j=1$ is the Christoffel problem, and the case where $j=n-1$ is the classical Minkowski problem. For $1<j<n-1$, it has been a long-standing open problem. Important progress was made recently by Guan–Ma [36]. See Guan–Guan [32] for a variant of this problem.

Extending Aleksandrov's work on the integral curvature and characterizing other curvature measures is also a major unsolved problem: *given an integer* $1\leqslant j\leqslant n-1$ *and a finite Borel measure* μ *on* S^{n-1}, *what are the necessary and sufficient conditions so that* μ *is the curvature measure* $C_j(K,\cdot)$ *of a convex body* K *in* $\mathbb{R}^n$. This is the *Minkowski problem for curvature measures* which can also be called the *general Aleksandrov problem*. See Guan–Lin–Ma [35] and the recent work of Guan–Li–Li [33] on this problem.

Cone-volume measure and logarithmic Minkowski problem. In addition to the surface area measure of a convex body, another fundamental measure associated with a convex body K in $\mathbb{R}^n$ that contains the origin in its interior is the *cone-volume measure* V_K, also denoted by $V(K,\cdot)$, defined for Borel sets $\eta\subset S^{n-1}$ by

$$V_K(\eta)=\frac{1}{n}\int_{x\in\nu_K^{-1}(\eta)} x\cdot\nu_K(x)\,d\mathcal{H}^{n-1}(x)=V(K\cap c(\eta)), \tag{1.6}$$

which is the volume of the cone $K\cap c(\eta)$, where $c(\eta)$ is the cone of rays eminating from the origin such that $\partial K\cap c(\eta)=\nu_K^{-1}(\eta)$ for the Borel set $\eta\subset S^{n-1}$.

A very important property of the cone-volume measure is its $\mathrm{SL}(n)$ invariance, or simply called affine invariance. The area and curvature measures are all $\mathrm{SO}(n)$ invariant. The $\mathrm{SL}(n)$ invariance of the cone-volume measure makes the measure a useful notion in the geometry of normed spaces; see [10], [60], [61], [65], [66], [69]. The Minkowski problem for the cone-volume measure is called the *logarithmic Minkowski problem*. It asks for necessary and sufficient conditions for a given measure on the unit sphere to be the cone-volume measure of a convex body. The existence part of the logarithmic Minkowski problem has been solved recently for the case of even measures within the class of origin-symmetric convex bodies, see [14]. A sufficient condition for discrete (not-necessarily even) measures was given by Zhu [82]. It was shown in [13] that the solution to both the existence and uniqueness questions for the logarithmic Minkowski problem for even measures would lead to a stronger Brunn–Minkowski inequality. It was shown in [15] that the necessary and sufficient conditions for the existence of a solution to the logarithmic Minkowski problem for even measures are identical to the necessary and sufficient conditions for the existence of an affine transformation that maps the given measure into one that is isotropic. The problem has strong connections with curvature flows; see Andrews [8], [9].

L_p surface area measure and L_p Minkowski problem. The L_p Brunn–Minkowski theory is an extension of the classical Brunn–Minkowski theory; see [41], [47], [51], [54], [56], [57], [59], [62], [75]. The L_p surface area measure, introduced in [54], is a fundamental notion in the L_p-theory. For fixed $p\in\mathbb{R}$, and a convex body K in $\mathbb{R}^n$ that contains the origin in its interior, the L_p *surface area measure* $S^{(p)}(K,\cdot)$ of K is a Borel measure on S^{n-1} defined, for a Borel set $\eta\subset S^{n-1}$, by

$$S^{(p)}(K,\eta)=\int_{x\in\nu_K^{-1}(\eta)}(x\cdot\nu_K(x))^{1-p}\,d\mathcal{H}^{n-1}(x). \tag{1.7}$$

The surface area measure and the cone-volume measure are the special cases $p=1$ and $p=0$, respectively, of L_p surface area measure. The L_p Minkowski problem, posed by Lutwak (see, e.g., [54]), asks for necessary and sufficient conditions that would guarantee that a given measure on the unit sphere would be the L_p surface area measure of a convex body; see, e.g., [18], [20], [43], [44], [54], [55], and [84]. The case of $p=1$ is the classical Minkowski problem, the case of $p=0$ is the logarithmic Minkowski problem (see [14]), and the case of $p=-n$ is the centro-affine Minkowski problem (see Chou–Wang [20], Lu–Wang [46], and Zhu [83]). The solution to the L_p Minkowski problem has been proven to be a critical tool in establishing sharp affine Sobolev inequalities via affine isoperimetric inequalities; see [21], [40], [56], [58], [79].

1.2. Geometric measures and their associated Minkowski problems in the dual Brunn–Minkowski theory

A theory analogous to the theory of mixed volumes was introduced in 1970s in [52]. It demonstrates a remarkable duality in convex geometry, and thus is called the *theory* *of dual mixed volumes*, or the *dual Brunn–Minkowski theory.* The duality, as a guiding principle, is conceptual in a heuristic sense and has motivated much investigation. A good explanation of this conceptual duality is given in Schneider [75, p. 507]. The aspect of the duality between projections and cross-sections of convex bodies is thoroughly discussed in Gardner [27]. The duality will be called the *conceptual duality* in convex geometry.

The main geometric functionals in the dual Brunn–Minkowski theory are the *dual quermassintegrals.* The following integral geometric definition of the dual quermassintegrals, via the volume of the central sections of the body, shows their amazing dual nature to the quermassintegrals defined in (1.4):

$$\widetilde{W}_{n-i}(K)=\frac{\omega_n}{\omega_i}\int_{G(n,i)}\mathrm{vol}_i(K\cap\xi)\,d\xi,\quad i=1,...,n. \tag{1.8}$$

The volume functional V is both the quermassintegral W_0 and the dual quermassintegral $\widetilde{W}_0$. Earlier investigations in the dual Brunn–Minkowski theory centered around finding isoperimetric inequalities involving dual mixed volumes that mirrored those for mixed volumes, see Schneider [75, §9.3 and §9.4] and Gardner [27]. It was shown in [80] that the fundamental kinematic formula for quermassintegrals in integral geometry has a dual version for dual quermassintegrals.

Exciting developments in the dual Brunn–Minkowski theory began in the late 1980s because of the duality between projection bodies and intersection bodies exhibited in [53]. The study of central sections of convex bodies by way of intersection bodies and the Busemann–Petty problem has attracted extensive attention in convex geometry; see, for example, [16], [26], [28], [53], [81], and see [27], [45] for additional references. Some of these works bring techniques from harmonic analysis, in particular, Radon transforms and the Fourier transform, into the dual Brunn–Minkowski theory, see [27], [45]. This is similar to the applications of cosine transform to the study of projection bodies and the Shephard problem in the Brunn–Minkowski theory (see [75, §10.11] and [27, §4.2]). However, the Busemann–Petty problem is far more interesting and is a problem whose isomorphic version is still a major open problem in asymptotic convex geometric analysis.

There were important areas where progress in the dual theory lagged behind that of the classical theory. Extending Aleksandrov's variational formula (1.1) from $\widetilde{W}_0$ to the dual quermassintegrals $\widetilde{W}_i$ is one of the main such challenges. This is critically needed in order to discover the duals of Federer's curvature measures of the classical theory. One purpose of this work is to establish this extension and thus to add key elements to the conceptual duality of the Brunn–Minkowski theory and the dual Brunn–Minkowski theory. The main concepts that will be introduced are the *dual curvature measures.*

Dual curvature measures. For each convex body K in $\mathbb{R}^n$ that contains the origin in its interior, we construct explicitly a set of geometric measures $\widetilde{C}_0(K,\cdot), ..., \widetilde{C}_n(K,\cdot)$, on S^{n-1} associated with the dual quermassintegrals, with

$$\widetilde{C}_j(K,S^{n-1}) = \widetilde{W}_{n-j}(K),$$

for $j=0,...,n$. These geometric measures can be viewed as the differentials of the dual quermassintegrals.

Our construction will show how these geometric measures, via conceptual duality, are the duals of the curvature measures, and thus warrant being called the *dual curvature measures* of K. While the curvature measures of a convex body depend closely on the body's boundary, its dual curvature measures depend more on the body's interior, but yet have deep connections with their classical counterparts. When $j=n$, the dual curvature measure $\widetilde{C}_n(K,\cdot)$ turns out to be the cone-volume measure of K. When $j=0$, the dual curvature measure $\widetilde{C}_0(K,\cdot)$ turns out to be Aleksandrov's integral curvature of the polar body of K (divided by n). When K is a polytope, $\widetilde{C}_j(K,\cdot)$ is discrete and concentrated on the outer unit normals of the facets of K with weights depending on the cones that are the convex hulls of the facets and the origin. Dual area measures are also defined. The new geometric measures we shall develop demonstrate yet again the amazing conceptual duality between the dual Brunn–Minkowski theory and the Brunn–Minkowski theory.

We establish dual generalizations of Aleksandrov's variational formula (1.1). Let K be a convex body in $\mathbb{R}^n$ that contains the origin in its interior, and let $f: S^{n-1}\to\mathbb{R}$ be continuous. For a sufficiently small $\delta>0$, define a family of *logarithmic Wulff shapes*,

$$[K,f]_t = \{x\in\mathbb{R}^n : x\cdot v \leqslant h_t(v) \text{ for all } v\in S^{n-1}\},$$

for each $t\in(-\delta,\delta)$, where $h_t(v)$, for $v\in S^{n-1}$, is given by

$$\log h_t(v) = \log h_K(v) + t f(v) + o(t,v),$$

and $\lim_{t\to 0} o(t,v)/t=0$ uniformly in v. The main formula to be presented is the following.

Variational formula for dual quermassintegrals. For $1\leqslant j\leqslant n$, and each convex body K that contains the origin in the interior, there exists a Borel measure $\widetilde{C}_j(K,\cdot)$ on S^{n-1} such that

$$\frac{d}{dt}\widetilde{W}_{n-j}([K,f]_t)\bigg|_{t=0} = j\int_{S^{n-1}} f(v)\, d\widetilde{C}_j(K,v), \tag{1.9}$$

for each continuous $f: S^{n-1}\to\mathbb{R}$.

Obviously (1.9) demonstrates that the dual curvature measures are differentials of the dual quermassintegrals. Clearly (1.9) is the dual of the variational formula (1.5), which is only known to hold in special cases. Aleksandrov's variational formula (1.1) is the special case $j=n$ of (1.9). Thus, our formula is a direct extension of Aleksandrov's variational formula for volume to dual quermassintegrals. Our approach and method of proof are very different from both Aleksandrov's proof of (1.1) and the proof of (1.5) for the case where the function involved is a support function.

The main problem to be solved is the following characterization problem for the dual curvature measures.

Dual Minkowski problem for dual curvature measures. *Let k be an integer, $1\leqslant k\leqslant n$. If μ is a finite Borel measure on S^{n-1}, find necessary and sufficient conditions on μ so that it is the k-th dual curvature measure $\widetilde{C}_k(K,\cdot)$ of a convex body K in $\mathbb{R}^n$.*

This will be called the *dual Minkowski problem*. For $k=n$, the dual Minkowski problem is just the logarithmic Minkowski problem. As will be shown, when the measure μ has a density function $g\colon S^{n-1}\to\mathbb{R}$, the partial differential equation that is the dual Minkowski problem is a Monge–Ampère type equation on S^{n-1}:

$$\frac{1}{n}h(v)|\overline{\nabla}h(v)+h(v)v|^{k-n}\det(\overline{\nabla}^2h(v)+h(v)I)=g(v), \tag{1.10}$$

where h is the unknown function on S^{n-1} to be found, $\overline{\nabla}h$ and $\overline{\nabla}^2h$ denote the gradient vector and the Hessian matrix of h with respect to an orthonormal frame on S^{n-1}, and I is the identity matrix.

If the factor

$$\frac{1}{n}h(v)|\overline{\nabla}h(v)+h(v)v|^{k-n}$$

were omitted in (1.10), then (1.10) would become the partial differential equation of the classical Minkowski problem. If only the factor $|\overline{\nabla}h(v)+h(v)v|^{k-n}$ was omitted, then equation (1.10) would become the partial differential equation associated with the logarithmic Minkowski problem. The gradient component in (1.10) significantly increases the difficulty of the problem when compared to the classical Minkowski problem or the logarithmic Minkowski problem.

In this paper we treat the important symmetric case when the measure μ is even and the solution is within the class of origin-symmetric bodies. As will be shown, the existence of solutions depends on how much of the measure's mass can be concentrated on great subspheres.

Let μ be a finite Borel measure on S^{n-1}, and $1 \leqslant k \leqslant n$. We will say that the measure μ satisfies the *k-subspace mass inequality* if

$$\frac{\mu(S^{n-1} \cap \xi_i)}{\mu(S^{n-1})} < 1 - \frac{k-1}{k} \frac{n-i}{n-1},$$

for each $\xi_i \in G(n,i)$ and each $i=1,...,n-1$.

The main theorem of the paper is the following.

Existence for the dual Minkowski problem. *Let μ be a finite even Borel measure on S^{n-1}, and $1 \leqslant k \leqslant n$. If the measure μ satisfies the k-subspace mass inequality, then there exists an origin-symmetric convex body K in $\mathbb{R}^n$ such that $\widetilde{C}_k(K,\cdot)=\mu$.*

The case of $k=n$ was proved in [14]. New ideas and more delicate estimates are needed to prove the intermediate cases. We remark that existence for the dual Minkowski problem is far easier to prove for the special case where the given measure μ has a positive continuous density (in which case the subspace mass inequality is trivially satisfied). The singular general case for measures is substantially more delicate. It involves measure concentration and requires far more powerful techniques to solve. The sufficient 1-subspace mass inequality is obviously necessary for the case of $k=1$. The sufficient n-subspace mass inequality is also necessary for the case $k=n$, except that certain equality conditions must be satisfied as well (see [14] for details). Discovering the necessary conditions for other cases would be of considerable interest.

2. Preliminaries

2.1. Basic concepts regarding convex bodies

Schneider's book [75] is our standard reference for the basics regarding convex bodies. The books [27] and [31] are also good references.

Let $\mathbb{R}^n$ denote n-dimensional Euclidean space. For $x \in \mathbb{R}^n$, let $|x| = \sqrt{x \cdot x}$ be the Euclidean norm of x. For $x \in \mathbb{R}^n \setminus \{0\}$, define $\bar{x} \in S^{n-1}$ by $\bar{x} = x/|x|$. For a subset E in $\mathbb{R}^n \setminus \{0\}$ we let $\bar{E} = \{\bar{x} : x \in E\}$. The origin-centered unit ball $\{x \in \mathbb{R}^n : |x| \leqslant 1\}$ is always denoted by B, and its boundary by S^{n-1}. Write ω_n for the volume of B and recall that its surface area is $n\omega_n$.

For the set of continuous functions defined on the unit sphere S^{n-1} write $C(S^{n-1})$, and for $f \in C(S^{n-1})$ write $\|f\| = \max_{v \in S^{n-1}} |f(v)|$. We shall view $C(S^{n-1})$ as endowed with the topology induced by this *max-norm*. We write $C^+(S^{n-1})$ for the set of strictly positive functions in $C(S^{n-1})$, and $C_e^+(S^{n-1})$ for the set of functions in $C^+(S^{n-1})$ that are even.

Let ∇ be the gradient operator in $\mathbb{R}^n$ with respect to the Euclidean metric and $\overline{\nabla}$ be the gradient operator on S^{n-1} with respect to the induced metric. Then for a function $h:\mathbb{R}^n\to\mathbb{R}$ which is differentiable at $v\in\mathbb{R}^n$, with $|v|=1$, we have

$$\nabla h(v)=\overline{\nabla} h(v)+h(v)v.$$

If $K\subset\mathbb{R}^n$ is compact and convex, the support function h_K, previously defined on S^{n-1}, can be extended to $\mathbb{R}^n$ naturally, $h_K:\mathbb{R}^n\to\mathbb{R}$, by setting $h_K(x)=\max\{x\cdot y:y\in K\}$ for $x\in\mathbb{R}^n$. This extended support function is convex and homogeneous of degree 1. A compact convex subset of $\mathbb{R}^n$ is uniquely determined by its support function.

Denote by $\mathcal{K}^n$ the space of compact convex sets in $\mathbb{R}^n$ endowed with the *Hausdorff metric*; i.e., the distance between $K,L\in\mathcal{K}^n$ is $\|h_K-h_L\|$. By a *convex body* in $\mathbb{R}^n$ we will always mean a compact convex set with non-empty interior. Denote by $\mathcal{K}_o^n$ the class of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors, and denote by $\mathcal{K}_e^n$ the class of origin-symmetric convex bodies in $\mathbb{R}^n$.

Let $K\subset\mathbb{R}^n$ be compact and star-shaped with respect to the origin. The *radial function* $\varrho_K:\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ is defined by

$$\varrho_K(x)=\max\{\lambda:\lambda x\in K\},$$

for $x\neq0$. A compact star-shaped (about the origin) set is uniquely determined by its radial function on S^{n-1}. Denote by $\mathcal{S}^n$ the set of compact star-shaped sets. A *star body* is a compact star-shaped set with respect to the origin whose radial function is continuous and positive. If K is a star body, then obviously

$$\partial K=\{\varrho_K(u)u:u\in S^{n-1}\}.$$

Denote by $\mathcal{S}_o^n$ the space of star bodies in $\mathbb{R}^n$ endowed with the *radial metric*; i.e., the distance between $K,L\in\mathcal{S}_o^n$ is $\|\varrho_K-\varrho_L\|$. Note that $\mathcal{K}_o^n\subset\mathcal{S}_o^n$ and that on the space $\mathcal{K}_o^n$ the Hausdorff metric and radial metric are equivalent, and thus $\mathcal{K}_o^n$ is a subspace of $\mathcal{S}_o^n$.

If $K\in\mathcal{K}_o^n$, then it is easily seen that the radial function and the support function of K are related by

$$h_K(v)=\max_{u\in S^{n-1}}(u\cdot v)\,\varrho_K(u)\qquad\text{for } v\in S^{n-1},\tag{2.1}$$

$$\frac{1}{\varrho_K(u)}=\max_{v\in S^{n-1}}\frac{u\cdot v}{h_K(v)}\qquad\text{for } u\in S^{n-1}.\tag{2.2}$$

For a convex body $K\in\mathcal{K}_o^n$, the *polar body* K^* of K is the convex body in $\mathbb{R}^n$ defined by

$$K^*=\{x\in\mathbb{R}^n : x\cdot y\leqslant 1 \text{ for all } y\in K\}.$$

From the definition of the polar body, we see that on $\mathbb{R}^n\setminus\{0\}$,

$$\varrho_K=\frac{1}{h_{K^*}} \quad\text{and}\quad h_K=\frac{1}{\varrho_{K^*}}. \tag{2.3}$$

For $K,L\subset\mathbb{R}^n$ that are compact and convex, and real $a,b\geqslant 0$, the *Minkowski combination*, $aK+bL\subset\mathbb{R}^n$, is the compact, convex set defined by

$$aK+bL=\{ax+by : x\in K \text{ and } y\in L\},$$

and its support function is given by

$$h_{aK+bL}=ah_K+bh_L. \tag{2.4}$$

For real $t>0$, and a convex body K, let $K_t=K+tB$ denote the *parallel body* of K. The volume of the parallel body K_t is a polynomial in t, called the *Steiner polynomial*,

$$V(K_t)=\sum_{i=0}^{n}\binom{n}{i}W_{n-i}(K)t^{n-i}.$$

The coefficient $W_{n-i}(K)$ is called the $(n-i)$-th *quermassintegral* of K which is precisely the geometric invariant defined in (1.4).

For $K,L\subset\mathbb{R}^n$ that are compact and star-shaped (with respect to the origin), and real $a,b\geqslant 0$, the *radial combination*, $aK\tilde{+}bL\subset\mathbb{R}^n$, is the compact star-shaped set defined by

$$aK\tilde{+}bL=\{ax+by : x\in K,\ y\in L \text{ and } x\cdot y=|x|\,|y|\}.$$

Note that the condition $x\cdot y=|x|\,|y|$ means that either $y=\alpha x$ or $x=\alpha y$ for some $\alpha\geqslant 0$. The radial function of the radial combination of two star-shaped sets is the combination of their radial functions; i.e.,

$$\varrho_{aK\tilde{+}bL}=a\varrho_K+b\varrho_L.$$

For real $t>0$, and star body K, let $\widetilde{K}_t=K\tilde{+}tB$ denote the *dual parallel body* of K. The volume of the dual parallel body $\widetilde{K}_t$ is a polynomial in t, called the *dual Steiner polynomial*,

$$V(\widetilde{K}_t)=\sum_{i=0}^{n}\binom{n}{i}\widetilde{W}_{n-i}(K)t^{n-i}.$$

The coefficient $\widetilde{W}_{n-i}(K)$ is the $(n-i)$-th *dual quermassintegral* of K which is precisely the geometric invariant defined in (1.8). For the $(n-i)$-th dual quermassintegral of K we have the easily established integral representation

$$\widetilde{W}_{n-i}(K) = \frac{1}{n}\int_{S^{n-1}} \varrho_K^i(u)\,du, \tag{2.5}$$

where such integrals should always be interpreted as being with respect to spherical Lebesgue measure.

In view of the integral representation (2.5), the dual quermassintegrals can be extended in an obvious manner: For $q\in\mathbb{R}$, and a star body K, the $(n-q)$-th dual quermassintegral $\widetilde{W}_{n-q}(K)$ is defined by

$$\widetilde{W}_{n-q}(K) = \frac{1}{n}\int_{S^{n-1}} \varrho_K^q(u)\,du. \tag{2.6}$$

For real $q\neq 0$, define the *normalized dual quermassintegral* $\overline{W}_{n-q}(K)$ by

$$\overline{W}_{n-q}(K) = \left(\frac{1}{n\omega_n}\int_{S^{n-1}} \varrho_K^q(u)\,du\right)^{1/q}, \tag{2.7}$$

and, for $q=0$, by

$$\overline{W}_n(K) = \exp\left(\frac{1}{n\omega_n}\int_{S^{n-1}} \log \varrho_K(u)\,du\right). \tag{2.8}$$

It will also be helpful to adopt the following notation:

$$\widetilde{V}_q(K) = \widetilde{W}_{n-q}(K) \quad \text{and} \quad \overline{V}_q(K) = \overline{W}_{n-q}(K), \tag{2.9}$$

called the *q-th dual volume* of K and the *normalized q-th dual volume* of K, respectively. Note, in particular, the fact that

$$\overline{V}_n(K) = \left(\frac{V(K)}{\omega_n}\right)^{1/n}. \tag{2.10}$$

2.2. The radial Gauss map of a convex body

Let K be a convex body in $\mathbb{R}^n$. For each $v\in S^{n-1}$, the hyperplane

$$H_K(v) = \{x\in\mathbb{R}^n : x\cdot v = h_K(v)\}$$

is called the *supporting hyperplane to K with unit normal v*.

For $\sigma\subset\partial K$, the *spherical image of* σ is defined by

$$\boldsymbol{\nu}_K(\sigma)=\{v\in S^{n-1}:x\in H_K(v)\text{ for some }x\in\sigma\}\subset S^{n-1}.$$

For $\eta\subset S^{n-1}$, the *reverse spherical image of* η is defined by

$$\boldsymbol{x}_K(\eta)=\{x\in\partial K:x\in H_K(v)\text{ for some }v\in\eta\}\subset\partial K.$$

Let $\sigma_K\subset\partial K$ be the set consisting of all $x\in\partial K$ for which the set $\boldsymbol{\nu}_K(\{x\})$, which we frequently abbreviate as $\boldsymbol{\nu}_K(x)$, contains more than a single element. It is well known that $\mathcal{H}^{n-1}(\sigma_K)=0$ (see Schneider [75, p. 84]). The function

$$\nu_K:\partial K\setminus\sigma_K\longrightarrow S^{n-1}, \tag{2.11}$$

defined by letting $\nu_K(x)$ be the unique element in $\boldsymbol{\nu}_K(x)$ for each $x\in\partial K\setminus\sigma_K$, is called the *spherical image map* of K and is known to be continuous (see Lemma 2.2.12 in Schneider [75]). In the introduction, $\partial K\setminus\sigma_K$ was abbreviated as $\partial' K$, something we will often do. Note that from definition (1.2) and the Riesz representation theorem, it follows immediately that, for each continuous $g:S^{n-1}\to\mathbb{R}$, one has

$$\int_{\partial' K} g(\nu_K(x))\,d\mathcal{H}^{n-1}(x)=\int_{S^{n-1}} g(v)\,dS(K,v). \tag{2.12}$$

Also, from definitions (1.2) and (1.6), it follows that, for the cone-volume measure $V(K,\cdot)$, we have

$$dV(K,\cdot)=\frac{1}{n}h_K\,dS(K,\cdot). \tag{2.13}$$

The set $\eta_K\subset S^{n-1}$ consisting of all $v\in S^{n-1}$ for which the set $\boldsymbol{x}_K(v)$ contains more than a single element, is of $\mathcal{H}^{n-1}$-measure zero (see Theorem 2.2.11 in Schneider [75]). The function

$$x_K:S^{n-1}\setminus\eta_K\longrightarrow\partial K, \tag{2.14}$$

defined for each $v\in S^{n-1}\setminus\eta_K$ by letting $x_K(v)$ be the unique element in $\boldsymbol{x}_K(v)$, is called the *reverse spherical image map*. The vectors in $S^{n-1}\setminus\eta_K$ are called the *regular normal vectors* of K. Thus, $v\in S^{n-1}$ is a regular normal vector of K if and only if the intersection $\partial K\cap H_K(v)$ consists of a single point. The function x_K is well known to be continuous (see Lemma 2.2.12 in Schneider [75]).

For $K\in\mathcal{K}_o^n$, define the *radial map* of K,

$$r_K:S^{n-1}\longrightarrow\partial K,\quad\text{by } r_K(u)=\varrho_K(u)u\in\partial K\text{ for }u\in S^{n-1}.$$

Note that $r_K^{-1}:\partial K\to S^{n-1}$ is just the restriction to ∂K of the map $x\mapsto\bar{x}$.

For $\omega\subset S^{n-1}$, define the *radial Gauss image of* ω by

$$\boldsymbol{\alpha}_K(\omega)=\boldsymbol{\nu}_K(r_K(\omega))\subset S^{n-1}.$$

Thus, for $u\in S^{n-1}$, one has

$$\boldsymbol{\alpha}_K(u)=\{v\in S^{n-1}:r_K(u)\in H_K(v)\}. \tag{2.15}$$

Define the *radial Gauss map* of the convex body $K\in\mathcal{K}_o^n$,

$$\alpha_K: S^{n-1}\setminus\omega_K\longrightarrow S^{n-1}, \quad \text{by } \alpha_K=\nu_K\circ r_K,$$

where $\omega_K=r_K^{-1}(\sigma_K)$. Since r_K^{-1} is a bi-Lipschitz map between the spaces ∂K and S^{n-1}, it follows that ω_K has spherical Lebesgue measure 0. Observe that if $u\in S^{n-1}\setminus\omega_K$, then $\boldsymbol{\alpha}_K(u)$ contains only the element $\alpha_K(u)$. Note that since both ν_K and r_K are continuous, α_K is continuous.

For $\eta\subset S^{n-1}$, define the *reverse radial Gauss image* of η by

$$\boldsymbol{\alpha}_K^*(\eta)=r_K^{-1}(\boldsymbol{x}_K(\eta))=\overline{\boldsymbol{x}_K(\eta)}. \tag{2.16}$$

Thus,

$$\boldsymbol{\alpha}_K^*(\eta)=\{\bar{x}:x\in\partial K \text{ where } x\in H_K(v) \text{ for some } v\in\eta\}.$$

Define the *reverse radial Gauss map* of the convex body $K\in\mathcal{K}_o^n$,

$$\alpha_K^*: S^{n-1}\setminus\eta_K\longrightarrow S^{n-1}, \quad \text{by } \alpha_K^*=r_K^{-1}\circ x_K. \tag{2.17}$$

Note that since both r_K^{-1} and x_K are continuous, α_K^* is continuous.

Note that, for a subset $\eta\subset S^{n-1}$,

$$\boldsymbol{\alpha}_K^*(\eta)=\{u\in S^{n-1}:r_K(u)\in H_K(v) \text{ for some } v\in\eta\}. \tag{2.18}$$

For $u\in S^{n-1}$ and $\eta\subset S^{n-1}$, it is easily seen that

$$u\in\boldsymbol{\alpha}_K^*(\eta) \quad \text{if and only if} \quad \boldsymbol{\alpha}_K(u)\cap\eta\neq\varnothing. \tag{2.19}$$

Thus, $\boldsymbol{\alpha}_K^*$ is monotone non-decreasing with respect to set inclusion.

If we abbreviate $\boldsymbol{\alpha}_K^*(\{v\})$ by $\boldsymbol{\alpha}_K^*(v)$, then (2.19) yields

$$u\in\boldsymbol{\alpha}_K^*(v) \quad \text{if and only if} \quad v\in\boldsymbol{\alpha}_K(u). \tag{2.20}$$

If $u \notin \omega_K$, then $\boldsymbol{\alpha}_K(u)=\{\alpha_K(u)\}$ and (2.19) becomes

$$u \in \boldsymbol{\alpha}^*_K(\eta) \quad \text{if and only if} \quad \alpha_K(u) \in \eta, \tag{2.21}$$

and hence (2.21) holds for almost all $u \in S^{n-1}$ with respect to spherical Lebesgue measure. It similarly follows that, if $v \notin \eta_K$ and $\omega \subset S^{n-1}$, then

$$v \in \boldsymbol{\alpha}_K(\omega) \quad \text{if and only if} \quad \alpha^*_K(v) \in \omega, \tag{2.22}$$

and hence (2.22) holds for almost all $v \in S^{n-1}$ with respect to spherical Lebesgue measure.

The following lemma consists of a basic fact regarding the reverse radial Gauss map. This fact is Lemma 2.2.14 in Schneider [75], an alternate proof of which is presented below.

LEMMA 2.1. *If $\eta \subset S^{n-1}$ is a Borel set, then $\boldsymbol{\alpha}^*_K(\eta)=\overline{\boldsymbol{x}_K(\eta)} \subset S^{n-1}$ is spherical Lebesgue measurable.*

Proof. The continuity of α_K assures that the inverse image $\alpha_K^{-1}(\eta)$, of the Borel set η in S^{n-1}, is a Borel set in the space $S^{n-1} \backslash \omega_K$ with relative topology. Since each Borel set in $S^{n-1} \backslash \omega_K$ is just the restriction of a Borel set in S^{n-1}, it follows that $\alpha_K^{-1}(\eta)$ is the restriction of a Borel set in S^{n-1} to $S^{n-1} \backslash \omega_K$, and is thus Lebesgue measurable in S^{n-1} (as ω_K has Lebesgue measure zero). Since $\boldsymbol{\alpha}^*_K(\eta)$ and $\alpha_K^{-1}(\eta)$ differ by a set of Lebesgue measure zero, the set $\boldsymbol{\alpha}^*_K(\eta)$ must be Lebesgue measurable in S^{n-1} as well. □

If $g: S^{n-1} \to \mathbb{R}$ is a Borel function, then $g \circ \alpha_K$ is spherical Lebesgue measurable because it is just the composition of a Borel function g and a continuous function α_K in $S^{n-1} \backslash \omega_K$ with ω_K having Lebesgue measure zero. Moreover, if g is a bounded Borel function, then $g \circ \alpha_K$ is spherical Lebesgue integrable. In particular, $g \circ \alpha_K$ is spherical Lebesgue integrable, for each continuous function $g: S^{n-1} \to \mathbb{R}$.

LEMMA 2.2. *Let $K_i \in \mathcal{K}^n_o$ be such that $\lim_{i\to\infty} K_i = K_0 \in \mathcal{K}^n_o$. Let $\omega = \bigcup_{i=0}^{\infty} \omega_{K_i}$ be the set (of $\mathcal{H}^{n-1}$-measure zero) off of which all of the α_{K_i} are defined. If $u_i \in S^{n-1} \backslash \omega$ are such that $\lim_{i\to\infty} u_i = u_0 \in S^{n-1} \backslash \omega$, then $\lim_{i\to\infty} \alpha_{K_i}(u_i) = \alpha_{K_0}(u_0)$.*

Proof. Since the sequence of radial maps r_{K_i} converges to r_{K_0}, uniformly, we have that $r_{K_i}(u_i) \to r_{K_0}(u_0)$. Let $x_i = r_{K_i}(u_i)$ and $v_i = \nu_{K_i}(r_{K_i}(u_i)) = \alpha_{K_i}(u_i)$.

Since $x_i \in \partial K_i$ and $u_i \notin \omega_{K_i}$, the vector v_i is the unique outer unit normal to the support hyperplane of K_i at x_i. Thus, we have

$$x_i \cdot v_i = h_{K_i}(v_i).$$

Suppose that a subsequence of the unit vectors v_i (which we again call v_i) converges to $v'\in S^{n-1}$. Since h_{K_i} converges to h_{K_0}, uniformly, $h_{K_i}(v_i)\to h_{K_0}(v')$. This, together with $x_i\to x_0$ and $v_i\to v'$ gives

$$x_0\cdot v' = h_{K_0}(v'). \tag{2.23}$$

But $x_0=r_{K_0}(u_0)$ is a boundary point of K_0, and since $u_0\notin\omega_{K_0}$, we conclude from (2.23) that v' must be the unique outer unit normal of K_0 at $x_0=r_{K_0}(u_0)$. And hence, $v'=\nu_{K_0}(r_{K_0}(u_0))=\alpha_{K_0}(u_0)$. Thus, all convergent subsequences of $v_i=\alpha_{K_i}(u_i)$ converge to $\alpha_{K_0}(u_0)$.

Consider a subsequence of $\alpha_{K_i}(u_i)$. Since S^{n-1} is compact, the subsequence has a subsequence that converges, and by the above it converges to $\alpha_{K_0}(u_0)$. Thus, every subsequence of $\alpha_{K_i}(u_i)$ has a subsequence that converges to $\alpha_{K_0}(u_0)$. □

LEMMA 2.3. *If $\{\eta_j\}_{j=1}^{\infty}$ is a sequence of subsets of S^{n-1}, then*

$$\boldsymbol{\alpha}_K^*\left(\bigcup_{j=1}^{\infty}\eta_j\right) = \bigcup_{j=1}^{\infty}\boldsymbol{\alpha}_K^*(\eta_j).$$

Proof. If $v\in\bigcup_{j=1}^{\infty}\eta_j$, then $v\in\eta_{j_1}$ for some j_1, and, by the monotonicity of $\boldsymbol{\alpha}_K^*$ with respect to set inclusion,

$$\boldsymbol{\alpha}_K^*(v)\subseteq\boldsymbol{\alpha}_K^*(\eta_{j_1})\subseteq\bigcup_{j=1}^{\infty}\boldsymbol{\alpha}_K^*(\eta_j).$$

Thus, $\boldsymbol{\alpha}_K^*\big(\bigcup_{j=1}^{\infty}\eta_j\big)\subseteq\bigcup_{j=1}^{\infty}\boldsymbol{\alpha}_K^*(\eta_j)$. Moreover, if $u\in\bigcup_{j=1}^{\infty}\boldsymbol{\alpha}_K^*(\eta_j)$, then for some j_2 we have that $u\in\boldsymbol{\alpha}_K^*(\eta_{j_2})\subseteq\boldsymbol{\alpha}_K^*\big(\bigcup_{j=1}^{\infty}\eta_j\big)$. Thus, $\boldsymbol{\alpha}_K^*\big(\bigcup_{j=1}^{\infty}\eta_j\big)\supseteq\bigcup_{j=1}^{\infty}\boldsymbol{\alpha}_K^*(\eta_j)$. □

LEMMA 2.4. *If $\{\eta_j\}_{j=1}^{\infty}$ is a sequence of pairwise disjoint sets in S^{n-1}, then*

$$\{\boldsymbol{\alpha}_K^*(\eta_j)\setminus\omega_K\}_{j=1}^{\infty}$$

is pairwise disjoint as well.

Proof. Suppose that there exists a u such that $u\in\boldsymbol{\alpha}_K^*(\eta_{j_1})\setminus\omega_K$ and $u\in\boldsymbol{\alpha}_K^*(\eta_{j_2})\setminus\omega_K$. As $u\notin\omega_K$, we know that $\boldsymbol{\alpha}_K(u)$ is a singleton. But (2.19), in conjunction with $u\in\boldsymbol{\alpha}_K^*(\eta_{j_1})$ and $u\in\boldsymbol{\alpha}_K^*(\eta_{j_2})$, yields $\boldsymbol{\alpha}_K(u)\cap\eta_{j_1}\neq\varnothing$ and $\boldsymbol{\alpha}_K(u)\cap\eta_{j_2}\neq\varnothing$, which contradicts the fact that $\eta_{j_1}\cap\eta_{j_2}=\varnothing$ since $\boldsymbol{\alpha}_K(u)$ is a singleton. □

The reverse radial Gauss image of a convex body and the radial Gauss image of its polar body are related.

LEMMA 2.5. *If $K\in\mathcal{K}_o^n$, then*

$$\boldsymbol{\alpha}_K^*(\eta) = \boldsymbol{\alpha}_{K^*}(\eta)$$

for each $\eta\subset S^{n-1}$.

Proof. It suffices to show that

$$\boldsymbol{\alpha}_K^*(v) = \boldsymbol{\alpha}_{K^*}(v)$$

for each $v \in S^{n-1}$. Fix $v \in S^{n-1}$. From (2.15), we see that, for $u \in S^{n-1}$,

$$u \in \boldsymbol{\alpha}_{K^*}(v) \quad \text{if and only if} \quad H_{K^*}(u) \text{ is a support hyperplane at } \varrho_{K^*}(v)v,$$

that is,

$$u \in \boldsymbol{\alpha}_{K^*}(v) \quad \text{if and only if} \quad h_{K^*}(u) = (u \cdot v)\varrho_{K^*}(v).$$

By (2.3), this is the case if and only if

$$h_K(v) = (v \cdot u)\, \varrho_K(u) = v \cdot r_K(u),$$

or equivalently, using (2.15), if and only if

$$v \in \boldsymbol{\alpha}_K(u).$$

But, from (2.20) we know that $v \in \boldsymbol{\alpha}_K(u)$ if and only if $u \in \boldsymbol{\alpha}_K^*(v)$. □

For almost all $v \in S^{n-1}$ we have $\boldsymbol{\alpha}_K^*(v) = \{\alpha_K^*(v)\}$, and for almost all $v \in S^{n-1}$ we have $\boldsymbol{\alpha}_{K^*}(v) = \{\alpha_{K^*}(v)\}$. These two facts combine to give the following lemma.

LEMMA 2.6. *If $K \in \mathcal{K}_o^n$, then*

$$\alpha_K^* = \alpha_{K^*}$$

almost everywhere on S^{n-1}, with respect to spherical Lebesgue measure.

2.3. Wulff shapes and convex hulls

Throughout, $\Omega \subset S^{n-1}$ will denote a closed set that is assumed not to be contained in any closed hemisphere of S^{n-1}. Let $h: \Omega \to (0, \infty)$ be continuous. The *Wulff shape* $[h] \in \mathcal{K}_o^n$, also known as the *Aleksandrov body*, determined by h is the convex body defined by

$$[h] = \{x \in \mathbb{R}^n : x \cdot v \leqslant h(v) \text{ for all } v \in \Omega\}.$$

Obviously, if $K \in \mathcal{K}_o^n$,

$$[h_K] = K.$$

For the radial function of the Wulff shape we have

$$\varrho_{[h]}(u)^{-1} = \max_{v \in \Omega} (u \cdot v) h(v)^{-1}, \tag{2.24}$$

which is easily verified by:

$$\begin{aligned}\varrho_{[h]}(u) &= \max\{r>0: ru\in[h]\} = \max\{r>0: ru\cdot v\leqslant h(v) \text{ for all } v\in\Omega\}\\ &= \max\{r>0: r\max_{v\in\Omega}(u\cdot v)h(v)^{-1}\leqslant 1\} = \frac{1}{\max_{v\in\Omega}(u\cdot v)h(v)^{-1}}\end{aligned}$$

for each $u\in S^{n-1}$.

Let $\varrho:\Omega\to(0,\infty)$ be continuous. Since $\Omega\subset S^{n-1}$ is assumed to be closed and ϱ is continuous, $\{\varrho(u)u:u\in\Omega\}$ is a compact set in $\mathbb{R}^n$. Hence, the convex hull $\langle\varrho\rangle$ generated by ϱ,

$$\langle\varrho\rangle = \operatorname{conv}\{\varrho(u)u : u\in\Omega\},$$

is compact as well (see Schneider [75, Theorem 1.1.11]). Since Ω is not contained in any closed hemisphere of S^{n-1} and ϱ is strictly positive, the compact convex set $\langle\varrho\rangle$ contains the origin in its interior. Obviously, if $K\in\mathcal{K}_o^n$,

$$\langle\varrho_K\rangle = K. \tag{2.25}$$

We shall make frequent use of the fact that

$$h_{\langle\varrho\rangle}(v) = \max_{u\in\Omega}(v\cdot u)\varrho(u) \tag{2.26}$$

for all $v\in S^{n-1}$.

LEMMA 2.7. *Let Ω be a closed subset of S^{n-1} that is not contained in any closed hemisphere of S^{n-1} and let $\varrho:\Omega\to(0,\infty)$ be continuous. If v is a regular normal vector of $\langle\varrho\rangle$, then $\boldsymbol{\alpha}^*_{\langle\varrho\rangle}(v)\subset\Omega$.*

Proof. By (2.26) there exists a $u_0\in\Omega$ such that

$$h_{\langle\varrho\rangle}(v) = (u_0\cdot v)\varrho(u_0).$$

This means that

$$\varrho(u_0)u_0 \in H_{\langle\varrho\rangle}(v) = \{x\in\mathbb{R}^n : x\cdot v = h_{\langle\varrho\rangle}(v)\}, \tag{2.27}$$

and since clearly $\varrho(u_0)u_0\in\langle\varrho\rangle$, it follows from (2.27) that $\varrho(u_0)u_0\in\partial\langle\varrho\rangle$ and $u_0\in\boldsymbol{\alpha}^*_{\langle\varrho\rangle}(v)$. But v is a regular normal vector of $\langle\varrho\rangle$, and hence

$$\boldsymbol{\alpha}^*_{\langle\varrho\rangle}(v) = \{\alpha^*_{\langle\varrho\rangle}(v)\}.$$

We conclude that $\alpha^*_{\langle\varrho\rangle}(v)=u_0\in\Omega$, which completes the proof. □

The Wulff shape of a function and the convex hull generated by its reciprocal are related.

LEMMA 2.8. *Let $\Omega\subset S^{n-1}$ be a closed set that is not contained in any closed hemisphere of S^{n-1}. Let $h:\Omega\to(0,\infty)$ be continuous. Then the Wulff shape $[h]$ determined by h and the convex hull $\langle 1/h\rangle$ generated by the function $1/h$ are polar reciprocals of each other*; *i.e.*,

$$[h]^*=\langle 1/h\rangle.$$

Proof. Let $\varrho=1/h$. Then, by (2.24) and (2.26), we see that for $u\in S^{n-1}$,

$$\varrho_{[h]}(u)^{-1}=\max_{v\in\Omega}(u\cdot v)h(v)^{-1}=\max_{v\in\Omega}(u\cdot v)\varrho(v)=h_{\langle\varrho\rangle}(u).$$

This and (2.3) give the desired identity. □

We recall Aleksandrov's convergence theorem for Wulff shapes (see Schneider [75, p. 412]): If a sequence of continuous functions $h_i:\Omega\to(0,\infty)$ converges uniformly to $h:\Omega\to(0,\infty)$, then the sequence of Wulff shapes $[h_i]$ converges to the Wulff shape $[h]$ in $\mathcal{K}_o^n$.

We will use the following convergence of convex hulls: If a sequence of positive continuous functions $\varrho_i:\Omega\to(0,\infty)$ converges uniformly to $\varrho:\Omega\to(0,\infty)$, then the sequence of convex hulls $\langle\varrho_i\rangle$ converges to the convex hull $\langle\varrho\rangle$ in $\mathcal{K}_o^n$. Lemma 2.8, together with Aleksandrov's convergence theorem for Wulff shapes, provides a quick proof.

Let $f:\Omega\to\mathbb{R}$ be continuous and $\delta>0$. Let $h_t:\Omega\to(0,\infty)$ be a continuous function defined for each $t\in(-\delta,\delta)$ and each $v\in\Omega$ by

$$\log h_t(v)=\log h(v)+tf(v)+o(t,v), \tag{2.28}$$

where, for each $t\in(-\delta,\delta)$, the function $o(t,\cdot):\Omega\to\mathbb{R}$ is continuous and $\lim_{t\to 0}o(t,\cdot)/t=0$ uniformly on Ω. Denote by $[h_t]$ the Wulff shape determined by h_t,

$$[h_t]=\{x\in\mathbb{R}^n: x\cdot v\leqslant h_t(v) \text{ for all } v\in\Omega\}.$$

We shall call $[h_t]$ a *logarithmic family of Wulff shapes formed by* (h,f). On occasion, we shall write $[h_t]$ as $[h,f,t]$, and if h happens to be the support function of a convex body K perhaps as $[K,f,t]$, or as $[K,f,o,t]$, if required for clarity.

Let $g:\Omega\to\mathbb{R}$ be continuous and $\delta>0$. Let $\varrho_t:\Omega\to(0,\infty)$ be a continuous function defined for each $t\in(-\delta,\delta)$ and each $u\in\Omega$ by

$$\log\varrho_t(u)=\log\varrho(u)+tg(u)+o(t,u), \tag{2.29}$$

where, for each $t\in(-\delta,\delta)$, the function $o(t,\cdot):\Omega\to\mathbb{R}$ is continuous and $\lim_{t\to 0}o(t,\cdot)/t=0$ uniformly on Ω. Denote by $\langle\varrho_t\rangle$ the convex hull generated by ϱ_t,

$$\langle\varrho_t\rangle=\operatorname{conv}\{\varrho_t(u)u: u\in S^{n-1}\}.$$

We will call $\langle\varrho_t\rangle$ a *logarithmic family of convex hulls generated by* (ϱ,g). On occasion, we shall write $\langle\varrho_t\rangle$ as $\langle\varrho,g,t\rangle$, and if ϱ happens to be the radial function of a convex body K as $\langle K,g,t\rangle$, or as $\langle K,g,o,t\rangle$, if required for clarity.

2.4. Two integral identities

LEMMA 2.9. *Let* $K\in\mathcal{K}_o^n$ *and* $q\in\mathbb{R}$. *Then for each bounded Lebesgue integrable function* $f\colon S^{n-1}\to\mathbb{R}$,

$$\int_{S^{n-1}} f(u)\varrho_K(u)^q\,du=\int_{\partial' K} x\cdot\nu_K(x)f(\bar{x})|x|^{q-n}\,d\mathcal{H}^{n-1}(x). \tag{2.30}$$

Proof. We only need to establish

$$\int_{S^{n-1}} f(u)\varrho_K(u)^n\,du=\int_{\partial' K} x\cdot\nu_K(x)f(\bar{x})\,d\mathcal{H}^{n-1}(x), \tag{2.31}$$

because replacing f with $f\varrho_K^{q-n}$ in (2.31) gives (2.30).

We begin by establishing equality (2.31) for C^1-functions f. To that end, define $F\colon\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ by letting $F(x)=f(\bar{x})$ for $x\neq 0$. Thus $F(x)$ is a C^1 homogeneous function of degree 0 in $\mathbb{R}^n\setminus\{0\}$. The homogeneity of F implies that $x\cdot\nabla F(x)=0$, and thus we have that $\operatorname{div}(F(x)x)=nF(x)$ for all $x\neq 0$.

Let $B_\delta\subset K$ be the ball of radius $\delta>0$ centered at the origin. Apply the divergence theorem for sets of finite perimeter (see [22, §5.8, Theorem 5.16]) to $K\setminus B_\delta$, and get

$$\begin{aligned}n\int_{K\setminus B_\delta} F(x)\,dx&=\int_{\partial' K} F(x)\,x\cdot\nu_K(x)\,d\mathcal{H}^{n-1}(x)-\int_{\partial B_\delta} F(x)\,x\cdot\nu_{B_\delta}(x)\,d\mathcal{H}^{n-1}(x)\\&=\int_{\partial' K} F(x)\,x\cdot\nu_K(x)\,d\mathcal{H}^{n-1}(x)-\delta^n\int_{S^{n-1}} F(u)\,du,\end{aligned}$$

and hence

$$\int_{\partial' K} F(x)\,x\cdot\nu_K(x)\,d\mathcal{H}^{n-1}(x)=n\int_{K\setminus\{0\}} F(x)\,dx.$$

Switching to polar coordinates gives

$$\int_{K\setminus\{0\}} F(x)\,dx=\int_{S^{n-1}}\int_0^{\varrho_K(u)} F(ru)r^{n-1}\,dr\,du=\frac{1}{n}\int_{S^{n-1}} F(u)\varrho_K^n(u)\,du,$$

which establishes (2.31) for C^1-functions.

Since every continuous function on S^{n-1} can be uniformly approximated by C^1 functions, (2.31) holds whenever f is continuous.

Define the measure $\widetilde{S}_n$ on S^{n-1} by

$$\widetilde{S}_n(\omega)=\frac{1}{n}\int_\omega \varrho_K^n(u)\,du$$

for each Lebesgue measurable set $\omega\subset S^{n-1}$, and define the measure $V_{\partial K}$ on $\partial' K$ by

$$V_{\partial K}(\sigma)=\int_\sigma x\cdot\nu_K(x)\,d\mathcal{H}^{n-1}(x)$$

for each $\mathcal{H}^{n-1}$-measurable $\sigma\subset\partial' K$.

It is easily seen that there exist constants $m_0, m_1, m_2>0$ such that

$$\mathcal{H}^{n-1}(r_K(\omega))\leqslant m_0\mathcal{H}^{n-1}(\omega),\ \ V_{\partial K}(\sigma)\leqslant m_1\mathcal{H}^{n-1}(\sigma)\ \text{ and }\ \widetilde{S}_n(\omega)\leqslant m_2\mathcal{H}^{n-1}(\omega) \tag{2.32}$$

for every spherical Lebesgue measurable set ω and $\mathcal{H}^{n-1}$-measurable σ.

Let $f\colon S^{n-1}\to\mathbb{R}$ be a bounded integrable function; say $|f(u)|\leqslant m$, for all $u\in S^{n-1}$. Lusin's theorem followed by the Tietze's extension theorem, guarantees the existence of an open subset $\omega_j\subset S^{n-1}$ and a continuous function $f_j\colon S^{n-1}\to\mathbb{R}$ so that $\mathcal{H}^{n-1}(\omega_j)<\frac{1}{j}$, while $f=f_j$ on $S^{n-1}\setminus\omega_j$, with $|f_j(u)|\leqslant m$ for all $u\in S^{n-1}$.

Observe that

$$\left|\int_{S^{n-1}}(f(u)-f_j(u))\varrho_K^n(u)\,du\right|\leqslant\left|\int_{S^{n-1}\setminus\omega_j}(f(u)-f_j(u))\varrho_K^n(u)\,du\right|+2mn\widetilde{S}_n(\omega_j),$$

where the integral on the right is zero, and that

$$\begin{aligned}&\left|\int_{\partial' K}(f(\bar{x})-f_j(\bar{x}))x\cdot\nu_K(x)\,d\mathcal{H}^{n-1}(x)\right|\\ &\qquad\leqslant\left|\int_{\partial' K\setminus r_K(\omega_j)}(f(\bar{x})-f_j(\bar{x}))x\cdot\nu_K(x)\,d\mathcal{H}^{n-1}(x)\right|+2mV_{\partial K}(r_K(\omega_j)),\end{aligned}$$

where the integral on the right is zero.

In light of (2.32), the above allows us to establish (2.31) for bounded integrable functions f, given that we had established (2.31) for the continuous functions f_j. □

LEMMA 2.10. *Let $K\in\mathcal{K}_o^n$ be strictly convex, and let $f\colon S^{n-1}\to\mathbb{R}$ and $F\colon\partial K\to\mathbb{R}$ be continuous. Then*

$$\int_{S^{n-1}}f(v)F(\nabla h_K(v))h_K(v)\,dS(K,v)=\int_{\partial' K}x\cdot\nu_K(x)f(\nu_K(x))F(x)\,d\mathcal{H}^{n-1}(x), \tag{2.33}$$

where the integral on the left is with respect to the surface area measure of K.

Proof. First observe that from the definition of the support function h_K and the definition of ν_K, it follows immediately that, for each $x\in\partial' K$,

$$h_K(\nu_K(x))=x\cdot\nu_K(x). \tag{2.34}$$

The assumption that K is strictly convex implies that ∇h_K always exists. But a convex function that is differentiable must be continuously differentiable and hence ∇h_K is continuous on S^{n-1}. We shall use the fact that the composition

$$\nabla h_K\circ\nu_K\colon\partial' K\longrightarrow\partial' K\quad\text{is the identity map} \tag{2.35}$$

(see, e.g., Schneider [75, p. 47]).

Now, from (2.34), (2.35), and (2.12), we have

$$\begin{aligned}\int_{\partial' K} x\cdot\nu_K(x) f(\nu_K(x)) F(x)\, d\mathcal{H}^{n-1}(x)\\ &= \int_{\partial' K} h_K(\nu_K(x)) f(\nu_K(x)) F(\nabla h_K(\nu_K(x))\, d\mathcal{H}^{n-1}(x)\\ &= \int_{S^{n-1}} f(v) F(\nabla h_K(v)) h_K(v)\, dS(K,v). \qquad \square\end{aligned}$$

3. Dual curvature measures

To display the conceptual duality between the Brunn–Minkowski theory and the dual Brunn–Minkowski theory, we first, following Schneider [75, Chapter 4], briefly develop the classical area and curvature measures for convex bodies in the Brunn–Minkowski theory. Then we introduce two new families of geometric measures: the dual curvature and dual area measures, in the dual Brunn–Minkowski theory. While curvature and area measures can be viewed as differentials of the quermassintegrals, dual curvature and dual area measures are viewed as differentials of the dual quermassintegrals.

3.1. Curvature and area measures

Let K be a convex body in $\mathcal{K}_o^n$. For $x\notin K$, denote by $d(K,x)$ the distance from x to K. Define the *metric projection map* $p_K:\mathbb{R}^n\setminus K\to\partial K$ so that $p_K(x)\in\partial K$ is the unique point satisfying

$$d(K,x) = |x-p_K(x)|.$$

Denote by $v_K:\mathbb{R}^n\setminus K\to S^{n-1}$ the *outer unit normal vector of* ∂K *at* $p_K(x)$, defined by

$$d(K,x)v_K(x) = x-p_K(x),$$

for $x\in\mathbb{R}^n\setminus K$.

For $t>0$, and Borel sets $\omega\subset S^{n-1}$ and $\eta\subset S^{n-1}$, let

$$\begin{aligned}A_t(K,\omega) &= \{x\in\mathbb{R}^n : 0<d(K,x)\leqslant t \text{ and } p_K(x)\in r_K(\omega)\},\\ B_t(K,\eta) &= \{x\in\mathbb{R}^n : 0<d(K,x)\leqslant t \text{ and } v_K(x)\in\eta\},\end{aligned}$$

which are the so-called *local parallel bodies* of K. There are the following Steiner-type formulas:

$$V(A_t(K,\omega)) = \frac{1}{n}\sum_{i=0}^{n-1}\binom{n}{i}t^{n-i}C_i(K,\omega),$$

$$V(B_t(K,\eta)) = \frac{1}{n}\sum_{i=0}^{n-1}\binom{n}{i}t^{n-i}S_i(K,\eta),$$

where $C_i(K,\cdot)$ is a Borel measure on S^{n-1}, called the *i-th curvature measure* of K, and $S_i(K,\cdot)$ is a Borel measure on S^{n-1}, called the *i-th area measure* of K. For all this, see Schneider [75, §4.2].

Note that the classical curvature measures are defined on the boundary ∂K, and are the image measures of the $C_i(K,\cdot)$ under the radial map $r_K: S^{n-1}\to\partial K$. Since, for $K\in\mathcal{K}_o^n$, the radial map r_K is bi-Lipschitz, one can define the curvature measures equivalently on either the space ∂K or the space S^{n-1}.

The $(n-1)$-th area measure $S_{n-1}(K,\cdot)$ is the usual surface area measure $S(K,\cdot)$ which can be defined, for each Borel $\eta\subset S^{n-1}$, directly by

$$S_{n-1}(K,\eta) = \mathcal{H}^{n-1}(\boldsymbol{x}_K(\eta)). \tag{3.1}$$

The $(n-1)$-th curvature measure $C_{n-1}(K,\cdot)$ on S^{n-1} can be defined, for each Borel $\omega\subset S^{n-1}$, by

$$C_{n-1}(K,\omega) = \mathcal{H}^{n-1}(r_K(\omega)). \tag{3.2}$$

From (3.1), (3.2), and the fact that $\boldsymbol{\alpha}_K^* = r_K^{-1}\circ\boldsymbol{x}_K$, we see that the $(n-1)$-th curvature measure $C_{n-1}(K,\cdot)$ on S^{n-1} and the $(n-1)$-th area measure $S_{n-1}(K,\cdot)$ on S^{n-1} are related by

$$C_{n-1}(K,\boldsymbol{\alpha}_K^*(\eta)) = S_{n-1}(K,\eta), \tag{3.3}$$

for each Borel $\eta\subset S^{n-1}$. See Schneider [75, Theorem 4.2.3].

The zeroth area measure $S_0(K,\cdot)$ is just spherical Lebesgue measure on S^{n-1}; i.e.,

$$S_0(K,\eta) = \mathcal{H}^{n-1}(\eta),$$

for Borel $\eta\subset S^{n-1}$. The zeroth curvature measure $C_0(K,\cdot)$ on S^{n-1} can be defined, for Borel $\omega\subset S^{n-1}$, by

$$C_0(K,\omega) = \mathcal{H}^{n-1}(\boldsymbol{\alpha}_K(\omega)); \tag{3.4}$$

that is, $C_0(K,\omega)$ is the spherical Lebesgue measure of $\boldsymbol{\alpha}_K(\omega)$. The zeroth curvature measure is also called the *integral curvature of* K. It was first defined by Aleksandrov. Obviously, (3.4) can be written as

$$C_0(K,\omega) = S_0(K,\boldsymbol{\alpha}_K(\omega)) \tag{3.5}$$

(see Schneider [75, Theorem 4.2.3]). If $K\in\mathcal{K}_o^n$ happens to be strictly convex, then (3.5) can be extended to

$$C_i(K,\omega)=S_i(K,\boldsymbol{\alpha}_K(\omega)),\quad i=0,1,...,n-1 \tag{3.6}$$

(see Schneider [75, Theorem 4.2.5]).

3.2. Definition of dual curvature and dual area measures

We first define the dual notions of the metric projection map p_K and the distance function $d(K,\cdot)$. Suppose $K\in\mathcal{K}_o^n$. Define the *radial projection map* $\tilde{p}_K:\mathbb{R}^n\setminus K\to\partial K$ by

$$\tilde{p}_K(x)=\varrho_K(x)x=r_K(\bar{x}),$$

for $x\in\mathbb{R}^n\setminus K$. For $x\in\mathbb{R}^n$, the *radial distance* $\tilde{d}(K,x)$ of x to K, is defined by

$$\tilde{d}(K,x)=\begin{cases}|x-\tilde{p}_K(x)|, & \text{if } x\notin K,\\ 0, & \text{if } x\in K.\end{cases}$$

Let

$$\tilde{v}_K(x)=\bar{x}.$$

For $t\geqslant 0$, a Lebesgue measurable set $\omega\subset S^{n-1}$, and a Borel set $\eta\subset S^{n-1}$, define

$$\tilde{A}_t(K,\eta)=\{x\in\mathbb{R}^n:0\leqslant\tilde{d}(K,x)\leqslant t \text{ and } \tilde{p}_K(x)\in\boldsymbol{x}_K(\eta)\}, \tag{3.7}$$

$$\widetilde{B}_t(K,\omega)=\{x\in\mathbb{R}^n:0\leqslant\tilde{d}(K,x)\leqslant t \text{ and } \tilde{v}_K(x)\in\omega\}, \tag{3.8}$$

to be the *local dual parallel bodies*. These local dual parallel bodies also have Steiner-type formulas as shown in the following theorem.

THEOREM 3.1. *Let $K\in\mathcal{K}_o^n$. For $t\geqslant 0$, a Lebesgue measurable set $\omega\subset S^{n-1}$, and a Borel set $\eta\subset S^{n-1}$,*

$$V(\tilde{A}_t(K,\eta))=\sum_{i=0}^{n}\binom{n}{i}t^{n-i}\widetilde{C}_i(K,\eta), \tag{3.9}$$

$$V(\widetilde{B}_t(K,\omega))=\sum_{i=0}^{n}\binom{n}{i}t^{n-i}\widetilde{S}_i(K,\omega), \tag{3.10}$$

where $\widetilde{C}_i(K,\cdot)$ and $\widetilde{S}_i(K,\cdot)$ are Borel measures on S^{n-1} given by

$$\widetilde{C}_i(K,\eta)=\frac{1}{n}\int_{\boldsymbol{\alpha}_K^*(\eta)}\varrho_K^i(u)\,du, \tag{3.11}$$

$$\widetilde{S}_i(K,\omega)=\frac{1}{n}\int_{\omega}\varrho_K^i(u)\,du. \tag{3.12}$$

Proof. Write (3.8) as

$$\widetilde{B}_t(K,\omega)=\{x\in\mathbb{R}^n:0\leqslant|x|\leqslant\varrho_K(\bar{x})+t \text{ with } \bar{x}\in\omega\}. \tag{3.13}$$

Writing $x=\varrho u$, with $\varrho\geqslant 0$ and $u\in S^{n-1}$, we find that

$$\begin{aligned}V(\widetilde{B}_t(K,\omega)) &= \int_{u\in\omega}\biggl(\int_0^{\varrho_K(u)+t}\varrho^{n-1}\,d\varrho\biggr)du\\ &= \frac{1}{n}\int_{u\in\omega}(\varrho_K(u)+t)^n\,du=\frac{1}{n}\sum_{i=0}^{n}\binom{n}{i}t^{n-i}\int_\omega\varrho_K^i(u)\,du.\end{aligned}$$

This gives (3.10) and (3.12).

In (3.7), the condition that $\tilde{p}_K(x)\in\boldsymbol{x}_K(\eta)$, or equivalently $r_K(\bar{x})\in\boldsymbol{x}_K(\eta)$, is by (2.16) the same as $\bar{x}\in r_K^{-1}(\boldsymbol{x}_K(\eta))=\boldsymbol{\alpha}_K^*(\eta)$. Thus, (3.7) can be written as

$$\tilde{A}_t(K,\eta)=\{x\in\mathbb{R}^n:0\leqslant|x|\leqslant\varrho_K(\bar{x})+t \text{ with } \bar{x}\in\boldsymbol{\alpha}_K^*(\eta)\ \}. \tag{3.14}$$

Since $\eta\subset S^{n-1}$ is a Borel set, $\boldsymbol{\alpha}_K^*(\eta)$ is a Lebesgue measurable subset of S^{n-1} by Lemma 2.1. Therefore, a glance at (3.13) and (3.14) immediately gives

$$\tilde{A}_t(K,\eta)=\widetilde{B}_t(K,\boldsymbol{\alpha}_K^*(\eta)).$$

Now (3.10) yields

$$V(\tilde{A}_t(K,\eta))=V(\widetilde{B}_t(K,\boldsymbol{\alpha}_K^*(\eta)))=\sum_{i=0}^{n}\binom{n}{i}t^{n-i}\widetilde{S}_i(K,\boldsymbol{\alpha}_K^*(\eta)),$$

and by defining

$$\widetilde{C}_i(K,\eta)=\widetilde{S}_i(K,\boldsymbol{\alpha}_K^*(\eta)) \tag{3.15}$$

we get both (3.9) and (3.11).

Obviously, $\widetilde{S}_i(K,\cdot)$ is a Borel measure. Note that since the integration in the integral representation of $\widetilde{S}_i(K,\cdot)$ is with respect to spherical Lebesgue measure, the measure $\widetilde{S}_i(K,\cdot)$ will assume the same value on sets that differ by a set of spherical Lebesgue measure zero.

We now show that $\widetilde{C}_i(K,\cdot)$ is a Borel measure as well. For the empty set $\varnothing$,

$$\widetilde{C}_i(K,\varnothing)=\widetilde{S}_i(K,\boldsymbol{\alpha}_K^*(\varnothing))=\widetilde{S}_i(K,\varnothing)=0.$$

Let $\{\eta_j\}_{j=1}^\infty$ be a sequence of pairwise disjoint Borel sets in S^{n-1}. From Lemmas 2.1 and 2.4, together with the fact that ω_K has spherical Lebesgue measure 0, we know that

$\{\boldsymbol{\alpha}_K^*(\eta_j)\backslash\omega_K\}_{j=1}^\infty$ is a sequence of pairwise disjoint Lebesgue measurable sets. Using (3.15), Lemma 2.3, the fact that ω_K has measure 0, the fact that the sets $\boldsymbol{\alpha}_K^*(\eta_j)\backslash\omega_K$ are pairwise disjoint, again the fact that ω_K has measure 0, and (3.15), we have

$$\begin{aligned}
\widetilde{C}_i\big(K,\textstyle\bigcup_{j=1}^\infty \eta_j\big) &= \widetilde{S}_i\big(K,\boldsymbol{\alpha}_K^*\big(\textstyle\bigcup_{j=1}^\infty \eta_j\big)\big)\\
&= \widetilde{S}_i\big(K,\textstyle\bigcup_{j=1}^\infty \boldsymbol{\alpha}_K^*(\eta_j)\big)\\
&= \widetilde{S}_i\big(K,\big(\textstyle\bigcup_{j=1}^\infty \boldsymbol{\alpha}_K^*(\eta_j)\big)\backslash\omega_K\big)\\
&= \widetilde{S}_i\big(K,\textstyle\bigcup_{j=1}^\infty (\boldsymbol{\alpha}_K^*(\eta_j)\backslash\omega_K)\big)\\
&= \textstyle\sum_{j=1}^\infty \widetilde{S}_i(K,\boldsymbol{\alpha}_K^*(\eta_j)\backslash\omega_K)\\
&= \textstyle\sum_{j=1}^\infty \widetilde{S}_i(K,\boldsymbol{\alpha}_K^*(\eta_j))\\
&= \textstyle\sum_{j=1}^\infty \widetilde{C}_i(K,\eta_j).
\end{aligned}$$

This shows that $\widetilde{C}_i(K,\cdot)$ is a Borel measure. □

We call the measure $\widetilde{S}_i(K,\cdot)$ the *i-th dual area measure of* K and the measure $\widetilde{C}_i(K,\cdot)$ the *i-th dual curvature measure of* K. From (2.5), (3.11) and (3.12), we see that the total measures of the ith dual area measure and the ith dual curvature measure are the $(n-i)$-th dual quermassintegral $\widetilde{W}_{n-i}(K)$; i.e.,

$$\widetilde{W}_{n-i}(K) = \widetilde{S}_i(K,S^{n-1}) = \widetilde{C}_i(K,S^{n-1}). \tag{3.16}$$

The integral representations (3.11) and (3.12) show that the dual curvature and dual area measures can be extended.

Definition 3.2. Let $K\in\mathcal{K}_o^n$ and $q\in\mathbb{R}$. Define the *q-th dual area measure* $\widetilde{S}_q(K,\cdot)$ by

$$\widetilde{S}_q(K,\omega) = \frac{1}{n}\int_\omega \varrho_K^q(u)\,du$$

for each Lebesgue measurable $\omega\subset S^{n-1}$, and the *q-th dual curvature measure* $\widetilde{C}_q(K,\cdot)$ by

$$\widetilde{C}_q(K,\eta) = \frac{1}{n}\int_{\boldsymbol{\alpha}_K^*(\eta)} \varrho_K^q(u)\,du = \frac{1}{n}\int_{S^{n-1}} \mathbf{1}_{\boldsymbol{\alpha}_K^*(\eta)}(u)\varrho_K^q(u)\,du \tag{3.17}$$

for each Borel $\eta\subset S^{n-1}$.

The verification that each $\widetilde{C}_q(K,\cdot)$ is a Borel measure is the same as for the cases where $q=1,...,n$ as can be seen by examining the proof of this fact in Theorem 3.1.

Obviously, the total measures of the q-th dual curvature measure and the qth dual area measure are the $(n-q)$-th dual quermassintegral; i.e.,

$$\widetilde{W}_{n-q}(K) = \widetilde{S}_q(K,S^{n-1}) = \widetilde{C}_q(K,S^{n-1}). \tag{3.18}$$

It follows immediately from their definitions that the qth dual curvature measure of K is the "image measure" of the qth dual area measure of K under $\boldsymbol{\alpha}_K$; i.e.,

$$\widetilde{C}_q(K,\eta)=\widetilde{S}_q(K,\boldsymbol{\alpha}_K^*(\eta)), \tag{3.19}$$

for each Borel $\eta\subset S^{n-1}$.

3.3. Dual curvature measures for special classes of convex bodies

LEMMA 3.3. *Let $K\in\mathcal{K}_o^n$ and $q\in\mathbb{R}$. For each function $g\colon S^{n-1}\to\mathbb{R}$ that is bounded and Borel,*

$$\int_{S^{n-1}} g(v)\,d\widetilde{C}_q(K,v)=\frac{1}{n}\int_{S^{n-1}} g(\alpha_K(u))\varrho_K^q(u)\,du. \tag{3.20}$$

In the integral on the right in (3.20), the integration is with respect to spherical Lebesgue measure (recall that α_K is defined a.e. with respect to spherical Lebesgue measure).

Proof. Let ϕ be a simple function on S^{n-1} given by

$$\phi=\sum_{i=1}^m c_i\mathbf{1}_{\eta_i}$$

with $c_i\in\mathbb{R}$ and Borel $\eta_i\subset S^{n-1}$. By using (3.17) and (2.21), we get

$$\begin{aligned}\int_{S^{n-1}}\phi(v)\,d\widetilde{C}_q(K,v)&=\int_{S^{n-1}}\sum_{i=1}^m c_i\mathbf{1}_{\eta_i}(v)\,d\widetilde{C}_q(K,v)\\&=\sum_{i=1}^m c_i\widetilde{C}_q(K,\eta_i)\\&=\frac{1}{n}\int_{S^{n-1}}\sum_{i=1}^m c_i\mathbf{1}_{\boldsymbol{\alpha}_K^*(\eta_i)}(u)\varrho_K^q(u)\,du\\&=\frac{1}{n}\int_{S^{n-1}}\sum_{i=1}^m c_i\mathbf{1}_{\eta_i}(\alpha_K(u))\varrho_K^q(u)\,du\\&=\frac{1}{n}\int_{S^{n-1}}\phi(\alpha_K(u))\varrho_K^q(u)\,du.\end{aligned}$$

Now that we have established (3.20) for simple functions, for a bounded Borel g, we choose a sequence of simple functions ϕ_k that converge uniformly to g. Then $\phi_k\circ\alpha_K$ converges to $g\circ\alpha_K$ a.e. with respect to spherical Lebesgue measure. Since g is a Borel function on S^{n-1} and the radial Gauss map α_K is continuous on $S^{n-1}\setminus\omega_K$, the composite function $g\circ\alpha_K$ is a Borel function on $S^{n-1}\setminus\omega_K$. Thus, g and $g\circ\alpha_K$ are Lebesgue integrable on S^{n-1} because g is bounded and ω_K has Lebesgue measure zero. Taking the limit $k\to\infty$ establishes (3.20). □

Lemma 3.4. *Let $K\in\mathcal{K}_o^n$ and $q\in\mathbb{R}$. For each bounded Borel function $g: S^{n-1}\to\mathbb{R}$,*

$$\int_{S^{n-1}} g(v)\,d\widetilde{C}_q(K,v)=\frac{1}{n}\int_{\partial' K} x\cdot\nu_K(x)g(\nu_K(x))|x|^{q-n}\,d\mathcal{H}^{n-1}(x). \tag{3.21}$$

Let $f=g\circ\alpha_K$. Then, as shown in the proof of Lemma 3.3, f is bounded and Lebesgue integrable on S^{n-1}. Thus, the desired (3.21) follows immediately from (2.30) and (3.20).

Lemma 3.5. *Let $K\in\mathcal{K}_o^n$ and $q\in\mathbb{R}$. For each Borel set $\eta\subset S^{n-1}$,*

$$\widetilde{C}_q(K,\eta)=\frac{1}{n}\int_{x\in\nu_K^{-1}(\eta)} x\cdot\nu_K(x)|x|^{q-n}\,d\mathcal{H}^{n-1}(x). \tag{3.22}$$

Taking $g=\mathbf{1}_\eta$ in (3.21) immediately yields (3.22).

We conclude with three observations regarding the dual curvature measures.

(i) Let $P\in\mathcal{K}_o^n$ be a polytope with outer unit normals $v_1,...,v_m$. Let Δ_i be the cone that consists of all of the rays emanating from the origin and passing through the facet of P whose outer unit normal is v_i. Then, recalling that we abbreviate $\boldsymbol{\alpha}_P^*(\{v_i\})$ by $\boldsymbol{\alpha}_P^*(v_i)$, we have

$$\boldsymbol{\alpha}_P^*(v_i)=S^{n-1}\cap\Delta_i. \tag{3.23}$$

If $\eta\subset S^{n-1}$ is a Borel set such that $\{v_1,...,v_m\}\cap\eta=\varnothing$, then $\boldsymbol{\alpha}_P^*(\eta)$ has spherical Lebesgue measure zero. Therefore, the dual curvature measure $\widetilde{C}_q(P,\cdot)$ is discrete and concentrated on $\{v_1,...,v_m\}$. From the definition of dual curvature measures (3.17), and (3.23), we see that

$$\widetilde{C}_q(P,\cdot)=\sum_{i=1}^m c_i\delta_{v_i}, \tag{3.24}$$

where δ_{v_i} denotes the delta measure concentrated at the point v_i on S^{n-1}, and

$$c_i=\frac{1}{n}\int_{S^{n-1}\cap\Delta_i}\varrho_P(u)^q\,du. \tag{3.25}$$

(ii) Suppose that $K\in\mathcal{K}_o^n$ is strictly convex. If $g: S^{n-1}\to\mathbb{R}$ is continuous, then (3.21) and (2.33) give

$$\begin{aligned}\int_{S^{n-1}} g(v)\,d\widetilde{C}_q(K,v)&=\frac{1}{n}\int_{\partial K} x\cdot\nu_K(x)\,g(\nu_K(x))|x|^{q-n}d\mathcal{H}^{n-1}(x)\\&=\frac{1}{n}\int_{S^{n-1}} g(v)|\nabla h_K(v)|^{q-n}h_K(v)\,dS(K,v).\end{aligned}$$

This shows that

$$d\widetilde{C}_q(K,\cdot)=\frac{1}{n}h_K|\nabla h_K|^{q-n}\,dS(K,\cdot). \tag{3.26}$$

(iii) Suppose that $K\in\mathcal{K}_o^n$ has a C^2 boundary with everywhere positive curvature. Since in this case $S(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure, it follows that $\widetilde{C}_q(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measure, and from (3.26) and (1.3) we have

$$\frac{d\widetilde{C}_q(K,v)}{dv}=\frac{1}{n}h_K(v)|\nabla h_K(v)|^{q-n}\det(h_{ij}(v)+h_K(v)\delta_{ij}), \tag{3.27}$$

where (h_{ij}) denotes the Hessian matrix of h_K with respect to an orthonormal frame on S^{n-1}.

3.4. Properties of dual curvature measures

The weak convergence of the qth dual curvature measure is critical and is contained in the following lemma.

LEMMA 3.6. *Let $q\in\mathbb{R}$. If $K_i\in\mathcal{K}_o^n$ with $K_i\to K_0\in\mathcal{K}_o^n$, then $\widetilde{C}_q(K_i,\cdot)\to\widetilde{C}_q(K_0,\cdot)$, weakly.*

Proof. Let $g: S^{n-1}\to\mathbb{R}$ be continuous. From (3.20) we know that

$$\int_{S^{n-1}} g(v)\,d\widetilde{C}_q(K_i,v)=\frac{1}{n}\int_{S^{n-1}} g(\alpha_{K_i}(u))\varrho_{K_i}^q(u)\,du,$$

for all i. Since $K_i\to K_0$, with respect to the Hausdorff metric, we know that $\varrho_{K_i}\to\varrho_{K_0}$ uniformly, and using Lemma 2.2 that $\alpha_{K_i}\to\alpha_K$ almost everywhere on S^{n-1}. Thus,

$$\frac{1}{n}\int_{S^{n-1}} g(\alpha_{K_i}(u))\varrho_{K_i}^q(u)\,du\to\frac{1}{n}\int_{S^{n-1}} g(\alpha_{K_0}(u))\varrho_{K_0}^q(u)\,du,$$

from which it follows that $\widetilde{C}_q(K_i,\cdot)\to\widetilde{C}_q(K_0,\cdot)$, weakly. □

LEMMA 3.7. *If $K\in\mathcal{K}_o^n$ and $q\in\mathbb{R}$, then the dual curvature measure $\widetilde{C}_q(K,\cdot)$ is absolutely continuous with respect to the surface area measure $S(K,\cdot)$.*

Proof. Let $\eta\subset S^{n-1}$ be such that $S(K,\eta)=0$, or equivalently, $\mathcal{H}^{n-1}(\nu_K^{-1}(\eta))=0$. In this case, using (3.22), we conclude that

$$\widetilde{C}_q(K,\eta)=\frac{1}{n}\int_{x\in\nu_K^{-1}(\eta)}|x|^{q-n}x\cdot\nu_K(x)\,d\mathcal{H}^{n-1}(x)=0,$$

since we are integrating over a set of measure zero. □

The following lemma tells us that the nth dual curvature measure of a convex body is the cone-volume measure of the body, while the zeroth dual curvature measure of the convex body is essentially Aleksandrov's integral curvature of the polar of the body.

LEMMA 3.8. *If* $K\in\mathcal{K}_o^n$, *then*

$$\widetilde{C}_n(K,\cdot)=V_K, \tag{3.28}$$

$$\widetilde{C}_0(K,\cdot)=\frac{1}{n}C_0(K^*,\cdot). \tag{3.29}$$

Proof. Let $\eta\subset S^{n-1}$ be a Borel set. From (3.22), with $q=n$, and (1.6), we have

$$\widetilde{C}_n(K,\eta)=\frac{1}{n}\int_{x\in\nu_K^{-1}(\eta)} x\cdot\nu_K(x)\,d\mathcal{H}^{n-1}(x)=V_K(\eta),$$

which establishes (3.28).

From the definition of the zeroth dual curvature measure (3.11), with $i=0$, and Lemma 2.5, followed by (3.4), we have

$$\widetilde{C}_0(K,\eta)=\frac{1}{n}\mathcal{H}^{n-1}(\boldsymbol{\alpha}_K^*(\eta))=\frac{1}{n}\mathcal{H}^{n-1}(\boldsymbol{\alpha}_{K^*}(\eta))=\frac{1}{n}\,C_0(K^*,\eta),$$

which gives (3.29). □

From equations (2.13), (3.28), and (3.21), we see that, for each bounded Borel function $g\colon S^{n-1}\to\mathbb{R}$, one has

$$\int_{\partial' K} g(\nu_K(x))\,d\mathcal{H}^{n-1}(x)=\int_{S^{n-1}} g(v)\,dS(K,v). \tag{3.30}$$

The theory of valuations has witnessed explosive growth during the past two decades (see, e.g., [4]–[7], [12], [37]–[39], [47]–[51], and [76]–[77]). Let $\mathcal{M}(S^{n-1})$ denote the set of Borel measures on S^{n-1}. That the dual area measures are valuations whose codomain is $\mathcal{M}(S^{n-1})$ is easily seen. But it turns out that the dual curvature measures are valuations (whose codomain is $\mathcal{M}(S^{n-1})$) as well. We now show that, for fixed index q, the functional that associates the body $K\in\mathcal{K}_o^n$ with $\widetilde{C}_q(K,\cdot)\in\mathcal{M}(S^{n-1})$ is a valuation.

LEMMA 3.9. *For each real* q, *the dual curvature measure* $\widetilde{C}_q\colon\mathcal{K}_o^n\to\mathcal{M}(S^{n-1})$ *is a valuation*; *i.e., if* $K,L\in\mathcal{K}_o^n$ *are such that* $K\cup L\in\mathcal{K}_o^n$, *then*

$$\widetilde{C}_q(K,\cdot)+\widetilde{C}_q(L,\cdot)=\widetilde{C}_q(K\cap L,\cdot)+\widetilde{C}_q(K\cup L,\cdot).$$

Proof. Since for $Q\in\mathcal{K}_o^n$, the function $r_Q\colon S^{n-1}\to\partial Q$ is a bijection, we have the following disjoint partition of $S^{n-1}=\Omega_0\cup\Omega_L\cup\Omega_K$, where

$$\begin{aligned}\Omega_0&=r_K^{-1}(\partial K\cap\partial L)=r_L^{-1}(\partial K\cap\partial L)=\{u\in S^{n-1}:\varrho_K(u)=\varrho_L(u)\},\\ \Omega_L&=r_K^{-1}(\partial K\cap\operatorname{int}L)=r_L^{-1}((\mathbb{R}^n\setminus K)\cap\partial L)=\{u\in S^{n-1}:\varrho_K(u)<\varrho_L(u)\},\\ \Omega_K&=r_K^{-1}(\partial K\cap(\mathbb{R}^n\setminus L))=r_L^{-1}(\operatorname{int}K\cap\partial L)=\{u\in S^{n-1}:\varrho_K(u)>\varrho_L(u)\}.\end{aligned}$$

Since $K\cup L$ is a convex body, for $\mathcal{H}^{n-1}$-almost all $u\in\Omega_0$ we have

$$\begin{aligned}\varrho_K(u)=\varrho_L(u)=\varrho_{K\cap L}(u)=\varrho_{K\cup L}(u),\\ \alpha_K(u)=\alpha_L(u)=\alpha_{K\cap L}(u)=\alpha_{K\cup L}(u);\end{aligned}$$

For $\mathcal{H}^{n-1}$-almost all $u\in\Omega_L$ we have

$$\begin{aligned}\varrho_K(u)=\varrho_{K\cap L}(u), \quad \varrho_L(u)=\varrho_{K\cup L}(u),\\ \alpha_K(u)=\alpha_{K\cap L}(u), \quad \alpha_L(u)=\alpha_{K\cup L}(u);\end{aligned}$$

For $\mathcal{H}^{n-1}$-almost all $u\in\Omega_K$ we have

$$\begin{aligned}\varrho_K(u)=\varrho_{K\cup L}(u), \quad \varrho_L(u)=\varrho_{K\cap L}(u),\\ \alpha_K(u)=\alpha_{K\cup L}(u), \quad \alpha_L(u)=\alpha_{K\cap L}(u).\end{aligned}$$

From this it follows that if $g: S^{n-1}\to\mathbb{R}$ is continuous, then

$$\begin{aligned}\int_{\Omega_0} g(\alpha_K(u))\varrho_K^q(u)\,du&=\int_{\Omega_0} g(\alpha_{K\cap L}(u))\varrho_{K\cap L}^q(u)\,du,\\ \int_{\Omega_L} g(\alpha_K(u))\varrho_K^q(u)\,du&=\int_{\Omega_L} g(\alpha_{K\cap L}(u))\varrho_{K\cap L}^q(u)\,du,\\ \int_{\Omega_K} g(\alpha_K(u))\varrho_K^q(u)\,du&=\int_{\Omega_K} g(\alpha_{K\cup L}(u))\varrho_{K\cup L}^q(u)\,du,\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega_0} g(\alpha_L(u))\varrho_L^q(u)\,du&=\int_{\Omega_0} g(\alpha_{K\cup L}(u))\varrho_{K\cup L}^q(u)\,du,\\ \int_{\Omega_L} g(\alpha_L(u))\varrho_L^q(u)\,du&=\int_{\Omega_L} g(\alpha_{K\cup L}(u))\varrho_{K\cup L}^q(u)\,du,\\ \int_{\Omega_K} g(\alpha_L(u))\varrho_L^q(u)\,du&=\int_{\Omega_K} g(\alpha_{K\cap L}(u))\varrho_{K\cap L}^q(u)\,du.\end{aligned}$$

Summing up both sides of the integrals above gives

$$\begin{aligned}&\int_{S^{n-1}} g(\alpha_K(u))\varrho_K^q(u)\,du+\int_{S^{n-1}} g(\alpha_L(u))\varrho_L^q(u)\,du\\ &\qquad=\int_{S^{n-1}} g(\alpha_{K\cap L}(u))\varrho_{K\cap L}^q(u)\,du+\int_{S^{n-1}} g(\alpha_{K\cup L}(u))\varrho_{K\cup L}^q(u)\,du.\end{aligned}$$

Since this holds for each continuous g, we may appeal to (3.20) to obtain the desired valuation property. □

4. Variational formulas for the dual quermassintegrals

When using the variational method to solve the Minkowski problem, one of the crucial steps is to establish the variational formula for volume which gives an integral of a continuous function on the unit sphere integrated with respect to the surface area measure. The variational formula is the key to transforming the Minkowski problem into the Lagrange equation of an optimization problem. Since the variational method needs to deal with convex bodies that are not necessarily smooth, finding variational formulas of geometric invariants of convex bodies is difficult. In fact, for either quermassintegrals or dual quermassintegrals, a variational formula was known for only one—namely, the volume. This variational formula was established by Aleksandrov.

Let $K\in\mathcal{K}_o^n$ and let $f: S^{n-1}\to\mathbb{R}$ be continuous. For some $\delta>0$, let $h_t: S^{n-1}\to(0,\infty)$ be defined, for $v\in S^{n-1}$ and each $t\in(-\delta,\delta)$, by

$$h_t(v)=h_K(v)+tf(v)+o(t,v),$$

where $o(t,\cdot): S^{n-1}\to\mathbb{R}$ is continuous and $o(t,\cdot)/t\to 0$, as $t\to 0$, uniformly on S^{n-1}.

Let $[h_t]$ be the Wulff shape determined by h_t. Aleksandrov's variational formula states that

$$\lim_{t\to 0}\frac{V([h_t])-V(K)}{t}=\int_{S^{n-1}} f(v)\,dS(K,v).$$

The proof makes critical use of the Minkowski mixed-volume inequality. Such a variational formula is not known for the surface area or the other quermassintegrals.

In this section we shall take a completely different approach. Instead of considering Wulff shapes, we consider convex hulls. We establish variational formulas for all dual quermassintegrals. In particular, Aleksandrov's variational principle will be established without using the Minkowski mixed-volume inequality.

Let $\Omega\subset S^{n-1}$ be a closed set that is not contained in any closed hemisphere of S^{n-1}. Let $\varrho_0: \Omega\to(0,\infty)$ and $g: \Omega\to\mathbb{R}$ be continuous. For some $\delta>0$, let $\varrho_t: \Omega\to(0,\infty)$ be defined, for $u\in\Omega$ and each $t\in(-\delta,\delta)$, by

$$\log\varrho_t(u)=\log\varrho_0(u)+tg(u)+o(t,u), \tag{4.1}$$

where $o(t,\cdot): \Omega\to\mathbb{R}$ is continuous on Ω and $o(t,\cdot)/t\to 0$, as $t\to 0$, uniformly on Ω. Recall that the logarithmic family of convex hulls $\langle\varrho_t\rangle$ of (ϱ_0, g), indexed by $t\in(-\delta,\delta)$, is just the family of convex bodies $\operatorname{conv}\{\varrho_t(u)u: u\in\Omega\}$, indexed by $t\in(-\delta,\delta)$.

Since $\varrho_t\to\varrho_0$ uniformly on Ω, for the associated bodies we have

$$\langle\varrho_t\rangle\to\langle\varrho_0\rangle,\quad \text{as } t\to 0, \tag{4.2}$$

in $\mathcal{K}_o^n$. But (2.26) tells us that, for each $v\in S^{n-1}$,

$$h_{\langle\varrho_t\rangle}(v)=\max_{u\in\Omega}(u\cdot v)\varrho_t(u) \tag{4.3}$$

for each $t\in(-\delta,\delta)$.

The following lemma shows that the support functions of a logarithmic family of convex hulls are differentiable with respect to the variational variable.

LEMMA 4.1. *Let $\Omega\subset S^{n-1}$ be a closed set that is not contained in any closed hemisphere of S^{n-1}. Let $\varrho_0:\Omega\to(0,\infty)$ and $g:\Omega\to\mathbb{R}$ be continuous. If $\langle\varrho_t\rangle$ is a logarithmic family of convex hulls of (ϱ_0,g), then*

$$\lim_{t\to0}\frac{\log h_{\langle\varrho_t\rangle}(v)-\log h_{\langle\varrho_0\rangle}(v)}{t}=g(\alpha^*_{\langle\varrho_0\rangle}(v)), \tag{4.4}$$

for all $v\in S^{n-1}\setminus\eta_{\langle\varrho_0\rangle}$; i.e., for all regular normals v of $\langle\varrho_0\rangle$. Hence (4.4) holds a.e. with respect to spherical Lebesgue measure. Moreover, there exist $\delta_0>0$ and $M>0$ so that

$$|\log h_{\langle\varrho_t\rangle}(v)-\log h_{\langle\varrho_0\rangle}(v)|\leqslant M|t|, \tag{4.5}$$

for all $v\in S^{n-1}$ and all $t\in(-\delta_0,\delta_0)$.

Proof. Recall that $\eta_{\langle\varrho_0\rangle}$ is the set of measure zero off of which $\boldsymbol{\alpha}^*_{\langle\varrho_0\rangle}$ is single valued, and by (2.17) the function $\alpha^*_{\langle\varrho_0\rangle}:S^{n-1}\setminus\eta_{\langle\varrho_0\rangle}\to S^{n-1}$ is defined by $\alpha^*_{\langle\varrho_0\rangle}=r^{-1}_{\langle\varrho_0\rangle}\circ x_{\langle\varrho_0\rangle}$ or $\boldsymbol{\alpha}^*_{\langle\varrho_0\rangle}(v)=\{\alpha^*_{\langle\varrho_0\rangle}(v)\}$.

Let $v\in S^{n-1}\setminus\eta_{\langle\varrho_0\rangle}$ be fixed throughout the proof. From (4.3) we know that there exist $u_t\in\Omega$ such that

$$h_{\langle\varrho_t\rangle}(v)=(u_t\cdot v)\varrho_t(u_t),\quad \text{while } h_{\langle\varrho_t\rangle}(v)\geqslant(u\cdot v)\varrho_t(u), \tag{4.6}$$

for all $u\in\Omega$. Note that $u_t\cdot v>0$, for all t.

From (4.6) we have $h_{\langle\varrho_0\rangle}(v)=(u_0\cdot v)\,\varrho_0(u_0)$, and hence we know that

$$\varrho_0(u_0)u_0\in H_{\langle\varrho_0\rangle}(v)=\{x\in\mathbb{R}^n:x\cdot v=h_{\langle\varrho_0\rangle}(v)\}.$$

But $\varrho_0(u_0)u_0\in\langle\varrho_0\rangle$, so $\varrho_0(u_0)u_0\in\partial\langle\varrho_0\rangle$, and hence $u_0\in\boldsymbol{\alpha}^*_{\langle\varrho_0\rangle}(v)=\{\alpha^*_{\langle\varrho_0\rangle}(v)\}$. Thus

$$u_0=\alpha^*_{\langle\varrho_0\rangle}(v). \tag{4.7}$$

We first show that

$$\lim_{t\to0}u_t=u_0, \tag{4.8}$$

where the u_t come from (4.6); i.e., are such that $h_{\langle\varrho_t\rangle}(v)=(u_t\cdot v)\varrho_t(u_t)$. To see this, we consider any sequence $t_k\to 0$ and show that the bounded sequence u_{t_k} in the compact set Ω converges to u_0. It is sufficient to show that any convergent subsequence of u_{t_k} converges to u_0. Pick a convergent subsequence of u_{t_k}, which we also denote by u_{t_k}, such that

$$u_{t_k}\to u'\in\Omega.$$

Since $\varrho_{t_k}\to\varrho_0$ uniformly on Ω,

$$h_{\langle\varrho_{t_k}\rangle}(v)=(u_{t_k}\cdot v)\,\varrho_{t_k}(u_{t_k})\to(u'\cdot v)\,\varrho_0(u'). \tag{4.9}$$

The fact that $\varrho_{t_k}\to\varrho_0$ uniformly on Ω implies that $\langle\varrho_{t_k}\rangle\to\langle\varrho_0\rangle$ in $\mathcal{K}_o^n$, and hence that $h_{\langle\varrho_{t_k}\rangle}(v)\to h_{\langle\varrho_0\rangle}(v)$, which with (4.9) shows that $h_{\langle\varrho_0\rangle}(v)=(u'\cdot v)\,\varrho_0(u')$. It follows that

$$\varrho_0(u')u'\in H_{\langle\varrho_0\rangle}(v)=\{x\in\mathbb{R}^n:x\cdot v=h_{\langle\varrho_0\rangle}(v)\}.$$

But $\varrho_0(u')u'\in\langle\varrho_0\rangle$, so $\varrho_0(u')u'\in\partial\langle\varrho_0\rangle$, and hence $u'\in\boldsymbol{\alpha}^*_{\langle\varrho_0\rangle}(v)=\{\alpha^*_{\langle\varrho_0\rangle}(v)\}$. But from (4.7) we know that $\alpha^*_{\langle\varrho_0\rangle}(v)=u_0$, and thus $u'=u_0$. This establishes (4.8).

From (2.1) we see that, for all t,

$$h_{\langle\varrho_0\rangle}(v)\geqslant(u_t\cdot v)\,\varrho_{\langle\varrho_0\rangle}(u_t). \tag{4.10}$$

Since $\langle\varrho_0\rangle=\operatorname{conv}\{\varrho_0(u)u:u\in\Omega\}$, we have

$$\varrho_{\langle\varrho_0\rangle}(u_t)\geqslant\varrho_0(u_t). \tag{4.11}$$

From (4.6), (4.10), (4.11), and (4.1), we have

$$\begin{aligned}\log h_{\langle\varrho_t\rangle}(v)-\log h_{\langle\varrho_0\rangle}(v)&=\log\varrho_t(u_t)+\log(u_t\cdot v)-\log h_{\langle\varrho_0\rangle}(v)\\&\leqslant\log\varrho_t(u_t)-\log\varrho_{\langle\varrho_0\rangle}(u_t)\\&\leqslant\log\varrho_t(u_t)-\log\varrho_0(u_t)\\&=tg(u_t)+o(t,u_t).\end{aligned} \tag{4.12}$$

Since $\langle\varrho_t\rangle=\operatorname{conv}\{\varrho_t(u)u:u\in\Omega\}$, it follows that $\varrho_{\langle\varrho_t\rangle}(u)\geqslant\varrho_t(u)$ for all $u\in\Omega$. Using the case $t=0$ in (4.6), followed by the fact that $h_{\langle\varrho_t\rangle}(v)\geqslant(u_0\cdot v)\varrho_t(u_0)$, and then again (4.6) and (4.1), we get

$$\begin{aligned}\log h_{\langle\varrho_t\rangle}(v)-\log h_{\langle\varrho_0\rangle}(v)&=\log h_{\langle\varrho_t\rangle}(v)-\log\varrho_0(u_0)-\log(u_0\cdot v)\\&\geqslant\log\varrho_t(u_0)-\log\varrho_0(u_0)\\&=tg(u_0)+o(t,u_0).\end{aligned} \tag{4.13}$$

Let $M_0=\max_{u\in\Omega}|g(u)|$. Since $o(t,\cdot)/t\to 0$, as $t\to 0$, uniformly on Ω, we may choose $\delta_0>0$ so that for all $t\in(-\delta_0,\delta_0)$ we have $|o(t,\cdot)|\leqslant|t|$ on Ω. From (4.12), (4.13), and the definition of M_0, we immediately see that

$$|\log h_{\langle\varrho_t\rangle}(v)-\log h_{\langle\varrho_0\rangle}(v)|\leqslant(M_0+1)|t|. \tag{4.14}$$

Combining (4.12) and (4.13), we have

$$0\leqslant\log h_{\langle\varrho_t\rangle}(v)-\log h_{\langle\varrho_0\rangle}(v)-tg(u_0)-o(t,u_0)\leqslant t(g(u_t)-g(u_0))+o(t,u_t)-o(t,u_0).$$

When $t>0$, this gives

$$\frac{o(t,u_0)}{t}\leqslant\frac{\log h_{\langle\varrho_t\rangle}(v)-\log h_{\langle\varrho_0\rangle}(v)}{t}-g(u_0)\leqslant g(u_t)-g(u_0)+\frac{o(t,u_t)}{t}.$$

From (4.8) and the continuity of g, we can conclude that

$$\lim_{t\to 0^+}\frac{\log h_{\langle\varrho_t\rangle}(v)-\log h_{\langle\varrho_0\rangle}(v)}{t}=g(u_0). \tag{4.15}$$

On the other hand, when $t<0$, we have

$$\frac{o(t,u_0)}{t}\geqslant\frac{\log h_{\langle\varrho_t\rangle}(v)-\log h_{\langle\varrho_0\rangle}(v)}{t}-g(u_0)\geqslant g(u_t)-g(u_0)+\frac{o(t,u_t)}{t},$$

from which we can also conclude that

$$\lim_{t\to 0^-}\frac{\log h_{\langle\varrho_t\rangle}(v)-\log h_{\langle\varrho_0\rangle}(v)}{t}=g(u_0). \tag{4.16}$$

Together with (4.7), we now obtain the desired result:

$$\lim_{t\to 0}\frac{\log h_{\langle\varrho_t\rangle}(v)-\log h_{\langle\varrho_0\rangle}(v)}{t}=g(u_0)=g(\alpha^*_{\langle\varrho_0\rangle}(v)).$$

Now (4.14) holds for all $v\in S^{n-1}\setminus\eta_{\langle\varrho_0\rangle}$, i.e. almost everywhere on S^{n-1} with respect to spherical Lebesgue measure. Since the support functions in (4.14) are continuous on S^{n-1}, it follows that (4.14) holds for all $v\in S^{n-1}$, which gives (4.5). □

LEMMA 4.2. *Let $\Omega\subset S^{n-1}$ be a closed set that is not contained in any closed hemisphere of S^{n-1}. Let $\varrho_0:\Omega\to(0,\infty)$ and $g:\Omega\to\mathbb{R}$ be continuous. If $\langle\varrho_t\rangle$ is a logarithmic family of convex hulls of (ϱ_0,g), then, for $q\in\mathbb{R}$,*

$$\lim_{t\to 0}\frac{h^{-q}_{\langle\varrho_t\rangle}(v)-h^{-q}_{\langle\varrho_0\rangle}(v)}{t}=-qh^{-q}_{\langle\varrho_0\rangle}(v)g(\alpha^*_{\langle\varrho_0\rangle}(v)), \tag{4.17}$$

for all $v\in S^{n-1}\setminus\eta_{\langle\varrho_0\rangle}$. Moreover, there exist $\delta_0>0$ and $M>0$ such that

$$|h^{-q}_{\langle\varrho_t\rangle}(v)-h^{-q}_{\langle\varrho_0\rangle}(v)|\leqslant M|t|, \tag{4.18}$$

for all $v\in S^{n-1}$ and all $t\in(-\delta_0,\delta_0)$.

Proof. Obviously,

$$\lim_{t\to 0}\frac{h^{-q}_{\langle\varrho_t\rangle}(v)-h^{-q}_{\langle\varrho_0\rangle}(v)}{t}=-qh^{-q}_{\langle\varrho_0\rangle}(v)\lim_{t\to 0}\frac{\log h_{\langle\varrho_t\rangle}(v)-\log h_{\langle\varrho_0\rangle}(v)}{t},$$

provided the limit on the right exists. Thus, Lemma 4.1 gives (4.17).

Since $\langle\varrho_0\rangle$ is a convex body in $\mathcal{K}^n_o$ and $\langle\varrho_t\rangle\to\langle\varrho_0\rangle$ as $t\to 0$, there exist $m_0, m_1\in(0,\infty)$ and $\delta_1>0$ such that

$$0<m_0<h_{\langle\varrho_t\rangle}<m_1\quad\text{on } S^{n-1},$$

for each $t\in(-\delta_1,\delta_1)$. From this it follows that there exists $M_1>1$ so that

$$0<\frac{h^{-q}_{\langle\varrho_t\rangle}}{h^{-q}_{\langle\varrho_0\rangle}}<M_1\quad\text{on } S^{n-1}.$$

It is easily seen that $s-1\geqslant\log s$ whenever $s\in(0,1)$, whereas $s-1\leqslant M_1\log s$ whenever $s\in[1,M_1]$. Thus,

$$|s-1|\leqslant M_1|\log s|\quad\text{when } s\in(0,M_1).$$

It follows that

$$\left|\frac{h^{-q}_{\langle\varrho_t\rangle}}{h^{-q}_{\langle\varrho_0\rangle}}-1\right|\leqslant M_1\left|\log\frac{h^{-q}_{\langle\varrho_t\rangle}}{h^{-q}_{\langle\varrho_0\rangle}}\right|,$$

that is,

$$|h^{-q}_{\langle\varrho_t\rangle}-h^{-q}_{\langle\varrho_0\rangle}|\leqslant h^{-q}_{\langle\varrho_0\rangle}M_1|\log h_{\langle\varrho_t\rangle}-\log h_{\langle\varrho_0\rangle}|\leqslant\frac{M_1}{\min\{m_0^q,m_1^q\}}|\log h_{\langle\varrho_t\rangle}-\log h_{\langle\varrho_0\rangle}|$$

on S^{n-1}, whenever $t\in(-\delta_1,\delta_1)$. This and (4.5) give (4.18). □

The derivative of radial functions of Wulff shapes is contained in the following lemma.

LEMMA 4.3. *Let $\Omega\subset S^{n-1}$ be a closed set not contained in any closed hemisphere of S^{n-1}. Let $h_0:\Omega\to(0,\infty)$ and $f:\Omega\to\mathbb{R}$ be continuous. If $[h_t]$ is a logarithmic family of Wulff shapes associated with (h_0,f), where*

$$\log h_t(v)=\log h_0(v)+tf(v)+o(t,v)$$

for $v\in\Omega$, then for almost all $u\in S^{n-1}$, with respect to spherical Lebesgue measure,

$$\lim_{t\to 0}\frac{\log\varrho_{[h_t]}(u)-\log\varrho_{[h_0]}(u)}{t}=f(\alpha_{[h_0]}(u)).$$

Proof. Let $\varrho_t = 1/h_t$. Then

$$\log \varrho_t(v) = \log \varrho_0(v) - tf(v) - o(t,v),$$

and from Lemma 4.1 we know that

$$\lim_{t\to 0} \frac{\log h_{\langle \varrho_t\rangle}(v) - \log h_{\langle \varrho_0\rangle}(v)}{t} = -f(\alpha^*_{\langle \varrho_0\rangle}(v)) \tag{4.19}$$

for almost all $v\in S^{n-1}$, with respect to spherical Lebesgue measure. From Lemma 2.8 we have

$$[h_t] = \langle \varrho_t\rangle^*. \tag{4.20}$$

From (4.20) and (2.3) we have

$$\log \varrho_{[h_t]} - \log \varrho_{[h_0]} = -(\log h_{\langle \varrho_t\rangle} - \log h_{\langle \varrho_0\rangle}). \tag{4.21}$$

But (4.20), when combined with Lemma 2.6, gives

$$\alpha^*_{\langle \varrho_t\rangle} = \alpha_{\langle \varrho_t\rangle^*} = \alpha_{[h_t]} \tag{4.22}$$

almost everywhere on S^{n-1}, with respect to spherical Lebesgue measure.

When (4.19) is combined with (4.21) and (4.22), we obtain the desired result. □

The following theorem gives a variational formula for a dual quermassintegral in terms of its associated dual curvature measure and polar convex hull.

THEOREM 4.4. *Let $\Omega\subset S^{n-1}$ be a closed set not contained in any closed hemisphere of S^{n-1}, and $\varrho_0:\Omega\to(0,\infty)$ and $g:\Omega\to\mathbb{R}$ be continuous. If $\langle\varrho_t\rangle$ is a logarithmic family of convex hulls of (ϱ_0, g), then, for $q\neq 0$,*

$$\lim_{t\to 0} \frac{\widetilde{V}_q(\langle \varrho_t\rangle^*) - \widetilde{V}_q(\langle \varrho_0\rangle^*)}{t} = -q\int_\Omega g(u)\, d\widetilde{C}_q(\langle \varrho_0\rangle^*, u),$$
$$\lim_{t\to 0} \frac{\log \overline{V}_0(\langle \varrho_t\rangle^*) - \log \overline{V}_0(\langle \varrho_0\rangle^*)}{t} = -\frac{1}{\omega_n}\int_\Omega g(u)\, d\widetilde{C}_0(\langle \varrho_0\rangle^*, u),$$

or equivalently, for each $q\in\mathbb{R}$,

$$\frac{d}{dt}\log \overline{V}_q(\langle \varrho_t\rangle^*)\Big|_{t=0} = -\frac{1}{\widetilde{V}_q(\langle \varrho_0\rangle^*)}\int_\Omega g(u)\, d\widetilde{C}_q(\langle \varrho_0\rangle^*, u).$$

Proof. Abbreviate $\eta_{\langle\varrho_0\rangle}$ by η_0. Recall that η_0 is the set of spherical Lebesgue measure zero that consists of the complement, in S^{n-1}, of the regular normal vectors of the convex body $\langle\varrho_0\rangle = \operatorname{conv}\{\varrho_0(u)u : u\in\Omega\}$. Recall also that the continuous function

$$\alpha^*_{\langle\varrho_0\rangle} : S^{n-1}\setminus\eta_0 \longrightarrow S^{n-1}$$

is well defined by $\alpha^*_{\langle\varrho_0\rangle}(v)\in\boldsymbol{\alpha}^*_{\langle\varrho_0\rangle}(v)=\{\alpha^*_{\langle\varrho_0\rangle}(v)\}$ for all $v\in S^{n-1}\setminus\eta_0$.

Let $v\in S^{n-1}\setminus\eta_0$. To see that $\boldsymbol{\alpha}^*_{\langle\varrho_0\rangle}(v)\subset\Omega$, let

$$h_{\langle\varrho_0\rangle}(v) = \max_{u\in\Omega} \varrho_0(u)u\cdot v = \varrho_0(u_0)u_0\cdot v$$

for some $u_0\in\Omega$. But this means that

$$\varrho_0(u_0)u_0 \in H_{\langle\varrho_0\rangle}(v),$$

and hence $\varrho_0(u_0)u_0\in\partial\langle\varrho_0\rangle$ because in addition to $\varrho_0(u_0)u_0$ obviously belonging to $\langle\varrho_0\rangle$, it also belongs to $H_{\langle\varrho_0\rangle}(v)$. But v is a regular normal vector of $\langle\varrho_0\rangle$, and therefore $\alpha^*_{\langle\varrho_0\rangle}(v)=u_0\in\Omega$. Thus,

$$\boldsymbol{\alpha}^*_{\langle\varrho_0\rangle}(S^{n-1}\setminus\eta_0)\subset\Omega. \tag{4.23}$$

But (4.23) and Lemma 2.5 now yield the fact that

$$\boldsymbol{\alpha}_{\langle\varrho_0\rangle^*}(S^{n-1}\setminus\eta_0)\subset\Omega. \tag{4.24}$$

As Ω is closed, we can, by using the Tietze extension theorem, extend the continuous function $g:\Omega\to\mathbb{R}$ to a continuous function $\hat{g}: S^{n-1}\to\mathbb{R}$. Therefore, using (4.24) we see that

$$g(\alpha_{\langle\varrho_0\rangle^*}(v)) = (\hat{g}\mathbf{1}_\Omega)(\alpha_{\langle\varrho_0\rangle^*}(v)) \tag{4.25}$$

for $v\in S^{n-1}\setminus\eta_0$.

Using (2.6), (2.9), (2.3), the fact that η_0 has measure zero, (4.18), the dominated convergence theorem, (4.17), Lemma 2.6, (2.3), (4.25), (3.20), and again (4.25), we have

$$\begin{aligned}
\lim_{t\to0}\frac{\widetilde{V}_q(\langle\varrho_t\rangle^*)-\widetilde{V}_q(\langle\varrho_0\rangle^*)}{t} &= \lim_{t\to0}\frac{1}{n}\int_{S^{n-1}}\frac{\varrho^q_{\langle\varrho_t\rangle^*}(v)-\varrho^q_{\langle\varrho_0\rangle^*}(v)}{t}\,dv \\
&= \lim_{t\to0}\frac{1}{n}\int_{S^{n-1}}\frac{h^{-q}_{\langle\varrho_t\rangle}(v)-h^{-q}_{\langle\varrho_0\rangle}(v)}{t}\,dv \\
&= \frac{1}{n}\int_{S^{n-1}\setminus\eta_0}\lim_{t\to0}\frac{h^{-q}_{\langle\varrho_t\rangle}(v)-h^{-q}_{\langle\varrho_0\rangle}(v)}{t}\,dv \\
&= -\frac{q}{n}\int_{S^{n-1}\setminus\eta_0} g(\alpha^*_{\langle\varrho_0\rangle}(v))h^{-q}_{\langle\varrho_0\rangle}(v)\,dv
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{q}{n}\int_{S^{n-1}\setminus\eta_0} g(\alpha_{\langle\varrho_0\rangle^*}(v))\varrho^q_{\langle\varrho_0\rangle^*}(v)\,dv\\
&= -\frac{q}{n}\int_{S^{n-1}} (\hat g\mathbf{1}_\Omega)(\alpha_{\langle\varrho_0\rangle^*}(v))\varrho^q_{\langle\varrho_0\rangle^*}(v)\,dv\\
&= -q\int_{S^{n-1}} (\hat g\mathbf{1}_\Omega)(u)\,d\widetilde C_q(\langle\varrho_0\rangle^*,u)\\
&= -q\int_{\Omega} g(u)\,d\widetilde C_q(\langle\varrho_0\rangle^*,u).
\end{aligned}
$$

From (2.8), (2.9), (2.3), the fact that η_0 has measure zero together with Lemma 4.1, Lemma 2.6, (4.25), (3.20), and again (4.25), we have

$$
\begin{aligned}
\lim_{t\to 0}\frac{\log \overline V_0(\langle\varrho_t\rangle^*)-\log \overline V_0(\langle\varrho_0\rangle^*)}{t} &= \lim_{t\to 0}\frac{1}{n\omega_n}\int_{S^{n-1}}\frac{\log\varrho_{\langle\varrho_t\rangle^*}(v)-\log\varrho_{\langle\varrho_0\rangle^*}(v)}{t}\,dv\\
&= -\lim_{t\to 0}\frac{1}{n\omega_n}\int_{S^{n-1}}\frac{\log h_{\langle\varrho_t\rangle}(v)-\log h_{\langle\varrho_0\rangle}(v)}{t}\,dv\\
&= -\frac{1}{n\omega_n}\int_{S^{n-1}\setminus\eta_0} g(\alpha^*_{\langle\varrho_0\rangle}(v))\,dv\\
&= -\frac{1}{n\omega_n}\int_{S^{n-1}\setminus\eta_0} g(\alpha_{\langle\varrho_0\rangle^*}(v))\,dv\\
&= -\frac{1}{n\omega_n}\int_{S^{n-1}} (\hat g\mathbf{1}_\Omega)(\alpha_{\langle\varrho_0\rangle^*}(v))\,dv\\
&= -\frac{1}{\omega_n}\int_{S^{n-1}} (\hat g\mathbf{1}_\Omega)(u)\,d\widetilde C_0(\langle\varrho_0\rangle^*,u)\\
&= -\frac{1}{\omega_n}\int_{\Omega} g(u)\,d\widetilde C_0(\langle\varrho_0\rangle^*,u). \qquad\square
\end{aligned}
$$

The following theorem gives a variational formula for a dual quermassintegral in terms of its associated dual curvature measure and Wulff shapes.

THEOREM 4.5. *Let $\Omega\subset S^{n-1}$ be a closed set not contained in any closed hemisphere of S^{n-1}. If $h_0:\Omega\to(0,\infty)$ and $f:\Omega\to\mathbb{R}$ are continuous, and $[h_t]$ is a logarithmic family of Wulff shapes associated with (h_0,f), then, for $q\neq 0$,*

$$
\lim_{t\to 0}\frac{\widetilde V_q([h_t])-\widetilde V_q([h_0])}{t} = q\int_\Omega f(v)\,d\widetilde C_q([h_0],v),
$$

and

$$
\lim_{t\to 0}\frac{\log \overline V_0([h_t])-\log \overline V_0([h_0])}{t} = \frac{1}{\omega_n}\int_\Omega f(v)\,d\widetilde C_0([h_0],v),
$$

or equivalently, for all $q\in\mathbb{R}$,

$$
\frac{d}{dt}\log \overline V_q([h_t])\bigg|_{t=0} = \frac{1}{\widetilde V_q([h_0])}\int_\Omega f(v)\,d\widetilde C_q([h_0],v).
$$

Proof. The logarithmic family of Wulff shapes $[h_t]$ is defined as the Wulff shape of h_t, where h_t is given by

$$\log h_t = \log h_0 + tf + o(t,\cdot).$$

Let $\varrho_t = 1/h_t$. Then

$$\log \varrho_t = \log \varrho_0 - tf - o(t,\cdot).$$

Let $\langle \varrho_t \rangle$ be the logarithmic family of convex hulls associated with $(\varrho_0, -f)$. But from Lemma 2.8 we know that

$$[h_t] = \langle \varrho_t \rangle^*,$$

and the desired conclusions now follow from Theorem 4.4. $\square$

The variational formulas above imply variational formulas for dual quermassintegrals of convex hull perturbations of a convex body in terms of dual curvature measures.

Let $K \in \mathcal{K}_o^n$ and $f: S^{n-1} \to \mathbb{R}$ be continuous. We shall write $[K, f, t]$ for the Wulff shape $[h_t]$ where $h_t: S^{n-1} \to \mathbb{R}$ is given by

$$\log h_t = \log h_K + tf + o(t,\cdot).$$

If $K \in \mathcal{K}_o^n$ and $g: S^{n-1} \to \mathbb{R}$ is continuous, we shall write $\langle K, g, t \rangle$ for the convex hull $\langle \varrho_t \rangle$, where $\varrho_t: S^{n-1} \to \mathbb{R}$ is given by

$$\log \varrho_t = \log \varrho_K + tg + o(t,\cdot).$$

Corollary 4.6. *Let $K \in \mathcal{K}_o^n$ and $g: S^{n-1} \to \mathbb{R}$ be continuous. Then, for $q \neq 0$,*

$$\lim_{t\to 0} \frac{\widetilde{V}_q(\langle K^*, g, t\rangle^*) - \widetilde{V}_q(K)}{t} = -q \int_{S^{n-1}} g(v)\, d\widetilde{C}_q(K, v)$$

and

$$\lim_{t\to 0} \frac{\log \overline{V}_0(\langle K^*, g, t\rangle^*) - \log \overline{V}_0(K)}{t} = -\frac{1}{\omega_n} \int_{S^{n-1}} g(v)\, d\widetilde{C}_0(K, v),$$

or equivalently, for all $q \in \mathbb{R}$,

$$\frac{d}{dt} \log \overline{V}_q(\langle K^*, g, t\rangle^*)\bigg|_{t=0} = -\frac{1}{\widetilde{V}_q(K)} \int_{S^{n-1}} g(v)\, d\widetilde{C}_q(K, v).$$

Proof. In Theorem 4.4, let $\varrho_0 = 1/h_K = \varrho_{K^*}$. Then $\langle \varrho_t \rangle^* = \langle K^*, g, t \rangle^*$, and in particular, from (2.25) we have $\langle \varrho_0 \rangle^* = \langle \varrho_{K^*} \rangle^* = K^{**} = K$. $\square$

The variational formulas for convex hulls above imply variational formulas for Wulff shapes.

COROLLARY 4.7. *Let $K\in\mathcal{K}_o^n$ and $f\colon S^{n-1}\to\mathbb{R}$ be continuous. Then, for $q\neq 0$,*

$$\lim_{t\to 0}\frac{\widetilde{V}_q([K,f,t])-\widetilde{V}_q(K)}{t}=q\int_{S^{n-1}}f(v)\,d\widetilde{C}_q(K,v)$$

and

$$\lim_{t\to 0}\frac{\log\overline{V}_0([K,f,t])-\log\overline{V}_0(K)}{t}=\frac{1}{\omega_n}\int_{S^{n-1}}f(v)\,d\widetilde{C}_0(K,v),$$

or equivalently, for all $q\in\mathbb{R}$,

$$\frac{d}{dt}\log\overline{V}_q([K,f,t])\bigg|_{t=0}=\frac{1}{\widetilde{V}_q(K)}\int_{S^{n-1}}f(v)\,d\widetilde{C}_q(K,v).$$

Proof. The logarithmic family of Wulff shapes $[K,f,o,t]$ is defined by the Wulff shape $[h_t]$, where

$$\log h_t=\log h_K+tf+o(t,\cdot).$$

This, and the fact that $1/h_K=\varrho_{K^*}$, allows us to define

$$\log\varrho_t^*=\log\varrho_{K^*}-tf-o(t,\cdot),$$

and ϱ_t^* will generate a logarithmic family of convex hulls $\langle K^*,-f,-o,t\rangle$. Since $\varrho_t^*=1/h_t$, Lemma 2.8 gives

$$[K,f,o,t]=\langle K^*,-f,-o,t\rangle^*.$$

The lemma now follows directly from Corollary 4.6. □

The following gives variational formulas for dual quermassintegrals of Minkowski combinations.

COROLLARY 4.8. *Let $K\in\mathcal{K}_o^n$ and L be a compact convex set in $\mathbb{R}^n$. Then, for $q\neq 0$,*

$$\lim_{t\to 0^+}\frac{\widetilde{V}_q(K+tL)-\widetilde{V}_q(K)}{t}=q\int_{S^{n-1}}\frac{h_L(v)}{h_K(v)}\,d\widetilde{C}_q(K,v),$$

and

$$\lim_{t\to 0^+}\frac{\log\overline{V}_0(K+tL)-\log\overline{V}_0(K)}{t}=\frac{1}{\omega_n}\int_{S^{n-1}}\frac{h_L(v)}{h_K(v)}\,d\widetilde{C}_0(K,v),$$

or equivalently, for all $q\in\mathbb{R}$,

$$\frac{d}{dt}\log\overline{V}_q(K+tL)\bigg|_{t=0^+}=\frac{1}{\widetilde{V}_q(K)}\int_{S^{n-1}}\frac{h_L(v)}{h_K(v)}\,d\widetilde{C}_q(K,v).$$

Proof. From (2.4), we have

$$h_{K+tL}=h_K+th_L. \tag{4.26}$$

From (4.26), it follows immediately that, for sufficiently small $t>0$,

$$\log h_{K+tL}=\log h_K+t\frac{h_L}{h_K}+o(t,\cdot).$$

Since K and L are convex, the Wulff shape $[h_{K+tL}]=K+tL$. The desired result now follows directly from Corollary 4.7. □

The following variational formula of Aleksandrov for the volume of a convex body is a critical ingredient in the solution of the classical Minkowski problem. The proof given by Aleksandrov depends on the Minkowski mixed-volume inequality, see Schneider [75, Lemma 7.5.3]. The proof presented below is different and does not depend on inequalities for mixed volumes.

COROLLARY 4.9. *Let $\Omega\subset S^{n-1}$ be a closed set not contained in any closed hemisphere of S^{n-1}. Let $h_0:\Omega\to(0,\infty)$ and $f:\Omega\to\mathbb{R}$ be continuous, and let $\delta_0>0$. If for each $t\in(-\delta_0,\delta_0)$ the function $h_t:\Omega\to(0,\infty)$ is defined by*

$$h_t=h_0+tf+o(t,\cdot), \tag{4.27}$$

where $o(t,\cdot)$ is continuous in Ω and $\lim_{t\to0}o(t,\cdot)/t=0$ uniformly in Ω, and $[h_t]$ is the Wulff shape of h_t, then

$$\lim_{t\to0}\frac{V([h_t])-V([h_0])}{t}=\int_\Omega f(v)\,dS([h_0],v).$$

Proof. From (4.27), it follows immediately that, for sufficiently small t,

$$\log h_t(v)=\log h_0(v)+t\frac{f(v)}{h_0(v)}+o(t,v).$$

Since $\widetilde{V}_n=V$, the case $q=n$ in Theorem 4.5 gives

$$\lim_{t\to0}\frac{V([h_t])-V([h_0])}{t}=n\int_\Omega\frac{f(v)}{h_0(v)}\,d\widetilde{C}_n([h_0],v).$$

But from (3.28) we know that, on S^{n-1},

$$d\widetilde{C}_n([h_0],\cdot)=dV_{[h_0]}=\frac{1}{n}h_{[h_0]}\,dS([h_0],\cdot)=\frac{1}{n}h_0\,dS([h_0],\cdot).$$

The last of these equalities follows from the well-known fact that the set of points on Ω where $h_{[h_0]}\neq h_0$ has $S([h_0],\cdot)$-measure zero (see statement (7.100) in Schneider [75]). □

5. Minkowski problems associated with quermassintegrals and dual quermassintegrals

Roughly speaking, Minkowski problems are characterization problems of the differentials of geometric functionals of convex bodies. Two families of fundamental geometric functionals of convex bodies are quermassintegrals and dual quermassintegrals. Minkowski problems associated with quermassintegrals have a long history and have attracted much attention from convex geometry, differential geometry, and partial differential equations. We first mention these Minkowski problems, and then pose a Minkowski problem for dual quermassintegrals, that we call the *dual Minkowski problem*.

Area measures come from variations of quermassintegrals and can be viewed as differentials of quermassintegrals. The Minkowski problem associated with quermassintegrals is the following.

The Minkowski problem for area measures. *Given a finite Borel measure μ on the unit sphere S^{n-1} and an integer $1\leqslant i\leqslant n-1$, what are necessary and sufficient conditions for the existence of a convex body K in $\mathbb{R}^n$ satisfying*

$$S_i(K,\cdot)=\mu\,?$$

The case of $i=n-1$ is the classical Minkowski problem. The case of $i=1$ is the classical Christoffel problem.

As is established in previous sections, dual curvature measures come from variations of dual quermassintegrals and can be viewed as differentials of dual quermassintegrals. Thus, the Minkowski problem associated with dual quermassintegrals is dual to the Minkowski problem for area measures which is associated with quermassintegrals. Therefore, it is natural to pose the following dual Minkowski problem.

The Minkowski problem for dual curvature measures. *Given a finite Borel measure μ on the unit sphere S^{n-1} and a real number q, what are necessary and sufficient conditions for the existence of a convex body $K\in\mathcal{K}_o^n$ satisfying*

$$\widetilde{C}_q(K,\cdot)=\mu\,?$$

Now, (3.28) reminds us that the nth dual curvature measure $\widetilde{C}_n(K,\cdot)$ is the cone-volume measure V_K. So, the case $q=n$ of the dual Minkowski problem is the logarithmic Minkowski problem for cone-volume measure. Also (3.29) reminds us that the zeroth dual curvature measure $\widetilde{C}_0(K,\cdot)$ is Aleksandrov's integral curvature of K^* (with the constant factor $1/n$). Thus, the case $q=0$ of the dual Minkowski problem is the Aleksandrov problem. The logarithmic Minkowski problem and the Aleksandrov problem were thought to be two entirely different problems. It is amazing that they are now seen to be special cases of the dual Minkowski problem.

We will use the variational method to obtain a solution to the dual Minkowski problem for the symmetric case. The first crucial step is to associate the dual Minkowski problem with a maximization problem. By using the variational formulas for dual quermassintegrals, we can transform the existence problem for the Minkowski problem for dual curvature measures into a maximization problem.

Let μ be a finite Borel measure on S^{n-1} and let $K\in\mathcal{K}_o^n$. Define

$$\Phi_\mu(K)=-\frac{1}{|\mu|}\int_{S^{n-1}}\log h_K(v)\,d\mu(v)+\log\overline{V}_q(K). \tag{5.1}$$

Recall that $\overline{V}_q(K)$ is the normalized $(n-q)$-th dual quermassintegral of K. Since $\overline{V}_q$ is homogeneous of degree 1, it follows that Φ_μ is homogeneous of degree 0; i.e., for $Q\in\mathcal{K}_o^n$ and $\lambda>0$,

$$\Phi_\mu(\lambda Q)=\Phi_\mu(Q). \tag{5.2}$$

The following lemma shows that a solution to the dual Minkowski problem for the measure μ is also a solution to a maximization problem for the functional Φ_μ.

Lemma 5.1. *Let $q\in\mathbb{R}$ and μ be a finite even Borel measure on S^{n-1} with $|\mu|>0$. When $q=0$, we further require that $|\mu|=\omega_n$. If $K\in\mathcal{K}_e^n$ with $\widetilde{V}_q(K)=|\mu|$ is such that*

$$\sup\{\Phi_\mu(Q):\widetilde{V}_q(Q)=|\mu| \text{ and } Q\in\mathcal{K}_e^n\}=\Phi_\mu(K),$$

then

$$\widetilde{C}_q(K,\cdot)=\mu.$$

Proof. On $C_e^+(S^{n-1})$, the class of strictly positive continuous even functions on S^{n-1}, define the functional $\Phi\colon C_e^+(S^{n-1})\to\mathbb{R}$ by letting, for each $f\in C_e^+(S^{n-1})$,

$$\Phi(f)=\frac{1}{|\mu|}\int_{S^{n-1}}\log f\,d\mu+\log\overline{V}_q(\langle f\rangle^*). \tag{5.3}$$

Note that since f is even, it follows that $\langle f\rangle=\operatorname{conv}\{f(u)u:u\in S^{n-1}\}\in\mathcal{K}_e^n$. We first observe that Φ is homogeneous of degree 0, in that for all $\lambda>0$ and all $f\in C_e^+(S^{n-1})$,

$$\Phi(\lambda f)=\Phi(f).$$

To see this, first recall the fact that, from its definition, $\overline{V}_q$ is obviously homogeneous of degree 1, while clearly $\langle\lambda f\rangle=\lambda\langle f\rangle$ and thus $\langle\lambda f\rangle^*=\lambda^{-1}\langle f\rangle^*$.

To see that $\Phi: C_e^+(S^{n-1})\to\mathbb{R}$ is continuous, recall that if $f_0, f_1, ...\in C_e^+(S^{n-1})$ are such that $\lim_{k\to\infty} f_k=f_0$, uniformly on S^{n-1}, then $\langle f_k\rangle\to\langle f_0\rangle$, and thus $\langle f_k\rangle^*\to\langle f_0\rangle^*$. Since $\overline{V}_q: \mathcal{K}_o^n\to(0,\infty)$ is continuous, the continuity of Φ follows.

Consider the maximization problem

$$\sup\{\Phi(f): f\in C_e^+(S^{n-1})\}. \tag{5.4}$$

For the convex hull $\langle f\rangle=\operatorname{conv}\{f(u)u: u\in S^{n-1}\}$, of $f\in C_e^+(S^{n-1})$, we clearly have $\varrho_{\langle f\rangle}\geqslant f$ and also $\langle\varrho_{\langle f\rangle}\rangle=\langle f\rangle$, from (2.25), and thus $\langle\varrho_{\langle f\rangle}\rangle^*=\langle f\rangle^*$. Thus, directly from (5.3), we see that

$$\Phi(f)\leqslant\Phi(\varrho_{\langle f\rangle}).$$

This tells us that in searching for the supremum in (5.4) we can restrict our attention to the radial functions of bodies in $\mathcal{K}_e^n$; i.e.,

$$\sup\{\Phi(f): f\in C_e^+(S^{n-1})\}=\sup\{\Phi(\varrho_Q): Q\in\mathcal{K}_e^n\}.$$

Therefore, a convex body $K_0\in\mathcal{K}_e^n$ satisfies

$$\Phi_\mu(K_0)=\sup\{\Phi_\mu(Q): Q\in\mathcal{K}_e^n\}$$

if and only if

$$\Phi(\varrho_{K_0^*})=\sup\{\Phi(f): f\in C_e^+(S^{n-1})\}.$$

From (5.2) we see that we can always restrict our search to bodies $Q\in\mathcal{K}_e^n$ for which $\widetilde{V}_q(Q)=|\mu|$, when $q\neq0$. When $q=0$, note that $\widetilde{V}_0(Q)=|\mu|$ requires that $|\mu|=\omega_n$, since $\widetilde{V}_0(Q)=\omega_n$ for all bodies Q. Thus, we can restrict our attention to bodies such that $\widetilde{V}_q(Q)=|\mu|$.

Suppose that $K_0\in\mathcal{K}_e^n$ is a maximizer for Φ_μ, or equivalently that $\varrho_{K_0^*}$ is a maximizer for Φ; i.e.,

$$\Phi_\mu(K_0)=\sup\{\Phi_\mu(Q): \widetilde{V}_q(Q)=|\mu| \text{ and } Q\in\mathcal{K}_e^n\}.$$

Fix an arbitrary continuous even function $g: S^{n-1}\to\mathbb{R}$. For $\delta>0$ and $t\in(-\delta,\delta)$, define ϱ_t by

$$\varrho_t=\varrho_{K_0^*}e^{tg},$$

or equivalently by

$$\log\varrho_t=\log\varrho_{K_0^*}+tg. \tag{5.5}$$

Let $\langle\varrho_t\rangle=\langle K_0^*, g, t\rangle$ be the logarithmic family of convex hulls associated with (K_0^*, g). Since $\langle\varrho_0\rangle=\langle\varrho_{K_0^*}\rangle=K_0^*$, Corollary 4.6 gives

$$\frac{d}{dt}\log \overline{V}_q(\langle K_0^*, g, t\rangle^*)\bigg|_{t=0} = -\frac{1}{\widetilde{V}_q(K_0)}\int_{S^{n-1}} g(v)\, d\widetilde{C}_q(K_0, v). \tag{5.6}$$

From the fact that $\varrho_0=\varrho_{K_0^*}$ is a maximizer for Φ and definition (5.3), we have

$$0=\frac{d}{dt}\Phi(\varrho_t)\bigg|_{t=0} = \frac{d}{dt}\left(\frac{1}{|\mu|}\int_{S^{n-1}}\log\varrho_t(v)\, d\mu(v)+\log\overline{V}_q(\langle K_0^*, g, t\rangle^*)\right)\bigg|_{t=0}.$$

This, together with (5.5) and (5.6), shows that

$$\frac{1}{|\mu|}\int_{S^{n-1}} g(v)\, d\mu(v)-\frac{1}{\widetilde{V}_q(K_0)}\int_{S^{n-1}} g(v)\, d\widetilde{C}_q(K_0, v)=0. \tag{5.7}$$

Since $\widetilde{V}_q(K_0)=|\mu|$, and since (5.7) must hold for all continuous even $g: S^{n-1}\to\mathbb{R}$, we conclude that $\mu=\widetilde{C}_q(K_0,\cdot)$. □

Since this paper aims at a solution to the dual Minkowski problem for origin-symmetric convex bodies, Lemma 5.1 is stated and proved only for even measures and origin-symmetric convex bodies. However, a similar result holds for general measures and convex bodies that contain the origin in their interiors. The above proof works, mutatis mutandis. However, note that the maximization problem in Lemma 5.1 may not have a solution for a general measure and convex bodies that are not origin-symmetric. For example, this can occur when the measure is discrete and the origin approaches one of the vertices of the polytope, since the supremum then becomes arbitrarily large.

6. Solving the maximization problem associated with the dual Minkowski problem

In the previous section, by using a variational argument, we showed that the existence of a solution to a certain maximization problem would imply the existence of a solution to the dual Minkowski problem. In this section we show that the maximization problem does indeed have a solution. The key is to prove compactness and non-degeneracy, that is, the convergence of a maximizing sequence of convex bodies to a convex body (a compact convex set with non-empty interior). This requires delicate estimates of dual quermassintegrals of polytopes and entropy-type integrals with respect to the given measure in the dual Minkowski problem.

Throughout this section, for real $p>0$, we shall use p' to denote the Hölder conjugate of p. Also, the expression $c_1=c(n,k,N)$ will be used to mean that c_1 is a "constant" depending on only the values of n, k, and N.

6.1. Dual quermassintegrals of cross polytopes

Let $e_1, ..., e_n$ be orthogonal unit vectors and $a_1, ..., a_n \in (0, \infty)$. The convex body

$$P = \{x \in \mathbb{R}^n : |x \cdot e_i| \leqslant a_i \text{ for all } i\}$$

is a rectangular parallelotope centered at the origin. The parallelotope P is the Minkowski sum of the line segments whose support functions are $x \mapsto a_i |x \cdot e_i|$, and hence the support function of P is given by

$$h_P(x) = \sum_{i=1}^{n} a_i |x \cdot e_i|,$$

for $x \in \mathbb{R}^n$. The polar body P^* is a cross polytope. From (2.3) we know that the radial function of P^* is given by

$$\varrho_{P^*}(x) = \frac{1}{h_P(x)} = \left(\sum_{i=1}^{n} a_i |x \cdot e_i| \right)^{-1}$$

for $x \in \mathbb{R}^n \setminus \{0\}$. From (2.6) we know that for the $(n-q)$-th dual quermassintegral of the cross polytope P^* we have

$$\widetilde{W}_{n-q}(P^*) = \frac{1}{n} \int_{S^{n-1}} \left(\sum_{i=1}^{n} a_i |u \cdot e_i| \right)^{-q} du. \tag{6.1}$$

From (2.6) we see that, when $q=n$, (6.1) becomes the (well-known) volume of a cross-polytope:

$$V(P^*) = \frac{2^n}{n!} (a_1 \dots a_n)^{-1}, \tag{6.2}$$

and thus

$$\int_{S^{n-1}} \left(\sum_{i=1}^{n} a_i |u \cdot e_i| \right)^{-n} du = \frac{2^n}{(n-1)!} (a_1 \dots a_n)^{-1}. \tag{6.3}$$

When some of the a_i are small, the dual quermassintegral $\widetilde{W}_{n-q}(P^*)$ becomes large. The following lemma gives a critical estimate for the size of the dual quermassintegral.

LEMMA 6.1. *Let $q \in (0, n]$ and let k be an integer such that $1 \leqslant k < n$. Let $e_1, ..., e_n$ be an orthonormal basis in $\mathbb{R}^n$, and let $\varepsilon_0 > 0$. If $a_1, ..., a_n \in (0, \infty)$ with $a_{k+1}, ..., a_n \in (\varepsilon_0, \infty)$, then*

$$\frac{1}{q} \log \int_{S^{n-1}} \left(\sum_{i=1}^{n} a_i |u \cdot e_i| \right)^{-q} du \leqslant -\frac{1}{N} \log(a_1 \dots a_k) + c_0, \tag{6.4}$$

where

$$N = \begin{cases} n, & \text{when } q = n, \\ \infty, & \text{when } 0 < q < 1, \\ \theta, & \text{when } 1 \leqslant q < n, \end{cases}$$

where θ can be chosen to be any real number such that

$$\frac{q-1}{(n-1)q} < \frac{1}{\theta} < \frac{1}{n},$$

and $c_0>0$ is

$$c_0 = \begin{cases} c(k,n,\varepsilon_0), & \textit{when } q=n, \\ c(q,k,n,\varepsilon_0), & \textit{when } 0<q<1, \\ c(q,k,n,\varepsilon_0,N), & \textit{when } 1\leqslant q<n. \end{cases}$$

Proof. When $q=n$, inequality (6.4) follows directly from (6.3).

Now consider the case where $0<q<n$. Write $\mathbb{R}^n=\mathbb{R}^k\times\mathbb{R}^{n-k}$, with $\{e_1,...,e_k\}\subset\mathbb{R}^k$ and $\{e_{k+1},...,e_n\}\subset\mathbb{R}^{n-k}$. Consider the general spherical coordinates

$$u=(u_2\cos\varphi, u_1\sin\varphi),$$

with $u_2\in S^{k-1}\subset\mathbb{R}^k$, $u_1\in S^{n-k-1}\subset\mathbb{R}^{n-k}$, and $0\leqslant\varphi\leqslant\frac{1}{2}\pi$.

For spherical Lebesgue measures on S^{n-1} and its subspheres we have (see [30])

$$du=(\cos\varphi)^{k-1}(\sin\varphi)^{n-k-1}\,d\varphi\,du_2\,du_1. \tag{6.5}$$

Let

$$h_1(u_1)=\sum_{i=k+1}^{n} a_i|u_1\cdot e_i| \quad \text{and} \quad h_2(u_2)=\sum_{i=1}^{k} a_i|u_2\cdot e_i|$$

be the support functions of the corresponding rectangular parallelotopes in $\mathbb{R}^{n-k}$ and $\mathbb{R}^k$. Throughout N will be chosen so that $N\geqslant n$. Let $p=N/k>1$. From (6.5) and Young's inequality, we have

$$\begin{aligned} \int_{S^{n-1}}\biggl(\sum_{i=1}^{n} a_i|u\cdot e_i|\biggr)^{-q} du &= \int_{S^{n-1}}\Bigl(h_2(u_2)\cos\varphi+h_1(u_1)\sin\varphi\Bigr)^{-q} du \\ &\leqslant \int_0^{\pi/2}\!\!\int_{S^{n-k-1}}\int_{S^{k-1}} (p'h_1(u_1)\sin\varphi)^{-q/p'}(ph_2(u_2)\cos\varphi)^{-q/p} \\ &\qquad\qquad \times(\sin\varphi)^{n-k-1}(\cos\varphi)^{k-1}\,d\varphi\,du_1\,du_2 \\ &= c_2\int_{S^{n-k-1}} h_1(u_1)^{-q/p'}\,du_1\int_{S^{k-1}} h_2(u_2)^{-q/p}\,du_2, \end{aligned} \tag{6.6}$$

where

$$\begin{aligned} c_2 &= ({p'}^{1/p'}p^{1/p})^{-q}\int_0^{\pi/2}(\sin\varphi)^{n-k-1-q/p'}(\cos\varphi)^{k-1-q/p}\,d\varphi \\ &= \frac{1}{2}({p'}^{1/p'}p^{1/p})^{-q}\,\mathrm{B}\biggl(\frac{1}{2}\biggl(n-k-\frac{q}{p'}\biggr),\frac{1}{2}\biggl(k-\frac{q}{p}\biggr)\biggr). \end{aligned}$$

The integral above lies in $(0,\infty)$ provided that both

$$k-\frac{q}{p}>0 \quad \text{and} \quad n-k-\frac{q}{p'}>0, \tag{6.7}$$

which, since $N=pk$, can be written as

$$\frac{1}{N}<\frac{1}{q} \quad \text{and} \quad \frac{1}{N}>\frac{1}{k}\left(1-\frac{n}{q}\right)+\frac{1}{q}. \tag{6.8}$$

From $1\leqslant k<n$ and $0<q<n$, we know that

$$\frac{1}{q}>\frac{1}{n} \quad \text{and} \quad \frac{1}{k}\left(1-\frac{n}{q}\right)+\frac{1}{q}\leqslant\frac{1}{n-1}\left(1-\frac{n}{q}\right)+\frac{1}{q}=\frac{1}{(n-1)q'}.$$

Thus, the inequalities in (6.8), and hence the inequalities in (6.7), will be satisfied whenever N can be chosen so that

$$\frac{1}{(n-1)q'}<\frac{1}{N}<\frac{1}{n}. \tag{6.9}$$

Since we are dealing with the case where $0<q<n$, such a choice of N is always possible. Note that all of this continues to hold in the cases where $N=\infty=p$ and where $p=1$, mutatis mutandis.

Consider first the subcase where $1\leqslant q<n$. Jensen's inequality, together with the left inequality in (6.7), and (6.3) in $\mathbb{R}^k$, gives

$$\left(\frac{1}{k\omega_k}\int_{S^{k-1}} h_2(u_2)^{-q/p}\,du_2\right)^{p/q}\leqslant\left(\frac{1}{k\omega_k}\int_{S^{k-1}} h_2(u_2)^{-k}\,du_2\right)^{1/k}=c_3(a_1\ldots a_k)^{-1/k}. \tag{6.10}$$

Jensen's inequality, together with the right inequality in (6.7), and (6.3) in $\mathbb{R}^{n-k}$, when combined with the fact that $a_{k+1},\ldots,a_n>\varepsilon_0$, gives

$$\begin{aligned}\left(\int_{S^{n-k-1}} h_1(u_1)^{-q/p'}\,du_1\right)^{p'/q}&\leqslant c_4\left(\int_{S^{n-k-1}} h_1(u_1)^{k-n}\,du_1\right)^{1/(n-k)}\\&=c_5(a_{k+1}\ldots a_n)^{1/(k-n)}\leqslant c_6,\end{aligned} \tag{6.11}$$

where $c_3,\ldots,c_6$ are $c(q,k,n,N,\varepsilon_0)$ constants.

Since $p>1$, we know that p' is positive. Using (6.6), (6.11), and (6.10), we get

$$\log\int_{S^{n-1}}\left(\sum_{i=1}^n a_i|u\cdot e_i|\right)^{-q}du\leqslant\log c_2+\frac{q}{p'}\log c_6+\frac{q}{p}\log c_3+\log(k\omega_k)-\frac{q}{pk}\log(a_1\ldots a_k).$$

Since $N=pk$, this gives (6.4) for the case where $1\leqslant q<n$.

Finally, we treat the subcase where $0<q<1$. In this case, the Hölder conjugate $q'<0$, and to satisfy (6.9) we may take N to be arbitrary large. Taking the limit $p\to\infty$ (and hence $p'\to 1$) turns (6.6) into

$$\int_{S^{n-1}}\biggl(\sum_{i=1}^n a_i|u\cdot e_i|\biggr)^{-q}du\leqslant c_2'\int_{S^{n-k-1}}h_1(u_1)^{-q}\,du_1\int_{S^{k-1}}du_2, \tag{6.12}$$

where

$$c_2'=\frac{1}{2}\,\mathrm{B}\biggl(\frac{n-k-q}{2},\frac{k}{2}\biggr),$$

which is positive and depends only on q, k, and n. In this subcase, inequalities (6.11) become

$$\begin{aligned}\biggl(\int_{S^{n-k-1}}h_1(u_1)^{-q}\,du_1\biggr)^{1/q}&\leqslant c_4'\biggl(\int_{S^{n-k-1}}h_1(u_1)^{k-n}\,du_1\biggr)^{1/(n-k)}\\&=c_5'(a_{k+1}\dots a_n)^{1/(k-n)}\leqslant c_6'.\end{aligned}\tag{6.13}$$

Thus, using (6.12) and (6.13), gives

$$\frac{1}{q}\log\int_{S^{n-1}}\biggl(\sum_{i=1}^n a_i|u\cdot e_i|\biggr)^{-q}du\leqslant\frac{1}{q}\log c_2'+\log c_6'+\frac{1}{q}\log(k\omega_k),$$

which gives (6.4) in the subcase where $0<q<1$. □

6.2. An elementary entropy-type inequality

As a technical tool, the following elementary entropy-type inequality is needed.

LEMMA 6.2. *Let $N\in(0,\infty)$ and $\alpha_1,\dots,\alpha_n\in(0,\infty)$ be such that*

$$\alpha_i+\dots+\alpha_n<1-\frac{i-1}{N}\ \textit{ for all } i>1,\quad \textit{and}\quad \alpha_1+\dots+\alpha_n=1. \tag{6.14}$$

Then there exists a small $t>0$ such that

$$\sum_{i=1}^n\alpha_i\log a_i\leqslant\frac{1+t}{N}\log(a_1\dots a_n)+\biggl(1-\frac{n(1+t)}{N}\biggr)\log a_n,$$

for all $a_1,\dots,a_n\in(0,\infty)$ with $a_1\leqslant a_2\leqslant\dots\leqslant a_n$.

Note that $t=t(\alpha_1,\dots,\alpha_n,N)$ is independent of all of the $a_1,\dots,a_n\in(0,\infty)$.

Proof. Let $t>0$ be sufficiently small such that, for all $1<i\leqslant n$,

$$\alpha_i+\ldots+\alpha_n<1-\frac{i-1}{N}(1+t)=1-(i-1)\lambda, \tag{6.15}$$

where $\lambda=(1+t)/N$. Let

$$\beta_i=\alpha_i-\lambda, \quad \text{for } i<n, \quad \text{and} \quad \beta_n=\alpha_n+(n-1)\lambda-1.$$

Let also

$$s_i=\beta_i+\ldots+\beta_n, \quad \text{for } i=1,\ldots,n, \quad \text{and} \quad s_{n+1}=0.$$

Then, not only is $s_{n+1}=0$, but also

$$s_1=\beta_1+\ldots+\beta_n=\alpha_1+\ldots+\alpha_n-(n-1)\lambda+(n-1)\lambda-1=0,$$

while, for $1<i\leqslant n$,

$$s_i=\beta_i+\ldots+\beta_n=\alpha_i+\ldots+\alpha_n-(n-i+1)\lambda+n\lambda-1=\alpha_i+\ldots+\alpha_n+(i-1)\lambda-1<0,$$

by (6.15). Now,

$$\begin{aligned}\sum_{i=1}^n\beta_i\log a_i&=\sum_{i=1}^n(s_i-s_{i+1})\log a_i\\&=\sum_{i=1}^n s_i\log a_i-\sum_{i=1}^n s_{i+1}\log a_i\\&=\sum_{i=1}^{n-1}s_{i+1}\log a_{i+1}+s_1\log a_1-\sum_{i=1}^{n-1}s_{i+1}\log a_i-s_{n+1}\log a_n\\&=\sum_{i=1}^{n-1}s_{i+1}(\log a_{i+1}-\log a_i)\\&\leqslant 0,\end{aligned}$$

since the a_i are monotone non-decreasing. Therefore,

$$\begin{aligned}\sum_{i=1}^n\alpha_i\log a_i&=\sum_{i=1}^n(\beta_i+\lambda)\log a_i+(1-n\lambda)\log a_n\\&=\sum_{i=1}^n\beta_i\log a_i+\lambda\log(a_1\ldots a_n)+(1-n\lambda)\log a_n\\&\leqslant\frac{1+t}{N}\log(a_1\ldots a_n)+\left(1-\frac{n(1+t)}{N}\right)\log a_n.\end{aligned}$$

□

6.3. Estimation of an entropy-type integral with respect to a measure

We first define a partition of the unit sphere. Then we use the partition to estimate an entropy-type integral by the entropy-type finite sum treated in §6.2.

Let $e_1, ..., e_n$ be a fixed orthonormal basis for $\mathbb{R}^n$. Relative to this basis, for each $i=1, ..., n$, define $S^{n-i}=S^{n-1}\cap\text{span}\{e_i, ..., e_n\}$. For convenience, define $S^{-1}=\varnothing$.

For small $\delta\in(0, 1/\sqrt{n}\,)$, define a partition of S^{n-1}, with respect to the orthonormal basis $e_1, ..., e_n$, by letting

$$\Omega_{i,\delta} = \{u\in S^{n-1} : |u\cdot e_i| \geqslant \delta, \text{ and } |u\cdot e_j| < \delta \text{ for } j<i\}, \quad i=1,2,...,n. \tag{6.16}$$

Explicitly,

$$\begin{aligned}
\Omega_{1,\delta} &= \{u\in S^{n-1} : |u\cdot e_1| \geqslant \delta\},\\
\Omega_{2,\delta} &= \{u\in S^{n-1} : |u\cdot e_1| < \delta, |u\cdot e_2| \geqslant \delta\},\\
\Omega_{3,\delta} &= \{u\in S^{n-1} : |u\cdot e_1| < \delta, |u\cdot e_2| < \delta, |u\cdot e_3| \geqslant \delta\},\\
&\;\;\vdots\\
\Omega_{n,\delta} &= \{u\in S^{n-1} : |u\cdot e_1| < \delta, ..., |u\cdot e_{n-1}| < \delta, |u\cdot e_n| \geqslant \delta\}.
\end{aligned}$$

These sets are non-empty since $e_i\in\Omega_{i,\delta}$. They are obviously disjoint. For $\delta\in(0, 1/\sqrt{n}\,)$ and each $u\in S^{n-1}$, there is an e_i such that $|u\cdot e_i|\geqslant\delta$ and for the smallest such i, say i_0, we will have $u\in\Omega_{i_0,\delta}$. Thus, the union of $\Omega_{i,\delta}$ covers S^{n-1}.

If we let

$$\begin{aligned}
\Omega'_{i,\delta} &= \{u\in S^{n-1} : |u\cdot e_i| \geqslant \delta, \text{ and } |u\cdot e_j| = 0 \text{ for } j<i\},\\
\Omega''_{i,\delta} &= \{u\in S^{n-1} : |u\cdot e_i| > 0, \text{ and } |u\cdot e_j| < \delta \text{ for } j<i\},
\end{aligned}$$

then

$$\Omega'_{i,\delta} \subset \Omega_{i,\delta} \subset \Omega''_{i,\delta}. \tag{6.17}$$

As δ decreases to 0, the set $\Omega'_{i,\delta}$ increases (with respect to set inclusion) to $S^{n-i}\setminus S^{n-i-1}$, while $\Omega''_{i,\delta}$ decreases to $S^{n-i}\setminus S^{n-i-1}$.

Suppose now that μ is a finite Borel measure on S^{n-1}. From the definitions of $\Omega'_{i,\delta}$ and $\Omega''_{i,\delta}$, we conclude that

$$\lim_{\delta\to0^+} \mu(\Omega'_{i,\delta}) = \mu(S^{n-i}\setminus S^{n-i-1})$$

and also

$$\lim_{\delta\to0^+} \mu(\Omega''_{i,\delta}) = \mu(S^{n-i}\setminus S^{n-i-1}).$$

This, together with (6.17), gives

$$\lim_{\delta\to 0^+}\mu(\Omega_{i,\delta})=\mu(S^{n-i}\setminus S^{n-i-1}). \tag{6.18}$$

It follows that, for each integer $1\leqslant k\leqslant n$,

$$\lim_{\delta\to 0^+}\mu\bigg(\bigcup_{i=k}^n\Omega_{i,\delta}\bigg)=\lim_{\delta\to 0^+}\sum_{i=k}^n\mu(\Omega_{i,\delta})=\sum_{i=k}^n\mu(S^{n-i}\setminus S^{n-i-1})=\mu(S^{n-k}), \tag{6.19}$$

where $S^{n-k}=S^{n-1}\cap\operatorname{span}\{e_k,...,e_n\}$.

LEMMA 6.3. *Let μ be a finite Borel measure on S^{n-1}. Let $a_{1l},...,a_{nl}$ be n sequences in $(0,\infty)$ indexed by $l=1,2,...$. Similarly, let $e_{1l},...,e_{nl}$ be a sequence of orthonormal bases in $\mathbb{R}^n$, which converges to the orthonormal basis $e_1,...,e_n$. For each $i=1,...,n$, and small $\delta\in(0,1/\sqrt{n}\,)$, let*

$$\Omega_{i,\delta}=\{u\in S^{n-1}:|u\cdot e_i|\geqslant\delta,\ \text{and}\ |u\cdot e_j|<\delta\ \text{for}\ j<i\}.$$

Then, for each small $\delta>0$, there exists an integer L such that, for all $l>L$,

$$\frac{1}{|\mu|}\int_{S^{n-1}}\log\sum_{i=1}^n\frac{|u\cdot e_{il}|}{a_{il}}\,d\mu(u)\geqslant\log\frac{\delta}{2}-\sum_{i=1}^n\frac{\mu(\Omega_{i,\delta})}{|\mu|}\log a_{il}.$$

Proof. Since $e_{1l},...,e_{nl}$ converge to $e_1,...,e_n$, for the given $\delta>0$, there exists an L such that, for all $l>L$,

$$|e_{il}-e_i|<\tfrac{1}{2}\delta,$$

for all i. Then, for $l>L$ and $u\in\Omega_{i,\delta}$,

$$|u\cdot e_{il}|\geqslant|u\cdot e_i|-|u\cdot(e_{il}-e_i)|\geqslant|u\cdot e_i|-|e_{il}-e_i|\geqslant\tfrac{1}{2}\delta \tag{6.20}$$

for all i. Therefore, for $l>L$, by using the partition $S^{n-1}=\bigcup_{i=1}^n\Omega_{i,\delta}$, together with (6.20), we have

$$\begin{aligned}\int_{S^{n-1}}\log\sum_{j=1}^n\frac{|u\cdot e_{jl}|}{a_{jl}}\,d\mu(u)&=\sum_{i=1}^n\int_{\Omega_{i,\delta}}\log\sum_{j=1}^n\frac{|u\cdot e_{jl}|}{a_{jl}}\,d\mu(u)\\&\geqslant\sum_{i=1}^n\int_{\Omega_{i,\delta}}\log\frac{|u\cdot e_{il}|}{a_{il}}\,d\mu(u)\\&=\sum_{i=1}^n\int_{\Omega_{i,\delta}}\log|u\cdot e_{il}|\,d\mu-\sum_{i=1}^n\mu(\Omega_{i,\delta})\log a_{il}\\&\geqslant\mu(S^{n-1})\log\frac{\delta}{2}-\sum_{i=1}^n\mu(\Omega_{i,\delta})\log a_{il}.\end{aligned}$$

□

6.4. The subspace mass inequality and non-degeneracy

Let μ be a non-zero finite Borel measure on S^{n-1}. Fix an ordered orthonormal basis $\beta=\{e_1, ..., e_n\}$ in $\mathbb{R}^n$. Let $\xi_{n-i+1}=\operatorname{span}(e_i, ..., e_n)$ be the subspace spanned by $e_i, ..., e_n$.

Let $q\in[1,n]$. We will say that μ satisfies the *q-th subspace mass inequality with respect to the basis* β if

$$\frac{\mu(S^{n-1}\cap\xi_{n-i})}{|\mu|}<1-\frac{i}{(n-1)q'} \tag{6.21}$$

for all positive $i<n$. We will say that μ satisfies the *q-th subspace mass inequality* if it does so with respect to every orthonormal basis.

For $q\in(0,1)$, we will say that μ satisfies the *q-th subspace mass inequality with respect to the basis* β if

$$\frac{\mu(S^{n-1}\cap\xi_{n-1})}{|\mu|}<1, \tag{6.22}$$

and if this is the case for every orthonormal basis, we shall say that μ satisfies the *q-th subspace mass inequality.*

When $q=1$, obviously $q'=\infty$, and thus the measure μ satisfies the qth subspace mass inequality (6.21) if

$$\frac{\mu(S^{n-1}\cap\xi_{n-i})}{|\mu|}<1$$

for all positive $i<n$ and each basis β. Here, obviously the case for $i=1$ implies all the cases for $i<n$. Observe that this is equivalent to the definition for $q\in(0,1)$.

When $q=n$, the subspace mass inequality (6.21) is also called the *strict subspace concentration* condition, see [14],

$$\frac{\mu(S^{n-1}\cap\xi)}{|\mu|}<\frac{\dim(\xi)}{n}$$

for each subspace ξ.

The following lemma will be used to show that the limit of a maximizing sequence, for the maximization problem associated with the dual Minkowski problem, will not be a degenerate compact convex set provided that the given measure satisfies the subspace mass inequality.

LEMMA 6.4. *Let μ be a non-zero finite Borel measure on S^{n-1} and $q\in(0,n]$. Let $a_{1l}, ..., a_{nl}$ be n sequences in $(0,\infty)$ for which there exist $\varepsilon_0>0$ and M_0 such that*

$$a_{1l}\leqslant a_{2l}\leqslant ... \leqslant a_{nl}\leqslant M_0 \quad \textit{for all } l,$$

and, for some integer $1\leqslant k<n$,

$$a_{1l}, ..., a_{kl}\to 0 \;\; \textit{as } l\to\infty, \quad \textit{and} \quad a_{k+1,l}, ..., a_{n,l}>\varepsilon_0 \;\; \textit{for all } l.$$

Let $e_{1l},...,e_{nl}$ be a sequence of orthonormal bases in $\mathbb{R}^n$ that converges to an orthonormal basis $e_1,...,e_n$. If μ satisfies the q-th subspace mass inequality, with respect to the orthonormal basis $e_1,...,e_n$, then

$$-\frac{1}{|\mu|}\int_{S^{n-1}}\log\sum_{i=1}^{n}\frac{|u\cdot e_{il}|}{a_{il}}\,d\mu(u)+\frac{1}{q}\log\int_{S^{n-1}}\left(\sum_{i=1}^{n}a_{il}|u\cdot e_{il}|\right)^{-q}du\to-\infty, \tag{6.23}$$

as $l\to\infty$.

Proof. First consider the case $q\in[1,n]$.

When $q\in[1,n)$, we use the fact that μ satisfies the qth subspace mass inequality, with respect to the orthonormal basis $e_1,...,e_n$, to deduce the existence of an $N\in(n,(n-1)q')$ such that, for all $i>1$,

$$\frac{\mu(S^{n-1}\cap\xi_{n-i+1})}{|\mu|}<1-\frac{i-1}{N}\leqslant 1-\frac{i-1}{(n-1)q'}, \tag{6.24}$$

where $\xi_{n-i+1}=\operatorname{span}\{e_i,...,e_n\}$. When $q=n$, take $N=n$ and note that (6.24) still holds.

For a small $\delta>0$, and each $i=1,...,n$, let $\Omega_{i,\delta}$ be the partition defined in (6.16), with respect to the orthonormal basis $e_1,...,e_n$, and let

$$\alpha_i=\alpha_i(\delta)=\frac{\mu(\Omega_{i,\delta})}{|\mu|}$$

for each i. From (6.19), we see that, as $\delta\to0^+$,

$$\alpha_i+...+\alpha_n\to\frac{\mu(S^{n-1}\cap\xi_{n-i+1})}{|\mu|}$$

for each i. This and (6.24) tell us that we can choose $\delta>0$ sufficiently small so that

$$\alpha_i+...+\alpha_n<1-\frac{i-1}{N}$$

for each $i>1$. Note that the α_i satisfy the conditions of (6.14) in Lemma 6.2.

The fact that $\alpha_i=\mu(\Omega_{i,\delta})/|\mu|$, combined with Lemma 6.3, followed by the fact that $a_{k+1,l},...,a_{nl}>\varepsilon_0$, together with Lemma 6.1, the given monotonicity $a_{1l}\leqslant a_{2l}\leqslant...\leqslant a_{nl}$, and lastly Lemma 6.2, yields the existence of a $t>0$ such that

$$\begin{aligned}
-\frac{1}{|\mu|}&\int_{S^{n-1}}\log\sum_{i=1}^{n}\frac{|u\cdot e_{il}|}{a_{il}}\,d\mu(u)+\frac{1}{q}\log\int_{S^{n-1}}\left(\sum_{i=1}^{n}a_{il}|u\cdot e_{il}|\right)^{-q}du\\
&\leqslant\sum_{i=1}^{n}\alpha_i\log a_{il}-\log\frac{\delta}{2}-\frac{1}{N}\log(a_{1l}\,...\,a_{kl})+c_0\\
&\leqslant\frac{1+t}{N}\log(a_{1l}\,...\,a_{nl})+\left(1-\frac{n(1+t)}{N}\right)\log a_{nl}-\log\frac{\delta}{2}-\frac{1}{N}\log(a_{1l}\,...\,a_{kl})+c_0.
\end{aligned} \tag{6.25}$$

Since the $a_{k+1,l}, ..., a_{n,l}$ are bounded, for all l, from below by ε_0 and from above by M_0, the last expression in (6.25) is bounded from above by $(t/N)\log(a_{1l} ... a_{kl})$ plus a quantity independent of l. But, since by hypothesis $a_{1l}, ..., a_{kl} \to 0$ as $l\to\infty$, obviously as $l\to\infty$ the last quantity in (6.25) tends to $-\infty$. This establishes the desired result for the case where $q\in[1, n]$.

Now suppose $q\in(0,1)$. For small $\delta>0$, and each $i=1, ..., n$, let $\Omega_{i,\delta}$ be the partition defined in (6.16), with respect to the orthonormal basis $e_1, ..., e_n$, and let

$$\alpha_i = \alpha_i(\delta) = \frac{\mu(\Omega_{i,\delta})}{|\mu|},$$

for each i, but choose $\delta>0$ so that $\alpha_1=\alpha_1(\delta)>0$. Since $\alpha_1+...+\alpha_n=1$, we have

$$\alpha_1 > 0 \quad \text{and} \quad \alpha_2+...+\alpha_n < 1.$$

From Lemmas 6.3 and 6.1 we have, for sufficiently large l,

$$\begin{aligned} -\frac{1}{|\mu|}\int_{S^{n-1}} \log \sum_{i=1}^n \frac{|u\cdot e_{il}|}{a_{il}}\, d\mu(u) + \frac{1}{q}\log \int_{S^{n-1}} \biggl(\sum_{i=1}^n a_{il}|u\cdot e_{il}|\biggr)^{-q} du \\ \leqslant \sum_{i=1}^n \alpha_i \log a_{il} - \log\frac{\delta}{2} + c_0, \end{aligned}$$

where c_0 is a constant independent of the sequences $a_{1l}, ..., a_{nl}$.

Since α_1 is positive and $a_{1l}\to 0^+$ as $l\to\infty$, we have $\alpha_1\log a_{1l}\to-\infty$. This and the assumption that $a_{1l}, ..., a_{nl}$ are bounded from above, allows us to conclude that, as $l\to\infty$,

$$\sum_{i=1}^n \alpha_i \log a_{il} \to -\infty.$$

This establishes the desired result for the case where $q\in(0,1)$. □

6.5. Existence of a solution to the maximization problem

The following lemma establishes the existence of solutions to the maximization problem associated with the dual Minkowski problem.

LEMMA 6.5. *Let $q\in(0,n]$, and let μ be a non-zero finite Borel measure on S^{n-1}. Suppose that the functional $\Phi: \mathcal{K}_e^n \to \mathbb{R}$ is defined for $Q\in\mathcal{K}_e^n$ by*

$$\Phi(Q) = \frac{1}{|\mu|}\int_{S^{n-1}} \log \varrho_Q(u)\, d\mu(u) + \frac{1}{q}\log\int_{S^{n-1}} h_Q(u)^{-q}\, du.$$

If μ satisfies the q-th subspace mass inequality, then there exists a $K\in\mathcal{K}_e^n$ so that

$$\sup_{Q\in\mathcal{K}_e^n} \Phi(Q) = \Phi(K).$$

Proof. Let Q_l be a maximizing sequence of origin-symmetric convex bodies; i.e. $Q_l \in \mathcal{K}_e^n$ such that

$$\lim_{l\to\infty} \Phi(Q_l) = \sup_{Q\in\mathcal{K}_e^n} \Phi(Q).$$

Since $\Phi(\lambda Q)=\Phi(Q)$ for $\lambda>0$, we may assume that the diameter of each Q_l is 1. By the Blaschke selection theorem, Q_l has a convergent subsequence, denoted again by Q_l, whose limit we call K. Note that K must be an origin-symmetric compact convex set. We will prove that K is not degenerate; i.e., K has non-empty interior.

Let $E_l \in \mathcal{K}_e^n$ be the John ellipsoid associated with Q_l, that is, the ellipsoid of maximal volume contained in Q_l. Then, as is well known (see Schneider [75, p. 588]),

$$E_l \subset Q_l \subset \sqrt{n} E_l.$$

Observe that for each ellipsoid E_l, there is a right parallelotope P_l so that

$$P_l \subset E_l \subset \sqrt{n} P_l.$$

This is easily seen when E_l is a ball. The general case is established as follows: transform E_l into a ball using an affine transformation whose eigenvectors are along the principal axes of E_l and whose eigenvalues are chosen so that E_l is transformed into a ball.

Therefore,

$$P_l \subset Q_l \subset nP_l. \tag{6.26}$$

But this means that

$$\varrho_{Q_l} \leqslant \varrho_{nP_l} = n\varrho_{P_l} \quad \text{and} \quad h_{P_l} \leqslant h_{Q_l}. \tag{6.27}$$

The support function of the right parallelotope P_l can be written, for $u\in S^{n-1}$, as

$$h_{P_l}(u) = \sum_{i=1}^{n} a_{il} |u\cdot e_{il}|, \tag{6.28}$$

where the orthonormal basis $e_{1l}, \dots .e_{nl}$ is ordered so that $0<a_{1l}\leqslant\dots\leqslant a_{nl}$. The radial function of P_l, for $u\in S^{n-1}$, is given by

$$\varrho_{P_l}(u) = \min_{1\leqslant i\leqslant n} \frac{a_{il}}{|u\cdot e_{il}|}.$$

Thus,

$$\varrho_{P_l}(u) \leqslant \left(\frac{1}{n}\sum_{i=1}^{n} \frac{|u\cdot e_{il}|}{a_{il}}\right)^{-1}. \tag{6.29}$$

Therefore, from the definition of Φ, (6.27), (6.29), and (6.28), we have

$$\begin{aligned}\Phi(Q_l) &\leqslant \Phi(P_l)+\log n \\ &\leqslant \frac{1}{|\mu|}\int_{S^{n-1}}\log\left(\sum_{i=1}^{n}\frac{|u\cdot e_{il}|}{a_{il}}\right)^{-1}d\mu(u)+\frac{1}{q}\log\int_{S^{n-1}}\left(\sum_{i=1}^{n}a_{il}|u\cdot e_{il}|\right)^{-q}du+2\log n.\end{aligned} \tag{6.30}$$

Since the diameter of each Q_l is 1, the parallelotopes P_l are bounded. Using the Blaschke selection theorem, we conclude that sequence P_l has a convergent subsequence, denoted again by P_l, whose limit we call P.

As $Q_l\to K$ and $P_l\to P$, while $P_l\subset Q_l\subset nP_l$, we must have $P\subset K\subset nP$. First note that P cannot be a point since the diameter of each P_l is at least $1/n$. Suppose that K has empty interior in $\mathbb{R}^n$. Hence, P is a degenerate right parallelotope, and hence there exists a k such that $1\leqslant k<n$ and a $\varepsilon_0>0$ so that $a_{1l},...,a_{kl}\to 0^+$, while $a_{k+1,l},...,a_{nl}\geqslant\varepsilon_0>0$, for all l. Moreover, as $l\to\infty$, taking subsequences as necessary, the orthonormal basis $e_{1l},...,e_{nl}$ converges to an orthonormal basis $e_1,...,e_n$ derived from P.

From (6.30) and Lemma 6.4, $\Phi(Q_l)\to-\infty$ as $l\to\infty$. Since Q_l is a maximizing sequence, we have that

$$\lim_{l\to\infty}\Phi(Q_l)\geqslant\Phi(B)=\frac{1}{q}\log(n\omega_n).$$

Thus we have the contradiction, which shows that K must have non-empty interior. □

6.6. Existence of a solution to the dual Minkowski problem

The main existence theorem for the dual Minkowski problem stated in the introduction is implied by the following theorem.

THEOREM 6.6. *Let μ be a non-zero finite even Borel measure on S^{n-1} and let $q\in(0,n]$. If the measure μ satisfies the q-th subspace mass inequality, then there exists an origin-symmetric convex body K in $\mathbb{R}^n$ such that $\widetilde{C}_q(K,\cdot)=\mu$.*

The proof follows directly from Lemmas 5.1 and 6.5.

The theorem above shows that the subspace mass inequality is a sufficient condition for the existence of a solution to the dual Minkowski problem. When $0<q\leqslant 1$, the subspace mass inequality means that the given even measure is not concentrated on any great hypersphere. This condition is obviously necessary. When $q=n$, if the given even measure is not concentrated in two complementary subspaces, it was proved in [14] that the subspace mass inequality is also necessary. Progress regarding the intermediate cases $1<q<n$ would be most welcome.

Acknowledgement

The authors thank Rolf Schneider, Guangxian Zhu, and Yiming Zhao for their comments on various drafts of this work. The authors thank the referees for detailed reading and comments that were both helpful and insightful.

References

[1] Aleksandrov, A. D., On the theory of mixed volumes of convex bodies. III. Extension of two theorems of Minkowski on convex polyhedra to arbitrary convex bodies. *Mat. Sb.*, 3 (1938), 27–46 (Russian); English translation in Aleksandrov, A. D., Selected Works, Part 1, Chapter 4, pp. 61–97, Gordon and Breach, Amsterdam, 1996.

[2] — On the area function of a convex body. *Mat. Sb.*, 6 (1939), 167–174 (Russian); English translation in Aleksandrov, A. D., Selected Works, Part 1, Chapter 9, pp. 155–162, Gordon and Breach, Amsterdam, 1996.

[3] — Existence and uniqueness of a convex surface with a given integral curvature. *Dokl. Akad. Nauk SSSR*, 35 (1942), 131–134.

[4] Alesker, S., Continuous rotation invariant valuations on convex sets. *Ann. of Math.*, 149 (1999), 977–1005.

[5] — Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture. *Geom. Funct. Anal.*, 11 (2001), 244–272.

[6] — The multiplicative structure on continuous polynomial valuations. *Geom. Funct. Anal.*, 14 (2004), 1–26.

[7] Alesker, S., Bernig, A. & Schuster, F. E., Harmonic analysis of translation invariant valuations. *Geom. Funct. Anal.*, 21 (2011), 751–773.

[8] Andrews, B., Gauss curvature flow: the fate of the rolling stones. *Invent. Math.*, 138 (1999), 151–161.

[9] — Classification of limiting shapes for isotropic curve flows. *J. Amer. Math. Soc.*, 16 (2003), 443–459.

[10] Barthe, F., Guédon, O., Mendelson, S. & Naor, A., A probabilistic approach to the geometry of the l_p^n-ball. *Ann. Probab.*, 33 (2005), 480–513.

[11] Berg, C., Corps convexes et potentiels sphériques. *Mat.-Fys. Medd. Danske Vid. Selsk.*, 37 (1969), 1–64.

[12] Bernig, A. & Fu, J. H. G., Hermitian integral geometry. *Ann. of Math.*, 173 (2011), 907–945.

[13] Böröczky, K. J., Lutwak, E., Yang, D. & Zhang, G., The log-Brunn–Minkowski inequality. *Adv. Math.*, 231 (2012), 1974–1997.

[14] — The logarithmic Minkowski problem. *J. Amer. Math. Soc.*, 26 (2013), 831–852.

[15] — Affine images of isotropic measures. *J. Differential Geom.*, 99 (2015), 407–442.

[16] Bourgain, J., On the Busemann–Petty problem for perturbations of the ball. *Geom. Funct. Anal.*, 1 (1991), 1–13.

[17] Caffarelli, L. A., Interior $W^{2,p}$ estimates for solutions of the Monge–Ampère equation. *Ann. of Math.*, 131 (1990), 135–150.

[18] Chen, W., L_p Minkowski problem with not necessarily positive data. *Adv. Math.*, 201 (2006), 77–89.

[19] Cheng, S. Y. & Yau, S. T., On the regularity of the solution of the n-dimensional Minkowski problem. *Comm. Pure Appl. Math.*, 29 (1976), 495–516.

[20] Chou, K. S. & Wang, X.-J., The L_p-Minkowski problem and the Minkowski problem in centroaffine geometry. *Adv. Math.*, 205 (2006), 33–83.

[21] Cianchi, A., Lutwak, E., Yang, D. & Zhang, G., Affine Moser–Trudinger and Morrey–Sobolev inequalities. *Calc. Var. Partial Differential Equations*, 36 (2009), 419–436.

[22] Evans, L. C. & Gariepy, R. F., *Measure Theory and Fine Properties of Functions*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2015.

[23] Federer, H., Curvature measures. *Trans. Amer. Math. Soc.*, 93 (1959), 418–491.

[24] Fenchel, W. & Jessen, B., Mengenfunktionen und konvexe Körper. *Danske Vid. Selskab. Mat.-Fys. Medd.*, 16 (1938), 1–31.

[25] Firey, W. J., Christoffel's problem for general convex bodies. *Mathematika*, 15 (1968), 7–21.

[26] Gardner, R. J., A positive answer to the Busemann–Petty problem in three dimensions. *Ann. of Math.*, 140 (1994), 435–447.

[27] — *Geometric Tomography*. Encyclopedia of Mathematics and its Applications, 58. Cambridge Univ. Press, New York, 2006.

[28] Gardner, R. J., Koldobsky, A. & Schlumprecht, T., An analytic solution to the Busemann–Petty problem on sections of convex bodies. *Ann. of Math.*, 149 (1999), 691–703.

[29] Goodey, P., Yaskin, V. & Yaskina, M., A Fourier transform approach to Christoffel's problem. *Trans. Amer. Math. Soc.*, 363 (2011), 6351–6384.

[30] Grinberg, E. & Zhang, G., Convolutions, transforms, and convex bodies. *Proc. London Math. Soc.*, 78 (1999), 77–115.

[31] Gruber, P. M., *Convex and Discrete Geometry*. Grundlehren der Mathematischen Wissenschaften, 336. Springer, Berlin–Heidelberg, 2007.

[32] Guan, B. & Guan, P., Convex hypersurfaces of prescribed curvatures. *Ann. of Math.*, 156 (2002), 655–673.

[33] Guan, P., Li, J. & Li, Y. Y., Hypersurfaces of prescribed curvature measure. *Duke Math. J.*, 161 (2012), 1927–1942.

[34] Guan, P. & Li, Y. Y., $C^{1,1}$ estimates for solutions of a problem of Alexandrov. *Comm. Pure Appl. Math.*, 50 (1997), 789–811.

[35] Guan, P., Lin, C. S. & Ma, X.-N., The existence of convex body with prescribed curvature measures. *Int. Math. Res. Not. IMRN*, 11 (2009), 1947–1975.

[36] Guan, P. & Ma, X.-N., The Christoffel–Minkowski problem. I. Convexity of solutions of a Hessian equation. *Invent. Math.*, 151 (2003), 553–577.

[37] Haberl, C., Minkowski valuations intertwining with the special linear group. *J. Eur. Math. Soc.* (*JEMS*), 14 (2012), 1565–1597.

[38] Haberl, C. & Parapatits, L., The centro-affine Hadwiger theorem. *J. Amer. Math. Soc.*, 27 (2014), 685–705.

[39] — Valuations and surface area measures. *J. Reine Angew. Math.*, 687 (2014), 225–245.

[40] Haberl, C. & Schuster, F. E., Asymmetric affine L_p Sobolev inequalities. *J. Funct. Anal.*, 257 (2009), 641–658.

[41] — General L_p affine isoperimetric inequalities. *J. Differential Geom.*, 83 (2009), 1–26.

[42] Hernández Cifre, M. A. & Saorín, E., On differentiability of quermassintegrals. *Forum Math.*, 22 (2010), 115–126.

[43] Huang, Y., Liu, J. & Xu, L., On the uniqueness of L_p-Minkowski problems: the constant p-curvature case in $\mathbb{R}^3$. *Adv. Math.*, 281 (2015), 906–927.

[44] Jian, H., Lu, J. & Wang, X.-J., Nonuniqueness of solutions to the L_p-Minkowski problem. *Adv. Math.*, 281 (2015), 845–856.

[45] Koldobsky, A., *Fourier Analysis in Convex Geometry*. Mathematical Surveys and Monographs, 116. Amer. Math. Soc., Providence, RI, 2005.
[46] Lu, J. & Wang, X. J., Rotationally symmetric solutions to the L_p-Minkowski problem. *J. Differential Equations*, 254 (2013), 983–1005.
[47] Ludwig, M., Ellipsoids and matrix-valued valuations. *Duke Math. J.*, 119 (2003), 159–188.
[48] — Intersection bodies and valuations. *Amer. J. Math.*, 128 (2006), 1409–1428.
[49] — Minkowski areas and valuations. *J. Differential Geom.*, 86 (2010), 133–161.
[50] — Valuations on Sobolev spaces. *Amer. J. Math.*, 134 (2012), 827–842.
[51] Ludwig, M. & Reitzner, M., A classification of $\mathrm{SL}(n)$ invariant valuations. *Ann. of Math.*, 172 (2010), 1219–1267.
[52] Lutwak, E., Dual mixed volumes. *Pacific J. Math.*, 58 (1975), 531–538.
[53] — Intersection bodies and dual mixed volumes. *Adv. in Math.*, 71 (1988), 232–261.
[54] — The Brunn–Minkowski–Firey theory. I. Mixed volumes and the Minkowski problem. *J. Differential Geom.*, 38 (1993), 131–150.
[55] Lutwak, E. & Oliker, V., On the regularity of solutions to a generalization of the Minkowski problem. *J. Differential Geom.*, 41 (1995), 227–246.
[56] Lutwak, E., Yang, D. & Zhang, G., L_p affine isoperimetric inequalities. *J. Differential Geom.*, 56 (2000), 111–132.
[57] — A new ellipsoid associated with convex bodies. *Duke Math. J.*, 104 (2000), 375–390.
[58] — Sharp affine L_p Sobolev inequalities. *J. Differential Geom.*, 62 (2002), 17–38.
[59] — Volume inequalities for subspaces of L_p. *J. Differential Geom.*, 68 (2004), 159–184.
[60] — Orlicz centroid bodies. *J. Differential Geom.*, 84 (2010), 365–387.
[61] — Orlicz projection bodies. *Adv. Math.*, 223 (2010), 220–242.
[62] Lutwak, E. & Zhang, G., Blaschke–Santaló inequalities. *J. Differential Geom.*, 47 (1997), 1–16.
[63] Minkowski, H., Allgemeine Lehrsätze über die convexen Polyeder. *Nachr. Ges. Wiss. Göttingen*, (1897), 198–219.
[64] — Volumen und Oberfläche. *Math. Ann.*, 57 (1903), 447–495.
[65] Naor, A., The surface measure and cone measure on the sphere of l_p^n. *Trans. Amer. Math. Soc.*, 359 (2007), 1045–1079.
[66] Naor, A. & Romik, D., Projecting the surface measure of the sphere of l_p^n. *Ann. Inst. H. Poincaré Probab. Statist.*, 39 (2003), 241–261.
[67] Nirenberg, L., The Weyl and Minkowski problems in differential geometry in the large. *Comm. Pure Appl. Math.*, 6 (1953), 337–394.
[68] Oliker, V., Embedding $\mathbf{S}^n$ into $\mathbf{R}^{n+1}$ with given integral Gauss curvature and optimal mass transport on $\mathbf{S}^n$. *Adv. Math.*, 213 (2007), 600–620.
[69] Paouris, G. & Werner, E. M., Relative entropy of cone measures and L_p centroid bodies. *Proc. Lond. Math. Soc.*, 104 (2012), 253–286.
[70] Pogorelov, A. V., The existence of a closed convex hypersurface with a given function of the principal radii of a curve. *Dokl. Akad. Nauk SSSR*, 197 (1971), 526–528 (Russian); English translation in *Soviet Math. Dokl.*, 12 (1971), 494–497.
[71] — *The Minkowski Multidimensional Problem*. Winston & Sons, Washington, DC, 1978.
[72] Schneider, R., Das Christoffel-Problem für Polytope. *Geom. Dedicata*, 6 (1977), 81–85.
[73] — Curvature measures of convex bodies. *Ann. Mat. Pura Appl.*, 116 (1978), 101–134.
[74] — Bestimmung konvexer Körper durch Krümmungsmasse. *Comment. Math. Helv.*, 54 (1979), 42–60.
[75] — *Convex Bodies*: *the Brunn–Minkowski Theory*. Encyclopedia of Mathematics and its Applications, 151. Cambridge Univ. Press, Cambridge, 2014.

[76] Schuster, F. E., Crofton measures and Minkowski valuations. *Duke Math. J.*, 154 (2010), 1–30.
[77] Schuster, F. E. & Wannerer, T., Even Minkowski valuations. *Amer. J. Math.*, 137 (2015), 1651–1683.
[78] Trudinger, N. S. & Wang, X.-J., The Monge–Ampère equation and its geometric applications, in *Handbook of Geometric Analysis*, Adv. Lect. Math., 7, pp. 467–524. Int. Press, Somerville, MA, 2008.
[79] Zhang, G., The affine Sobolev inequality. *J. Differential Geom.*, 53 (1999), 183–202.
[80] — Dual kinematic formulas. *Trans. Amer. Math. Soc.*, 351 (1999), 985–995.
[81] — A positive solution to the Busemann–Petty problem in $\mathbf{R}^4$. *Ann. of Math.*, 149 (1999), 535–543.
[82] Zhu, G., The logarithmic Minkowski problem for polytopes. *Adv. Math.*, 262 (2014), 909–931.
[83] — The centro-affine Minkowski problem for polytopes. *J. Differential Geom.*, 101 (2015), 159–174.
[84] — The L_p Minkowski problem for polytopes for $0<p<1$. *J. Funct. Anal.*, 269 (2015), 1070–1094.

Yong Huang
Institute of Mathematics
Hunan University
Changsha 410082
China
huangyong@hnu.edu.cn

Erwin Lutwak
Department of Mathematics
New York University
Tandon School of Engineering
6 Metrotech Center
Brooklyn, NY 11201
U.S.A.
lutwak@nyu.edu

Deane Yang
Department of Mathematics
New York University
Tandon School of Engineering
6 Metrotech Center
Brooklyn, NY 11201
U.S.A.
deane.yang@nyu.edu

Gaoyong Zhang
Department of Mathematics
New York University
Tandon School of Engineering
6 Metrotech Center
Brooklyn, NY 11201
U.S.A.
gaoyong.zhang@nyu.edu

Received June 10, 2015
Received in revised form June 10, 2016